BSc Thesis Applied Mathematics

Choosing Two Facilities Hotelling Model

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Preface

This paper is written as part of my Bachelor assignment for Applied Mathematics at the University of Twente. This was a difficult, challenging and beautiful process, where I learned a lot about how to apply mathematics, but also about myself. I really enjoyed working on this project in this specific research field and using mathematics more in reasoning logically instead of calculations.

I want to thank Alexander Skopalik for supervising me during this project. Without his feedback, his time and mental support, I would not be able to achieve this. Next to that, I want to thank my family and friends and my study advisor, who were a listening ear every time I needed that.

With pride, I hope you, the reader, will enjoy reading this article and maybe can use it, if applicable.

Sam Baak
Enschede, 2025

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Abstract

In this paper, we introduce a game-theoretic model that represents the facility location problem where clients need to choose two facilities. This is based on the classic *Hotelling-Downs model*, with the change from choosing one facility to choosing two facilities. We develop tools that state that if one facility, or two facilities on the same location, are on the outside, that this strategy profile cannot be an equilibrium. This is useful for characterizing the possible equilibria for two, three and four facilities, and for the result that there does not exist an equilibrium for five facilities.

Keywords: Choosing Two Facilities, Hotelling-Downs model, equilibria, game-theoretic, facility location problem

1 Introduction

When a facility chooses a location, it wants to make sure that this location will maximize their own profit. For example, the clothing business STING wants to open a new facility in the center of Enschede. What is the best location to place this facility? That depends among others on the behavior of the clients and the location of other similar facilities. This is an example of facility location problems, which are studied in Economics, Mathematics, Operations Research, Theoretical Computer Science and Artificial Intelligence [5]. To solve these kinds of competitive situations, we can use methods from Game Theory, and represent these situations using mathematical location models.

1.1 Related work

The first model that was introduced in this research field is the *Hotelling-Downs model*, which was first found by Hotelling [4] and later by Downs [2]. In this model there are two facilities and infinitely many clients on a continuous line. These clients have to choose between these two facilities, only based on the shortest distance. They have shown that the only stable situation, which is the case when the facilities cannot move for their own benefit, also called equilibrium, is when both facilities are placed in the middle of the line.

Later, Lerner [7] has shown that in the *Hotelling-Downs model* there is an equilibrium for any number of facilities at a line, except if there are three facilities. Eaton and Lipsey [3] introduced the principle of local clustering in the *Hotelling-Downs model*, where a new facility (or when a facility relocates) has a strong tendency to locate as close as possible

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to the other firm(s). This results in local clusters in many situations. A little more than a decade later, Cox [1] showed in the *Hotelling-Downs model* that in equilibria, there can be no more than two facilities at any given point.

1.2 My contribution

Most of the research that is currently done in this research field includes the assumption that every client chooses exactly one facility. But there are also situations where clients need to choose two facilities. For example, a factory where a certain product needs to be made and there are two similar machines (which refer to the facilities in our situation) which can make this product. We can choose to let them both work, to avoid the situation that you do not have a working machine left if one of the machines breaks down. This is an application of the situation we will investigate and this can be translated into a lot of different areas. In all these different areas, the goal is still that facilities want to maximize their own profit. The question then arises, how can the facilities do that? That brings me to my main research question: *"What happens when clients have to choose two facilities?"*. To answer this main research question, I will first answer two sub-research questions, namely: *"Do equilibria exist while choosing two facilities?"* and *"Can we translate the existing results directly to the two-facility case?"*. The overall objective in this field is to extend and improve the existing models and this research is a small piece of this goal.

The model that will be described in this paper, is a game-theoretic model for non-cooperative facility location and is based on the *Hotelling-Downs model*. The model will be explained in the next chapter. After that, the results regarding this model will be explained. We developed two tools that are useful for characterizing the possible equilibria for two, three, four and five facilities. The results of Cox [1] and Eaton and Lipsey [3] are related to these tools and Lerner [7] is related to the equilibria. This paper will end with a conclusion and recommendations for further research.

2 Model and notation

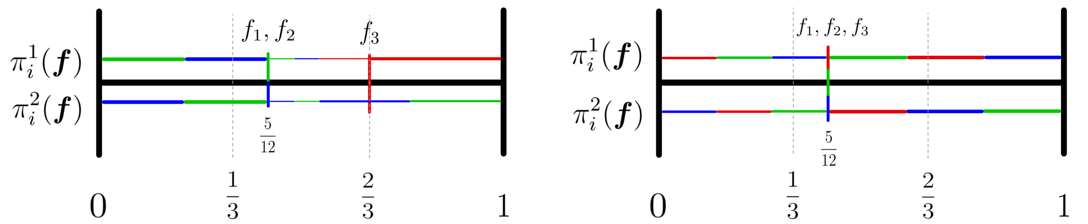
We consider a non-cooperative facility location model, which is similar to the *Hotelling-Downs model* [4, 2], but in this new model, clients have to choose two facilities instead of one. We call this model the *Choosing Two Facilities Hotelling Model*. In this model we have the set of m facilities $M := \{1, ..., m\}$. The facilities will choose strategically a location on the interval $I := [0, 1]$, and we have infinitely many clients that are uniformly distributed over this interval. The facilities are non-cooperative, so they do not choose a location together, but on their own. We define the strategy profile to be the vector $\boldsymbol{f} = (f_1, ..., f_m)$ where $f_i \in I$ represents the location of facility i. If we want to remove the location of facility i and replace this by a new location, we can represent this new strategy profile with the vector $(f_i^{'}, f_{-i})$, where f_{-i} means that the location of facility i is removed, and $f_i^{'}$ is the new location of facility i.

The payoff for facility i is defined as the amount of clients that choose facility i when facility i has strategy profile $\boldsymbol{f}$, which is denoted by $\pi_i(\boldsymbol{f})$. This is the sum of the payoffs from the first choice and second choice of the clients, i.e. $\pi_i(\boldsymbol{f}) = \pi_i^1(\boldsymbol{f}) + \pi_i^2(\boldsymbol{f})$, where $\pi_i^1(\boldsymbol{f})$ represents the first choice payoff for facility i and $\pi_i^2(\boldsymbol{f})$ represents the second choice payoff for facility i. The total payoff per facility has a maximum of 1, since each client chooses a facility at most once.

Every client $c \in I$ chooses the closest two facilities, where the facility that is the closest will be the first choice. When there are $k \geq 2$ facilities on the same location, such that for an amount of clients, let us call this x, this is the closest location, then the payoff for facility i per choice of the clients is equal to $\pi_i^1(\boldsymbol{f}) = \frac{1}{k} \cdot x = \pi_i^2(\boldsymbol{f})$ for all $2 \leq m \leq k$ where $i \in M$.

