Credit Valuation Adjustment: An Empirical Application to Interest Rate Swaps

Timon J. Ambergen

October 2024

Master Thesis

MSc Industrial Engineering & Management
Specialization in Financial Engineering & Management

University of Twente
Faculty of Behavioural, Management and Social Sciences

Supervisors
Prof. Dr. Spierdijk (University of Twente)
Dr. Roorda (University of Twente)
MSc Goncalves Hilario (Deloitte)
MSc Gorter (Deloitte)

Abstract

This thesis explores the development and implementation of a Credit Valuation Adjustment (CVA) model specifically for interest rate swaps (IRS), in response to challenges faced in the financial risk management (FRM) department of Deloitte Netherlands. The work addresses the limitations of using Bloomberg for CVA valuation, such as its inability to handle certain derivative types and lack of transparency in the valuation process. A literature review on the methodologies for CVA calculation is conducted, comparing equilibrium and no-arbitrage interest rate models, including the Vasicek, Hull-White, and Cox-Ingersoll-Ross models. Based on this comparison a CVA model is developed using the Hull-White one-factor model for simulating interest rates. The CVA model developed in this thesis provides a viable alternative to Bloomberg's valuation methodology for interest rate swaps, as the results were consistent with those produced by Bloomberg.

Acknowledgements

This thesis has been written as part of my Master's degree in Financial Engineering & Management at the University of Twente. The thesis is executed at the financial risk management (FRM) department of Deloitte Netherlands. I would like to express my sincere gratitude to Deloitte for providing me with the opportunity to work on this project. My time at Deloitte has been immensely rewarding, thanks in large part to the talented and supportive team that I had the pleasure of working with. The collaborative environment and the engaging after-work activities made my experience truly enjoyable.

I am grateful to my supervisor Amiel Goncalves Hilario at Deloitte for his guidance, insights, and encouragement throughout this project. His expertise and support was invaluable in shaping this thesis and ensuring its relevance to both the academic and professional realms.

Additionally, I would like to extend my heartfelt thanks to my supervisors Prof. Dr. Spierdijk and Dr. Roorda from the University of Twente for their support and feedback. Their guidance has been instrumental in completing this thesis and achieving my academic goals.

Timon Ambergen

Table of contents

List of Abbreviations

CVA	-	Credit valuation adjustment
U.S.	-	United States
MBS	-	Mortgage-backed securities
CDO	-	Collateral debt obligation
SPV	-	Special purpose vehicles
OTC	-	Over-the-counter
CCP	-	Central clearing parties
IFRS	-	International Financial Reporting Standards
IRS	-	Interest rate swap
FRM	-	Financial Risk Management
FX	-	Foreign exchange
DS	-	Design Science
DSRM	-	Design Science Research Methodology
EE	-	Expected Exposure
PFE	-	Positive Future Exposure
NFE	-	Negative Future Exposure
PD	-	Probability of Default
LGD	-	Loss given default
MTM	-	Mark-to-market
CDS	-	Credit default swap
DVA	-	Debt valuation adjustment
ENE	-	Expected negative exposure
BCVA	-	Bilateral credit valuation adjustment
CIR	-	Cox-Ingersoll-Ross equilibrium model
MLE	-	Maximum Likelihood Estimator
LSM	-	Least Squares Method
CIR++	-	Cox-Ingersoll-Ross no-arbitrage model
JCIR	-	Jump Cox-Ingersoll-Ross model (no-arbitrage)
CIR#	-	Cox-Ingersoll-Ross with ARIMA jump (no-arbitrage)
ARIMA	-	Autoregressive Integrated Moving Average
BDT	-	Black-Derman-Toy model
HW1	-	Hull-White one-factor model (Extended Vasicek model)
CDF	-	Cumulative distribution function
PV01	-	Price Value of a Basis Point
RPV01	-	Risky Price Value of a Basis Point
AHP	-	Analytical Hierarchy Process
NV	-	Naamloze Vennootschap (Public Limited Company in English)
ATM	-	At the money
ESTR	-	Euro Short-Term Rate
EURIBOR	-	Euro Interbank Offer Rate
EPE	-	Expected Positive Exposure
ENE	-	Expected Negative Exposure

1 Introduction

This chapter introduces the topic and the aim of this thesis. After reading this chapter the reader should understand the relevance of introducing credit valuation adjustments (CVA) after the financial crisis. Additionally, the problem context of Deloitte, the company for which this thesis is executed, should be clear. The problem context will be translated into a problem statement with corresponding research objectives. Finally, based on the research objectives, research questions are formulated which will be answered throughout the thesis. The thesis is structured according to a specific research design, which will also be explained in this section.

1.1 Background

From 2004 to 2006, the United States (U.S.) witnessed a significant increase in interest rates, soaring from 1% to surpass 5%. This surge in interest rates played a pivotal role in causing a deceleration within the U.S. housing market. A substantial number of homeowners, who had struggled to meet their mortgage payments during the period of low-interest rates, found themselves unable to cope and began defaulting on their mortgages. This predicament was particularly pronounced in the realm of subprime loans, which were extended to individuals with a subpar or non-existent credit history, as default rates reached unprecedented highs (Gregoy, 2012).

Many subprime loans in the U.S. were held by domestic retail banks and mortgage providers such as Fannie Mae and Freddie Mac. The problem escalated as these loans were packaged into complex financial products through advanced financial engineering techniques. These structured products, such as mortgage-backed securities (MBSs), received favourable credit ratings from rating agencies. Consequently, institutions that did not originate the underlying mortgages, including investment banks and international institutional investors, ended up holding these securities (Gregory, 2012).

In the middle of 2007, the dawn of a credit crisis evolved, primarily stemming from the systematic misvaluation of U.S. mortgages and MBSs. By the end of 2007, certain insurance companies, commonly referred to as "monolines", found themselves in a precarious situation. Monoline insurers provided guarantees to debt issuers, often in the form of credit swaps that enhance the credit of the issuer. These monoline insurers started with providing wraps for municipal bond issues, but later on expanded its offering by providing credit enhancements for other types of bonds, such as MBSs and collateral debt obligations (CDO). The banks, in their willingness to ignore the risk that their counterparties might default (hereafter referred to as counterparty credit risk), accumulated significant exposures to monoline insurers. These banks did not require the monolines to post collateral, as long as they maintained their top-tier Triple-A credit ratings. However, as monolines began reporting significant losses, it became evident that any downgrade in their credit ratings could prompt collateral demands they were unable to fulfil. Such downgrades happened in December 2007, compelling banks to incur substantial losses amounting to billions of dollars due to the substantial counterparty risk they were now confronting. This type of counterparty risk was particularly detrimental, known as wrong-way risk, as the exposure to the counterparty and their default probability were correlated. Wrong-way risk is considered undesirable because it increases credit risk, raises concerns about counterparty credit quality, complicates risk management efforts, and contributes to systemic risk (Gregory, 2012).

In September 2008, an unprecedented event occurred as Lehman Brothers, a prominent global investment bank and the fourth largest in the United States with a century-long legacy, filed for bankruptcy protection—the largest in history. The reluctance of the U.S. government to intervene and rescue Lehman stemmed from concerns about the moral hazard associated with such bailouts. The bankruptcy of Lehman took the financial community by surprise, as all major rating agencies (Moody’s, Standard & Poor’s, and Fitch) had assigned at least a Single-A rating until the moment of Lehman's collapse. Furthermore, the credit derivative market had not priced in an actual default (Gregory, 2012).

Many counterparties likely did not perceive their exposure to Lehman's counterparty risk as a significant concern, nor did they comprehend that the failure of counterparty risk mitigation methods, such as collateral and special purpose vehicles (SPVs), would result in legal complications.

Severe liquidity issues resulted in bailouts of other high-profile banks in 2008. Bear Stearns was acquired by JP Morgan Chase, AIG received a bailout from the U.S. government and Merrill Lynch agreed to be acquired by Bank of America (Adinarayan, 2023).

Counterparty credit risk, commonly referred to as counterparty risk, involves the potential that the party with whom an individual has engaged in a financial arrangement (referred to as the counterparty) may not fulfil their obligations as outlined in the contractual agreement, for instance, by defaulting. This risk is commonly associated with two primary categories of financial instruments: over-the-counter (OTC) derivatives and securities financing transactions (Gregory, 2012).

Derivatives are financial instruments whose values are derived from the performance of underlying assets, indices, or other financial instruments. They play a crucial role in modern financial markets, enabling participants to manage risk, speculate on price movements, and enhance portfolio performance. Derivatives come in various forms, including options, futures, forwards, and swaps. Each serving specific purposes and exhibiting unique characteristics (Quail & Overdahl, 2002).

Since the credit crisis of 2008, significant alterations have been implemented in the trading and clearing processes of derivatives within the OTC market. Standard derivatives exchanged among financial institutions now necessitate clearing through central clearing parties (CCPs). This brings about a resemblance to the handling of exchange-traded contracts, resulting in diminished counterparty credit risk. On the other hand, nonstandard derivatives traded between two financial institutions may undergo bilateral clearing, adhering to an agreement between the involved parties. However, there are stipulations mandating both sides to provide collateral, surpassing the previously established norms, to ensure the fulfilment of their obligations. These nonstandard derivatives are traded in the OTC market, where counterparty credit risk remains a factor (Hull, 2012).

As a consequence of the financial crisis, IFRS 13 (International Financial Reporting Standards) came into effect for annual periods starting after January 1, 2013. According to IFRS 13, fair value in derivatives valuations must be determined using the assumptions of market participants by pricing in counterparty risk (EY, 2014). Credit valuation adjustment, abbreviated as CVA, results in a decrease in the valuation of a basket of derivatives with a counterparty. This adjustment is made to account for the potential scenario in which the counterparty might fail to meet its obligations (Hull & White 2012). The calculation of the CVA "charge" should be conducted with sophistication, considering all relevant aspects that contribute to the definition of CVA:

- the default probability of the counterparty;
- the recovery rate;
- the expected exposure of the derivative;
- the transaction in question;
- netting of existing transactions with the same counterparty;
- collateralisation;
- hedging aspects

Given the absence of a specific method prescribed in accounting literature, derivatives dealers and end users employ a range of approaches in practice to assess the impact of credit risk on the fair value of OTC derivatives (EY, 2014). The purpose of this thesis is to conduct a literature review on common market practices to calculate CVA. This goal extends to the development of a CVA model for interest rate swaps (IRS) to demonstrate the effectiveness of the CVA valuation methodology.

1.2 Problem context

This thesis is carried out for the financial risk management (FRM) team of Deloitte Netherlands. Deloitte is a global provider of audit & assurance, consulting, financial advisory, risk advisory, tax, and related services. At the time of writing this thesis, Deloitte has approximately 455,000 fulltime employees in more than 150 countries (Deloitte, 2024). The FRM team of Deloitte Netherlands is part of the risk advisory department. The main responsibility of FRM is helping banks and other financial institutions to manage their risks by developing or validating risk models.

Financial institutions state the value of derivative positions on their financial statements. The primary duty of audit teams lies in examining the financial statements of clients, encompassing the assessment of derivative valuations. Audit procedures, such as substantive testing, analytical procedures, and other established auditing techniques, are undertaken by the audit team to accumulate evidence regarding the correctness of derivative valuations. Frequently, audit teams engage in collaboration with external financial experts or specialists to assess the value of derivatives, particularly when the valuation process is intricate or demands specialized expertise.

The audit team of Deloitte consults the FRM team of Deloitte for this procedure, leveraging their proficiency in financial modelling, derivatives pricing, and specialized knowledge related to the industry and derivates under assessment. Deloitte's FRM team employs Bloomberg for the valuation of these derivatives. However, the utilization of Bloomberg as a valuation model has three drawbacks:

1. First, not all types of derivates can be valued with Bloomberg. Bloomberg can only be used to calculate CVA for interest rate swaps, cross currency swaps and foreign exchange (FX) forwards, but is not able to calculate CVA for other derivates like options, floors and caps. Derivatives that cannot be valued with Bloomberg are currently valued with Deloitte's own methodology and models (in Excel). This process is time consuming and sensitive to errors.

2. Second, Bloomberg does provide functionalities to value a portfolio of derivates, but contracts need to be put in one by one. Resulting in a time consuming and error prone procedure.

3. Third, there is little transparency in the valuation methodology used by Bloomberg. Especially the way Bloomberg calibrates the model is unknown. If there is a significant discrepancy between the derivative value delivered by the client and Bloomberg, it is difficult to determine the root cause of this discrepancy.

An overview of the problem context and the causal relationships between problems is visualized in *Figure 1*. The FRM team already developed an inhouse valuation model for interest rate swaps. However, the valuation of CVA for interest rate swaps is not yet incorporated.

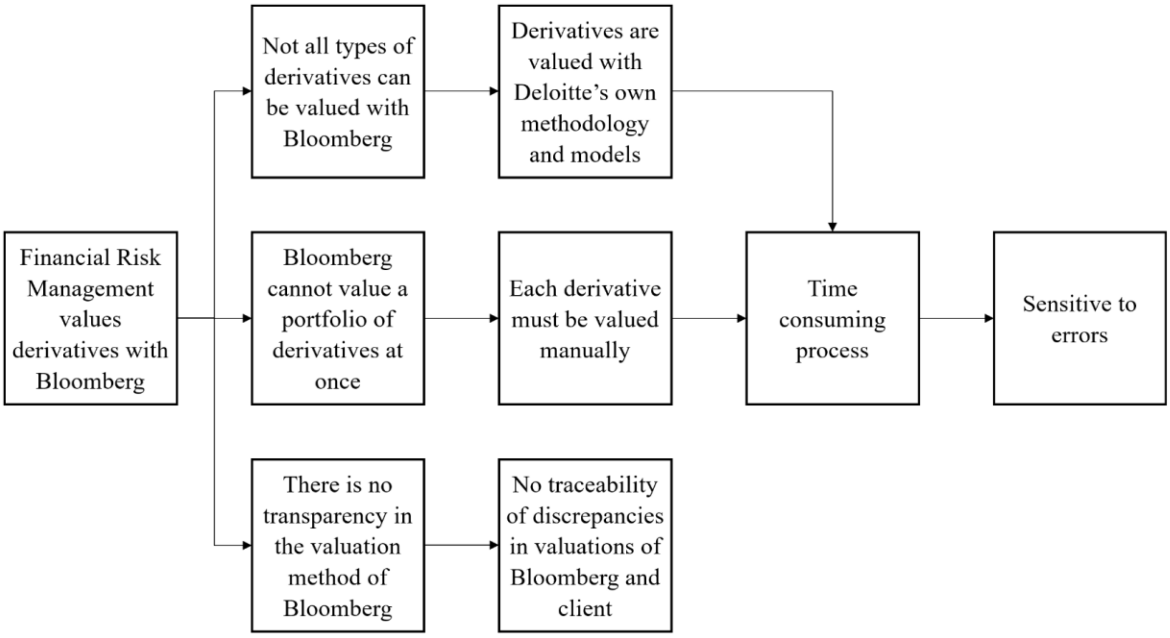


Figure 1. Problem cluster.

1.3 Problem statement

The problem cluster in *Figure 1* reveals that the utilization of Bloomberg as valuation tool leads to an error prone valuation procedure of CVA for derivatives that cannot be valued with Bloomberg. The second problem arising here is that Bloomberg as valuation tool leads to limited traceability of discrepancies between the CVA valuation of Bloomberg and client. These problems are both caused by the utilization of Bloomberg as valuation tool for derivatives, which is consequently the core problem in this thesis. The problem cluster also reveals that the solution should have a transparent CVA valuation methodology.

1.4 Research objectives

To solve the core problem a new valuation model should be used by the team. To develop this valuation model, the CVA should be modelled to account for counterparty credit risk. The initial scope for this thesis is narrowed down to only interest rate swaps (IRS). The reason why the model is initially developed for interest rate swaps is that they are one of the most traded derivatives globally. According to the Bank for International Settlements (2023), interest rate swaps alone account for approximately 35-40% of the total notional amounts outstanding in the global derivatives market. When the model can accurately value interest rate swaps, the model can easily be extended to value cross currency swaps as well. The knowledge problem arising here is that the best way to calculate CVA is unknown. Therefore the first research objective is to create a framework that summarizes all the used methodologies for CVA valuation in the existing literature. To gather this information a literature study will be conducted. There is a lot of literature on modelling methodologies of individual components of CVA, however one study that compares all modelling methodologies is missing. This thesis aims to fill this gap in the literature.

In addition, based on this literature review the best methodology for Deloitte will be chosen. To demonstrate the effectiveness of the chosen methodology, a CVA valuation model will be developed. This extends the purpose of the thesis to also serve as a CVA valuation guide. The CVA valuation model will be used in an empirical application to value a specific interest rate swap. The quantitative data resulting from the empirical application will be compared to results from the

Bloomberg valuation methodology. The results should not be significant different, because that means that the results from the model are aligned with common market practice in CVA valuation. Additionally, a sensitivity analysis will be conducted to see how sensitive the outcomes are to changing input variables.

1.5 Research questions

Based on the research objectives mentioned in section *1.4 Research objectives*, the following main research question is defined:

What is the most suitable method to calculate CVA for interest rate swaps and how can this be developed into a valuation model for Deloitte?

This research question is divided into multiple sub research questions.

1. How is CVA calculated for interest rate swaps in the existing literature?
    a. What are the components of CVA?
    b. How does the literature calculate/model each component of CVA for interest rate swaps?

To answer the main research question, a comprehensive understanding of CVA and its components need to be established. After understanding CVA and its components, a framework will be created stating all the methodologies used in the literature to calculate these individual components of CVA. The answer to this research question is given in chapter *2 Literature review*.

2. What is the most suitable methodology for Deloitte to use in their CVA valuation model for interest rate swaps?

Based on the framework created in the literature review that answers the first research question, the best methodology for Deloitte to model CVA for interest rate swaps is chosen. The answer to this question is given by internal discussions with experts from the FRM team, based on the results from the theoretical framework created in the first research question. The answer to this research question is given in chapter *3 Methodology*.

3. How can the CVA methodology be implemented to develop a CVA valuation model for interest rate swaps?

Based on the chosen CVA valuation methodology for interest rate swaps, the model needs to be developed. Additionally, the parameters of the model should be calibrated. The answer to this research question is given in chapter *4 Implementation*.

4. How accurate is the valuation of interest rate swaps with the CVA model compared to common market practice valuations?

The model will be used in an empirical application to value an interest rate swap to demonstrate the model's effectiveness. The results from the empirical application should be interpretated and validated. This is done with a sensitivity analysis and a comparison of the results with the commonly used valuation tool Bloomberg. The answer to this research question is given in chapter *5 Results*.

1.6 Research design

Selecting an appropriate research methodology is a critical decision for any research, as it lays the foundation for the entire research process and significantly influences the reliability and validity of the study. The importance of choosing a suitable research methodology encompasses several key aspects, including alignment with the specific research objective and data collection techniques. Additionally, the research methodology should be flexible and adaptable to changes, because research is an iterative process and unexpected challenges may arise.

