An Analysis of Extremal Periodic Water Wave Profile

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Preface

This report is the result of nine months work at the group of Applied Analysis and Mathematical Physics (AAMP) University of Twente. This work is presented as part of the curriculum of Mathematical Sciences at the University of Twente. Behind this final report, there are a lot of efforts, supports and concerns from many people to whom I owe a debt of gratitude.

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Chapter 1

Introduction

An investigation of Extremal water wave problem was initiated by Stokes in 1847. He found the so-called Stokes waves, the nonlinear variants of monochromatics waves. As one specific result he showed that for infinite depth water, the wave with largest amplitude which can travel at a certain speed without change of form will have 120^0 angle at the crest position. Rainey and Longuet Higgins in 2005 approximated Stokes extreme wave by a simple function. A numerical simulation for a steady water wave problem is done by Rienecker and Fenton in 1979. The idea behind Rienecker and Fenton simulation is Fourier expansion in approximating potential function which satisfies Laplace equation and solving a system of non-linear equations by using Newton's method.

This thesis will give another contribution in the extremal water wave area by using a different approach of modeling. Using physical integral constraints, namely Momentum and Hamiltonian, the maximal high crest profile that satisfies such constraints will be determined. As a governing equation, this thesis considers the linearized KdV equation and the nonlinear AB-equation on deep water.

The name of AB-equation itself comes from a skew-symmetric operator A and a symmetric operator B which appear in the equation. The AB-equation is exact in the dispersion and is second order accurate in wave height. Another fact about AB-equation is that this equation conserves Momentum and Hamiltonian. Some limiting cases and approximations of AB-equation can be found in the paper of E. van Groesen and Andonowati in [2]. In that paper, they show that the derivation of AB-equation is also valid for water wave problem on finite bottom depth and shallow water equation.

In this chapter, a brief history of the problem and AB-equation is given. After some formal definitions and basic concepts in Chapter 2, we start our discussion (in Chapter 3) by modeling extremal water wave problem using momentum and hamiltonian for a rather simple formulation, namely the linearized KdV equation. Chapter 4 will deal with deep water waves and a derivation of works done by some previous scientists is also presented in this chapter. Conclusions and further investigations that can be done will close this thesis.

Chapter 2

Basic Concepts

This chapter will give a brief overview about the concepts that will be used for the discussion in this thesis. We refer to [3, 4, 8, 9] for the proof of some results presented here.

2.1 Definition of Function Spaces

Let Ω be a nonempty open set in $\mathbf{R}^N, N \geq 1$, then:

1. $C^k(\Omega)$ is the set of all real functions

$$u : \Omega \to \mathbf{R}$$

   that have continuous partial derivatives of orders $m = 0, 1, ..., k$.

2. $C_0^\infty(\Omega)$ is the set of all function $u \in C^\infty(\Omega)$ that vanish outside a compact subset V of Ω, that is $u(x) = 0$ for all $x \in \Omega - V$.

2.2 First Variation and Variational Derivative

The idea of the variational derivative is basically the same as the derivative of a multivariable function in the calculus course, but instead of dealing with a finite dimensional space, variational derivative will work in the functional space which is infinite.

Definition 1: (First variation)

The first variation of a functional L at u in the direction v is denoted by $\delta L(u;v)$ and defined as

$$\delta L(u;v) = \frac{d}{d\epsilon} L(u+\epsilon v)\bigg|_{\epsilon=0}.$$

From the definition of variational derivative given above, it follows that a linear approximation of $L(u+\epsilon v)$ is given as

$$L(u+\epsilon v) = L(u) + \epsilon \delta L(u;v) + o(\epsilon)$$

where the $o(\epsilon)$ means a higher order terms in ϵ : $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$. This definition applies to all kinds of functional.

It is usually possible to perform a partial integration and rewrite $\delta L(u;v)$ as the $L_2(\Omega)$-inner product of v and some function which will be denoted by $\delta L(u)$. This may require a smoothness criteria of function u and involve an integration over the boundary as an addition,i.e:

$$\delta L(u;v) = \int_\Omega \delta L(u).v + \int_{\partial\Omega} b(u;v)$$

If the function v are considered to be vanish on the boundary, the boundary contribution vanishes as well. Therefore, using the class of test function $C_0^\infty(\Omega)$ can avoid this boundary contribution. The following definition now can be established.

Definition 2: (Variational derivative)

The function $\delta L(u)$ on Ω defined by the condition:

$$\delta L(u;\eta) = \langle \delta L(u), \eta \rangle \equiv \int_\Omega \delta L(u).\eta dx,$$

for all $\eta \in C_0^\infty(\Omega)$ is called the variational derivative of the functional L at the point u.

The notation $\delta L(u)$ for variational derivative is also denoted as $\delta L(u) \equiv \frac{\delta L}{\delta u}(u)$. But, for convenience, the variational derivative will be denoted by $\delta L(u)$ in this thesis.

The derivative operator that acts on a functional space also has the same properties like derivative for functions on the finite dimensional spaces.

1. Linearity : $\delta(L_1 + L_2) = \delta L_1 + \delta L_2$.

2. Product Rule : $\delta(L_1.L_2) = L_2\delta L_1 + L_1\delta L_2$.

3. Quotient Rule : $\delta(\frac{L_1}{L_2}) = \frac{L_2\delta L_1 - L_1\delta L_2}{L_2^2}$.

4. Chain Rule : for $h : \mathbf{R} \to \mathbf{R}$, $\delta h(L) = h'(L)\delta L$.

2.3 Stationarity Condition

Definition 3: A point $\hat{u}$ is called a critical point or a stationary point of the functional L on the set M if the following holds:

$$\delta L(\hat{u}; v) = 0 \text{ for all } v \in T_{\hat{u}}M,$$

where $T_{\hat{u}}M$ denotes the tangent space to M at $\hat{u}$.

As in finite dimensions, local extrema, i.e. points at which L has an extreme value (maximal or minimal) when compared to neighboring points in M are critical points. This gives a necessary condition for a point to be a local maximizer or local minimizer.

Preposition 1: If a functional L has a local maximizer or local minimizer at $\overline{u}$, then $\overline{u}$ is a critical point of L.

2.4 Constrained Variational Problems

The problem is stated as follow: for a given finite number of functional constraints, say $K_1, K_2, ..., K_n$, what is the critical point of functional L? First, we define U as the space of functions satisfying the boundary conditions and let U_0 be the tangent space of U, i.e. U_0 consists of elements v such that $u + \epsilon v \in U$ whenever $u \in U$: v are the functions that satisfy the homogeneous boundary conditions. The set M of admissible elements will be defined as the intersection of the levelsets of certain functionals $K_1, K_2, ..., K_n$:

$$M = \{u \in U \mid K_1(u) = k_1, K_2(u) = k_2, \ldots, K_n(u) = k_n\}$$

with $k_1, k_2, ...k_n$ are given values. We assume that for the given values of the constraints $k_1, k_2, ...k_n$ this set is non-empty. Extremizing a functional L on a set M, we arrive at the stationary condition for a critical point:

$$\delta L(\hat{u}; v) = 0, \text{ for all } v \in T_{\hat{u}}M,$$

as derived in the previous section. The tangent space of the set M is given by the following lemma.

Lemma 1: (Lyusternik Lemma)

The tangent space to M at a point $u \in M$ is the set

$$T_u M := \{v \in U_0 \mid \delta K_1(u; v) = \ldots = \delta K_n(u; v) = 0\}.$$

This Lemma says that the tangent space consists of elements v that satisfy n linear constraints: it is a hyperplane in the function space. Applying Lyusternik lemma for the specific set M under consideration, the condition for a critical point can be reformulated to:

$$\delta L(\hat{u}; v) = 0, \text{ for all } v \in U_0, \delta K_k(u; v) = 0, 1 \leq k \leq n. \quad (2.1)$$

In another word, the null-space of the functional $\delta L(\hat{u}; .)$ on U_0 contains the intersection of the null-space of the functional $\delta K_k(\hat{u}; .)$. It is obvious that (2.1) is satisfied if $\delta L(\hat{u}; .)$ is a linear combination of the $\delta K_k(\hat{u}; .), 1 \leq k \leq n$. This is expressed in the next proposition.

Proposition 2: (Lagrange's Multiplier Rule)

A point $\hat{u} \in M$ is a constrained critical point L on M, i.e. satisfied (2.1), if and only if there are real numbers, called Lagrange multipliers, $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$\delta L(\hat{u}; v) = \sum_k \lambda_k \delta K_k(\hat{u}; v), \text{ for all } v \in U_0. \quad (2.2)$$

It is also possible to formulate this result in a different way.

Proposition 3: (Lagrange's Multiplier Rule)

A point $\hat{u} \in M$ is a constrained critical point L on the constrained set M, if and only if for some multipliers, $\lambda_1, \lambda_2, ..., \lambda_n$, the element $\hat{u}$ is an unconstrained critical point of the constrained functional

$$L(u) - \sum_m \lambda_m \delta K_m(u). \tag{2.3}$$

This functional is called the Lagrangian functional of the constrained problem.

2.5 Fourier Series

The idea of Fourier expansion is to rewrite a complicated function as a series of simple functions, which is in this case are harmonic functions. The question become: which function $f(x)$ can be written as a sum of harmonic functions, and if so, how can these harmonic functions be determined from $f(x)$. That is the central question of this section. The famous result states that every L-periodic function $f(x)$ can be expressed as a superposition of L-periodic harmonic functions,

$$f(x) = \sum_{j \in Z} f_j e^{ij\frac{2\pi}{L}x}, f_j \in \mathbf{C}. \tag{2.4}$$

The right hand side is known as the Fourier series expansion of $f(x)$. Notice that the harmonic functions in the right hand side of the expression indeed are all L-period:

$$e^{ij\frac{2\pi}{L}(x+L)} = e^{ij\frac{2\pi}{L}x} \underbrace{e^{ij2\pi}}_{1} = e^{ij\frac{2\pi}{L}x}, \text{ for all } x \in \mathbf{R}.$$

Defining $k = \frac{2\pi}{L}$ then equation(2.4) can be rewritten as

$$f(x) = \sum_{j \in Z} f_j e^{ijkx}, f_j \in \mathbf{C}. \tag{2.5}$$

The last equation shows that the wave number jk of the harmonic function that together form $f(x)$ are integer multiples of k. For this reason, $k = \frac{2\pi}{L}$ is called the fundamental wave number of L-period functions.

Definition 4: (Convergence of Fourier series)

For a given $x \in \mathbf{R}$ the fourier series, in (2.5) is said to converge pointwise with limit $f(x)$ if

$$\lim_{N \to \infty} \sum_{j=-N}^{N} f_j e^{ijkx} = f(x). \tag{2.6}$$

To simplify the notation,the notation $f(x) = \sum_{j=-\infty}^{\infty} f_j e^{ijkx}$ is sometimes used. If in a Fourier series $f(x) = \sum_{j=-\infty}^{\infty} f_j e^{ijkx}$ only finitely many coefficients f_j are nonzero, then obviously the Fourier series converges for every $x \in \mathbf{R}$. For a more general case, the convergence of the Fourier series is ensured if the f_j are absolutely summable, that is,if $\sum_{j=\infty}^{\infty} |f_j|$ converges.

Theorem 1:(Existence of Infinite Fourier series)
If $\sum_{j=-\infty}^{\infty} |f_j| < \infty$ then the Fourier series $\sum_{j=-\infty}^{\infty} f_j e^{ijkx}$ converges to a continuous function.
Proof: Since $\left|e^{ijkx}\right| = 1$ it follows that

$$\sum_{j=-\infty}^{\infty} f_j e^{ijkx} < \sum_{j=-\infty}^{\infty} |f_j| < \infty.$$

From here, we conclude that the Fourier series converge absolutely for any x. Hence, the Fourier series itself converges for any x. The proof of the continuity is given in the appendix.

Previously, it is assumed that the Fourier coefficient f_j are given and that the function $f(x)$ is the resulting sum. But how are the coefficients f_j determined for a given $f(x)$? The claim is that the Fourier coefficients are uniquely determined by $f(x)$ through the integral:

$$f_j = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-ijkx}\, dx. \tag{2.7}$$

This is infact established by following argument:

$$\begin{aligned} \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-ijkx}\, dx &= \frac{1}{L} \int_{-L/2}^{L/2} \sum_{n=-\infty}^{\infty} f_n e^{inkx} e^{-ijkx}\, dx \\ &= \sum_{n=-\infty}^{\infty} f_n \frac{1}{L} \int_{-L/2}^{L/2} e^{i(n-j)kx}\, dx \\ &= f_j. \end{aligned}$$

In the last integral, we made use of the fact that the integral over a period of a harmonic function $e^{i(n-j)k}$ is zero, unless the harmonic function is a constant (when $n = j$). So only for $n = j$ does the above integral contribute to the sum. In that case the integrand $e^{i(n-j)kx}$ is 1 for all x and its integral over $[-L/2, L/2]$ hence equals L. One of the deep theorem in Fourier analysis stated in the next theorem. It tells about the convergence of the series or in another word to what function that this series converges. The proof is omitted here and can be found in the reference.

Theorem 2: (The Fourier Series Theorem)

If $f(x)$ is an L-periodic and piecewise smooth function then

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{j=\infty}^{\infty} f_j e^{ijkx}, \text{ for every } x \in \mathbf{R},$$

where f_j are the Fourier coefficients of $f(x)$ defined as

$$f_j = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-ijkx}\, dx. \tag{2.8}$$

If $f(x)$ has no discontinuities then $\frac{f(x^+)+f(x^-)}{2} = f(x)$ everywhere and hence the result then says that

$$f(x) = \sum_{j=\infty}^{\infty} f_j e^{ijkx}.$$

Lemma 2: (Riemann-Lebesgue Lemma)

If $f(x)$ is a piecewise smooth function on $[a, b]$ then

$$\lim_{|j|\to\infty} \int_a^b f(x) e^{ijx}\, dx = 0.$$

Proof: Suppose first that $f(x)$ is a continuously differentiable on $[a, b]$, then using integration by parts, we obtain:

$$\int_a^b f(x) e^{ijx}\, dx = \frac{1}{ij} \left[f(x) e^{ijx} \right]_a^b - \frac{1}{ij} \int_a^b f'(x) e^{ijx}\, dx.$$

Since $|e^{ijx}| = 1$, we get that:

$$\left| \int_a^b f(x) e^{ijx}\, dx \right| \leq \frac{|f((b)| + |f(a)|)}{|ij|} + \frac{1}{|ij|} \int_a^b |f'(x)|\, dx.$$

It is immediate that the right hand side goes to zero as $|j| \to \infty$, which proves the claim.

