Multi-Objective Road Pricing Problem:
A Cooperative and Competitive Bilevel Optimization Approach

Master thesis

by

Anthony E. Ohazulike
University of Twente
Applied Mathematics Department
(Industrial Engineering and Operations Research)
Enschede, The Netherlands

Graduation Committee:
- Prof. Mark Uetz
  University of Twente, Applied Mathematics Department
- Dr. Georg Still
  University of Twente, Applied Mathematics Department
- Dr. Michiel Bliemer
  Goudappel Coffeng, Transport Innovation and Modelling
- Dr. Walter Kern
  University of Twente, Applied Mathematics Department
- Ir. Luc Wismans
  Goudappel Coffeng, Transport Innovation and Modelling

University of Twente
Enschede - The Netherlands

Goudappel Coffeng
Adviseurs verkeer en vervoer

August, 2009

Acknowledgements

I am grateful to the Almighty God for keeping me fit and healthy in body and mind throughout my two years of studies in the Netherlands.

My heartfelt gratitude goes to my able supervisors, Georg Still, Michiel Bliemer, Walter Kern and Luc Wismans. Their professionalism, expertise, suggestions and technical advice made this work a successful one. The motivational encouragement I received from them kept me going throughout my master’s assignment period. They were always there for me; to answer my questions and to sit for meetings without complaining, even when it involved travelling out of their primary duty posts. Thank you all for such devotion and kindness. George, the memories of the fatherly love you showed me will never go out of my mind. I could always reach you even at your home and we talked about things outside academics. Thank you so much.

I remain grateful to Goudappel Coffeng BV for providing the friendly and enabling environment for me throughout my assignment in the company. I will not stop until I mentioned few names that made my stay there a memorable one. Luuk Brederode you are like a brother and a friend, I thank you so much for your devoted supervision during my internship and valuable assistance during my master’s assignment. Jantine Boxum and Robert, I have never seen such wonderful office mates throughout my working life. I really appreciate your love and kindness. I owe special thanks to Klaas Friso and Peter Kant who first introduced me to the field of transportation modeling. Wim Kover, your words of encouragement are part of what kept me going. Thank you. Also, I thank the staff and management of Goudappel Coffeng for providing me with such an enabling and motivational environment.

I would like to thank my family; my parents, Mr and Mrs Cyprian Ohazulike, my sisters, Ozii, Koko, Oluu and Nneo for their moral and warm support during my study time here in the Netherlands.

I would also like to thank all my friends; Tiblets, Husna, Wenny, Paul, Umar, Olusegun, Ezekiel, Ekpo, Hazel, Obioma, Chris, Mike and Malisa. Your company meant so much to me. I would not have been emotionally fit if not for you guys. Thank you.

Finally, my deep appreciation goes to Hurink Johann for his advice and encouraging words during my study years. I also thank all the staff of Applied Mathematics department, University of Twente for the job well done.

Abstract

Traffic externalities are increasingly becoming unbearable in most cities of the world. Congestion has caused many countries huge loss in man-hours which translates to huge amounts of money being lost every single day. Air qualities in most cities are not in accordance with the stipulated standards. This results in a severe health issues to innocent residents and endangers the existence of future generations. Restlessness and nuisance caused by unacceptable high noise level associated with road traffic are also increasingly becoming a point of concern by both inhabitants and the government. In most cities, accident and fatality rates resulting from road traffic are on the increase. By allowing much traffic or heavy duty vehicles on small pavements dilapidates these pavements, and this translates to the cost of road (or pavement) maintenance which is generally high. It has been shown from the famous Braess' paradox that combating congestion by adding more infrastructures can worsen the situation if not properly planned. In most places in the world today, expansion of the existing infrastructures is practically impossible due to geographical and financial constraints. It is in view of the aforementioned among others that road traffic analysts suggest the use of road pricing to battle congestion. Economists suggest that by charging every user a toll equal to the cost he or she imposes on other users will lead to most efficient use of transportation network with respect to travel costs. This is what they call marginal social cost pricing (MSCP). Others claim that MSCP is not obtainable in practice and suggest the use of so called second best congestion toll pricing to battle congestion. In their views, most of them claim that battling congestion will take care of other externalities since they (other externalities) have direct link to congestion. This thesis argues that this is not the case in general. We show that the so called first best congestion charging does not give the most efficient use of transportation network in general. In all the models seen in literature to the best of our knowledge, they always assume that the transportation system is managed by a single decision maker, usually the government. In other words, they assume that only one body is capable of imposing tolls on roads. In this thesis, we look at a more general and realistic situation where different agencies or stakeholders set tolls on the road network to maximize their selfish interests. Specifically, for example, insurance companies may set tolls to minimize road accidents and have no interest in congestion, whereas the ministry of economics may be interested in minimizing man-hour loss in the traffic so as to boost productivity. Congestion charging may create negative benefits for society and thus, its main purpose (increasing transportation efficiency and social welfare) may be defeated. Because of this and increasing attention for problems resulting from road traffic externalities, we redefine the first best toll to include not only congestion cost but also cost arising from all other road traffic externalities. We design a flexible toll pattern to incorporate all the aforementioned externalities in our model, and in this way, road pricing will help to internalize these road traffic externalities. By internalizing externalities, we mean that the negative effects on the society caused by a user joining a road segment are accounted for in the user's decision making process. Numerical results show that this yields a better result than the existing models that consider only congestion externality.

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Introduction into Road Pricing

1.1 Introduction

Over the past years, vehicle ownership has increased tremendously. It has been realized that the social cost of owning a vehicle does not only include the purchase, fuel, and maintenance fees, but also the cost of man hour loss due to congestion and road maintenance. In addition to the aforementioned costs, costs of health issues resulting from accidents, inhalation of poisonous compounds emitted from vehicle exhaust pipes, and exposure to high noise level from vehicles are on the high side. In this thesis, we develop an optimization model which is capable of finding Pareto optimal flow pattern that optimizes (in Pareto sense) all the mentioned traffic externalities at once. To implement this result in practice, we design a flexible toll pattern which will compel the users to behave in an 'optimal' way.
Sometimes in practice, toll setting is on hands of several stakeholders instead of a central governing body. This means that transportation network can now be tolled by more than one body. The toll setting competition among these stakeholders is also a part this thesis.

1.2 Overview

This thesis is broadly divided into two main parts, namely; multiobjective road pricing problem with a single descision maker and multidescision maker multiobjective road pricing problem.
In this chapter, we look at the state-of-the-art of congestion and road pricing problem. We give the full descriptions of so called first best and second best pricing schemes. We also give the setbacks and limitations of these schemes. We conclude the chapter by citing where road pricing has successfully been implemented and where it has failed.
In chapter two, we give a detailed mathematical review of the toll pricing problem for fixed and variable (elastic) demand with respect to first and second best pricing schemes. The chapter also points out limitations of these schemes in an economic sense.
Chapter three discusses multi-objective problems and solution concepts in the first section. Secondly, we discuss multi-objectives in transportation systems with a single decision maker. We give the mathematical formulation of the problem. After this we introduce the first best pricing, solution methods and the optimality conditions. Limitations of the first best pricing come immediately and this leads to second best pricing with multi-objective.
Chapter four explores the effect of having different stakeholders that manage the affairs of the upper level objectives. In this regard, we assume that each (or some) upper level objective(s) is (are) in the hands of different people. We give the economic intuition and mathematical representation of the problem. The chapter also deals with the existence of Nash equilibrium, partial and grand coalition of stakeholders (or 'objective owners'). We also look at possible profit allocation rules that will make everyone better off than in Nash scenario. We conclude the chapter with numerical examples to demonstrate the strength of our models.
Chapter five summarizes our findings and gives future research directions.

1.3 Literature review

1.3.1 Congestion and Road Pricing

Economists found that when a resource that is vital and scarce is free or underpriced, then demand for such resource will outstrip the supply, resulting in shortages. This phenomenon is readily seen in the transportation sector. When the demand or number of vehicles using a certain road exceeds the road's capacity, then congestion begins to build up. This using up of road capacity mostly occurs during the so called peak hours. Congestion, especially in the city ways and urban roads is increasingly becoming unbearable. In the most populous and industrialised African city Lagos (Nigeria), it is estimated that each resident, on average looses approximately three hours every day to road congestion. In 2006, it was estimated that there were 41,118 traffic jams and approximately 60 million vehicle lost hours at Dutch highways (38). The negative effects of congestion range from time and man-hour losses to damages to pavements, environment and residents in the urban areas. Owing to these negative effects and others, many economists, scientists and engineers propose the use of road pricing to tackle this problem. As road is a valuable and scarce resource, they suggest that it ought to be rationed by a price mechanism. Road users should pay for using the road network to make correct allocation decisions between transport and other activities. On the other hand, imposing tolls on (certain) roads will make users change their planned routes, and by so doing, traffic is more 'evenly' distributed in the network. They argued that this reduces the entire network travel time. With the advent of electronic road pricing techniques, it is now easier than ever to implement road pricing since cars no longer have to stop before been charged. This will ensure that road pricing does not create unnecessary congestion. Others (2), (22), (27) propose the use of vehicle tax and differential parking charges to combat congestion. They argued that imposing taxes on vehicles will discourage people from buying cars. By charging users of certain roads to specific parking areas, which they argue, will reduce the number of cars entering a congested area while not interfering with business activities and shopping. For an initial short amount of time, users are charged relatively smaller amounts, and then for longer periods they are charged more. Thus users who need short term parking benefit from such a parking scheme while others are encouraged to use public transportation and commuting. The benefits of owing a car outweighs the taxes, and the subsidies received by employees from their employers for transportation fares cushion the effect of parking charges, thus making it impossible to tackle congestion problems with such schemes. Again, parking fees do not depend on the traffic volume or distance travelled, neither do they depend on the environmental characteristics of the vehicle. Traffic in transit through a congested area is not affected by parking fees at all. These show that parking fees are not efficient ways to battle traffic externalities. In literature, road pricing has been broadly classified into two main schemes as given in the following definitions:

Definition 1
First best road pricing scheme: this is a tolling scheme that leads to the optimal use of transportation network in terms of travel costs. Ofcourse, there is possibility of tolling all links.

Definition 2
Second best pricing scheme: this a tolling scheme that does not allow every link to be tolled, and when this happens, it may be that the resulting flow is not optimal. The toll vector that achieves this sub optimal flow is called second best toll vector.
In the following sections we will discuss in detail these two schemes and their limitations.

1.3.2 First Best (Congestion) Pricing and its limitation

First best pricing, also known as Marginal Social Cost Pricing (*MSCP*) in transportation systems is defined to include private costs plus external ones, such as delay, air pollution, noise pollution, road damage and accidents. Marginal Cost Pricing (*MCP*) on a congested transportation network dates back to Beckmann et al.(4). They argue that if there were a way to collect tolls from the users of congested roads at rates that would measure the (delay) cost an average road user inflicts on others, a better use of the highway system would be obtained. They also added that the collected revenue should be invested back into the transportation system or to the society so that the social cost or welfare loss is not increased as a result of such tolls. *MSCP* sets on each link a toll which is equal to the difference between the marginal social cost, and the average cost. With this set of tolls, efficient use of the

transportation system is obtained. Ferrari Paolo (10), and Yang and Huang (39) discuss the first best congestion pricing scheme with capacity and environmental constraints. Bergendorff (5), Bergendorff et al. (6), Hearn and Ramana (12), and Hearn and Yildirim (13) introduce the concept of congestion toll pricing framework using the optimality conditions. They transform the optimal flow pattern of the road pricing problems into their equivalent conditions. With such transformation, they prove that other congestion toll vectors exist which will achieve the optimal usage (in terms of travel time) of the transport system. In fact, their results show that when there is no restrictions on tolls, then, there exist infinite congestion toll vectors that can be used to achieve the most efficient use of the transportation system (with respect to travel time). Infinite toll vectors can be genarated from marginal social cost toll vector alone. This leads to a redefinition of first best congestion pricing to include not only *MCP*, but also those congestion toll vectors that can be used to achieve the most efficient use of the transportation network. Suh-Wen Chiou (32) proposes the use of non-smooth optimization approach and a general bundle method combined with sub gradient projection to solve first best congestion pricing with variable demands. Yang and Huang (39), Yang and Huang (40), Chen, M. et al. (8) and Gou and Yang (11) extend the single class first best congestion pricing to include heterogeneous users.
In the following, we give the graphical interpretation of marginal social cost pricing. But, before we proceed, it it important we give the following definitions:

Definition 3
System Optimum (**SO**): This is a state of a transportation network which achieves minimum system costs.