Peffers et al. (2007) conducted a thorough literature study on determining the appropriate elements of a design science (DS) research. Based on the framework that the authors created, a new research methodology has been created called Design Science Research Methodology (DSRM). The DSRM is a process model consisting of six activities in a nominal sequence. As the research of this thesis is strongly focused on designing a solution based on a problem-centered initiation, the DSRM is chosen as research methodology.

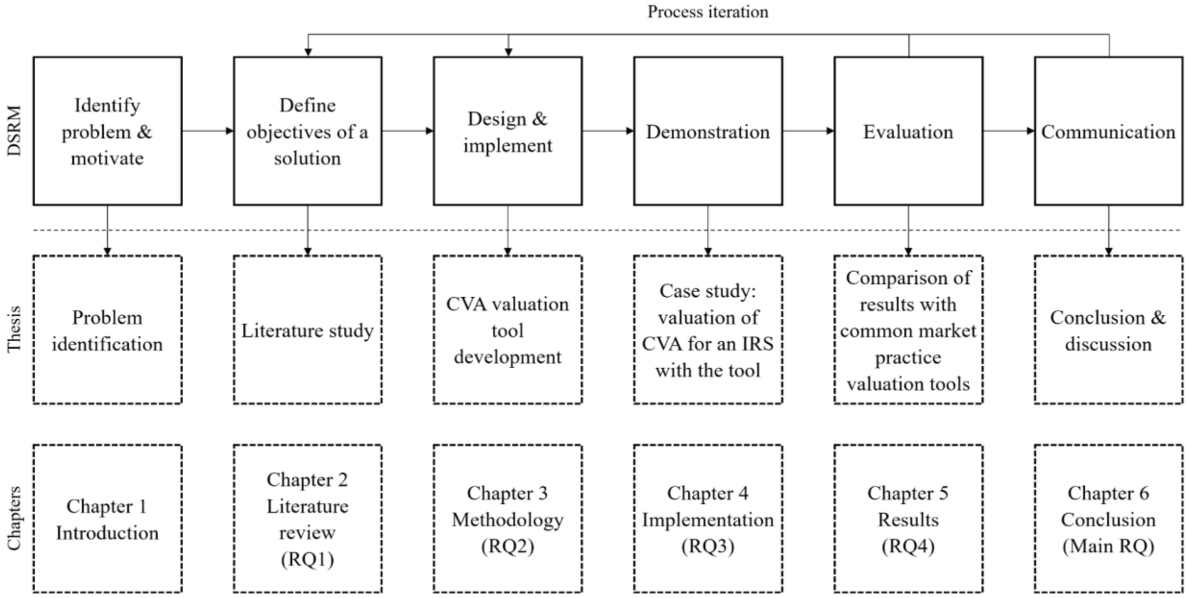


Figure 2. The steps of the Design Science Research Methodology (DSRM) translated to this thesis and chapters.

How the DSRM translates to our specific research is visualized in *Figure 2*, and further elaborated below.

Phase 1: Problem identification and motivation. This phase is focused on defining the specific research problem and justifying the value of a solution. That includes creating a problem cluster showing the relationship of problems and choosing the corresponding core problem. This core problem is translated into a knowledge problem with corresponding research questions.

Phase 2: Define the objectives for a solution. This phase is focused on inferring the objectives of a solution for the problem definition and knowledge of what is possible and feasible. For this thesis that translates to solving the knowledge problem by conducting a literature study.

Phase 3: Design and development. In this phase the design of the solution is chosen that will later be developed. Here phase 2 serves as the foundation of the design. Concretized for this thesis that means that based on the literature study conducted in phase 2, the methodology of the model that salves the core problem will be chosen.

Phase 4: Demonstration. In this phase the use of the artifact to solve the problem is demonstrated. This could involve its use in experimentation, simulation, case study, proof or other appropriate activity. The demonstration phase in this thesis includes an empirical application that values CVA for a specific interest rate swap. This valuation includes model calibration and determination of the parameters in order to use the model effectively.

Phase 5: Evaluation. The evaluation phase is used to observe and measure how well the artifact supports a solution to the problem. That means that for this thesis the results of the CVA model will be compared to Bloomberg CVA valuations. Additionally, a sensitivity analysis will be executed to evaluate how sensitive the model is to changes in the values of parameters.

Phase 6: Communication. This phase is used to communicate the effectiveness of the solution to researchers and practicing professionals that will use the CVA valuation model. This phase will translate to the discussion and conclusion chapter of this thesis.

2 Literature review

This chapter corresponds with the second step of the DSRM: *define objectives of a solution*. This means that in this chapter the literature review will be conducted to answer the research question: '*how is CVA calculated for interest rate swaps in the existing literature?*'. This research question is divided into two sub research questions. The first sub research question is focused on the definition of CVA and is defined as: '*what are the components of CVA?*'. The second sub research question will be a literature review that creates a theoretical framework showing all modelling methodologies used in the literature to calculate CVA for interest rate swaps. The second sub research question is defined as: '*how does the literature model/calculate each component of CVA for interest rate swaps?*'.

2.1 The components of credit valuation adjustments (CVA)

This section formulates an answer to the sub research question: '*what are the components of CVA?*'. The main literature used in this section is the research of Gregory (2012), Canabarro & Duffie (2003) and Zhu & Pykhtin (2007). A general introduction to counterparty credit risk is given by Gregory. A more in depth discussion on the quantitative measures for counterparty credit risk is given by Canabarro & Duffie. Furthermore, the paper of Zhu & Pykhtin provides an excellent discussion on modelling and pricing of counterparty credit risk.

Counterparty credit risk, commonly referred to as counterparty risk, involves the potential that the party with whom an individual has engaged in a financial arrangement (referred to as the counterparty) may not fulfil their obligations as outlined in the contractual agreement, i.e., by defaulting. Credit valuation adjustment (CVA) is the quantification, or the market value, of this counterparty risk. This can also be interpreted as the expected loss from a default by the counterparty. As a response to the 2007-2008 financial crisis, IFRS 13 was introduced in 2011 and became effective on the 1st of January 2013. IFRS 13 obliged dealers to calculate CVA for each counterparty with whom they have bilaterally cleared OTC derivatives and adjust the derivative value for this CVA. The adjusted derivative value f_0^* is the value of the derivative contract today assuming no defaults f_0 minus the CVA.

$$f_0^* = f_0 - CVA \tag{1}$$

The formula to calculate CVA on the interval of $[0, T]$ is given by Gregory (2012)

$$CVA = (1 - R) \int_0^T EE^*(t) dPD(0, t) \tag{2}$$

We can see that CVA consists of three different components: the loss (of the derivative) given default (by the counterparty) $(1 - R)$, the risk neutral expected exposure of the derivative $EE^*(t)$ and the probability of default by the counterparty $dPD(0, t)$ (Zhu & Pykhtin, 2007). Each component will be explained in more detail in the remainder of this section.

2.1.1 Loss given default (LGD)

When a company declares bankruptcy, creditors who are owed money by the company submit claims. Occasionally, there is a reorganization where these creditors agree to receive partial payment of their claims. Alternatively, the liquidator sells off assets, and the proceeds are utilized to settle the claims to the extent possible. Certain claims typically hold priority over others and are satisfied to a greater extent (Hull, 2012).

The recovery rate, denoted as R, for a derivative is conventionally defined as the price at which it trades approximately 30 days following default, expressed as a percentage of its face value. This value is just a constant and the recovery rate used in price calculations for credit default swaps is by default 40% (Bloomberg, 2024). Consequently, the loss given default by the counterparty equals 60% can be calculated as

$$LGD = (1 - R) \tag{3}$$

For credit default swaps concerning the Japanese Yen, the recovery rate is often set to 35% (Bloomberg, 2024). For each credit default swap the recovery rate used for price calculations is specified beforehand and can be find on Bloomberg. Therefore it is important to consult Bloomberg for the recovery rate before calculating CVA.

2.1.2 Risk neutral expected exposure (EE)

Counterparty exposure is the amount that a company could potentially lose in the event of a default by the counterparty in the absence of recovery. In other words, counterparty exposure equals the maximum of the market value of a derivative and zero. Let $V_i(t)$ be the value of contract i at time t , then the contract-level exposure of contract i at time t is denoted as $E_i(t)$ and is quantified as

$$E_i(t) = \max\{V_i(t), 0\} \tag{4}$$

The value of the contract is only known at the current time, future exposure is uncertain because the value of the contract unpredictably changes over time. In practice, it is common to have more than one trade with a specific counterparty. Here the counterparty exposure $E(t)$ can be calculated as the sum of all exposures at contract-level

$$E(t) = \sum_i E_i(t) = \sum_i \max\{V_i(t), 0\} \tag{5}$$

The exposure can be significantly minimized through the implementation of netting agreements. These legally binding agreements enable the consolidation of transactions in the event of a default. Essentially, transactions with negative values can be set off against those with positive values, resulting in only the net positive value representing the credit exposure during a default. Consequently, the overall credit exposure arising from all transactions within a netting agreement is limited to the maximum of the net portfolio value and zero

$$E(t) = \max\left\{\sum_i V_i(t), 0\right\} \tag{6}$$

In addition to netting agreements, also collateral agreements must be incorporated into the calculation of $E(t)$. Mark-to-market (MTM) is an accounting method used to record the value of assets or liabilities based on their current market prices or fair values. Collateral agreements require counterparties to periodically mark-to-market their positions and to provide collateral (i.e., to transfer the ownership of assets) to each other as exposures exceed pre-established thresholds. Collateral agreements do not eliminate all counterparty risk, market movements can increase the exposure between the time of the last collateral exchange and the time when default is determined and the trades are closed out. Usually the threshold amount, which determines when collateral obligations are triggered, is typically determined based on the credit ratings of the parties involved in the derivatives transaction. Suppose that $C(t)$ is defined as the collateral posted by the counterparty at the time t and H is the threshold value, then the $C(t)$ can be denoted as

$$C(t) = \max\{E(t-s) - H, 0\} \quad (7)$$

Here s is the margin period of risk, which is the time interval from the last exchange of collateral until the defaulting counterparty is closed out and the resulting market risk is re-hedged. Finally, the exposure $E(t)$, including netting and collateral agreements, will be calculated as

$$E(t) = \max\left\{\sum_i V_i(t) - C_i(t), 0\right\} \quad (8)$$

In the context of derivatives and financial modelling, being "risk-neutral" refers to the assumption that investors do not have any preference for or aversion to risk. They are solely concerned with maximizing expected returns from their investments and are indifferent to the level of risk associated with those investments. The risk-neutral probability refers to a situation where the expected return on an investment is equal to the risk-free rate of return, regardless of the level of risk associated with the investment.

Expected Exposure EE^* refers to the risk-neutral expected financial loss that a company might face if the counterparty defaults, discounted at the risk-free rate. That is the average exposure weighted by their risk-neutral probabilities (the distinction between risk-neutral and actual expectations is emphasized with an asterisk), see *Figure 3*. The curve of $EE^*(t)$, as t varies over future dates, provides the expected exposure profile and is denoted by

$$EE^*(t) = \mathbb{E}[E(t)] \quad (9)$$

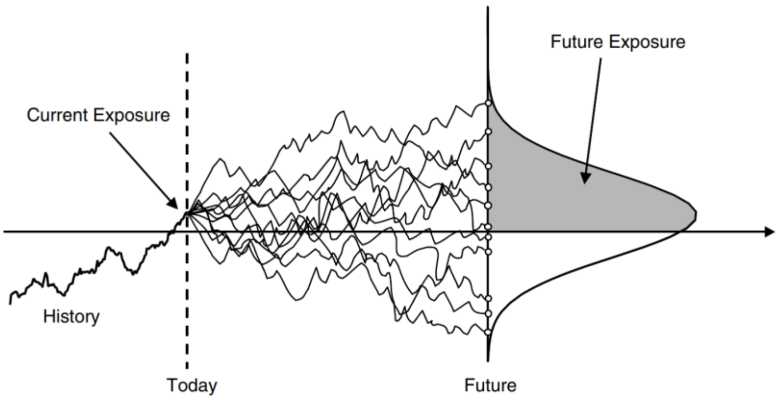


Figure 3. Illustration of future exposure with the grey area representing the PFE and white represents the NFE (Gregory, 2012).

There are multiple ways to calculate this expected exposure of an interest rates swap. The most simplistic approach is called the "mark-to-market + add-on" approach. This approach determines the expected exposure of an interest rate swap by adding a component that represents that uncertainty to the current exposure. This add-on component should incorporate the maturity of the interest rate swap; the volatility of the underlying; and characteristics of the underlying. The major drawback of the "mark-to-market + add-on" approach is that it is too simplistic to take other effects into account, like netting agreements, collateral agreements, payment frequencies, payer versus receiver swaps and floating reference rates (Gregory, 2012).

The semi-analytical approach overcomes the drawbacks of the "mark-to-market + add-on" approach. This approach involves formulating basic assumptions about the risk factors influencing the exposure. Based on these risk factors the probability distribution of the exposure is determined and semi-analytical approximation is derived. Despite being more accurate (and complex) than the “mark-to-market + add-on” approach, there are still several drawbacks. First of all, this approach relies on simplifying assumptions about the underlying risk factors, which neglects complex dynamics like mean reversion. Consequently, these methods calculate exposures independently over time, potentially overlooking path-dependent features. Typically, they yield a single risk measure, such as Potential Future Exposure (PFE), rather than the complete distribution. Furthermore, netting and collateral agreements are not incorporated in this approach (Gregory, 2012).

The most accurate (but also time-consuming and complex) approach is Monte Carlo simulation. This method is highly versatile and addresses numerous complexities that “mark-to-market + add-on” and the semi-analytical approach often overlook, such as transaction specifics, path dependency and netting and collateral agreements. The accuracy of the Monte Carlo simulation is dependent on the number of simulation runs. More simulation runs will also increase the computation time, therefore the ideal number of simulation runs should be investigated during the Monte Carlo study. The Monte Carlo method involves the following steps:

1. *Factor choice*. It is essential to identify the risk factors that will impact the exposure of the transaction. This identification includes the selection of an appropriate model for these risk factors, which subsequently determines the PFE. Examples of risk factors include variables like spot interest rates and spot FX rates, or more complex factors such as implied volatilities. The chosen model for the risk factor can range from a simple one-factor model to a more intricate multi-factor model.

2. *Scenario generation*. Future market scenarios are simulated by employing evolutionary models of risk factors for a predetermined set of simulation dates. This predetermined set of simulations dates must be reasonably large to accurately capture the PFE, however too many simulation dates will increase the computation time. According to Gregory (2012), a typical value for the number of simulation dates ranges between 50 and 200. The risk factor that influences the price of interest rate swaps is the interest rate. Therefore, the future market scenarios that will be generated in this study will heavily rely on interest rate simulation models.

3. *Instrument valuation*. Instrument valuation is conducted for each trade in the counterparty portfolio on every simulation date and for each realization of the underlying market risk factors.

4. *Portfolio aggregation*. On each simulation date and for every realization of the underlying market risk factors, the exposure at the counterparty level is determined using *Equation (8)*, applying the required netting rules and incorporating collateral agreements.

2.1.3 Probability of default (PD)

The market value of the counterparty risk also depends on the risk-neutral probability of a loss from a default by the counterparty. The risk neutral probability of counterparty default between times s and t is denoted as $PD(s,t)$. Mathematically, default is represented by means of the default time, which is defined as the first jump time of a Poisson process that models the occurrence of default as a random event. The default time, typically denoted by τ , is a random variable that can be modelled in several ways. There are essentially two paradigms that emerged over the years: reduced and structural form models. Reduced form models rely on credit default swaps (CDS) market quotations. This approach is in line with risk-neutral valuation, therefore the probability of default does not represent the real-world probability of default. Structural form models determine the probability of default based on the internal structure of the firm using historical data, representing a real-world default probability instead of a risk-neutral default probability (Brigo & Mercurio, 2006).

2.1.4 Other aspects of CVA worth noting

In addition to the different components of CVA, there are also other important aspects to keep in mind. These aspects are wrong-way risk and bilateral CVA and are explained in this section.

Wrong-way risk

In the CVA formula given in *Equation (2)* the assumption is made that there is an independence between expected exposure and the counterparty's probability of default. A scenario characterized by a positive dependence between the two, so a high (low) probability of default by the counterparty when the dealer's exposure to the counterparty is high (low), is referred to as "wrong-way risk". Conversely, when there is a negative dependence, so a high (low) probability of default by the counterparty when the dealer's exposure is low (high), is referred to as "right-way risk" (Hull & White, 2012).

Wrong-way risk often arises when a counterparty engages in selling credit protection to the dealer. This is due to the correlated nature of credit spreads. When credit spreads are high, the value of the protection to the dealer increases, leading to a substantial exposure to its counterparty. Simultaneously, the counterparty's credit spreads are likely to be high, indicating a comparatively higher probability of default for the counterparty (Hull & White, 2012).

Wrong-way risk, as suggested by the name, is considered as a negative risk. The reason for this is that a positive correlation between expected exposure and probability of default will lead to a higher CVA than obtained by *Equation (2)*. However, because of time constraints the incorporation of wrong-way risk is neglected in this thesis and the assumption is made that there is an independence between expected exposure and the counterparty's probability of default.

Bilateral CVA

Up to now, CVA is calculated as unilateral CVA. For unilateral CVA the assumption is made that only the counterparty can default, while the dealer is considered non-defaultable. In reality this assumption does not hold true because there is also an expected cost to the counterparty, this cost is referred to as *debit valuation adjustment* or DVA. DVA is the mirror image of CVA and is a cost to the counterparty and must be a benefit for to the dealer. DVA is defined as follows

$$DVA = (1-R)\int_0^T ENE^*(t)\, dPD(0,t) \tag{10}$$

Where $ENE^*(t)$ is defined as the expected negative exposure at time t (in other words, the $EE^*(t)$ from the point of view of the counterparty). Accounting standards recognize both CVA and DVA and including both CVA and DVA is referred to as bilateral CVA (BCVA). The formula for BCVA is given by

$$BCVA = CVA - DVA \tag{11}$$

2.2 Methodologies used in the literature to model CVA

In this section a literature review is conducted to answer the sub research question: '*how does the literature model each component of CVA for interest rate swaps?*'. The purpose of this section is the create a theoretical framework displaying an comparison of all methodologies used in the literature. This theoretical framework can be used later in the thesis to choose the final methodology that will be implemented in the CVA valuation model. The previous section showed that the LGD component of CVA is just a constant that can be retrieved from Bloomberg. Therefore only the determination of the risk-neutral expected exposure and the probability of default require modelling methodologies.

2.2.1 Interest rate models (expected exposure)

As explained in section *2.1.2 Risk neutral expected exposure (EE)*, the expected exposure of interest rate swaps depends on different interest rate scenarios. Therefore this section dives into the different interest rate simulation models that are used in the literature (for expected exposure modelling).

