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Geometric integration of stochastic Lotka-Volterra equations

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Abstract

In this thesis, Stratonovich noise terms are introduced into the Lotka-Volterra equations while still maintaining the Poisson property of the flow. Three stochastic Poisson models are proposed. The first stochastic model adds stochasticity to the interaction of the species, the second stochastic model adds stochasticity to the death and birth rates of the populations and the last proposed stochastic model adds stochasticity to the frequency of the periodic solutions. Additionally a stochastic Poisson integrator is proposed and its performance is tested against a non-geometric integrator.

Keywords: Lotka-Volterra equations, Stratonovich calculus, stochastic differential equations, geometric integrators, Hamiltonian systems

1 Introduction

The Lotka-Volterra equations (described in [15]) is a two-dimensional differential equation that captures the interplay between predators and prey in ecological settings. While this model is very basic and does not perfectly mirror real-world ecosystem behaviors due to the assumptions made, it remains an important instrument for ecologists and mathematicians exploring the interactions among species in nature. A very important feature of the Lotka-Volterra equations is their non-canonical Hamiltonian structure, or to be more specific, Poisson structure. In general, the Hamiltonian structure describes the evolution of the system in terms of a conserved quantity. This structure can be utilized to construct a class of numerical schemes called *geometric integrators*. A geometric integrator is a numerical method that is designed to preserve the underlying geometric properties of Hamiltonian systems, which results in a more stable result than non-geometric integrators. *Symplectic integrators* are a subclass of geometric integrators, specifically designed for canonical Hamiltonian systems. A two-dimensional canonical Hamiltonian system is a system which takes the form

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(x_1, x_2), \tag{1}$$

where $H(x_1, x_2)$ is the Hamiltonian. Symplectic integrators preserve the symplectic structure of the phase space which results in good long-time behaviour of the numerically integrated canonical Hamiltonian system. We say that a numerical integrator is a symplectic integrator if the following definition holds.

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Definition 1.1. *Let $\Omega : U \to \mathbb{R}^{2d}$ be a map, where U is an open set in $\mathbb{R}^{2d}$ and $2d$ is the dimension of the system. The timestep is defined by*

$$y_{n+1} = \Omega_{\Delta t}(y_n).$$

We say that Ω is a symplectic integrator if the condition

$$\left[\frac{\partial \Omega(y)}{\partial y}\right] J \left[\frac{\partial \Omega(y)}{\partial y}\right]^T = J$$

holds for all $y \in U$, where J is given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and I is the identity matrix of dimension d.

In reality we know that nature does not only consist of predators and prey. External factors might influence the populations of certain species. It makes sense to rewrite the Lotka-Volterra equations with a stochastic term which models uncertainties and external influences. This can be done by extending the Lotka-Volterra equations to a stochastic differential equation. A stochastic differential equations is a differiential equation where one or more terms is a stochastic process, which results in a solution which is also a stochastic process (see [16]). The stochastic extension of the Lotka-Volterra equations has been explored in several papers, for instance in [13], [10] and [14]. It is important to note that, in these papers the stochastic noise introduced to the Lotka-Volterra equations follows the principles of Itô calculus. The goal of this thesis is to add Stratonovich noise to the Lotka-Volterra equations in such a way that the stochastic flow is Poisson, and then investigate their solutions via numerical simulations using stochastic geometric integrators

In stochastic calculus, $\int_0^T h(t)\, dW(t)$ and $\int_0^T h(t) \circ dW(t)$ denote the Itô and Stratonovich integral respectively. In these integrals, we integrate $h(t)$ with respect to a Wiener process $W(t)$. A Wiener process is given by the following definition according to page 27 of ([4]).

Definition 1.2. *A stochastic process $W(t)$ for $t \in [0,T]$ is called a Brownian motion, or a Wiener process, over $[0,T]$ if it satisfies the following three conditions.*

- $W(0) = 0$
- *For $0 \le s < t \le T$, the random variable given by the increment $W(t) - W(s)$ is $N(0, t-s)$.*
- *For $0 \le s \le t \le u \le v \le T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.*

A concept that is closely related to a Wiener process is *white noise*. Consider the following definition. ([4])

Definition 1.3. *A zero-mean Gaussian stochastic process $\dot{W}(t)$ for $t \in [0,T]$ is called white noise, if*

$$\mathbb{E}[\dot{W}(t)\dot{W}(s)] = \delta_0(s-t),$$

holds for all $t, s \in [0,T]$, where δ_0 is the Dirac-delta function.