If $f(x)$ is piecewise smooth, we may split $[a, b]$ into a finite set of subintervals $[x_n, x_{n+1}](n = 1, 2, ...)$ such that $f(x)$ is continuously differentiable on each of this subintervals. Similarly as done above (using integration by parts), it follows that $\lim_{|j|\to\infty} \int_{x_n}^{x_{n+1}} f(x)e^{ijx}\,dx = 0.$. Hence $\lim_{|j|\to\infty} \int_a^b f(x)e^{ijx}\,dx = 0$.

A consequence of the previous lemma is that a piecewise continuous function $f(x)$ satisfies

$$\lim_{|j|\to\infty} \int_{-L/2}^{L/2} f(x)e^{-ijkx}\,dx = 0,$$

i.e. the Fourier coefficients converge to zero as $|j| \to \infty$. If a function $f(x)$ is n-times piecewise continuously differentiable, then the Fourier coefficients of $f^{(n)}$, n^{th} derivative of f, can be obtained by a relation: $f_k^{(n)} = (ik)^n f_k$. From this fact, we state following lemma.

Lemma 3: For any positive integer p, if $f(x)$ is C^p and piecewise C^{p+1}, then

$$\lim_{k\to\infty} \left|k^{p+1} f_k\right| = 0.$$

Proof: Since $k^{(p+1)} f_k = i^{-(p+1)} f_k^{(p+1)}$ and $f^{(p+1)}(x)$ is a piecewise continuous function ($\lim_{k\to\infty} f_k^{(p+1)} = 0$) then $\left|k^{(p+1)} f_k\right| \to 0$ as $k \to \infty$.

From this lemma, we conclude that Fourier coefficients of a continuous function are bounded by $\frac{1}{k^{(1+q)}}, q > 0$, i.e $|f_k| < \frac{C}{|k^{(1+q)}|}$, $C > 0$.

2.6 Real Fourier series

Preposition 3: For a real value function $f(x)$, the Fourier coefficients obey the symmetry rule $f_{-j} = f_j^*$.

Proof:

$$\begin{aligned} f_{-j} &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{ijkx}\,dx \\ &= \frac{1}{L}\int_{-L/2}^{L/2} f^*(x)e^{ijkx}\,dx \text{ since f(x) is real} \\ &= \left(\frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-ijkx}\,dx\right)^* \\ &= f_j^* \end{aligned}$$

The complex fourier series of a real-valued function can then be rewritten as a real Fourier series, that is, as a sum of sinusoids. This goes as follows. First define real a_j and b_j by:

$$a_j = 2Re(f_j),\ b_j = -2Im(f_j), j = 0, 1, 2, ...$$

That is to say

$$f_0 = \frac{a_0}{2}, f_j = \frac{a_j - ib_j}{2}, j > 0.$$

Using this definition of a_j and b_j, it gives:

$$\begin{aligned}
f(x) &= \sum_{j=\infty}^{\infty} f_j e^{ijkx} \\
&= f_0 + \sum_{j=1}^{\infty} \left(f_{-j}e^{-ijkx} + f_j e^{ijkx}\right) \\
&= f_0 + \sum_{j=1}^{\infty} \left(\left(f_j e^{ijkx}\right)^* + f_j e^{ijkx}\right),\ f_{-j} = f_j^* \\
&= f_0 + \sum_{j=1}^{\infty} Re\left(2f_j e^{ijkx}\right) \\
&= \frac{a_0}{2} + \sum_{j=1}^{\infty} Re\left[(a_j - ib_j)(\cos(jkx) + i\sin(jkx))\right] \\
&= \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j\cos(jkx) + b_j\sin(jkx)).
\end{aligned}$$

This finally establishes that $f(x)$ is indeed a sum of sinusoids. The last equation is known as the real Fourier series and the coefficients a_j and b_j are the real Fourier coefficients. The terms $a_j\cos(jkx) + b_j\sin(jkx)$ is sometimes referred to the j^{th} harmonic of $f(x)$. In summary, we have following theorem:

Theorem 3: (Fourier series theorem, real-valued case)

If $f(x)$ is real-valued, L-periodic and piecewise smooth function, then

$$\frac{f(x^+) + f(x^-)}{2} = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j\cos(jkx) + b_j\sin(jkx)) \text{ for all } x \in \mathbf{R}$$

where a_j and b_j are the real Fourier coefficients of $f(x)$ given by

$$a_j = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(jkx)\,dx\ \ j = 0, 1, ... \tag{2.9}$$

$$b_j = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin(jkx)\,dx \;\; j = 1, 2, ... \tag{2.10}$$

Proof: Let f_j be the complex Fourier coefficients of $f(x)$. In the above, we show that $a_j = 2Re(f_j),\ b_j = -2Im(f_j)$. Therefore,

$$\begin{aligned} a_j &= 2Re(f_j) = 2Re\frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-ijkx}\,dx \\ &= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(jkx)\,dx. \end{aligned}$$

and,

$$\begin{aligned} b_j &= -2Im(f_j) = -2Im\frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-ijkx}\,dx \\ &= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin(jkx)\,dx. \end{aligned}$$

2.7 Parseval's theorem

Theorem 4: If $f(x)$ is a piecewise continuous L-periodic function then

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\,dx = \sum_{j=-\infty}^{\infty} |f_j|^2 .$$

Proof:

$$\begin{aligned} \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\,dx &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\left(\sum_{j=-\infty}^{\infty} f_j e^{ijkx}\right)^* dx \\ &= \sum_{j=-\infty}^{\infty} f_j^*\left(\frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-ijkx}\,dx\right) \\ &= \sum_{j=-\infty}^{\infty} f_j^* f_j \\ &= \sum_{j=-\infty}^{\infty} |f_j|^2 . \end{aligned}$$

2.8 Poincaré-Friedrichs Inequality

Proposition 4: There exists a constant $\gamma \in \mathbf{R}$ such that for all $u \in C_0^1(a, b)$,

$$\int_a^b u^2 dx \le \gamma \int_a^b u'^2 dx$$

Proof: Since $u \in C_0^1(a,b)$, then by fundamental theorem in calculus u can be written as follows:

$$u(x) = \int_a^x u'(y)dy,$$

for all $x \in (a,b)$ but

$$u(x)^2 \leq \left(\int_a^b 1. \left| u'(y) \right| dy \right)^2 \leq \int_a^b dy \int_a^b u'^2 dy.$$

The last inequality is obtained by applying Schwarz inequality in the L_2 space which is a Hilbert Space. Hence,

$$\int_a^b u^2 dx \leq (b-a)^2 \int_a^b u'^2 dx.$$

Choose $\gamma = (b-a)^2$, completes the prove.

Chapter 3

Extremal Periodic Waves for Linearized KdV-type of Equation

3.1 Introduction

In this chapter, we derive the profile of extremal periodic wave for a very simple case namely the linearized KdV. As constraints, the value of Momentum and Hamiltonian are prescribed. The result is that there is a certain area in the Momentum-Hamiltonian plane, for which the solution exists and uniquely determined by the value of those functional constraints. On the boundary curve, the profiles of extremal periodic wave can be written as the pure harmonic function. Inside this area, the solution will have a corner.

3.2 Korteweg de Vries Model

Unidirectional waves on a layer of depth D above flat bottom are well described by the Korteweg de Vries (KdV) equation which are of the form

$$\partial_t \eta = -\partial_x \delta H(\eta).$$

Including exact dispersion property and simplified nonlinearity, the hamiltonian for surface wave is given by:

$$H_{sw} = \int \left[\frac{1}{2}\eta C_p \eta + \frac{c_0}{4D}\eta^3 \right] dx,$$

with the pseudo differential operator has as symbol the phase velocity of dispersive waves on a layer of depth D:

$$c_p = c_0\sqrt{\frac{tanh(kD)}{kD}} \text{ with } c_0 = \sqrt{gD}.$$

Here g is the gravitational acceleration. The dynamic equation now given as follows

$$\partial_t \eta = -\partial_x \left[C_p \eta + \frac{3c_0}{4D}\eta^2 \right].$$

To arrive at the classical KdV-equation, the pseudo differential operator C_p is approximated by the lowest order differential operator, i.e:

$$c_p = c_0\sqrt{\frac{tanh(kD)}{kD}} \approx c_0 \left[1 - \frac{1}{6}(kD)^2 \right].$$

Using above approximation, it gives us the hamiltonian for KdV, H_{KdV}:

$$H_{KdV} = c_0 \int \left[\frac{1}{2}\eta^2 - \frac{D^2}{12}(\partial_x \eta)^2 + \frac{1}{4D}\eta^3 \right] dx. \tag{3.1}$$

The governing equation reads

$$\partial_t \eta = -c_0 \partial_x \left[\eta + \frac{D^2}{6}\partial_x^2 \eta + \frac{3}{4D}\eta^2 \right]. \tag{3.2}$$

A wave that travel undisturbed at speed V are of the form $\eta(x - Vt)$ and satisfy the equation

$$\mu_1 + V\eta = c_0 \left[\eta + \frac{D^2}{6}\partial_x^2 \eta + \frac{3}{4D}\eta^2 \right].$$

Scaling the spatial variable and the wave height with respect to the water depth, $\widetilde{x} = x/D, \widetilde{\eta} = 9\eta/(2D)$ leads to the normalized equation:

$$\mu + \lambda\widetilde{\eta} = -\partial_x^2 \widetilde{\eta} - \widetilde{\eta}^2, \tag{3.3}$$

with $\mu = -27\mu_1/(Dc_0)$ and $\lambda = 6(c_0 - V)/c_0$.

The corresponding hamiltonian for equation(3.2) is given by (without tildes):

$$H = \int \left[\frac{1}{2}(\partial_x \eta)^2 - \frac{1}{3}\eta^3 \right] dx. \tag{3.4}$$

In this chapter, we analyze the first term of the hamiltonian and neglecting the cubic terms which produces a nonlinearity in the equation(3.2). An analysis involving the nonlinearity term can be found in the work of E. van Groesen and Andonowati in [1].

3.3 Modeling of Extremal Crest Formulation

Denote the maximal crest height functional for functions η by

$$C(\eta) := \max_x \eta(x).$$

Then the extremal-periodic problem can be written as

$$\max_\eta \{C(\eta) \mid \eta \in \mathcal{C}\} \tag{3.5}$$

where the constraint set is given by

$$\mathcal{C}(h, m) = \{\eta \mid \eta\, \text{is} L - \text{periodic},\ av(\eta) = 0, H(\eta) = h, M(\eta) = m\}$$

with the average, Momentum and Hamiltonian respectively defined as follows:

$$av(\eta) = \int_{-L/2}^{L/2} \eta(x)\, dx$$

$$M(\eta) = \int_{-L/2}^{L/2} \frac{1}{2}(\eta(x))^2\, dx$$

$$H(\eta) = \int_{-L/2}^{L/2} \frac{1}{2}(\partial_x \eta(x))^2\, dx$$

To make this formulation well defined, it is necessary that the constraint set is nonempty and this is determined by the value of (m, h) that should satisfy a certain conditions.

3.4 Feasible Area

We define feasible area as the area in the Momentum Hamiltonian plane such that there exists function satisfying a given value of Momentum and Hamiltonian. And the curve that bounds that area will be called as feasible curve.

Recalling the Poincaré-Friedrichs inequality in chapter 2, it is guaranteed that for a continuous differentiable function in the interval $(-L/2, L/2)$ there must be a certain constant γ, such that

$$\int_{-L/2}^{L/2} u^2 dx \leq \gamma \int_{-L/2}^{L/2} (\partial_x u)^2\, dx.$$

Or stated differently, we must prescribe the value of the Momentum has to be smaller or equal than the value of the Hamiltonian multiply by some constant ($M \leq \gamma H$). To make it rather precise, another way to look at the condition that should be satisfied by the constraint set is by transforming Momentum and Hamiltonian functional into spectral space. Since we assume η is an L-periodic continuous real function, then according to Parseval's theorem in chapter 2, we obtain that:

$$\begin{aligned} M &= \int_{-L/2}^{L/2} \frac{1}{2} (\eta(x))^2 \, dx \\ &= L/2 \sum_{k=-\infty}^{\infty} \eta_k^2. \end{aligned}$$

$$\begin{aligned} H &= \int_{-L/2}^{L/2} \frac{1}{2} (\partial_x \eta(x))^2 \, dx \\ &= L/2 \sum_{k=-\infty}^{\infty} k^2 \eta_k^2. \end{aligned}$$

But since the $av(\eta) = \int_{-L/2}^{L/2} \eta \, dx = 0$,which means that $\eta_0 = 0$ and using the fact that

$$\eta_k^2 \leq k^2 \eta_k^2, \text{ for all } k \in \mathbf{Z}$$

then:

$$\sum_{k=-\infty}^{\infty} \eta_k^2 \leq \sum_{k=-\infty}^{\infty} k^2 \eta_k^2 \tag{3.6}$$

or equivalently,

$$M \leq H.$$

The feasible curve in the parameter space can be obtained from a separate extremal formulation. Namely, by the minimization problem of the Hamiltonian on the level set of the Momentum for which the average vanishes, i.e.

$$\min_{\eta} \{ H \mid M(\eta) = m, av(\eta) = 0 \}.$$

Or equivalently,

$$\min_{\eta} \left\{ \int_{-L/2}^{L/2} \frac{1}{2} (\partial_x \eta)^2 \, dx \,\middle|\, \int_{-L/2}^{L/2} \frac{1}{2} \eta^2 \, dx = m, \int_{-L/2}^{L/2} \eta \, dx = 0 \right\}.$$

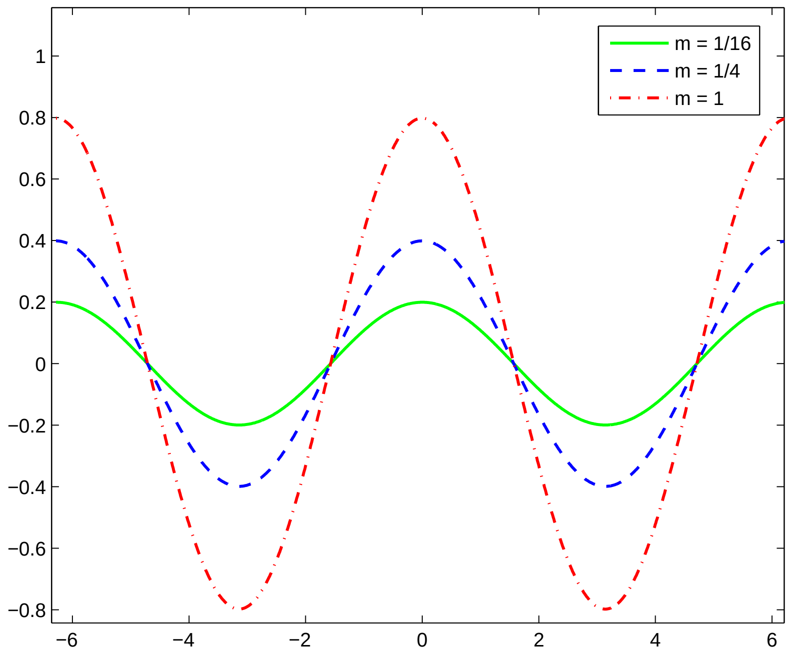

Figure 3.1: Maximal periodic wave profiles along feasible curve for $M = \frac{1}{16}, \frac{1}{4}, 1$.