There are two type of one-factor models in the literature: equilibrium and no-arbitrage models. Equilibrium models are time-homogeneous and produce an endogenous term structure, where the interest rate curve is derived from the model's assumptions. In contrast, no-arbitrage models assume time-varying parameters resulting in an exogeneous term structure, allowing the model to exactly match the yield curve observed in the market. In less technical terms the difference between equilibrium and no-arbitrage models can be explained by the perspective from which they predict interest rates. Equilibrium models are focused on understanding the broader market dynamics and the behaviour of market participants. With interest rates the equilibrium is the interest rate such that at that rate the total amount of banks and other groups are willing to lend is equal to the total amount of money that people want to borrow. The problem with equilibrium models is that it requires some knowledge about the preferences of the market participants. To retrieve the supply and demand curves the risk preferences of both side of the market need to be known. No-arbitrage models are focussed on eliminating opportunities for risk-free profits in predicting interest rates. Resulting in the possibility to calibrate the model with observed market prices, and overcoming the disadvantage of equilibrium models (Lawson, 2015).

In this section both of these models are elaborated. In addition to one-factor models, there also exist two-factor interest rate models. However, two-factor models are out of the scope for this thesis because of their increased complexity to calibrate. Additionally, one-factor models are more often used in the literature to calculate CVA for interest rate swaps because of this increased complexity in two-factor models and they are sufficiently accurate to use in CVA valuations (Fabozzi, 2014).

2.2.1.1 Equilibrium models

The first short rate models proposed in the literature were time-homogeneous models whose diffusion coefficients are constant. The advantage of these models is that closed-from formulas of bonds and bond options prices can be derived by using the dynamics of the models. These formulas make it easy to calibrate the model. In addition, they can be used to evaluate all interest rate contingent claims in a consistent way. However, the classic problem with these models is they cannot reproduce the yield curve. This is caused by their endogenous nature.

Vasicek Model

In 1977, Vasicek is the first in the literature that assumed that the instantaneous spot rate under the real world measure evolves as an Ornstein-Uhlenbeck process with constant coefficients. The Vasicek model can be written down as

$$dr_t = \beta(\alpha - r_t)dt + \sigma dW_t \tag{12}$$

Where β, α and σ are non-negative constants. With β representing the long-term mean interest rate, α representing the mean reversion speed, r_t being the instantaneous interest rate or current level of interest rates, σ being the volatility of interest rate changes and W_t is Wiener process representing the random component of interest rate changes.

The main advantage of this model is that it incorporates mean reversion because the drift will become negative (positive) if the interest rate r_t is bigger (smaller) than the mean reversion speed α. This is caused by the fact that long-term mean interest rate β is a nonnegative constant. Furthermore, the model assumes r_t to be normally distributed resulting in the possibility of negative interest rates. This used to thought of as a drawback of the Vasicek model (Vasicek, 1977) (Bernal, 2016) (Yolcu, 2005), however in the current economic state with a long period of negative interest rates, this is rather an advantage than a drawback. In addition to that, the analytical tractability that is implied by a Gaussian density is hardly achieved when assuming other distributions for the process of the instantaneous spot rate (Joheski & Apostolov, 2021).

Moreover, there exist an analytical zero-coupon bond pricing formula based on the Vasicek model. Therefore, model parameters are easily calibrated by minimizing the error of market prices and model prices of zero-coupon bonds (Brigo & Mercurio, 2006).

Dothan Model

In 1978, Dothan introduced a model with lognormal interest rate to overcome the “drawback” of negative interest rates in the Vasicek model. The formula of this model equals

$$dr_t = \beta r_t dt + \sigma r_t dW_t \tag{13}$$

Where β and σ are non-negative constants. The disadvantage of this model, unlike Vasicek, is that it does not incorporate mean reversion (Yolcu, 2005). The Dothan model is the only lognormal short rate model in the literature with analytical formulas for pure discount bonds. Albeit, the formula is rather complex since it depends on two integrals of functions involving hyperbolic sines and cosines (Brigo & Mercurio, 2006).

Rendleman-Barttner Model

In 1980, Rendleman and Bartter assumed that the short-term interest rate behaves like a stock price. The following expression is suggested by the Rendleman-Barttner Model

$$dr_t = \mu r_t dt + \sigma r_t dW_t \tag{14}$$

Where μ and σ are non-negative constants. With μ representing the expected rate of return of the underlying asset. The disadvantage of this model is that interest rates behave different than stock prices and do not have an expected rate of return. Interest rates, unlike stock prices, tend to converge back to specific mean (Yolcu, 2005).

Marsh-Rosenfeld Model

In 1983, Marsh and Rosenfeld incorporated constant elasticity of the variance diffusion process, which is nested within the typical diffusion-Poisson jump model. The constant elasticity of variance process includes the "square root" and normal processes, and as a limiting case, the lognormal model.

$$dr_t = \left(\beta r_t^{-(1-\gamma)} + \alpha r_t\right) dt + \sigma r_t^{\frac{\gamma}{2}} dW_t \tag{15}$$

Where α, β, σ and γ are non-negative constants. If $\gamma = 1$, the model becomes a square root process with mean reverting drift (the model turns into a Cox-Ingersoll-Ross Model).

If $\gamma = 0$, it becomes,

$$dr_t = \left(\frac{\beta}{r_t} + \alpha r_t\right) dt + \sigma dW_t \tag{16}$$

When r_t becomes small in *Equation (15)*, and $\beta > 0$, the first term dominates, and large positive changes are expected. If r_t is very large, the diffusion term dominates, and the process behaves like an Ornstein-Uhlenbeck process (with proportional drift). *Equation (15)*, has the undesirable feature that mean reversion is not built into the drift (Yolcu, 2005).

Cox-Ingersoll-Ross Model

In 1985, Cox-Ingersoll and Ross presented the following interest rate model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t^{\frac{1}{2}} dW_t \tag{17}$$

Where β, α and σ are non-negative constants. The instantaneous short rate dynamics corresponds to a continuous time first-order autoregressive process with the randomly moving interest rate being elastically pulled towards a long term value α, meaning that the model incorporates mean reversion. Additionally, this model does not face negative interest rates. The diffusion term ensures that negative interest rates will not occur if the initial interest rate is nonnegative. This results from the fact that r_t can reach zero if $\sigma^2 > 2\alpha\beta$ and conversely the upward drift is sufficiently large to make the initial interest rate unreachable if $\sigma^2 \leq 2\alpha\beta$ (Joheski & Apostolov, 2021).

Furthermore, Cox-Ingersoll and Ross suggested a non-central chi-square distribution to represent interest rate changes over time. This assumption results in more realistic interest rate with skewness and a fatter tail with respect to normal distribution (Di Francesco & Kamm, 2022).

There does exist an analytical pricing formula for zero-coupon bonds under the CIR model. However, the instantaneous spot rate in the CIR formula and in the pricing formula of zero-coupon bonds do not follow the same distribution, and therefore classical optimization algorithms such as maximum likelihood estimation (MLE) and least squares methods (LSM) will fail (Amin, 2012). In 2001, Brigo and Mercurio concluded that there will exist failures in the calibration of the CIR model because the zero-coupon bond curve is quite likely to be badly reproduced (Brigo & Mercurio 2001).

2.2.1.2 No-arbitrage models

The shortcoming of one-factor equilibrium models is that due to their endogenous nature, they cannot reproduce the yield curve. To overcome this problem one-factor no-arbitrage are introduced. These models have an exogeneous nature caused by time-varying parameters, which can therefore reproduce the yield curve. Also financial instruments with different time to maturities can be fitted to these no-arbitrage models, which results in more accurate calibration. Another disadvantage of equilibrium models is that option pricing leads to arbitrage because today's term structure is an output rather than an input (Yolcu, 2005). This section elaborates on the one-factor no-arbitrage models described in the literature.

Ho-Lee Model

In 1986, Ho and Lee introduced the first no-arbitrage interest rate model

$$dr_t = \theta(t)dt + \sigma dW_t \tag{18}$$

with $\theta(t)$ being a random function of t and σ being a non-negative constant. In the Ho-Lee model, $\theta(t)$ is better understood as the drift term or the trend component that guides the direction of the interest rates over time. Here $\theta(t)$ is actually not a stochastic process. It is typically a deterministic function of time, which means it is a known, predictable function that can vary over time but does not have randomness associated with it. The stochastic component in the Ho-Lee model comes from the σdW_t term, where dW_t represents the random shocks from a Wiener process (or Brownian motion).

The advantage of this model is that the whole term structure can be used to price contingent claims. Because the short rates follow a normal distribution, it is possible that the model could produce negative interest rates. A disadvantage of this model is that it does not incorporate mean reversion (Lawson, 2015).

Hull-White (Extended Vasicek) Model

In 1990, Hull and White introduced the following model assuming a normal distribution for the short-rate

$$dr_t = [\theta(t) - \alpha(t)r_t]dt + \sigma(t)dW_t \tag{19}$$

Where $\theta(t)$, $\alpha(t)$ and $\sigma(t)$ are non-random functions of t. This model is an extension of the Vasicek Model. However, Hull (1996) showed that this time-dependency in the parameter $\theta(t)$ and $\sigma(t)$ can yield a nonstationary volatility term structure, which is undesirable when pricing instruments whose value depends on the term structure of future volatility (Kozpinar, 2022) (Svoboda, 2004). To overcome this problem, the extension of Hull and White (1994) can be used as well

$$dr_t = [\theta(t) - \alpha r_t]dt + \sigma dW_t \tag{20}$$

The volatility of the instantaneous short rate can be a function of time as well. This model does incorporate mean reversion, in contrast to the Ho-Lee model.

This model is analytically tractable in the sense that analytical formulas for zero-coupon bonds and options on them can be derived because of the Gaussian distribution of continuously-compounded rates. The Gaussian distribution also makes it possible to generate negative interest rates (Brigo & Mercurio, 2006).

A drawback of the model is its lack of state-dependent volatility. In reality, one would anticipate that a high short rate would exhibit greater volatility compared to a short rate approaching zero. However, the model assumes a fixed (or deterministic) level of volatility.

Extensions of the CIR Model

In 1990, Hull and White also introduced an no-arbitrage extension to the Cox-Ingersoll-Ross (CIR++) model.

$$dr_t = [\theta(t) - \alpha r_t]dt + \sigma\sqrt{r_t}dW_t \tag{21}$$

Where $\theta(t)$ is a non-random function of t and α and σ are non-negative constants. This process follows a non-central chi-square distribution. Accordingly, analytical formulas for prices of zero-coupon bond options, caps and floors, and, through Jamshidian's decomposition, coupon-bearing bond options and swaptions, can be derived.

Positive interest rates can be guaranteed in this model, by imposing restrictions on parameters. This might worsen the quality of the calibration to caps/floors or swaption prices. A drawback is that this model does not include jumps (e.g. caused by government fiscal and monetary policies and by release of corporate instruments) (Brigo & Mercurio, 2001). These are the reasons why this extension has been less successful than the Hull-White extended Vasicek model (Brigo & Mercurio, 2006).

In 2006, Brigo and El-Bachir added a jump component to the CIR++ model, referred to as the JCIR++ model. This ensures that the model may attain high implied volatilities (for swaptions or caps, in the present context) when the basic CIR++ model fails to do so. The model can be written down as

$$dr_t = [\theta(t) - \alpha r_t]dt + \sigma\sqrt{r_t}dW_t + dJ_t \tag{22}$$

Here J is a pure jump process, which is a type of stochastic process where changes occur only at discrete points in time, with no changes occurring continuously between these points. In other words, the process remains constant most of the time and jumps to new values at random times. In this model the pure jump process J has a jump arrival rate $\alpha > 0$ and jump sizes distribution π on $\mathbb{R}^+$. Here, π is an exponential distribution with mean $\gamma > 0$, and

$$J_t = \sum_{i=1}^{M_t} Y_i \tag{23}$$

Where M is a time-homogeneous Poisson process with intensity α. The Y is exponentially distributed with parameter γ.

In 2019, Orlando et al. introduced the CIR# model. This model has a different approach of capturing jumps than the JCIR++ model, here the available market data sample is divided into sub-samples to capture all statistically significant changes of variance in real spot rates. An "optimal" autoregressive integrated moving average (ARIMA) model will be fitted to each sub-sample of market data. The parameters are calibrated to the shifted market interest rates allowing to overcome the disadvantage of the instantaneous volatility σ being constant in the CIR models mentioned so far (Orlando et al., 2019).

Black-Derman-Toy Model

In 1990, Black, Derman and Toy introduced their interest rate model (BDT). The stochastic differential equation, assuming that the short rate follows a lognormal process, can be written down as

$$d \ln r_t = \theta(t)dt + \sigma dW_t \tag{24}$$

The model is algorithmically created by constructing a short-rate binomial tree to match the existing term structure of interest rates and volatilities. The lognormal process ensures that negative interest rates are not possible (Radhakrishnan, 1998). The main drawback is that the BDT model does not allow for a time-varying volatility parameter, which may distort interest rate forecasts over long time horizons (or over shorter time horizons if rates and spreads are sufficiently volatile) (Joshi & Swertloff, 1999)

Black-Karasinski Model

In 1991, Black and Karasinski introduced a generalization of the BDT model.

$$d \ln r_t = [\alpha(t) - \beta(t) \ln r_t]dt + \sigma dW_t \tag{25}$$

Where $\alpha(t)$ and $\beta(t)$ are non-random functions of t and σ is a non-negative constant. Brigo and Mercurio (2006) observed that “the rather good fitting quality of the model to market data, and especially to the swaption volatility surface, has made the model quite popular among practitioners and financial engineers”. However, Tourrucôo et al. (2007) addressed the shortcoming that this model turns out to be less tractable which renders the model calibration to market data than in the HW1 model because no analytical formulas for bonds are available. For this reason the Black-Karasinski model has been used less in the literature than the HW1 and CIR++ model.

2.2.1.4 Conclusion

The literature review showed that there are a lot of different interest rate models. Each model with its benefits and drawbacks. The single factor equilibrium models Vasicek (1977), Dothan (1978), Rendleman-Barttner (1980), Marsh-Rosenfeld (1983) and Cox-Ingersoll-Ross (1985) all have the advantage that they are analytically tractable because close-from pricing formulas can be derived. The drawback is that they do not reproduce the yield curve, making them a less accurate reflection of reality. Single factor no-arbitrage models do overcome this drawback by introducing time-varying parameters. The single factor no-arbitrage models Ho-Lee (1986), Hull White Extended Vasicek, CIR++, JCIR++, CIR#, Black-Derman-Toy (1990) and Black-Karasinski (1991) are analysed. Still these models have the drawback that they do not accommodate for twists in the term structure of interest rates and are limited to generating only increasing, decreasing, or slightly humped curves (Russo & Torri, 2019). This problem is tackled by multi-factor models, however these models are out of scope for this thesis. The main conclusion of the literature review is that there is a trade-off between economic realism and complexity. The more economically realistic the model is, the higher the computational complexity and calibration. An overview of single factor equilibrium models, together with their benefits and drawbacks are shown in *Table 1*. Contrarily, an overview of single factor no-arbitrage models and their benefits and drawbacks are given in *Table 2*. The scale of calibration accuracy ranges between poor, good, and excellent. Where ‘calibration accuracy’ can be divided into fit to historical data, stability of parameters, predictive power and sensitivity to market conditions. For an overview of the definition of calibration accuracy and the definition of the scale, see *Table 16* in *Appendix A – Definitions of the scale for ranking interest rate models*. The scale of computation complexity ranges from efficient, moderate, and complex.

Where ‘computation complexity’ can be divided into calibration time, resource usage and convergence reliability. For an overview of the definition of computation efficiency and the definition of the scale, see *Table 17* in *Appendix A – Definitions of the scale for ranking interest rate models*.

.

Table 1. Overview of single factor equilibrium models.

	Vasicek	**Dothan**	**Rendleman-Barttner**	**Marsh-Rosenfeld**	**Cox-Ingersoll-Ross**
Model type	Equilibrium	Equilibrium	Equilibrium	Equilibrium	Equilibrium
Drift term	$\beta(\alpha - r_t)$	βr_t	μr_t	$(\beta r_t^{-(1-\gamma)} + \alpha r_t)$	$\beta(\alpha - r_t)$
Diffusion term	σ	σr_t	σr_t	$\sigma r_t^{\gamma/2}$	$\sigma r_t^{1/2}$
Distribution	Normal	Lognormal	Lognormal	Lognormal	Non-central chi-square
Mean reversion	Yes	No	No	No	Yes
Positive interest rate	No	Yes	Yes	Yes	Yes, but can be zero
Volatility	Constant	Constant	Constant	Constant	Constant
Calibration accuracy	Poor	Poor	Poor	Good	Good
Computation	Efficient	Moderate	Moderate	Moderate	Complex
Major benefits	Mean reversion	Lognormal distribution (economic realism)	Lognormal distribution	Constant elasticity of the variance	Mean reversion
Major drawbacks	Lack of flexibility	No mean reversion	Interest rates behave as stock prices, no mean reversion	No mean reversion	Failures in calibration

Table 2. Overview of single factor no-arbitrage models.

	Ho-Lee	**Hull White**	**CIR++**	**JCIR++**	**CIR#**	**Black-Derman-Toy**	**Black-Karasinski**
Model type	No-arbitrage	No-arbitrage	No-arbitrage	No-arbitrage	No-arbitrage	No-arbitrage	No-arbitrage
Drift term	$\theta(t)$	$\theta(t) - \alpha r_t$	$\theta(t) - \alpha r_t$	$\theta(t) - \alpha r_t$	$\theta(t) - \alpha r_t$	$\theta(t)$	$\alpha(t) - \beta(t) \ln r_t$
Diffusion term	σ	σ	$\sigma\sqrt{r_t}$	$\sigma\sqrt{r_t} dW_t + dJ_t$	$\sigma\sqrt{r_t}$	σ	σ
Distribution	Normal	Normal	Non-central chi-square	Non-central chi-square	Non-central chi-square	Lognormal	Lognormal
Mean reversion	No	Yes	Yes	Yes	Yes	No	Yes
Positive interest rate	No	No	Yes, but can be zero	Yes, but can be zero	Yes, but can be zero	Yes	Yes
Volatility	Constant	Constant, but can be tir	Constant	Constant, but adds jum	Time-varying	Constant	Constant
Calibration accuracy	Poor	Good	Poor	Good	Excellent	Poor	Good
Computation	Efficient	Efficient	Moderate	Complex	Complex	Efficient	Moderate
Major benefits	Computationally efficient	Mean reversion, analytically tractable	Mean reversion	Includes jumps	Time-varying volatility by fitting	Easily constructed and computationally	Good fit to market data
Major drawbacks	No mean reversion	Volatility is not state-dependent	Positive interest rates need be forced by imposing restrictions	Constant volatility, computationally cumbersome	Computationally cumbersome	No mean reversion, constant volatility	Less analytical tractable

2.2.2 Hazard rate (probability of default)

As described in section *2.1.3 Probability of default (PD)*, the market value of the counterparty risk also depends on the risk-neutral probability of a loss from a default by the counterparty. There are essentially two paradigms that emerged over the years: reduced and structural form models. Reduced form models rely on credit default swaps (CDS) market quotations. This approach is in line with risk-neutral valuation, therefore the probability of default does not represent the real-world probability of default. Structural form models determine the probability of default based on the internal structure of the firm using historical data, representing a real-world default probability instead of a risk-neutral default probability. This section dives deeper into the methodologies used in the literature to estimate the risk-neutral probability of default for counterparties (reduced form models).