In a generalized sense, white noise is the time derivative of a Wiener process, that is

$$\frac{dW(t)}{dt} := \dot{W}(t).$$

Now, let T be the total time up to which the integral is evaluated, L be the number of subintervals, and t_n a discrete time point in the partition of $[0,T]$. Then the Itô integral, which is defined as the mean-square limit of Riemann sum approximated from the left endpoint, is given by

$$\int_0^T h(t)\,dW(t) = \lim_{L\to\infty}\sum_{n=0}^{L-1} h(t_n)(W(t_{n+1}) - W(t_n)), \tag{2}$$

and the Stratonovich integral is defined as the MidPoint Riemann sum,

$$\int_0^T h(t)\circ dW(t) = \lim_{L\to\infty}\sum_{n=0}^{L-1} h(\frac{1}{2}(t_{n+1}+t_n))(W(t_{n+1}) - W(t_n)). \tag{3}$$

Interestingly, the left endpoint method (2) and the midpoint method (3) do not converge to the same value as $L \to \infty$ and $\max(t_{n+1} - t_n) \to 0$ (see [4]). Another difference between Stratonovich integrals and Itô integrals is that the Itô integral satisfies the martingale property ([4], page 45), but the Stratonovich integral satisfies the chain-rule used in deterministic calculus ([8], page 76). The different ways of defining stochastic integrals, choosing between the Itô or Stratonovich integral, highlights the purpose of this thesis.

In this study we are particularly interested in stochastic Lotka-Volterra equations which can be written as stochastic Poisson systems. Stochastic Poisson systems take the form

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = B(x_1,x_2)\nabla H(x_1,x_2)dt + \sum_{i=1}^{M} B(x_1,x_2)\nabla h_i(x_1,x_2)\circ dW_i(t), \tag{4}$$

where $H(x_1,x_2)$ is the drift Hamiltonian, $h_i(x_1,x_2)$ are the diffusion Hamiltonians, $W_i(t)$ are independent Wiener processes (with $i = 1,...,M$), $\circ$ defines the Stratonovich integral, and $B(x_1,x_2)$ is the Poisson structure matrix given by

$$B(x_1,x_2) = \begin{pmatrix} 0 & x_1x_2 \\ -x_1x_2 & 0 \end{pmatrix}.$$

When we numerically integrate stochastic Poisson systems we want to maintain the Poisson structure for the system. In order words, we want to find a geomtetric integrator that satisfies the following definition (see [3]) .

Definition 1.4. *Let $\phi : U \to \mathbb{R}^d$ be a map, where U is an open set in $\mathbb{R}^d$ and d is the dimension of the system. The timestep is defined by*

$$y_{n+1} = \phi_{\Delta t}(y_n).$$

We say that ϕ is a Poisson integrator if for a given Poisson structure matrix function $B : U \to \mathbb{R}^{d\times d}$, the condition

$$\left[\frac{\partial\phi(y)}{\partial y}\right] B(y) \left[\frac{\partial\phi(y)}{\partial y}\right]^T = B(\phi(y))$$

holds for all $y \in U$, where $\frac{\partial\phi(y)}{\partial y}$ denotes the Jacobian matrix of ϕ at y.

A stochastic numerical integrator that satisfies definition 1.4 is called a stochastic Poisson integrator. Hence, in this thesis we are interested in the research questions:

- "Can Stratonovich noise terms be introduced into the Lotka-Volterra equation in a way that maintains the Poisson property of its flow?"

- "Is it possible to construct a suitable stochastic Poisson integrator for this system?"

2 Stochastic Models

In this this work, we consider a simple case of the determinstic Lotka-Volterra equations. The classical Lotka-Volterra equations are given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(a - bx_2), \\ \frac{dx_2}{dt} &= x_2(cx_1 - d). \end{aligned} \tag{5}$$

Here, x_1 and x_2 are the prey and predator populations, respectively. The parameters a, b, c and d are constants, and in this work we assume that $b = c = \tau$. The parameter τ represents the interaction between the two populations, the parameter a represents the birth rate of the prey and the parameter d represents the death rate of the predators.

This system is a Poisson system, with the Hamiltonian function $H(x_1, x_2)$ described as

$$H(x_1, x_2) = a\ln(x_2) + d\ln(x_1) - \tau(x_1 + x_2). \tag{6}$$

An important property of the determistic Lotka-Volterra equations is that the Hamiltonian function is preserved. To see this we suppose that $(x_1(t), x_2(t))$ is a solution of (5), and then when we differentiate the Hamiltonian (6) with respect to time, we get

$$\begin{aligned} \frac{dH(x_1, x_2)}{dt} &= \frac{\partial H}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial H}{\partial x_2}\frac{dx_2}{dt} \\ &= \frac{\partial H}{\partial x_1} x_1 x_2 \frac{\partial H}{\partial x_2} + \frac{\partial H}{\partial x_2}(-x_1 x_2)\frac{\partial H}{\partial x_1} \\ &= 0. \end{aligned}$$

Another property of the Lotka-Volterra equations is that the solutions are periodic, as seen in Figure 1 (here a=2, d=1 and $\tau = 0.05$, and the initial conditions are $x_1(0) = 150$ and $x_2(0) = 50$). According to Section 8.2 in [12], the equilibrium points for (5) are $(0,0)$ and $(\frac{d}{c}, \frac{a}{b})$. Since in this case $b = c = \tau$, we can write this as $(\frac{d}{\tau}, \frac{a}{\tau})$.