Now that we have introduced the model, let us first define an equilibrium properly and give an example of what can happen in this model, see Figure 1, before diving into the results.

Definition 1. *The strategy profile* $\boldsymbol{f}$ *is called an equilibrium if* $\pi_i(\boldsymbol{f}) \geq \pi_i(f_i', f_{-i})$ *for all* $i \in \{1, ..., m\}$ *and* $f_i' \neq f_i$.



(a) Example of $m = 3$ in an unstable situation where $\boldsymbol{f} = (f_1, f_2, f_3) = (\frac{5}{12}, \frac{5}{12}, \frac{2}{3})$.

(b) Example of $m = 3$ in an equilibrium where $\boldsymbol{f} = (\frac{5}{12}, \frac{5}{12}, \frac{5}{12})$.

FIGURE 1: Examples of strategy profiles in the *Choosing Two Facilities Hotelling Model* with three facilities. Figures a and b show an example where the facilities are in an unstable situation and an example where the facilities are in an equilibrium, respectively. The horizontal colored lines represent the payoff per facility per different color. The highest colored line represents the first choice payoff and the lower colored line represents the second choice payoff. Facility 1 is green, facility 2 is blue and facility 3 is red. Facility 3 can move to the left in Figure a to maximize his own payoff. This payoff is maximized if facility 3 is at the same location as the other two facilities, as we have shown in Figure b. The maximized payoff is then equal to $\pi_i(\boldsymbol{f}) = \frac{2}{3}$ for all $i \in \{1, 2, 3\}$.

3 Results

Our first result already shows that our situation is fundamentally different from the classical *Hotelling-Downs model*, where for two facilities the only equilibrium is both facilities exactly in the middle of the interval I. In our model, there are infinitely many equilibria when there are two facilities at the interval I.

Theorem 1. *For* $m = 2$*, every* $\boldsymbol{f}$ *is an equilibrium.*

Proof. Let $\boldsymbol{f}$ be arbitrary and let $x \coloneqq \pi_1^1(\boldsymbol{f})$, which represents the amount of clients who choose facility 1 first. That means that $x = \pi_2^2(\boldsymbol{f})$, which represents the amount of clients who choose facility 2 as second, since that is the only other choice. Then $1 - x$ clients choose facility 2 as first and facility 1 as second, i.e., $\pi_2^1(\boldsymbol{f}) = 1 - x = \pi_1^2(\boldsymbol{f})$. Therefore, both facilities get a payoff of 1, i.e., $\pi_i(\boldsymbol{f}) = 1$ for all i, and it is clear that this is independent of the location of the facilities. Therefore, $\pi_1(\boldsymbol{f}) = \pi_1^1(\boldsymbol{f}) + \pi_1^2(\boldsymbol{f}) = x + 1 - x = 1$ and $\pi_2(\boldsymbol{f}) = \pi_2^1(\boldsymbol{f}) + \pi_2^2(\boldsymbol{f}) = 1 - x + x = 1$. Because 1 is the maximum payoff per facility, $\pi_i(\boldsymbol{f}) = \pi_i(f_i^{'}, f_{-i})$ for all i and $f_i^{'} \neq f_i$, thus by definition we can state that $\boldsymbol{f}$ is an equilibrium. Since $\boldsymbol{f}$ was chosen arbitrarily, every $\boldsymbol{f}$ is an equilibrium. □

Now that we have proved the first result, it is interesting to study cases of more than two facilities placed on the interval I. Before we state and proof various theorems regarding more facilities, we first need the following tools. These are useful for characterizing the equilibria for cases with more than two facilities. The first tool describes that it is not possible to have one facility on the outside in an equilibrium.

Lemma 1. *For $m \geq 3$, if for the strategy profile $\boldsymbol{f}$ there is one facility outside, i.e., if $\exists k \in M$, such that $f_k > f_i$ (or $f_k < f_i$) for all $i \neq k$, then $\boldsymbol{f}$ is not an equilibrium.*

Proof. Let $m \geq 3$, assume that there is one facility $k \in M$ such that $f_k > f_j$ (the case $f_k < f_j$ is equivalent) for all $j \in M$ and $j \neq k$, and assume that $\boldsymbol{f} = (f_1, ..., f_k, ..., f_m)$ is an equilibrium. From the equilibrium condition it follows that $\pi_i(\boldsymbol{f}) \geq \pi_i(f_i^{'}, f_{-i})$, where $f_i^{'} \neq f_i$, for all $i \in M$.

There is at least one facility located in I, let us call it $l \in M$, such that $|f_k - f_l|$ is the smallest. Then $|f_k - f_l|$ is the distance between facility k and facility l and represents the amount of clients between these two facilities. Because $f_k > f_l$, all clients c, for which it holds that $c \geq f_k$, choose facility k as their first choice, which is the amount $1 - f_k$. Half of the clients that are in between f_k and f_l, that are closest to facility k, also choose for facility k as their first choice. This results in the following payoff, $\pi_k^1(\boldsymbol{f}) = 1 - f_k + \frac{f_k - f_l}{2} = 1 - \frac{f_k + f_l}{2}$. Now we make a case distinction. First, the case where there are two facilities located at the location of facility l, and second the case where facility l is alone at this location.

Case 1. If $f_l = f_j$ for at least one $j \in M$ and $j \neq l$, the second choice payoff for facility k is zero, i.e., $\pi_k^2(\boldsymbol{f}) = 0$. This is because the distance between the clients and facility l is the same as the distance between the clients and facility j, so $|c - f_l| = |c - f_j|$. So clients that chose facility l as first choice, because it is the closest, choose facility j as second choice and vice versa.
Now let facility k move $\delta := \frac{f_k - f_l}{2}$ to facility l and let this be the new location of k, i.e., $f_k^{'} = f_k - \delta$. For the new location of facility k the second choice payoff is still zero by the same argument as above, so $\pi_k^2(f_k^{'}, f_{-k}) = 0$. Therefore, we are only interested in the new payoff of the first choice of facility k. This is $\pi_k^1(f_k^{'}, f_{-k}) = 1 - \frac{f_k^{'} + f_l}{2}$. Since $f_k^{'} = f_k - \delta$, this implies that $\pi_k^1(f_k^{'}, f_{-k}) = 1 - \frac{f_k - \delta + f_l}{2}$. The payoff of the first choice for the original location of facility k was $\pi_k^1(\boldsymbol{f}) = 1 - \frac{f_k + f_l}{2}$. Note that since $\delta > 0$, $\frac{f_k - \delta + f_l}{2} < \frac{f_k + f_l}{2}$ and we can conclude that $1 - \frac{f_k - \delta + f_l}{2} > 1 - \frac{f_k + f_l}{2}$. Therefore, $\pi_k^1(f_k^{'}, f_{-k}) > \pi_k^1(\boldsymbol{f})$ and since $\pi_k^2(f_k^{'}, f_{-k}) = \pi_k^2(\boldsymbol{f}) = 0$ this implies that $\pi_k(f_k^{'}, f_{-k}) > \pi_k(\boldsymbol{f})$, which is a contradiction. Thus, our assumption was wrong and therefore $\boldsymbol{f}$ is not an equilibrium.

