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On the Daniell integral

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Abstract

The goal of this paper is presenting the construction of the Daniell integral together with the proof of the Riesz-Markov-Kakutani representation theorem. Moreover, this article also presents results related to the so-called Bochner integral.

Like any other notion of integration, the Daniell integral is a functional that acts on a space of functions. However, the approach is different from the one used for the Lebesgue integral as it does not require a predefined notion of a measure. Therefore, one may alternatively define the measure of a set A (given that certain conditions are satisfied) by applying the Daniell integral on the indicator function of A.

This construction provides a natural proof of the Riesz-Markov-Kakutani representation theorem, which states that the dual space of continuous functions defined on a locally-compact Hausdorff space Ω is the set of all measures on Ω.

One may further define the integral as a functional acting on the space of functions mapping a subset of $\mathbb{R}^n$ to an arbitrary Banach space X. This is the so-called Bochner integral and will be presented in this paper as well.

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1 Introduction

We start by considering L, a linear space of functions which is closed under taking the maximum and the minimum. On this space, we consider a functional that maps positive functions to positive real numbers, while ensuring that it converges to 0 point-wise, as the function approaches 0. Such a functional is called an integral. We will then extend the integral on the space of monotone increasing limits of sequences in L, denoted by U. This is done by defining the $I(f) \coloneqq \lim_{n\to\infty} I(f_n)$ for all $f \in U$, where f_n is a monotone increasing sequence in L that converges to f. A similar process will be done to extend the integral on the space $-U$, defined as $-U \coloneqq \{f | -f \in U\}$.

We will then call a function f integrable if it "fits" between a function $g \in -U$ and a function $h \in U$ for which the integral yields a finite result and such that $I(h-g)$ is arbitrarily small. We can then define the integral of f as $\inf I(h)$ or, equivalently $\sup I(g)$, where h and g are prescribed as above. The space of integrable functions will be called L^1.

We continue by defining the space of Baire functions. This is the smallest monotone family of functions containing our vector space L. Then we consider the space L^1 to be the intersection of the previously defined L^1 with the space of Baire functions. Now, we call a function f $\infty-$integrable if either $f^+ \coloneqq \max\{f, 0\}$ or $f^- \coloneqq \min\{-f, 0\}$ are integrable functions and define its integral as $I(f) \coloneqq I(f^+) - I(f^-)$, with $\pm\infty$ allowed as possible values.

In contrast to the Lebesgue approach, we did not need the notion of a measure so far. Hence, this concept can be defined by using the already constructed integral, as the argument will not be circular. Therefore, we will call a set A whose indicator function is in L^1 a measurable set and define its measure by $\mu(A) \coloneqq I(\mathbb{1}_A)$. If the measure of A is of finite value, we further call A a summable set. We will then present a few results which will allow us to conclude that for any $f \in \mathbb{B}^+$ we have that $I(f) = \int f \mathrm{d}\mu$, with $\int$ representing the standard Lebesgue integral. This statement is a big part of the Riesz-Markov-Kakutani representation theorem, which is the main point of interest of this paper.

The Riesz-Markov-Kakutani representation theorem states that for any functional ϕ on $C([a,b])$ there exists a unique measure μ such that $\phi(f) := \int_a^b f \mathrm{d}\mu$ for all $f \in C([a,b])$. The proof follows in a natural manner by exploiting our new approach of integration.

We will then turn our attention to the so-called Bochner integral. Namely, we will formally define a method of integrating functions which map to arbitrary Banach spaces. This construction starts by considering a $\sigma-$finite measure space (Ω, Σ, μ). In the case of functions mapping from $\Omega \subseteq \mathbb{R}^n$ to a Banach space X, f is called simple if the image of Ω under f is a finite set and the preimage of all $x \in X \setminus \{0\}$ is a set of finite measure in the $\sigma-$algebra Σ.

Furthermore, we will say that a function f is Bochner-measurable if it can be approximated almost everywhere by a sequence of simple functions. We then construct the Bochner-Lebesgue L^p spaces by defining $L^p := \mathcal{L}^p/_{\sim}$, where $\mathcal{L}^p$ is the space of Bochner-measurable functions with $\|f\|_X < \infty$ and the equivalence relation $\sim$ is given by $f \sim g \iff f = g$ almost everywhere. This spaces will turn out to be Banach spaces, with the space of simple functions being a dense subspace of these Bochner-Lebesgue spaces. Using these results, we can finally define the Bochner integral as the continuous linear extension of the mapping $f \to \sum_{x \in X} x\mu(f^{-1}(x))$, defined on the space of simple functions from Ω to X and taking values in X.

As we have constructed the Bochner integral by employing a similar approach as for the one used for defining the Lebesgue integral, we expect that the Daniell approach can also be used to construct an alternative, equivalent notion of the Bochner integral. Therefore, we will conclude the paper by presenting Pettis' theorem, a result which indeed supports this idea.

2 Preliminaries

This paper makes use of a few notions from measure theory and functional analysis. However, as those courses are not part of the regular Bachelor's programme, the relevant notions will be very briefly introduced. Furthermore, as we will also require some results from these two courses, the essential theorems and lemmas will be presented.

Definitions and results related to functional analysis, such as functionals and boundedness come from [1], while for measure theoretic aspects we use notions from [2].

Definition 2.1. A *linear functional* is a linear map from an arbitrary Banach space X to its underlying scalar field (in general $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.2. A linear functional ϕ on a Banach space X is said to be *bounded* if
$\exists\, a \in \mathbb{R}$ such that $\forall x \in X : \|\phi(x)\| \leq a \cdot |x|$
where $\|\cdot\|$ is the norm on X.

A useful characterization of continuity of a functional is presented in the next theorem.

Theorem 2.3. *A linear functional is continuous if and only if it is bounded.*

Proof. Let ϕ be a continuous linear functional defined on the Banach space X. As this implies continuity at $0 \in X$, we know that $\exists \delta > 0$ such that $\forall y \in X$ with $\|y\| \leq \delta$ the inequality $\|\phi(y) - \phi(0)\| < 1$. By linearity, we know that $\phi(0) = 0$, therefore the inequality

becomes $\|\phi(y)\| < 1$ for all $y \in X$ with $\|y\| \leq \delta$.
Now for any point $x \in X$, by linearity of ϕ it holds that $\|\phi(x)\| = \|\frac{\|x\|}{\delta}\phi(\delta\frac{x}{\|x\|})\| = \frac{\|x\|}{\delta}\|\phi(\delta\frac{x}{\|x\|})\|$. As $\|\delta\frac{x}{\|x\|}\| = \delta$, we can apply the previously found result and hence obtain $\|\phi(x)\| < \frac{\|x\|}{\delta}1 = \frac{\|x\|}{\delta}$. Therefore, the functional is bounded.

Conversely, assuming that ϕ is a bounded linear functional, we know that $\|\phi(x+y) - \phi(x)\| = \|\phi(y)\| < C\|y\|$ for every $x, y \in X$ and some constant $C > 0$. By letting $y \to 0$ we conclude that the functional is continuous at any $x \in X$. □

Definition 2.4. A *simple function* ϕ defined on Ω and mapping to $S \subseteq \mathbb{R}$ is defined to be a linear combination of indicator functions of intervals. To be precise, this means that a simple function is of the form $\phi(x) = c_i$ if $x \in A_i$ where A_i are countably many disjoint sets such that their union is precisely Ω, and $c_i \in \mathbb{R}$.

Definition 2.5. The *integral of a simple* function θ over a domain Ω with respect to a measure μ is defined as $\int_\Omega \theta \mathrm{d}\mu = \sum_{i=1}^{N} c_i\mu(A_i)$, where the sets A_i are measurable sets (as defined in [2]) and they are as in definition 2.4.

Definition 2.6. The *integral of a function* f over a domain Ω with respect to a measure μ is defined as $\int_\Omega f \mathrm{d}\mu := \sup\{\int_\Omega \theta \mathrm{d}\mu | \theta \leq f, \theta \text{ simple function}\}$.