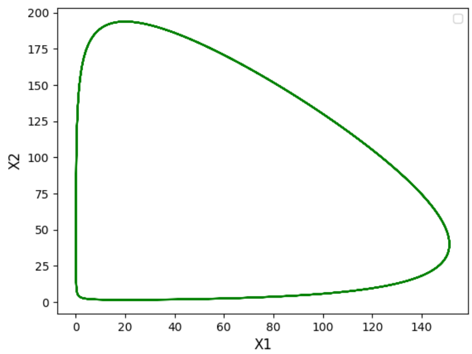


Figure 1: Phase space portrait of the solutions of the Lotka-Volterra equations.

2.1 Stochastic Model 1

Consider a modification of the parameter which represents the interaction of the two species

$$\tau \to \tau + \sigma \circ \dot{W}(t),$$

where $\dot{W}(t)$ is white noise and $\sigma \geq 0$. This modification of the parameter τ introduces stochasticity into the interactions between the prey and predator populations. When applying this transformation of τ the deterministic Lotka-Volterra equations (5) become a stochastic differential equation with Stratonovich noise:

$$dx_1 = x_1(a - \tau x_2)dt - \sigma x_1 x_2 \circ dW(t),$$
$$dx_2 = x_2(\tau x_1 - d)dt + \sigma x_1 x_2 \circ dW(t).$$

The Itô version of this stochastic differential equation has been studied by Vadillo [14]. The Itô version is, however, not Poisson. Furthermore, this system is a stochastic Poisson system with the drift Hamiltonian (6) and the diffusion Hamiltonian described by

$$h = -\sigma(x_1 + x_2).$$

2.2 Stochastic Model 2

In the second model distinct perturbations are introduced for both predators and prey. Consider the following modification of the parameters a and d:

$$a \to a + \sigma_1 \circ \dot{W}_1(t),$$
$$d \to d - \sigma_2 \circ \dot{W}_2(t),$$

where $\dot{W}_1(t)$ and $\dot{W}_2(t)$ are independent white noise processes. These modifications introduce stochasticity to the birthrate for the prey population and deathrate for the predator population. The stochastic Lotka-Volterra equations are then written as

$$dx_1 = x_1(a - \tau x_2)dt + \sigma_1 x_1 \circ dW_1(t),$$
$$dx_2 = x_2(\tau x_1 - d)dt + \sigma_2 x_2 \circ dW_2(t).$$

The Itô version of this system has been studied in detail in [13], [10] and [14]. The Stratonovich system is a stochastic Poisson system as described in equation (4) with the drift Hamiltonian (6) and the diffusion Hamiltonians

$$h_1 = \sigma_1 \ln(x_2),$$
$$h_2 = -\sigma_2 \ln(x_1).$$

2.3 Stochastic Model 3

The following theorem lays the foundation for the third and final model.

Theorem 2.1. *Let $\tilde{x}_1$ and $\tilde{x}_2$ be solutions to the deterministic Lotka-Volterra* (5)*, with Hamiltonian H. If*

$$\begin{aligned} x_1(t) &= \tilde{x}_1(t+\alpha W(t)), \\ x_2(t) &= \tilde{x}_2(t+\alpha W(t)), \end{aligned}$$

then $x_1(t)$ and $x_2(t)$ satisfy the stochastic Lotka-Volterra equations (4) *with the Hamiltonian H and diffusion Hamiltonian $h = \alpha H$.*

Proof. When we apply formula 2.27 in [9] to the solution (x_1, x_2) of the stochastic Lotka-Volterra equations we can transform the system to

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \dot{\tilde{x}}_1(t+\alpha W(t)) \\ \dot{\tilde{x}}_2(t+\alpha W(t)) \end{pmatrix} dt + \begin{pmatrix} \alpha \dot{\tilde{x}}_1(t+\alpha W(t)) \\ \alpha \dot{\tilde{x}}_2(t+\alpha W(t)) \end{pmatrix} \circ dW(t).$$

Since $\tilde{x}_1$ and $\tilde{x}_2$ satisfy the Lotka-Volterra equations, we can write the transformed system as

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & x_1x_2 \\ -x_1x_2 & 0 \end{pmatrix} \nabla H(x_1,x_2)dt + \begin{pmatrix} 0 & x_1x_2 \\ -x_1x_2 & 0 \end{pmatrix} \nabla h(x_1,x_2) \circ dW(t),$$

with $h(x_1,x_2) = \alpha H(x_1,x_2)$. □

In the models described in Section 2.1 and Section 2.2 we introduced stochasticity in the parameters corresponding to the interactions between the species and the birth/death rates of the species. It is important to note that in this section we introduced stochasticity to the frequency of the periodic solution. Adding stochasticity to the periodic solution of the Lotka-Volterra equations results in the global positivity of the populations of the predators and prey for positive initial conditions.

Theorem 2.2. *The predator and prey populations of the solution of the resulting system of Theorem 2.1 will never go extinct for positive initial conditions.*

Proof. Since the species in the deterministic Lotka-Volterra equations (5) do not go extinct, it directly follows that the species of the resulting system described in Theorem 2.1 also avoid extinction, as it represents a stochastic time shift of the original system. □

Besides the global positivity of the populations of the predators and prey, this system has another important property.

