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Positivity and LTI systems

Omar Fazry

Supervisors: Sahiba Arora & Felix Schwenninger

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Department of Applied Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Science

UNIVERSITY OF TWENTE.

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Abstract

This paper investigates the external and internal positivity of standard linear systems, in which the input function is restricted to be positive. By using cone-invariant matrix operators, we find parametric conditions on a system such that its state is contained within a proper convex cone. Then, by use of the dual of said proper cone, we first establish sufficient conditions for external positivity. From this result, we find conditions for internal positivity based on the external positivity of an analog system. Lastly, we investigate internal positivity independently of its external counterpart.

Keywords: LTI systems, external positivity, internal positivity, second-order cones, polynomial cones, cone-invariant.

1 Introduction

In many fields of science, models are given by input dependent differential equations, some of which describe and manipulate physical quantities that are positive by nature [7, 1, 6, 4, 3, 2]. One can think of quantities such as concentrations, temperatures expressed in Kelvin, masses, densities, volumes, areas, and population amounts.

Example 1.1 (Lotka-Volterra model). A predator-prey model [2] which describes, at time $t \geq 0$, the interaction of two populations $N(t)$ and $P(t)$ in a sealed ecosystem:

$$\begin{cases} \dfrac{dN}{dt} = rN(t) - aN(t)P(t), \quad N(0) \geq 0, \\ \dfrac{dP}{dt} = bN(t)P(t) - sP(t), \quad P(0) \geq 0, \end{cases}$$

where r is the prey's growth rate, s the predator's mortality rate, a the predation efficiency, and b the conversion efficiency of predation into offspring. □

Because of this positivity constraint in the modelization of this family of problems, one could find useful to have a way, let it be numerical or theoretical, to verify that a proposed model generates results that are positivity preserving during the simulation or conceptualization process.

In this paper, we will constrain our investigation to a specific type of system. Namely, the ones with continuous time-axis $\mathbb{R}_{\geq 0}$ and for which we can find an equivalent behaviour of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^{n_x}, \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1.1}$$

with input $u \in \mathbb{R}^{n_u}$, state $x \in \mathbb{R}^{n_x}$, output $y \in \mathbb{R}^{n_y}$, and real matrices A, B, C, D with dimensions $n_x \times n_x$, $n_x \times n_u$, $n_y \times n_x$, and $n_y \times n_u$ respectively. Then, by defining vectors to be positive when element-wise positive, we will investigate the **external** positivity [7, 5] of said systems, which refers to the positivity of the output when provided a positive input for $t \geq 0$. Similarly, we will also take a look at the positivity of the state when provided a positive input for $t \geq 0$, which we will refer to as **internal** positivity [7, 5].

To do so we will rely the notion of cone invariant operators [7, 9, 11, 10], which, although quickly noted as important by Luenberger [7, 8], has only recently gained notable attention [7, 1, 5]. That is, provided that the input is positive for $t \geq 0$, we will look for conditions on matrices A, B, C and D such that we can find a **proper** convex cone [7, 9, 11, 10] that contains the state for $t \geq 0$ [7, 9]. Then, we will show that if the transpose of the rows of matrix C are in the dual of that same cone, then external positivity is achieved [7]. We will also show internal positivity is attained if the analog system with $C = I$ and $D = 0$ is externally positive, and, additionally, to not be constrained by this initial condition $x_0 = 0$, we will investigate results on internal positivity that are independent of its external counterpart.

2 Convex cones

The goal of this section is to introduce to the reader the notion of convex cones and their proper variant. Not only that, but we will also define the dual of said cones, as well as investigating conditions on proper convex cones for them to be invariant under certain matrix operators. Note that this section is mainly written as an attempt to group and harmonize the various concepts and notions formulated in [7, 11, 10].

For the mathematical context, We will be working in real vector spaces represented as $\big(\mathbb{R}^{n_x}, +, \cdot\big)$, with scalar field $\mathbb{R}$ and where for $w, z \in \mathbb{R}^{n_x}$ and $\alpha \in \mathbb{R}$ we have

$$w + z = \begin{pmatrix} w_1 +_{\mathbb{R}} z_1 \\ \vdots \\ w_{n_x} +_{\mathbb{R}} z_{n_x} \end{pmatrix}, \qquad \alpha \cdot z = \begin{pmatrix} \alpha \cdot_{\mathbb{R}} z_1 \\ \vdots \\ \alpha \cdot_{\mathbb{R}} z_{n_x} \end{pmatrix}.$$

Furthermore, we equip those vector spaces with the 2-norm and the resulting euclidean topology $\mathcal{O}$. That is, $S \subset \mathbb{R}^{n_x}$ is an element of $\mathcal{O}$, and is thus an open subset of $\mathbb{R}^{n_x}$, if and only if for all points $z \in S$ there is a radius $r > 0$ such that

$$B_r(z) = \big\{ w \in \mathbb{R}^{n_x} \mid \|z - w\|_2 < r \big\} \subset S.$$

Lastly, for convenience, we will sometimes denote the positive quadrant of $\mathbb{R}^{n_x}$ as $\mathbb{R}^{n_x}_{\geq 0}$.