Definition 4
User Equilibrium (**EU**): This is a state of a transportation network when no user can increase his utility by unilaterally switching to another route.

Graphical Representation
In figure 1.1 below, *AC* represents the average or private cost perceived by a user entering a road segment. *MC* is the marginal or social cost felt or perceived on the road segment (or the entire system) due to a user entering the road segment. Observe that marginal cost includes the private cost perceived by a user entering a road and the cost this marginal user imposes on others already using the network. The demand curve (inverted) represents the benefit a user gets by travelling. Naturally, users have incentives to travel until their benefits of travel equal their travel expenses or costs. When the travel cost equals the benefit, then, it is no more profitable to travel and the system is said to be in User Equilibrium. This occurs when the total demand d is equal to d_{UE}. Observe that with $d < d_{UE}$, a user will still have incentive to travel since the benefit is more than the cost he or she perceives. When $d = d_{UE}$, the benefit of travel becomes equal to the cost users perceive before entering the stream, so, it is very likely that they will not make the trip since making the trip will make them to pay more than they will benefit. So from users perspective, $d = d_{UE}$ is ideal and optimal. Observe also from the graph that the d_{UE}^{th} user is enjoying a benefit equal to jh while inflicting a cost equal to jg on the system. In general, traffic beyond d_{SO} is enjoying a benefit equal to $jhei$ while generating a system cost of $jgei$, leaving a deadweight system loss or social welfare loss of hge. The *MSCP* principle states that for efficient utilization of the transport system, each user should pay for all the externalities he is causing to the system. By imposing tolls equal to *MSCP* on each link, users will now perceive cost equal to that of *MC*. Self optimizing route choices by the users will now lead to user equilibrium point of e or d_{SO} instead of h or d_{UE}. It can easily be shown that the most efficient use of the transportation system is achieved at this point. *MSCP* for congestion (or first best congestion tolls) can easily be calculated with a simple formula which will be given in the next chapter. $febd$ is equal to the so called Pigouvian tax or

toll revenue. $C1 - C3$ or *fe* is the amount of optimal toll levied on users. *eab* is the consumers surplus, which is the amount users benefit by being able to travel for a cost less than they would be willing to pay.

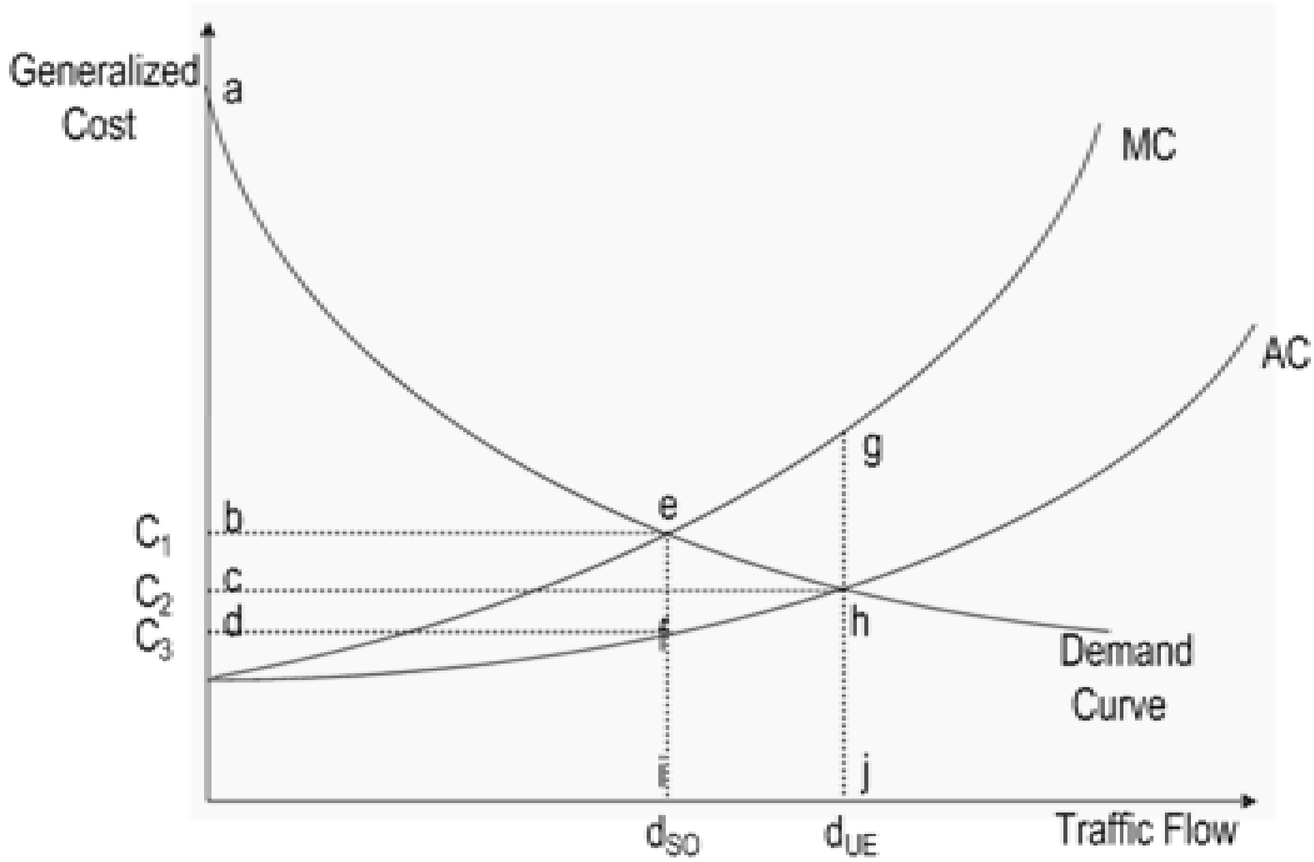


Figure 1.1

The first best pricing schemes discussed in all the literatures above are with respect to congestion. Since they fail to consider all other externalities such as noise and air pollutions, accidents and damages to infrastructure, it simply means that the first best congestion toll does not make every user to pay 'completely' for the costs he or she is imposing on the system. Hence, the claim that such congestion schemes achieve the most efficient use of the transportation network does not hold in general.

It turns out that, though the first best pricing scheme has a perfect theoretical basis and very easy to compute (even in the case of heterogeneous users), it fails to meet practical feasibility. Due to political reasons, huge setup cost, technological constraints, tradition of free access roads and technical complexity, it may me be required that some of the roads in the transportation network cannot be tolled. Moreover, in practice, it is more natural not to toll all links. When this is the case, then it simply means that *MSCP* has failed since it requires that all 'used' roads be tolled. On the other hand, it may be that the toll pricing framework proposed by Hearn et al. (12, 13) will satisfy this constraint (of not tolling all roads) and still achieve the most efficient use of the network, but this does not hold in general. These limitations lead to the so called second best pricing scheme discussed in the next section.

1.3.3 Second Best Pricing and its limitation

As mentioned in the previous section, it may be practically impossible to toll all roads due to poor public acceptance, cost of setting up toll booths, political and technical reasons among others. This means that the most efficient use of the transportation system may not be achieved. With such constraints in place, we look for a solution or a toll vector that will lead to a near optimal usage of the transportation system and hence the name second best pricing. The key questions to answer in a second best pricing scheme in general include: where to levy the toll and how much? There is scientific literature on second best pricing schemes. Yang and Lam (41) model the second best pricing with elastic demand as a bilevel programming problem. The upper level represents the system controller (or the decision maker) that determines the tolls that optimize a given system's performance while considering users' route choice behavior. On the other hand, the lower level represents the users. Since the users will always minimize their perceived cost in their route choice behavior, it means that the system will eventually settle at user equilibrium. The lower level problem therefore translates to finding user equilibrium flows. This problem setting can be seen as a Stackelberg game where the system controller is the leader and the network users are the followers. Hearn and Yildirim (14), May and Milne (23), and Verhoef (33) study second best pricing with different model formulations, but with the same bilevel approach as Yang and Lam (41). Verhoef (33) uses the assumption of existence of set of relevant paths per OD. Yildirim (42) arrives at the same result as Verhoef (33) by assuming the existence of Lagrangian multipliers for the associated Krush-Kuhn-Tucker (*KKT*) conditions. Yang and Lam (41) and Verhoef (34) develop algorithms for solving the second best pricing problem. Sumalee et al.(24) prove the convergence of Verhoef's algorithm when applied to small network section of the city of Leeds. Hearn et al. (15) propose two methods for solving congestion pricing; the first is the use of toll set approximation to obtain the alternative toll vectors, and the second is the cutting plane method for the minimum toll revenue problem. Zhang and Ge (44) discuss a class of parameterized second best road pricing scheme, where the toll on each tolled link is proportional to Pigouvian charge. They also propose a convex mathematical program to study the effect of road pricing scheme of this kind on travel behavior in congested road networks with elastic demand. Lawphongpanich et al.(19) and Yildirim (42) formulate the second best pricing problem as a mathematical problem with an equilibrium constraint expressed as variational inequality for both fixed and elastic demand. They assume existence of Lagrange multipliers. Lawphongpanich et al.(19) further investigate the properties associated with the second best pricing problem and derived an algorithm for it. Liu and McDonald (21) investigate the economic efficiency of second-best congestion pricing schemes in urban highway systems where they compared the results of first best tolling scheme and that of second best tolling scheme. They also investigate the impact of second best pricing policy on the allocation of traffic volumes for two time windows. Owing to the fact that drivers may differ with respect to value of time and with respect to marginal impact on others, Verhoef and Small (36) consider second best congestion pricing with heterogeneous drivers on three links, assuming elastic demand and a continuum of values of time. Verhoef et al (35) look at the road pricing problem from a multi-displinary perspective. Their work imclude optimal design of road pricing schemes, the behavioural effects that may be induced among individuals and firms, and acceptability of road pricing. Game theoritical analogy of road pricing is also part of their research output.

Due to the constraint on the tolls, the upper level problem is generally solved simultaneously with the lower level problem. This type of problem falls into a special class of mathematical problem called the Bi-level Programming Problem (*BLPP*). BLPP is discussed in Appendix *A*. Second best pricing problem is usually formulated as a *BLPP* called Mathematical Program with Equilibrium Constraint (*MPEC*). *MPECs* are generally hard mathematical problems, as we will see. As mentioned earlier, all of the above literature have common shortcomings; the first is that they fail to explicitly define all the externalities caused by road users, and the second is, that they all assume the existence of only one decision maker or system controller, usually the government. Both aspects will be analysed within this research.

1.3.4 Application of Road Pricing.

In this section we review some places where road pricing scheme has been implemented successfully, the advantages and disadvantages and where it has failed.

Singapore in 1975 introduced the Area Licensing scheme (*ALS*) making it the first country to design and implement a practical (low-tech) congestion pricing. This was later replaced by Electronic Road Pricing (*ERP*) in 1998. The aim was to check traffic (at peak periods) into the Central Business District, so that congestion is minimised. The tolls would vary according to average speed on the network.

Norwegian cities of Bergen, Oslo and Trondheim, in 1986, 1990 and 1991 respectively, introduced cordon pricing scheme to raise revenue for financing road projects and to small extent, public transport. Though the scheme was not originally designed to reduce traffic, some impact on travel behaviour and traffic volumes were noticed. One drawback of the Trondheim toll ring as a financing mechanism is that about one-third of the region's drivers live inside the ring and therefore seldom pay charges, yet they benefit from some of the road improvements.

Autoroute A1 is an expressway connecting Paris to Lille, about 200 km to the north. Vehicles receive a ticket upon entering the expressway and pay at a toll booth upon exiting, the amount depending on the length of the trip. The A1 is subject to heavy traffic near Paris on Sunday afternoons and evenings. In April 1992, after a period of extensive public consultation and publicity, this congestion problem was confronted by implementing a time varying toll scheme for Sundays only. A special 'red tariff' is charged during the Sunday peak period (16:30–20:30), with toll rates 25 to 56 percent higher than the normal toll. Before and after the peak there is a 'green tariff' with rates 25 to 56 percent lower than the normal toll. These hours and rates were designed so that total revenues are nearly identical to those collected with the normal tariff. This property was believed essential for public acceptance, which in fact has been largely favourable.