Theorem 2.3. *The drift Hamiltonian of the resulting system of Theorem 2.1 is almost surely preserved.*

Proof. Recall that the diffusion Hamiltonian $h(x_1,x_2) = \alpha H(x_1,x_2)$. Then the Poisson bracket $\{H,h\}$ is given by

$$\begin{aligned} \{H,h\} &= \sum_{i,j=1}^{2} \frac{\partial H}{\partial x_i} B(x_1,x_2)_{ij} \frac{\partial h}{\partial x_j} \\ &= \frac{\partial H}{\partial x_1} x_1x_2 \frac{\partial h}{\partial x_2} - \frac{\partial H}{\partial x_2} x_1x_2 \frac{\partial h}{\partial x_1} \\ &= \frac{\partial H}{\partial x_1} x_1x_2 \alpha \frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_2} x_1x_2 \alpha \frac{\partial H}{\partial x_1} = 0 \end{aligned}$$

Since the Poisson bracket equals zero, the drift Hamiltonian is almost surely preserved according to Section 2.2 in [1]. □

3 Numerical Methods

In this section, we propose a numerical method to ensure that the numerical solutions of the models discussed in Section 2 preserve the Poisson structure. In general, finding a stochastic Poisson integrator is not easy; however, according to Theorem 3.1 in [6], we can do the following.

1. Find a specific transformation $z = \vartheta(y)$, with $y = (x_1, x_2)$, that converts the stochastic Poisson system (defined as in equation (4)) into stochastic canonical Hamiltonian form. A stochastic two-dimensional canonical Hamiltonian system is an extension of (1), that is

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(x_1, x_2)dt + \sum_{i=1}^{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla h_i(x_1, x_2) \circ dW_i(t),$$

   where $H(x_1, x_2)$ is the Hamiltonian, $h_i(x_1, x_2)$ are the diffusion Hamiltonians and $W_i(t)$ are independent Wiener processes with $i = 1, 2, \ldots, M$.

2. Calculate $z_n = \vartheta(y_n)$.

3. Apply a symplectic integrator ($\psi_{\Delta t}$) on the transformed system to obtain $z_{n+1} = \Psi_{\Delta t}(z_n)$.

4. Find y_{n+1} by $y_{n+1} = \vartheta^{-1}(z_{n+1})$.

This then results in a stochastic Poisson integrator.The upcoming theorem shows that the systems described in Section 2, more generally stochastic Poisson systems (4), can be transformed to stochastic canonical form.

Theorem 3.1. *The coordinate transformation* $\vartheta(x_1, x_2) = (\ln(x_2), \ln(x_1)) = (q, p)$ *brings the stochastic Poisson system* (4)*, with a M-dimensional Wiener process, to canonical Hamiltonian form, with:*

$$K(q, p) = -H(x_1, x_2) = -H(e^p, e^q),$$
$$k_i(q, p) = -h_i(x_1, x_2) = -h_i(e^p, e^q) \quad \forall i = 1, 2, 3, \ldots, M.$$

Here, K and k_i represent the drift Hamiltonian and the diffusion Hamiltonians for the transformed system, respectively.

Proof. Notice that the model can be written as:

$$dx_1 = x_1 x_2 \frac{\partial H(x_1, x_2)}{\partial x_2} dt + x_1 x_2 \sum_{i=1}^{M} \frac{\partial h_i(x_1, x_2)}{\partial x_2} \circ dW_i(t),$$
$$dx_2 = -x_1 x_2 \frac{\partial H(x_1, x_2)}{\partial x_1} dt - x_1 x_2 \sum_{i=1}^{M} \frac{\partial h_i(x_1, x_2)}{\partial x_1} \circ dW_i(t).$$

According to formula 2.27 in [9] we can transform the system to:

$$dq = -\frac{1}{x_2} x_1 x_2 \frac{\partial H(x_1, x_2)}{\partial x_1} dt - \frac{1}{x_2} x_1 x_2 \sum_{i=1}^{M} \frac{\partial h_i(x_1, x_2)}{\partial x_1} \circ dW_i(t),$$
$$dp = \frac{1}{x_1} x_1 x_2 \frac{\partial H(x_1, x_2)}{\partial x_2} dt + \frac{1}{x_1} x_1 x_2 \sum_{i=1}^{M} \frac{\partial h_i(x_1, x_2)}{\partial x_2} \circ dW_i(t).$$

When we simplify this, the system becomes

$$dq = -x_1 \frac{\partial H(x_1,x_2)}{\partial x_1} dt - x_1 \sum_{i=1}^{M} \frac{\partial h_i(x_1,x_2)}{\partial x_1} \circ dW_i(t),$$
$$dp = x_2 \frac{\partial H(x_1,x_2)}{\partial x_2} dt + x_2 \sum_{i=1}^{M} \frac{\partial h_i(x_1,x_2)}{\partial x_2} \circ dW_i(t).$$