2.1 Proper convex cones

Proper convex cones will be at the center of our approach to both internal and external positivity of standard linear systems. Thus, we start by defining convex cones.

> **Definition 2.1.1 (Convex cones).** A set $\mathcal{K} \subset \mathbb{R}^{n_x}$ is said to be a convex cone if for any elements $x, z \in \mathcal{K}$ and scalars $\alpha, \beta \in \mathbb{R}_{\geq 0}$ it follows that $\alpha x + \beta z \in \mathcal{K}$.

Naturally, there are many different types of convex cones one can construct from this definition. We provide a few examples and show that they indeed satisfy the definition:

Example 2.1.2 (Convex cones)

1. The set $\mathbb{R}^{n_x}$ itself, as any linear combination of elements in $\mathbb{R}^{n_x}$ is again an element of $\mathbb{R}^{n_x}$, which then also includes positive linear combinations.

2. The positive span of finitely many vectors in $\mathbb{R}^{n_x}$,

$$\mathcal{K} = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R}_{\geq 0}\},$$

where $v_1, \ldots, v_k \in \mathbb{R}^{n_x}$. To show that this construct indeed satisfies Definition 2.1.1, let us pick $x, z \in \mathcal{K}$ arbitrarily. Then, we can find coefficients $\beta_1, \gamma_1, \ldots, \beta_k, \gamma_k \in \mathbb{R}_{\geq 0}$ such that $x = \beta_1 v_1 + \cdots + \beta_k v_k$ and $z = \gamma_1 v_1 + \cdots + \gamma_k v_k$. Now, for $\lambda, \mu \in \mathbb{R}_{\geq 0}$ we know that $\lambda\beta_i + \mu\gamma_i \in \mathbb{R}_{\geq 0}$ for all indices $i \in \{1, \ldots, k\}$, and, therefore, we see that

$$\lambda x + \mu z = (\lambda\beta_1 + \mu\gamma_1)v_1 + \cdots + (\lambda\beta_k + \mu\gamma_k)v_k \in \mathcal{K}.$$

3. The positive quadrant of $\mathbb{R}^{n_x}$. Indeed, for indices $i, j \in \{1, \ldots, n_x\}$ we define $e_i \in \mathbb{R}^{n_x}$ such that $(e_i)_j = 1$ for $j = i$ and $(e_i)_j = 0$ otherwise. We then note that the positive quadrant of $\mathbb{R}^{n_x}$ is the positive span of those unit vectors.

4. The n_x-dimensional ice cream cone [11], which we define as

$$\mathcal{K} = \Big\{ z \in \mathbb{R}^{n_x} \Big| \sum_{i=1}^{n_x - 1} z_i^2 \leq z_{n_x}^2 \ , \ z_{n_x} \geq 0 \Big\}.$$

To show that this is indeed a convex cone, we consider the following equivalent representation:

$$\tilde{\mathcal{K}} = \Big\{ (z, w) \in \mathbb{R}^{n_x - 1} \times \mathbb{R}_{\geq 0} \ \Big| \ \|z\|_2 \leq w \Big\}.$$

Then, we arbitrarily pick $(z_1, w_1), (z_2, w_2) \in \tilde{\mathcal{K}}$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$. From there, we define

$$y = \lambda(z_1, w_1) + \mu(z_2, w_2) = (\lambda z_1 + \mu z_2, \lambda w_1 + \mu w_2),$$

where we can immediately see that $\lambda w_1 + \mu w_2 \in \mathbb{R}_{\geq 0}$. Furthermore, by the triangle inequality on the 2-norm, we also have

$$\|\lambda z_1 + \mu z_2\|_2 \leq \lambda \|z_1\|_2 + \mu \|z_2\|_2 \leq \lambda w_1 + \mu w_2,$$

and, therefore, it follows that $y \in \tilde{\mathcal{K}}$. □

Now, for the family of systems mentioned in the introduction, our primary goal with convex cones is to contain the state x produced by a positive input u. Then, we can focus on **proper** convex cones.

Definition 2.1.3 (Proper convex cones). Let $\mathcal{K} \subset \mathbb{R}^{n_x}$ be a convex cone. If $\mathcal{K}$ satisfies all of the following conditions:

1. $\mathcal{K}$ is a closed subset of $\mathbb{R}^{n_x}$,

2. $\mathcal{K}$ is pointed, i.e. $\mathcal{K} \cap -\mathcal{K} = \{0\}$, and

3. $\mathcal{K}$ is solid, i.e. $\mathcal{K}$ has a non-empty interior,

then $\mathcal{K}$ is said to be a proper convex cone.

The reason of this choice lies in the many advantages the conditions of Definition 2.1.3 bring. Indeed, as we will see later, computing the state will require taking the limit of a convergent sequence whose terms all belong to a convex cone $\mathcal{K}$. By making this cone closed, we ensure that the limiting value is also an element of $\mathcal{K}$. Furthermore, solidity makes the confinement for a given system more tractable. That is, by the euclidean topology, a solid cone has an interior that exploits locally all the dimensions of its space.

For $n_x \geq 3$, there are exactly two types of proper convex cones one can encounter. First, we have the **second-order** case, which is defined using quadratic inequalities and hyperplane separations.