San Diego I-15 Express in 1996 introduced high occupancy toll (*HOT*). The spare capacity of the *HOT* lanes is now more efficiently used, and moreover, many users are enthusiastic about the scheme and are ready to pay provided they get an enhanced service.

In 1997, 407 Express Toll Road, Toronto started operation. Traffic demand rose from 11,000 to 12,000 cars in peak hours, and at the same time, average speed is about double that of the nearby congested public highways.

London, on February 17, 2003, kicked off congestion pricing to reduce traffic congestion, increase journey time reliability and decrease air pollution. The benefits of London congestion charging include; 15% traffic reduction, 30% congestion reduction, 12% pollution reduction (*NOx*, *PM*10), journeys had become more reliable, buses significantly gain reliability, and substantial reduction of road accidents among others.

On January 1, 2005, the German Federal Government introduced distance-related tolling of heavy trucks (>= 12 tons) using Autobahns. The technology is based on GPS/ GSM in order to have the option to extend tolling to all kinds of roads and vehicles later on. The amount of toll is based on the internal / direct costs caused by heavy trucks - calculated according to a special EU-directive. The net toll-revenue was decided to be used exclusively for the transportation infrastructure – 50 % for the Federal Highways, 50 % for the Federal railways and the inland waterways. Since the inception of the scheme, it has been working without any problems.

Stockholm in Sweden introduced a trial system with 19 toll plazas from 3rd January to 31st July 2006. After successful trials, the system was continued from fall 2007.

On 2nd January 2008, Milan became the first metropolitan area in Italy to introduce a congestion charge for the city center. The scheme aims at reducing congestion and air pollution.

On 1st October 2008, an extension of the current charging zone around Oslo was established to cover the suburbia Baerum west of Oslo. The new zone is operated separately, which means that vehicles going to Oslo from west will be charged twice. The system is based on the existing system in Oslo, which is fully automatic operated. The motivation for the new charging zone is to provide capital for the operation of the public transport in Oslo and Akershus the next twenty years. Moreover it will provide capital for expansion of the west corridor out of the Norwegian capital.

Though road pricing has successfully been implemented in some countries, it has failed in some countries due to poor public acceptance among other factors. Hong Kong, in 1986 proposed a congestion pricing scheme which was never implemented due to some factors which include improved traffic flows and the mid 1980's recession in Hong Kong.

Edinburgh city in 2002 decided to carry out extensive public hearings during 2003, followed by a referendum. The referendum was only meant to be guiding. The public hearings lead to some adjustments of the design of the road pricing system. The referendum took place in February 2005. The result was that 74% voted no to introducing road pricing while only 26% voted yes. This resulted in the City Council's decision to immediately drop all plans to introduce road pricing. Most of the planned investments will still be carried out but will now be financed by ordinary taxes.

Though Trondheim road pricing reduced the inbound traffic by up to 10%, the charging scheme as mentioned earlier was not operated with the intention of reducing congestion but was implemented over a 15 year period in order to gain funds largely for road investments. This period ended in 2005 and as such Trondheim is the first city ever to stop collecting tolls. Proposals are currently under debate however, even though traffic conditions are not been viewed as problematic.

New York congestion pricing was a proposed traffic congestion fee for vehicles traveling into or within the Manhattan central business district of New York City. The congestion pricing charge was one component of New York City Mayor Michael Bloomberg's plan to improve the city's future environmental sustainability while planning for population growth, entitled PlaNYC 2030: A Greener, Greater New York. If approved and implemented, it would have been the first such fee scheme enacted in the United States. The deadline to approve the plan by the State Assembly was April 7, 2008, for the city to be eligible to receive US$ 354 million in federal assistance for traffic congestion relief and mass transit improvements. On April 7, 2008, after a closed-door meeting, the Democratic Conference of the State Assembly decided not to vote on the proposal, "...the opposition was so overwhelming,...that he would not hold an open vote of the full Assembly," Sheldon Silver, the Assembly Speaker said. Afterwards, the US Department of Transport (*UDOT*) announced that they will seek to allocate those funds to relief traffic congestion in other cities. Chances for the bill to return in the near future to the State Assembly are considered dim, so long as Sheldon Silver remains the Speaker.

The following countries have plans of implementing road pricing in the near future; The Netherlands, some cities in the USA, Hong Kong, AUS, and other cities in Britain e.g. Bristol, Leeds, Derby, Edinburgh, Leicester etc.

Netherlands aims to start with freight transport in 2011. This will require intensive technical and policy-related cooperation with Belgium, France and Germany. Passenger cars will follow a year after the launch of freight transport. The complete system roll-out will be scheduled for 2016 and beyond.

Chapter 2
Single-Objective Road Pricing Problem

2.1 Introduction

In this chapter, we give an insight into mathematical theory of toll pricing problems, solution methods and limitations of the mentioned approaches. We kick off by reviewing the mathematical models in section 2.2. In sections 2.3 and 2.4, we discuss first best congestion pricing theory for fixed and elastic demand respectively. Owing to limitations of first best congestion pricing schemes, we extend the discussion to second best congestion pricing for fixed and elastic demand in sections 2.5 and 2.6 respectively.

2.2 Review of mathematical models of toll pricing problems

In this section, we summarize general mathematical models for toll pricing problems. We start by defining some terms, followed by defining a network and its variables. We then give the mathematical models for first and second best congestion pricing, and finally, we discuss the toll pricing framework. As stated before, the toll pricing problem is equivalent to a Stackelberg game, where the leader is the system controller capable of setting tolls and the followers are the road users who react according to perceived costs. In the toll pricing game, we assume that the leader can predict the followers' reaction and this will help him in choosing optimal strategies. In transportation terms, the system controller knows that users will always want to minimize their perceived cost and hence set optimal tolls in such a way that users' behavior will result in system optimal flows.

John Glen Wardrop, an English transport analyst, in 1952 introduced the notion of user equilibrium in the transportation sector via his first principle. His idea is similar to that of Nash equilibrium in game theory. However, in transportation networks there are many players, making the analysis more difficult than in games with a 'smaller' number of players. In his second principle, he looks at system problem as cooperation among road users which he claims will result in most efficient use of the transportation system. These notions remained not mathematically stated until in 1956, when Beckmann, McGuire and Winsten (4) first translated Wardrop's principles into mathematical models for network equilibrium. We state below the Wardrop's first and second principles:

Wardrop's First Principle: In equilibrium, the journey times (or cost) on all routes actually used between a given origin-destination pair are equal and less than those which would be experienced by a single vehicle on any unused route. It actually states that, users' behaviour in a transportation network is such that after some time no road user will be better off by unilaterally switching to another route. When a traffic network is in this state, then we say that the system is in **User Equilibrium** (**UE**), or that the traffic flow is in User Equilibrium flow. It can also be seen as a non-cooperative game where players (users) do not cooperate and selfishly aim at minimizing their respective travel cost.

Wardrop's Second Principle: The average journey time is minimum when no user can lower the total travel time (or cost) in the system by changing route. It actually states here that when users of a transport network cooperate, their average travel cost will be minimal. This implies that users cooperation will lead to minimization of total network cost. The mathematical problem of minimizing the system's cost is called the System Problem or the **System Optimum** (**SO**) problem. The famous Braess' Paradox shows that User Equilibrium can differ from System Optimum. Economists argue that charging users of a road a toll equal to MSCP, will transform UE into SO. This means that the system optimum can be achieved by altering users' behaviour via tolling the network. In the following, we give the mathematical formulations of Wardrop's principles;

2.2.1 Notations

We define $G = (N,A)$ to be a network, where N is the set of all nodes in the network and A is the set of all arcs or links or edges in the network. Throughout we will assume directed arcs. We also define the folllowing network variables and attributes: Let

a be used as an index for links or arcs in the network.We will also use (i,j) to represent two end points of an arc a. We assume the arc direction, $i \rightarrow j$.

r be used as an index for routes or paths in the network.

w be used as an index for OD pairs. We will also use (p,q) as an OD pair to mean origin p and destination q.

R be the set of all paths in the network.

V be the set of feasible flow pattern.

F be the set of feasible path flows.

R_w be the set of all paths connecting OD pair w.

R_A^r be the set of all arcs in path r.

R_{pq} be the set of paths going from origin p to destination q.

W be the set of all OD pairs in the network.

$\Gamma \epsilon \mathbb{R}^{|W|\times|R|}$ be the OD-path incident matrix.

$P \epsilon \mathbb{R}^{|W|\times|N|}$ be the OD-node incident matrix.

$\Lambda \epsilon \mathbb{R}^{|A|\times|R|}$ be the arc-path incident matrix.

$\Pi \epsilon \mathbb{R}^{|N|\times|A|}$ be the node-arc incident matrix that describes the entire network.

λ_w be the least cost to transverse the w^{th} OD pair.

$d \epsilon \mathbb{R}_+^{|W|}$ be travel demand vector whose element, d_w is the demand for the w^{th} OD pair.

$D(\lambda)$ be the vector of demand functions whose element , $D_w(\lambda_w)$ is the demand function for the w^{th} OD pair.

$B(d) \epsilon \mathbb{R}^{|W|}$ be the inverse demand vector whose element , $B_w(d_w)$ is the inverse demand (or benefit) function for the w^{th} OD pair.

$P_w \epsilon \mathbb{R}^{|N|}$ be the vector that defines the origin and destination nodes of the w^{th} OD pair. Where $P_k = e_q - e_p$, e_q and e_p are respectively the q^{th} and p^{th}unit vectors.

$f \epsilon \mathbb{R}_+^{|R|}$ be the path flow vector whose element, f_r, is the flow on path r.

$f_{r,w}$ be flow on path r for OD pair w. Or just flow on path r where $r \epsilon R_w$.

$v\ \varepsilon\ \mathbb{R}_+^{|A|}$ be the vector of link flows whose element, v_a, is the flow on link a, where $v \epsilon V$ and $a \epsilon A$.

$v^w\ \varepsilon\ \mathbb{R}_+^{|A|}$ be the vector of link flows for the w^{th} OD pair whose element, v_a^w, is the flow on link a for the w^{th} OD pair.

$\delta_{ar} \epsilon \mathbb{R}^{|A|\times|R|}$ be a binary variable that is 1 if link a is in path r and 0 otherwise.

$$\text{OR that } \delta_{ar} = \begin{matrix} 1 & \text{if } r \epsilon R_A^r \\ 0 & \text{otherwise} \end{matrix}$$

$t(v)$ be the vector of link travel time (cost) function whose element, $t_a(v)$, is the travel time function for link a.

-Observations:

1) $v = \sum_{w \epsilon W} v^w = \Lambda f \quad ; \quad f \epsilon F.$

2) $v_a = \sum_{w \epsilon W} v_a^w = \sum_{r \epsilon R} f_r \delta_{ar} = \sum_{w \epsilon W} \sum_{r \epsilon R_w} f_r \delta_{ar} \quad ; \forall a \epsilon A.$

3) $v_a^w = \sum_{r \in R_w} f_r \delta_{ar}$; $\forall a \in A,\ \forall w \in W$

4) $d_w = \sum_{a \in A} v_a^w = \sum_{r \in R_w} f_r$; $\forall w \in W.$

5) $\Pi v^w = P_w d_w$; $\forall w \in W.$

6) $d = \Gamma f$

Observations 1, 2and 3 are equivalent and they are the flow conservaton conditions. 4, 5 and 6 are also equivalent and they are used to conserve the travel demand. This implies that any of 1, 2 and 3 with any of 4 and 5 can define a feasible flow together with non negativity constraint of the link flows. In the following sections, we study the toll pricing problem for fixed and elastic demand.

2.3 First Best Princing scheme for Fixed Demand (FBP_FD)

For fixed demand, the following system defines a feasible flow:

$$v = \Lambda f$$
$$\Gamma f = d^*$$
$$f \geq 0$$

The first equation states that the flow on a link is equal to the sum of all path flows that passes through this link.