Since $(x_1,x_2) = (e^p,e^q)$, $K(q,p) = -H(x_1,x_2) = -H(e^p,e^q)$ and $k_i(q,p) = -h_i(x_1,x_2) = -h_i(e^p,e^q)$ (for all $i = 1,2,3,...,M$), the system can be written as:

$$dq = \frac{\partial K(q,p)}{\partial p} dt + \sum_{i=1}^{M} \frac{\partial k_i(q,p)}{\partial p} \circ dW_i(t),$$
$$dp = -\frac{\partial K(q,p)}{\partial q} dt - \sum_{i=1}^{M} \frac{\partial k_i(q,p)}{\partial q} \circ dW_i(t),$$

where we use that fact that

$$\frac{\partial K(q,p)}{\partial p} = \frac{\partial}{\partial p}(-H(e^p,e^q)) = -e^p \frac{\partial H}{\partial x_1}(e^p,e^q) = -x_1 \frac{\partial H}{\partial x_1}(x_1,x_2),$$

the same argument can be used for $\frac{\partial K(q,p)}{\partial q}, \frac{\partial k_i(q,p)}{\partial q}$ and $\frac{\partial k_i(q,p)}{\partial q}$ for i= $1,2,3,...,M$. The system can now be written in the stochastic canonical form, that is

$$\begin{pmatrix} dq \\ dp \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla K(q,p) dt + \sum_{i=1}^{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla k_i(q,p) \circ dW_i(t).$$

□

Finding a symplectic integrator is considerably easier, see, for instance [5], [3]. In this work, we will use the Störmer-Verlet method as our symplectic integrator proposed in ([5], Section 3.4.1). This method is particularly advantageous because for a separable Hamiltonian it becomes explicit. However, one could see that the method proposed is only for stochastic processes with one dimensional Wiener process, hence for a model with an M-dimensional Wiener process (notice that the stochastic model described in Section 2.2 has a 2-dimensional Wiener process) the stochastic Störmer-Verlet method can be extended to

$$\begin{aligned}
P_1 &= p_n - \frac{1}{2}\frac{\partial H}{\partial q}(q_n,P_1)\Delta t - \sum_{i=1}^{M} \frac{1}{2}\frac{\partial h_i}{\partial q}(q_n,P_1)\Delta W^i, \\
q_{n+1} &= q_n + \frac{1}{2}\frac{\partial H}{\partial p}(q_n,P_1)\Delta t + \frac{1}{2}\frac{\partial H}{\partial p}(q_{n+1},P_1)\Delta t + \sum_{i=1}^{M}\left(\frac{1}{2}\frac{\partial h_i}{\partial p}(q_n,P_1)\Delta W^i + \frac{1}{2}\frac{\partial h_i}{\partial p}(q_{n+1},P_1)\Delta W^i\right), \\
p_{n+1} &= P_1 - \frac{1}{2}\frac{\partial H}{\partial q}(q_{n+1},P_1)\Delta t - \sum_{i=1}^{M} \frac{1}{2}\frac{\partial H}{\partial q}(q_{n+1,P_1})\Delta W^i.
\end{aligned} \tag{7}$$

Here H is the drift Hamiltonian, h_i are the diffusion Hamiltonians, Δt is the time step size and ΔW^i is an independent increment of a Wiener process given by $\Delta W^i \sim N(0,\Delta t)$. In this thesis we will refer to the method described at the start of this section, when applied together with the Störmer-Verlet method, as the Poisson Störmer-Verlet method (PSVM).

3.1 Order of convergence

Convergence of stochastic schemes has been discussed in [4] (mainly Itô integrals), [8] and [7]. The mean-square convergence and strong convergence focus on pathwise approximations of the exact solution. In this paper we will focus on the mean-square convergence, whose definition is as follows ([5]).

Definition 3.2. *Let $\bar{z}(t) = (\bar{q}(t), \bar{p}(t))$ denote the exact solution to a stochastic differential equation with initial conditions q_0 and p_0. Let $z_n = (q_n, p_n)$ represent the numerical solution at time t_n, obtained by iteratively applying a suitable numerical integrator n times with a constant time step Δt.*

The numerical solution is said to converge in the mean-square sense with a global order r if there exist $\delta > 0$ and a constant $C > 0$ such that for all $\Delta t \in (0, \delta)$, we have:

$$\sqrt{\mathbb{E}(\|z_N - \bar{z}(T)\|^2)} \leq C(\Delta t)^r$$

where T is the total time duration and N is the number of discrete time steps such that $\Delta t = \dfrac{T}{N}$.

Despite the general difficulty of proving mean-square convergence, specific cases have been successfully proven, such as the extended Störmer-Verlet method (7). According to Theorem 1.1 of [11] the extended Störmer-Verlet method has an mean-square order of 1/2 for multidimensional non-commutative noise, and order 1 for a one-dimensional noise, or a multi dimensional noise that satisfies commutativity conditions. Let us verify whether the noise for the stochastic model described in Section 2.2 is commutative.