Theorem 2.7. *A functional ϕ over a space of functions $\mathbb{F}$ which is closed under addition, function multiplication and division by functions that are non-zero at every point, defined on a domain Ω can be split into a positive and a negative part such that $\phi = \phi^+ - \phi^-$, such that both ϕ^+ and ϕ^- are positive.*

Proof. Let us define a new functional $\phi^+ \colon \mathbb{F}^+ \to \mathbb{R}$, by $\phi^+(f) := \sup\{\phi(h) | h \in \mathbb{F}^+, h \leq f\}$ for all functions $f \in \mathbb{F}^+$. Here, $\mathbb{F}^+$ denotes the space of all positive functions in $\mathbb{F}$. Moreover, we wish to extend the definition of this functional on the entire function space $\mathbb{F}$. We will do so by defining $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$ for all $f \in \mathbb{F}$, where the functions f^+ and f^- represent the unique decomposition of the function f, namely $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$ for every $x \in \Omega$.
Moreover, we can simply define $\phi^- := \phi^+ - \phi$. Hence, we have constructed a decomposition of ϕ according to the statement.

It remains to be proven that these functionals are indeed linear. To this end consider $f, g \in \mathbb{F}^+$. Then $\phi^+(f) = \sup\{\phi(h_1) | h_1 \in \mathbb{F}^+, h_1 \leq f\}$ and $\phi^+(g) = \sup\{\phi(h_2) | h_2 \in \mathbb{F}^+, h_2 \leq g\}$. If $h_1 \leq f, h_2 \leq g$ then $h_1 + h_2 \leq f + g$ and $h_1 + h_2 \in \mathbb{F}^+$, therefore $\phi(h_1 + h_2) \leq \sup\{\phi(h) | h \in \mathbb{F}^+, h \leq f + g\} = \phi^+(f + g)$. We know that ϕ is linear, and hence $\phi(h_1) + \phi(h_2) \leq \phi^+(f + g)$. Taking the supremum over $h_1 \leq f, h_2 \leq g$, we obtain that for all $f, g \in \mathbb{F}^+$, $\phi^+(f) + \phi^+(g) \leq \phi^+(f + g)$.

Consider now $l \in \mathbb{F}^+$, $l \leq f + g$ and define $l_1 := \frac{f}{f+g} l$ and $l_2 := \frac{g}{f+g} l$. Clearly, it holds that

- $l_1, l_2 \in \mathbb{F}^+$
- $l_1 + l_2 = l$
- $l_1 \leq f, l_2 \leq g$

By linearity of ϕ, we can write $\phi(l) = \phi(l_1) + \phi(l_2) \leq \phi^+(f) + \phi^+(g)$. This inequality hold $\forall l \in \mathbb{F}^+, l \leq f + g$, thus by taking the supremum we obtain $\phi^+(f + g) \leq \phi^+(f) + \phi^+(g)$ for all $f, g \in \mathbb{F}^+$.

Hence we conclude that $\forall f, g \in \mathbb{F}^+, \phi^+(f + g) = \phi^+(f) + \phi^+(g)$. In a similar fashion, one can prove that the equality holds also for $f, g \in \mathbb{F}^-$, and hence it holds for all functions $f, g \in \mathbb{F}$.

For a scalar $c > 0$ and $f \in \mathbb{F}^+$, we have that $\phi^+(cf) = \sup\{\phi(h)|h \in \mathbb{F}^+, h \leq cf\} = \sup\{\phi(ch)|h \in \mathbb{F}^+, h \leq f\} = c \sup\{\phi(h)|h \in \mathbb{F}^+, h \leq f\} = c\phi^+(f)$. This equality extends to $c \in \mathbb{R}$ by extension to the whole space $\mathbb{F}$.

Therefore, the functional ϕ^+ is linear on $\mathbb{F}^+$. Moreover, the functional ϕ^- is also linear on $\mathbb{F}^+$, as it is defined as the difference of two linear functionals. Similarly, one shows linearity on $\mathbb{F}^-$, and hence $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$ and $\phi^-(f) = \phi^+(f) - \phi(f)$ are indeed linear on $\mathbb{F}$.

□

Theorem 2.8. *If ϕ is bounded, then the functionals ϕ^+ and ϕ^- are also bounded.*

Proof. By construction, $|\phi^+(f)| \leq |\phi(f)|$, hence $|\phi^+(f)| \leq |\phi(f)| \leq a\|f\|, \forall f \in \mathbb{F}$. Therefore, ϕ^+ is a bounded functional. Moreover, using the triangle inequality, we obtain $|\phi^-| = |\phi^+ - \phi| \leq |\phi| + |\phi^+| \leq 2a\|f\|$. Therefore, ϕ^- is also a bounded functional. □

Definition 2.9. Consider a set Ω and its power set $P(\Omega)$. A subset $\Sigma \subseteq P(\Omega)$ is called a *σ-algebra* if the following conditions hold:

- $\Omega \in \Sigma$
- $X \in \Sigma \implies \Omega \setminus X \in \Sigma$
- $X_1, X_2, ... \in \Sigma \implies \bigcup X_n \in \Sigma$

Definition 2.10. A *measure space* is a triple (Ω, Σ, μ) where:

- Ω is a set
- Σ is a σ-algebra on Ω
- μ is a measure on (Ω, Σ)

Definition 2.11. A measure μ on (Ω, Σ) is called *σ-finite* if one of the following hold equivalent criteria holds:

- $\exists X_1, X_2, ...$ with $\mu(X_n) < \infty, \forall n \in \mathbb{N}$ and $\forall i \neq j, X_i \cap X_j = \emptyset$ such that $\bigcup X_n = \Omega$
- $\exists$ strictly positive function f such that $\int f \mathrm{d}\mu < \infty$.

A measure space on which the measure is σ-finite will be called a *σ-finite measure space.*

Definition 2.12. A sequence of functions f_n defined on a measure space (Ω, Σ, μ) is said to converge to f *almost everywhere* (or shortly denoted as *a.e.*) if the set $A := \{x \in \Omega | \lim_{n\to\infty} f_n(x) \neq f(x)\}$ has $\mu(A) = 0$.

Definition 2.13. A set X is said to be a *μ-null set* if $\mu(X) = 0$.

Definition 2.14. A function f will be said to be Lebesgue-measurable if the preimage under f of any interval is a measurable set.

Theorem 2.15 (The monotone convergence theorem)**.** *If a monotone increasing sequence of Lebesgue-measurable, non-negative functions f_n defined on a measurable set X converges to some function f point-wise a.e., then* $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$

The proof of the monotone convergence theorem will be skipped, as it requires further results that are not needed in this paper. So is the case for the dominated convergence theorem, which is stated below.

Theorem 2.16 (The dominated convergence theorem)**.** *Assume that for the sequence of measurable functions f_n defined on a measurable set Ω there exists a function g which is integrable over Ω and such that $|f_n| \leq g$ a.e. on Ω. If $f_n \to f$ a.e. as $n \to \infty$, then f is integrable on Ω and* $\lim_{n\to\infty} \int_\Omega f_n(\omega)\mathrm{d}\mu = \int_\Omega f(\omega)\mathrm{d}\mu$.

Throughout the paper we will prove equivalent versions of the monotone convergence theorem and dominated convergence theorem.

3 Construction of the Daniell integral

We wish to construct a notion of integration which does not depend on measures, as was done in the original paper, cf. [3].

Let us start by considering a vector space L consisting of bounded real-valued functions on a set Ω. Moreover, assume that L is closed under the following *lattice operations*:

- $f \vee g := \max(f, g)$
- $f \wedge g := \min(f, g)$

Definition 3.1. An *integral* is a linear functional I on L that satisfies that following conditions:

- C1: $f \geq 0 \Rightarrow I(f) \geq 0$
- C2: $f_n \downarrow 0 \Rightarrow I(f_n) \downarrow 0$ point-wise.