The second equation is the flow-OD balance contraint, it preserves flow for each OD. It states that, the sum the flows on all paths originating from node p and ending at destination node q for an OD pair p, q equals the demand for this OD pair. The (*) on d^* signifies that the demand d is fixed. The third inequality simply states that path flows are non negatives. The non negativity of link flows follows directly from this last equation. Before we proceed, let us first state the system problem and the users problem. The system problem is concerned about minimizing the entire network cost, whereas the user problem is about minimizing user's cost.

2.3.1 System Problem_Fixed Demand (SP_FD)

The system problem is mathematically stated as follows:

$\min v^T t(v)$

$s.t$

$$\left.\begin{array}{lc} v = \Lambda f & \psi \\ \Gamma f = d^* & \lambda \\ g(v) \leq 0 & \xi \\ f \geq 0 & \rho \end{array}\right\} \quad FeC\_FD$$

Where v^T is the traspose vector of v, $g(v) \leq 0$ are the possible side constraints on the link flow vector v which we assume to be linear in v, and $\{\psi \in \mathbb{R}^{|A|},\ \lambda \in \mathbb{R}^{|W|},\ \xi \in \mathbb{R}^{|Z|},\ \rho \in \mathbb{R}^{|R|}\}$ are the *KKT* multipliers associated with the constraints. The above system has a convex objective (if the cost functions are link separable in v or monotone and this implies that $(v_a t_a(v_a))'' \geq 0$) and the constraints are linear in v. We will call these constraints, feasibility conditions for fixed demad which we will always refer to with the abbreviation *FeC_FD* which reads, *Fesibility Conditions for Fixed Demand*. Let us quikly look at the *KKT* first order optimality conditions;

Suppose L is the Lagrangian, and $\bar{v}$ is the solution to the above system, then, there exists $(\psi, \lambda, \xi, \rho)$ such that the following hold:

$$
\begin{aligned}
L &= t(v)^T v + (\Lambda f - v)^T \psi + (d^* - \Gamma f)^T \lambda + (g(v))^T \xi - f^T \rho \\
\frac{\partial L}{\partial v} &= t(\bar{v}) + (\nabla t(\bar{v}))^T \bar{v} + (\nabla g(\bar{v}))^T \xi - \psi = 0 \\
&\Rightarrow t_a(\bar{v}) + \sum_{b \in A} \bar{v}_b \frac{\partial (t_b(\bar{v}))}{\partial v_a} + \sum_{z \in Z} \xi_z \frac{\partial (g_z(\bar{v}))}{\partial v_a} - \psi_a = 0 \qquad \forall a \in A \qquad (2.1) \\
\frac{\partial L}{\partial f} &= \Lambda^T \psi - \Gamma^T \lambda - \rho = 0 \\
&\Rightarrow \sum_{a \in A} \psi_a \, \delta_{ar} - \lambda^w - \rho^r = 0 \qquad \forall r \in R_w, \ \forall w \in W \qquad (2.2) \\
\xi g(\bar{v}) &= 0 \qquad (2.3) \\
&\Rightarrow \xi_z g_z(\bar{v}) = 0 \qquad \forall z \in Z \\
f^T \rho &= 0 \qquad (2.4) \\
\xi, \ \rho &\geq 0, \qquad OR \qquad \xi^z, \ \rho^r \geq 0 \qquad \forall z \in Z \quad \forall r \in R \\
&\Rightarrow \rho^r f_r = 0 \qquad \forall r \in R
\end{aligned}
$$

Equations (2.3) and (2.4) are called complementarity equations.
From (2.1) and (2.2) follows that

$$
t_a(\bar{v}) + \sum_{b \in A} \bar{v}_b \frac{\partial (t_b(\bar{v}))}{\partial v_a} + \sum_{z \in Z} \xi_z \frac{\partial (g_z(\bar{v}))}{\partial v_a} = \psi_a \qquad \forall a \in A \qquad (2.5)
$$

$$
\sum_{a \in A} \psi_a \, \delta_{ar} = \lambda^w + \rho^r \qquad \forall r \in R_w, \ \forall w \in W \qquad (2.6)
$$

Substituting (2.5) into (2.6) yields:

$$
\sum_{a \in A} \left(t_a(\bar{v}) + \sum_{b \in A} \bar{v}_b \frac{\partial (t_b(\bar{v}))}{\partial v_a} + \sum_{z \in Z} \xi_z \frac{\partial (g_z(\bar{v}))}{\partial v_a} \right) \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W
$$

Since all the side contraints discussed in this thesis are all linear in v, constraint qualification conditions are automatically satisfied and multipliers will exist at local optima. If the side constraints $g_z(v) \leq 0$ are not present, then, the above reduces to:

$$
\sum_{a \in A} \left(t_a(\bar{v}) + \sum_{b \in A} \bar{v}_b \frac{\partial (t_b(\bar{v}))}{\partial v_a} \right) \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W
$$

NB: If we assume separable link travel time functions for link a (i.e, the travel time on link a depends on the flow only on link a), then, the equation above simplifies to

$$
\sum_{a \in A} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{d v_a} \right) \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W
$$

As a result of the optimality conditions above, let us derive the following which will serve as a powerful tool in this thesis:

$$\bar{v}_a = \sum_{w \in W} \sum_{r \in R_w} f_r\, \delta_{ar} \qquad \forall a \in A$$

$$\sum_{a \in A} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{dv_a} \right) \bar{v}_a = \sum_{a \in A} \sum_{w \in W} \sum_{r \in R_w} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{dv_a} \right) f_r\, \delta_{ar}$$

$$= \sum_{w \in W} \sum_{r \in R_w} f_r \sum_{a \in A} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{dv_a} \right) \delta_{ar}$$

$$= \sum_{w \in W} \sum_{r \in R_w} f_r(\lambda^w + \rho^r)$$

$$= \sum_{w \in W} \sum_{r \in R_w} f_r \lambda^w + \sum_{w \in W} \sum_{r \in R_w} \rho^r f_r$$

$$= \sum_{w \in W} \sum_{r \in R_w} f_r \lambda^w = \sum_{w \in W} \lambda^w \sum_{r \in R_w} f_r = \sum_{w \in W} \lambda^w d_w^*$$

$$\therefore \sum_{a \in A} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{dv_a} \right) \bar{v}_a = \sum_{w \in W} \lambda^w d_w^* \qquad (2.7)$$

2.3.2 User Problem_Fixed Demand (UP_FD)

The user problem can be mathematically formulated as a variational inequality. A link flow pattern $v^* \in V$ is in equilibrium if and only if it solves the variational inequality problem:

$$t(v^*)^T(v - v^*) \geq 0 \qquad \forall v \in V \qquad (2.8)$$

$$OR \quad t(v^*)^T v \geq t(v^*)^T v^* \qquad \forall v \in V$$

Any solution of the above variational inequality is UE flow pattern.

If the link travel time functions are separable and monotonic in flows, then, the user problem can be stated as a convex mathematical program:

$$\min \sum_{a \in A} \int_0^{v_a} t_a(x)dx$$

$$s.t \quad FeC\_FD$$

The proof that any flow pattern that solves the above program satisfies Wardrop's user equilibrium conditions can be seen in Yang and Huang (*f*). Observe that the variational inequality (2.8) is equivalent to stating that v^* is the minimizer of

$$\arg\min_{v} \ t(v^*)^T v$$

$$s.t$$

$$FeC\_FD$$

Thus we write the user problem as, find a user equilibrium link flow vector v^* such that it solves the following problem

$$\min_{v} \ t(v^*)^T v$$

$$s.t$$

$$FeC\_FD$$

Again, we give the *KKT* first order optimality conditions: Now, suppose L is the Lagrangian, and v^* is the solution of the above problem, then there exists multipliers $(\psi, \lambda, \xi, \rho)$ such that:

$$
\begin{aligned}
L &= t(v)^T v + (\Lambda f - v)^T \psi + (d^* - \Gamma f)^T \lambda + (g(v))^T \xi - f^T \rho \\
\frac{\partial L}{\partial v} &= t(v^*) + (\nabla g(v^*))^T \xi - \psi = 0 \\
&\Rightarrow t_a(v^*) + \sum_{z \in Z} \xi_z \frac{\partial (g_z(v^*))}{\partial v_a} - \psi_a = 0 && \forall a \in A \\
\frac{\partial L}{\partial f} &= \Lambda^T \psi - \Gamma^T \lambda - \rho = 0 \\
&\Rightarrow \sum_{a \in A} \psi_a \, \delta_{ar} - \lambda^w - \rho^r = 0 && \forall r \in R_w, \ \forall w \in W \\
\xi g(v^*) &= 0 && (2.9) \\
&\Rightarrow \xi_z g_z(v^*) = 0 && \forall z \in Z \\
f^T \rho &= 0 && (2.10) \\
\xi, \, \rho &\geq 0, \qquad OR \qquad \xi^z, \, \rho^r \geq 0 && \forall z \in Z \quad \forall r \in R \\
&\Rightarrow \rho^r f_r = 0 && \forall r \in R
\end{aligned}
$$

As in system problem, equations (2.9) and (2.10) are called complementarity equations.
Observations

$$
t_a(v^*) + \sum_{z \in Z} \xi_z \frac{\partial (g_z(v^*))}{\partial v_a} = \psi_a \qquad \forall a \in A
$$

$$
\sum_{a \in A} \psi_a \, \delta_{ar} = \lambda^w + \rho^r \qquad \forall r \in R_w, \ \forall w \in W
$$

$$
\sum_{a \in A} \left(t_a(v^*) + \sum_{z \in Z} \xi_z \frac{\partial (g_z(v^*))}{\partial v_a} \right) \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W \quad \text{since } \rho^r \geq 0
$$

Suppose the side constraints $g(v) \leq 0$ are not present, then the above observations reduce to:

$$
\sum_{a \in A} t_a(v^*) \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W
$$

If $f_r > 0$, then the complementarity equation forces ρ^r to be zero, thus we have

$$
\sum_{a \in A} t_a(v^*) \, \delta_{ar} = \lambda^w \qquad \forall \, f_r > 0, \ r \in R_w, \ \forall w \in W
$$

Observe that the *LHS* is the total travel time on route $r \in R_w$ by a single user, and the *RHS* does not depend on r.
Interpretation: At equilibrium, the travel times on all used routes for a given OD pair $w \in W$ are the same and equal to λ^w. The following equation holds in general:

$$
\boxed{\sum_{a \in A} t_a(v^*) \, \delta_{ar} = \lambda^w + \rho^r \geq \lambda^w \qquad \forall r \in R_w, \ \forall w \in W} \quad \text{since } \rho^r \geq 0 \ \forall r \in R
$$

Interpretation:
At equilibrium, the travel time on all routes for a given OD pair $w \in W$ is greater or equal to λ^w. This means that λ^w must be the least path travel time between the OD pair $w \in W$ when the system is in equilibrium. Recall that λ^w is the travel time for all used path of $w \in W$. We thus state the following: at equilibrium, the travel times on all used paths for a given OD pair are the same and less or equal to those on unused paths (Wardrop's first principle). We thus conclude that, any flow vector v^* that solves the variational inequality (2.3) is a user equilibrium flow.

Since equation (2.7) holds, we summarize our observations below:

$$\boxed{\begin{array}{l} \sum_{a \in A} \left(t_a(v^*) + \sum_{z \in Z} \xi_z \frac{\partial (g_z(v^*))}{\partial v_a} \right) \delta_{ar} \geq \lambda^w \qquad \forall r \in R_w,\ \forall w \in W \\ \qquad \sum_{a \in A} t_a(v^*) v_a^* = \sum_{w \in W} \lambda^w d_w^* \end{array}} \qquad (2.11a)$$

OR

$$\sum_{a \in A} t_a(v^*)\, \delta_{ar} \geq \lambda^w \qquad \forall r \in R_w,\ \forall w \in W$$

$$\sum_{a \in A} t_a(v^*) v_a^* = \sum_{w \in W} \lambda^w d_w^*$$

if $g_z(v) \leq 0$ are not present $\forall z \in Z$.