Reduced form models, sometimes also referred to as intensity models, describe default occurrences through an exogeneous jump process. Specifically, the default time (τ) is the first jump time of the Poisson process. This Poisson process can exhibit either deterministic or stochastic intensities. In reduced form models, default is not activated by basic market metrics; instead, it stems from an exogeneous component independent of all default-free market information. Therefore, monitoring market variables like interest rates and exchange rates do not provide comprehensive insight into the default mechanism, as default lacks an inherent economic rationale.

There are two main types of Poisson processes: homogeneous and inhomogeneous processes. The intensity function $\lambda(t)$ in a Poisson process determines the rate of event occurrence. In a Poisson homogeneous process, the intensity function $\lambda(t)$ is constant over time (deterministic), i.e., it remains constant for all t: $\lambda(t) = \lambda$. In a Poisson in-homogeneous process, the intensity function $\lambda(t)$ fluctuates over time.

Both Poisson homogeneous as Poisson in-homogeneous processes are used in the literature for the purpose of estimating the probability of default for CVA valuation. Van Vuuren and Esterhuysen (2014), Reghai and Kettani (2015), Wu (2015) and Xiao (2017) all used a deterministic hazard rate for the probability of default estimation in CVA valuation. The main reason for choosing a deterministic hazard rate for defaults is the simplicity and analytically tractability. On the other side, Hoffman (2011) and Brigo & Mercurio (2006) used a time-dependent hazard rate for defaults because of the increased accuracy. Especially in papers focused on CVA valuation including wrong-way risk, a time-dependent hazard rate is often used. In the remainder of this section, both approaches are elaborated in more detail.

2.2.2.1 Deterministic hazard rate

Let us consider a time homogeneous Poisson process to model the first jump, where the intensity function equals the moment the counterparty defaults. Recall that τ denotes the first jump in the Poisson process, i.e. the time until default. The number of jumps X_t in the interval $[0, t]$ is denoted as $X_t \sim P(\lambda t)$. Now the probability that there is no jump on the interval $[0, t]$ can be denoted as:

$$Q(\tau > \mathrm{t}) = \mathrm{P}(X_t = 0) = e^{-\lambda t} \tag{26}$$

Consequently, the probability that a default occurs can be written down mathematically as

$$Q(\tau \leq t) = 1 - P(X_t = 0) = 1 - e^{-\lambda t} \tag{27}$$

The conclusion can be made that the time until the first jump (i.e. the default time) follows an exponential distribution, because $Q(\tau \leq t)$ equals the exponential cumulative distribution function (CDF) (Brigo & Mercurio, 2006).

For a CDS with a deterministic hazard rate and maturity time T, the CDS pays $(1 - R)$ at the time of a credit event if default occurs before maturity. To secure protection, the purchaser makes a series of payments based on a spread denoted as S until either default or maturity, whichever comes first. This information can be used for the valuation of the premium and protection legs. For the premium leg, the protection buyer makes a payment of Sdt between time t and $t + dt$ if the credit remains has not defaulted. This payment is then discounted using the Libor factor and aggregated over the duration of the contract to yield (Brigo & Mercurio, 2006)

$$Value\ premium\ leg = S * \int_0^T Z(0,t)Q(0,t)dt \tag{28}$$

Here, $Z(t,T)$ is the (Libor) discount curve and $Q(t,T)$ is the survival probability up to T. For the protection leg, a payment of $(1 - R)$ is made if default occurs

$$Value\ protection\ leg = (1 - R) * \int_0^T Z(0,t) * \lambda(t)Q(0,\mathrm{t})\mathrm{dt} \tag{29}$$

Because the assumption is made that the hazard rate is deterministic, we can rewrite the value of the protection leg as

$$Value\ protection\ leg = \lambda(1 - R)\int_0^T Z(0,t)Q(0,\mathrm{t})\mathrm{dt} \tag{30}$$

Finally, the breakeven spread is determined as the CDS spread paid upon a new contract. Meaning that the CDS spread can be calculated by setting the premium leg equal to the protection leg

$$S * \int_0^T Z(0,t)Q(0,t)dt = \lambda(1 - R)\int_0^T Z(0,t)Q(0,t)dt \tag{31}$$

Looking at the equation, the $\int_0^T Z(0,t)Q(0,\mathrm{t})\mathrm{dt}$ cancels out and we are left with the following formula for the CSD spread

$$S = \lambda(1 - R) \tag{32}$$

Or

$$\lambda = \frac{S}{1 - R} \tag{33}$$

Recall that we derived the formula for the survival probabilities in *Equation 26*, now filling in the formula for the deterministic hazard rate we obtain

$$Q(\tau > t) = e^{-\lambda t} = e^{-\frac{S}{1-R}t} \tag{34}$$

And consequently, the formula for the probability of default, is given by

$$Q(\tau \leq t) = 1 - e^{-\lambda t} = 1 - e^{-\frac{S}{1-R}t} \tag{35}$$

2.2.2.2 Time-dependent hazard rate

The same approach as in section *2.2.2.1 Deterministic hazard rate* can be used to derive the time-dependent function $\lambda(t)$ (i.e. the hazard rate of default). Here, we can say that:

$$X_t \sim P(\Lambda(t)) \tag{36}$$

Where

$$\Lambda(t) = \int_0^t \lambda(u)du \tag{37}$$

The term $\Lambda(t)$ is called the cumulated intensity, cumulated hazard rate, or hazard function. Then, the CDF of the time to the first jump (i.e. the CDF of the default time) becomes:

$$Q(\tau \leq t) = F(t) = 1 - \exp(-\Lambda(t)) = 1 - \exp\left(-\int_0^t \lambda(u)du\right) \tag{38}$$

Consequently, the formula for the survival probability from time 0 to time T is obtained as

$$Q(\tau > t) = Q(0,t) = \exp(-\Lambda(t)) = \exp\left(-\int_0^t \lambda(u)du\right) \tag{39}$$

This formula shows that the hazard rate $\lambda(t)$ needs to be obtained in order to calculate the probability of default. This is done by deriving an expression for the par CDS spread, denoted by S_0. Which is achieved by evaluating the present value of the payment received when defaulting (protection leg) and the cost of paying for this protection (premium leg) (Brigo & Mercurio, 2006).

Because the S_0 will be retrieved by using an iterative algorithm, from now on we will look at the discretized formulas instead of continuous formulas. The discretized formulas below will evaluate the valuation of the protection and premium leg over N number of discrete time steps or payment periods. Assuming that the hazard rate process, interest rates, and recovery rates are independent, the discretized present value of the protection leg is given by

$$Protection\ PV = \frac{1-R}{2}\sum_{n=1}^{N}[Z(t,t_n) + Z(t,t_{n-1})][Q(t,t_{n-1}) - Q(t,t_n)] \tag{40}$$

The discretized present value of the premium leg is given by

$$Premium\ PV = S_0 \sum_{n=1}^{N} \Delta(t_{n-1},t_n)Q(t,t_n)Z(t,t_n)$$

$$+\frac{1}{2}\Delta(t_{n-1},t_n)\sum_{n=1}^{N}Z(t,t_n)[Q(t,t_{n-1})-Q(t,t_n)] \tag{41}$$

Here, S_0 represents $S(0,T)$, i.e., the fixed contractual spread of a contract traded at time 0 which matures at time T, and $\Delta(t_{n-1},t_n)$ is the day count fraction between dates t_{n-1} and t_n in the appropriate day count convention, typically Actual/360.

The present value of the premium leg can also be denoted as $S_0 * RPV01(0,T)$, where RPV01 is the risky PV01, i.e., the expected present value of 1bp paid on the premium leg until default or maturity. The variable T represents the maturity or termination time of the credit derivative, which is the point at which the contract ends. While T is not explicitly present in the formula below, it is implicitly embedded within the summation limits. Specifically, T defines the final time period t_N, which serves as the upper bound in the calculation of the RPV01. This risky PV01 can be denoted mathematically as

$$\begin{aligned} RPV01(t,T) = &\sum_{n=1}^{N}\Delta(t_{n-1},t_n)Q(t,t_n)Z(t,t_n) \\ &+\frac{1}{2}\Delta(t_{n-1},t_n)\sum_{n=1}^{N}Z(t,t_n)[Q(t,t_{n-1})-Q(t,t_n)] \end{aligned} \tag{42}$$

Now, the mark-to-market value of a CDS is given by the difference between the protection leg and the premium leg. With a face value of \$1, the mark-to-market value of a CDS $V(t)$ at time t is given below. Here N is the number of discrete time intervals between current time t and maturity T.

$$\begin{aligned} V(t,T) &= Protection\ PV - Premium\ PV \\ &= \frac{1-R}{2}\sum_{n=1}^{N}[Z(t,t_n)+Z(t,t_{n-1})][Q(t,t_{n-1})-Q(t,t_n)] - \\ &\qquad S_0 * RPV01(0,T) \end{aligned} \tag{43}$$

Finally, the breakeven spread is determined as the CDS spread paid upon a new contract. Meaning that the CDS spread can be calculated by solving $V(0)=0$, doing so result in the following formula (Pereira, 2014)

$$S_0 = \frac{1-R}{2}\frac{\sum_{n=1}^{N}[Z(0,t_n)+Z(0,t_{n-1})][Q(0,t_{n-1})-Q(0,t_n)]}{RPV01(0,T)} \tag{44}$$

In section *2.2.2.1 Deterministic hazard rate* it was possible to cancel out the $Z(0,t)$ and $Q(0,t)$ terms. However, by assuming an time-dependent hazard rate, this is not the case anymore and there is no analytical formula for the hazard rate. To retrieve the default probabilities we should perform an iterative algorithm. This algorithm builds the survival curve step-by-step by considering the shortest-dated instrument first and moving to the longest-dated one. At each stage, the price of the subsequent instrument is used to infer a parameter, extending the survival curve to the next maturity point. Through this process, a survival curve is derived that accurately reflects the market dynamics. The algorithm to build the survival curve is created by O'Kane (2008) and includes the following steps:

1. Initialize the survival curve with $Q(T_0=0)=1$.In other words, the survival probability at time equals 0 is 1 because there is no default at time equals 0.

2. Set $m = 1$.

3. Determine the survival probability at time T_m (denoted as $Q(T_m)$) such that the mark-to-market valuation of the CDS maturing at T_m, with a market spread S_m, equals zero. Utilizing the following formula

$$S_0 = \frac{1-R}{2} \; \frac{\sum_{n=1}^{N}[Z(0,t_n) + Z(0,t_{n-1})][Q(0,t_{n-1}) - Q(0,t_n)]}{RPV01(0,T)} \tag{45}$$

   Where all required discount factors for the CDS mark-to-market calculation are interpolated from the known values $Q(T_1), \dots, Q(T_{m-1})$, with the exception of $Q(T_m)$, which is the value being determined.

4. Upon identifying the value of $Q(T_m)$ required to reprice the CDS maturing at T_m, we incorporate this time and corresponding value into our survival curve.

5. Set $m = m + 1$. If $m \leq M$ return to step (3).

6. We possess a survival curve consisting of $M + 1$ data points, with time intervals ranging from $0, T_1, T_2, \dots, T_M$ and values $1, Q(T_1), Q(T_2), \dots, Q(T_M)$.

The number of points M used in the survival curve impacts the accuracy of the default probability estimation. A higher M provides a more detailed survival curve, potentially increasing accuracy, but also requires more computational effort. The choice of M depends on the desired balance between accuracy and computational feasibility. Typically, M is chosen based on the granularity of available market data and the computational resources at hand.

2.2.2.3 Conclusion

To conclude, there are two ways to extract default probabilities from CDS. One way is by calibrating CDS assuming $\lambda(t)$ to be time-dependent. This approach is more advanced and accurate to find the risk-neutral survival probabilities, however this approach is also computationally cumbersome and relies on numerical (iterative) methods. The other way assumes $\lambda(t)$ to be deterministic and leads to a closed-form formula for λ. The advantage of this methodology for extracting default probabilities from CDS is that the interest rate curve is not needed. However, the results are less accurate than assuming $\lambda(t)$ to be time-dependent. An overview of both methodologies is given in *Figure 4*.

	Deterministic hazard rate	**Time-dependent hazard rate**
Assumption	$\lambda(t) = \lambda$	$\lambda(t)$
Major benefit	Interest rate curve is not needed	More accuracte
Major drawback	Less accurate	Computationally cumbersome and relies on numerical (iterative) methods
Implied hazard rate	$\lambda = \frac{S}{1-R}$	$S_0 = \frac{1-R}{2} \frac{\sum_{k=1}^{K}[Z(0,t_k) + Z(0,t_{k-1})][Q(0,t_{k-1}) - Q(0,t_k)]}{RPV01(0,T)}$
Probability of default	$Q(\tau \leq t) = 1 - e^{-\lambda t}$	$Q(\tau \leq t) = 1 - \exp\left(-\int_0^t \lambda(u)du\right)$

Figure 4. Overview of PD estimation methodologies.

3 Methodology

This chapter corresponds with the third step of the DSRM: *design and development*. Therefore, this chapter formulates an answer to the sub research question: '*what is the best methodology for Deloitte to use in their CVA valuation model for interest rate swaps?*'. This methodology will be chosen by internal discussions with experts from the FRM team, based on the literature study conducted in chapter *2 Literature review*. Additionally, further research is conducted on the calibration of the chosen methodology, which serves as the foundation of the final model.

3.1 The most suitable methodology for the CVA valuation model

In this section, we determine the methodology to be employed in Deloitte's CVA valuation model. In this section we decide on the interest rate model that is going to be used in the determination of the expected exposure. This decision will be made based on a comparison of the interest rate models from section *2.2.1 Interest rate models (expected exposure)* and on a predefined set of criteria. In this section, we also decide whether to assume a deterministic hazard rate or a time-dependent hazard rate for estimating the probability of default.

3.1.1 The most suitable interest rate model

Together with the specialists from the FRM team of Deloitte, a predefined set of criteria is created. The following criteria are of importance in determining the most suitable interest rate model to implement in the CVA valuation model.

Computational efficiency: It should be computationally efficient to handle large portfolios of interest rate swaps and perform calculations in a reasonable time frame.

Economic realism (accuracy): The model should accurately capture the behaviour of interest rates and their impact on the valuation of interest rate swaps. The model should be robust enough to handle various market scenarios, including stressed market conditions and extreme movements in interest rates.

Calibration: The model should be easily calibrated to market data, ensuring that it reflects current market conditions accurately. This ease of calibration is crucial because the CVA tool is designed to function automatically, without requiring manual intervention. An automated calibration process ensures that the model can continuously adapt to new market data, allowing for efficient and accurate CVA calculations with minimal user input.

Flexibility: The model's flexibility should allow for future expansion to accommodate different derivatives while enhancing accuracy. This may include incorporating features such as time-varying volatility parameters or extending from a single-factor to a multi-factor model.

The criteria are not equally important. Therefore weights are assigned to each criterium. The weights are determined based on the Analytical Hierarchy Process (AHP). In line with the AHP, the specialists of the FRM team of Deloitte conducted a pairwise comparison of each criterium. The scale of this pairwise comparison ranges from one to nine, where one implies that the two criteria are the same or are equally important. On the other hand, nine implies that one criterium is extremely more important than the other one. For the complete steps in this process see *Appendix B – AHP weights for interest rate model selection*. For now, the final weights for each criterium given Deloitte's perspective is shown in *Table 3*.

Table 3. Requirements of the interest rate model.

Requirement	Weight
Computational efficiency	0.166
Economic realism	0.510
Calibration	0.287
Flexibility	0.037

Finally, a score ranging from 1 to 5 is given to all criteria by the experts of Deloitte for each interest rate model. The interpretation of the scale is shown in *Table 20*, see *Appendix C – Definition of the scores for all criteria on the implemented interest rate model*.

The results of the comparison between interest rate models based on our predefined set of criteria is shown in *Table 4*. The interest rate models are ranked from best suited to worst suited for our CVA valuation model according to Deloitte's experts perspective. *Table 4* shows that the Hull-White Extended Vasicek (HW1) model emerges as standout choice for our CVA valuation model according to the specialists of Deloitte. This model demonstrates a clear advantage over its counterparts due to its balance between analytical tractability and economic realism.

Table 4. Best suited interest rate model for the CVA model according to Deloitte's FRM experts, ranked from best to worst.

	Computational efficiency	Economic realism	Calibration	Flexibility	Overall
Hull-White	3	4	4	5	3.87
Ho-Lee	3	3	5	1	3.50
CIR++	2	4	3	5	3.42
CIR	3	3	4	4	3.32
Black-Karasinski	2	4	3	1	3.27
Black-Derman-Toy	3	3	4	1	3.21
Marsh-Rosenfeld	4	2	5	1	3.16
JCIR++	1	5	1	3	3.11
CIR#	1	5	1	2	3.08
Vasicek	5	1	5	3	2.89
Dothan	4	1	5	1	2.65
Rendleman-Barttner	4	1	5	1	2.65

While acknowledged that single-factor models, including HW1, possess limitations in replicating the entirety of yield curve shapes, they exhibit notable strengths in capturing the term structure of interest rates effectively. The HW1 model, despite its simplicity, offers a robust framework capable of accommodating various market conditions, including the increasingly relevant scenario of negative interest rates. Moreover, its compatibility with closed-form pricing formulas for bonds and options enhances its appeal for our purpose.

Although multi-factor models promise greater flexibility and more nuanced representation of interest rate dynamics, the HW1 model's computational efficiency, ease of interpretation, and reduced parameter uncertainty make it an optimal choice for our objectives. Importantly, its adaptability allows for future enhancements, potentially leveraging extensions to the multi-factor Hull-White framework for further refinement of our CVA valuation model.

3.1.2 The most suitable PD estimation methodology

The literature study showed that there are two ways to extract default probabilities from CDS. One way is by calibrating CDS assuming $\lambda(t)$ to be time-dependent. This approach is more advanced and accurate to find the risk-neutral survival probabilities, however this approach is also computationally cumbersome and relies on numerical (iterative) methods. The other way assumes $\lambda(t)$ to be deterministic and leads to a closed-form formula for λ. The advantage of this methodology for extracting default probabilities from CDS is that the interest rate curve is not needed. However, the results are less accurate than assuming $\lambda(t)$ to be time-dependent.

The research of van Schuppen (2014) compared the outcomes of the two approaches in terms of default probabilities using CDS spreads as input and assuming a recovery rate of 40%. *Table 5* shows that the results of assuming a deterministic hazard rate are relatively close to the results of assuming a time-dependent hazard rate.

Table 5. Comparison of default probabilities with a deterministic hazard rate and a time-dependent hazard rate, assuming a recovery rate of 40% (van Schuppen, 2014).