Case 2. Let $f_l \neq f_j$ for all $l \neq j$. The first choice payoff for facility k is the same as in *Case 1*, namely $\pi_k^1(\boldsymbol{f}) = 1 - f_k + \frac{f_k - f_l}{2}$. On the other hand, the second choice payoff for facility k is different from *Case 1*. Since $m \geq 3$ and we already have facility k and l, we know that there must be at least one other facility on I, let us call this facility h. Since $f_k > f_j$ for all $j \in M$ and $j \neq k$ and $f_l \neq f_j$ for all $l \neq j$, facility h is located somewhere at the left of facility l, i.e., $f_h \in [0, f_l)$. Facility h is the next facility, so there is no other facility between f_l and f_h.

Let us look at the clients. Clients who will choose facility k as their second choice are clients who chose facility l as their first choice. The amount of clients that choose facility k as their second choice is half of the amount that is between facility k and h, is closest to facility k, and depends on the distance between them. In other words, the location that is exactly at the middle of facility h and k is equal to $f_{k,h} = f_h + \frac{f_k - f_h}{2} = \frac{f_k + f_h}{2}$. Therefore, it is at least $\pi_k^2(\boldsymbol{f}) = \frac{f_k - f_h}{2}$. But the payoff is not complete yet, because part of this amount of clients overlaps with part of the clients that chose facility k as their first choice, which is half of the amount of clients between f_l and f_k (the half that is closest to f_k). Therefore, the correct amount of clients that choose facility k as their second choice is $\pi_k^2(\boldsymbol{f}) = \frac{f_k - f_h}{2} - \frac{f_k - f_l}{2} = \frac{f_l - f_h}{2}$. To get the total payoff for facility k we add the payoff of the first and second choice, which results in: $\pi_k(\boldsymbol{f}) = \pi_k^1(\boldsymbol{f}) + \pi_k^2(\boldsymbol{f}) = (1 - f_k + \frac{f_k - f_l}{2}) + (\frac{f_l - f_h}{2}) = 1 - \frac{f_k + f_h}{2}$.
Now, let facility k move $\delta \coloneqq \frac{f_k - f_l}{2}$ to facility l and let this be the new location of k, i.e., $f_k' = f_k - \delta$. The new payoff of the first choice for facility k is the same as in *Case 1*, so $\pi_k^1(f_k', f_{-k}) = 1 - \frac{f_k - \delta + f_l}{2}$. Since facility k moved towards facility l, and therefore also moved closer to facility h, the distance between facility k and h has become smaller, i.e., $f_k' - f_h < f_k - f_h$. This implies that the location that is exactly at the middle of facility h and k moves to the left, i.e., $f_{k,h}' = \frac{f_k' + f_h}{2} < \frac{f_k + f_h}{2} = f_{k,h}$. Since the original second choice payoff of facility k is independent of the location of f_k, the new second choice payoff with f_k' is the same, i.e. $\pi_k^2(f_k', f_{-k}) = \frac{f_l - f_h}{2}$. To get the new total payoff for facility k we add the payoff of the first choice and of the second choice, which results in: $\pi_k(f_k', f_{-k}) = \pi_k^1(f_k', f_{-k}) + \pi_k^2(f_k', f_{-k}) = (1 - \frac{f_k - \delta + f_l}{2}) + (\frac{f_l - f_h}{2})$. When we rewrite this, we end up with $\pi_k(f_k', f_{-k}) = 1 - \frac{f_k + f_h - \delta}{2}$. Since $\frac{\delta}{2} > 0$ and $\pi_k(\boldsymbol{f}) + \frac{\delta}{2} = \pi_k(f_k', f_{-k})$, we can conclude that $\pi_k(f_k', f_{-k}) > \pi_k(\boldsymbol{f})$. This is a contradiction which implies that our assumption was wrong. Therefore, $\boldsymbol{f}$ is not an equilibrium. □

Now that we know that one facility cannot be on the outside in an equilibrium, the second tool will describe that it is also not possible to have two facilities, which are on the same location, on the outside.

Lemma 2. *For $m \geq 3$, if for the strategy profile $\boldsymbol{f}$ there are two facilities $k \neq l$ outside, i.e., $f_k = f_l > f_i$ (or $f_k = f_l < f_i$) for all $i \neq k, l$, $\boldsymbol{f}$ is not an equilibrium.*

Proof. Let $m \geq 3$, assume that there are two facilities $k \neq l$ such that $f_k = f_l > f_j$ ($f_k = f_l < f_j$, respectively) for all $j \in M$ and $j \neq k, l$, and assume that $\boldsymbol{f} = (f_1, ..., f_k, ..., f_l, ..., f_m)$ is an equilibrium. From the equilibrium condition it follows that $\pi_i(\boldsymbol{f}) \geq \pi_i(f_i', f_{-i})$, where $f_i' \neq f_i$, for all $i \in M$.

There is at least one facility located in I, let's call it $h \in M$, such that $|f_k - f_h|$ is the smallest. Then $|f_k - f_h|$ is the distance between facility k (and facility l) and facility h and represents the amount of clients between these two facilities. Because $f_k > f_h$, all clients c, for which it holds that $c \geq f_k$ (and $c \geq f_l$), choose one of these facilities for the first choice, which is the amount $\frac{1 - f_k}{2}$ per facility. Half of the clients that are in between f_k (and f_l) and f_h, that are closest to facility k (and l), also choose for facility k (or l) for the first choice. This results in the following payoff, $\pi_k^1(\boldsymbol{f}) = \frac{1 - f_k}{2} + \frac{f_k - f_h}{4} = \pi_l^1(\boldsymbol{f})$. Now we make a case distinction. First the case where there are more than one facility located at the location of facility h (only possible when $m > 3$), and second the case where facility h is alone at this location.