Note that the condition C1 is in fact the same as $f \geq g \Rightarrow I(f) \geq I(g)$ as the integral is a linear functional. In fact, in [4] it is even considered as a third condition in the definition.

To better visualize these new concepts, consider the standard Riemann integral of continuous functions. In this case, the Riemann integral is our linear operator I, acting on the space $C(\Omega)$ of continuous real-valued functions on the set Ω, which in turn is our vector space L. Another example would be taking the Lebesgue integral as our I, while for the vector space L we may have the span of all indicator functions of intervals on Ω.

Now we are interested in extending I to a larger class of functions that has the same properties as L, namely closure under the lattice operations. This is motivated by the fact that, for example, there are functions which are not continuous, yet they are Riemann integrable. To this end we will now consider a new vector space U that we define to be the set of point-wise limits of monotone increasing sequences of functions in L, equipped with the standard addition and non-negative scalar multiplication. In this paper, infinity will be allowed as a possible value of the limit function.

Trivially, $L \subseteq U$, as for any function $f \in L$ we may consider the sequence $f_n := f$ $\forall n \in \mathbb{N}$. This sequence is clearly increasing (non-strictly) and converges to f.

Due to the properties of taking limits, this newly constructed vector space U is closed under addition, non-negative scalar multiplication and the lattice operations "$\wedge$" and "$\vee$".

In order to extend the integral I to the space U, we will define

$$I_{new}(f) := \lim_{n\to\infty} I(f_n)$$

for some sequence f_n of elements in L that is monotone increasing and converges to f. Furthermore, we will allow ∞ as a possible value of I_{new}.

Because of C2, this construction is well defined as, by the uniqueness of limits, it does not matter which monotone increasing sequence that converges to f we choose.

For convenience, we will from now on refer to I_{new} simply as I.

Theorem 3.2. *If f_n is a monotone increasing sequence in U that converges point-wise to f, then $f \in U$ and $I(f_n) \uparrow I(f)$ as $n \to \infty$.*

Proof. As $f_n \in U$, we know by the definition of the space U that there exists monotone increasing sequences $g_n^m \in L$ such that $\lim_{m\to\infty} g_n^m = f_n$. Defining $h^n := g_1^n \vee g_2^n \vee ... \vee g_n^n$ we obtain a sequence in L, as L is closed under the lattice operations. Moreover, this sequence is increasing. Furthermore, note that $\forall m \leq n$, $g_m^n \leq h_n \leq f_n$. Letting $n \to \infty$, we obtain $f_m \leq \lim_{n\to\infty} h_n \leq f$. Moreover, taking the limit as m goes to infinity, we obtain $f \leq \lim_{n\to\infty} h_n \leq f$, hence $f \in U$. Applying the same trick for the inequality $I(g_m^n) \leq I(h_n) \leq I(f_n)$, we conclude that $I(f_n) \uparrow I(f)$ as $n \to \infty$. □

We will now look at a natural extension, as we wish to further extend the integral such that it also works for negative values. To this end we will define the set $-U := \{f | -f \in U\}$.

On this new set we further define

- $I^{new}(f) := -I(-f)$, $\forall f \in -U$
- $I^{new}(f) := I(f)$, $\forall f \in U$

Note that $-U$ is closed under taking monotone decreasing limits, standard addition, non-negative scalar multiplication and the lattice operators. Again, for our convenience, we will from now on refer to I^{new} as I.

Definition 3.3. A function f will be said to be *I-integrable* if $\forall \epsilon > 0, \exists g \in -U, h \in U$ such that:

- $g \leq f \leq h$

- $I(g) < \infty$, $I(h) < \infty$
- $I(h) - I(g) < \epsilon$

By varying g an h in the above definition, we may further define $I(f) \coloneqq \inf I(h) = \sup I(g)$.

We will denote by L^1 the set of all I-integrable functions.

Let us check that this new definition of I is the same as I^{new} for all $f \in U$ and $f \in -U$. Hence, we start by considering the case $f \in U$. By definition of the space U, we know that there exists a monotone increasing sequence $f_n \in L$ such that $f_n \uparrow f$ as $n \to \infty$. Moreover, we have that $I^{new}(f) \geq \inf_{h \geq f} I(h) \geq \sup_{g \leq f} I(g) \geq I(f_n)$ for all $n \in \mathbb{N}$. However, by Theorem 3.2, it holds that $I(f_n) \uparrow I(f)$. Thus $I^{new}(f) \geq \inf_{h \geq f} I(h) \geq \sup_{g \leq f} I(g) \geq I(f_n) \to I^{new}(f)$ as $n \to \infty \implies \inf I(h) = \sup I(g)$ and hence f is $I-$integrable and $I^{new}(f) = I(f)$.

It is very important to note that when considering $L = C(\Omega)$ for some compact $\Omega \subset \mathbb{R}$, we have $\mathbb{1}_{(a,b)} \in L^1$. This will be used later in the main proof of this paper. To show this, by definition of I-integrable functions we need to find two sequences of functions g_n, h_n which approximate $\mathbb{1}_{(a,b)}$ from below and above respectively. To this end define the sequence

$$g_n(x) \coloneqq \begin{cases} (x-a)n, & \text{if } x \in \left[a,\, a+\frac{1}{n}\right) \\ 1, & \text{if } x \in \left[a+\frac{1}{n},\, b-\frac{1}{n}\right) \\ (b-x)n, & \text{if } x \in \left[b-\frac{1}{n},\, b\right) \end{cases} \tag{1}$$

Now $g_n \in C(\Omega)$, $g_n < \mathbb{1}_{(a,b)}$ for all $n \in \mathbb{N}$ and $g_n \to \mathbb{1}_{(a,b)}$ as $n \to \infty$. Similarly, one can construct a sequence $h_n \in C(\Omega)$ with $h_n > \mathbb{1}_{(a,b)}$ and $h_n \to \mathbb{1}_{[a,b]}$ as $n \to \infty$ by defining

$$h_n(x) \coloneqq \begin{cases} (x-a+\frac{1}{n})n, & \text{if } x \in \left[a-\frac{1}{n},\, a\right) \\ 1, & \text{if } x \in [a,\, b) \\ (b+\frac{1}{n}-x)n, & \text{if } x \in \left[b,\, b+\frac{1}{n}\right) \end{cases}$$

Note that g_n is a monotone increasing sequence in L, while h_n is monotone decreasing. Thus, the indicator function $\mathbb{1}_{(a,b)} \in U$ and $\mathbb{1}_{(a,b)} \in -U$. Therefore, as we have shown $I(f) = I^{new}(f)$ on U and $-U$, we have that $\mathbb{1}_{(a,b)} \in L^1$ and $\mathbb{1}_{[a,b]} \in L^1$.

Lemma 3.4. *L^1 is a linear space.*

Proof. Consider arbitrary f_1 and f_2 in L^1, together with an arbitrary constant $c \in \mathbb{R}$. Now, by definition of the space L^1, we can choose g_1, g_2 in $-U$ and h_1, h_2 in U such that

- $g_1 \leq f_1 \leq h_1$, $g_2 \leq f_2 \leq h_2$
- $I(h_1) - I(g_1) \leq \epsilon$, $I(h_2) - I(g_2) \leq \epsilon$.

Clearly, if $c \geq 0$, we have

$$g_1 + cg_2 \leq f_1 + cf_2 \leq h_1 + ch_2$$

and

$$I(h_1 + ch_2) - I(g_1 + cg_2) = I(h_1) - I(g_1) + cI(h_2) - cI(g_2) \leq (1+c)\epsilon$$

where ϵ can be made arbitrarily small. Thus, for $c \geq 0$, it holds that $f_1 + cf_2 \in L^1$. If $c < 0$, we can then write

- $g_1 + ch_2 \leq f_1 + cf_2 \leq h_1 + cg_2$
- $I(h_1 + cg_2) - I(g_1 + ch_2) = I(h_1) - I(g_1) + cI(g_2) - cI(h_2) \leq (1-c)\epsilon$

and thus we also conclude that $f_1 + cf_2 \in L^1$ for negative c. □

Note that the previous result also holds for the lattice operations, i.e., f_1, f_2 in L^1 implies that $f_1 \wedge f_2 \in L^1, f_1 \vee f_2 \in L^1$. The proof is very much the same as before.