The last set of observations can be condensed to:

$$\boxed{\begin{array}{c} \Lambda^T t(v^*) \geq \Gamma^T \lambda \\ t(v^*)^T v^* = (d^*)^T \lambda \end{array}} \qquad (2.11b)$$

Where λ is a free scalar. With a separable travel time functions, $t_a(v^*)$ becomes $t_a(v_a^*)$.

Lemma 1

Let $v^* \in V$, then, the following are equivalent:

1) v^* is user equilibrium, $i.e.,\ \ t(v^*)^T(v - v^*) \geq 0, \quad \forall v \in V.$

2) $v^* \in$ arg min$\{t(v^*)^T v \mid v \in V\}$, $i.e.,$ min$\{t(v^*)^T v \mid v \in V\} = t(v^*)^T v^*$

3) There exists λ such that

$$\Lambda^T t(v^*) \geq \Gamma^T \lambda$$
$$t(v^*)^T v^* = (d^*)^T \lambda$$

Proof

The flow vector is user equilibrium if and only if $t(v^*)^T v \geq t(v^*)^T v^*$, $\forall v \in V$, and since $\forall v^* \in V$, the equivalence of 1 and 2 follows. By using $\forall v^* \in V$, and the feasibility conditions, 2 can be written

$$\min_f \ \left\{ t(v^*)^T \Lambda f \mid \Gamma f = d^*, f \geq 0 \right\} \geq t(v^*)^T v^*$$

By *LP* duality (or by KKT optimality conditions), this is the same as

$$\max\left\{ (d^*)^T \lambda \mid \Lambda^T t(v^*) \geq \Gamma^T \lambda \right\} \geq t(v^*)^T v^*$$

where λ is a free scalar associated with the constraint $\Gamma f = d^*$.

The above holds if and only if there exists λ such that

$$\Lambda^T t(v^*) \geq \Gamma^T \lambda$$
$$(d^*)^T \lambda \geq t(v^*)^T v^*$$

Suppose we let $f^* \geq 0$ be such that $v^* = \Lambda f^*, \Gamma f^* = d^*$, see that

$$(d^*)^T \lambda \geq t(v^*)^T v^* = f^{*T}(\Lambda^T t(v^*)) \geq f^{*T}(\Gamma^T \lambda) = (d^*)^T \lambda$$

The lemma now follows. ☐

Notice from $(2.11b)$ that the difference between these lines of observations and those of system problem is the term $\sum_{b \in A} v_b^* \frac{\partial (t_b(v^*))}{\partial v_a}$. Therefore, by adding the term $\sum_{b \in A} v_b \frac{\partial (t_b(v))}{\partial v_a}\Big|_{v=v^*}$ to the link travel time function $t_a(v^*), \forall a \in A$, the first order optimality conditions of user problem will be exactly the same as those of system problem. This means that any flow pattern that solves the system problem will also solve the user problem ($i.e\ \bar{v} = v^*$). It is in view of this that road traffic analysts for intelligent transportation believe that imposing a toll equal to $\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=v^*}$ on link a $\forall a \in A$ will help to achieve the most efficient (optimal w.r.t travel time) use of transportation system. This is the basis of the so called Marginal Cost Pricing (MCP) for congestion. The term $\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=v^*}$ is called the marginal cost toll for congestion or the congestion externality in pricing theory. By definition, $\left(\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\right)$ is the additonal congestion cost or travel time imposed on all the existing road users by an additional user on link a. In particular, $\frac{\partial(t_b(v))}{\partial v_a}$ is the additional congestion cost imposed on a single user of link b by an additional user on link a, and $v_b \frac{\partial(t_b(v))}{\partial v_a}$ is the additional total congestion cost imposed on (all users of) link b by an additional user on link a. Note that if link a has a separable trave time function, then the term $\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=v^*}$ becomes $v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=v_a^*}$.

For the rest of this chapter, we will assume that the side constraints $g_z(v) \le 0\ \forall z \in Z$ are not present.

2.3.3 Algorithm to determine the first best congestion pricing for fixed demand

Let us formally state the steps involve in solving first best congestion pricing or MCP for fixed demand assuming link separable travel time function:

1) Solve the system problem to get the optimal link flow vector $\bar{v}$

2) For each link a, compute $v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a}$

3) Update the new link travel time function to $t_a(v) + v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a} \quad \forall a \in A$

With this new link cost funtion, the users will be forced to 'play ' according to SO, and thus, the most efficient use of the transportation system is achieved.

If we let φ to be the congestion toll vector whose element φ_a is the congestion toll on link a, then, any congestion toll satisfying the following conditions will also result in an optimal flow pattern:

$$\boxed{\begin{aligned} &\sum_{a \in A} (t_a(\bar{v}) + \varphi_a)\,\delta_{ar} \ge \lambda^w \qquad \forall r \in R_w,\ \forall w \in W \\ &\sum_{a \in A} (t_a(\bar{v}) + \varphi_a)\bar{v}_a = \sum_{w \in W} \lambda^w d_w^* \end{aligned}} \qquad (2.12a)$$

Note that $\varphi_a = v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a}$ $(= MCP\ 'congestion'\ toll\ for\ link\ a)$ $\forall a \in A$ is a valid solution.

The above conditions can be writen in a matrix or condensed form as:

$$\boxed{\begin{aligned} \Lambda^T(t(\bar{v}) + \varphi) \ge \Gamma^T\lambda \\ (t(\bar{v}) + \varphi)^T\bar{v} = (d^*)^T\lambda \end{aligned}} \qquad EqC\_FD \qquad (2.12b)$$

From now on, we will refer to equation (2.12) as the equlibrium constraint for fixed demand which we will often abreviate as *EqC_FD* and read as *Equilibrium Constraint for Fixed Demand*

Thus, we can restate the **steps involve in finding optimal first best congestion tolls for fixed demand**:

1) Solve the system problem to get the optimal link flow vector $\bar{v}$

2) Find any congestion toll vector φ satisfying (2.12)

3) Update the link cost function to $t_a(v) + \varphi_a \quad \forall a \in A$

Again, with this new link travel time function, users individual attempts to minimize percieve travel time will end up being system optimum.

2.3.4 Secondary objectives on the tolls when demand is fixed.

In addition to the flexibility in calculating optimal tolls, the above transformation of the first other optimality conditions (equation (2.12)) has the following advantages:

1) It makes it easy to achieve secondary objectives on the tolls without deviating from the optimal flows(in all we denote the SO flow by $\bar{v}$). Some of the secondary objectives include:

i) Minimize the total toll collected. This can be stated as a linear mathematical program:

$$\min \bar{v}^T \varphi$$
$$s.t$$
$$EqC\_FD$$

ii) Minimize the total number of toll booths. This objective will be of very great importance where the cost of setting a toll booth is costly. The integer linear program which is given as folllows:

$$\min \sum_{a \in A} x_a$$
$$s.t$$
$$EqC\_FD$$
$$\varphi_a \leq M x_a \qquad \forall a \in A$$
$$x_a = \{0, 1\} \qquad \forall a \in A$$

Where M is a large number.

iii) Set the total toll collected equal to a specific amount Q. This objective can be used to target a revenue figure Q, where Q is greater or equal to the minimum possible revenue. The linear mathematical program can be formulated as follows:

$$\min 1$$
$$s.t$$
$$EqC\_FD$$
$$\bar{v}^T \varphi = Q$$

iv) Minimize the maximum toll on a link. This objective is used to maintain tolls as low as possible on each link. In real life application, low tolls per link is more acceptable by users when compared with higher tolls on fewer links. The two may yield the same total toll revenue, but the fact remains that, most users do not look at it in this way. With this objective, the tolls are evenly spread throughout the network. The mathematical program that solves this problem is again linear, as seen below:

$$\min(\max \varphi_a)$$
$$s.t$$
$$EqC\_FD$$

The min max problem can be transformed into a minimization problem as follows:

$$\min(k)$$
$$s.t$$
$$EqC\_FD$$

$$k \geq \varphi_a, \qquad \forall a \in A$$

The above formulations remain valid even when $\varphi_a \geq 0\ , \forall a \in A.$ Also observe that the solution sets of $(i), (ii), (iii), (iv)$ are respectively non empty since MCP is already a valid solution to all. For this reason, the multipliers exist. Note that without the secondary objectives $(i), (ii), (iii), and\ (iv)$ on the tolls, infinite number of different valid toll vectors exist.

2.4 First Best Princing scheme for Elastic Demand (FBP_ED)

In the foregoing, we demonstarted how to compute optimal first best congestion tolls for fixed demand. The fact that economists argue that as demand of a given commodity decreases as the cost increases leads to the introduction of variable demand in the intelligent transportation. It is intuitive to argue that users will be reluctant to use a certain facility if the cost they bear is greater than the benefit they derive in using such facility. In the transportation sector, it is likely that users who drive to a certain city to shop will look for an alternative shoping center immediately they notice that the cost of travel exceeds the benefit they derive from the shoping. When demand is fixed, it is assumed that users will still travel even when the travel cost on all routes grow to infinity. This is not the case when demand is elastic. If demand is made flexible, then the route flow will be zero for all routes with infinite costs, unless the benefit of travel grows to infinity as these costs grow to infinity.

2.4.1 System Problem_Elastic Demand (SP_ED)

When demand is elastic, we can no longer solve the system problem as in the case of fixed demand. The reason been that the mathematical formulation of SP_FD will result in an unrealistic optimal zero flow for all links. In other words, the pricing scheme will impose unacceptable tolls on links to achieve the zero flow traffic. This means that in any event, there is a toll threshold beyond which transportation cost, inevitably increases. We thus seek for another way to define the system problem when demand is elastic. As briefly mentioned before, it is very likely that users will still be willing to travel provided that the cost they perceive is not more than the benefit they derive. With this in mind, the user problem for elastic demand can now be seen as maximizing the user benefit and SP_ED will be to maximize the entire network social benefit or welfare. This leads to redefinition of Wardrop's principles in terms of social (or user) benefit instead of travel cost. The second Wardrop's principle can now be stated as follows: the average user benefit is maximum when no user can increse the economic benefit by unilaterally switching routes. It actually means that when users cooperate, their average benefit will be maximized. Let the OD demand in an OD pair be a function of the equilibrium OD travel cost, namely $d_w = D_w(\lambda_w)$. Here we assume that the demand function is a separable function of the shortest travel cost λ_w between OD pair $w \in W$. In this thesis, we further assume that d_w is invertible, strictly monotone and decreasing function of travel time. Denote the inverse demand or the benefit function by $\lambda_w = D_w^{-1}(d_w) = B_w(d_w),\ w \in W.$ As we described in fig 1.1, the inverse demand function can be seen as the cost a user is willing to bear for his travel. In other words, it is the benefit a user derives from traveling between OD pair $w \in W$.