Case 1. If $f_h = f_j$ for at least one facility $j \in M$ such that $j \neq h$, the second choice payoff for facility k does not depend on the clients that chose facility h or j as their first choice. This is because $|c - f_h| = |c - f_j|$, which is the distance from client c to a facility. Therefore, clients that chose as first choice for h, because it is the closest, choose for j as their second choice and vice versa.
Because there are two facilities at location f_k, the clients that chose for facility k the first time, now choose facility l and vice versa (because these are still the closest), i.e., $\pi_k^1(\boldsymbol{f}) = \pi_k^2(\boldsymbol{f}) = \frac{1}{2}(1 - f_k + \frac{f_k - f_h}{2}) = \frac{1}{2} - \frac{f_k + f_h}{4} = \pi_l^1(\boldsymbol{f}) = \pi_l^2(\boldsymbol{f})$. This implies that $\pi_k(\boldsymbol{f}) = 2(\frac{1}{2} - \frac{f_k + f_h}{4}) = 1 - \frac{f_k + f_h}{2}$.
Now let facility k move $\delta \coloneqq \frac{f_k - f_h}{2}$ to facility h and let this be the new location of k, i.e., $f_k' = f_k - \delta$. Therefore, $f_k' < f_l$, which implies that all clients for which it holds that $c \geq f_l$, choose for facility l as their first choice, instead of half of these clients, like in the original situation. This amount is $1 - f_l$. Also half of the clients that are in between f_k' and f_l, that are closest to facility l also choose for facility l for the first choice. This amount is $\frac{f_l - f_k'}{2}$. This implies that $\pi_l^1(f_k', f_{-k}) = 1 - f_l + \frac{f_l - f_k'}{2} = 1 - \frac{f_l + f_k'}{2}$. The other half of the clients that are between (the new location of) facility k and facility l, that are closest to facility k, choose for facility k as their first choice. Similar holds for the clients between facility h and k. Therefore, $\pi_k^1(f_k', f_{-k}) = \frac{f_l - f_k'}{2} + \frac{f_k' - f_h}{2} = \frac{f_l - f_h}{2}$.

The second choice payoff still does not depend on the clients that chose facility h or j as their first choice, because of the same argument as before moving. Since $f_l > f_i$ for all $i \neq l$, $|f_k - f_h|$ is the smallest and facility k moves towards facility h, clients who chose for facility l in their first choice, choose for facility k in their second choice, since that is the closest one. That implies that $\pi_k^2(f_k', f_{-k}) = 1 - \frac{f_l + f_k'}{2}$. To get the total new payoff for facility k, we add the first and the second choice, which results in: $\pi_k(f_k', f_{-k}) = \pi_k^1(f_k', f_{-k}) + \pi_k^2(f_k', f_{-k}) = \frac{f_l - f_h}{2} + 1 - \frac{f_l + f_k'}{2} = 1 - \frac{f_h + f_k'}{2}$. Since $f_k' = f_k - \delta$, that implies that $\pi_k(f_k', f_{-k}) = 1 - \frac{f_h + f_k - \delta}{2} = 1 - \frac{f_h + f_k}{2} + \frac{\delta}{2}$. Since $\frac{\delta}{2} > 0$ and $\pi_k(f_k', f_{-k}) = \pi_k(\boldsymbol{f}) + \frac{\delta}{2}$, we can conclude that there exists an f_k' such that $\pi_k(f_k', f_{-k}) > \pi_k(\boldsymbol{f})$. This is a contradiction thus our assumption was wrong. Therefore, $\boldsymbol{f}$ is not an equilibrium.

Case 2. Let $f_h \neq f_j$ for all $h \neq j$. The first choice payoff for facility k is the same as in *Case 1*, namely $\pi_k^1(\boldsymbol{f}) = \frac{1}{2} - \frac{f_k + f_h}{4}$. For the second choice, it still holds from *Case 1* that for two facilities at location f_k, the clients that chose for facility k the first time, now choose facility l and vice versa (because these are still the closest), but there is also something different for the second choice. Therefore, we make a case distinction. The first case is that there are two or more facilities next to facilities k and l (for which the prove will follow) and the second case that there is exactly one more facility next to facilities k and l, which is a special case and we will prove in *Case 3*.

Let $g, h, k, l \in M$ where $f_k = f_l$ still holds and $f_g < f_h < f_k$. Because there are two facilities at location f_k, clients c for which it holds that $c \geq f_k$, that chose facility l the first time, now choose facility k and vice versa. This is the amount $\frac{1 - f_k}{2}$. Therefore, the second choice payoff for facility k is at least $\frac{1 - f_k}{2}$. Since facility h is the first next facility if we look from facility k (and l), for a part of the clients that chose facility h as their first choice, since it is the closest, facility k and l are the second closest and are the facilities that will be chosen for their second choice. This amount depends on the distance between the next facility, which is facility g, and facility k. Half of the clients that are in between these facilities, that are closest to facility k and l, also choose either for facility k or for

facility l as their second choice. This is the amount $\frac{f_k - f_g}{4}$, since we divide by two to get half of the clients, and again by two to get the amount per facility. This results in the following second choice payoff for facility k, $\pi_k^2(\boldsymbol{f}) = \frac{1-f_k}{2} + \frac{f_k - f_g}{4} = \frac{1}{2} - \frac{f_k + f_g}{4}$. Add this to the first choice payoff of facility k (or l) and we get the total payoff of facility k, i.e., $\pi_k(\boldsymbol{f}) = \pi_k^1(\boldsymbol{f}) + \pi_k^2(\boldsymbol{f}) = \frac{1}{2} - \frac{f_k + f_h}{4} + \frac{1}{2} - \frac{f_k + f_g}{4} = 1 - \frac{f_k}{2} - \frac{f_h + f_g}{4}$.
Now let facility k move $\delta \coloneqq \frac{f_k - f_h}{2}$ to facility h and let this be the new location of k, i.e., $f_k' = f_k - \delta$. The new payoff for facility k of the first choice is the same as in *Case 1*: $\pi_k^1(f_k', f_{-k}) = \frac{f_l - f_h}{2}$. The new payoff for facility k of the second choice is different. Clients that chose facility l as their first choice, choose facility k as their second since it is the second closest. This amount is $1 - \frac{f_l + f_k'}{2}$. Since $f_h < f_k' < f_l$, part of the clients that chose facility h as their first choice, also choose facility k as their second. This amount depends on the distance between the next facility, which is facility g, where $f_g < f_h$, and facility k and is equal to $\frac{f_k' - f_g}{2}$. But in this amount, part overlaps with clients who have already chosen facility k as their first choice. That amount is half of the clients that are in between facility k and h, which are closest to facility k. If we subtract this amount we get $\frac{f_k' - f_g}{2} - \frac{f_k' - f_h}{2} = \frac{f_h - f_g}{2}$. This results in a new payoff for facility k for the second choice of $\pi_k^2(f_k', f_{-k}) = 1 - \frac{f_l + f_k'}{2} + \frac{f_h - f_g}{2}$. To get the new total payoff for facility k we add the payoff of the first choice and of the second choice, which results in: $\pi_k(f_k', f_{-k}) = \pi_k^1(f_k', f_{-k}) + \pi_k^2(f_k', f_{-k}) = \frac{f_l - f_h}{2} + 1 - \frac{f_l + f_k'}{2} + \frac{f_h - f_g}{2} = 1 - \frac{f_k' + f_g}{2}$. If we plug in $f_k' = f_k - \delta$ and $\delta = \frac{f_k - f_h}{2}$ we get $\pi_k(f_k', f_{-k}) = 1 - \frac{f_k - \delta + f_g}{2} = 1 - \frac{f_k - \frac{f_k - f_h}{2} + f_g}{2}$. If we rewrite this we get $\pi_k(f_k', f_{-k}) = 1 - \frac{f_k + f_h}{4} - \frac{f_g}{2}$.