Lemma 3.3. *When the system described in Section 2.2 is transformed to its canonical form using Theorem 3.1, and the symplectic numerical method* (7) *is applied, then the noise is commutative and the symplectic numerical method for the canonical system has mean-square convergence of order 1.*

Proof. Let the vectors Γ_{ij} and Λ_{ij} for each i,j=1,...,M be defined the same as (3.49) in [5], that is

$$\Gamma_{ij} = \frac{\partial^2 k_j}{\partial p \partial q}\frac{\partial k_i}{\partial p} - \frac{\partial^2 k_j}{\partial p^2}\frac{\partial k_i}{\partial q}, \quad \lambda_{ij} = -\frac{\partial^2 k_j}{\partial q^2}\frac{\partial k_i}{\partial p} + \frac{\partial^2 k_j}{\partial q \partial p}\frac{\partial k_i}{\partial q}.$$

As seen in the proof of Theorem 3.1, $k_1 = -\sigma_1 q$ and $k_2 = \sigma_2 p$, hence

$$\Lambda_{ij} = 0 = \Lambda_{ji},$$
$$\Gamma_{ij} = 0 = \Gamma_{ji}.$$

Theorem 1.1 of [11] then implies that the numerical integrator of the transformed canonical system has mean-square order 1. □

By the previous lemma and Theorem 1.1 from [11] we know the mean-square order of the symplectic integrator (7) applied to the systems which are transformed to canonical Hamiltonian form. However, now the question arises whether the mean-square order of the stochastic Poisson integrator can be found. The following theorem assures that the resulting stochastic Poisson integrator has the same mean-square order as the symplectic integrator applied to the transformed system.

Theorem 3.4. *If a symplectic integrator with mean-square convergence order r is applied to the transformed Poisson systems found by Theorem 3.1, and if $0 < m \leq x_1(t) \leq M < \infty$ and $0 < m \leq x_2(t) \leq M < \infty$ almost surely for $t \in [0, T]$ and some $m, M \in \mathbb{R}$, then the resulting Poisson integrator will exhibit mean-square convergence of order r.*

Proof. Let $\bar{z}(t) = (\bar{p}(t), \bar{q}(t))$ be the exact solution of the system in canonical coordinates, T the total time duration and N the number of discrete time steps. Furhtermore, let $z_n = (p_n, q_n)$ be the numerical solution at time t_n and assume that the mean-square order of convergence of the scheme applied to the system in canonical coordinates with a symplectic integrator (such as (7)) is r, that is

$$\mathbb{E}(\|z_N - \bar{z}(T)\|^2) \leq C(\Delta t)^{2r}.$$

To get the mean-square order of the stochastic Poisson integrator we have to transform the system back with the transformation of theorem 3.1, hence:

$$\begin{aligned}\mathbb{E}(\|\varphi^{-1}(z_N) - \varphi^{-1}(\bar{z}(T))\|^2) &= \mathbb{E}(\|\varphi^{-1}(p_N, q_N) - \varphi^{-1}(\bar{p}(T), \bar{q}(T))\|^2) \\ &= \mathbb{E}(\|\varphi^{-1}(p_N, q_N) + \varphi^{-1}(\bar{p}(T), q_N) - \varphi^{-1}(\bar{p}(T), q_N) - \varphi^{-}1(\bar{p}(T), \bar{q}(T))\|^2) \\ &\leq 2(\mathbb{E}(\|\varphi^{-1}(p_N, q_N) - \varphi^{-1}(\bar{p}(T), q_N)\|^2) + \mathbb{E}(\|\varphi^{-1}(\bar{p}(T), q_N) - \varphi^{-}1(\bar{p}(T), \bar{q}(T))\|^2))\end{aligned}$$

Since the solutions of the resulting canonical system are almost surely bounded, we know that φ^{-1} is locally Lipschitz continuous, hence

$$\begin{aligned}2\left(\mathbb{E}\left(\|\varphi^{-1}(p_N, q_N) - \varphi^{-1}(\bar{p}(T), q_N)\|^2\right) + \mathbb{E}\left(\|\varphi^{-1}(\bar{p}(T), q_N) - \varphi^{-1}(\bar{p}(T), \bar{q}(T))\|^2\right)\right) \\ \leq 2B\left(\mathbb{E}\left(\|p_N - \bar{p}(T)\|^2\right) + \mathbb{E}\left(\|q_N - \bar{q}(T)\|^2\right)\right)\end{aligned}$$

for a sufficiently large B. By the assumption that the mean-square order of convergence of the method applied to the system in canonical coordinates with a symplectic integrator is equal to r, we have

$$2B(\mathbb{E}(\|p_N - \bar{p}(T)\|^2) + \mathbb{E}(\|q_N - \bar{q}(T)\|^2)) \leq 2B(C(\Delta t)^{2r}),$$

where B and C are sufficiently large and independent of Δt.
Thus, since we assumed that the solution is almost surely bounded, the resulting stochastic Poisson integrator exhibits the same order of convergence as the symplectic integrator applied to the transformed system. □