According to equation(3.6), this minimization is achieved for $\eta_1 = \sqrt{m/L}$ and $\eta_k = 0$ for $k \neq 1$. Hence, the profile related to this minimization is given by

$$\eta(x) = \sqrt{\frac{4m}{L}} \cos x.$$

Figure(3.1) illustrates this kind of profile related to a certain Momentum with the wave length $L = 2\pi$.

The previous analysis proves the statement that the feasible area in the Momentum-Hamiltonian plane of the linearized KdV equation is given by $M \leq H$ and the feasible curve is a straight line $M = H$ for which the maximal crest wave along this feasible line is a pure harmonic function.

3.5 Profile Inside Feasible Area

Applying Lagrange Multiplier rule to the formulation(3.5), the wave profile inside feasible area should satisfy:

$$\delta C = \lambda_M \delta M + \lambda_H \delta H + \mu. \tag{3.7}$$

Since the first variational derivative of maximal height crest functional is dirac delta function (the derivation is given is the appendix), equation (3.7) can be rewritten as:

$$\delta_{dirac}(x - x_{\max}) = \lambda_M \eta - \lambda_H \eta_{xx} + \mu. \tag{3.8}$$

The effect of appearance of the dirac delta in the last expression is that the profile η that fulfills this equation will have a corner at the maximum position $x_{\max}$. This can be seen from the property of dirac delta function that outside the maximum position the profile η has to satisfy $\lambda_M \eta - \lambda_H \eta_{xx} + \mu = 0$. But this non-homogeneous ordinary differential equation is indeed a formulation of an eigen value problem for which the solution will be trigonometric functions or a hyperbolic functions depending on the sign of $\left(\frac{\lambda_M}{\lambda_H}\right)$. Therefore, a cornered profile will consist of a suitable composition of these class of functions.

It is also possible to use Fourier expansion to conclude the statement about cornered solution inside the feasible area. Suppose that the wave length being considered is 2π and the domain of interest is $[-\pi, \pi]$. Transforming equation(3.8) into Fourier space leads to:

$$\sum_k \left(\frac{1}{2\pi} e^{-ikx_{max}} - \left(k^2 \lambda_H + \lambda_M\right) \eta_k \right) e^{ikx} = \mu. \tag{3.9}$$

The solution of this equation is given by

$$\eta(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{\lambda_M + \lambda_H k^2} \cos k(x - x_{\max}).$$

Since $\sum_{k=1}^{\infty} \left| \frac{1}{\lambda_M + \lambda_H k^2} \right| < \infty$ then according to theorem(1) in chapter(2), the Fourier series will converge to a continuous function η. But $\lim_{k \to \infty} \left| \frac{k^2}{\lambda_M + \lambda_H k^2} \right| = \frac{1}{\lambda_H} \neq 0$ and lemma(3) guarantees that $\eta \notin C^1$. Hence η, the solution of equation(3.9), must form corners.

Recalling equation(3.8) and integrating around the neighborhood of the maximum position,i.e. $[x_{\max} - \epsilon, x_{\max} + \epsilon]$ with $\epsilon > 0$, it leads to:

$$\int_{x_{\max}-\epsilon}^{x_{\max}+\epsilon} \delta_{dirac}(x - x_{\max}) dx = \int_{x_{\max}-\epsilon}^{x_{\max}+\epsilon} \left[\lambda_M \eta - \lambda_H \eta_{xx} + \mu\right] dx \tag{3.10}$$

$$\begin{aligned} 1 &= \lambda_M \int_{x_{\max}-\epsilon}^{x_{\max}+\epsilon} \eta \, dx - \lambda_H \, \eta_x|_{\hat{x}_{\max}-\epsilon}^{\hat{x}_{\max}+\epsilon} + \mu(2\epsilon) \\ &= -\lambda_H \, \eta_x|_{x_{\max}-\epsilon}^{x_{\max}+\epsilon}, \text{ as } \epsilon \to 0. \end{aligned}$$

or equivalently,

$$\eta_x|_{x_{\max}-\epsilon}^{x_{\max}+\epsilon} = -\frac{1}{\lambda_H}$$

The last expression shows that the corner at the maximum position actually measures the difference of the angle formed by the tangent line approaches from the left and the right side of x_{max}.

As mentioned earlier, the profile inside the feasible area is basically a solution of eigen value problem of equation (3.8). This solution has a form of either trigonometric function or hyperbolic(catenary) function depending on the sign of $\left(\frac{\lambda_M}{\lambda_H}\right)$. Since one of the constraints in the formulation is to make the average of the profile vanishes, then the profile can be written as follows:

$$\eta_{tri} = a\left[\frac{1}{q\pi}\sin(q\pi) - \cos(qx)\right], a > 0, -1 < q < 0. \tag{3.11}$$

$$\eta_{cat} = A\left[-\frac{1}{q\pi}\sinh(q\pi) + \cosh(qx)\right], A > 0, q > 0. \tag{3.12}$$

For $q \to 0$ the trigonometric and catenary profiles meet at the parabolic profile given by: $\eta_{sing} = a(x^2 - \pi^2/3)$. The value of a (or A) and q can be determined by the constraint values of the Hamiltonian and Momentum.

If we maximize the restricted functional

$$H(\eta_{tri}) = \frac{(aq)^2}{2}\pi - \frac{a^2 q}{4}\sin(2q\pi)$$

on the level set

$$M(\eta_{tri}) = \frac{-a^2}{q^2\pi}\sin^2(q\pi) + \frac{a^2\pi}{2} + \frac{a^2}{4q}\sin(2q\pi) = m,$$

for $q \in (-1, 0)$, we obtain that the maximum of $H(\eta_{tri})$ is achieved at the position $q = 0$ and the related value of Hamiltonian is $H = \left(\frac{15}{\pi^2}\right) m$. This analysis shows that the solution of the trigonometric form, η_{tri}, only exists for $m < h < \left(\frac{15}{\pi^2}\right) m$,

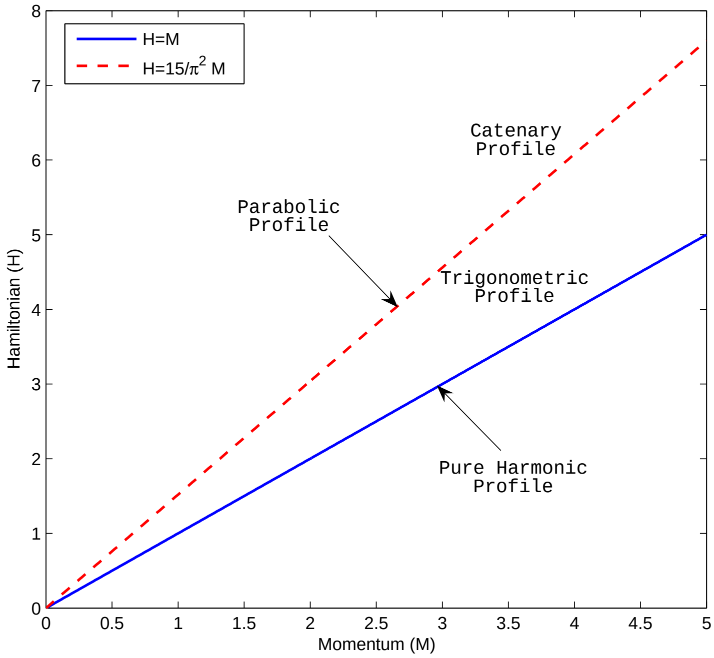

Figure 3.2: MH-plane for linearized KdV type of equation.

with m is a given value of Momentum. Above $h = (15/\pi^2)m$, the profile will have a hyperbolic form and the profile of the wave that satisfies $h = (15/\pi^2)m$ is given by a parabolic shape. See figure(3.2) for the illustration.

If we define half-corner angle, ϕ by $\tan(\phi) = \left[\frac{d\eta}{dx}\right]_{x=\pi}$, then the steepness of the wave at the corner position for both branches, trigonometric and catenary, are given by:

$$\phi_{tri} = \arctan(aq\sin(q\pi)),\ \phi_{cat} = \arctan(Aq\sinh(q\pi)).$$

Some profiles of the trigonometric and catenary branch are given in figure(3.3). Notice that at $q = 0$ the plot shows that the profile of the catenary and trigonometry branch meets. Figure(3.4) shows plots of the hamiltonian and the maximal crest height as function of q and figure(3.5) shows the steepness of the profile as function of q by assuming that the position of the crest is at $x = \pi$.

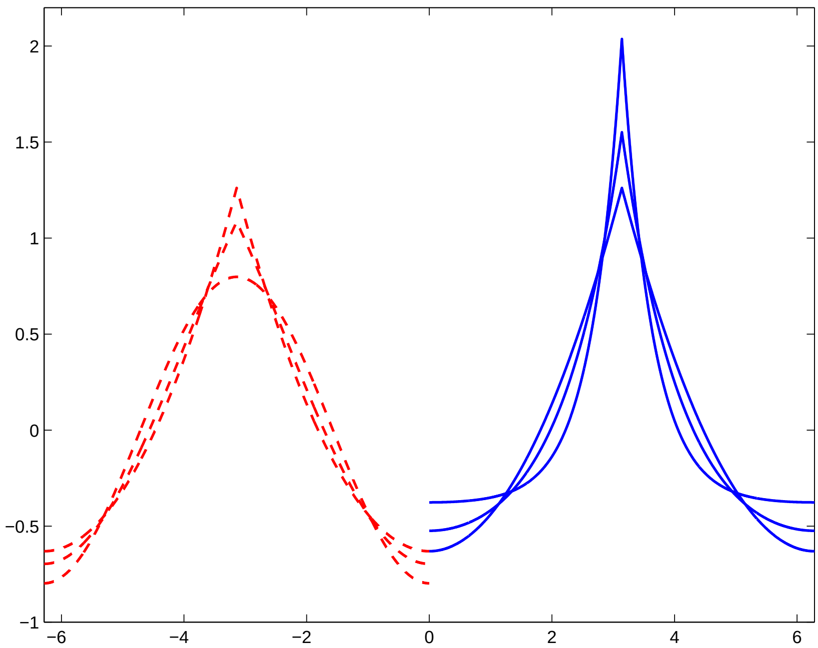

Figure 3.3: Plot of one period (2π) of the extremal profile for normalized Momentum $M = 1$. The dash line indicates the trigonometric branch $(q = 0, -\frac{2}{3}$ and $-1)$ and the solid line shows catenary branch $(q = 0, 1, 2)$.

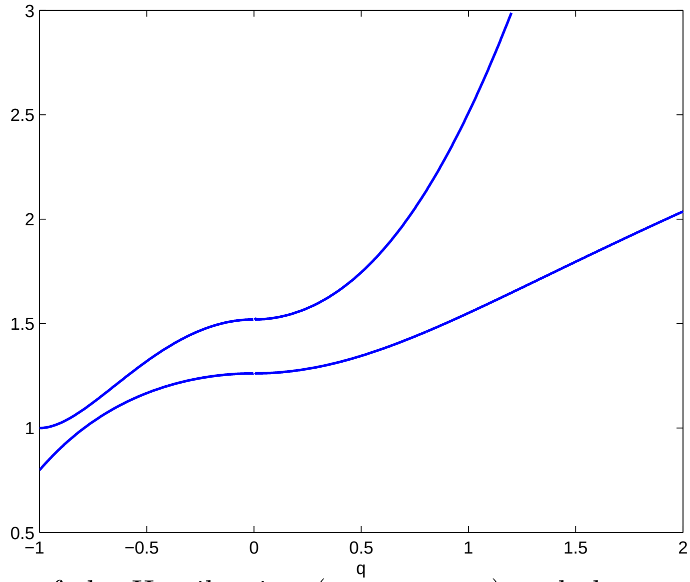

Figure 3.4: Plot of the Hamiltonian (upper curve) and the maximal crest height (lower curve) for a normalized value of Momemtum, $M = 1$.

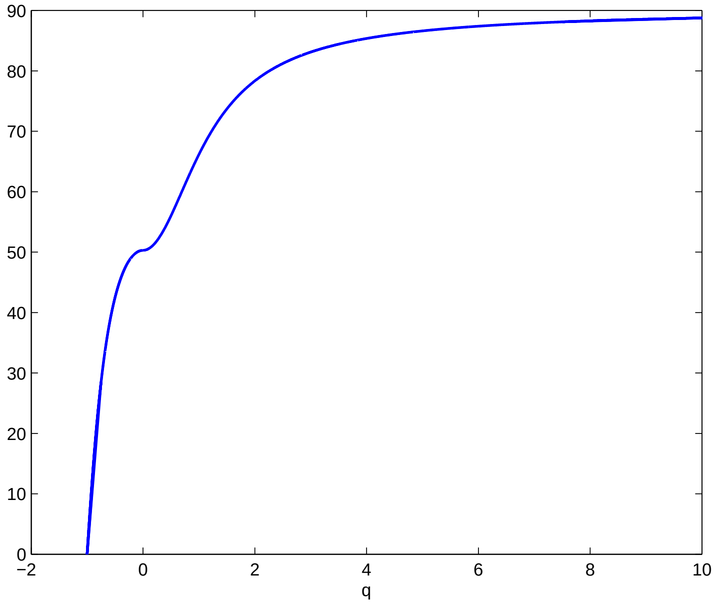

Figure 3.5: Plot of the half corner angle (in degrees) at the crest position for $M = 1$.

Chapter 4

Extremal Periodic Waves on Deep Water

This chapter is concerned with extremal waves on deep water. Starting from works by previous scientists, we analyze a new governing equation, the so called AB-equation, which gives another approach to understand an extremal high crest profile on deep water. We also make a comparison of the result from AB-equation with others.