Lemma 3.5. *If $f \geq g$ and f, g are I-integrable, then $I(f) \geq I(g)$.*

Proof. Given $f \geq 0$, we have that $h \geq 0$ and $I(f) = \inf I(h) \geq 0$ by the definition of the extended I together with the property of the previously defined one. Therefore, if $f \geq g$, we then have $f - g \geq 0$ and thus $I(f-g) \geq 0$. By linearity of I, we conclude that $I(f) \geq I(g)$. □

Theorem 3.6. *Assume that f_n is a monotone increasing sequence in L^1 that converges to some function f and $\lim_{n\to\infty} I(f_n) < \infty$. Then $f \in L^1$ and $I(f_n) \uparrow I(f)$.*

Proof. Assume without loss of generality that $f_0 = 0$. Consider a sequence h_n in U such that

- $0 \leq f_i - f_{i-1} \leq h_i$
- $I(h_i) \leq I(f_i - f_{i-1}) + \frac{\epsilon}{2^i}$

Such a sequence exists because the sequence f_n is, by assumption, I-integrable. These two properties yield, by summing over i, the following inequalities:

- $f_n \leq \sum_{i=1}^{n} h_i$
- $\sum_{i=1}^{n} I(h_i) \leq I(f_n) + \epsilon \sum_{i=1}^{n} \frac{1}{2^i}$

The first inequality follows from the fact that the right-hand side is a telescopic sum. For the second inequality the same argument was used, together with the linearity of I. The second inequality may be extended by recognizing that, for finite m, the sum is strictly less than 1. Therefore, it now becomes

$$\sum_{i=1}^{n} I(h_i) < I(f_n) + \epsilon$$

Now, looking at $h := \sum_{i=1}^{\infty} h_i$, we can say that the series converges in U by theorem 3.2, as partial sums are elements of U by the spaces' closure under addition, together with the fact that $\sum_{i=1}^{n} h_i$ is monotone in n, as $h_i \geq 0$ by definition. The sequence consisting of partial sums is increasing and hence, as infinity is allowed as a possible limit value, it converges. Furthermore, by theorem 3.2 we may also conclude that $I(h) = \lim_{n\to\infty} \sum_{i=1}^{n} I(h_i) = \sum_{i=1}^{\infty} I(h_i)$, where we again used the linearity of I. Moreover, we know that $f \leq h$ and $I(h) \leq \lim_{n\to\infty} I(f_n) + \epsilon$ for arbitrary $\epsilon > 0$.

In a similar fashion we may also consider a sequence g_n in U and its corresponding $g := \sum_{i=1}^{\infty} g_i$ that will satisfy $g \leq f$ and $\lim_{n\to\infty} I(f_n) - \epsilon \leq I(g)$. The second inequality implies that $I(h) - I(g) \leq 2\epsilon$, which can be made arbitrarily small, and thus $f \in L^1$ by the definition of this space. As the sequence f_n converges to f and it is monotonic increasing, we know that there exists a $N \in \mathbb{N}$ for which $g \leq f_n \leq f \leq h, \forall n \geq N$. By using Lemma 1.5 together with the squeeze theorem, we may now conclude that $I(f) = \lim_{n\to\infty} I(f_n)$ and thus the proof is complete. □

Therefore, L^1 and I have all the properties of the initially considered L and I.

Note that in theorem 3.6 we may also consider a sequence of monotone decreasing functions converging to f and $\lim_{n\to\infty} I(f_n) > -\infty$. By setting $g_n := -f_n$, we may use the original version of the theorem to show that $f \in L^1$ and $I(f_n) \downarrow I(f)$. Theorem 3.6 is also known as the *Beppo Levi* theorem.

Looking back at the indicator function $\mathbb{1}_{(a,b)}$ for $L = C(\Omega)$, we observe that the previously constructed sequences g_n and h_n are monotone increasing and decreasing respectively. Hence, by using the Beppo Levi theorem, one can indeed verify that $I(\mathbb{1}_{(a,b)}) = b - a$, if I is the standard Riemann integral.

We will now show a similar version of the dominated convergence theorem. This result is a really important one in measure theory, and hence we already expect it to play a big role for the Daniell integration approach as well.

Theorem 3.7 (Dominated convergence theorem)**.** *Given a sequence $f_n \in L^1$ that converges to a function f point-wise and such that $|f_n| \leq g$ for some $g \in L^1$ for all $n \in \mathbb{N}$, then $f \in L^1$.*

Proof. Let us start by considering the sequence $F_n := \sup_{k\geq n} f_k$. One can observe that F_n is a monotone decreasing sequence by construction which converges point-wise to f as $n \to \infty$.

Now let us have a look at $\tilde{f}_{n,m} := \max_{n\leq k\leq m} f_k = f_n \vee f_{n+1} \vee ... \vee f_m$. Clearly, $\tilde{f}_{n,m} \leq g$ for all $n, m \in \mathbb{N}$ and $\tilde{f}_{n,m} \in L^1$. Moreover, $\tilde{f}_{n,m}$ is monotone increasing in m. Furthermore,

$$I(\tilde{f}_{n,m}) = I(\max_{n\leq k\leq m} f_k) \leq I(g) < \infty$$

since $g \in L^1$. Observe that $\lim_{m\to\infty} \tilde{f}_{n,m} = F_n$, thus by Theorem 3.6 we know that $F_n \in L^1$.

Now, since F_n is a monotone decreasing sequence in L^1 that converges point-wise to f, Theorem 3.6 lets us conclude our desired result if we manage to show that $\lim_{n\to\infty} I(F_n) > -\infty$. One can quickly observe that this indeed holds, as

$$|f_n| \leq g, \forall n \in \mathbb{N} \implies f_n > -g, \forall n \in \mathbb{N} \implies I(\sup_{n\leq k} f_n) \geq I(-g) > -\infty, \forall n \in \mathbb{N}$$

thus $\lim_{n\to\infty} F_n > -\infty$. Therefore, we have shown that $f \in L^1$. □

Definition 3.8. A function $f \in \mathbb{B}$ will be said to be ∞-*integrable* if either $f^+ := f \vee 0$ or $f^- := -f \wedge 0$ is in L^1. Furthermore, we may now define $I(f) := I(f^+) - I(f^-)$. Note that $I(f)$ may have ∞ or $-\infty$ as possible values.

A function f will now be called integrable if it is ∞-integrable and $|I(f)| < \infty$.

Remark 3.9. If two functions are integrable, then so is any linear combination of them. There is a slight ambiguity regarding the value of $I(f + g) = I(f) + I(g)$ when the two integrals are of infinite values, with opposite signs. However, this does not concern us in this paper, and thus it will be neglected.

Remark 3.10. If f_n is an integrable increasing sequence with $I(f_1) > \infty$ that converges to a function f, then the limit is integrable and $I(f_n) \uparrow I(f)$ as $n \to \infty$.

4 Measures

Let us now consider a set $A \subset \Omega$. We can define the indicator function of A as $\mathbb{1}_A(x) := 1$ if $x \in A$ and $\mathbb{1}_A(x) := 0$ if $x \notin A$.

For simplicity, we will assume that $I(1) < \infty$. For the Riesz-Markov-Kakutani representation theorem, this assumption holds. If one prefers not using this assumption, then eventually one will need to make use of the fact that $1 \in L^1 \iff m.v.\{f, 1, -f\} \in L^1$ for all $f \in L^1$, where $m.v.$ represents the median value defined as $m.v.(f, g, h)(x) := f(x) + g(x) + h(x) - \max\{f(x), g(x), h(x)\} - \min\{f(x), g(x), h(x)\}$.