3.2 Heun Method

To evaluate the performance of our stochastic Poisson integrator, we will benchmark it against the Heun method, which is a non-geometric integrator. Consider the following stochastic differential equation

$$d\mathbf{X} = f(\mathbf{X})dt + g(\mathbf{X}) \circ dW(t), \tag{8}$$

here $f(\mathbf{X})$ is an m-vector-valued function, $g(\mathbf{X})$ is an m x d matrix-valued function, $\circ$ denotes the Stratonovich integral and $W(t) = (W_1(t), ..., W_d(t))$ is a d-dimensional process having independent scalar Wiener process components for $t \geq 0$. According to Section 3 in [2] the Heun method for a stochastic differential equation of the form (8) is given by

$$\begin{aligned} Y_1 &= y_n + \sqrt{h} f(y_n) + \Delta W_n g(y_n), \\ y_{n+1} &= y_n + \frac{1}{2}h(f(y_n) + f(Y_1)) + \frac{1}{2}\Delta W_n(g(y_n) + g(Y_1)), \end{aligned} \tag{9}$$

This method has the same order of as the stochastic Poisson Störmer-Verlet method (see [2]).

4 Numerical Results

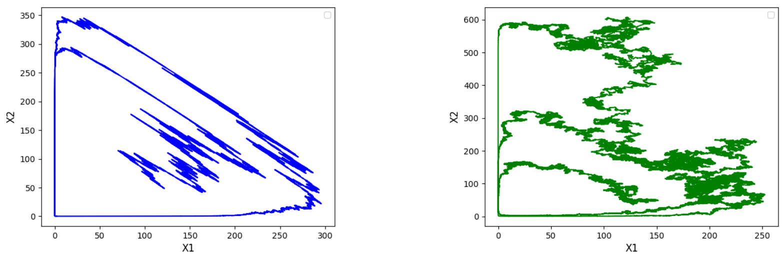
In this section we compare the mean extinction times of model 2.1 and 2.2. Specifically, we examine the extinction times obtained by the Heun method, as described in (9), and the Poisson Störmer-Verlet method. We define the extinction time to be the smallest positive time such that either $x_1 \leq 2.2204 \cdot 10^{-16}$ or $x_2 \leq 2.2204 \cdot 10^{-16}$. The means and confidence intervals presented in both Table 1 and Table 2 are calculated based on 1000 trials. For each numerical method, running 1000 trials on my laptop with a 13th Gen Intel(R) Core(TM) i7-1355U CPU took approximately 48 hours. Moreover, for the stochastic model described in Section 2.3 we will compare the behaviour of the (preserved) Hamiltonian when we numerically integrate the system with the Heun method and the Poisson Störmer-Verlet method.

4.1 Stochastic Model 1

For this stochastic model the following parameters are used: $a = 2, \tau = 0.05$, $d = 1$ and $\sigma = 0.01$, with a stepsize of $\Delta t = 10^{-5}$. Furthermore, the initial conditions are: $x_1(0) = 120$ and $x_2(0) = 50$. When we compare the mean extinction times in Table 1 to the mean extinction times in Table 1 of [14], we see that the mean-extinction time of the stochastic model with Stratonovich noise is higher than the mean-extinction time found for the Itô version of this system. A phase plot of a solution obtained via the Heun method and stochastic Poisson Störmer-Verlet method is given in Figure 2 for a total time of $T = 20$.

Table 1: Comparing extinction times of stochastic model 2.1

Method	**Mean**	**95% Confidence Interval**
Poisson Störmer-Verlet method	352.5461	[337.2481; 367.8440]
Heun Method	347.7841	[332.7013; 362.8670]



(a) Phase plane portrait by Heun method

(b) Phase plane portrait by Poisson Störmer-Verlet method

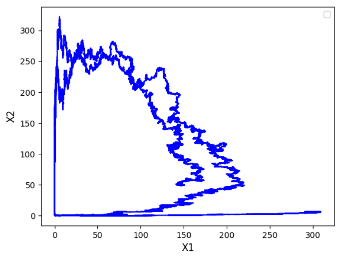
Figure 2: Comparison of the phase portraits with step size $\Delta t = 10^{-5}$

4.2 Stochastic Model 2

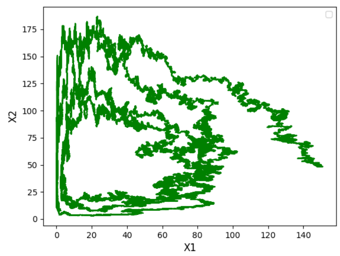
For this stochastic model the following parameters are used: $a = 2, \tau = 0.05$, $d = 1$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.5$, with a stepsize of $\Delta t = 10^{-5}$. Furthermore, the initial conditions are: $x_1(0) = 150$ and $x_2(0) = 50$. In Table 2 we can see that, surprisingly, the mean-extinction time of the Heun method is higher than the designed stochastic Poisson integrator. A possible explanation for this is that the Heun method is less accurate. This can be checked by decreasing the step size and seeing where the mean-extinction time converges to. Unfortunately due to time constraints of this project we were not able to verify this. A phase plot of a solution obtained via the Heun method and stochastic Poisson Störmer-Verlet method is given in Figure 3 for a total time of $T = 20$.