Definition 2.1.4 (Second order cone). A proper convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ is said to be of second-order type if and only if there exist a symmetric matrix $K \in \mathbb{R}^{n_x \times n_x}$ and vector $c \in \mathbb{R}^{n_x}$ such that

$$\mathcal{K} = \Big\{ z \in \mathbb{R}^{n_x} \mid z^T K z \leq 0 \ , \ c^T z \geq 0 \Big\}, \tag{2.1}$$

which we alternatively also denote as $\mathcal{K}_{K,c}$.

An example of such construct would be the n_x-dimensional ice cream cone that we've investigated in Example 2.1.1. We now show that it is of second-order type and proper:

Example 2.1.5 (Second-order proper convex cone) Consider the convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ defined as

$$\mathcal{K} = \Big\{ z \in \mathbb{R}^{n_x} \Big| \sum_{i=1}^{n_x - 1} z_i^2 \leq z_{n_x}^2 \ , \ z_{n_x} \geq 0 \Big\}. \tag{2.2}$$

Then, we define the matrix $Q \in \mathbb{R}^{n_x \times n_x}$ such that for indices $i, j \in \{1, \dots, n_x\}$ we have the following:

$$Q_{i,j} = \begin{cases} 1 & i = j \leq n_x - 1, \\ -1 & i = j = n_x, \\ 0 & i \neq j. \end{cases}$$

We can immediately notice that Q is symmetric, and, furthermore, we can now rewrite our original convex cone as

$$\mathcal{K} = \Big\{ z \in \mathbb{R}^{n_x} \mid z^T Q z \leq 0 \ , \ (0 \ \dots \ 0 \, 1) z \geq 0 \Big\}, \tag{2.3}$$

which confirms that $\mathcal{K}$ is a convex cone of second-order type. Now, since the inequalities defining $\mathcal{K}$ in (2.3) are not strict, it follows that $\mathcal{K}$ is a closed subset of $\mathbb{R}^{n_x}$. To investigate the solidity of $\mathcal{K}$, we consider the point $q = (0 \ \dots \ 0 \, 1)^T \in \mathcal{K}$. If we define the perturbation $\epsilon \in \mathbb{R}^{n_x}_{\geq 0}$ such that we have at least one $\epsilon_i > 0$ for some index $i \in \{1, \dots, n_x\}$, then we observe that $q + \epsilon \in \mathcal{K}$ if

$$(\epsilon_1)^2 + \dots + (\epsilon_{n_x - 1})^2 \leq (1 + \epsilon_{n_x})^2. \tag{2.4}$$

Hence, no matter how much we initially perturbate the first $n_x - 1$ entries of q, we can take ϵ_{n_x} to be large enough such that (2.4) holds. If, instead, we initially perturbate the last entry of q, then $\epsilon_1, \dots, \epsilon_{n_x - 1}$ can be taken to be small enough such that, again, (2.4) holds. Thus, for any direction along which we move away from q, we can always find a step size small enough such that we remain within $\mathcal{K}$, and, therefore, we can find a $r > 0$ such that the open ball of radius r centered at q is a subset of $\mathcal{K}$. Because of this, q is in the interior of $\mathcal{K}$, which implies that the interior is non-empty, and, therefore, that $\mathcal{K}$ is solid. Lastly, to investigate pointedness, we must take a look at the set $\mathcal{K} \cap -\mathcal{K}$, where

$$-\mathcal{K} = \Big\{ z \in \mathbb{R}^{n_x} \ \Big|\ z^T Q z \leq 0 \ ,\ (0 \ \dots \ 0\, 1) z \leq 0 \Big\}.$$

Then, $z \in \mathcal{K} \cap -\mathcal{K}$ if $(0 \ \dots \ 0\, 1) z = 0$ and $z^T Q z \leq 0$. This indicates that $z_{n_x} = 0$, which implies that $z_1^2 + \cdots + z_{n_x - 1}^2 = 0$. Hence, $z_i = 0$ for $i \in \{1, \dots, n_x\}$, confirming that $\mathcal{K}$ is pointed. □

Some might notice that verifying if a second order cone is proper can quickly become cumbersome when relying solely on Definitions 2.1.3 and 2.1.4. Since those cones will be slightly preferred when working on the positivity of systems, we make this whole process simpler by using the following observation from Section 2.2 of [7].

Theorem 2.1.6 (Proper second order cones). Let $\mathcal{K}_{K,c} \subset \mathbb{R}^{n_x}$ be a second-order convex cone. Then, $\mathcal{K}_{K,c}$ is proper if K has exactly $n_x - 1$ strictly positive eigenvalues, and exactly one strictly negative eigenvalue.

Note that for the previously stated observation, [7] also makes mention of c being separating. That is,

$$\big\{ z \in \mathbb{R}^{n_x} \mid c^T z = 0 \big\} \cap \big\{ z \in \mathbb{R}^{n_x} \ \big|\ z^T K z \leq 0 \big\} = \{0\}.$$

However, in our case, this condition is redundant, as we preemptively require the set $\mathcal{K}_{K,c}$ to be a convex cone.