Definition 4.1. A set $A \in \Omega$ for which its indicator function $\mathbb{1}_A \in L^1$ will be called an *measurable set*. Furthermore, we define the *measure* of A by $\mu(A) := I(\mathbb{1}_A)$.

Theorem 4.2. *If A and B are two measurable sets, then so are $A \cap B, A \cup B$ and $A \setminus B$.*

Proof. Clearly, one can see that $\mathbb{1}_{A \cup B} = \mathbb{1}_A \vee \mathbb{1}_B$, $\mathbb{1}_{A \cap B} = \mathbb{1}_A \wedge \mathbb{1}_B$ and $\mathbb{1}_{A \setminus B} = \mathbb{1}_A - \mathbb{1}_{A \cap B}$. As L^1 is closed under the lattice and the algebraic operations, we conclude that the indicator functions of the sets $A \cap B, A \cup B$ and $A \setminus B$ are all functions in L^1. □

Theorem 4.3. *If $\{A_n\}$ is a sequence of disjoint measurable sets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\bigcup_{n=1}^{\infty} A_n$ is measurable and $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.*

Proof. The sequence consists of disjoint sets, and thus $\mathbb{1}_{\cup_{n=1}^{\infty} A_n} = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}$. This is because any point $x \in \Omega$ belongs to at most one set of the sequence, and thus the series is either 0 (when $\forall i \in \mathbb{N}^*, x \notin A_i$) or 1 (when $x \in A_i$ for one $i \in \mathbb{N}^*$). When x does not lie in any set, it obviously will not lie in their union, and thus the indicator will be evaluated as 0 at such a point, whereas when x lies in one set it will also lie in the union, meaning that the indicator will again be evaluated as the series. Moreover, due to theorem 2.2, the elements of the sequence $\{U_k\}$ defined as $U_k := \cup_{n=1}^{k} A_n$ are measurable sets and the sequence is increasing and converging to $U := \cup_{n=1}^{\infty} A_n$. Now, by applying theorem 1.6 for $f_k = \mathbb{1}_{U_k}$, we obtain that $I(\cup_{n=1}^{\infty} \mathbb{1}_{A_n}) = \lim_{k\to\infty} I(\cup_{n=1}^{k} A_n) = \lim_{k\to\infty} \sum_{n=1}^{k} I(\mathbb{1}_{A_n}) = \sum_{n=1}^{\infty} I(\mathbb{1}_{A_n})$ and thus, by the definition of measure, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. □

Remark 4.4. Note that we have now shown that the measurable sets form a $\sigma-$algebra, with μ being a measure on it.

Theorem 4.5. *For any function $f \in L^1$ and any constant $a > 0$, the set $A := \{x | f(x) > a\}$ is a measurable set.*

Proof. In order to prove the first statement we will start by considering the sequence of functions $\{f_n\}$ defined by $f_n := [n(f - f \wedge a)] \wedge 1$. The elements of this sequence are functions in L^1 and they are constructed such that $\lim_{n\to\infty} f_n = \mathbb{1}_A$. This convergence is easy to check by applying the functions on a point $x \in \Omega$. If x is such that $f(x) > a$, then $f_n(x) = [n(f(x) - a)] \wedge 1$. As $f(x) - a > 0$ and is constant with respect to n, for n sufficiently large we have that $n[f(x) - a] > 1$. Therefore, when letting n go to infinity, we obtain $\lim_{n\to\infty} f_n(x) = 1$. On the other hand if x is such that $f(x) \leq a$, then $f_n(x) = 0$, thus its limit is 0. These two results lead us to the conclusion that $\lim_{n\to\infty} f_n = \mathbb{1}_A$. Hence, the indicator function of A is in L^1, which implies that A is indeed a measurable set and $I(f_n) \leq I(1)$.

□

Furthermore, theorem 4.5 yields the following corollary

Corollary 4.6. *If $f \in L^{1+}$, then $I(f) = \int f \mathrm{d}\mu$. Here, the integral is as in definition 2.6.*

This result will turn out to be of very high importance later in this paper.

Proof. We will construct, for given $a > 1$, the sets

$$A_n^a := \{x | a^n < f(x) \leq a^{n+1}\}$$

with $n \in \mathbb{Z}$. These sets are integrable as they can be seen as the set difference between $\{x|f(x) > a^n\}$ and $\{x|f(x) > a^{n+1}\}$ which, by assumption, are measurable sets. Furthermore, we define the function

$$f_a := \sum_{n=-\infty}^{\infty} a^n \mathbb{1}_{A_n^a}$$

As the sets are integrable, the indicator functions are in L^1, hence $f_a \in L^1$ by monotonicity. For an arbitrary point $x \in \Omega$, we can evaluate $f_a(x) = \sum_{n=-\infty}^{\infty} a^n \mathbb{1}_{A_n^a}(x)$. The elements of the sequence $\{A_n^a\}$ are disjoint sets, and thus $f_a(x) = 0$ if $\forall n \in \mathbb{Z}, x \notin A_n^a$ or $f_a(x) = a^m$ if $x \in A_m^a$ for some unique $m \in \mathbb{Z}$. In both cases, as $f \geq 0$ and $f(x) > a^m$ if $x \in A_m^a$, we get that $f_a \leq f$, hence $f_a \in L^1$.
Applying the functional I on f_a, we obtain $I(f_a) = \sum_{-\infty}^{\infty} a^n I(\mathbb{1}_{A_n^a}) = \sum_{-\infty}^{\infty} a^n \mu(A_n^a)$. Since f_a is defined as a simple function, we can see that applying I of it yields precisely its integral with respect to the measure μ, as per definition 2.6. Hence $I(f_a) = \int f_a \mathrm{d}\mu$.

By the same argument, we further conclude that $I(af_a) = \int af_a \mathrm{d}\mu$. Note that μ is the measure induced by I as per Definition 4.1.

By the properties of I, we know that $I(f_a) \leq I(f) \leq I(af_a)$. Moreover, it is also true that $\int f_a \mathrm{d}\mu \leq \int f \mathrm{d}\mu \leq \int af_a \mathrm{d}\mu$. Moreover, $|I(f) - \int f \mathrm{d}\mu| \leq (a-1)I(f_a) \leq (a-1)I(f)$ for every $a > 1$. Let us now take the limit as a goes to 1. The right-hand side of the inequality will converge to 0 as $I(f)$ is independent of a. Therefore, by the squeeze theorem, we conclude that $|I(f) - \int f \mathrm{d}\mu| \to 0$ as $a \downarrow 1$. However, $|I(f) - \int f \mathrm{d}\mu|$ does not depend on a, hence we have proven that $I(f) = \int f \mathrm{d}\mu$. □

Corollary 4.6 essentially states that the functional I is equal to the Lebesgue integral, given they both exist on the considered space of functions. Moreover, it is known that the Lebesgue and the Riemann integrals of some function f are also the same, given that f is both Riemann and Lebesgue integrable. Hence, for a continuous function, these three different notions of integration are in fact the same, cf. [5].

Remark 4.7 (Equivalence to the Lebesgue approach). Note that when given a measure space (Ω, Σ, μ), using the Lebesgue approach we define the integral with respect to μ of a simple function f as $I(f) := \sum_i \alpha_i \mu(A_i)$, where $f(x) = \alpha_i$ for all $x \in A_i$. This definition verifies the conditions $C1$ and $C2$, therefore it is an integral also in the sense of Definition 3.1. Following the Lebesgue approach, we reach a definition of integration equivalent to the one we have presented throughout section 3.

5 The Riesz-Markov-Kakutani representation theorem

In the previous sections we have gathered enough knowledge to complete the main proof of this paper, namely the proof of the Riesz-Markov-Kakutani representation theorem.