Table 2: Comparing extinction times of stochastic model 2.2

Method	**Mean**	**95% Confidence Interval**
Poisson Störmer-Verlet method	116.3741	[111.0347; 121.7136]
Heun Method	119.9850	[114.4001; 125.5700]



(a) Phase plane portrait by Heun method

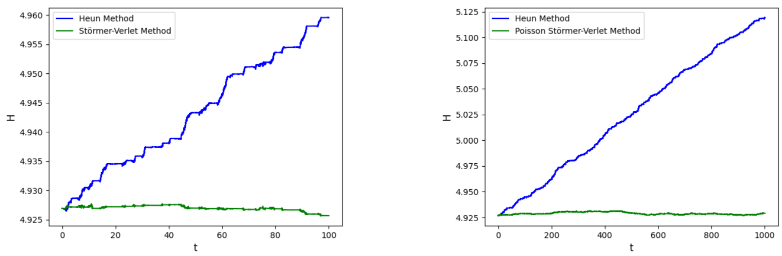


(b) Phase plane portrait by Poisson Störmer-Verlet method

Figure 3: Comparison of the phase portraits of a solution of stochastic model 1

4.3 Stochastic Model 3

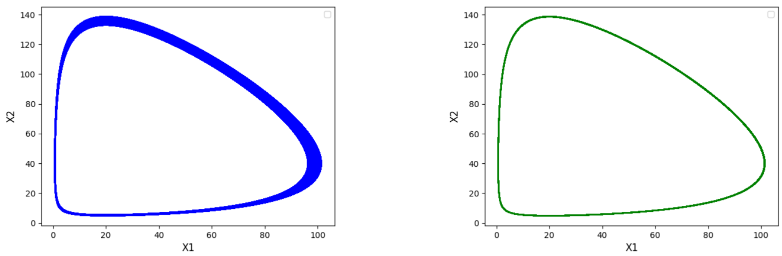
In Theorem 2.2 we have proven that the drift Hamiltonian is preserved for the stochastic model described in Section 2.3. Figure 4 shows that the Heun method fails to preserve the Hamiltonian, and additionally, the phase plane plot using the Heun method appears more unstable than the Poisson Störmer-Verlet method as seen in figure 5 . In contrast, the solution obtained with the Poisson Störmer-Verlet method oscillates around one of the equilibtium points, indicating better stability. Figure 6 shows that with the Poisson Störmer-Verlet method, smaller step sizes result in better preservation of the Hamiltonian. For the plots we have used the parameters $a = 2, \tau = 0.05$, $d = 1$, $\alpha = 1$ and a total time of $T = 1000$.



(a) Time interval [0, 100]

(b) Time interval [0, 1000]

Figure 4: Comparison of the evolution of the Hamiltonian with $\Delta t = 10^{-5}$



(a) Phase plane portrait by Heun method

(b) Phase plane portrait by Poisson Störmer-Verlet method

Figure 5: Comparison of the phase portraits of a solution of stochastic model 2

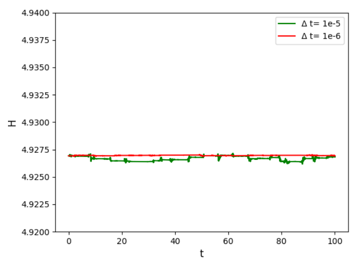


Figure 6: Comparison of the drift Hamiltonian using PSVM with different step sizes

5 Conclusions

In this thesis, Stratonovich noise has been introduced to three distinct models while maintaining the Poisson property of the flow. Both stochastic models described in 2.1 and 2.2 are inspired by the paper by Vadillo [14] and the stochastic model described in 2.3 is based on Theorem 2.1. Moreover, we have designed a stochastic Poisson integrator in Section 3 using Theorem 3.1 in [6] and the Störmer-Verlet method (7), and we called this the Poisson Störmer-Verlet method.

As seen in Table 1 we can conclude that the mean extinction times of the model with Stratonovich noise are higher than the mean extinction times of the Itô version of this system described in [14]. Furthermore, in Table 2 we observed that the mean extinction time of the Heun method is greater than the mean extinction time of the Poisson Störmer-Verlet method. A possible explanation for this is that the Heun method is less accurate and requires a smaller time step. Finally, we have looked at the behaviour of a non-geometric integrator in comparison to a stochastic Poisson integrator in context of the preservation of the Hamiltonian of stochastic model described in Section 2.3. We observe that the preservation is highly dependent on the type of integrator used, with the found Poisson integrator being superior to the Heun method.

In future research, we suggest to take a deeper dive in the comparison of different integrators and the effect of different parameters for the diffusion Hamiltonians. Especially, we suggest to lower the step size for the Heun method to see where the mean extinction time converges to, as this can further highlight the power of the stochastic Poisson integrator designed in this thesis. Moreover, one could investigate the performance of the proposed numerical integrator for different stochastic Poisson systems.

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