4.1 Rainey and Longuet-Higgins Approximation

In 1880, Stokes claims that the maximal amplitude profile that appears on deep water will form a sharp angle of 120^0 at the crest position. This wave travels without changing its form at a certain speed. Rainey and Longuet-Higgins approximated such a profile using a simple function in 2005. They proved in their publication [6], that their approximation is accurate up to 99.7%. Here, we give the derivation of the approximation that they proposed.

A simple curve with a sharp angle of 120^0 at $x = 0$ is given by

$$y = Ce^{-|x|}, C = \frac{1}{\sqrt{3}} \tag{4.1}$$

Let us superpose an infinite sequence of such sharp angles at the points $x = n - \frac{1}{2}$

where n is any integer. Thus, let

$$y = C \sum_{n=-\infty}^{\infty} e^{-\left|x-n+\frac{1}{2}\right|}, C = \frac{1}{\sqrt{3}}. \tag{4.2}$$

In the interval $-\frac{1}{2} < x < \frac{1}{2}$ the above series can be rewritten as

$$\left[e^{-(x+\frac{1}{2})} + e^{-(x+\frac{3}{2})} + e^{-(x+\frac{5}{2})} + \ldots\right] + \left[e^{(x-\frac{1}{2})} + e^{(x-\frac{3}{2})} + e^{(x-\frac{5}{2})} + \ldots\right] \tag{4.3}$$

that is,

$$\begin{aligned} e^{-(x+\frac{1}{2})}\left(1 + e^{-1} + e^{-2} + \ldots\right) + e^{(x-\frac{1}{2})}\left(1 + e^{-1} + e^{-2} + \ldots\right) &= \frac{e^{-\frac{1}{2}}\left(e^{x} + e^{-x}\right)}{\left(1 - e^{-1}\right)} \\ &= \frac{e^{x} + e^{-x}}{e^{\frac{1}{2}} - e^{-\frac{1}{2}}} \\ &= \frac{\cosh x}{\sinh(1/2)}. \end{aligned}$$

So eq(4.2) can be written as

$$y = A \cosh x \tag{4.4}$$

where

$$A = \left(\sqrt{3} \sinh \frac{1}{2}\right)^{-1} = 1.107, 955$$

For convenience, we shift the origin of y to the level of the wave trough by subtracting a constant, so the free surface is given by

$$Y = A(\cosh X - 1), \tag{4.5}$$

where $X = x$ and $Y = y - A$. This approximation is valid for a wavelength L as unity, i.e $L = 1$, but the same procedure can be applied for an arbitrary wave length L and we get an expression as follow:

$$y = AL\left[\cosh\left(\frac{X}{L}\right) - 1\right], -\frac{1}{2} < \frac{X}{L} < \frac{1}{2} \tag{4.6}$$

where A is the same constant as in the previous expression. The profile of Rainey and Longuet Higgins approximation for the wave length 2π can be seen in fig(4.4) of the next sections.

4.2 Rienecker Fenton Approximation

The idea behind Rienecker and Fenton approximation is solving full non-linear equation of steady wave using Fourier expansion. Following are the details of their method. The problem considered is that two-dimensional periodic waves propagating without changing its form over a layer of fluid on a horizontal bed. Defining the horizontal coordinate x and the vertical coordinate y and assuming that the fluid is incompressible and irotational, then it is possible to define a stream function $\psi(x, y)$ that satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \tag{4.7}$$

and such that the velocity components (u, v) are given by

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}.$$

The boundary conditions to be satisfied are

$$\psi(x, 0) = 0, \tag{4.8}$$

$$\psi(x, \eta(x)) = -Q, \tag{4.9}$$

where $y = \eta(x)$ on the free surface and Q is a positive constant denoting the total volume rate of flow underneath the steady wave per unit length in a direction normal to the x, y plane. On the free surface, the pressure is constant so that Bernoulli's equation gives

$$\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] + \eta = R \tag{4.10}$$

where R is a constant. In these equations , all variables have been non-dimensionalised with respect to the average depth, $\overline{\eta}$, and the gravitational acceleration.

If the symmetry of the wave about the crest is exploited, a series for $\psi(x, y)$ can be written as follow:

$$\psi(x, y) = B_0 y + \sum_{j=1}^{N} B_j \frac{\sinh jky}{\cosh jkD} \cos jkx, \tag{4.11}$$

This expansion satisfies equation(4.7)and boundary condition(4.8) directly. The $B_0, B_1, \ldots, B_N$ are constants to be determined for a particular wave. The assumption, for computational purposes, that N is finite is the only approximation made in this method of solution. Now, the boundary condition (4.9) can be written as:

$$B_0\eta + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta}{\cosh jkD} \cos jkx = -Q \tag{4.12}$$

and equation (4.10) becomes

$$\frac{1}{2}\left\{B_0 + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta}{\cosh jkD} \cos jkx\right\}^2 + \frac{1}{2}\left\{k\sum_{j=1}^{N} jB_j \frac{\sinh jk\eta}{\cosh jkD} \sin jkx\right\}^2 + \eta = R. \tag{4.13}$$

To solve the problem numerically, equations (4.12) and (4.13) are to be satisfied at $2N$ points equally space over one wavelength, though by symmetry only $(N+1)$ points, from the wave crest to the trough, need to be considered. Let $\eta_m = \eta(x_m)$ where $x_m = m\lambda/2N, m = 0, 1, \ldots, N$ and $kx_m = \pi m/N$ that equation (4.12) and (4.13) are become

$$B_0\eta_m + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta_m}{\cosh jkD} \cos jm\pi/N + Q = 0, form = 0, 1, .., N \tag{4.14}$$

and

$$\frac{1}{2}(u_m)^2 + \frac{1}{2}(v_m)^2 + \eta_m - R = 0, form = 0, 1, .., N \tag{4.15}$$

where

$$u_m = B_0 + k\sum_{j=1}^{N} jB_j \frac{\cosh jk\eta_m}{\cosh jkD} \cos\left(jm\pi/N\right)$$

and

$$v_j = k\sum_{j=1}^{N} jB_j \frac{\sinh jk\eta_m}{\cosh jkD} \sin\left(jm\pi/N\right).$$

These $(2N+2)$ nonlinear equation involve $(2N+5)$ variables $\eta_j, B_j(j = 0, 1, \ldots, N)$, k, Q and R. To obtain a solution 3 more equations must be specified. Since variables have been non-dimensionalized with respect to $\overline{\eta}$, an equation for the unit mean depth may be written as:

$$\frac{1}{2N}\left[\eta_0 + \eta_N + 2\sum_{j=1}^{N-1} \eta_j\right] - 1 = 0. \tag{4.16}$$

This is a simple trapezoid rule for numerical integration of the periodic η. With the value of R and Q specified, these equations may be solve for the remaining unknowns. However, for practical problems, the values of wave height H and period τ are defined. Two addition equations which specify these two physical quantities are

$$\eta_0 - \eta_N - H = 0 \tag{4.17}$$

where η_0 and η_N are the surface elevation at the crest and at the trough respectively and

$$kc\tau - 2\pi = 0. \tag{4.18}$$

The later equation introduces one more variable, namely the wave speed c. The mean Eulerian velocity c_E can be prescribed to add one more equation. In the steady frame the mean velocity at each level within the fluid is B_0. To consider motion relative to any frame through which the waves move, a uniform wave speed c is superimposed, so that in this frame, the mean Eulerian velocity $c_E = c + B_0$.

$$c - c_E = B_0. \tag{4.19}$$

The $(2N+6)$ equations, (4.14)-(4.19) form a closed system for the unknown variables $\eta_j, B_j (j = 0, \ldots, N), c,\ k, Q$ and R. The system of these nonlinear equations is solved using Newton's method with following initial guess:

$$\eta_m = 1 + \frac{1}{2} H \cos(m\pi/N) \text{ for } m = 0, \ldots, N$$

$$B_0 = -c,\ B_1 = -\frac{1}{4} H/ck,\ B_j = 0, j = 2, \ldots, N$$

$$R = 1 + \frac{1}{2} c^2$$

$$Q = c$$

$$k = 2\pi/\tau c$$

$$c = 1$$

4.3 *AB*-equation

In 2006, van Groesen and Andonowati derived a new class of equation so called *AB*-equation. The idea behind this equation is by approximating the kinetic energy functional that appears in the Hamiltonian functional and restrict the action principle to a unidirectional waves. The fact of *AB*-equations is that it is valid for finite depth water and also infinite depth water (with adapted operator A and B). The *AB*-equation is also accurate up to second order in the wave height, is exact in dispersion and conserves the Momentum and Hamiltonian. A brief derivation of *AB*-equation for infinite depth water can be seen in the appendix and we refer to [2] for a detailed discussion about this equation.

4.3.1 *AB Equation on Infinitely Deep Water*

As derived in the appendix, the Hamiltonian (H) and Momentum (M) functional for *AB*-equation in infinite depth are given as follows:

$$\begin{aligned} H(\eta) &= g\int_{-L/2}^{L/2}\left[\eta^2+\frac{1}{2}\eta\left\{(B\eta)^2-(A\eta)^2\right\}\right]dx \\ &= H_2(\eta)+H_3(\eta), \end{aligned}$$

where H_2 and H_3 are the quadratic and cubic part of the Hamiltonian.

$$M(\eta)=\sqrt{g}\int_{-L/2}^{L/2}\eta B\eta\,dx$$

here A and B denote the pseudo differential operator with $\check{A}(k)=i*sign(k)\sqrt{\mid k\mid}$ and $\check{B}(k)=\sqrt{\mid k\mid}$ in the Fourier space. Note that A is a skew-symmetric operator and B is a symmetric operator.

Now then the *AB*-equation becomes

$$\partial_t\eta=-\frac{A}{2\sqrt{g}}\delta H(\eta)$$

or explicitly,

$$\partial_t\eta=-\sqrt{g}A\left[\eta+\frac{1}{4}\left\{(B\eta)^2-(A\eta)^2\right\}+\frac{1}{2}\left\{A(\eta A\eta)+B(\eta B\eta)\right\}\right] \tag{4.20}$$

The pseudo differential operator A and B cannot be easily approximated by ordinary differential operators.

4.3.2 *Relative Equilibrium of AB equation on Infinite Depth*

In this section, we restrict the waves with fundamental period of 2π and zero average, $\int \eta \; dx = 0$, to investigate an extremal value of hamiltonian for a given value of momentum, i.e:

$$ext_\eta \left\{ H(\eta) \mid \; M(\eta) = m \right\}, \tag{4.21}$$

for which the H and M are given in the previous section.

According to Lagrange Multiplier rule, formulation(4.21) is equivalent to solve $\delta H = \mu \delta M$ for a prescribed value of momentum. By recalling the dynamics of AB-equation, we obtain

$$\begin{aligned} \partial_t \eta &= -\frac{A}{2\sqrt{g}} \delta H \\ &= -\frac{A}{2\sqrt{g}} \mu \delta M \\ &= -\frac{A}{2\sqrt{g}} \mu 2\sqrt{g} B\eta \\ &= -\mu A B \eta \\ &= -\mu \partial_x \eta. \end{aligned}$$

This observation show that formulation(4.21) is actually a relative equilibria problem, meaning that the extremum profile η will travel undisturbed with a certain speed which is the lagrange multiplier, μ, for this case. As a matter of writing and for a convenience, we write the multiplier rule as $\delta H_2 + \delta H_3 = \mu \delta M$,where H_2 and H_3 are the quadratic and cubic part of the hamiltonian respectively.

Computing the variational derivative of the momentum and hamiltonian functional leads to the problem of finding η such that η satisfies:

$$2\frac{\mu}{\sqrt{g}} B\eta = \left\{ 2\eta + \frac{1}{2} \left\{ (B\eta)^2 - (A\eta)^2 \right\} + \left\{ A(\eta A\eta) + B(\eta B\eta) \right\} \right\}. \tag{4.22}$$

The linear part of equation(4.22) comes from the quadratic part of the Hamiltonian (H_2). The linear equation

$$\delta H_2 = \mu \delta M,$$

will have a pure harmonic function as a solution, namely $\eta(x) = a\cos(kx), k = 0, 1, 2, ...$ Further analysis shows that $H_3(a\cos(kx)) = 0$ and $\delta H_3(a\cos(kx)) =$

$\frac{a^2}{2}\cos(2kx)$. In other words, the value of $\delta[H_2 + H_3](acos(kx))$ shows Stokes wave phenomenon: The crest of the wave is steeper and the trough is flatter, see fig(4.1) below. This motivates us to try Fourier expansion,namely

$$\eta(x) = a\cos(x) + a^2 \sum_{k>1} \beta_k(a)\cos(kx),$$

as our ansatz for the *AB*-relative equilibrium profile. With this expression for η, the Momentum and Hamiltonian functionals are now given by:

$$M = 4\pi\sqrt{g}\sum_{k=1}^{\infty} \sqrt{k} \mid \eta_k \mid^2$$

$$H = 4\pi g \left\{ \sum_{k=1}^{\infty} \mid \eta_k \mid^2 + \sum_{k,l=1}^{\infty} \sqrt{kl}\eta_k\eta_l\eta_{k+l} \right\}$$

where η_k denoted the k^{th} Fourier coefficient at our series expansion and given explicitly as $\eta_1 = \frac{a}{2}$, $\eta_k = \frac{a^2\beta_k}{2}$ for $k > 1$. Furthermore, equation(4.22) can now be expressed as:

$$\begin{aligned} \frac{\mu}{\sqrt{g}} \sum_{k>0} \beta_k \sqrt{k}\cos(kx) &= \sum_{k>0} \beta_k \cos(kx) + \frac{1}{4}\sum_{k=1}^{\infty} k\beta_k^2 a^2 \cos(2kx) \\ &+ \frac{1}{4} \sum_{\substack{m,n>0 \\ m\neq n}} \sqrt{mn}\beta_m\beta_n a^2 \cos((m+n)x) \\ &+ \frac{1}{2}\sum_{m>1}\sum_{\substack{k>0 \\ k<m}} \sqrt{k(m-k)}\beta_k\beta_m a^2 \cos((m-k)x). \end{aligned}$$

The problem is to determine the value of $\{\mu, \beta_k, k = 2, 3, ...\}$ which satisfy the last equation for a given amplitude (a). Truncating the number of modes up to k and projecting to the space span $\{\cos(nx), n = 2, 3, ..., k\}$ for the same value of n at the left and right hand side of the last equation then it gives us a following set of equation:

$$\begin{aligned} \frac{\mu_k\sqrt{s}\beta_s}{\sqrt{g}} &= \beta_s + \frac{1}{2}\sum_{j=1}^{(s-1)/2} \sqrt{j(s-j)}\beta_j\beta_{s-j}a^2 \\ &+ \frac{1}{2}\sum_{j=1}^{k-s} \sqrt{js}\,\beta_{j+s}\beta_s a^{-s+j+2} \qquad \text{s is odd} \end{aligned} \tag{4.23}$$