The only thing that is left to prove is that $\pi_k(f_k', f_{-k}) > \pi_k(\boldsymbol{f})$, i.e., $1 - \frac{f_k + f_h}{4} - \frac{f_g}{2} > 1 - \frac{f_k}{2} - \frac{f_h + f_g}{4}$. Let us first rewrite the new payoff of facility k; $\pi_k(f_k', f_{-k}) = 1 - \frac{f_k + f_h}{4} - \frac{f_g}{2} = 1 - \frac{f_h}{4} - \frac{f_k}{2} + \frac{f_k}{4} - \frac{f_g}{4} - \frac{f_g}{4}$. Then, $1 - \frac{f_h}{4} - \frac{f_k}{2} + \frac{f_k}{4} - \frac{f_g}{4} - \frac{f_g}{4} = \pi_k(\boldsymbol{f}) + \frac{f_k}{4} - \frac{f_g}{4}$. Since $f_k > f_g$, we can state that $\frac{f_k}{4} - \frac{f_g}{4} > 0$. This implies $\pi_k(f_k', f_{-k}) = \pi_k(\boldsymbol{f}) + \frac{f_k}{4} - \frac{f_g}{4} > \pi_k(\boldsymbol{f})$. This is a contradiction, which implies that our assumption was wrong. Therefore, $\boldsymbol{f}$ is not an equilibrium.

Case 3. Let $M = \{h, k, l\}$ where $f_k = f_l$ still holds and $f_h < f_k$. Like mentioned in *Case 2*, this is a special case. Still, the first choice payoff for facility k is the same as in *Case 1*, namely $\pi_k^1(\boldsymbol{f}) = \frac{1}{2} - \frac{f_k + f_h}{4}$. The second choice is different, which we will explain now.

Clients who chose facility h as their first choice do not have any other choice for their second choice than facility k or l. Since $f_k = f_l$, we know that $|c - f_k| = |c - f_l|$. Therefore, the second choice payoff for facility k (or l) is at least half of the first choice payoff of facility h, which is independent of the location of facilities k and l, as long as these facilities are at the same location. This amount is $\frac{1}{2}(1 - 2\pi_k^1(\boldsymbol{f})) = \frac{1}{2} - (\frac{1}{2} - \frac{f_k + f_h}{4}) = \frac{f_k + f_h}{4}$. Besides this, there are still clients for which facilities k and l are the closest. Because there are two facilities at location f_k, the clients that chose facility k the first time, now choose facility l and vice versa. This is the amount $\pi_k^1(\boldsymbol{f}) = \frac{1}{2} - \frac{f_k + f_h}{4}$. Therefore, the total second choice payoff of facility k (or l) is equal to $\pi_k^2(\boldsymbol{f}) = \frac{f_k + f_h}{4} + \frac{1}{2} - \frac{f_k + f_h}{4} = \frac{1}{2}$. Add this to the first choice payoff of facility k (or l) and we get the total payoff of facility k, i.e., $\pi_k(\boldsymbol{f}) = \pi_k^1(\boldsymbol{f}) + \pi_k^2(\boldsymbol{f}) = \frac{1}{2} - \frac{f_k + f_h}{4} + \frac{1}{2} = 1 - \frac{f_k + f_h}{4}$.
Now let facility k move $\delta \coloneqq \frac{f_k - f_h}{2}$ to facility h and let this be the new location of k, i.e., $f_k' = f_k - \delta$. The new payoff for facility k of the first choice is the same as in *Case 1*:

$\pi^1_k(f'_k, f_{-k}) = \frac{f_l - f_h}{2}$. The new payoff for facility k of the second choice is different. Since $f_h < f'_k < f_l$, the clients that chose facility h as their first choice, choose facility k as their second, since it is the closest, and clients that chose facility l as their first choice, also choose facility k as their second since it is the closest. This results in a new payoff for facility k for the second choice of $\pi^2_k(f'_k, f_{-k}) = \pi^1_h(f'_k, f_{-k}) + \pi^1_l(f'_k, f_{-k}) = \frac{f_h + f'_k}{2} + 1 - \frac{f_l + f'_k}{2} = 1 - \frac{f_l - f_h}{2}$. To get the new total payoff for facility k we add the payoff of the first choice and of the second choice, which results in: $\pi_k(f'_k, f_{-k}) = \pi^1_k(f'_k, f_{-k}) + \pi^2_k(f'_k, f_{-k}) = \frac{f_l - f_h}{2} + 1 - \frac{f_l - f_h}{2} = 1$. Since $1 > 1 - \frac{f_k + f_h}{4}$ we can conclude that there exists an f'_k such that $\pi_k(f'_k, f_{-k}) > \pi_k(\boldsymbol{f})$. This is a contradiction thus our assumption was wrong. Therefore, $\boldsymbol{f}$ is not an equilibrium. □

Now that we have developed the two tools in the *Choosing Two Facilities Hotelling Model*, we will use these tools to characterize the equilibria for three facilities on the interval I. The main difference with the *Hotelling-Downs model* [4, 2] is that in the *Choosing Two Facilities Hotelling Model* there exist equilibria for three facilities, in contrast to the *Hotelling-Downs model*. Next to the proof, there is an example below, see Figure 2, to also get a visual idea of this situation.

Theorem 2. *For $m = 3$, if $\boldsymbol{f} = (f, f, f)$ where $f \in [\frac{1}{3}, \frac{2}{3}]$, then $\boldsymbol{f}$ are all possible equilibria.*

Proof. Let us first prove that all facilities should be placed at the same location. Assuming this is not the case, the possibilities of placing three facilities in the interval I are as follows. Either each facility at a different location (i) or two facilities at the same location and the one at a different location (ii).

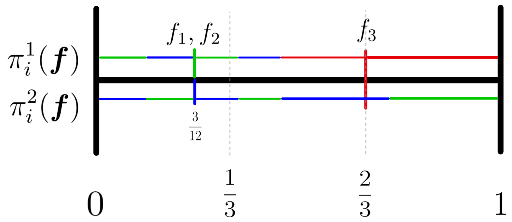
In Case (i), there are two different places that have a facility on the outside. By Lemma 1 this cannot be an equilibrium. In Case (ii), there is one facility located outside and there are two facilities together located outside. By Lemma 1 and by Lemma 2, respectively, this cannot be an equilibrium. Therefore, we proved that $\boldsymbol{f} = (f, f, f)$.