Now, alternatively to the second-order case, there is the **polynomial** proper convex cone, which is constructed by taking the positive span of finitely many vectors.

Definition 2.1.7 (Polynomial proper convex cone). A proper convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ is said to be of polynomial type if and only if there exists a matrix $N \in \mathbb{R}^{n_x \times l}$ such that

$$\mathcal{K} = \Big\{ N w \in \mathbb{R}^{n_x} \ \Big|\ w \in \mathbb{R}^{l}_{\geq 0} \Big\}, \tag{2.5}$$

which we alternatively also denote as $\mathcal{K}_N$.

The only instance of proper polynomial cone we will explicitly use during our investigation of positivity will be the positive quadrant of $\mathbb{R}^{n_x}$. Since we've already shown in Example 2.1.2 that it is a convex cone, let us now prove that it is of polynomial type and proper.

Example 2.1.8 (polynomial proper convex cone) First, for indices $i, j \in \{1, \dots, n_x\}$ let us define the unit vectors $e_i \in \mathbb{R}^{n_x}$ such that $(e_i)_j = 1$ for $j = i$ and $(e_i)_j = 0$ otherwise. Then, we notice that $\mathbb{R}^{n_x}_{\geq 0}$ is the positive span of those vectors. Furthermore, we can write

$$\mathbb{R}^{n_x}_{\geq 0} = \Big\{ z \in \mathbb{R}^{n_x} \ \Big|\ z \in \mathbb{R}^{n_x}_{\geq 0} \Big\} = \Big\{ I z \in \mathbb{R}^{n_x} \ \Big|\ z \in \mathbb{R}^{n_x}_{\geq 0} \Big\}, \tag{2.6}$$

and, thus, it follows that $\mathbb{R}^{n_x}_{\geq 0}$ is indeed of polynomial type. Now, we immediately observe that $\mathbb{R}^{n_x}_{\geq 0}$ is a closed subset of $\mathbb{R}^{n_x}$. Additionally, to show that this convex cone is solid, take $q = (1 \ \dots \ 1)^T \in \mathbb{R}^{n_x}_{\geq 0}$. Then, we note that the open ball of radius $1/2$ centered at q is within $\mathbb{R}^{n_x}_{\geq 0}$, as any of it elements has strictly positive entries. This implies that q is in the interior of our cone, which is then non-empty. To investigate pointedness, consider the following equation:

$$\alpha_1 e_1 + \cdots + \alpha_{n_x} e_{n_x} = (-\alpha_1)e_1 + \cdots + (-\alpha_{n_x})e_{n_x}, \tag{2.7}$$

where $\alpha_1, \dots, \alpha_{n_x} \in \mathbb{R}_{\geq 0}$. By linear independence of those unit vectors, it follows that Equation 2.7 holds only if $\alpha_i = -\alpha_i = 0$ for $i \in \{1, \dots, n_x\}$. Thus, the only element of $\mathbb{R}^{n_x}_{\geq 0} \cap -\mathbb{R}^{n_x}_{\geq 0}$, where

$$-\mathbb{R}^{n_x}_{\geq 0} = \Big\{(-1)z \in \mathbb{R}^{n_x} \;\Big|\; z \in \mathbb{R}^{n_x}_{\geq 0}\Big\},$$

is the zero vector. □

2.2 Cone invariance

The notion of cone-invariance is one that involves a cone and some matrix operator. It is a form of set invariance which wants elements of the cone to still be within the cone after the said matrix operator is applied.

Definition 2.2.1 (Cone-invariance). Let $A \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{K} \subset \mathbb{R}^{n_x}$ be a convex cone. Then $\mathcal{K}$ is said to be A-invariant if for any element $z \in \mathcal{K}$ we have that $Az \in \mathcal{K}$.

We can then immediately extend this definition to a form that will be the cornerstone of our approach to positivity.

Definition 2.2.2 (Exponential cone-invariance). Let $A \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{K} \subset \mathbb{R}^{n_x}$ be a convex cone. Then $\mathcal{K}$ is said to be exponentially A-invariant if for any element $z \in \mathcal{K}$ and $t \geq 0$ we have that $e^{At}z \in \mathcal{K}$.

Before we formulate results that enable to verify that Definition 2.2.2 is satisfied for a proper cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ and a matrix $A \in \mathbb{R}^{n_x \times n_x}$, we introduce the concept of the **dual** of a convex cone.

Definition 2.2.3 (Dual of a cone). Let $\mathcal{K} \subset \mathbb{R}^{n_x}$ be a convex cone. Then the set

$$\mathcal{K}^* = \Big\{y \in \mathbb{R}^{n_x} \;\Big|\; y^T z \geq 0 \quad \forall z \in \mathcal{K}\Big\}.$$

is called the dual of $\mathcal{K}$.

An interesting example of such construct would be the dual of the previously examined n_x-dimensional ice cream cone for $n_x = 2$.