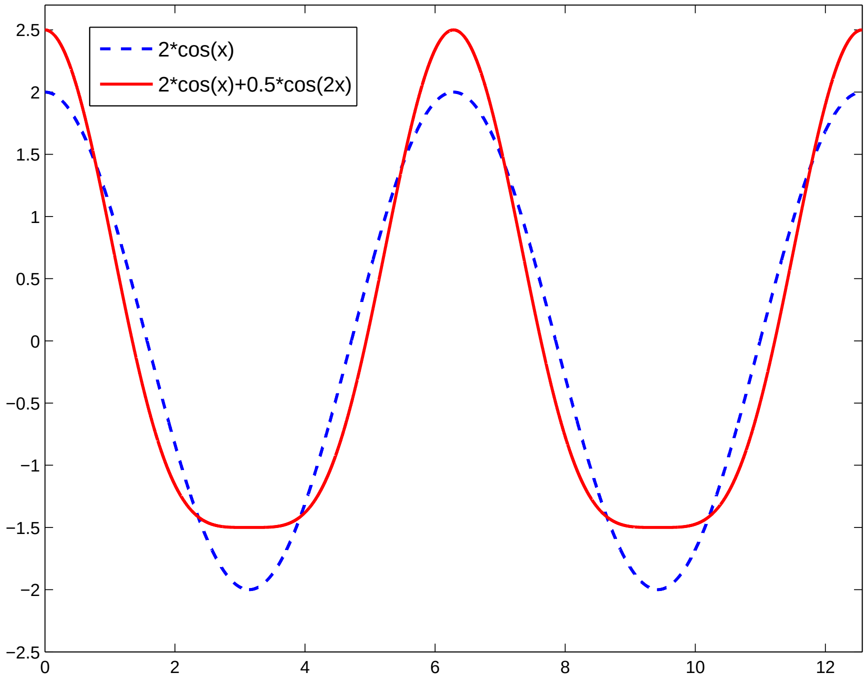

Figure 4.1: Stokes wave phenomenon: the crest of the wave becomes steeper and the trough becomes flatter.

$$\frac{\mu_k\sqrt{s}\beta_s}{\sqrt{g}} = \beta_s + \frac{1}{8}s\beta_{(s/2)}^2 a^2 + \frac{1}{2}\sum_{j=1}^{s/2-1}\sqrt{j(s-j)}\beta_j\beta_{s-j}a^2 + \frac{1}{2}\sum_{j=1}^{k-s}\sqrt{js}\,\beta_{j+s}\beta_j a^{-s+j+2} \qquad \text{s is even} \tag{4.24}$$

This system of equation consists of k equations with k unknows, $\{\mu, \beta_j, j = 2, .., k\}$. We solve the last system of equation using Newton's method.

In fact, we can get exactly the same system of equation as in $(4.23 - 4.24)$ by using following approach as well. Suppose A is defined as $(H - \lambda M)$,where H and M are the Hamiltonian and Momentum defined as in section(4.3.1). If A is restricted on the space $L = \text{span}\{\cos(x), \cos(2x), ..., \cos(kx)\}$ then the extremum of A for $\eta(x) = a\cos(x) + a^2\sum_{n=1}^{n=k}\beta_n(a)\cos(nx)$ has to satisfy the necessary condition which is given by $\nabla A(\eta_k) = 0$, or explicitly $\partial_\lambda A = \partial_{\beta_j}A = 0, j = 2, 3, .., k$. These lead to the equation (4.23-4.24). In other words, instead of calculating variational derivative of Hamiltonian and Momentum, we can just calculate the partial derivative of the

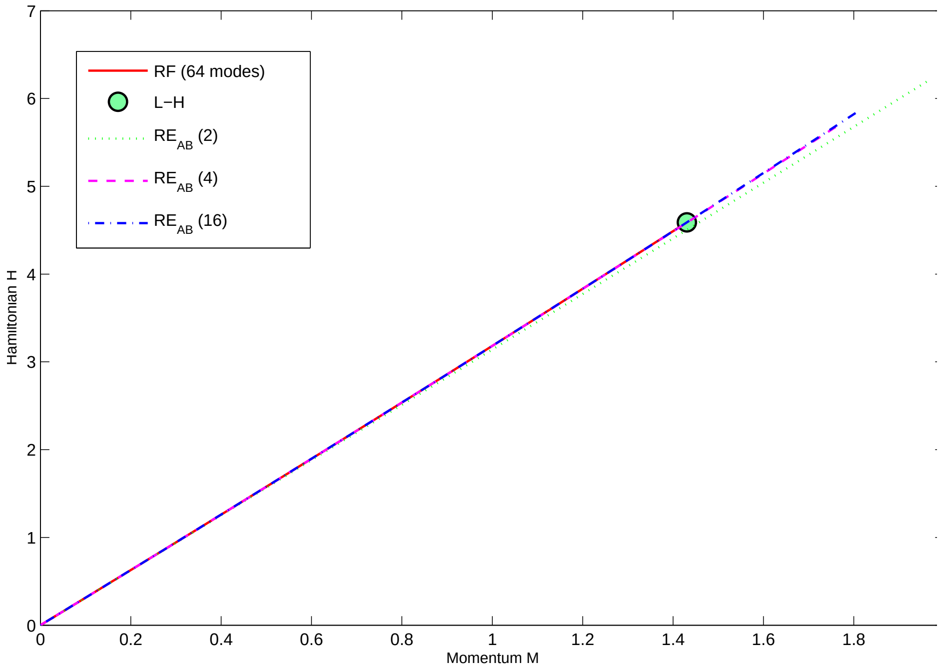

Figure 4.2: Curves of some wave profiles in MH-plane

restricted H and M as what we emphasized recently and got the system that need to be solved.

4.3.3 *Comparison of Rainey and Longuet-Higgins Approximation, Rienecker Fenton Approximation, AB-Relative Equilibria*

Figure (4.2) and (4.3) show the curves of relative equilibria of $2, 4$ and 16 terms expansion together with the curve of the simulation of Rienecker-Fenton for 64 fourier modes in MH-plane.

From the plot, we observed that a curve of Fourier expansion as a candidate of the relative equilibria for AB-equation close to the curve of Rieneker Fenton. By taking more terms in the series expansion, the relative equilibria curve will be closer to the Rieneker Fenton's.

The value of the Momentum for the Rainey and Longuet-Higgins approximation is 1.4306. Using this value of Momentum, we make a comparison of AB-relative equilibria profile, Longuet-Higgins approximation and Rienecker Fenton approximation, as can be seen in figure (4.4). This plot shows that the relative equilibria of

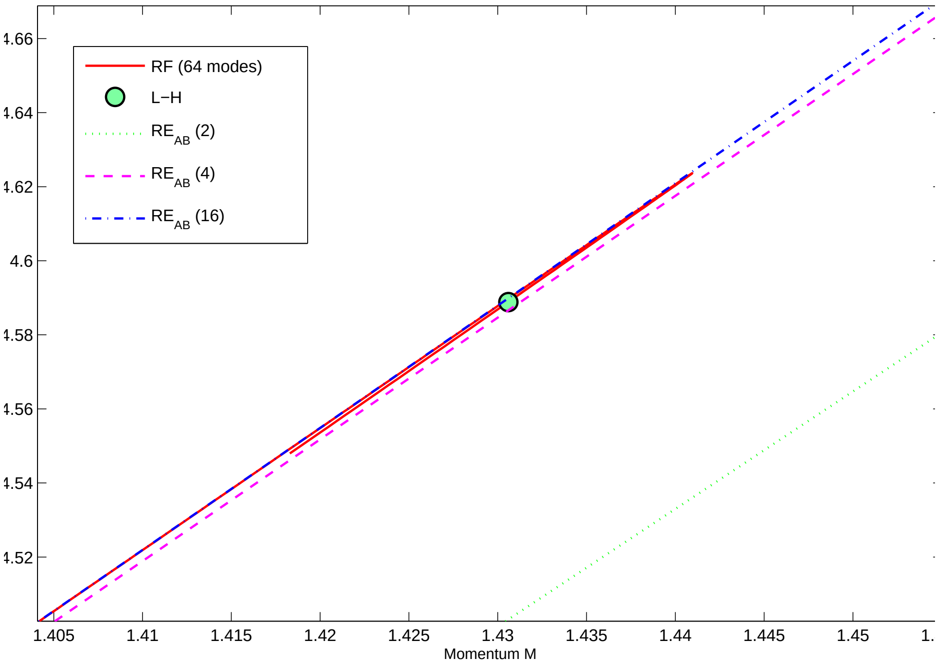

Figure 4.3: Zooming of Curves of some wave profiles in MH-plane

AB-equation using 16 modes is exactly the same as the profile proposed by Rainey and Longuet Higgins except on the small neighborhood of the maximal crest position. The plot of the Fourier coeffficients, β_k, using 16 modes expansion and for the value of $m = 1.4306$ are presented in figure(4.5)

4.3.4 *A Non-well posed problem*

The optimization model in chapter 3 for finding maximal high crest profile is no longer valid for deep water (by using AB-equation) at least for the quadratic part of the Hamiltonian. We get that an unbounded profile will be a solution for such optimization. This is not the solution that is expected. The following analysis clarifies this statement.

Considering the following problem: Denote the maximal crest height functional for functions η by

$$C(\eta) := \max_x \eta(x).$$

Then the extremal problem can be written as

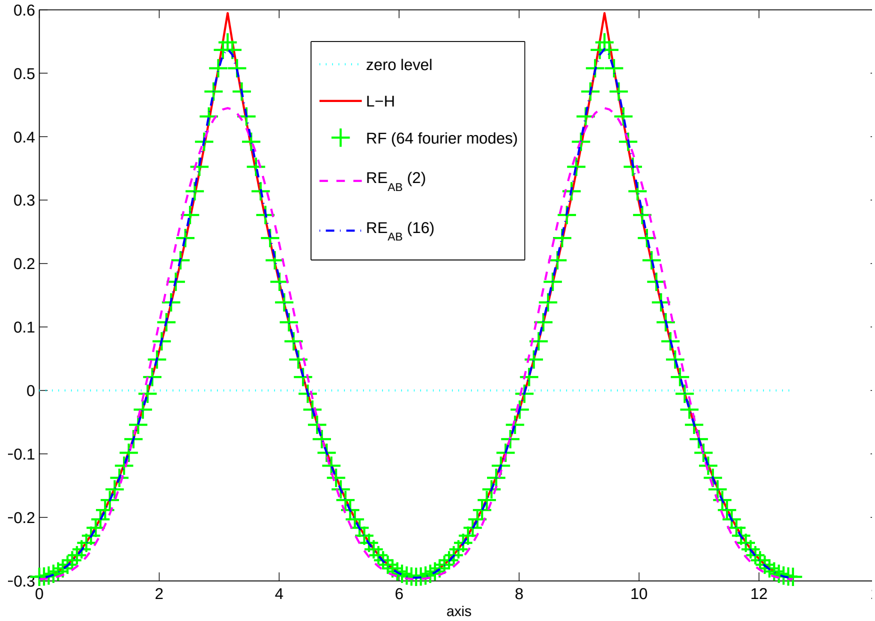

Figure 4.4: Some wave profiles for $M = 1.4306$

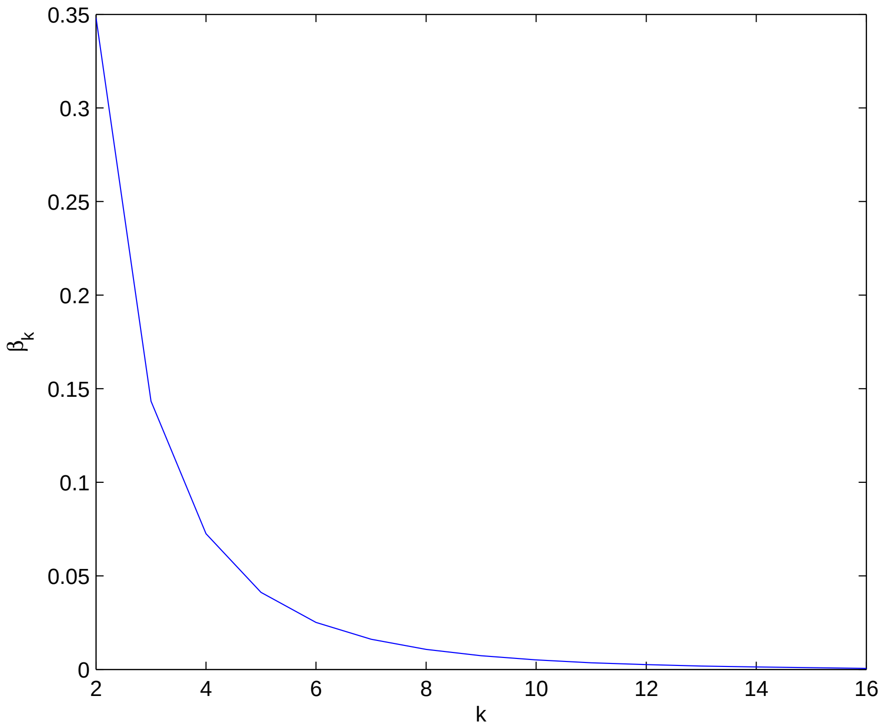

Figure 4.5: Plot of Fourier coefficients, β_k, for the $RE_{AB}(16)$ $M = 1.4306$

$$\max_{\eta} \{C(\eta) \mid \eta \in \mathcal{C}\}, \tag{4.25}$$

where the constraint set is given by

$$\mathcal{C}(h,m) = \{\eta \text{ is} L - \text{periodic}, \, av(\eta) = 0, H_2(\eta) = h, M(\eta) = m\}$$

with the average, Momentum and Hamiltonian defined respectively as follows:

$$av(\eta) = \int_{-L/2}^{L/2} \eta dx$$

$$M(\eta) = \sqrt{g} \int_{-L/2}^{L/2} \eta B \eta \, dx$$

$$H_2(\eta) = g \int_{-L/2}^{L/2} \eta^2 \, dx$$

here B denotes the pseudo differential operator which has symbol $\check{B}(k) = \sqrt{\mid k \mid}$ in the Fourier space.

Following the same technique as in section 3.3, the feasible curve in the MH-plane is given by a straight line $M\sqrt{g} = H_2$ and the feasible area requires $M\sqrt{g} \geq H_2$. Since a profile along the feasible curve is a solution of

$$\min_{\eta} \{M \mid H_2 = h, av(\eta) = 0\},$$

we get that such profiles are described by pure harmonic functions, namely $a\cos(kx)$.