A general assumption that will be used in all the results of this section is that

$$\Omega \text{ is a compact set}$$

Lemma 5.1. *Consider a sequence of functions $f_n \in L^1$ such that $f_n \downarrow 0$. Then $f_n \downarrow 0$ uniformly.*

Proof. For any given $\epsilon > 0$, consider the set $S_n := \bigcup_{k \geq n} \{x \in \Omega | f_k(x) \geq \epsilon\}$. Clearly, these sets are nested i.e, $\forall m > n \; S_m \subset S_n$. Now assume by contradiction that $f_n \to 0$ not uniformly. The there exists $\epsilon > 0$ and a sequence $x_k \in \Omega$ such that $|f_n(x_k)| > \epsilon$. Therefore, $x_k \in S_n$. However, S_n is a closed subset of the compact set Ω, hence it is compact as well. By Bolzano-Weierstrass, x_k has a convergent subsequence x_{k_p} which converges to $x_0 \in \Omega$.

Now we have that $\bigcap_{n=1}^{\infty} S_n$ is not empty, as it must contain x_0. Therefore, for infinitely many n we have that $|f_n(x_0)| > \epsilon$ and clearly this implies that f_n does not converge to 0, contradicting the assumption. Thus, we conclude that $f_n \downarrow 0$ uniformly. □

Lemma 5.2. *All non-negative linear functionals on $C([a, b])$ are bounded with respect to $\|\cdot\|_\infty$.*

Proof. Consider the non-negative linear functional I and the arbitrary functions $f \in L^1$ and $g \in L^+$ with $g \geq 1$. By definition of the supremum norm, $|f| \leq \|f\|_\infty$, hence $|f| \leq g\|f\|_\infty$. Then, $|I(f)| \leq \|f\|_\infty I(g)$. Therefore $\|I\| = \sup_{f \in L^1, f \neq 0} \frac{|I(f)|}{\|f\|_\infty} \leq I(g)$ and thus I is bounded. □

Theorem 5.3. *All non-negative linear functionals on $C([a, b])$ are integrals.*

Proof. Consider the non-negative linear functional I. All that is needed to be verified is condition C2 in definition 3.1. To this end consider a sequence of functions $f_n \in L^1$ such that $f_n \downarrow 0$. By lemma 5.1 this already tells us that $f_n \downarrow 0$ uniformly. By lemma 5.2 we know that I is bounded, thus $|I(f_n)| \leq c\|f_n\|_\infty$ for some constant $c > 0$. We can now conclude by using the squeeze theorem that $I(f_n) \downarrow 0$, thus I is indeed an integral. □

Theorem 5.4 (The Riesz-Markov-Kakutani representation theorem)**.** *Let ϕ be a bounded functional on the space of continuous functions defined on the interval $[a, b]$, generally denoted by $C([a, b])$. Then there exists a unique measure μ on the Borel subsets of $[a, b]$ such that $\phi(f) = \int_a^b f \mathrm{d}\mu, \forall f \in C([a, b])$.*

Proof. Given a functional ϕ on $C([a, b])$ we will start by showing the existence of a measure that satisfies $\phi(f) = \int_a^b f \mathrm{d}\mu, \forall f \in C([a, b])$. To this end, we will consider the decomposition of ϕ in two positive functionals, as presented in theorem 2.7. By theorem 5.3, we know that these functionals are integrals, as per definition 3.1.
Applying the extensions described in section 3, together with corollary 4.6, we know that there exist two measures μ^+ and μ^- such that, for all $f \in C([a, b])$ $\phi^+(f) = \int f \mathrm{d}\mu^+$ and $\phi^-(f) = \int f d\mu^-$. Moreover, these measures are positive. This is true since for all Borel sets $A \subseteq [a, b]$ we have $\mu^+(A) = \phi^+(\mathbb{1}_A) \geq 0$, as ϕ^+ is positive. The same holds for μ^-. Note that the indicator function of a set A is not necessarily continuous on $[a, b]$ (unless $A = [a, b]$). However, in section 3 we have shown that $\mathbb{1}_A$ is in L^1 and hence the extended

functionals may act on it.
Therefore, we have shown that $\forall f \in C([a,b])$, $\phi(f) = \phi^+(f) - \phi^-(f) = \int_a^b f \mathrm{d}\mu^+ - \int_a^b f d\mu^- = \int_a^b f d\mu$, where μ is the unique signed measure which has the Hahn decomposition[1] $\mu = \mu^+ - \mu^-$. Hence, we have proved the existence part of the theorem.

As $\phi(f) = \int_a^b f \mathrm{d}\mu = \int_a^b f \mathrm{d}\nu$ for all functions f, we are free to choose any function f on which the functional and the integral may act. Moreover, as discussed before, this function does not need to be in $C([a,b])$, as it may be chosen in the extension L^1. Therefore, let us choose $f = \mathbb{1}_{[c,d]}$ for some $[c,d] \subseteq [a,b]$. For this choice of f we get that $\phi(\mathbb{1}_{[c,d]}) = \mu([c,d]) = \nu([c,d])$. As the intervals generate the Borel sets, we have that $\mu(A) = \nu(A)$ for all intervals $A \implies \mu(B) = \nu(B)$ for all Borel sets B. Hence, we have shown that the measures are equal. This concludes the proof for the uniqueness part of the Riesz-Markov-Kakutani representation theorem. □

Definition 5.5. A topological space X is a *Hausdorff space* if $\forall x, y \in X$ $\exists O_x, O_y$ open sets such that $x \in O_x, y \in O_y$ and $O_x \cap O_Y = \emptyset$.

Definition 5.6. A topological space X is said to be *locally compact* if $\forall x \in X$ there exist O open set and K compact set such that $x \in O \subseteq K$.

Remark 5.7. The Riesz-Markov-Kakutani representation theorem can be generalized for the space of continuous functions defined on any arbitrary locally-compact Hausdorff space X. The proof of this stronger version is similar to the one that we have provided, and therefore will be skipped in this paper.

6 The Bochner integral

In this section we aim to extend the concept of integration as a function from a subset $\Omega \subseteq \mathbb{R}^n$ to an arbitrary Banach space X, as done in [6]. We will thus consider a σ-finite measure space (Ω, Σ, μ). As in the construction of the Daniell and Lebesgue integrals, we will first have a look at the simple functions defined on Ω which map to X.

Definition 6.1. A function $f : \Omega \to X$ is said to be a simple functions if:

- the image of Ω under f is a finite set
- $\forall x \in X \setminus \{0\}$ it holds that $f^{-1}(x) \in \Sigma$ and $\mu(f^{-1}(x)) < \infty$.

It is easy to check that when $X \subseteq \mathbb{R}$ this new definition of simple functions is equivalent to the old.

Definition 6.2. A function $f : \Omega \to X$ is said to be *Bochner-measurable* if there exists a sequence of simple functions f_n on Ω mapping to X such that $\lim_{n\to\infty} f_n = f$ almost everywhere.

For $p \in [1, \infty]$ we may define the spaces $\mathcal{L}^p(\Omega, X) := \{f : \Omega \to X | f$ Bochner-measurable, $\|f\|_X \in \mathcal{L}^p(\Omega)\}$. Here $\mathcal{L}^p(\Omega)$ denotes the set of all Lebesgue-integrable functions mapping from Ω to a subset of $\mathbb{R}$.

As in the case of the Lebesgue spaces, we are interested in the equivalence classes of these $\mathcal{L}^p(\Omega, X)$ spaces, where the equivalence relation is defined as $f \sim g \iff B := \{x \in \Omega | f(x) \neq g(x)\}$ has $\mu(B) = 0$.