Example 2.2.4 (polynomial proper convex cone) Consider the two dimensional ice cream cone

$$\mathcal{K} = \left\{ z \in \mathbb{R}^2 \,|\, z_1^2 \leq z_2^2,\ z_2 \geq 0 \right\} = \left\{ z \in \mathbb{R}^2 \,|\, z_1 = \alpha - \beta,\ z_2 = \alpha + \beta \quad \text{for } \alpha, \beta \in \mathbb{R}_{\geq 0} \right\}.$$

Then, for $y \in \mathbb{R}^2$ and $z \in \mathcal{K}$ we note the following:

$$y^T z = y_1 z_1 + y_2 z_2 = \|y\|_2 \, \|z\|_2 \cos\big(\theta(y, z)\big), \tag{2.8}$$

where $\theta : \mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2 \backslash \{0\} \to (-\pi, \pi]$ returns the angle formed by two vectors in $\mathbb{R}^2 \backslash \{0\}$. We can now note that, for both y and z non-zero, $y^T z \geq 0$ if and only $\theta(y, z) \in [-\pi/2, \pi/2]$, and, therefore, $y^T z \geq 0$ for all $z \in \mathcal{K} \backslash \{0\}$ if and only if $y \in \mathcal{K}$. Hence, $\mathcal{K}^* = \mathcal{K}$. □

Now, as mentioned in the previous subsection, we will later have a preference for second-order cones. Therefore, the first result we formulate enables us to verify that an element y of $\mathbb{R}^{n_x}$ is also an element of the dual of a given proper second-order cone $\mathcal{K}$.

Theorem 2.2.5 (Dual cones). Let $\mathcal{K}_{K,c} \subset \mathbb{R}^{n_x}$ be a proper second-order convex cone. If for some $y \in \mathbb{R}^{n_x}$ there exists a $\zeta \in \mathbb{R}$ such that $K + \zeta y y^T$ is positive definite, then $y \in \mathcal{K}^*$.

Proof. Let $\mathcal{K}_{K,c} \subset \mathbb{R}^{n_x}$ be a proper second-order cone. Then, we know, from Appendix A of [7], that $y \in \mathbb{R}^{n_x}$ is in the interior of $\mathcal{K}^*_{K,c}$, and therefore in $\mathcal{K}^*_{K,c}$, if there exists a $\zeta \in \mathbb{R}$ such that for all $z \in \mathcal{K}_{K,c} \backslash \{0\}$ we have

$$z^T K z + \zeta z^T y y^T z = z^T \big(K + \zeta c c^T\big) z > 0.$$

Again from Appendix A of [7], this is the same as requiring the matrix $K + \zeta y y^T$ to be positive definite for some $\zeta \in \mathbb{R}$. Thus concluding the proof. ■

From Lemma 2.4 of [7] and Theorem 3.5 of [11], we obtain our main result on exponential invariance for proper second-order cones.

Theorem 2.2.6 (Exponential invariance of proper second-order cones). Let $\mathcal{K}_{K,c} \subset \mathbb{R}^{n_x}$ be a proper second-order convex cone and $A \in \mathbb{R}^{n_x \times n_x}$. If there exist $\xi, \zeta \in \mathbb{R}$ such that

1. $A^T K + K A + 2\xi K$ is negative semi-definite,
2. and $K + \zeta c c^T$ is positive definite,

then $\mathcal{K}$ is exponentially A-invariant.

Furthermore, [9] and (2.7) of [7] offer us a way to verify exponential invariance for the proper polynomial cone $\mathbb{R}^{n_x}_{\geq 0}$.

Theorem 2.2.7 (Exponential invariance of $\mathbb{R}^{n_x}_{\geq 0}$). Let $A \in \mathbb{R}^{n_x \times n_x}$. If there exists a $\lambda \in \mathbb{R}$ such that $A + \lambda I$ is element-wise positive, then $\mathbb{R}^{n_x}_{\geq 0}$ is exponentially A-invariant.

Proof. From Equation (2.7) of [7], we know, for a matrix $A \in \mathbb{R}^{n_x \times n_x}$, that a proper polynomial cone $\mathcal{K}_N \subset \mathbb{R}^{n_x}$ with $N \in \mathbb{R}^{n_x \times l}$ is exponentially A-invariant if there exist a $\lambda \in \mathbb{R}$ and an element-wise positive matrix $P \in \mathbb{R}^{l \times l}$ such that $(A + \lambda I)N = NP$. For the proper polynomial cone $\mathbb{R}^{n_x}_{\geq 0}$ we find $N = I$, concluding the proof. ■

3 Positivity

Recall that, we will only consider systems with time axis $\mathbb{R}_{\geq 0}$, and for which we can find an equivalent behaviour of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^{n_x}, \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3.1}$$

with input $u \in \mathbb{R}^{n_u}$, state $x \in \mathbb{R}^{n_x}$, output $y \in \mathbb{R}^{n_y}$, and real matrices A, B, C, D with dimensions $n_x \times n_x$, $n_x \times n_u$, $n_y \times n_x$, and $n_y \times n_u$ respectively. Then, for convenience, such systems will exclusively be referred to using the tuple (A, B, C, D). Furthermore, for any matrix $M \in \mathbb{R}^{n \times k}$ and index $i \in \{1, \dots, n\}$, we denote by $M_{[i]}$ the vector that contains the elements of the i^{th} row of M. Then, similarly, for index $j \in \{1, \dots, k\}$ we define the vector $M_{(j)}$ to contain the elements of the j^{th} column of M. Lastly, a vector or matrix will be called positive if it is element-wise positive.