In order to get the profile inside the feasible area, we need the governing equation which involves the variational derivative of the crest height functional, which is given by Dirac's delta function centered at the position of maximal crest height, say at x_{max},

$$\delta C(\eta) = \delta_{Dirac}(x - x_{max}).$$

Then the Lagrange's multiplier rule discussed in Chapter (2) leads to the equation

$$\delta_{dirac}(x - x_{\max}) = \{2g\lambda_H + 2\sqrt{g}\lambda_M B\} \, \eta + \mu. \tag{4.26}$$

Transforming the last equation into the Fourier space, we get:

$$\sum_k \left(\frac{1}{2\pi} e^{-ikx_{max}} - \left(2g\lambda_H + 2\sqrt{g}\lambda_M \sqrt{\mid k \mid}\right) \eta_k \right) e^{ikx} = \mu. \tag{4.27}$$

According to this equation, for $k \neq 0$, $\left(\frac{1}{2\pi}e^{-ikx_{max}} - \left(2g\lambda_H + 2\sqrt{g}\lambda_M\sqrt{|\,k\,|}\right)\eta_k\right) = 0$. Or equivalently, $\eta_k = \frac{e^{-ikx_{max}}}{4\pi\left(g\lambda_H+\lambda_M\sqrt{g}\sqrt{|k|}\right)}$ But since it is assumed that the mean water level is zero then $\eta_0 = 0$ and the solution of equation(4.25), is then given by

$$\eta(x) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{2g\lambda_H + 2\lambda_M\sqrt{g}\sqrt{k}}\cos k(x - x_{\max})$$

Note that at the position of the maximal crest height x_{max}, the solution becomes

$$\eta(x) = \frac{1}{4\pi\sqrt{g}}\sum_{k=1}^{\infty}\frac{1}{\sqrt{g}\lambda_H + \lambda_M\sqrt{k}}$$

This summation diverges as $k \to \infty$.

4.3.5 *Bootstrap argument*

In this section, we use a method, the so called bootstrap argument, to analyze the relative equilibrium of *AB*-equation and see whether it is possible or not to conclude that the solutions of the equation belong to the class of a smooth function space. The Following simple example illustrates the method.

Suppose $u \in C(\Omega)$, and we are interested in investigating the function class that solutions of the following differential equation belong to:

$$u_{xx} = \lambda(u + u^2),\ \lambda \in \mathbf{R}. \tag{4.28}$$

Since u is a continuous function then u^2 is also a continuous function and adding it with a continuous function and multiply it with some constant (namely λ) will give us a continuous function. Hence, the right hand side of equation(4.28) is a continuous function and equation(4.28) implies that the second derivative of u has to be a continuous, which also means that $u \in C^2(\Omega)$. Using the fact that $u \in C^2(\Omega)$ and doing the same method as before, we obtain that u now belongs to the class of $C^4(\Omega)$. Applying this method n times gives us the conclusion that $u \in C^{2n}(\Omega)$. If we let $n \to \infty$ then we obtain the result that a function u that satisfies equation(4.28) is infinitely continuous differentiable.

From the previous section, we find that the profiles that satisfy the linearized relative equilibrium equation are given by pure harmonic functions which are in the class of C^∞. Here, we argue differently by using bootstrap argument and see that this method successfully proves the continuity requirement for the profile of this linear equation. First, we define a class of function

$$F^p := \left\{ f \,\middle|\, \sum_{k=-\infty}^{\infty} |k^p f_k| < \infty \right\},$$

with f_k denotes the k^{th} Fourier coefficient. Notice that $F^p \subset C^p$ (a generalization of theorem(1) in chapter 2 which states that F^0 is a subset of C^0).

Suppose $\eta \in C^0$, then $\eta \in F^0$ by the remark of lemma(3) in chapter 2. Hence the right hand side of the linearized relative equilibrium equation, $2\frac{\mu}{\sqrt{g}}B\eta = 2\eta$ with $\check{B}(k) = \sqrt{|\, k \,|}$, is also a function in F^0. It means that $B\eta$ has to be in F^0, or equivalently $\sum_{k=-\infty}^{\infty} |\sqrt{|k|}\eta_k| < \infty$. Therefore $\eta \in F^{1/2}$. Using the fact that $\eta \in F^{1/2}$, then $B\eta \in F^{1/2}$ which means that $\sum_{k=-\infty}^{\infty} |k\eta_k| < \infty$. Therefore, we conclude that $\eta \in F^1$. Continuing this argument n times will give the fact that $\eta \in F^{n/2}$. If we let $n \to \infty$, then we obtain that $\eta \in F^\infty$. Since $F^\infty \subset C^\infty$, the profiles for the linearized relative equilibrium is also an element in C^∞. So we conclude that the solution of the linearized relative equilibrium of AB-equation will be in C^∞.

We use the same assumption that $\eta \in C^0$ (which implies $\eta \in F^0$) to analyze the full nonlinear relative equilibrium of AB-equation,

$$2\frac{\mu}{\sqrt{g}}B\eta = \left\{ \underbrace{2\eta}_{\delta H_2} + \underbrace{\frac{1}{2}\left\{(B\eta)^2 - (A\eta)^2\right\} + \left\{A(\eta A\eta) + B(\eta B\eta)\right\}}_{\delta H_3} \right\}. \tag{4.29}$$

The assumption $\eta \in F^0$ implies that $B\eta \in F^{-1/2}$. Hence the left hand side of the equation is an element of $F^{-1/2}$. The nonlinear term, δH_3, which involves convolution in Fourier space is not so easy to be determined to which function space it belongs. Here we give a conjecture that $\delta H_3 \in F^{-1/2}$, for $\eta \in F^0$.

For a specific continuous function $\eta(x) = \sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} e^{ikx}$, which is an element in F^0 with $\eta_k = \frac{1}{1+k^2}$, the k^{th} Fourier coefficient of δH_3 is given by (see appendix for the derivation):

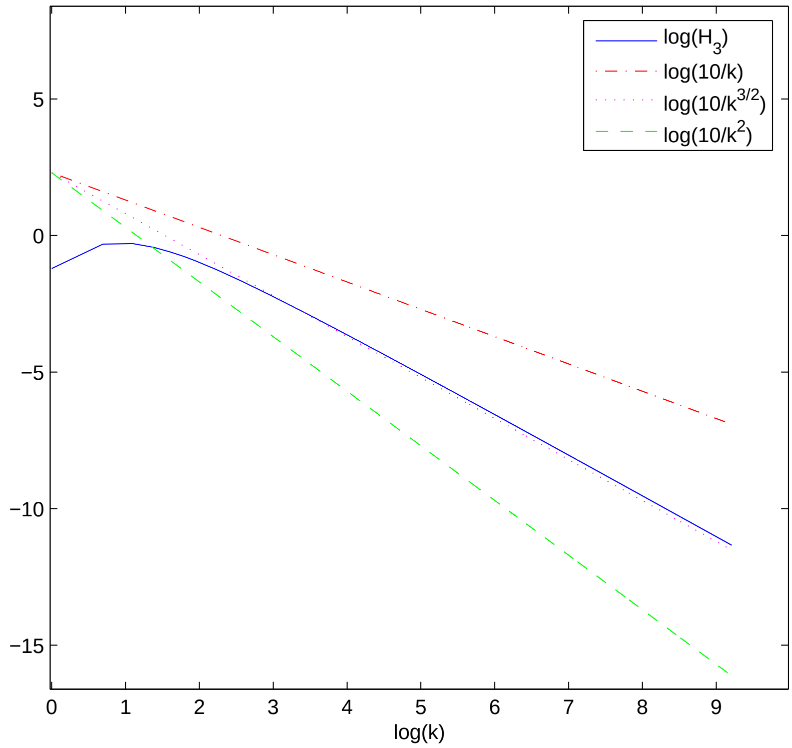

Figure 4.6: Plot of $\{\delta H_3\}_k$ for a function in F^0 with $\eta_k = \frac{1}{(1+k^2)}$

$$\begin{aligned}\{\delta H_3\}_k &= 2\left\{\sum_{l>0}\sqrt{kl}\,\eta_{k+l}\eta_{-l} + \sum_{\substack{l>0\\l<k}}\sqrt{l(k-l)}\,\eta_{k-l}\eta_l\right\}\\ &= 2\left\{\sum_{l>0}\sqrt{kl}\frac{1}{(1+(k+l)^2)(1+l^2)} + \sum_{\substack{l>0\\l<k}}\sqrt{l(k-l)}\frac{1}{(1+(k-l)^2)(1+l^2)}\right\}.\end{aligned}$$

Figure(4.6) shows plot of $\{\delta H_3\}_k$ in the logarithmic scale. We observe that for large value of k, $\{\delta H_3\}_k$ behaves in the same order as $\frac{10}{k^{3/2}}$.

According to this conjecture, we have that $\delta H = \delta H_2 + \delta H_3$ is an element of $F^{-1/2}$. Hence the bootstrap argument cannot be applied in this case, since the left and right hand side equation(4.29) has the same order ($p = -1/2$). Therefore we cannot conclude anything from this method about the continuity requirement of the exact solution of the equation.

4.3.6 *Functional analysis in MH-plane*

In this section, we will exploit functional H and M to see what we can say about lower bound for $\max\{H|M = m\}$. First notice that:

$$H(\alpha\eta) = \alpha^2 H_2(\eta) + \alpha^3 H_3(\eta) \tag{4.30}$$

$$M(\alpha\eta) = \alpha^2 M(\eta). \tag{4.31}$$

Suppose m_* is an arbitrary positive value of Momentum, with η_* the relative equilibrium profile correspondent to m_* and the related speed μ_*,we denote

$$\mathcal{H}(m) := \max\left\{ H|M = \underbrace{\alpha^2 m_*}_{=m} \right\}.$$

Using equation(4.30) and (4.31), then we can establish the following estimation,

$$\begin{aligned}
\mathcal{H}(m) \geq H(\alpha\eta_*) &= \alpha^2 H_2(\eta_*) + \alpha^3 H_3(\eta_*) \\
&= \frac{m}{m_*} H_2(\eta_*) + \left(\frac{m}{m_*}\right)^{3/2} H_3(\eta_*) \\
&:= h(m).
\end{aligned}$$

It shows we conclude that $h(m)$ is a lower bound for $\mathcal{H}(m)$. Figure(4.7) shows plot of $h(m)$ together with Relative equilibrium curve (above) and their difference (below), for the value $m_* = m_{LH} = 1.4306$ and using 16 modes. Notice that from the plot of figure(4.7), we see that $(H - h)(m) \geq 0$ for all m. It means that there is a possibility that relative equilibrium curve in MH-plane is in fact a constraint maximizer of H.

If we denote $< f, g >$ as an L^2 inner-product of function f and g, we obtain that

$$\langle \delta H_2, \eta \rangle = 2H_2(\eta),$$

$$\langle \delta H_3, \eta \rangle = 3H_3(\eta),$$

$$\langle \delta M, \eta \rangle = 2M(\eta).$$

From the relative equilibrium equation we know that

$$< \delta H, \eta > = < \delta H_2, \eta > + < \delta H_3, \eta > = \mu < \delta M, \eta > .$$

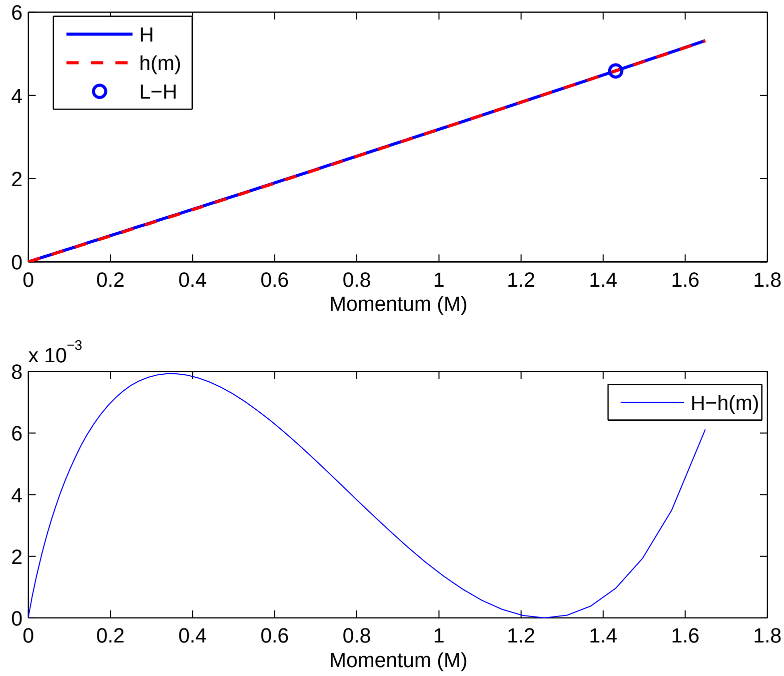

Figure 4.7: Plot of the lower bound $h(m)$ together with H for RE_{AB} using 16 modes and $m_* = m_{LH}$(above). The plot below shows their difference.