[1]This is defined in [2], namely decomposition of signed measures

Definition 6.3. For $p \in [1,\infty]$ the so-called *Bochner-Lebesgue p-space* are defined as $L^p(\Omega, X) := \mathcal{L}^p(\Omega, X)/_\sim$, equipped with the well-known p-norms

- $\|f\|_p := (\int_\Omega \|f(\omega)\|_X^p \mathrm{d}\mu(\omega))^{1/p}$ for any $p < \infty$
- $\|f\|_\infty := \sup_{\omega\in\Omega} \|f(\omega)\|_X$.

Note that these spaces are linear spaces, hence they are closed to addition and scalar multiplication.

For the following lemma we will omit the proof, as it requires further results which will not be used for other purposes. However, the lemma is very important as it will allow us to prove a not so surprising theorem.

Lemma 6.4. *Consider a sequence of functions $f_n : \Omega \to X$ and $f : \Omega \to X$. If f_n is Bochner-measurable for every $n \in \mathbb{N}$ and $f_n \to f$ a.e. as $n \to \infty$, then f is Bochner-measurable.*

For the Lebesgue p-spaces, it is a well known result that $L^p(\Omega)$ is a Banach space for all $p \in [1,\infty]$ and furthermore $L^2(\Omega)$ is even a Hilbert space. One expects similar results for the Bochner-Lebesgue p-spaces, hence we will have a look at the following theorem.

Theorem 6.5. *The Bochner-Lebesgue p-spaces $(L^p(\Omega, X), \|\cdot\|_p)$ are Banach spaces. Moreover, if X is in fact a Hilbert space, then so is $(L^2(\Omega, X), \|\cdot\|_2)$.*

Proof. Recall that a normed space is said to be Banach if and only if it is complete. To this end, consider a Cauchy sequence $f_n \in L^p(\Omega, X)$. As f_n is Cauchy, there exists a subsequence f_{n_k} such that $\|f_{n_{k+1}} - f_{n_k}\|_p < 2^{-k}$. Now define the sequence $g_k := f_{n_{k+1}} - f_{n_k}$ in $L^p(\Omega, X)$ which clearly has the property $\sum_{k=1}^\infty \|g_k\|_p < \sum_{k=1}^\infty 2^{-k} < \infty$.

We can further define a sequence of functions mapping from Ω to $\mathbb{R}$ such that $h_k(\omega) := \|g_k(\omega)\|_X$. We can see that $h_k \in L^p(\Omega)$, as $\|h_k\|_{L^p\Omega}^p = \int_\Omega \|g_k(\omega)\|_X^p \mathrm{d}\mu(\omega) = \|g_k\|_{L^p(\Omega,X)}^p < \infty \ \forall p \in [1,\infty)$. For $p = \infty$, this result also trivially holds. This new sequence has the property that $\sum_{k=1}^\infty \|h_k\|_{L^p(\Omega)} = \sum_{k=1}^\infty \|g_k\|_{L^p(\Omega,X)} < \infty$. It is well known that the space $L^p(\Omega)$ is complete, hence $\exists h \in L^p(\Omega)$ such that $h = \sum_{k=1}^\infty h_k$ and $h(\omega) < \infty$ *a.e.*

Therefore we can observe that $\sum_{k=1}^\infty \|g_k(\omega)\|_X = \sum_{k=1}^\infty h_k(\omega) = h(\omega) < \infty$ *a.e.* As X is assumed to be a complete normed space, there exists a function $g : \Omega \to X$ such that $g(\omega) = \sum_{k=1}^\infty g_k(\omega)$ for almost all $\omega \in \Omega$. By lemma 6.4, g is Bochner-measurable.

It remains to be shown that $g \in L^p(\Omega, X)$ and that $\sum_{k=1}^m g_k \to g$ as $m \to \infty$ in the $L^p(\Omega, X)$ norm. This is equivalent to showing that $\sum_{k=m}^\infty g_k \in L^p(\Omega, X)$ and that $\sum_{k=m}^\infty g_k \to 0$ as $m \to \infty$, since this would mean that $g - \sum_{k=1}^m g_k \in L^p(\Omega, X)$ and $g - \sum_{k=1}^m g_k \to 0$ as $m \to \infty$.

- Case 1: $p = \infty$
  As $g_k \in L^\infty(\Omega, X)$, we have that $\|g_k(\omega)\|_X \leq \|g_k\|_\infty$ for almost all $\omega \in \Omega$. Moreover, we have that
  $$\|\sum_{k=m}^\infty g_k(\omega)\|_X \leq \sum_{k=m}^\infty \|g_k(\omega)\|_X \leq \sum_{k=m}^\infty \|g_k\|_\infty < \infty$$
  hence $\sum_{k=m}^\infty g_k(\omega) \in L^\infty(\Omega, X)$. Moreover, as the series $\sum_{k=1}^\infty \|g_k\|_\infty < \infty$, its tail converges i.e. $\sum_{k=m}^\infty \|g_k\|_\infty \to 0$ as $m \to \infty$. Thus, by the squeeze theorem and the definiteness of the norm $\|\cdot\|_X$, $\sum_{k=m}^\infty g_k(\omega) \to 0$ as $m \to \infty$.

- Case 2: $p < \infty$
  Consider now

$$\|\sum_{k=m}^{\infty} g_k(\omega)\|_p = (\int_\Omega \|\sum_{k=m}^{\infty} g_k(\omega)\|_X^p \mathrm{d}\mu(\omega))^{1/p} = (\int_\Omega \|\lim_{j\to\infty} \sum_{k=m}^{j} g_k(\omega)\|_X^p \mathrm{d}\mu(\omega))^{1/p}$$

  By the monotone convergence theorem, this implies that

$$\|\sum_{k=m}^{\infty} g_k(\omega)\|_p = \lim_{j\to\infty} (\int_\Omega \|\sum_{k=m}^{j} g_k(\omega)\|_X^p \mathrm{d}\mu(\omega))^{1/p} \leq \lim_{j\to\infty} (\sum_{k=m}^{j} \int_\Omega \|g_k(\omega)\|_X^p \mathrm{d}\mu(\omega))^{1/p}$$
$$= \lim_{j\to\infty} \sum_{k=m}^{j} \|g_k\|_p = \sum_{k=m}^{\infty} \|g_k\|_p$$

  Here, we have use the triangle inequality to swap the integral with the sum. This inequality, together with the arguments presented in case 1, allows us to conclude that $\sum_{k=m}^{\infty} g_k(\omega) \in L^p(\Omega, X)$ and $\sum_{k=m}^{\infty} g_k(\omega) \to 0$ as $m \to \infty$.

We will now go back to the original Cauchy sequence f_n in $L^p(\Omega, X)$. We have constructed a subsequence such that $f_{n_k} = f_{n_1} + \sum_{i=1}^{k-1} g_k$. Taking the limit as $k \to \infty$, we can see that the subsequence converges to $f := f_{n_1} + \sum_{i=1}^{\infty} g_i \in L^p(\Omega, X)$. Hence, we have obtained a candidate limit for the sequence f_n. Let us now finally show that f_n indeed converges to f in $L^p(\Omega, X)$.

As f_n is Cauchy in $L^p(\Omega, X)$, $\forall \epsilon > 0\ \exists N_1 \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \frac{\epsilon}{2}, \forall n, m > N_1$. Moreover, as the subsequence f_{n_k} converges to f, $\forall \epsilon > 0\ \exists N_2 \in \mathbb{N}$ such that $\|f_{n_k} - f\|_p < \frac{\epsilon}{2}$ $\forall n_k > N_2$. Therefore, by choosing $N = \max\{N_1, N_2\}$, we have that for all $n, n_k > N$ $\|f_n - f\|_p \leq \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p < \epsilon$. Therefore, $f_n \to f$ in $L^p(\Omega, X)$ as $n \to \infty$.