Maturity (years)	**CDS spread**	**Deterministic hazard rate**	**Time-dependent hazard rate**	**Difference (%)**
1	192.5	3.16%	3.29%	0.13%
3	215	10.19%	10.45%	0.26%
5	225	17.1%	17.49%	0.39%
7	235	23.98%	24.64%	0.66%
10	235	32.41%	32.92%	0.51%

While a deterministic hazard rate may offer slightly less accuracy, it remains sufficiently reliable for estimating the probability of default. Additionally, opting for a deterministic hazard rate enhances the computational efficiency of the model, because there is no need for an interest rate curve. Hence, we have chosen to employ a deterministic hazard rate for estimating the probability of default.

3.2 Hull-White one-factor model

The HW1 is an extension of the Vasicek (1977) model, that assumes that short-term interest rate follows a stochastic process driven by a single source of uncertainty. The HW1 process can be described by the following stochastic differential equation

$$dr(t) = [\theta(t) - \alpha r(t)]dt + \sigma dW(t) \tag{46}$$

Where $r(t)$ represents the short-term interest rate at time t, the speed of the mean reversion is given by α, the long-run mean interest rate at time t is given by $\theta(t)$, the volatility of interest rates is denoted by σ and $W(t)$ is a Wiener process.

This model is analytically tractable in the sense that analytical formulas for zero-coupon bonds and options on them can be derived because of the Gaussian distribution of continuously-compounded rates. In this section, we derive analytical formulas for zero-coupon bonds and options, followed by an explanation of how these formulas can be utilized for model calibration.

3.2.1 Pricing of zero-coupon bonds and options

The analytical formulas from the HW1 model are derived according to Brigo and Mercurio (2006) and Russo and Fabozzi (2016). The assumption of one-dimensional dynamics for the instantaneous spot rate process r is very convenient for the derivation of analytical formulas for rates and bonds. These analytical formulas are defined, by using no-arbitrage arguments, as the expectation of the process r. The arbitrage-free price of a contingent claim at time t with payoff H_T at time T is, because of the existence of a risk-neutral measure, given by

$$H_t = E_t\{D(t,T)\, H_T\} = E_t\left\{e^{-H_T \int_t^T r(s)ds}\right\} \tag{47}$$

Here E_t denotes the expectation conditional on t under that measure, and $D(t,T)$ denotes the stochastic discount factor at time t for maturity T. A zero-coupon bond is characterized by a unit amount of currency available at time T, or mathematically $H_T = 1$. Therefore, the price of a zero-coupon bond $P(t,T)$ at time t with maturity T is given by

$$P(t,T) = E_t\left\{e^{-\int_t^T r(s)ds}\right\} \tag{48}$$

This expression shows that bond prices can by computed if the distribution of $e^{-\int_t^T r(s)ds}$ can be categorized in terms of the dynamics of r conditional on the information available at time t (Brigo & Mercurio, 2006).

Now, we will show that the short rate $r(t)$ in the HW1 process is normally distributed. Let us denote $f^M(0,T)$ as the market instantaneous forward rate at time 0 for maturity T. The market instantaneous forward rate is the partial derivative of $P^M(0,T)$ with respect to T, the market discount factor for maturity T.

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T} \tag{49}$$

Furthermore, the long-run mean interest rate at time t, is denoted as

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + a f^M(0,t) = \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{50}$$

By integrating the stochastic differential equation of the HW1 process (*Equation 46*), we get the formula below. Here s is the point in time that includes all the observable market data (in this case interest rates) and historical information available up to that point.

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dW(u)$$

$$= r(s)e^{-a(t-s)} + a(t) - a(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u) \tag{51}$$

With

$$a(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \tag{52}$$

Therefore, $r(t)$ conditional on the $\mathcal{F}_s$ σ-field, A σ-field is a mathematical structure that represents a collection of events (subsets of a probability space) that are measurable with respect to a given filtration up to time s. When $r(t)$ is said to be conditional on $\mathcal{F}_s$, it means that the value of the interest rate $r(t)$ is dependent on the information that is available up to time s. This reflects the notion that future values of rates are uncertain and can only be predicted based on past information. Brigo & Mercurio (2006) showed that $r(t)$ conditional on the $\mathcal{F}_s$ σ-field is normally distributed with mean and variance given respectively by

$$E\{r(t)|\mathcal{F}_s\} = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} \tag{53}$$

$$Var\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right] \tag{54}$$

Zero-coupon bonds

The valuation of a zero-coupon bond at time t is determined by the expected value, as indicated in *Equation 48*. This expected value computation is relatively straightforward within the framework of the HW1 process, as depicted in *Equation 46*. Notice that, because we concluded that $r(T)$ conditional on $\mathcal{F}_t, t \leq T$ follows a Gaussian distribution, we can conclude that $\int_t^T r(u)du$ is itself normally distributed (Brigo & Mercurio, 2006). Therefore, we can show that

$$\int_t^T r(u)du|\mathcal{F}_t \sim \mathcal{N}\left(B(t,T)[r(t) - \alpha(t)] + \ln\frac{P^M(0,t)}{P^M(0,T)} + \frac{1}{2}[V(0,T) - V(0,t)], V(t,T)\right) \tag{55}$$

Where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{56}$$

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \tag{57}$$

So that we obtain

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{58}$$

Where

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\right\} \tag{59}$$

European options on zero-coupon bonds

Brigo and Mercurio (2006) showed that the price of an European call option $ZBC(t,T,S,X)$ at time t, with strike X, maturity T and written on a zero-coupon bond maturing at time S is given by the expectation

$$ZBC(t,T,S,X) = E\left(e^{-\int_t^T r(s)ds}(P(T,S) - X)^+|\mathcal{F}_t\right) \tag{60}$$

Or, equivalent, by

$$ZBC(t,T,S,X) = P(t,T)E^T((P(T,S) - X)^+|\mathcal{F}_t) \quad (61)$$

The latter expectation can be computed if the distribution of the process r under the T-forward measure Q^T is known. Brigo & Mercurio (2006) showed that the short rate $r(t)$ conditional on $\mathcal{F}_s$ under the measure Q^T, is Gaussian with mean and variance given by the formulas below. The T-forward measure Q^T is a probability measure that is used to evaluate future cash flows as if they were known at a future time T. It effectively allows market participants to price cash flows that occur after time T by adjusting for the time value of money. In the formula, $M^T(s,t)$ adjusts the mean of the conditional expectation $E^T\{r(t)|\mathcal{F}_s\}$ based on how the short rate is expected to evolve from time s to t under the forward measure Q^T. This term encapsulates the dynamics of the short rate process, including factors such as the mean reversion or the impact of any shocks to the interest rate.

$$E^T\{r(t)|\mathcal{F}_s\} = x(s)e^{-a(t-s)} - M^T(s,t) + a(t) \quad (62)$$

$$Var^T\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right] \quad (63)$$

As a consequence the European call-option price is

$$ZBC(t,T,S,X) = P(t,S)\Phi(h) - XP(t,T)\Phi(h - \sigma_p) \quad (64)$$

Where

$$\sigma_p = \sigma\sqrt{\frac{1 - e^{-2a(T-t)}}{2a}}B(T,S) \quad (65)$$

$$h = \frac{1}{\sigma_p}\ln\frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2} \quad (66)$$

Similarly, the prize $ZBP(t,T,S,X)$ at time t of a European put option with strike X, maturity T and written on a zero-coupon bond maturing at time S is given by

$$ZBP(t,T,S,X) = XP(t,T)\Phi(-h + \sigma_p) - P(t,S)\Phi(-h) \quad (67)$$

European options on coupon-bearing bonds

The price of European options on coupon-bearing bonds can be determined with the Jamshidian's (1989) decomposition. The Jamshidian decomposition does not offer a closed-form solution for pricing coupon-bearing bonds. Consider a European option with a strike price X and maturity T written on a bond that pays out n coupons after the option expires. For each cash flow after T, denoted by T_i (where $T_i > T$), the value of this cash flow is denoted by c_i. Let $\mathcal{T} := \{T_1, \dots, T_n\}$ and $c := \{c_1, \dots, c_n\}$. Let X_i be the value of a pure-discount bond at time T maturing at T_i when the spot rate for which the coupon-bearing bond price equals the strike. Now, the Jamshidian decomposition solves the value of X_i numerically by changing the spot rate until the coupon-bearing bond prices equals the strike (Russo & Fabozzi, 2016). Then the option price at time $t < T$ is

$$CBO(t, T, \mathcal{T}, c, X) = \sum_{i=1}^{n} c_i ZBO(t, T, T_i, X_i) \quad (68)$$

European swaptions

Now that we have the analytical formula for pricing options on coupon-bearing bonds (*Equation 68*), we can use this for the analytical formula to price European swaptions. Because a European swaption can be seen as an option on a coupon-bearing bond. To illustrate this, consider a payer swaption with strike X, maturity T and nominal value N. The holder of this swaption has the right to enter into an interest rate swap at time $t_0 = T$ with payment frequencies $\mathcal{T} = \{t_1, \dots, t_n\}$, for which $t_1 > T$. For this interest rate swap the holder will pay the fixed rate X and receives the floating LIBOR rate in arears. The year fraction is denoted by τ_i from t_{i-1} to t_i for which $i = 1, \dots, n$. Then we can denote the cashflow associated with the interest rate swap by $c_i \coloneqq X\tau_i$ for $i = 1, \dots, n-1$ and $c_n \coloneqq 1 + X\tau_i$. Next, the spot rate at time T that equals the present value of the cash flows from the interest rate swap to the value of the swaption is denoted by r^*. Mathematically, this is the value of r^* for which the following equation holds

$$\sum_{i=1}^{n} c_i A(T, t_i) e^{-B(T, t_i) r^*} = 1 \quad (69)$$

Finally, we can set $X_i \coloneqq A(T, t_i) e^{-B(T, t_i) r^*}$, then the swaption price at time $t < T$ is given by

$$PSwpt(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i ZBP(t, T, t_i, X_i) \quad (70)$$

Similarly, the price of the corresponding receiver swaption is given by

$$RSwpt(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i ZBC(t, T, t_i, X_i) \quad (71)$$

3.2.2 Calibration

With the closed formulas derived in the previous section, the calibration of the HW1 model can be established using market swaption prices. The primary goal of the calibration process is to determine the model parameters in a manner that ensures the consistency of model prices with swaptions quoted in the market. This can be achieved through a numerical optimization technique, aiming to minimize the square root of the sum of the squares of the relative differences between market and model swaption prices,

$$\arg \min_{\beta} \sqrt{\sum_{i=1}^{N} \left(\frac{Swpt_i - Swpt_i^M}{Swpt_i^M} \right)^2} \quad (72)$$

Here, $Swpt_i^M$ denotes the market-quoted value of the swaption, and $Swpt_i$ represents the theoretical price of the swaption under the HW1 model. The calibration process involves N calibrated instruments, with β representing the parameter vector. In the case of the HW1 model, calibration is necessary for 2 parameters, α and σ.

It is worth noting that this approach is just one among various options available. Some authors suggest estimating the mean reversion parameter through historical data rather than integrating it into the optimization procedure mentioned above. Others opt for a two-step estimation process. For example, Schlenkrich (2012) suggests calibrating the mean reversion parameter using Bermudian swaptions and determining the volatility parameter using European swaptions.

In this thesis we will use the Levenberg-Marquardt algorithm to find the local minimum of the objective function (*Equation 72*). The Levenberg-Marquardt algorithm is the most widely used optimization algorithm because it outperforms simple gradient descent and other conjugate gradient methods. One disadvantage of the Levenberg-Marquardt is that it has the tendency to get stuck in local minima and fail to find the global minimum (Vollrath & Wendland, 2009). However, because of the non-convex nature of the objective function, finding a local minimum sufficiently calibrates the HW1 process. Additionally, convergence typically occurs fast and stable under the Levenberg-Marquardt algorithm (Hull & White, 2001).

4 Implementation

This chapter corresponds with the fourth step of the DSRM: *demonstration*. Therefore this chapter formulates an answer to the sub research question: *how can the chosen methodology be implemented to develop a CVA valuation model for interest rate swaps?*'. The CVA valuation model will be programmed in Python and its effectiveness will be demonstrated by valuing CVA of the interest rate swap shown in *Table 6*. This chapter starts with calibrating the Hull-White one-factor model (HW1). After calibration, the model can be used to simulate interest rate paths which will be used to calculate the expected exposure of the interest rate swap. Furthermore, the probability of default will be extracted from CDS in the market. All these steps are executed according to the methodology chosen in chapter *3 Methodology*. Finally, the value of CVA for this interest rate swap is calculated.

The details of the interest rate swap used in this chapter are shown in *Table 6*. Koninklijke Philips NV is the counterparty of this 5 year maturity fix-floating interest rate swap issued by ABN AMRO Bank NV. The interest rate swap has a notional of €10 million, an effective date of 31/12/2023,

- a fixed leg with a 3% rate, annual payment frequencies and a 30U/360 day count convention,
- and a floating leg with a 6M EUR EURIBOR rate, semi-annual payment frequencies and an ACT/360 day count convention.

Table 6. Interest rate swap under valuation.

Interest rate swap		
	Fixed leg	*Floating leg*
Notional	10MM	10MM
Currency	EUR	EUR
Effective date	31/12/2023	31/12/2023
Maturity	31/12/2028	31/12/2028
Rate	3%	6M EUR EURIBOR
Payment frequency	Annual	Semi Annual
Day count	30U/360	ACT/360
Counterparty	Koninklijke Philips NV	
Dealer	ABN AMRO Bank NV	

4.1 Calibration of the Hull-White one-factor model

The HW1 model can be calibrated using co-maturity swaptions. However, this approach may potentially lead to overfitting and parameter instability. To address this concern, the calibration process is executed using a volatility grid of European swaptions, characterized by various combinations of tenor and maturity. The volatility grid of swaptions consists of at the money (ATM) swaptions. A swaption is considered ATM when the strike of the underlying option equals the forward swap rate for the same swap maturity. The swaptions volatility grid displays normal volatilities rather than lognormal volatilities. Normal volatilities are computed using the Bachelier formula, whereas the Black formula relies on lognormal volatility. Nowadays, market preference leans towards normal volatilities due to the challenge posed by negative rates, where Black (lognormal) quotes become undefined. The European swaption volatilities are obtained from Bloomberg, see *Figure 12* in *Appendix D – Bloomberg data*.

The ESTR curve is used to discount the cash flows of the fixed and floating legs to price the swaption. The curve date is 12/31/2023, which is the date for which we want to calculate the value of CVA. The ESTR curve is obtained from Bloomberg, see *Figure 13* in *Appendix D – Bloomberg data*.

In order to calculate the theoretical price of swaptions under the HW1 model, the Jamshidian decomposition is used. The Jamshidian decomposition consists of dividing the swap into a set of zero-coupon bonds with strikes, so all are exercised under the same conditions, see chapter *3 Methodology* section *3.2.2 Calibration*.

Finally, the parameters of set $\beta = \{\alpha, \sigma\}$ are simultaneously changed to minimize the objective function (*Equation 72*) with the Levenberg-Marquardt optimization algorithm. The model showed that for $\alpha = 0.00001$ and $\sigma = 0.00801$ the square root of the sum of squares of the relative difference between market and model swaption prices is at a minimum. The RMSE of for these parameters is 0.305 and the full calibration report is depicted in *Table 7*.

Table 7. Calibration report based on the Levenberg-Marquardt optimization algorithm.

Maturity	Tenor	Market Volatility	Model Volatility	Relative Error
1	30	39.52	36.10	-0.0866
2	25	37.46	34.93	-0.0675
3	20	35.51	34.24	-0.0357
4	15	33.60	33.21	-0.0116
5	12	33.16	32.63	-0.0160
7	10	31.60	32.97	0.0432
10	7	30.62	32.20	0.0515
12	5	31.35	32.99	0.0524
15	4	33.71	37.32	0.1069
20	3	45.78	44.72	-0.0231
25	2	47.81	51.97	0.0871
30	1	79.70	61.40	-0.2297

Another commonly used method in the literature is calibrating the HW1 model with a predefined fixed value for α, usually derived from historical data, while allowing σ to vary Russo & Fabozzi (2019). The benefit of calibrating the HW1 model with a fixed α is the reduction of complexity and reducing the risk of overfitting. On the other hand, it may sacrifice some accuracy in fitting the short-term dynamics. This approach is often used in the literature when computational efficiency or stability is important, or when the mean reversion parameter α is assumed to be relatively stable over time.

Because the accuracy of fitting the short-term dynamics is considered important for Deloitte's CVA valuation model, this thesis will calibrate the HW1 model with a varying mean reversion α and volatility σ.

4.2 Interest rate simulation

In essence, the complexity of accurately modelling future exposures, the need to incorporate counterparty credit risk dynamically, and the flexibility required to handle various market conditions make simulation the only suitable methodology for calculating CVA in the case of an interest rate swap. Simulations capture the random nature of interest rates and default events more effectively than closed-form models, ensuring the CVA reflects the true risk of the position.

The HW1 model, once calibrated, is used to simulate the future evolution of the 6M Euribor, the underlying of the floating leg of the interest rate swap. This is crucial for calculating the expected exposure of the interest rate swap over time. The short-rate $r(t)$ is simulated using the calibrated parameters α and σ retrieved in section *4.1 Calibration of the Hull-White one-factor model*. The simulation is carried out over 500 paths using the Monte Carlo method, each path represents a possible future trajectory of the short-term interest rates based on the HW1 model. A Gaussian random number generator is used to simulate paths for the short-rate.

Based on these short-term interest rates, the yield curve can be derived from zero-coupon bonds. For each simulated short-term interest rate path, the zero-coupon bond prices $P(t,T)$ for various maturities T are calculated. The formula for the price of a zero-coupon bond under the HW1 model is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \quad (73)$$

Where $A(t,T)$ and $B(t,T)$ are functions dependent on the model parameters α and σ, and $r(t)$ is the simulated short rate at time t, see chapter *3 Methodology*.

Now, we compute the average zero-coupon bond price $\bar{P}(t,T)$ across all 500 paths for each maturity T. This step corresponds to the risk-neutral expectation of the bond price, as the HW1 model assumes the simulated paths follow a risk-neutral probability measure. The yield for each maturity T can be calculated using the following formula

$$y(t,T) = -\frac{1}{T-t} log\bar{P}(t,T) \quad (74)$$

Finally, the yield curve is obtained by repeating the above process for a range of maturities T. The yield curve represents the term structure of interest rates based on the average behaviour of the 500 simulated short-rate paths.

4.3 Expected exposure of the interest rate swap

For each simulated short-rate path, the net present value (NPV) of the interest rate swap is computed at each time step (simulation dates). The code uses weeks for the predetermined set of simulation dates, meaning that there are 260 simulation dates for the 5 year maturity interest rate swap, which is in line with the typical value of simulation dates according to Gregory (2012), as explained in chapter *2 Literature review*. The NPV depends on the floating and fixed legs of the interest rate swap, which are discounted using the simulated interest rates. For each simulated path, the exposure is calculated and visualized in *Figure 5*.