What is left to prove is that if $f \in [\frac{1}{3}, \frac{2}{3}]$, that $\boldsymbol{f}$ are all the possible equilibria. Before we do that, note that because $\boldsymbol{f} = (f, f, f)$, the payoffs for all facilities are the same. For the first choice, every facility gets a third of the clients c for which it holds that $c \leq f$, which is equal to $\frac{f}{3}$, and a third of the clients c for which it holds that $c > f$, which is equal to $\frac{1-f}{3}$. The same holds for the second choice. Therefore, the payoff is equal to $\pi_i(\boldsymbol{f}) = \pi^1_i(\boldsymbol{f}) + \pi^2_i(\boldsymbol{f}) = (\frac{f}{3} + \frac{1-f}{3}) \cdot 2 = \frac{2}{3}$ for all $i \in \{1, 2, 3\}$. We see that this is independent of the specific location of f, as long as the three locations are the same.

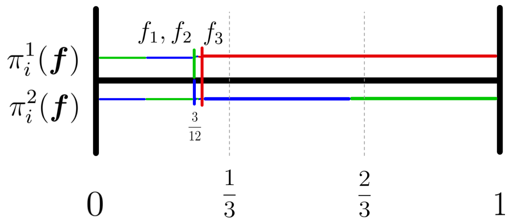
Assume that $f \in [\frac{1}{3}, \frac{2}{3}]$. Let, without loss of generality, facility 1 move an arbitrary distance, call it δ, away from the other two facilities and let this be the new location of facility 1, i.e., $f'_1 = f + \delta$ such that $f'_1 \in I$. Note that if δ is positive, $f'_1 > f$ and if δ is negative, $f'_1 < f$. For now, we assume that δ is positive and thus $f'_1 > f$. The other case is equivalent. We are now in the *Case 1* of the proof of Lemma 1. Therefore, the new payoff for facility 1 is equal to $\pi_1(f'_1, f_{-1}) = \pi^1_1(f'_1, f_{-1}) + \pi^2_1(f'_1, f_{-1}) = 1 - \frac{f'_1 + f}{2} + 0 = 1 - \frac{f'_1 + f}{2}$. Since $f'_1 = f + \delta$, this implies that $\pi_1(f'_1, f_{-1}) = 1 - \frac{f + \delta + f}{2} = 1 - f - \frac{\delta}{2}$. Since $f \in [\frac{1}{3}, \frac{2}{3}]$, if we rewrite the new payoff, we get $\frac{1}{3} - \frac{\delta}{2} \leq \pi_1(f'_1, f_{-1}) \leq \frac{2}{3} - \frac{\delta}{2}$. Since δ is positive, the new payoff is always smaller than $\frac{2}{3}$, which was the original payoff. This means that there does not exist an f'_1 such that $\pi_1(f'_1, f_{-1}) > \pi_1(\boldsymbol{f})$. Therefore, if $f \in [\frac{1}{3}, \frac{2}{3}]$ for $\boldsymbol{f} = (f, f, f)$, then these are possible equilibria.

The only question that is left, is if these possible equilibria are the only equilibria, or if there are more. Assume that for the strategy profile $\boldsymbol{f} = (f, f, f)$ where $f < \frac{1}{3}$ and assume that $\boldsymbol{f}$ are possible equilibria. The payoff is the same as above, namely $\pi_i(\boldsymbol{f}) = \pi_i^1(\boldsymbol{f}) + \pi_i^2(\boldsymbol{f}) = (\frac{f}{3} + \frac{1-f}{3}) \cdot 2 = \frac{2}{3}$ for all $i \in \{1, 2, 3\}$. Let, without loss of generality, facility 1 move $\delta := \frac{\frac{1}{3} - f}{2}$, away from the other two facilities and let this be the new location of facility 1, i.e., $f_1' = f + \delta$. We are now in the *Case 1* of the proof of Lemma 1. Therefore, the new payoff for facility 1 is equal to $\pi_1(f_1', f_{-1}) = \pi_1^1(f_1', f_{-1}) + \pi_1^2(f_1', f_{-1}) = 1 - \frac{f_1' + f}{2} + 0 = 1 - \frac{f_1' + f}{2}$.

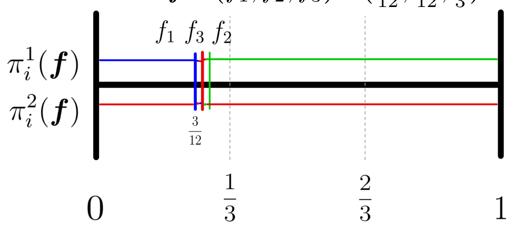
Since $f_1' = f + \delta$ where $\delta = \frac{\frac{1}{3} - f}{2}$, this implies that $\pi_1(f_1', f_{-1}) = 1 - \frac{f + \frac{\frac{1}{3} - f}{2} + f}{2} = 1 - f - \frac{\frac{1}{3} - f}{4} = 1 - \frac{3}{4} f - \frac{1}{12}$. Since $f < \frac{1}{3}$, we can state that $1 - \frac{3}{4} f - \frac{1}{12} > 1 - \frac{3}{4} \cdot \frac{1}{3} - \frac{1}{12} = \frac{2}{3}$. Since the original payoff is equal to $\frac{2}{3}$, we can conclude that $\pi_1(f_1', f_{-1}) > \pi_1(\boldsymbol{f})$ and this is a contradiction. Therefore, our assumption was wrong and thus $\boldsymbol{f}$ where $f < \frac{1}{3}$ is not an equilibrium. The case where $f > \frac{2}{3}$ is equivalent. This concludes that $\boldsymbol{f} = (f, f, f)$ are all the possible equilibria if $f \in [\frac{1}{3}, \frac{2}{3}]$. □



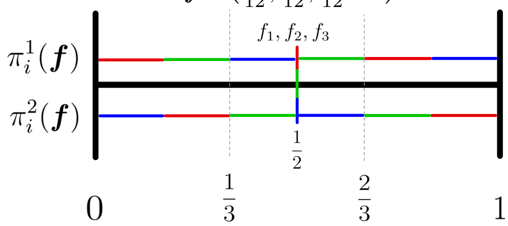
(a) Example of $m = 3$ in an unstable situation where $\boldsymbol{f} = (f_1, f_2, f_3) = (\frac{3}{12}, \frac{3}{12}, \frac{2}{3})$.



(b) Example of $m = 3$ in an unstable situation where $\boldsymbol{f} = (\frac{3}{12}, \frac{3}{12}, \frac{3}{12} + \epsilon)$.