Hence, $L^p(\Omega, X)$ are Banach spaces $\forall p \in [1, \infty]$. If X is a Hilbert space, we know that the norm on X comes from an inner-product. One may easily check that the inner-product $\langle f, g\rangle_{L^2(\Omega,X)} := \int_\Omega \langle f, g\rangle_X \mathrm{d}\mu(\omega)$ is well-defined and that indeed $\langle f, f\rangle_{L^2(\Omega,X)}^{1/2} = \|f\|_{L^2(\Omega,X)}$, thus $L^2(\Omega, X)$ is in fact a Hilbert space. □

We will now prove a density result which will enable us to formulate a clear definition of the Bochner integral.

Theorem 6.6. *The space of all simple functions defined on Ω and mapping to X is dense in $L^p(\Omega, X)$ for all $p \in [1, \infty]$.*

Proof. Consider a function $f \in L^p(\Omega, X)$. Then, f is Bochner-measurable and hence, by definition, there exists a sequence of simple functions f_n such that $f_n \to f$ as $n \to \infty$ *a.e.* Moreover, assume that the functions $\|f_n(\omega)\|_X$ and $\|f(\omega)\|_X$ are Lebesgue-measurable for almost all $\omega \in \Omega$.

Consider now the sets $I_n := \{\omega \in \Omega |$ the functions $\|f_n(\omega)\|_X, \|f(\omega)\|_X)$ are Lebesgue-measurable and $\|f_n(\omega)\|_X \leq 2\|f(\omega)\|_X\}$ which are measurable sets. Now we can define the sequence of functions $\tilde{f_n} := f_n \mathbb{1}_{I_n}$. These functions are simple functions mapping from Ω to X. We will now show that $\tilde{f_n} \to f$ as $n \to \infty$, which will prove the theorem.

If $f(\omega) = 0$, then the sequence $\tilde{f}_n(\omega) = 0$ trivially converges to $f(\omega)$. For $f(\omega) \neq 0$ there exists an $N \in \mathbb{N}$ large enough such that $\|f_n(\omega)\|_X \leq 2\|f(\omega)\|_X$ for all $n > N$, since $f_n \to f$ as $n \to \infty$ and the norm on X is continuous. Therefore, we can see that $\forall \omega \in \Omega$ such that $f(\omega) \neq 0$ we have that $\omega \in I_m$ for some $m \in \mathbb{N}$. Therefore, $\tilde{f}_n(\omega) = f_n(\omega)$ and the sequence converges point-wise to $f(\omega)$. Hence, applying the dominated convergence theorem, it yields that $\|\tilde{f}_n - f\|_{L^p(\Omega,X)} = \int_\Omega \|\tilde{f}_n(\omega) - f(\omega)\|_X^p \mathrm{d}\mu(\omega) \to 0$ as $n \to \infty$. □

Theorem 6.7. *The mapping $\int_\Omega \mathrm{d}\mu$ defined on the space of simple functions from Ω to X with values in X which maps f to $\sum_{x\in X} x\mu(f^{-1}(x))$ is linear and continuous.*

Proof. We will start by proving linearity. To this end, consider simple functions f, g and a constant $c \in \mathbb{R}$. Note that

$$\begin{aligned}(f + cg)^{-1}(x) &= \{\omega \in \Omega \mid f(\omega) + cg(\omega) = x\} \\ &= \bigcup_{y\in X} \left(\{\omega_1 \in \Omega \mid f(\omega_1) = y\} \cap \{\omega_2 \in \Omega \mid g(\omega_2) = \tfrac{x-y}{c}\}\right) \\ &= \bigcup_{y\in X} f^{-1}(y) \cap g^{-1}\left(\tfrac{x-y}{c}\right)\end{aligned}$$

Hence, we have

$$\int_\Omega (f + cg)\mathrm{d}\mu = \sum_{x\in X} x\mu(\bigcup_{y\in X} f^{-1}(y) \cap g^{-1}(\frac{x-y}{c})) = \sum_{x\in X}\sum_{y\in X} x\mu(f^{-1}(y) \cap g^{-1}(\frac{x-y}{c}))$$

Moreover, we may write $x = y + c\frac{x-y}{c}$, conveniently switch the summation order and obtain

$$\int_\Omega (f+cg)\mathrm{d}\mu = \sum_{y\in X}\sum_{x\in X} y\mu(f^{-1}(y)\cap g^{-1}(\frac{x-y}{c})) + \sum_{x\in X}\sum_{y\in X} c\frac{x-y}{c}\mu(f^{-1}(y)\cap g^{-1}(\frac{x-y}{c}))$$

Interchanging the measures with the sum in x we obtain

$$\sum_{x\in X} y\mu(f^{-1}(y) \cap g^{-1}(\frac{x-y}{c})) = y\mu(f^{-1}(y) \cap (\bigcup_{x\in X} g^{-1}(\frac{x-y}{c}))) = y\mu(f^{-1}(y))$$

and similarly,

$$\sum_{y\in X} \frac{x-y}{c}\mu(f^{-1}(y) \cap g^{-1}(\frac{x-y}{c})) = \sum_{y\in X} cz\mu(f^{-1}(y) \cap g^{-1}(z)) = cz\mu(g^{-1}(z))$$

Therefore, we have shown that

$$\int_\Omega (f + cg)\mathrm{d}\mu = \sum_{y\in X} y\mu(f^{-1}(y)) + c\sum_{z\in X} z\mu(g^{-1}(z)) = \int_\Omega f\mathrm{d}\mu + c\int_\Omega g\mathrm{d}\mu$$

For continuity, one may use an extended version of theorem 2.3, stating that the same characterization of continuity holds for functions mapping to arbitrary Banach spaces, so not just $\mathbb{R}$. To this end, we have

$$\|\int_\Omega f\mathrm{d}\mu\|_X = \|\sum_{x\in f(\Omega)} x\mu(f^{-1}(x))\|_X \leq \sum_{x\in f(\Omega)} \|x\|_X\mu(f^{-1}(x)) = \int_\Omega \sum_{x\in f(\Omega)} \|x\|_X \mathbb{1}_{f^{-1}(x)}\mathrm{d}\mu$$

We recognize the right-hand side as $\int_\Omega \|f\|_X \mathrm{d}\mu$, therefore we conclude

$$\|\int_\Omega f\mathrm{d}\mu\|_X \leq \|f\|_{L^1(\Omega,X)}$$

and this proves the statement. □

By the Hahn-Banach extension principle (see [1]), this mapping has a unique continuous linear extension on $L^1(\Omega, X)$, as it is linear, continuous and defined on a dense subset of this space. The extension is called the *Bochner integral.*

Moreover, for a set A in the $\sigma-$algebra Σ, we define $\int_A f \mathrm{d}\mu := \int_\Omega f \mathbb{1}_A \mathrm{d}\mu$.

We will now look at a theorem which links the Bochner integral defined via the "measure-theoretical" approach with the Daniell integral. This result hints at a possible construction of the Bochner integral by using the method presented in sections 3 and 4, which may have further benefits, as it was the case of the proof of Riesz-Markov-Kakutani representation theorem when defining integration the way we did.

Theorem 6.8 (Pettis)**.** *A function $f\colon \Omega \to X$ is Bochner-measurable if and only if:*

- *$\forall \phi$ functional over the space X, it holds that $\phi \circ f\colon \Omega \to \mathbb{R}$ is Bochner-measurable*
- *f is almost separably-valued*

We will not look at the proof of this result, as it requires further definitions and results. Moreover, we are not interested in the second condition, as it is not of high relevance for this paper. Nevertheless, we can observe the strong connection with the dual space of X, space which we have analyzed throughout the paper.

If the reader is interested in further results regarding the Bochner integral, paper [7] is recommended.

7 Conclusion

Throughout the paper we have seen a new approach to integration that provides a very natural and elegant proof of the Riesz-Markov-Kakutani representation theorem. Without this approach, another proof may be provided, but it requires more effort and even more advanced knowledge in functional analysis.

Even if measure theory is not required, we have made use of various results in order to properly compare the two approaches to integration. There may be even more applications where the Daniell approach provides easier proofs or more clear answers. It is up to the reader to later decide which method of integration best suits certain goals.

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