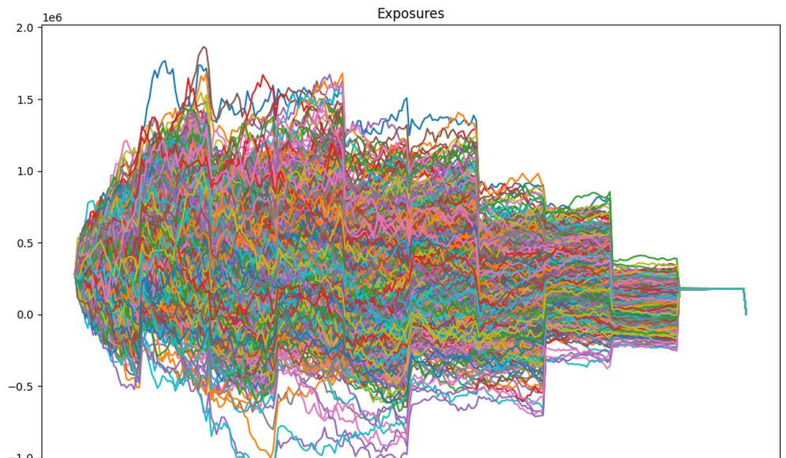


Figure 5. 500 simulation exposure paths of the interest rate swap.

Now, we will make a distinction between the expected positive exposure (EPE) and expected negative exposure (ENE). The reason why we split the exposure into EPE and ENE is that the EPE is needed for calculating the CVA, while the ENE is used for calculation DVA. The logic behind this is simple, the EPE represents the potential loss if the counterparty (Koninlijke Philips NV) defaults and the ENE reflects the exposure that the dealer (ABN AMRO Bank NV) owes to the counterparty (Koninklijke Philips NV), and is considered to be the dealer's own potential default.

The EPE equals, as the name suggests, all exposures with a positive NPV. The EPE is calculated as the average of the NPV of the exposures across all simulated paths. The EPE for the interest rate swap under valuation is visualized in *Figure 6*.

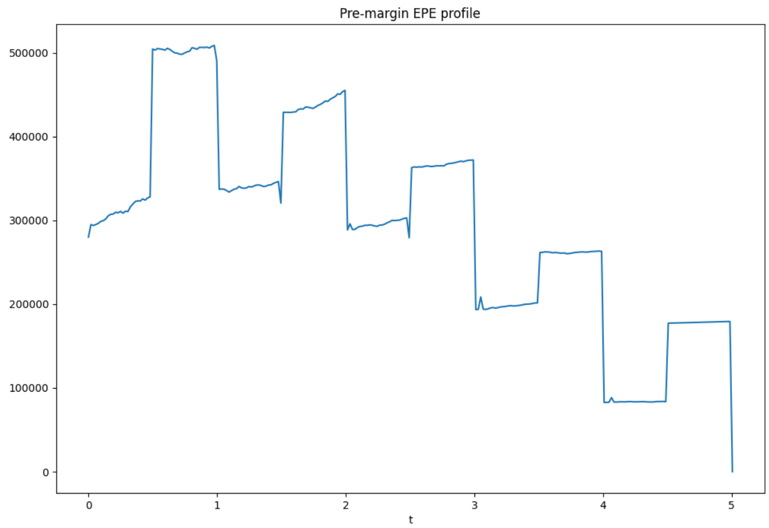


Figure 6. EPE profile of the interst rate swap based on 500 simulation runs.

In the case of this interest rate swap, several features make the EPE profile intuitive. The swap has a notional of 10MM EUR, with fixed annual payments on one leg and semi-annual floating payments on the other. As time progresses, exposure increases periodically in alignment with payment dates. After each payment, the risk associated with future interest rate movements builds up again until the next settlement date. This results in step-like increases in the EPE profile.

The floating leg, tied to the 6M EURIBOR, resets semi-annually. These rate resets introduce variability in the exposure. When interest rates rise, the floating leg payments increase, which could lead to higher exposure for the fixed payer, as they may owe more after each reset. This variability is reflected in the upward movements of the EPE profile after each reset, corresponding to the increased exposure from changes in floating rates.

As the swap approached maturity, the outstanding time decreases, reducing overall exposure. This explains the gradual downward steps after each major payment, as fewer rate resets remain, lowering future interest rate risk. Spikes in the EPE profile just before settlements reflect the accumulation of risk since the last payment, as interest rate risk builds up over time.

Consequently, the ENE equals all exposures with a negative NPV and is also calculated as the average across all simulated paths. The ENE for the interest rate swap under valuation is visualized in *Figure 7*. The elaboration of the intuition behind the ENE profile is neglected, because the same reasoning of the EPE profile holds true but in reverse.

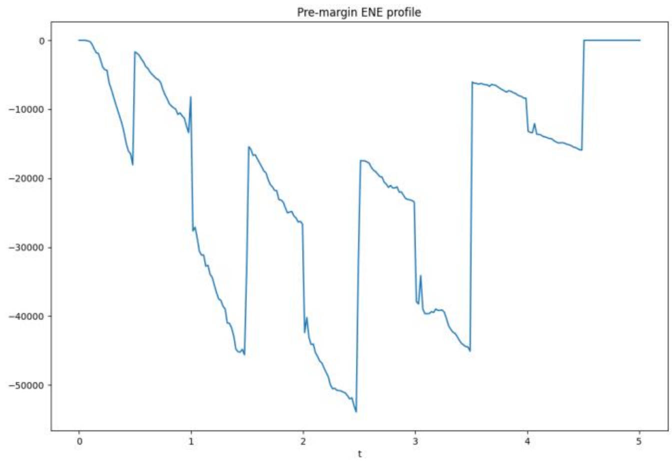


Figure 7. ENE profile of the interest rate swap based on 500 simulation runs.

4.4 Probability of default estimation

Another unknown that we need for the calculation of CVA is the probability of default. As explained in chapter *3 Methodology*, we assume an Poisson homogeneous process, where the intensity function $\lambda(t)$ (i.e., hazard rate) is constant over time (deterministic). The advantage of this methodology is that the default probabilities can be extracted from a CDS without the need for the interest rate curve. By assuming a deterministic hazard rate, the default probabilities can be calculated with the following formula

$$Q(\tau \leq \mathrm{t}) = 1 - e^{-\lambda t} = 1 - e^{-\frac{S}{1-R}t} \tag{75}$$

With S being the spread of the CDS, which is an market-implied indicator of default risk and R being the recovery rate, which is 40%.

Counterparty (Koninklijke Philips NV)

The probability of default for the counterparty is derived from the CDS curve for Koninklijke Philips NV. In the code, CDS spreads for various maturities are used to construct a piecewise-flat hazard rate curve, which provides the cumulative probability of default at each point in the future. The difference in cumulative probabilities between two time steps gives the incremental probability of default, which is used for the CVA calculation. In the code a piecewise-flat hazard curve is used to calculate the probability of default for each week, however for clarity and visual simplicity, we present the default probabilities in tabular form rather than displaying the full curve, see *Table 8*.

Table 8. Probability of default for counterparty (Koninklijke Philips NV) based on a recovery rate of 40% and curve date of 31/12/2023. (Source: Bloomberg)

Term	**Spread**	**Probability of default**
6 MO	19.72	0.16%
1 YR	21.33	0.35%
2 YR	31.27	1.04%
3 YR	43.93	2.20%
4 YR	52.68	3.52%
5 YR	61.95	5.17%
7 YR	82.55	9.57%
10 YR	93.20	15.06%

Dealer (ABN AMRO Bank NV)

Similarly, the default probability for the dealer is constructed using CDS spreads for ABN AMRO. The dealers' default risk impact the calculation of DVA, as it represents the potential loss to the counterparty if the dealer defaults. A default term structure is created to estimate the probability of default over time and the default probabilities are shown in *Table 9*.

Table 9. Probability of default for dealer (ABN AMRO Bank NV) based on a recovery rate of 40% and curve date of 31/12/2023. (Source: Bloomberg)

Term	**Spread**	**Probability of default**
6 MO	36.54	0.29%
1 YR	39.05	0.64%
2 YR	43.73	1.45%
3 YR	49.40	2.46%
4 YR	55.34	3.67%
5 YR	61.37	5.08%
7 YR	76.13	8.78%
10 YR	85.51	13.81%

If we compare both tables, we can see that the probability of default for the counterparty and dealer are close to each other. For ABN AMRO the probability of default is higher in the first 4 years, but after a maturity of 4 years the probability of default for Koninklijke Philips is higher.

4.5 Bilateral CVA valuation

Finally the bilateral CVA can be calculated. Recall from chapter *2 Literature review* that CVA can be calculated according to the following formula

$$CVA = (1 - R) \int_0^T EPE^*(t) dPD(0, t)_{counterparty} \tag{76}$$

And recall that DVA can be calculated according to the following formula

$$DVA = (1 - R) \int_0^T ENE^*(t)\, dPD(0, t)_{dealer} \tag{77}$$

The bilateral CVA considers both the credit risk of the counterparty (CVA) and the credit risk of the dealer (DVA) and can be calculated by subtracting DVA from CVA

$$BCVA = CVA - DVA \tag{78}$$

The CVA for Koninklijke Philips according to the HW1 model for interest rates and a 5 year maturity interest rate swap with a notional of €10 million, an effective date of 31/12/2023, a fixed leg with a 3% rate, annual payment frequencies and a 30U/360 day count convention, and a floating leg with a 6M EUR EURIBOR rate, semi-annual payment frequencies and an ACT/360 day count convention equals €7,771. This holds for a recovery rate of 40%. Conversely, the DVA for ABN AMRO equals €272. Resulting in a bilateral CVA of €7,499.

5 Results

This chapter corresponds with the fifth step of the DSRM: *evaluation*. Therefore, this chapter formulates an answer to the sub research question: '*how accurate is the valuation of interest rate swaps with the CVA model compared to common market practice valuations?*'. The most widely used valuation tool is Bloomberg, and therefore the results of the developed CVA model will be compared to the results from Bloomberg. Additionally, this chapter also serves the purpose of testing how reliable the CVA model is to changes in input parameters like the calibration parameters and the number of simulation runs.

5.1 Comparison to Bloomberg

In this section, we compare the outcomes of the CVA model with those generated by Bloomberg. While Bloomberg does not disclose the specifics of its CVA calculation methodology, it is reasonable to assume that it employs a sophisticated multi-factor interest rate model, given its widespread use. In contract, our model relies on the single-factor Hull-White framework. Despite these differences, the comparison remains valuable as it highlights how closely our model aligns with Bloomberg's valuation. This comparison does not aim to prove one model as correct, but rather to assess the similarity in results. For this comparison, different results will be compared, namely: the present value of the interest rate swap under valuation; the expected exposure profiles; the probability of default for the counterparty and dealer; and finally the bilateral CVA value.

5.1.1 The present value of the interest rate swap

First of all, let us evaluate the accuracy of pricing the interest rate swap of the model. Pricing an interest rate swap involves calculating the present value of its future cash flows. This should be done for both the fixed leg and the floating leg. The present value of the fixed leg is calculated by discounting each fixed payment using the appropriate discount factors derived from the interest rate curve. The present value of the floating leg is calculated by discounting based on the future floating interest rate (for the interest rate swap under valuation the 6M EUR EURIBOR) and are projected based on the current forward rate curve. The difference between the present value (PV) of the interest rate swap retrieved from Bloomberg and the developed CVA model are displayed in *Table 10*.

Table 10. Present value of the interest rate swap according to Bloomberg and the CVA model.

Valuation method	PV
Bloomberg	€278,894
CVA model	€280,052

The slight difference between the PV calculated by Bloomberg (€278,894) and the CVA model (€280,052) could be attributed to several factors. First, differences in the interest rate curves used for discounting might arise, as both models may be calibrated on slightly different market data or assumptions. Additionally, small variations in the interpolation methods for the yield curve or forward curve could affect the projected floating leg cash flows. Lastly, differences in numerical methods for pricing interest rate swaps, such as Monte Carlo simulation settings or the treatment of convexity adjustments, might result in minor discrepancies.

However, a difference of €1,158 is relatively minor, representing a variation of less than 0.5%. In financial modelling, small discrepancies like this are expected due to the possible reasons mentioned above. Given that these differences are within the margin of typical market tolerances, they are not considered significant and do not materially impact the overall valuation of CVA.

5.1.2 The expected exposure profiles

In comparing the expected positive exposure (EPE) profiles between the CVA model and Bloomberg, there are notable similarities but also key differences.

Both EPE profiles exhibit a stepwise pattern, which reflects the nature of discrete reset dates or payment periods in the interest rate swap being valued, see *Figure 8* and *Figure 9*. This stepwise reduction over time is typical, as the exposure diminishes as the swap approaches maturity. The overall trend in both models indicate that exposure peaks early in the interest rate swap and then gradually decreases, which is expected as fewer future payments remain to be exchanged as time progresses.

However, some differences between the two profiles can be observed. In the CVA model, there are more pronounced fluctuations in the earlier stages of the swap, whereas the Bloomberg model shows smoother step transitions. This discrepancy could be due to Bloomberg employing a more sophisticated multi-factor interest rate model compared to the single-factor Hull-White model used in the CVA calculations. Bloomberg's model may account for more complex dynamics or market conditions that smooth out the exposure curve.

Additionally, Bloomberg's EPE profile shows larger step-downs at specific points, which could be attributed to different assumptions about interest rate volatility, market liquidity, or the forward rate curve used to project future cash flows. The CVA model may have higher early-stage exposure due to how it calibrates to historical interest rate movements or how it discounts future floating payments.

These differences, while minor, could also stem from variations in how the discount factors or yield curves are derived. Bloomberg may use a more refined or real-time market data-driven curve, while the CVA model may rely on a slightly simplified or less granular curve.

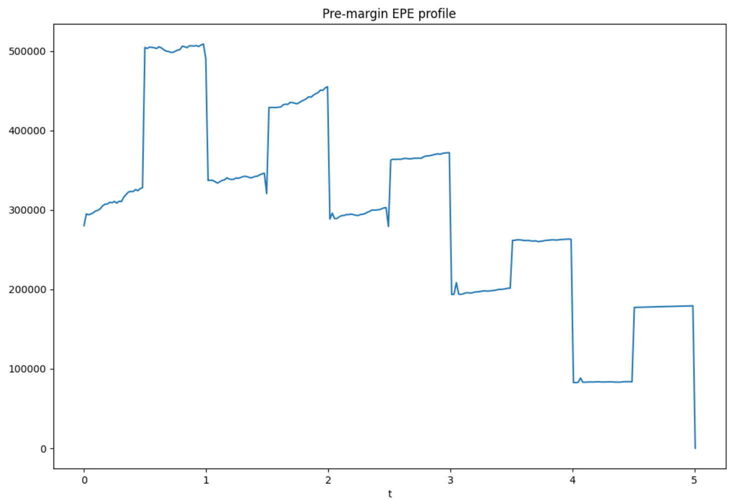


Figure 8. EPE profile of the interest rate swap based on 500 simulation runs.

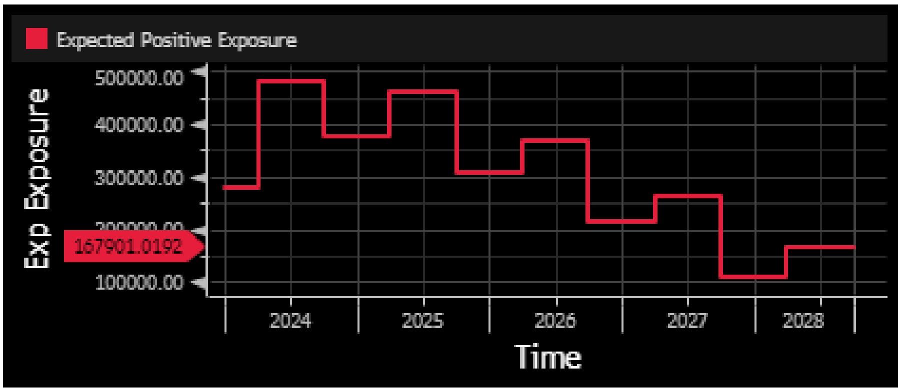


Figure 9. EPE profile of the interest rate swap from Bloomberg.

Furthermore, we will also compare the ENE outcomes of the CVA model with the outcomes from Bloomberg, see *Figure 10* and *Figure 11*. Both the CVA model and Bloomberg's ENE profiles show consistent patterns in exposure over time. In terms of magnitude, both ENE profiles generally show similar ranges of exposure. For instance, in the Bloomberg profile, ENE peaks at around €80,000, while in the CVA model the ENE peaks at around €50,000. The scale of the difference in the maximum ENE values is notable, with Bloomberg's ENE being around 60% larger than the CVA model's maximum. However, this difference is not extreme and can still be considered reasonably close, given the potential variations in model assumptions and numerical methods used in calculating ENE.

There are several factors that might explain these differences in the ENE values. First of all, the HW1 model used in the CVA model is sensitive to interest rate volatility, which can lead to sharper movements in exposure. Bloomberg may be using a different interest rate model or applying a more conservative approach that produces higher ENE values, particularly under extreme scenarios.

Second, Bloomberg's ENE profile shows a more structured, stepped pattern, while the CVA model displays sharper, more frequent changes. This difference suggests that Bloomberg might aggregate scenarios differently or smooth exposure transitions, leading to higher and more stable ENE values compared to the more dynamic exposure captured by the CVA model. Another reason can be that the CVA model calculates the ENE for every week, whereas Bloomberg seems to calculate it every half year or averages the ENE for every half year.

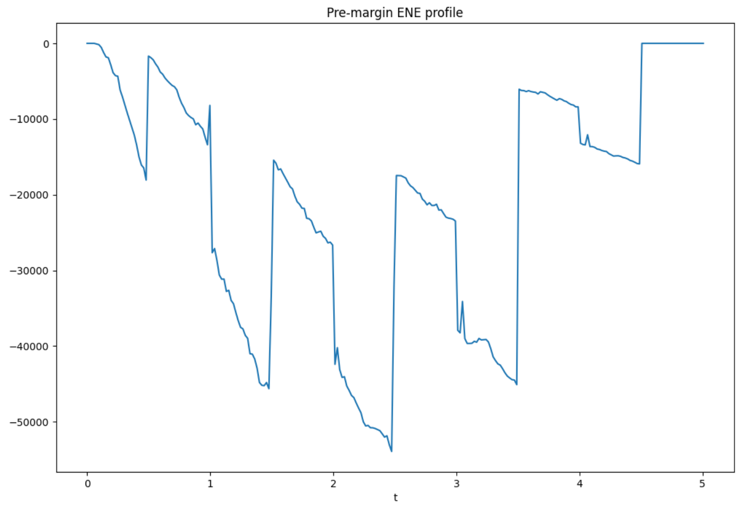


Figure 10. ENE profile of the interest rate swap based on 500 simulation runs.