The following function as in (39) can be used to evaluate the total network user benefit (UB):

$$UB = \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

Recall that the total system (or social) cost (SC) incurred by all network users is given by:

$$SC = \sum_{a \in A} v_a t_a(v)$$

Hence, the economic benefit (EB) is defined as:

$$EB = UB - SC$$
$$= \sum_{w \epsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma \; - t(v)^T v$$

We are now ready to define a mathematical program that solves SP_ED. Since the system controller would like to maximize the economic benefit (or social welfare), the following convex mathematical program can be used to achieve this objective:

$$\min Z = t(v)^T v \; - \sum_{w \epsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$\left.\begin{array}{ll} v = \Lambda f & \psi \\ \Gamma f = d & \lambda \\ f \geq 0 & \rho \\ d \geq 0 & \vartheta \end{array}\right\} \quad FeC\_ED$$

Where we have also assumed that there are no possible side constraints on OD demands $G(d) = 0$. Every other constraint remains as defined in fixed demand case, except that the (*) is no more on d which signifies the variability of demand. We will also call the above constraints the feasibility conditions for elastic demand which we will always refer to with the abbreviation *FeC_ED* which reads, *Fesibility Conditions for Elastic Demand.* Associated with the non negativity constraint of the OD demands is a multiplier $\vartheta \; \varepsilon \; \mathbb{R}^W$. The *KKT* first order optimality condition for the above program is given below:

Let L be the Lagrangian, and $\bar{v}$, $\bar{d}$ be the solution of the above program, then there exists multipliers $(\psi, \lambda, \rho, \vartheta)$ such that the following holds:

$$L \; = \; t(v)^T v - \sum_{w \epsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma + (\Lambda f - v)^T \psi + (d - \Gamma f)^T \lambda - f^T \rho - d^T \vartheta$$

$$\frac{\partial L}{\partial v} \; = \; t(v) + (\nabla t(\bar{v}))^T \bar{v} - \psi \; = \; 0$$

$$\Rightarrow t_a(\bar{v}) + \sum_{b \epsilon A} \bar{v}_b \frac{\partial (t_b(\bar{v}))}{\partial v_a} - \psi_a \; = \; 0 \qquad \forall a \epsilon A$$

$$\frac{\partial L}{\partial f} \; = \; \Lambda^T \psi - \Gamma^T \lambda - \rho \; = \; 0$$

$$\Rightarrow \sum_{a \epsilon A} \psi_a \, \delta_{ar} - \lambda^w - \rho^r \; = \; 0 \qquad \forall r \epsilon R_w, \; \forall w \epsilon W$$

$$\frac{\partial L}{\partial d} \; = \; \lambda - B(\bar{d}) - \vartheta \; = \; 0$$

$$\Rightarrow \lambda^w - B(\bar{d}_w) - \vartheta^w \; = \; 0 \qquad \forall w \epsilon W$$

$$f^T \rho \; = \; 0 \qquad \forall w \epsilon W \qquad (2.13)$$

$$d^T \vartheta \; = \; 0 \qquad \forall w \epsilon W \qquad (2.14)$$

$$\rho, \; \vartheta \; \geq \; 0, \qquad OR \qquad \rho^r, \vartheta^w \geq 0 \qquad \forall z \epsilon Z, \; \forall r \epsilon R \;\; , \forall w \epsilon W$$

Equations (2.13)and (2.14) are complementarity contraints.

Observations:

$$t_a(\bar{v}) + \sum_{b \epsilon A} \bar{v}_b \frac{\partial(t_b(\bar{v}))}{\partial v_a} = \psi_a \qquad \forall a \epsilon A$$

$$\sum_{a \varepsilon A} \psi_a \, \delta_{ar} = \lambda^w + \rho^r \qquad \forall r \epsilon R_w, \ \forall w \epsilon W$$

$$\sum_{a \epsilon A} \left(t_a(\bar{v}) + \sum_{b \epsilon A} \bar{v}_b \frac{\partial(t_b(\bar{v}))}{\partial v_a} \right) \delta_{ar} = [B(\bar{d}_w) + \vartheta^w] + \rho^r \qquad \forall r \epsilon R_w, \ \forall w \epsilon W$$

Since $\vartheta^w, \ \rho^r \ \geq 0, \ \forall r \epsilon R_w, \ \forall w \epsilon W$ the following holds:

$$\sum_{a \epsilon A} \left(t_a(\bar{v}) + \sum_{b \epsilon A} \bar{v}_b \frac{\partial(t_b(\bar{v}))}{\partial v_a} \right) \delta_{ar} \geq B(\bar{d}_w) \qquad \forall r \epsilon R_w, \ \forall w \epsilon W$$

When the link travel cost function is separable, then, the above reduces to

$$\sum_{a \epsilon A} \left(t_a(\bar{v}_a) + \bar{v}_a \frac{d(t_a(\bar{v}_a))}{dv_a} \right) \delta_{ar} \geq B(\bar{d}_w) \qquad \forall r \epsilon R_w, \ \forall w \epsilon W$$

2.4.2 User Problem_Elastic Demand (UP_ED)

When demand is elastic, users tend to maximize their individual benefit. Users will be willing to travel provided the benefit they derive is not less than the cost they incure. With this in mind, Wardrop's first principle can now be stated as follows: At equilibrium, the journey costs on all routes actually used are the same and equal to the benefit derived from making the trip, but less than those which would be experienced by a single vehicle on any unused route. The UP_ED can be mathematically formulated as a variational inequality. A link flow vector v^* and demand vector d^* form a user equilibrium flow if and only if they solve the following variational inequality:

$$t(v^*)^T(v - v^*) - B(d^*)^T(d' - d^*) \geq 0 \qquad \forall v \epsilon V, \ \forall d' \epsilon d \qquad (2.15)$$

$$OR \qquad t(v^*)^T v - B(d^*)^T d' \geq t(v^*)^T v^* - B(d^*)^T d^* \qquad \forall v \epsilon V, \ \forall d' \epsilon d$$

If the link and the benefit functions are separable and monotonic in flows, then, the user problem can be stated as a convex mathematical program:

$$\min_{v_a, d_w} \sum_{a \epsilon A} \int_0^{v_a} t_a(x) dx - \sum_{w \epsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t \qquad v \epsilon V$$

Any flow pattern that solves the above program satisfies Wardrop's user equilibrium conditions (see Yang and Huang (39) for proof). It can be readily seen that (v^*, d^*) solves the variational inequlity (vi). The UP_ED can be written as the following optimization problem, find (v^*, d^*) such that it solves

$$\min_{v,d} \ Z = \ t(v^*)^T v - B(d^*)^T d$$

$$s.t$$

$$FeC\_ED$$

Below is the first order optimality condition of the above system.

If (v^*, d^*) solves the above system, and L is the Lagrangian, then there exist multipliers $(\psi, \lambda, \rho, \vartheta)$ such that the following equations hold:

$$
\begin{aligned}
L &= t(v^*)^T v - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma + (\Lambda f - v)^T \psi + (d - \Gamma f)^T \lambda - f^T \rho - d^T \vartheta \\
\frac{\partial L}{\partial v} &= t(v^*) - \psi = 0 \\
&\Rightarrow t_a(v^*) - \psi_a = 0 && \forall a \in A \\
\frac{\partial L}{\partial f} &= \Lambda^T \psi - \Gamma^T \lambda - \rho = 0 \\
&\Rightarrow \sum_{a \in A} \psi_a \, \delta_{ar} - \lambda^w - \rho^r = 0 && \forall r \in R_w, \ \forall w \in W \\
\frac{\partial L}{\partial d} &= \lambda - B(d^*) - \vartheta = 0 \\
&\Rightarrow \lambda^w - B(d_w^*) - \vartheta^w = 0 && \forall w \in W \\
f^T \rho &= 0 && \forall w \in W \qquad (2.16) \\
d^T \vartheta &= 0 && \forall w \in W \qquad (2.17) \\
\rho, \ \vartheta &\geq 0, \qquad OR \qquad \rho^r, \vartheta^w \geq 0 && \forall z \in Z, \ \forall r \in R \ , \forall w \in W
\end{aligned}
$$

Equations (2.16) and (2.17) are the complementarity conditions.
Observations:

$$
\begin{aligned}
t_a(v^*) &= \psi_a && \forall a \in A \\
\sum_{a \in A} \psi_a \, \delta_{ar} &= \lambda^w + \rho^r && \forall r \in R_w, \ \forall w \in W \\
\sum_{a \in A} (t_a(v^*))\delta_{ar} &= [B(d_w^*) + \vartheta^w] + \rho^r && \forall r \in R_w, \ \forall w \in W
\end{aligned}
$$

If $f_r > 0, r \in R_w$ ($\Rightarrow d_w > 0$) then the complimentarity equations force ϑ^w and ρ^r to be zero, thus we have

$$
\boxed{\sum_{a \in A} t_a(v^*)\delta_{ar} = B(d_w^*) \qquad \forall f_r > 0, \ r \in R_w, \ \forall w \in W} \tag{2.18}
$$

Interpretation: At equilibrium, the travel times on all routes actually used for a given OD pair $w \,\varepsilon\, W$ are the same and equal to the benefit derived from making the trip. In general, the following equation holds:

$$
\begin{aligned}
\sum_{a \in A} t_a(v^*)\delta_{ar} &= B(d_w^*) + \vartheta^w + \rho^r && \forall r \in R_w, \ \forall w \in W \\
&\geq B(d_w^*) && \forall r \in R_w, \forall w \in W \text{ since } \vartheta^w, \ \rho^r \geq 0, \forall r \in R, \forall w \in W
\end{aligned}
$$

Interpretation: At equilibrium, the travel times on all routes for a given OD pair $w \in W$ are greater or equal to the benefit derived from making the trip. Recall that we have just shown that, at equilibrium, the travel times for all used routes for a given OD pair $w \in W$ are the same and equal to the benefit $B_w(d_w^*)$ derived from making the trip. In other words, at equilibrium, the journey times on all routes actually used are the same and equal to the benefit derived from making the trip, but also less than those which would be experienced by a single vehicle on any unused route (Wardrop's first principle). This leads to the following conclusion; any flow vector (v^*, t^*) that solves the variational inequality (2.15) is a user equilibrium flow.
From equation (2.7), the following holds:

$$\sum_{a \in A} t_a(v^*) v_a^* = \sum_{w \in W} \lambda^w d_w^*$$

$$= \sum_{w \in W} (B(d_w^*) + \vartheta^w) d_w^* = \sum_{w \in W} B(d_w^*) d_w^* + \sum_{w \in W} \vartheta^w d_w^* = \sum_{w \in W} B(d_w^*) d_w^*$$

In summary we have

$$\boxed{\begin{aligned} &\sum_{a \in A} t_a(v^*) \delta_{ar} \geq B(d_w^*) \qquad \forall r \in R_w,\ \forall w \in W \\ &\sum_{a \in A} t_a(v^*) v_a^* = \sum_{w \in W} B(d_w^*) d_w^* \end{aligned}} \tag{2.19}$$

OR

$$\Lambda^T t(v^*) \geq \Gamma^T B(d^*)$$
$$t(v^*)^T v^* = B(d^*)^T (d^*)$$

when written in matrix form.

Note that $t_a(v^*)$ becomes $t_a(v_a^*)$ when we assume the link travel time functions are separable. See that the only difference between this set of observations (first order optimality conditions) and that of SP_ED is the term $\sum_{b \in A} v_b^* \frac{\partial(t_b(v^*))}{\partial v_a}$. Everything said about this term in fixed demand case remains valid here (refer to UP_FD) .

2.4.3 Algorithm to determine the first best congestion pricing for Elastic demand

Formally, the following steps are involved in determining the first best congestion pricing tolls for elastic demand:

1) Solve the system problem (SP_ED) to get the optimal link flow $\bar{v}$ and demand vector $\bar{d}$

2) compute the optimal toll for each link a from $\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=\bar{v}}$

3) then update all link cost to $t_a(v) + \sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=\bar{v}} \qquad \forall a \in A$

With this link cost, users will be forced to behave in a way that achieves system optimal flow.

As shown in UP_FD, $\sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=\bar{v}}$ is not the only toll vector that achieves system optimal flow.

Any congestion toll vector φ whose element φ_a is the congestion toll on link a satisfying the following set of linear equations will also yield a system optimal flow:

$$\boxed{\begin{aligned} \Lambda^T (t(\bar{v}) + \varphi) \geq \Gamma^T B(\bar{d}) \\ (t(\bar{v}) + \varphi)^T \bar{v} = B(\bar{d})^T (\bar{d}) \end{aligned}} \qquad EqC\_ED \tag{2.20}$$

Where $(\bar{v}, \bar{d})$ is a solution set from SP_ED. The congestion toll vector φ whose element $\varphi_a = \sum_{b \in A} v_b \frac{\partial(t_b(v))}{\partial v_a}\Big|_{v=\bar{v}} = MCP$ congestion toll for link a is a valid congestion toll vector for the above system or $\varphi_a = v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a}$ when the link travel time functions are separable. We will call equation (2.20) the equilibrium constraint for elastic demand, which we will often refer with the abbreviation *EqC_ED* and read as *Equilibrium Constraint for Elastic Demand* .

Observe from the last equation that

$$\varphi^T\bar{v} = B(\bar{d})^T(\bar{d}) - t(\bar{v})\bar{v} \qquad (2.21)$$

The *RHS* of equation (2.21) is a constant. This shows that the total congestion toll collected in a network is constant no matter what toll vector one uses.