Definition 3.1 (Positive matrix). A matrix $M \in \mathbb{R}^{n \times k}$ is said to be positive, which is denoted as $M \geq 0$, if and only if $M_{ij} \geq 0$ for all indices $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$.

Now, recall that we previously mentioned the idea of containing the state of a system (A, B, C, D) within a proper cone $\mathcal{K}$. We expand on this by using cone-invariance.

Theorem 3.2 (Cone confinement). Let $\mathcal{K} \subset \mathbb{R}^{n_x}$ be a proper convex cone, and consider a system (A, B, C, D). If $x_0 \in \mathcal{K}$ and the following conditions are satisfied:

1. the columns of B are in $\mathcal{K}$,
2. and $\mathcal{K}$ is exponentially A-invariant,

then $x(t) \in \mathcal{K}$ for $t \geq 0$ and $u(t) \geq 0$.

Proof. Let (A, B, C, D) and $\mathcal{K}$ be such that the aforementioned conditions are satisfied. First, we can write the product of B and u as a linear combination of the columns of B, with the coefficients being the corresponding elements of u. But, provided that the input u is positive on $\mathbb{R}_{\geq 0}$, it follows, by the definition of convex cones, that $Bu(t)$ is an element of $\mathcal{K}$ for $t \geq 0$. Furthermore, we know that $\mathcal{K}$ is exponentially A-invariant, which implies that $e^{A(t-\tau)}Bu(\tau) \in \mathcal{K}$ and $e^{At}x_0 \in \mathcal{K}$ for all $t \geq 0$ and $\tau \in [0, t]$. Now, recall that the state of system (A, B, C, D) can be explicitly expressed as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau. \tag{3.2}$$

Then, by expanding the integral term as the limit of a Riemann sum, we can make use of the definition of convex cones and the fact that $\mathcal{K}$ is closed to conclude that $x(t) \in \mathcal{K}$ for all $t \geq 0$ and $u(t) \geq 0$. ■

Now that we have found parametric conditions for the containment of the state within a cone, we can start investigating the notion of **external** positivity.

Definition 3.3 (External positivity). A system (A, B, C, D) is said to be externally positive if for $x_0 = 0$ and $t \geq 0$ we have that $u(t) \geq 0$ implies that the output $y(t) \geq 0$.

Note that this restriction on x_0 is an assumption we take from Definition 2.8 of [7]. Then, by using the confinement of the state within a proper cone, together with the notion of dual cones, we are able to formulate the following.

Theorem 3.4 (External positivity). A system (A, B, C, D) with $x_0 = 0$ is externally positive if D is positive and if there exists a proper convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ that satisfies the following conditions:

1. $\mathcal{K}$ is exponentially A-invariant.
2. The columns of B are in $\mathcal{K}$.
3. The transpose of the rows of C are in $\mathcal{K}^*$.

Proof. Let (A, B, C, D) and $\mathcal{K}$ be such that the aforementioned conditions are satisfied, we also let $u(t)$ be an input positive for $t \geq 0$. Since $\mathcal{K}$ is proper, contains the columns of B and is exponentially A-invariant, it follows, using Theorem 3.2, that $x(t) \in \mathcal{K}$ for $t \geq 0$. We then note that the transpose of the rows of C being in $\mathcal{K}^*$ implies that

$$0 \leq \left(C_{[i]}^T\right)^T x(t) = C_{[i]} x(t) = \left(Cx(t)\right)_i$$

for $t \geq 0$ and any index $i \in \{1, \dots, n_y\}$. We can therefore write that $Cx(t) \geq 0$ for $t \geq 0$. Lastly, D being positive implies that $Du(t)$ also is positive for $t \geq 0$, and thus $y(t) = Cx(t) + Du(t) \geq 0$ for $t \geq 0$. ■

Note that this result is formulated in a way that makes it independent of the type of proper cone one must find, which can be either second-order or polynomial. However, for certain matrices, exponential invariance is achieved exclusively by proper second-order cones. Because of this observation, we will use the many results of Section 2 to reformulate the conditions of Theorem 3.4 in a way that addresses this limitation.

Theorem 3.5 (External positivity). A system (A, B, C, D) with $x_0 = 0$ is externally positive if there exists a symmetric matrix $K \in \mathbb{R}^{n_x \times n_x}$ such that the following conditions are satisfied:

1. $B_{(j)}^T K B_{(j)} \le 0$ for all $j \in \{1, \ldots, n_u\}$,
2. $\exists \zeta_i \in \mathbb{R} : K + \zeta_i C_{[i]}^T C_{[i]}$ is positive definite for all $i \in \{1, \ldots, n_y\}$,
3. $\exists \xi \in \mathbb{R} : A^T K + KA + 2\xi K$ is negative semi-definite,
4. K has exactly $n_x - 1$ strictly positive eigenvalues and exactly one strictly negative eigenvalue,
5. $D \ge 0$ and $CB \ge 0$.