Hence,

$$2H_2(\eta) + 3H_3(\eta) = 2\mu M(\eta). \tag{4.32}$$

We have that

$$\begin{aligned} h(m) &= \frac{m}{m_*}\overbrace{H_2(\eta_*)}^{h_2^*} + \left(\frac{m}{m_*}\right)^{3/2}\overbrace{H_3(\eta_*)}^{h_3^*} \\ &= \underbrace{h_2^* + h_3^*}_{=h(m_*):=h_*} + \left(\frac{m}{m_*} - 1\right)h_2^* + \underbrace{\left[\left(\frac{m}{m_*}\right)^{3/2} - 1\right]h_3^*}_{=\frac{3}{2}\frac{h_3^*}{m_*}(m-m_*)+\frac{3}{8}\frac{h_3^*}{m_*^2}(m-m_*)^2+\ldots} \\ &= h_* + \frac{h_2^*}{m_*}(m - m_*) + \frac{3}{2}\frac{h_3^*}{m_*}(m - m_*) + \frac{3}{8}\frac{h_3^*}{m_*^2}(m - m_*)^2 + \ldots \end{aligned}$$

Using equation(4.32), $h_2^* + \frac{3}{2}h_3^* = \mu_* m_*$, then the last expression can be rewritten as:

$$h(m) = h_* + \mu_*(m - m_*) + \frac{3}{8}\frac{h_3^*}{m_*^2}(m - m_*)^2 + \ldots,$$

hence

$$h(m) - \mu_* m = h_* - \mu_* m_* + \frac{3}{8}\frac{h_3^*}{m_*^2}(m - m_*)^2 + \ldots \tag{4.33}$$

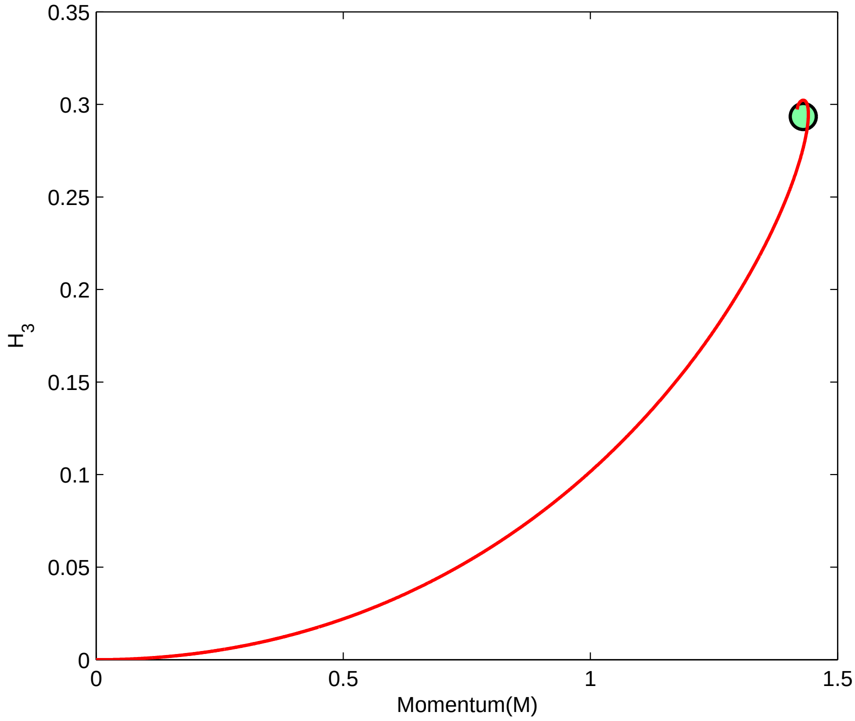

Figure 4.8: Plot of $H_3(\eta)$ for the Rienecker Fenton profiles.

The value of μ_* appears in this equation can be interpreted as the slope of the tangent line at m_* in the MH-plane (it can be seen from Lagrange's multiplier rule for finding relative equilibrium equation). Hence $h(m) - \mu_* m = h_* - \mu_* m_*$ is the equation of the tangent line of $h(m)$ curve in MH-plane at m_*. From equation(4.33), we conclude that for a positive value of h_3^*, $h(m)$ lies above its tangent line at every m and so does $\mathcal{H}(m)$.

For the case of Rienecker Fenton model, the value of H_3 is always positive (see figure(4.8)). Hence, using the above remark, both the curve of $h(m)$ and $\mathcal{H}(m)$ in MH-plane must lie above the tangent line of $h(m)$ in every position of m.

As we already noticed before, that μ can be seen as the slope of the tangent line, but from previous sections, the value of μ also describes the wave speed. Hence the tangent line of $h(m)$ at each position m is related to the wave speed at the same position. Figure(4.9) and (4.10) shows the plot of the wave speed as a function of Momentum. From this plot, we see that the curve of the wave speed of Rienecker Fenton is monotonically increasing before reaching $m = 1.415$ and rapidly decreasing after that point. This observation tells us that after $m = 1.415$, the related profile

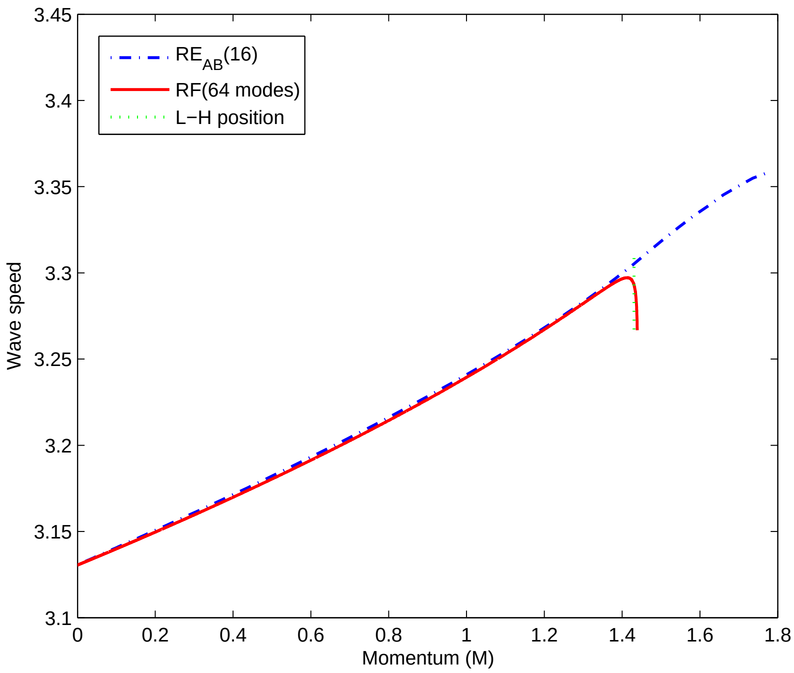

Figure 4.9: Plot of wave speed respect to Momentum for the Rienecker Fenton profiles and relative equilibrium AB equation using 16 modes.

of η cannot be a maximizer of Hamiltonian for a given value of Momentum, since a curve that has monotonically decreasing gradient on a certain domain $[a, b]$ but positive must lie below its tangent line.

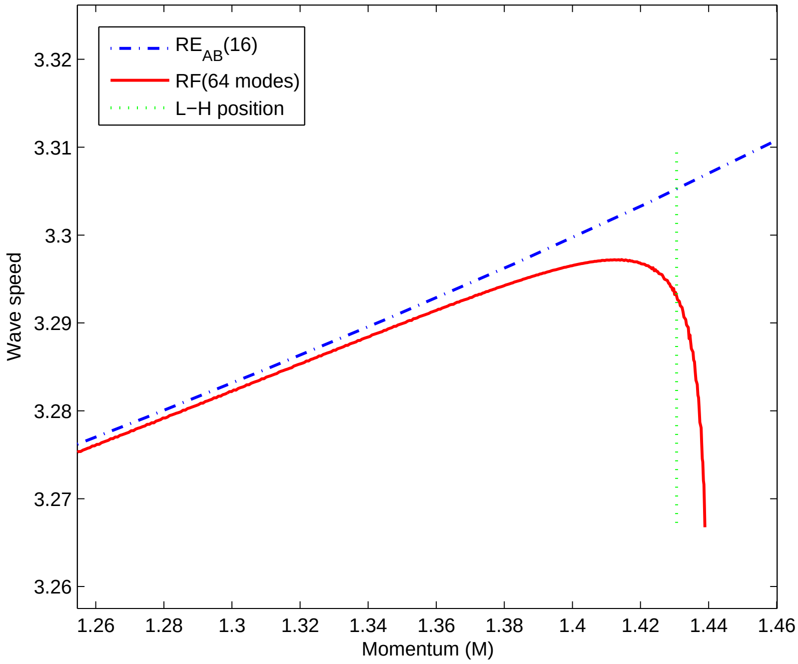

Figure 4.10: Zooming of the previous plot.

Chapter 5

Conclusions and Recommendations

This last chapter will sum up our discussion in this thesis and also give recommendations for further possible work and investigation.

A new water wave model for determining maximal high crest profile had been derived by using Momentum and Hamiltonian as constraints. We start from a very simple case model which is actually a linearized KdV equation and go further to infinitely depth water wave case.

In the linearized KdV model, we found that there is an area, so called feasible area, for which the wave profile will exist by prescribing the value of Momentum and Hamiltonian. Along the boundary curve in the MH-plane, the profile uniquely described by a pure harmonic function. Furthermore, the profile inside the feasible area has also already been obtained. It is known that the profile will have a certain corner at the maximal crest position.

AB-equation for infinitely depth water was considered to investigate a maximal high crest profile that occurred in deep water. As in the case of linearized KdV, we also get a certain curve in the MH-plane which is called as relative equilibria curve. By using a Fourier series technique, a profile along this curve can be approximated. We make a comparison of AB-equation, Longuet-Higgins approximation for Stokes 120^0 wave and also a simulation proposed by Rienecker-Fenton. As a result, we see in the MH-plane that the curve of Relative equilibria AB-equation are very close to the curve of Rienecker-Fenton (which is known as the exact numerical simulation for steady water wave problem). Another observation is that the profile of AB-relative equilibria (for the same value of Momentum as Longuet Higgins) is getting closer by increasing the value of modes to the profile of Longuet-Higgins (which is claimed exact up to 99.7% for the Stokes wave).

A further investigation for infinitely depth water wave using the approach given in this final report can be done. It is still not clear yet what kind of functional that is possible to be used to make the optimization related to maximal high crest profile well posed. In the linearized KdV model, there are two branches, catenary and trigonometric, for the profile inside the feasible area depending on the value of Momentum and Hamiltonian, maybe the same case appears in the study of deep water wave model. An further analysis for the turning back edges on the Rienecker-Fenton simulation can also be investigated even it seems that this phenomenon happened due to the error of numerical solver.

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Appendix A

Derivative of Maximal Crest Height Functional

Suppose $u(x)$ is a continuous function in a close interval $[-L, L]$ and the maximum is achieved at $x_{\max}$ and we denote $F(u) := \max u(x)$. By the definition of first variational derivative,

$$\begin{aligned}
\delta F &= \frac{d}{d\epsilon} F(u+\epsilon v)\bigg|_{\epsilon=0} \\
&= \lim_{\epsilon \to 0} \frac{F(u+\epsilon v) - F(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\max(u+\epsilon v) - \max(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{(u+\epsilon v)(x_{max}+\mu) - u(x_{max})}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{(u+\epsilon v)(x_{max}) + \mu(u+\epsilon v)'(x_{max}) - u(x_{max})}{\epsilon}
\end{aligned}$$

As $\epsilon \to 0$,$\mu \to 0$, and using the fact that $u'(x_{max}) = 0$ then

$$\delta F = \lim_{\epsilon \to 0} \frac{\epsilon v(x_{max})}{\epsilon} = v(x_{\max})$$

and by the definition of Delta dirac function, this last expression can be written as:

$$\delta F = \int_{-\infty}^{\infty} \delta(x - x_{max}) v(x) dx.$$

From here, we show that the variational derivative of maximal crest height functional is a delta dirac function centered at the position of maximal crest height.

Appendix B

Continuity of Fourier Series

Here the continuity of Fourier series in theorem(1) is proved. Let $f(x)$ denote the Fourier series $\sum_{j=-\infty}^{\infty} f_j e^{ijkx}$ and assume that the series is absolutely convergent, we prove that $f(x)$ is continuous for every $a \in \mathbf{R}$, that is, we show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$.

Define the partial sums

$$S_N(x) = \sum_{j=-N}^{N} f_j e^{ijkx}.$$

Then there holds that

$$|f(x) - S_N(x)| = \left| \sum_{|j|>N} f_j e^{ijkx} \right| \leq \sum_{|j|>N} |f_j|. \tag{B.0.1}$$

By assumption $\sum_{j=-\infty}^{\infty} |f_j| < \infty$, so that

$$\sum_{|j|>N} |f_j| = \sum_{j=-\infty}^{\infty} |f_j| - \sum_{j=-N}^{N} |f_j| \to 0 \text{ as } N \to \infty.$$

Consequently there is a large enough positive integer N_1 such that for every $N > N_1$, we get $\sum_{|j|>N} |f_j| < \epsilon/3$. Considering equation(B.0.1), we conclude that $|f(x) - S_{N_1}(x)| \leq \sum_{|j|>N_1} |f_j| < \epsilon/3$ for all x.

The partial sum $S_{N_1}(x)$ is a finite sum of continuous functions, hence is itself a continuous. For a given ϵ, therefore a $\delta > 0$ can be found such that $|S_{N_1}(x) -$

$S_{N_1}(a)| < \epsilon/3$ whenever $|x - a| < \delta$. Finally then,

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - S_{N_1}(t) + S_{N_1}(t) - S_{N_1}(a) - (f(a) - S_{N_1}(a))| \\ &\leq |f(x) - S_{N_1}(t)| + |S_{N_1}(t) - S_{N_1}(a)| + |(f(a) - S_{N_1}(a)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Appendix C

AB-equation in Infinite Water Depth

A work of van Groesen and Andonowati in [2] is reviewed for convenience. We consider the evolution of surface elevation of an ideal incompressible fluid in one horizontal direction x. The dynamics of the irrotational fluid is described by the surface elevation $\eta(x,t)$ and the fluid potential at the free surface, $\phi(x,t) = \Phi(x, z = \eta, t)$. The variation of the action principle, i.e:$\delta A(\eta,\phi) = 0$ with

$$A(\eta,\phi) = \int \left[\int \phi.\partial_t \eta dx - H(\eta,\phi) \right] dt \tag{C.0.1}$$

leads to the following dynamics equations:

$$\partial_t \eta = \delta_\phi H(\eta,\phi) \tag{C.0.2}$$

$$\partial_t \phi = -\delta_\eta H(\eta,\phi) \tag{C.0.3}$$

The Hamiltonian is the total energy, the sum of potential energy and kinetic energy, expressed in the variables η, ϕ. The potential energy is calculated with respect to the undisturbed water level and the Hamiltonian is then expressed as:

$$H(\eta,\phi) = \int \frac{1}{2} g \eta^2 dx + K(\phi,\eta). \tag{C.0.4}$$

The kinetic energy is formally given by:

$$K(\phi,\eta)=\int\int\frac{1}{2}\left|\nabla\Phi\right|^2 dzdx$$

where Φ satisfies the Laplace equation in the fluid interior and the surface condition $\Phi=\phi$ at $z=\eta$ and the impermeable bottom boundary condition. Denote the fluid potential and the vertical velocity at the still water as

$$\phi_0=\Phi(x,z=0,t),W_0=\partial_z\Phi(x,z=0,t)$$

respectively. Splitting the kinetic energy expression till the still water level and a term involves the actual surface elevation as

$$K=\int\left[\int^{z=0}+\int_{z=0}^{z=\eta}\right]\frac{1}{2}\left|\nabla\Phi\right|^2 dzdx,$$

we observe that

$$\int\int^{z=0}\frac{1}{2}\left|\nabla\Phi\right|^2 dzdx=\int\frac{1}{2}\phi_0 W_0 dx.$$