Since any toll vector satisfying (2.20) yields system optimal flow pattern, we then redefine the **steps involve in computing first best congestion tolls** as follows:

1) Solve SP_ED to obtain the optimal link flows $\bar{v}$ and demand vector $\bar{d}$.
2) find any toll vector φ that satisfies condition (2.20)
3) update the travel cost of each link to $t_a(v) + \varphi_a$

With φ_a added to each link travel time function, the non cooperative behaviour of users will end up producing a system optimal flow pattern.

We now enumarate again the advantages one can derive from the above tranformation of the *KKT* optimality conditions.

2.4.4 Secondary objectives on the tolls when demand is elastic

The secondary objectives on the tolls for elastic demand is the same as formulated in sub section 2.2.4 with *EqC_ED* replacing *EqC_FD*.

First best pricing scheme can be solved to optimality with any non linear optimization software like CPLEX.

We conclude this section by demonstrating that the above toll pricing framework still obeys the principle of marginal cost pricing. The principle states that every user should pay for the (congestion) externality he/she is causing. Recall from (2.18) that the following equation was derived for all used path for the w^{th} OD pair:

$$\sum_{a \in A} t_a(v^*)\delta_{ar} = B(d_w^*) \qquad \forall f_r > 0,\ r \in R_w,\ \forall w \in W$$

The tolled user equilibrium will now be

$$\sum_{a \in A} (t_a(v^*) + \varphi_a)\delta_{ar} = B_w(d_w^*) \qquad \forall f_r > 0,\ r \in R_w,\ \forall w \in W \quad (2.22)$$

Recall also that we noted above that $\varphi_a = v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a} = MCP$ congestion toll for link a is a valid congestion toll for link a. The following thus holds:

$$\sum_{a \in A} \left(t_a(v^*) + v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a} \right)\delta_{ar} = B_w(d_w^*) \qquad \forall f_r > 0,\ r \in R_w,\ \forall w \in W \quad (2.23)$$

$$OR \quad \sum_{a \in A} \left(v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a} \right)\delta_{ar} = B_w(d_w^*) - \sum_{a \in A} t_a(v^*)\delta_{ar} \qquad \forall f_r > 0,\ r \in R_w,\ \forall w \in W \quad (2.24)$$

It follows from equation (2.23) that $B_w(d_w^*)$ is the marginal cost between OD pair $w \in W$. Since $\sum_{a \in A} t_a(v^*)\delta_{ar}$ is the private cost of all users of route $r \in R_w$, it implies that $B_w(d_w^*) - \sum_{a \in A} t_a(v^*)\delta_{ar}$ is the total congestion externality by users of route $r \in R_w$. The term on the *LHS* of equation (2.24) confirms this result. We now rewrite equation (2.22) to get:

$$\sum_{a \in A} \varphi_a\delta_{ar} = B_w(d_w^*) - \sum_{a \in A} t_a(v^*)\delta_{ar} \qquad f_r > 0,\ r \in R_w, \forall w \in W; \quad w = (p,q) \quad (2.22')$$

Observe that the right hand sides of equations (2.24) and (2.22′)) are the same. This means that $\sum_{a \in A} (\varphi_a)\delta_{ar} = \sum_{a \in A} \left(v_a \frac{d(t_a(v_a))}{dv_a}\Big|_{v_a=\bar{v}_a} \right)\delta_{ar}$ showing that the toll pricing framework described in this section still obeys the principle of marginal cost pricing on path level.

Next, we look at the general mathematical overview of the so called second best pricing scheme.

2.5 Second Best Pricing scheme for Fixed Demand (SBP_FD)

As we mentioned in section 1.3.3, due to some reasons which range from political to technical constraints, first best pricing may not meet some practical requirements. It may be that tolls are not allowed in some sections of the network, (or that tolls are not allowed on some links). With this constraint in place, it is no more assured that there will exist a toll vector that achieves the most efficient use of the transportation network. In other words, optimal flow pattern is no longer guaranteed. When this is the case, we then seek for the toll vector that will give the 'best' use of the transportation network with the requirement that some links cannot be tolled. Since the flow pattern achieved by this toll vector is in general not optimal, we call it second best toll vector. We apply directly the results derived in section 2.3 in the second best toll pricing theory for fixed demand.

Denote by Y the set of links in the network that cannot be tolled. In general, second best pricing formulations for fixed demand take the form below:

$$\min \ Total\ System\ TravelCost$$
$$s.t$$

*The flows and demands are feasible

*The flows and demands are in constrained tolled user equilibrium

*Certain links cannot be tolled.

2.5.1 Algorithm for solving the second best congestion toll pricing for fixed demand

1) Solve SP_FD to obtain the optimal flow pattern $\bar{v}$

2) Find solution set $\mathcal{L}$ containing any congestion toll vector φ satisfying

$$EqC\_FD$$
$$\varphi_a = 0 \qquad \forall a \in Y$$

3) Check if $\mathcal{L}$ is empty, if NO, GOTO step 4 to compute the First Best congestion tolls, else GOTO step 5 to compute the second best congestion tolls

4) The vector φ is the first best congestion toll vector. Update the link cost function to $t_a(v) + \varphi_a \quad \forall a \in A$ where $\varphi_a = 0 \ \forall a \in Y$ STOP.

5) Using $\bar{v}$ as the initial flow vector, solve the following bilevel toll pricing problem (see appendix (A) for notes on bilevel)

$$\min t(v)^T v$$
$$s.t$$
$$[t(v) + \varphi]^T u \geq [t(v) + \varphi]^T v \qquad \forall u \in V$$
$$\varphi_a = 0 \qquad \forall a \in Y$$

In step 5, the first constraint ensures that the solution (feasible flow pattern) of the above system is a user equilibrium flow. This is why this bilevel program is commonly referred to as Mathematical Program for Equilibrium Constraint ($MPEC$). The above program can be written as a real bi-level problem:

$$\begin{aligned} &\min v^T t(v) \\ &s.t \\ &v\ solves \quad \min_{u_a} \ (t_a(v) + \varphi_a) u_a \\ &\qquad\qquad s.t \\ &\qquad\qquad FeC\_FD \\ &\qquad\qquad \varphi_a = 0 \qquad \forall a \in Y \end{aligned}$$

Observe that the constraint set of the 'mother' problem is the same as UP_FD except that the toll φ_a has been added to each link cost function and ofcourse the last condition. For this reason, all mathematical transformations in UP_FD remain valid. We now restate the above program as a single level program using the optimality conditions described in FBP_FD:

$$\min v^T t(v)$$

$$s.t$$

$$EqC\_FD$$

$$FeC\_FD$$

$$\varphi_a = 0 \qquad \forall a \in Y$$

The objective solves the system problem and the constraints solves the user problem. The last constraint ensures that only tollable arcs are tolled. It is this last requirement that differentiates SBP_FD from FBP_FD. (Note here that the above problem definition can be used to solve first best pricing without the last constraint). It is unfortunate to note also that the above problem has convex objective with non linear and non convex constraint. The second contraint is neither linear nor convex due to the term φv. This is why second best toll pricing problem is in the class of mathematically hard problems. The proof of this can be seen in (18) Note that if the algorithm terminates in step 4, then the toll vector φ is the first best toll, and so, can achieve the most efficient use of the transportation network. We call this type of first best toll vector, the constrained first best congestion toll for fixed demand (CFBT_FD). If the algorithm does not terminate in step 4, then the program in step 5 is a typical bilevel MPEC problem which is notoriously hard to solve to optimality even when the network is small. It can be solved with non linear commercial solvers like MINOS and CPLEX built-in solvers. Since the program has non linear and non convex constraint, it means that finding global optimum is not guaranteed with these solvers. Verhoef (34) proposed algorithms for finding second best optimal toll levels and toll points. Yildirim (42)enumarated setbacks of Verhoef's algorithm which include the failure of the agorithm to yield a flow pattern better or the same as the user equlibrium flow pattern in some special cases. We thus state the following lemma:

Lemma 2
If SP_FD and UP_FD have solutions $(\bar{v})$ and (v^*) respectively and the link travel time functions $t(v)$ are continuous and monotonic in v, then, the second best algorithm in step1-5 above has a global minimum in the interval $[\bar{v}t(\bar{v})\ ,\ v^*t(v^*)]$ provided the derivative $\nabla t(v)$ exists.

Proof:
The proof is a very simple one. Assume that $v't(v')$ is the solution of the algorithm above. If the algorithm terminates in step 4, then the feasible flow pattern is the same as the system optimum flow pattern (SP_FD) and the total system cost is $\bar{v}t(\bar{v})$ (i.e $v't(v') = \bar{v}t(\bar{v})$) which is the best one can get. On the other hand, if the algorithm does not terminate in step 4, then, the program in step 5 is solved. Since the solution of the user problem UP_FD ($v^*t(v^*)$) is a feasible solution of the program in step 5, then, it means that $\varphi_a = 0\ ;\forall a\ \varepsilon\ A$ is feasible toll vector. Search for a better solution will force some link tolls $\varphi_a\ \forall a\ \varepsilon\ A \setminus Y$ to be non zero. That is $v^*t(v^*) \geq v't(v')$. It simply means that $v't(v')$ is bounded below by $\bar{v}t(\bar{v})$ and above by $v^*t(v^*)$, or

$$\bar{v}t(\bar{v}) \leq v't(v') \leq v^*t(v^*) \qquad \square$$

2.6 Second Best Pricing scheme for Elastic Demand (SBP_ED)
The formulations we use in this section is similar to those used in SBP_FD. The difference is that the fixed demand results of section 2.3 is now replaced with elastic demand formulations of section 2.4. In

general, second best pricing formulations for elastic demand take the form below:

max *The Economic Benefit*

$s.t$

*The flows and demands are feasible

*The flows and demands are in constrained tolled user equilibrium

*Certain links cannot be tolled.

2.6.1 Algorithm for solving the second best congestion toll pricing for elastic demand

1) Solve the SP_ED to obtain the optimal flow pattern $\bar{v}, \bar{d}$

2) Find solution set $\mathcal{F}$ containing any congestion toll vector φ satisfying

$$EqC\_ED$$

$$\varphi_a = 0 \qquad \forall a \in Y$$

3) Check if $\mathcal{F}$ is empty, if NO, GOTO step 4 to compute the First Best Pricing congestion tolls, else GOTO step 5 to compute the Second Best congestion tolls

4) The vector φ is the first best congestion toll vector. Update the link cost function to $t_a(v) + \varphi_a \quad \forall a \in A$ where $\varphi_a = 0 \; \forall a \in Y$ STOP.

5) Using $\bar{v}, \bar{d}$ as the initial flow vectors, solve the following bilevel toll pricing problem

$$\min Z = vt(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$(t(v) + \varphi)^T (u - v) - B(d)^T (d' - d) \geq 0 \qquad \forall u \in V, \; \forall d' \in d$$

$$\varphi_a = 0 \qquad \forall a \in Y$$

$$FeC\_ED$$

The object is as defined in FBP_ED and the constraints are as defined in SBP_FD. This again is an *MPEC* problem since the first constraint ensures that the feasible flow pattern is a user equilibrium flow pattern. The program in step 5 can be restated as:

$$\min Z = vt(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$\min_{u,d'} \; (t(v) + \varphi)u - B(d)d'$$

$$s.t$$

$$FeC\_ED$$

$$\varphi_a = 0 \qquad \forall a \in Y$$

Which can be written as a single level problem using the optimality conditions described in FBP_ED:

$$\min Z = vt(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$EqC\_ED$$

$$FeC\_ED$$

$$\varphi_a = 0 \qquad \forall a \in Y$$

The first two equations ensure that the solution generated by the above system is a feasible user equilibrium flow, and the las constraint ensures feasibilty of the resulting toll. As discussed in SBP_FD, MPECs are hard to solve due to non linearity and non convexity of the first constraint. The algorithm above has global optimum solution if the system and the user problems have solutions. This global optimum may be difficult to achieve by non linear optimization tools. We again state the following lemma:

Lemma 3
If SP_ED and UP_ED have solutions $(\bar{v},\bar{d})$ and (v^*,d^*) respectively and the functions $t(v)$, $B(d)$ are continuous and monotonic in v and d respectively, then, the second best algorithm in step1-5 above has a global minimum in the interval $\left[v^*t(v^*) - \sum_{w \epsilon W} \int_0^{d_w^*} B_w(\varsigma)d\varsigma \ , \ \bar{v}t(\bar{v}) - \sum_{w \epsilon W} \int_0^{\bar{d}_w} B_w(\varsigma)d\varsigma \right]$ provided the derivatives $(\nabla t(v) \ , \nabla B(d))$ exist.