Proof. Let (A, B, C, D) and K be such that the aforementioned conditions are satisfied. First, we notice, for any indices $i \in \{1, \ldots, n_y\}$ and $j \in \{1, \ldots, n_u\}$, that the positivity of CB yields the following:

$$(CB)_{ij} = C_{[i]} B_{(j)} = \left(C_{[i]}^T\right)^T B_{(j)} \ge 0.$$

We now, again for any indices $i \in \{1, \ldots, n_y\}$ and $j \in \{1, \ldots, n_u\}$, combine this observation with the the fact that $B_{(j)}^T K B_{(j)} \le 0$, yielding via the symmetry of K that

$$B_{(j)} \in \mathcal{K}_{K, C_{[i]}^T} = \left\{ z \in \mathbb{R}^{n_x} \mid z^T K z \le 0 \ , \ C_{[i]}^T z \ge 0 \right\}.$$

By the disposition of the eigenvalues of K, those second-order cones constructed on K and $C_{[i]}^T$ can be verified to be proper using Theorem 2.1.6. Furthermore, the existence of scalars ξ and ζ_i such that $A^T K + KA + 2\xi K$ is negative semi-definite, and $K + \zeta_i C_{[i]}^T C_{[i]}$ is positive definite for any $i \in \{1, \ldots, n_y\}$, tells us, via Theorem 2.2.6, that these cones are all exponentially A-invariant. Now, for any $i, k \in \{1, \ldots, n_y\}$, there exist a $\zeta_i > 0$ such that $K + \zeta_i C_{[i]}^T C_{[i]}$ is positive definite, then by Theorem 2.2.5, it follows that $C_{[i]}^T \in \mathcal{K}^*_{K, C_{[k]}^T}$. Hence, for any $i \in \{1, \ldots, n_y\}$ and proper convex cone $\mathcal{K}_{K, C_{[i]}^T}$, the conditions of Theorem 3.4 are satisfied for our system, which is thus is externally positive. ■

Example 3.6 (External positivity) Consider the following system defined for $t \ge 0$:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} u(t), \quad x(0) = 0, \\ y(t) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 11 \\ 3 \end{pmatrix} u(t), \end{cases}$$

Now, to investigate whether it is externally positive or not, we will go over every conditions stated in Theorem 3.4. First, consider the symmetric matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whose diagonality indicates that it has eigenvalues 1 and -1. Now, we note that

$$A^T K + KA + 2\xi K = \begin{pmatrix} 2\xi & -1 \\ -1 & -2\xi \end{pmatrix}$$

is negative semi-definite for $\xi = 0$. Furthermore, for $\zeta_1 = \zeta_2 = 1$ we observe that the following matrices:

$$K + \zeta_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}, \quad K + \zeta_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 8 \end{pmatrix},$$

are positive definite. Finally, $CB \geq 0$, $C \geq 0$ and

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = (-8) < 0, \quad \begin{pmatrix} 2 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = (-45) < 0.$$

Therefore, by Theorem 3.5, it follows that the system is externally positive. □

Now that sufficient conditions to verify external positivity have been derived, we can also take a look at the notion of **internal** positivity.

Definition 3.7 (Internal positivity). A system (A, B, C, D) is said to be internally positive if for $t \geq 0$ we have that $u(t) \geq 0$ implies that the state $x(t) \geq 0$.

Then, for a system (A, B, C, D), one way to verify such property would be to apply Theorem 3.4 to the system $(A, B, I, 0)$.

Corollary 3.8 (Internal positivity). A system (A, B, C, D) with $x_0 = 0$ is internally positive if there exists a proper convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}$ that satisfies the following conditions:

1. $\mathcal{K}$ is exponentially A-invariant,
2. the columns of B are in $\mathcal{K}$, and
3. $\mathcal{K}^*$ contains the positive quadrant of $\mathbb{R}^{n_x}$.

Proof. Let (A, B, C, D) and $\mathcal{K}$ be such that the aforementioned conditions are satisfied. Then, we immediately notice that the transpose of the rows of I are in the positive quadrant of $\mathbb{R}^{n_x}$, and, therefore, in $\mathcal{K}^*$. Therefore, it follows by Theorem 3.4 that the system $(A, B, I, 0)$ is externally positive. Additionally, we note that the output of $(A, B, I, 0)$ is the state of (A, B, C, D). Hence, (A, B, C, D) with $x_0 = 0$ is internally positive. ■

You might notice that Corollary 3.8 is, similarly to Theorem 3.4, formulated such that the type of cone one must find matters not. However, for the investigation of internal positivity we will, surprisingly, prefer to use proper polynomial cones. Additionally, Corollary 3.8 has the default of being constructed on a result that investigates external positivity, whose definition limit x_0 to necessarily be null. However, from Theorem 3.2, we note that it might be possible to formulate internal positivity for any $x_0 \geq 0$. As a stepping stone for this new approach, consider the following corollary.

Corollary 3.9 (Internal positivity). A system (A, B, C, D) is internally positive if there exists a proper convex cone $\mathcal{K} \subset \mathbb{R}^{n_x}_{\geq 0}$ that is exponentially A-invariant and contains the columns of B and x_0.