Taking the approximation of the lowest order in the wave height:

$$\int\int_{z=0}^{z=\eta}\frac{1}{2}\left|\nabla\Phi\right|^2 dzdx\approx\frac{1}{2}\int\eta\left[(\partial_x\phi_0)^2+W_0^2\right]dx \tag{C.0.5}$$

Then we get the following approximation for the Hamiltonian:

$$H(\eta,\phi)\approx\int\left\{\frac{1}{2}g\eta^2+\frac{1}{2}\phi_0 W_0+\frac{1}{2}\eta\left[(\partial_x\phi_0)^2+W_0^2\right]\right\}dx. \tag{C.0.6}$$

As a result of solving Laplace problem at the still water level by prescribing the potential, we get:

$$W_0=-\frac{1}{g}C^2\partial_x^2\phi_0.$$

Here C is the phase velocity operator, i.e. the pseudo-differential operator with symbol $\hat{C}$ related to the dispersion relation $\Omega(k)$:

$$\hat{C}(k)=\frac{\Omega(k)}{k}$$

where $\Omega(k)=c_0 k\sqrt{\frac{\tanh(kh_0)}{kh_0}}$,$c_0=\sqrt{gh_0}$. For the case of infinitely depth water, we have:

$$\hat{C}_\infty(k)=\frac{\Omega_\infty(k)}{k}=\sqrt{\frac{g}{|k|}}.$$

For a unidirectionalisation purpose we take $\phi_0 = g\partial_x^{-1}C^{-1}\eta$ and by introducing a skew-symmetric operator A and a symmetric operator B as follows:

$$A = C\partial_x/\sqrt{g}, B = \sqrt{g}C^{-1},$$

we can rewrite ϕ_0 and W_0 as follows: $\phi_0 = \sqrt{g}A^{-1}\eta, W_0 = -\sqrt{g}A\eta$. Substituting these relations into hamiltonian (C.0.6), the result is explicitly given by

$$\overline{H}(\eta) = g\int \left[\eta^2 + \frac{1}{2}\eta\left\{(A\eta)^2 + (B\eta)^2\right\}\right] dx. \tag{C.0.7}$$

The same restriction in the action, using $\phi = \phi_0 + \eta W_0$ results into

$$\int\int \phi\partial_t\eta\, dxdt = \sqrt{g}\int\int \left[A^{-1}\eta - \eta\, A\eta\right]\partial_t\eta\, dxdt.$$

Taken together, the action functional now rewritten as:

$$\overline{A}(\eta) = \int\left[\int \sqrt{g}\left[A^{-1}\eta - \eta A\eta\right]\partial_t\eta dx - \overline{H}(\eta)\right] dt. \tag{C.0.8}$$

Vanishing the variational derivative of the above action functional lead to the evolution equation

$$\begin{aligned}\sqrt{g}\left[-2A^{-1}\partial_t\eta + A(\eta\partial_t\eta) + \eta A\partial_t\eta\right] &= \delta\overline{H}(\eta)\\ &= g\left[2\eta + \frac{1}{2}(A\eta)^2 - A(\eta A\eta) + \frac{1}{2}(B\eta)^2 + B(\eta B\eta)\right]\end{aligned}$$

Although this formulation is correct, but the expression involving $\partial_t\eta$ is rather complicated, we will simplify it by considering the lowest order of wave height the linear equation reads

$$-2\sqrt{g}A^{-1}\partial_t\eta = 2g\eta + O(\eta^2), \text{ i.e. } \partial_t\eta = -\sqrt{g}A\eta + O(\eta^2).$$

Substituting this approximation into the action, we get:

$$\int \sqrt{g}\left[A^{-1}\eta - \eta A\eta\right]\partial_t\eta dx = \int \left[\sqrt{g}A^{-1}\eta\partial_t\eta + g\eta(A\eta)^2\right] dx.$$

Then the total action functional is approximated correctly up to and including qubic terms in wave height. We write it as a modified action principle, reading explicitly

$$A_{mod}(\eta) = \sqrt{g}\int\left[\int A^{-1}\eta\partial_t\eta dx - \overline{H}_{mod}(\eta)\right] dt \tag{C.0.9}$$

where the modified hamiltonian contains a term from the original action and is given by

$$\overline{H}_{mod}(\eta) = \overline{H} - g\int \eta(A\eta)^2 dx = g\int \left[\eta^2 + \frac{1}{2}\eta\left\{-(A\eta)^2 + (B\eta)^2\right\}\right] dx. \quad \text{(C.0.10)}$$

The resulting equation $\delta A_{mod}(\eta) = 0$:

$$-2\sqrt{g}A^{-1}\partial_t\eta = \delta\overline{H}_{mod}(\eta) \quad \text{(C.0.11)}$$

will be called as the AB equation and is given by:

$$\partial_t\eta = -\sqrt{g}A\left[\eta - \frac{1}{4}(A\eta)^2 + \frac{1}{2}A(\eta A\eta) + \frac{1}{4}(B\eta)^2 + \frac{1}{2}B(\eta B\eta)\right] \quad \text{(C.0.12)}$$

Remark: the pseudo differential operator A and B can also be written as $\hat{A} = ik/\sqrt{|k|} = sign(k)\sqrt{|k|}, \hat{B} = \sqrt{|k|}$ in the fourier space.

Appendix D

AB-equation in Fourier space

The appearance of pseudo-differential operator in our equation of motion advice us to transform these equations into Fourier space. These transformation is presented here. For simplicity of writing, if it is not stated then the integral is taken over 1 period ,$[-\pi,\pi]$, and the summation is taken over the whole $spectrum(k)$. $\overline{u}$ denotes the conjugate of u. The real pseudo differential operator will be applied for the real value function with real Fourier coefficient.

$$\begin{aligned}
M &= g\int \eta C^{-1}\eta\, dx \\
&= g\int \eta\overline{\sum_k C^{-1}(k)\eta_k e^{ikx}}\, dx \\
&= \sqrt{g}\sum_k \overline{\eta_k}\sqrt{\mid k\mid}\int \eta e^{-ikx}\, dx \\
&= 2\pi\sqrt{g}\sum_k \sqrt{\mid k\mid}\overline{\eta_k}\eta_k \\
&= 4\pi\sqrt{g}\sum_{k>0}\sqrt{k}\,\eta_k^2
\end{aligned}$$

The last expression is done using the fact that for expansion in section(4.3.2), $\eta_k = \overline{\eta_k}$

$$\begin{aligned}
H &= g\int \left[\eta^2 + \frac{1}{2}\eta\left\{(B\eta)^2 - (A\eta)^2\right\}\right] dx \\
&= H_2(\eta) + H_3(\eta)
\end{aligned}$$

We derive the quadratic and the cubic terms separately as follows.

$$H_2(\eta) = g \int \eta^2 \, dx = 2\pi g \sum_k \eta_k^2$$

$$\begin{aligned}
\frac{g}{2}\int \eta(B\eta)^2 \, dx &= \frac{g}{2}\int \eta(B\eta)\overline{(B\eta)} \, dx \\
&= \frac{g}{2}\int \eta(B\eta)\sum_k \overline{\eta_k \sqrt{|\,k\,|} e^{ikx}} \, dx \\
&= \frac{g}{2}\sum_k \sqrt{|\,k\,|}\overline{\eta_k}\int \eta(B\eta)e^{-ikx} \, dx \\
&= \frac{g}{2}\sum_k \sqrt{|\,k\,|}\overline{\eta_k}\int (\eta e^{-ikx})\sum_m \sqrt{|\,m\,|}\overline{\eta_m} e^{-imx} \, dx \\
&= \frac{g}{2}\sum_k\sum_m \sqrt{|\,km\,|}\overline{\eta_k\eta_m}\int (\eta e^{-i(k+m)x}) \, dx \\
&= \pi g \sum_k\sum_m \sqrt{|\,km\,|}\overline{\eta_k\eta_m}\eta_{k+m} \\
&= \pi g \sum_k\sum_m \sqrt{|\,km\,|}\eta_k\eta_m\eta_{k+m}
\end{aligned}$$

$$\begin{aligned}
\frac{g}{2}\int \eta(A\eta)^2 \, dx &= \frac{g}{2}\int \eta(A\eta)\overline{(A\eta)} \, dx \\
&= \frac{g}{2}\int \eta(A\eta)\sum_k -i\, sign(k)\overline{\eta_k \sqrt{|\,k\,|} e^{ikx}} \, dx \\
&= \frac{g}{2}\sum_k -i\, sign(k)\sqrt{|\,k\,|}\overline{\eta_k}\int \eta(A\eta)e^{-ikx} \, dx \\
&= \frac{g}{2}\sum_k -i\, sign(k)\sqrt{|\,k\,|}\overline{\eta_k}\int (\eta e^{-ikx})\sum_m -i\, sign(m)\sqrt{|\,m\,|}\overline{\eta_m} e^{-imx} \, dx \\
&= -\frac{g}{2}\sum_k\sum_m sign(k)sign(m)\sqrt{|\,km\,|}\overline{\eta_k\eta_m}\int (\eta e^{-i(k+m)x}) \, dx \\
&= -\pi g \sum_k\sum_m sign(k)\, sign(m)\sqrt{|\,km\,|}\overline{\eta_k\eta_m}\eta_{k+m} \\
&= -\pi g \sum_k\sum_m sign(k)\, sign(m)\sqrt{|\,km\,|}\eta_k\eta_m\eta_{k+m}
\end{aligned}$$

$$\begin{aligned}
H &= H_2(\eta) + H_3(\eta) \\
&= \pi g \left\{ 2\sum_k \eta_k^2 + \sum_k\sum_m (1 + sign(k)\, sign(m)) \left(\sqrt{|\,km\,|}\eta_k\eta_m\eta_{k+m} \right) \right\} \\
&= 4\pi g \left\{ \sum_{k>0} \eta_k^2 + \sum_{k>0}\sum_{m>0} \sqrt{km}\,\eta_k\eta_m\eta_{k+m} \right\}
\end{aligned}$$

Relative equilibria for AB-equation in deep water is given by

$$2\mu C^{-1}\eta = \left\{2\eta + \frac{1}{2}\left\{(B\eta)^2 - (A\eta)^2\right\} + \left\{A(\eta A\eta) + B(\eta B\eta)\right\}\right\}.$$

If we take $\eta(x) = \sum_{k>0} a^k \alpha_k \cos(kx)$ as our ansatz, then transforming part by part of this equation gives us following equalities:

$$\begin{aligned} C^{-1}\eta &= \sum_k \frac{1}{\sqrt{g}}\sqrt{|\,k\,|}\eta_k e^{ikx} \\ &= \frac{1}{\sqrt{g}}\sum_{k>0}\sqrt{k}\alpha_k a^k \cos(kx) \end{aligned}$$

$$\begin{aligned} (A\eta)^2 - (B\eta)^2 &= \left(i\sum_k sign(k)\sqrt{|\,k\,|}\eta_k e^{ikx}\right)^2 - \left(\sum_k \sqrt{|\,k\,|}\eta_k e^{ikx}\right)^2 \\ &= -\left(\sum_k sign(k)\sqrt{|\,k\,|}\eta_k e^{ikx}\right)^2 - \left(\sum_k \sqrt{|\,k\,|}\eta_k e^{ikx}\right)^2 \\ &= -2\sum_k |\,k\,|\eta_k^2 e^{i2kx} - \sum_k\sum_{m\neq k}(1 + sign(k)\,sign(m))\sqrt{|\,km\,|}\eta_k\eta_m e^{i(k+m)x} \\ &= -\sum_{k>0} k\alpha_k^2 a^{2k}\cos(2kx) - \sum_{\substack{k,m>0\\ k\neq m}}\sqrt{km}\alpha_k\alpha_m a^{k+m}\cos((k+m)x) \end{aligned}$$

Using the fact that for $m > 0$,

$$A(\cos(mx)) = -\sqrt{|\,m\,|}\sin(mx)$$

$$B(\cos(mx)) = \sqrt{|\,m\,|}\cos(mx)$$

and that

$$2\sin(mx)\cos(nx) = \sin((m+n)x) + \sin((m-n)x)$$

$$2\cos(mx)\cos(nx) = \cos((m+n)x) + \cos((m-n)x)$$

we obtain:

$$\begin{aligned} \eta A\eta &= \left(\sum_{m>0}\alpha_m a^m\cos(mx)\right)\left(-\sum_{k>0}\alpha_k a^k\sqrt{k}\sin(kx)\right) \\ &= -\left(\sum_{k,m>0}\alpha_k\alpha_m a^{k+m}\sqrt{k}\sin(kx)\cos(mx)\right) \\ &= -\left(\sum_{k,m>0}\sqrt{k}\,\frac{\alpha_k\alpha_m a^{k+m}}{2}\left\{\sin((k+m)x) + \sin((k-m)x)\right\}\right) \end{aligned}$$

$$\begin{aligned}
\eta B\eta &= \left(\sum_{m>0}\alpha_m a^m\cos(mx)\right)\left(-\sum_{k>0}\alpha_k a^k\sqrt{k}\cos(kx)\right)\\
&= \left(\sum_{k,m>0}\alpha_k\alpha_m a^{k+m}\sqrt{k}\cos(kx)\cos(mx)\right)\\
&= \left(\sum_{k,m>0}\sqrt{k}\ \frac{\alpha_k\alpha_m a^{k+m}}{2}\left\{\cos((k+m)x)+\cos((k-m)x\right\}\right)
\end{aligned}$$

$$\begin{aligned}
A(\eta A\eta) = & -\sum_{k,m>0}\sqrt{k}\frac{\alpha_k\alpha_m a^{k+m}}{2}\\
&\left\{\sqrt{k+m}\cos((k+m)x)+\text{sign(k-m)}\sqrt{\mid k-m\mid}\cos((k-m)x)\right\}
\end{aligned}$$

$$B(\eta B\eta) = \sum_{k,m>0}\sqrt{k}\ \frac{\alpha_k\alpha_m a^{k+m}}{2}\left\{\sqrt{k+m}\cos((k+m)x)+\sqrt{\mid k-m\mid}\cos((k-m)x)\right\}$$

Now,we get that:

$$\begin{aligned}
A(\eta A\eta)+B(\eta B\eta) &= \sum_{m>1}\sum_{\substack{k>0\\k<m}}\sqrt{k\mid(k-m)\mid}\alpha_k\alpha_m a^{k+m}\cos((k-m)x)\\
&= \sum_{m>1}\sum_{\substack{k>0\\k<m}}\sqrt{k(m-k)}\alpha_k\alpha_m a^{k+m}\cos((m-k)x)
\end{aligned}$$