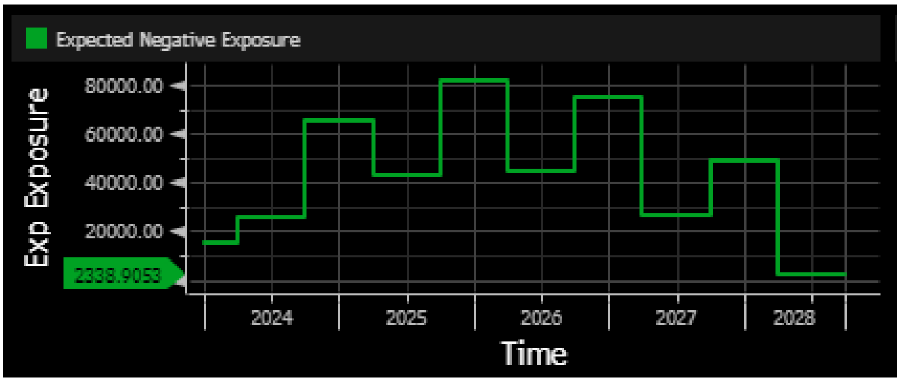


Figure 11. ENE profile of the interest rate swap from Bloomberg.

Additionally, it is notable that the difference of the EPE profiles are closer to each other than the difference of the ENE profiles. The close alignment of the EPE suggests that both models agree on the general risk exposure of the interest rate swap, and the difference in ENE can be attributed to Bloomberg potentially using a more conservative or smoothed approach for negative exposures. Given these factors, the differences in ENE are not excessively large and remain within a reasonable range, providing a meaningful basis of the CVA model for the final calculation of CVA compared to Bloomberg.

5.1.3 The probability of default

In the comparison of the default probabilities between the CVA model and Bloomberg, the outcomes are exactly the same as *Table 8* and *Table 9*. This suggests that Bloomberg, just like the CVA model, assumes a Poisson homogeneous process with a constant hazard rate over time. However, the comparison that can be made is the probability of default for different terms of the CDS (i.e., 6 MO, 1 YR, 2 YR etc.). Bloomberg does not disclose the exact methodology behind the construction of its hazard or probability of default curve, making it impossible to directly compare the interpolation techniques they employed. Common interpolation methods include linear and log-linear interpolation, both of which can influence the shape of the hazard curve. Nevertheless, regardless of the specific interpolation method used, the difference is likely negligible, as both models yield consistent values for the probability of default over the same time horizons.

5.1.4 Bilateral CVA

The final comparison between the bilateral CVA (BCVA) values of the CVA model and Bloomberg highlights some key differences. The CVA model estimates a bilateral CVA of €7,499, while Bloomberg reports a slightly lower value of €6,970, see *Table 11*. This €529 difference can be attributed to various factors, similar to those discussed earlier in this chapter in the comparison of the present value of the interest rate swap; the exposure profiles; and the probability of defaults.

Table 11. CVA, DVA and BCVA comparison of the CVA model and Bloomberg.

Outcomes	CVA model	Bloomberg
CVA	€7,771	€8,285
DVA	€272	€1,315
BCVA	€7,499	€6,970

Firstly, the differences in the interest rate models is likely to play a significant role. This a conclusion already made in the comparison of the EPE and ENE, and the same difference can be seen in the final CVA and DVA. Bloomberg likely uses a multi-factor interest rate model, which account for more complex interest rate movements and may provide a more conservative view of future market conditions. In contract, the CVA model relies on the single-factor Hull-White model, which could result in slightly different exposure and discounting dynamics over time, as seen in the difference of the EPE and ENE.

Furthermore, the significant difference between the calculated DVA values (€272 in the CVA model versus €1,315 in Bloomberg) can provide some insight. A higher DVA from Bloomberg suggests that Bloomberg places more weight on the counterparty's credit risk, potentially reflecting a more risk-averse approach or more comprehensive market data. This approach would, in turn, result in a lower bilateral CVA value due to the greater credit risk deduction on the dealer side.

Finally, in putting the difference between the calculated BCVA results into perspective, it is important to reflect on the overall present value (PV) of the swap, which is approximately €280,000 according to the CVA model. As explained in chapter *2 Literature review* the adjusted derivative value f_0^* is the value of the derivative contract today assuming no defaults f_0 minus the BCVA.

$$f_0^* = f_0 - BCVA \tag{79}$$

Therefore, a €529 difference in BCVA represents less than 0.2% of the total PV of the swap. This small discrepancy is within acceptable tolerances in financial modelling and does not materially impact the swap's overall risk profile or valuation. In summary, while differences exist between the CVA model and Bloomberg, they are relatively minor, especially when viewed in the context of the overall swap value.

5.2 Sensitivity analysis

In this section, sensitivity analyses are conducted to assess the robustness of the CVA model with respect to key parameters. The first analysis focuses on the number of simulation runs used in the Monte Carlo process, aiming to determine the point at which the model produces stable returns. This ensures that computational efficiency is balanced with accuracy. The second analysis evaluates the sensitivity of the CVA to changes in the α and σ parameters of the Hull-White calibration. These parameters directly influence the interest rate volatility and can significantly impact CVA valuations, highlighting the importance of model calibration. Through these sensitivity tests, we aim to better understand the conditions under which the model delivers reliable and consistent valuations.

5.2.1 Number of simulation runs

To conduct the sensitivity analysis for the number of simulation runs in the CVA model, we performed five simulations at four different levels of simulation runs: 500, 250, 100 and 50 runs. After each set of simulations, we compared the CVA and DVA outcomes to observe how varying the number of runs affected the stability of the results.

For CVA, we see that the average values across simulations vary between €7,936 (500 runs) and €8,497 (50 runs), with the standard deviation decreasing as the number of simulation runs increases, see *Table 12*. The smallest standard deviation is €368 for 500 runs, while the highest, €944, occurs with only 50 runs. This indicates that the CVA becomes more stable if the number of runs increases, with lower variation among individual simulation outcomes. At 100 and 50 runs, the fluctuations are more pronounced, showing more volatile CVA estimates.

Table 12. Sensitivity analysis of the number of simulation runs on the value of CVA.

	500 runs	**250 runs**	**100 runs**	**50 runs**
Simulation 1	€8,085	€8,515	€7,417	€7,071
Simulation 2	€8,112	€8,275	€8,317	€7,888
Simulation 3	€8,326	€7,013	€7,523	€9,610
Simulation 4	€7,250	€8,201	€7,669	€9,397
Simulation 5	€7,907	€8,095	€8,675	€8,523
Average	€7,936	€8,020	€7,920	€8,497
Standard deviation	€368	€522	€490	€944

Similarly, the DVA results show a similar pattern of stability with more simulation runs, see *Table 13*. At 50 runs, the average DVA is -€224 with a standard deviation of €84. When the number of simulation runs increases to 500, the average DVA is -€214, but with a much lower standard deviation of €44. This again demonstrates that a higher number of simulation runs introduces less uncertainty and variability in the results.

Table 13. Sensitivity analysis of the number of simulation runs on the value of DVA.

	500 runs	**250 runs**	**100 runs**	**50 runs**
Simulation 1	-€220	-€154	-€315	-€300
Simulation 2	-€182	-€260	-€180	-€320
Simulation 3	-€153	-€370	-€264	-€108
Simulation 4	-€235	-€258	-€201	-€147
Simulation 5	-€280	-€232	-€259	-€246
Average	-€214	-€255	-€244	-€224
Standard deviation	€44	€69	€48	€84

In summary, the analysis shows that as the number of simulation runs increases, both CVA and DVA values become more consistent, with less variability across simulations. While 250 runs provide a reasonable trade-off between consistency and computational cost, the results suggest that fewer than 100 runs lead to significant volatility, making it difficult to draw accurate conclusions. To be conservative with the CVA and DVA calculations in this thesis, 500 simulation runs is used for the final results.

5.2.2 HW1 model calibration parameters

Next, the sensitivity analysis of the α and σ parameters of the Hull-White one-factor model is essential to understand the impact of the interest rate volatility and mean reversion on the CVA and DVA. The parameter α controls the mean reversion speed of interest rates, while σ represents the volatility of interest rates. Both parameters directly influence the exposure profile and hence the valuation of CVA and DVA.

The purpose of this analysis is to assess how variations in α and σ affect the final CVA and DVA outcomes. By testing different parameter values, we can determine the robustness of the model's output to changes in the interest rate dynamics. This sensitivity is crucial because inaccurate calibration of these parameters may result in over- or under-estimation of credit risk, leading to mispricing of the interest rate swap under valuation.

From *Table 14* and *Table 15* below, we observe that increasing σ results in a notable rise in both CVA and DVA, especially for $\sigma = 0.02$, which pushes CVA up to €9,755 and DVA to -€2,274 for $\alpha = 0$. The sensitivity to α is more nuanced, for a smaller σ, a higher α appears to have a dampening effect on CVA. For instance, at $\sigma = 0.08$, increasing α from 0 to 0.2 leads to minimum change in CVA. This is in line with the findings of Russo & Fabozzi (2019) and explains why it is common practice in the literature to calibrate the HW1 model with a fixed value for α while allowing σ to vary.

Table 14. Sensitivity analysis of the calibration parameters from the HW1 model on the value of CVA.

		σ				
		0.004	**0.008**	**0.012**	**0.016**	**0.02**
α	**0**	€7,852	€7,771	€8,646	€8,952	€9,755
	0.05	€7,737	€7,999	€8,784	€8,690	€9,732
	0.1	€7,978	€8,228	€8,119	€8,691	€9,005
	0.15	€7,978	€8,257	€8,007	€8,749	€8,871
	0.2	€7,932	€7,819	€7,983	€8,129	€8,054

The DVA results display a similar pattern, where higher values for σ increase the DVA, reflecting greater potential loss to the dealer. However, higher α seems to mitigate these increases, as seen when $\alpha = 0.2$ yields more stable DVA results at smaller σ.

Table 15. Sensitivity analysis of the calibration parameters from the HW1 model on the value of DVA.

		σ				
		0.004	**0.008**	**0.012**	**0.016**	**0.02**
α	**0**	-€8	-€272	-€680	-€1,675	-€2,274
	0.05	-€2	-€154	-€565	-€1,222	-€1,764
	0.1	-€1	-€80	-€382	-€861	-€1,292
	0.15	€0	-€58	-€297	-€621	-€1,239
	0.2	€0	-€31	-€190	-€449	-€912

When comparing the sensitivity analysis results to the initial calibration parameters ($\alpha = 0$ and $\sigma = 0.008$), the differences appear moderate but potentially impactful, particularly for the CVA. For example, the initial calibrated parameters result in a CVA of €7,771. In the sensitivity analysis, deviations in σ of ±0.004 lead to changes of up to approximately €500 (or ~6.25%) in CVA. While these shifts are significant, they remain within a range that is manageable and reasonable, given typical market fluctuations.

In terms of DVA, the sensitivity results also show some variation, but these differences are somewhat smaller in magnitude compared to CVA. For instance, the initial calibrated parameters result in a DVA of -€272. Deviations in σ of ±0.004 lead to changes up to approximately €400 (or ~200%) in DVA. Showing that the impact of σ on DVA is more significant than for CVA.

6 Conclusion

This chapter corresponds with the sixth step of the DSRM: *communication*. Therefore, this chapter formulates an answer to the main research question: '*what is the most suitable method to calculate CVA for interest rate swaps and how can this be developed into a valuation model for Deloitte?*'. In other words, in this chapter we will evaluate to what extent the core problem of Deloitte as defined in chapter *1 Introduction,* is solved. Furthermore, this chapter reflects on how this thesis contributes to the scientific literature and the practice. The future work and limitations of this thesis are also discussed in this chapter.

6.1 Conclusion

This thesis addresses the challenges in Credit Valuation Adjustment (CVA) valuations for interest rate swaps within the Deloitte's Financial Risk Management (FRM) department. By reviewing the existing literature on CVA methodologies and models, a suitable framework was developed to guide the selection and implementation of a CVA model tailored for Deloitte. The Hull-White one-factor model was identified as the most appropriate for interest rate modelling due to its balance between computational complexity and calibration accuracy.

The CVA model developed in this thesis calibrates the Hull-White one-factor model with co-maturity swaptions in the market and simulates interest rates according to this calibrated model. Based on the simulated interest rates, the expected exposure of an interest rate swap is determined. The probability of default for the counterparty and dealer are extracted from credit default swaps by assuming a homogenous Poisson process and deterministic hazard rate. Finally the CVA, DVA and bilateral CVA are calculated to evaluate the effectiveness of the developed CVA model.

The implementation of the CVA model demonstrated that it could effectively replace Bloomberg as valuation tool for CVA of interest rate swaps, addressing key limitations such as transparency and flexibility. A comparison between the CVA model and Bloomberg showed that the results were consistent, validating the effectiveness of the CVA model. Furthermore, the sensitivity analysis proved insights into the robustness of the model under varying parameters, confirming its reliability in practical applications.

6.2 Contribution

This thesis makes several contributions to the existing body of literature on Credit Valuation Adjustment and financial risk management, particularly in the context of interest rate swaps.

By applying the Hull-White one-factor model in the empirical valuation of an interest rate swap, the thesis bridges a gap between theoretical model selection and real-world application. The literature lacks extensive studies that combine rigorous model comparison with an actual CVA model implementation for financial institutions (or for this thesis, Deloitte). This thesis demonstrates how theoretical models can be adapted and calibrated for use in industry settings, offering valuable insights into the practical challenges of implementing CVA models at financial institutions.

Furthermore, the sensitivity analysis conducted as part of this research extends the existing literature by providing empirical evidence of how CVA outcomes fluctuate with changes in key parameters, further enriching the discussion on model reliability and robustness.

Finally, this thesis adds to the discussion on the limitations of widely-used valuation tools, such as Bloomberg, by critically evaluating their transparency and applicability to interest rate swaps. This critique introduces new considerations for the ongoing evolution of CVA valuation practices, contributing to the academic conversation around improving risk management methodologies in the post-financial crisis era.

6.3 Discussion and limitations

In this section, the key aspects of the research methodology and findings are addressed, discussing their implications and acknowledging the limitations of the thesis.

First, a notable limitation of this research is the selection of the interest rate model used on conjunction with the CVA model. Given that the CVA model is intended for regular use at Deloitte, more complex and accurate interest rate models, such as multi-factor models, were excluded from the study. The primary reason for this exclusion is the increased complexity and calibration challenges associated with these models, which would require significant additional resources.

Furthermore, for empirical application of the Hull-White one-factor model, a time-varying parameter for volatility is not incorporated. This decision was driven by similar considerations fo complexity and practical limitations. While a time-varying volatility model might provide a more nuanced representation of interest rate dynamics, the additional complexity in calibration was deemed impractical for this study’s objectives.

Moreover, the default probabilities in the empirical application were extracted assuming a deterministic hazard rate. Although this approach is less precise compared to methods involving time-dependent hazard rates, the difference in accuracy is negligible, as demonstrated in chapter *3 Methodology*.

In addition to that, the research exclusively examines CVA for interest rate swaps. Consequently, the CVA model’s applicability and accuracy for other types of derivatives, such as FX swaps or equity derivatives, remain unaddressed. The choice of limiting the scope of this thesis is justified by the fact that 35-40% of the total notional amounts outstanding in the global derivatives market accounts for interest rate swaps alone.

Finally, wrong-way risk, which can significantly impact CVA calculations, is not incorporated in this thesis due to its limited scope. Although often neglected in practice, wrong-way risk can lead to substantial deviations in CVA, particularly in adverse market conditions. This limitation should be considered when interpreting the results, and future research could benefit from including an analysis of wrong-way risk to enhance the model’s robustness.

6.4 Future work

To build upon the thesis and enhance the practical applicability of the CVA model, several areas for future development and exploration are identified, both for Deloitte and the broader academic body.

First, future research could focus on implementing and comparing various interest rate models to assess their impact on CVA calculations. While this thesis assessed the advantages and disadvantages of different models based on a literature review, a practical comparison of results between these models was beyond the scope of this thesis. Researching how different models perform in real-world scenarios could yield valuable insights and guide the selection of the most appropriate model.

Furthermore, Deloitte can further explore more sophisticated calibration techniques for the Hull-White one-factor model, such as incorporating stochastic volatility or employing multi-factor models like the Hull-White two-factor model.

Moreover, Deloitte can further develop de CVA model where it employs netting and collateral agreements in the CVA and DVA calculations of a basket of interest rate swaps. Additionally, understanding how these factors influence the CVA model's outcomes would contribute to a more nuanced and practical approach to credit risk management.

In addition to that, an exploration of wrong-way risk and its implications for CVA models could be a valuable area of research and further development. Developing methodologies to accurately quantify and incorporate wrong-way risk could improve the performance of the CVA model for Deloitte.

Expanding the CVA model to include various types of derivatives will make Deloitte's FRM department less reliant on Bloomberg as valuation method. A logical next step is to incorporate FX swaps as well, as FX swaps account for approximately 50% of the global OTC derivatives market (Bank for International Settlements, 2023).

7 References

Adinarayan, T. (2023). Bank failures in modern history: Crises, mergers and bailouts. Bloomberg.com. Link

Amin, H. (2012). Calibration of Different Interest Rate Models for a Good Fit of Yield Curves. Link

Bank for International Settlements. (2023). Quarterly Review: International banking and financial market developments – September 2023. Link

Basel III. (2010). A global regulatory framework for more resilient banks and banking systems. Link

Bernal, V. (2016). Calibration of the Vasicek model: a step by step guide. Link

Black, F., Derman, E., & Toy, W. (1990). A one-factor model of interest rates and its application to treasury bond options. Financial analysts journal, 46(1), 33-39. Link

Black, F., & Karasinski, P. (1991). Bond and option pricing when short rates are lognormal. Financial Analysts Journal, 47(4), 52-59. Link

Bloomberg. (2024). SWPM <GO> Swap Manager Help Page.

Brigo, D., & Mercurio, F. (2001). A deterministic–shift extension of analytically–tractable and time–homogeneous short–rate models. Finance and Stochastics, 5, 369-387. Link

Brigo, D., & El-Bachir, N. (2006). Credit derivatives pricing with a smile-extended jump stochastic intensity model. Available at SSRN 950208. Link

Brigo, D., & Mercurio, F. (2006). Interest rate models-theory and practice: with smile, inflation and credit (Vol. 2). Berlin: Springer. Link

Brigo, D., Morini, M., & Pallavicini, A. (2013). Counterparty credit risk, collateral and funding: with pricing cases for all asset classes (Vol. 478). John Wiley & Sons. Link

Canabarro, E., & Duffie, D. (2003). Measuring and marking counterparty risk. Asset/Liability Management for Financial Institutions, Institutional Investor Books. Link

Cox, J. C., Ingersoll Jr, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. In Theory of valuation (pp. 129-164). Link

Deloitte. (2024). About Deloitte, our global organisation. Link

Di Francesco, M., & Kamm, K. (2022). On the deterministic-shift extended CIR model in a negative interest rate framework. International Journal of Financial Studies, 10(2), 38. Link

Dothan, L. U. (1978). On the term structure of interest rates. Journal of financial economics, 6(1), 59-69. Link

EY. (2014). Credit valuation adjustments for derivative contracts. Link

Fabozzi (2014). Bond Market Analysis & Strategies. Pearson, India.