(c) Example of $m = 3$ in an unstable situation where $\boldsymbol{f} = (\frac{3}{12}, \frac{3}{12} + 2\epsilon, \frac{3}{12} + \epsilon)$.



(d) Example of $m = 3$ in an equilibrium where $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

FIGURE 2: Examples of strategy profiles in the *Choosing Two Facilities Hotelling Model* with three facilities. The horizontal colored lines represent the payoff per facility per different color. The highest colored line represents the first choice payoff and the lower colored line represents the second choice payoff. Facility 1 is blue, facility 2 is green and facility 3 is red. Figure a is an example of an unstable situation. Here, facility 3 can improve his own payoff by moving to the left. An example of this is shown in Figure b. Since all facilities want to maximize their own profit, it is beneficial for the other two facilities to move to the right of facility 3. An example where one of the two facilities moves to the right of facility 3 is shown in Figure c. This process can be repeated with all facilities, until all facilities are at the same location within the interval $[\frac{1}{3}, \frac{2}{3}]$. An example of this is shown in Figure d. The maximized payoff is then equal to $\pi_i(\boldsymbol{f}) = \frac{2}{3}$ for all $i \in \{1, 2, 3\}$.

The result for four facilities, which we will prove below, is quite similar to the result for three facilities. After the proof, there is an example, see Figure 3, to get a visual idea of this situation.

Theorem 3. *For* $m = 4$, *if* $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, *then* $\boldsymbol{f}$ *is the only equilibrium.*

Proof. Let us first prove that all facilities should be placed at the same location. Assuming this is not the case, the possibilities of placing four facilities in the interval I are as follows. Each facility at a different location (i), two pairs of two facilities where the pairs are at different locations (ii), three facilities at one location and one at a different location (iii) or all four facilities at the same location (iv).

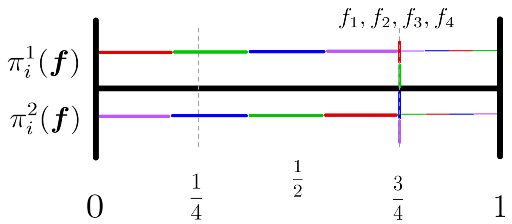
In Cases (i) and (iii), there is a facility outside. By Lemma 1 these cases do not contain equilibria. In Case (ii), there are two pairs of two facilities together located outside. By Lemma 2 this cannot be an equilibrium. Therefore, we proved that all facilities should be placed at the same location, i.e., $\boldsymbol{f} = (f, f, f, f)$.

What is left to prove is that $\boldsymbol{f}$ is the only equilibrium if $f = \frac{1}{2}$. Before we do that, note that because $\boldsymbol{f} = (f, f, f, f)$, the payoffs for all facilities are the same. For the first choice, every facility gets a fourth of the clients c for which it holds that $c \leq f$, which is equal to $\frac{f}{4}$, and a fourth of the clients c for which it holds that $c > f$, which is equal to $\frac{1-f}{4}$. The same holds for the second choice. Therefore, the payoff is equal to $\pi_i(\boldsymbol{f}) = \pi_i^1(\boldsymbol{f}) + \pi_i^2(\boldsymbol{f}) = (\frac{f}{4} + \frac{1-f}{4}) \cdot 2 = \frac{1}{2}$ for all $i \in \{1, 2, 3, 4\}$. We see that this is independent of the specific location of f, as long as the four locations are the same.

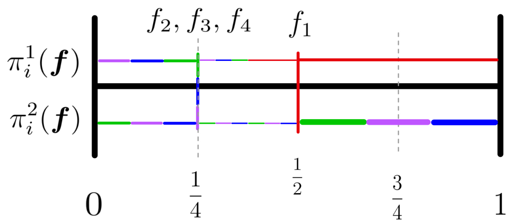
Assume that $f = \frac{1}{2}$. Let, without loss of generality, facility 1 move an arbitrary distance, call it δ, away from the other two facilities and let this be the new location of facility 1, i.e., $f_1' = f + \delta$ such that $f_1' \in I$. Note that if δ is positive, $f_1' > f$ and if δ is negative, $f_1' < f$. For now, we assume that δ is positive and thus $f_1' > f$. The other case is equivalent. We are now in the *Case 1* of the proof of Lemma 1. Therefore, the new payoff for facility 1 is equal to $\pi_1(f_1', f_{-1}) = \pi_1^1(f_1', f_{-1}) + \pi_1^2(f_1', f_{-1}) = 1 - \frac{f_1'+f}{2} + 0 = 1 - \frac{f_1'+f}{2}$. Since $f_1' = f + \delta$, this implies that $\pi_1(f_1', f_{-1}) = 1 - \frac{f+\delta+f}{2} = 1 - f - \frac{\delta}{2}$. Since $f = \frac{1}{2}$, if we rewrite the new payoff, we get $\pi_1(f_1', f_{-1}) = \frac{1}{2} - \frac{\delta}{2}$. Since δ is positive, the new payoff is always smaller than $\frac{1}{2}$, which was the original payoff. This means that there does not exist an f_1' such that $\pi_1(f_1', f_{-1}) > \pi_1(\boldsymbol{f})$. Therefore, if $f = \frac{1}{2}$ for $\boldsymbol{f} = (f, f, f, f)$, then this is an equilibrium.

The only question that is left, is if this is the only equilibrium. Assume that the strategy profile $\boldsymbol{f} = (f, f, f, f)$ where $f < \frac{1}{2}$ and assume that $\boldsymbol{f}$ is an equilibrium. The payoff is the same as above, namely $\pi_i(\boldsymbol{f}) = \pi_i^1(\boldsymbol{f}) + \pi_i^2(\boldsymbol{f}) = (\frac{f}{4} + \frac{1-f}{4}) \cdot 2 = \frac{1}{2}$ for all $i \in \{1, 2, 3, 4\}$. Let, without loss of generality, facility 1 move $\delta := \frac{\frac{1}{2}-f}{2}$, away from the other three facilities and let this be the new location of facility 1, i.e., $f_1' = f + \delta$. We are now in the *Case 1* of the proof of Lemma 1. Therefore, the new payoff for facility 1 is equal to $\pi_1(f_1', f_{-1}) = \pi_1^1(f_1', f_{-1}) + \pi_1^2(f_1', f_{-1}) = 1 - \frac{f_1'+f}{2} + 0 = 1 - \frac{f_1'+f}{2}$.