Proof. Let (A, B, C, D) and $\mathcal{K}$ be such that the aforementioned conditions are satisfied.

Then, by Theorem 3.2, we know that $x(t) \in \mathcal{K}$ for $t \geq 0$ and $u(t) \geq 0$. Lastly, since $\mathcal{K}$ is in the positive quadrant of $\mathbb{R}^{n_x}$, it follows that $x(t) \geq 0$ for $t \geq 0$ and $u(t) \geq 0$. ■

We can then use Corollary 3.9 to produce a result that is less restrictive and in practice much easier to use than Corollary 3.8. This is done by preemptively picking the cone we must find to be the positive quadrant of $\mathbb{R}^{n_x}$.

Theorem 3.10 (Internal positivity). A system (A, B, C, D) with $x_0 \geq 0$ is internally positive if $B \geq 0$ and if there exists a $\lambda \geq 0$ such that $A + \lambda I \geq 0$.

Proof. Let (A, B, C, D) and λ be such that the aforementioned conditions are satisfied. Notice now that $\mathcal{K} = \mathbb{R}^{n_x}_{\geq 0}$ is a proper polynomial cone. Hence, for all indices $i \in \{1, \ldots, n_u\}$ we can already observe that $B_{(i)} \in \mathcal{K}$. Furthermore, by Theorem 2.2.7, the existence of λ tell us that $\mathcal{K}$ is exponentially A-invariant. Hence, by Theorem 3.2, for all positive inputs we know that $x(t) \in \mathbb{R}^{n_x}_{\geq 0}$ for $t \geq 0$. ■

Example 3.11 (Internal positivity) For $t \geq 0$ consider the following system:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} -2 & 2 \\ 9 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 3 \end{pmatrix} u(t), \quad x(0) = 0, \\ y(t) = Cx(t), \end{cases}$$

where $C \in \mathbb{R}^{n_y \times 2}$. Then, we immediately note that

$$\begin{pmatrix} -2 & 2 \\ 9 & 1 \end{pmatrix} + 2I = \begin{pmatrix} 0 & 2 \\ 9 & 3 \end{pmatrix} \geq 0, \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} \geq 0.$$

Therefore, by Theorem 3.10, it follows that the system is internally positive. □

Note that, for an internally positive system (A, B, C, D), we can much more easily investigate the positivity of the output for any $x_0 \geq 0$ and positive input. In theory, this verification would only require to investigate the positivity of matrices C and D. However, even though many externally systems also are internally positive (see Example 3.13 for counter case), the construction of our approach puts an emphasis on studying those concepts separately. Nevertheless, for the sake of completeness, we also formulate the following.

Theorem 3.12 (Extended external positivity). Let (A, B, C, D) with $x_0 \geq 0$ be an internally positive system. If matrices C and D are positive, then for $t \geq 0$ and $u(t) \geq 0$ we have that the output $y(t) \geq 0$.

Proof. Let (A, B, C, D) be such that the aforementioned conditions are satisfied. Then, since the system is internally positive, it follows that for $t \geq 0$ and $u(t) \geq 0$ we have $x(t) \geq 0$. Furthermore, matrices C and D are positive, and, thus, for any index $i \in \{1, \ldots, n_y\}$ we can write

$$y(t)_i = \big(Cx(t) + Du(t)\big)_i = C_{[i]}x(t) + D_{[i]}u(t) = \sum_{j=1}^{n_x} \Big[C_{ij}x(t)_j\Big] + \sum_{j=1}^{n_u} \Big[D_{ij}u(t)_j\Big] \geq 0,$$

which then concludes the proof. ■

Example 3.13 (Externally positive but not internally positive). For $t \geq 0$ consider the following system

$$\begin{cases} \dot{x}(t) = -Iu(t), \quad x(0) = 0, \\ y(t) = -Ix(t), \end{cases}$$

with $I \in \mathbb{R}^{2\times 2}$. Then, we find the following expression for the state:

$$x(t) = \int_0^t -Iu(\tau)d\tau = -I\int_0^t u(\tau)d\tau.$$

Thus, for $t \geq 0$, the positive input $u(t) = (1 \;\; 1)^T$ produces the element-wise negative state $x(t) = (-t \;\; -t)^T$. Hence, it follows that our system is not internally positive. Now, we can also compute the output, yielding

$$y(t) = -Ix(t) = \int_0^t u(\tau)d\tau.$$

Therefore, any positive input for $t \geq 0$ provides a positive output, thus verifying external positivity. $\square$

4 Conclusion

We've taken a detour through convex analysis to define and illustrate proper convex cones and many notions related to them. From those, we determined conditions for the confinement of the state within a proper cone — a cone whose dual we then used to establish sufficient conditions for external positivity. By using the external positivity of an analog system, we then established conditions for internal positivity. Lastly, to liberate ourselves from the limitation on the initial condition of our system, we investigated internal positivity independently of its external counterpart.

For future research, we suggest investigating more tractable methods to find the proper convex cones required in the results on positivity we've established. Furthermore, we also suggest taking a closer look at the possible consequences positivity has on the reachability and observability of concerned systems.

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