Gregory, J. (2012). Counterparty credit risk and credit value adjustment: A continuing challenge for global financial markets. John Wiley & Sons.

Hull, J. (1994). Numerical procedures for implementing term structure models II: Two-factor models. J. Deriv., 2, 37-48.

Hull, J. (1996). Using Hull-White interest rate trees. Journal of Derivatives, 3(3), 26-36. Link

Hull, J. (2012). Risk management and financial institutions (Vol. 733). John Wiley & Sons.

Hull, J., & White, A. (1990). Pricing interest-rate-derivative securities. The review of financial studies, 3(4), 573-592. Link

Hull, J., & White, A. (2001). The general Hull–White model and supercalibration. Financial Analysts Journal, 57(6), 34-43. Link

Hull, J., & White, A. (2012). CVA and wrong-way risk. Financial Analysts Journal, 68(5), 58-69. Link

Ho, T. S., & Lee, S. B. (1986). Term structure movements and pricing interest rate contingent claims. the Journal of Finance, 41(5), 1011-1029. Link

Hoffman, F. (2011). Credit Valuation Adjustment. Link

Ingber, L. (1993). Adaptive simulated annealing (ASA). Global optimization C-code, Caltech Alumni Association, Pasadena, CA. Link

Jamshidian, F. (1995). A simple class of square-root interest-rate models. Applied Mathematical Finance, 2(1), 61-72. Link

Josheski, D., and Apostolov, M. (2021). Equilibrium short-rate models vs no-arbitrage models: Literature review and computational examples. Econometrics. Advances in Applied Data Analysis, 25(3). Link

Joshi, J., & Swertloff, L. (1999). A User's Guide to Interest Rate Models: Applications for Structured Finance. The Journal of Risk Finance, 1(1), 106-114. Link

Kozpınar, S. (2021). A Brief Look at OU, Vasicek, CIR and Hull-White Models Through Their Actuarial Applications. Başkent Üniversitesi Ticari Bilimler Fakültesi Dergisi, 5(2), 37-49. Link

Lawson, B. I. (2015). An Exposition and Calibration of the Ho-Lee Model of Interest Rates. Link

Marsh, T. A., & Rosenfeld, E. R. (1983). Stochastic processes for interest rates and equilibrium bond prices. The Journal of Finance, 38(2), 635-646. Link

O'Kane, D. (2008). Modelling single-name and multi-name credit derivatives (Vol. 545). John Wiley & Sons.

Orlando, G., Mininni, R. M., & Bufalo, M. (2019). Interest rates calibration with a CIR model. The Journal of Risk Finance, 20(4), 370-387. Link

Peffers, K., Tuunanen, T., Rothenberger, M. A., & Chatterjee, S. (2007). A design science research methodology for information systems research. Journal of management information systems, 24(3), 45-77.

Pelsser, A. (1996). A tractable yield-curve model that guarantees positive interest rates. Review of Derivatives Research, 1, 269-284. Link

Pereira, S. M. C. (2014). Pricing of a credit default swap (Doctoral dissertation). Link

Quail, R., & Overdahl, J. A. (2002). Financial derivatives (Vol. 127). John Wiley & Sons. Link

Radhakrishnan, A. R. (1998). Mispricing of Discount Bond Options in the Black-Derman-Toy Model Calibrated to Term Structure and Cap Volatilities: An Empirical Study. Link

Reghai, A., & Kettani, O. (2015). A Sensitivity Based Approach for CVA Computation. Available at SSRN 2576846. Link

Rendleman, R. J., & Bartter, B. J. (1980). The pricing of options on debt securities. Journal of Financial and Quantitative Analysis, 15(1), 11-24. Link

Russo, V., & Fabozzi, F. J. (2016). Pricing coupon bond options and swaptions under the one-factor Hull-White model. The Journal of Fixed Income, 25(4), 76. Link

Russo, V., & Torri, G. (2019). Calibration of one-factor and two-factor hull–white models using swaptions. Computational Management Science, 16(1-2), 275-295. Link

Schlenkrich, S. (2012). Efficient calibration of the Hull White model. Optimal Control Applications and Methods, 33(3), 352-362. Link

Svoboda, S. (2004). The Hull and White Model. In Interest Rate Modelling (pp. 103-120). London: Palgrave Macmillan UK. Link

Taherdoost, H. (2017). Decision making using the analytic hierarchy process (AHP); A step by step approach. International Journal of Economics and Management Systems, 2. Link

Tourrucôo, F., P. S. Hagan and G. F. Schleiniger (2007). Approximate Formulas for Zero-Coupon Bonds. Applied Mathematical Finance Vol. 14, pp. 107–226. Link

Treepongkaruna, S., & Gray, S. (2003). Short-term interest rate models: valuing interest rate derivatives using a Monte-Carlo approach. Accounting & Finance, 43(2), 231-259. Link

Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of financial economics, 5(2), 177-188. Link

Van Schuppen, B. (2014). Counterparty credit risk in interest-rate swaps: CVA, wrong-way risk, and the Basel III CVA capital charge. Link

Van Vuuren, G., & Esterhuysen, J. N. (2014). A primer on counterparty valuation adjustments in South Africa. South African Journal of Economic and Management Sciences, 17(5), 584-600. Link

Vollrath, I., & Wendland, J. (2009). Calibration of interest rate and option models using differential evolution. Available at SSRN 1367502. Link

Wu, L. (2015). CVA and FVA to derivatives trades collateralized by cash. International Journal of Theoretical and Applied Finance, 18(05), 1550035. Link

Xiao, T. (2017). The Valuation of Interest Rate Swap with Bilateral Counterparty Risk. FinPricing. Link

Yolcu, Y. (2005). One-factor interest rate models: analytical solutions and approximations. Link

Zarpellon, M. (2014). Comparing Basel III CVA with IFRS13 CVA. Available at SSRN 2446222. Link

Zhu, S. H., & Pykhtin, M. (2007). A guide to modeling counterparty credit risk. GARP Risk Review, July/August. Link

Appendix A – Definitions of the scale for ranking interest rate models

This part of the appendix shows the definitions of the scale poor-good-excellent for the criteria “calibration efficiency” on which the different interest rate models are rated, see *Table 16*. Moreover, the criteria “computation complexity” is ranked on the scale efficient-moderate-complex, and the definitions are given in *Table 17*.

Table 16. Definition of calibration efficiency and corresponding scale.

	Poor	**Good**	**Excellent**
Fit to historical data	The model fails to capture key patterns and trends in the historical data. Significant discrepancies are observed	The model captures most key patterns and trends in the data with minor discrepancies	The model perfectly fits historical data, capturing all key patterns and trends
Stability of parameters	Parameters are highly unstable and vary significantly over different calibration periods	Parameters are generally stable across different periods with some minor variations	Parameters are highly stable and show minimal variability across different periods
Predictive power	The model performs poorly in predicting out-of-sample data, with large errors	The model has good predictive accuracy for out-of-sample data	The model provides highly accurate predictions for out-of-sample data
Sensitivity to market conditions	The model cannot adapt to changing market conditions, leading to inaccurate results	The model adapts reasonably well to changes in market conditions	The model is highly adaptable to changing market conditions and maintains exceptional accuracy

Table 17. Definition of computation complexity and corresponding scale.

	Efficient	**Moderate**	**Complex**
Calibration time	The model calibrates rapidly, requiring minimal time	Requires a moderate amount of time for calibration	Takes a long time to calibrate, which can be impractical
Resource usage	Low computational resources are necessary, suitable for standard hardware	Reasonable computational resources, sometimes straining standard hardware	High computational resources needed, potentially requiring specialized hardware
Convergence reliability	Consistent and quick convergence of the calibration algorithm	Generally reliable convergence with occasional need for adjustments	Frequent difficulties in converging, necessitating extensive tuning

Appendix B – AHP weights for interest rate model selection

This section will show how the weights for the interest rate model selection procedure are determined. The weights are determined based on the Analytical Hierarchy Process (AHP).

The first step is creating a pairwise comparison matrix of the criteria. One of the FRM specialists of Deloitte will indicate how important one requirement is over the others. This is done on a scale ranging from one to nine, where one means that the criteria are the same or that they are of the same importance. Nine means that the one requirement is extremely more important over the other. The interpretation of the whole scale is shown in *Table 18*.

Table 18. Definitions of the importance scale.

Importance scale	**Definition of importance scale**
1	Equally important
2	Equally to moderately important
3	Moderately important
4	Moderately to strongly important
5	Strongly important
6	Strongly to very strongly important
7	Very strongly important
8	Very strongly to extremely important
9	Extremely important

The pairwise comparison of the criteria is shown in matrix A. The sequence of criteria along the rows is the same for the columns.

$$A = \begin{matrix} Com \\ Eco \\ Cal \\ Fle \end{matrix} \begin{bmatrix} 1 & 7/9 & 6/9 & 2 \\ 7 & 1 & 3 & 9 \\ 6 & 6/9 & 1 & 7 \\ 2/9 & 1/9 & 1/7 & 1 \end{bmatrix}$$

This matrix is transformed to a normalized matrix A_{norm}.

$$A_{norm} = \begin{bmatrix} 0.07 & 0.35 & 0.14 & 0.11 \\ 0.49 & 0.45 & 0.62 & 0.47 \\ 0.42 & 0.15 & 0.21 & 0.37 \\ 0.02 & 0.05 & 0.03 & 0.05 \end{bmatrix}$$

Finally, the weights for the criteria is the column vector $\boldsymbol{w}$ of the normalized matrix A_{norm}.

$$\boldsymbol{w} = \begin{bmatrix} 0.116 \\ 0.510 \\ 0.287 \\ 0.037 \end{bmatrix}$$

Validation of results

The results of the AHP should be validated using the consistency ratio (CR), using the formula $CR = \frac{CI}{RI}$. Here CI is the consistency index, which can be calculated with

$$CI = \frac{\lambda_{max} - n}{n - 1} \tag{80}$$

Where λ_{max} is the maximum eigen value of matrix A and n is the number of criteria (Taherdoost, 2017). Next, RI is the random consistency index is dependent on the number of criteria, the weights can be find in *Table 19.*

Table 19. Random consistency index.

Dimension	RI
1	0
2	0
3	0.5799
4	0.8921
5	1.1159
6	1.2358
7	1.3322
8	1.3952
9	1.4537
10	1.4882

Finally, a consistency ratio lower than 0.10 verifies the results and means that the comparison is acceptable.

To validate the results retrieved above, the following steps are executed.

1. $A\boldsymbol{w} = \begin{bmatrix} 0.83 \\ 2.87 \\ 1.71 \\ 0.17 \end{bmatrix}$

2. $\lambda_{max} = \frac{1}{n} \sum_{i=1}^{n} \frac{i\ entry\ in\ A\boldsymbol{w}}{i\ entry\ in\ \boldsymbol{w}} = \left(\frac{1}{4}\right)\left(\frac{0.83}{0.116} + \frac{2.87}{0.510} + \frac{1.71}{0.287} + \frac{0.17}{0.037}\right) = 4.197$

3. $CI = \frac{\lambda_{max} - n}{n-1} = \frac{4.197-4}{3} = 0.066$

4. $CR = \frac{CI}{RI} = \frac{0.066}{0.8921} = 0.073$

Because the consistency ratio is lower than 0.10, we can conclude that the degree of consistency is satisfactory and therefore the results are validated.

Appendix C – Definition of the scores for all criteria on the implemented interest rate model

The table in this section shows the definitions of the scores ranging from 1 to 5, on which all criteria are ranked to decide what interest rate model is used in the final CVA model.

Table 20. Definitions of the scores for all criteria for determining the best interest rate model for the CVA model.

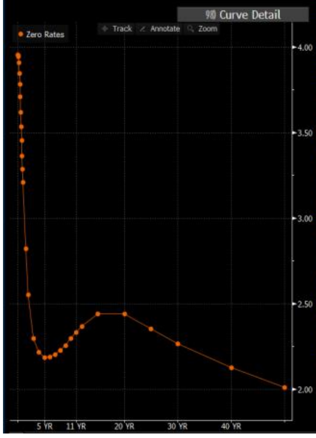
Criteria	Score 1	Score 2	Score 3	Score 4	Score 5
Computational efficiency	Extremely slow, impractical for large portfolios	Slow and inefficient; significant delays in processing	Moderate efficiency; manageable for moderate sized portfolios	Efficient for large portfolios; reasonable processing time	Highly efficient; performs well even with very large portfolios
Economic realism (accuracy)	Poorly captures interest rate behaviour; fails under various scenarios	Limited accuracy; struggles with extreme market conditions	Reasonable accuracy; handles typical scenarios but may struggle under stress	High accuracy; robust under most market conditions	Excellent accuracy; captures interest rate behaviour and extremes effectively
Calibration	Difficult to calibrate; requires extensive manual adjustments	Calibration is challenging and time-consuming	Calibration is manageable but still requires some manual input	Calibration is straightforward; minimal manual adjustments needed	Easily calibrated; highly automated with minimal manual intervention
Flexibility	Very rigid; cannot accommodate future changes or expansions	Limited flexibility; difficult to adapt new requirements	Moderate flexibility; can handle some changes with effort	High flexibility; can be adapted for various derivatives and features	Highly flexible; easily expanded and adaptable for diverse requirements

Appendix D – Bloomberg data

Expiry	1Yr	2Yr	3Yr	4Yr	5Yr	7Yr	10Yr	12Yr	15Yr	20Yr	25Yr	30Yr
1Mo	26.37	39.07	42.32	43.33	43.33	42.54	40.97	39.74	38.40	37.80	38.93	39.75
3Mo	32.06	42.13	44.75	45.39	45.06	43.74	41.38	40.13	38.81	38.19	39.19	40.28
6Mo	38.84	46.39	47.53	47.11	46.21	44.38	41.81	40.53	39.27	38.69	39.65	40.47
9Mo	44.31	49.74	49.36	48.22	46.82	44.44	41.48	40.09	38.73	38.49	39.17	40.10
1Yr	50.34	52.32	50.21	48.31	46.61	43.96	40.83	39.55	38.40	38.03	38.91	39.52
2Yr	51.71	49.50	46.69	45.05	43.62	41.46	38.87	37.65	36.56	36.69	37.46	38.56
3Yr	46.50	44.62	42.78	41.47	40.38	38.78	37.21	36.14	35.36	35.51	36.38	37.44
4Yr	42.37	41.21	39.76	38.51	37.49	36.23	34.86	34.25	33.60	34.07	34.83	35.74
5Yr	39.67	38.79	37.53	36.43	35.58	34.24	33.43	33.16	32.52	33.37	34.05	35.15
6Yr	37.79	37.06	35.93	34.76	33.89	32.80	32.46	32.20	31.93	32.68	33.43	34.41
7Yr	36.12	35.69	34.41	33.28	32.49	31.67	31.60	31.36	31.47	32.12	32.94	33.84
8Yr	35.08	34.39	33.15	32.20	31.62	30.89	31.04	30.75	31.20	31.81	32.73	33.62
9Yr	33.54	33.12	32.10	31.54	30.92	30.68	30.57	30.68	31.02	31.60	32.67	33.46
10Yr	32.55	32.38	31.74	31.14	30.44	30.62	30.20	30.74	30.94	31.37	32.53	33.28
12Yr	32.53	32.47	31.57	31.71	31.35	30.97	31.36	31.61	31.46	32.04	32.90	33.74
15Yr	36.19	36.00	34.88	33.71	32.50	33.43	33.54	33.25	32.41	33.21	33.57	34.60
20Yr	47.86	47.87	45.78	43.67	41.64	39.59	36.90	36.71	35.78	35.57	36.45	36.88
25Yr	47.89	47.81	45.55	42.70	40.19	40.23	38.91	38.13	36.59	37.65	37.95	37.88
30Yr	79.70	79.56	67.80	59.04	54.01	48.42	42.59	42.45	41.19	41.14	40.47	39.78

Figure 12. Swaption volatility surface on 31/12/2023 based on Black volatilities. (Source: Bloomberg)

Curve # 514 - EUR OIS ESTR
Curve Name EUR OIS ESTR
Curve Side Mid
Interpolation Step Forward (Cont)
Curve Date 12/31/2023
DV01 Calc Type Shifting



Curve Data Shift +0.00 bp

Term	Market Ra...	Shift	Shifted Rate	Zero Rate	Discount
1 WK	3.89900	+0.00	3.89900	3.95166	0.998918
2 WK	3.90250	+0.00	3.90250	3.95334	0.998160
1 MO	3.90256	+0.00	3.90256	3.94987	0.996112
2 MO	3.90100	+0.00	3.90100	3.94261	0.993111
3 MO	3.87100	+0.00	3.87100	3.90715	0.989988
4 MO	3.81425	+0.00	3.81425	3.84528	0.987022
5 MO	3.75800	+0.00	3.75800	3.78360	0.984061
6 MO	3.68800	+0.00	3.68800	3.70879	0.981378
7 MO	3.60100	+0.00	3.60100	3.61698	0.978629
8 MO	3.52150	+0.00	3.52150	3.53361	0.976371
9 MO	3.44600	+0.00	3.44600	3.45427	0.974126
10 MO	3.35800	+0.00	3.35800	3.36273	0.971933
11 MO	3.28350	+0.00	3.28350	3.28522	0.970036
1 YR	3.21000	+0.00	3.21000	3.20869	0.968082
18 MO	2.81400	+0.00	2.81400	2.82183	0.958371
2 YR	2.55300	+0.00	2.55300	2.55280	0.949827
3 YR	2.29900	+0.00	2.29900	2.29776	0.933096
4 YR	2.21800	+0.00	2.21800	2.21681	0.914923
5 YR	2.18560	+0.00	2.18560	2.18482	0.896246
6 YR	2.18700	+0.00	2.18700	2.18743	0.876739
7 YR	2.19950	+0.00	2.19950	2.20145	0.856927
8 YR	2.22200	+0.00	2.22200	2.22601	0.836518
9 YR	2.25000	+0.00	2.25000	2.25659	0.815900
10 YR	2.28760	+0.00	2.28760	2.29786	0.794404
11 YR	2.31930	+0.00	2.31930	2.33282	0.773373
12 YR	2.35000	+0.00	2.35000	2.36693	0.752450
15 YR	2.41695	+0.00	2.41695	2.44188	0.692983
20 YR	2.42080	+0.00	2.42080	2.44129	0.613325
25 YR	2.35450	+0.00	2.35450	2.35327	0.554867
30 YR	2.28920	+0.00	2.28920	2.26688	0.506176
40 YR	2.18200	+0.00	2.18200	2.12458	0.427163
50 YR	2.09800	+0.00	2.09800	2.01047	0.365636

Figure 13. ESTR OIS curve on 31/12/2023. (Source: Bloomberg)