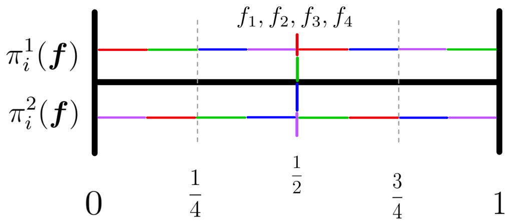
Since $f_1' = f + \delta$ where $\delta = \frac{\frac{1}{2}-f}{2}$, this implies that $\pi_1(f_1', f_{-1}) = 1 - \frac{f + \frac{\frac{1}{2}-f}{2} + f}{2} = 1 - f - \frac{\frac{1}{2}-f}{4} = 1 - \frac{3}{4}f - \frac{1}{8}$. Since $f < \frac{1}{2}$, we can state that $1 - \frac{3}{4}f - \frac{1}{12} > 1 - \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{8} = \frac{1}{2}$. Since the original payoff is equal to $\frac{1}{2}$, we can conclude that $\pi_1(f_1', f_{-1}) > \pi_1(\boldsymbol{f})$ and this is a contradiction. Therefore, our assumption was wrong and $\boldsymbol{f}$ where $f < \frac{1}{2}$ is not an equilibrium. The case where $f > \frac{1}{2}$ is equivalent. This concludes that $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the only equilibrium. □



(a) Example of $m = 4$ in an unstable situation where $\boldsymbol{f} = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})$. When one facility moves slightly to the left, the payoff for that facility will be higher than $\frac{1}{2}$, which is the payoff of all facilities in the current situation.



(b) Example of $m = 4$ in an unstable situation where $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. When facility 1 moves to the left, and stays on the right of the other three facilities, the payoff will get larger the more it moves and thus this situation is unstable.



(c) Representation of $m = 4$ in their only equilibrium, which is $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

FIGURE 3: Examples of strategy profiles in the *Choosing Two Facilities Hotelling Model* with four facilities. Figures a and b show examples of unstable situations. The horizontal colored lines represent the payoff per facility per different color. The highest colored line represents the first choice payoff and the lower colored line represents the second choice payoff. Facility 1 is red, facility 2 is blue, facility 3 is green and facility 4 is purple. Figure c shows the only equilibrium with four facilities, where all facility locations are equal to $\frac{1}{2}$ and the payoff is equal to $\pi_i(\boldsymbol{f}) = \frac{1}{2}$ for all $i \in \{1, 2, 3, 4\}$.

The proofs regarding three and four facilities are quite similar to each other. But with five facilities, something interesting happens. This result is similar to the result of the three facilities in the *Hotelling-Downs model* [4, 2].

Theorem 4. *For $m = 5$, there does not exist an $\boldsymbol{f}$ which is an equilibrium.*

Proof. The options for placing five facilities on the interval I are as follows. *Case 1.* All facilities are at the same location, i.e., $\boldsymbol{f} = (f, f, f, f, f)$. *Case 2.* Four facilities are at the same location and one is at a different location, i.e., $\boldsymbol{f} = (f, f, f, f, f')$ where $f \neq f'$. *Case 3.* Three facilities are at the same location and two are at the same different location, i.e., $\boldsymbol{f} = (f, f, f, f', f')$ where $f \neq f'$. *Case 4.* Two pairs of two facilities at two locations $f \neq f'$ and one at a different location, i.e., $\boldsymbol{f} = (f", f, f, f', f')$ where $f" \neq f, f'$. *Case 5.* All facilities are at a different location, i.e., $\boldsymbol{f} = (f_1, f_2, f_3, f_4, f_5)$ such that $f_i \neq f_k$ for all $i \neq k$ where $i, k \in \{1, ..., 5\}$. All these possibilities can be proved by the following three cases:

Assume that we are in *Case 1* and let $\epsilon > 0$. Then the payoff of every facility is $2/5$, because all clients choose two facilities out of the five facilities. If one of the facilities moves ϵ to the larger side of the interval, i.e., if $f < \frac{1}{2}$, $f_i' = f + \epsilon$ and if $f > \frac{1}{2}$, $f_i' = f - \epsilon$, the payoff of this facility will be at least $1/2 - \epsilon/2$, since this facility gets at least $1/2 - \epsilon/2$ of the first choices and zero second choices. Since ϵ is small, $1/2 - \epsilon/2 > 2/5$, and thus there exists at least one $f_i' \neq f_i$ in *Case 1* such that $\pi_i(f_i', f_{-i}) > \pi_i(\boldsymbol{f})$, which implies that this case cannot be an equilibrium.

Assume one facility is on the outside. This is true in *Case 2* and *Case 5*. Then by Lemma 1 these two cases cannot be equilibria.

Assume two facilities are on the outside. This is true in *Case 3* and *Case 4*. Then by Lemma 2 these two cases cannot be equilibria. □

Now that we proved the situations where the number of facilities was two, three, four and five, it would be interesting to generalize the results of this paper. We think that this can be generalized in three different cases.
The *first case* will be the number of facilities that is divisible by three. Then we think that the stable situation is that the facilities are placed at a location in groups of three.
The *second case* will be the number of facilities that results in groups of three and one facility left. We think the stable situation is that the facilities are again placed at a location in groups of three with one facility somewhere between these groups. It has to be between the groups, because otherwise this facility is on the outside and that cannot be an equilibrium by Lemma 1.
The *third case* will be the number of facilities that results in groups of three and two facilities left. We think that the stable situation is that the facilities are again placed at a location in groups of three with two facilities somewhere between these groups, either together or separate. Again, these two facilities have to be in between the groups of three, by Lemma 1 and Lemma 2.

4 Conclusion and further research

In this paper it became clear that the *Choosing Two Facilities Hotelling Model* is fundamentally different from the classical *Hotelling-Downs model*. The principle is the same as the classical *Hotelling-Downs model* but the results are different. Therefore, we could not translate the existing results literally to our model, but we could use the basis. Furthermore, we developed tools which were useful to show that there exist equilibria in our model.

For further research, these tools can also be used in generalizing our model, which was mentioned in the last paragraph of previous section. Further, in our model, clients only choose based on the distance. It would be interesting to add an assumption and investigate what will happen. For example, let clients choose based on the distance and the waiting time.

A different approach is, for example, to let go of the distance and the fact that the clients are uniformly distributed. There are models where this is the case, like the *Two-Sided Facility Location Game* [5] and the *Two-Stage Facility Location Fame with atomic clients* [6]. In these models, the clients are modeled as vertices of a graph and have a certain weight which they can spread between facilities. This makes the client itself deterministic, which is closer to reality, but the weight can be treated continuously. The facilities also have to be placed at a vertex, which makes it possible to let go of the distance like it was modeled in the classical *Hotelling-Downs model*, and replace it with the neighborhood of the client.

In this approach, also the line in the classical *Hotelling-Downs model* has changed to a graph. This also gives the possibility to translate the problem to a three-dimensional space instead of a two-dimensional, like in our model. Three-dimensional is interesting for further research, since this is closer to reality.

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