Proof:
The proof is very similar to that given in SBP_FD. □
See that all the pricing schemes mentioned above consider only one objective; the travel time. The system problems aim at minimizing total system travel time (or maximizing economic benefit in the case of elastic demand). They neglet the effect of other road traffic externalities which might have a significant effects on the social welfare. Economically, congestion tolling does not yield the optimal social welfare (with respect to transportation system) since other road traffic externalities have significant effects on poeple's welfare. Observe also that these models assume that transportation system is in control of one hand and thus only one body is capable of setting tolls on the roads. In general, this is not always the case as many stakeholders affected by transportation system may have the authority to place tolls for selfish or equity reasons. It is in regard of the first shortcoming that we discuss in the next chapter, multi-objective road pricing schemes.

Chapter 3
Single-Leader Multi-Objective Road Pricing Problem

3.1 Introduction and Overview of Transportation Externalities

Over the years, many countries have implemented the so called congestion pricing schemes. Following the positive results of such schemes on congestion, they have also claimed that the societal welfare has also been positively affected. In the literature, many spacial economists have continuously used a function that considers only the travel time and the user's benefit to optimize the social welfare in the transportation sector. They have over time neglected the impact of other traffic externalities such as air pollution, noise pollution, traffic safety and pavement damage among others. In this thesis, we claim that the negligence of these other externalities in pricing schemes does not lead in general to optimal social welfare. This is one of the motivations of this chapter. To capture almost all the damages a network user causes other users, the environment, residents and the future generations, we incorporate in our models the most important traffic externalities in the definition of the social welfare. In this section, we will briefly look at the mentioned traffic externalities. Since we have dealt with congestion externality in the previous chapter, we now go ahead to discuss other externalities.

Air Pollution (Emission)

Air pollution is the introduction of chemicals, particulate matter, or biological materials that cause harm or discomfort to humans or other living organisms, or damages the natural environment, into the atmosphere. Though it is always difficult for all countries of the world to agree on a common project, the issue of climate change is an exceptional case. Everybody agrees that our climate must be protected. Due to technological advancement and human actions, our climate and of course our environment deteriorate every second of the day. The negative effects of human actions on the environment as well as on humans include; global warming, ozone depletion, smog, haze, invisibility, acid rain, respiratory problems, eye irritation, restlessness and discomfort among others. A greenhouse gas, CO_2, though vital for living organisms, is one of the major causes of global warming and the so called acid rain. It traps the heat emitted by the earth's surface thus increasing the temperature of our environment. When it reacts with atmospheric water vapour or simply water, a very weak acid called carbonic acid (trioxocarbonate (*iv*) acid) is formed. This weak acid (though unstable) can slightly increase the acidity of an unpolluted rain.

Carbon monoxide (CO) is a colorless poisonous gas which forms a stable compound with hemoglobin in the blood when inhaled by living things. This causes a reduction in the oxygen transportation from the lungs to the body cells. High concentration of CO can increase the risk of cardiovascular problems and impede the psychomotor functions.

An organic compound methane (CH_4) is also one of the three main compounds that causes global warming besides CO_2 and water vapor ($H_2O(g)$).

Sulphur oxides (SO_x) when dissolved in atmospheric water vapour or rain cause acid rain. They also cause lung irritation.

Some nitrogen oxidesNO_x compounds are toxic. They cause eutrophication (nutrient overload in water bodies), contribute to the formation of smog, and known to be ground level ozone precursors. These toxic oxides of nitrogen also cause ill health in humans and other animals, and these include; decrease in pulmonary function, inflammation of the lungs and immunological changes. Reaction of NO_2 with water droplets results in nitric acid (HNO_3) which causes acid rain.

Volatile organic compounds ($VOCs$) are one of the major causes of aerosols. They are very dangerous to health and they are also known to be precursor to the formation of ground level ozone. When NO_x reacts with $VOCs$, ozone (O_3) is released. Ozone, which is beneficial in the upper atmosphere where it protects the Earth by filtering out ultraviolet radiation, has been identified as one of the leading causes of chronic respiratory diseases when found at ground level. Eye inflammation has over time been associated with ground level ozone.

Particulate matters (PM_s) are solid particles suspended in the atmosphere. They include; resuspended road dust, smoke, and liquid droplets. PM_s can cause chronic and acute bronchitis, lung cancer, chest illness and chronic respiratory diseases when inhaled.

The presence of NO_x, PM_{10}, SO_2, CO, CO_2, O_3, $VOC(HC)$ and lead (Pb) in the atmosphere has extremely been associated with road traffic;

NO_x are formed when fuel is burned at high pressure and temperature conditions. This induces the dissociation and subsequent recombination of atmospheric N_2 and O_2, an action that generates NO_x.

PM_{10} are released into the atmosphere from so many sources. Brake pads and tires of motor vehicles are examples of such sources. Reaction of gases (e.g. NO_x, SO_2, and NH_3) from burning fuel with atmospheric water vapour leads to suspension of particulate matter in the atmosphere. Solid carbons leaving the exhaust pipes of vehicles in the form of smoke, constitute part of the solid particles seen in the atmosphere.

SO_x is one of the principal emissions from diesel engines.

CO is released into the atmosphere when fuel combustion is incomplete.

Reaction between CO and atmospheric oxygen releases CO_2 into the atmosphere. In the Netherlands, traffic and transportation is responsible for approximately 20% of the emitted CO_2. CO_2 emission is proportional to the vehicle's fuel consumption rate, which in turn, depends on the smoothness of the traffic flow (38).

$VOCs$ (HC) can either be released into the atmosphere as a by-product of incomplete fuel combustion or as a vapour due to fuel evaporation.

Photochemical reactions which involve principally nitrogen oxides (NO_x), oxygen (O_2), and hydrocarbons (HC_s), in the presence of sunlight release (ground level) ozone (O_3) into the atmosphere.

Most fuels contain lead compounds which prevent knocking in the engine. When these fuels burn, the lead compounds are released into the atmosphere. Lead compounds when inhaled can be very injurious to health.

Noise pollution

Noise pollution is defined to be an annoying and potentially harmful environmental noise. It can be as a result of factory machines or road traffic among others. Road traffic noise results from two main factors; propulsion noise and tyre/road noise (rolling noise). Factors that influence the sound pressure level are traffic volume, speed, traffic composition (vehicle types), road design (i.e. slopes, crossings, speed bumps) and reflection, absorption and dispersion (i.e. road surface, walls, trees) (38). The adverse effects of traffic noise include; annoyance, disturbance, high blood pressure, certain cardiovascular diseases, limited mental illness, lethal heart attack and restlessness. These effects are already recognised by the world health organisation (WHO) as serious health problems on humans. Research has proved that sounds above 55dB are potentially dangerous to health. In 2000, it was recorded that more than 44% of the EU25* population (about 210 million people) were regularly exposed to road traffic noise level of over 55dB (7), a level, as stated above, that is dangerous to health. Research reveals that three quarter of Dutch houses experience a cumulative sound pressure of over 50dB (38). It might be interesting to note that, the social costs of traffic noise in the EU22** amount to at least € 40 billion per year (0.4% of total GDP). The bulk of these costs (about 90%) are caused by passenger cars and lorries (7).

* EU25 refers to EU27 except Cyprus and Malta.

**EU22 refers to EU27 except Cyprus, Estonia, Latvia, Lithuania and Malta.

Traffic safety

Mortality rate due to road traffic accidents is alarmingly increasing in most cities of the world. It has become an urgent point of attention for the government as well as individuals since it involves immediate lost of life and severe injuries. Amazingly, it is also a major concern for insurance companies who seek for less frequent accident occurrence since this would translate to making more profit. As stated in (38), traffic safety can be discussed under three subjects; objective and subjective traffic safety, and external safety. Objective safety quantifies the traffic safety. It divides the actual number of crashes into fatal, injuries (combined with casualties) and material damage. Factors influencing objective traffic safety include; human (e.g. use of alcohol and high speed), vehicle type (e.g. mass differences), and thirdly the road type/design. Full description of other subjects can be seen in (38).

Though the number of death and/or injury as a result of road accident is on the increase in most countries, it may be worth mentioning that despite the increase in traffic volume, an EU country, the Netherlands is becoming increasingly safer over the past years. The trend that describes the number of injuries per year in the Netherlands is steeply approaching 'zero'(38).

Road damage

Road pavements are built according to specifications. Factors that determine the type of a pavement include; geographical location and feasibility, type of vehicles that use the pavement, and financial availability among others. When heavy vehicles start using roads meant for small cars, then, it is very likely that the pavements start to dilapidate. The adverse effects of this road dilapidation range from high cost of repair to road accidents. Annoyance and discomfort to users are also fallouts of dilapidated roads. When the traffic flow keeps exceeding the construction strength or the capacity of a pavement, the pavement also starts to crumble. This means that, even when road pavements are used according to vehicle specifications, dense traffic can pose a serious danger to the durability of roads. Loads, which are the vehicle forces exerted on the pavements, can be characterized by tire loads, axle and tire configurations, vehicle speed, traffic distribution across the pavement and load repetition. Tire loads are the forces exerted due to tire-pavement contact. Axle and tire configuration describe how many tire contact points on a pavement, and how close they are to each other. When tires are close to each other, the pressure exerted per pavement area is increased. Slow and steady vehicles tend to create more damage to pavements. So, reducing traffic congestion will help preserve pavements to some extent. Since continuous heavy traffic on a road segment will cause this road segment to deteriorate, good traffic distribution is necessary to preserve the high cost infrastructures. In our thesis, we define the pavement structural design by quantifying all expected loads a pavement will encounter over its design life. We do this by using the equivalent single axle loads ($ESAL$) which converts the wheel loads of various magnitudes and repetitions, to an equivalent number of standard (or equivalent) loads based on the amount of damage they cause to the pavement. The commonly used standard load is the $18,000lb$ equivalent single axle load. As a rule of thumb, the load equivalent factor (LEF) of each vehicle (and also the pavement/infrastructure damage imparted by each vehicle) can be roughly determined from $\left(\frac{vehicle\ weight(lb)}{18,000}\right)^4$(45).

Since man-hour is precious, and the cost of road maintenance and health care very high, it means that huge amount of money is been lost every single second from the effects of road traffic externalities. We mention again, that the search for an optimal traffic flow that minimizes the mentioned traffic externalities is one of the main motives of this thesis. Minimizing the mentioned effects, of course, means reducing cost for road users and non users, for the government and for organizations like insurance companies. Above all, it also means protecting our planet Earth from excessive heat which poses threat to all living things. These in turn, imply minimizing the societal costs or maximizing the welfare of the general society. To achieve this aim, we must put into consideration all the mentioned effects and sort for a way to minimize the cost generated by each effect. A closer look tells one that we are already confronted with a multi-objective problem, and this leads to the introduction of overview of multi-objective problem in the next section.

3.2 Overview of multiobjective problems

More realistic optimization problems require the simultaneous optimization of more than one objective function. This is because many real life problems are defined in many objectives. In most cases, these objectives are in conflict with each other and may or may not be equally important. Examples of realistic optimization problems that involve more than one objective function are as follows:

1) In the bridge construction, a good design is characterized by low total mass and high stiffness.
2) Aircraft design requires simultaneous optimization of fuel efficiency, payload and weight.
3) A good road pricing model that maximizes social or economic welfare must involve the simultaneous minimization of travel time, noise and air pollution, road accident, road damage and maximization of user benefit.
4) The traditional portfolio optimization problem attempts to simultaneously minimize the risk and maximize the fiscal return.
5) In chemical design, or in design of a ground water remediation facility, objectives to look out for include total investment and net operating costs.
6) In car manufacturing industries, two objectives; minimization of the noise a driver hears and maximization of ventilation are used to define a good sun roof design.
7) In logistics and supply chain, a good scheduling of truck routes minimizes dead head miles while equalizing work load among drivers.
8) Optimization problem in radiation therapy planning always aim at minimizing radiation dose on the normal tissues while maximizing dose on the tumor regions.

In these and many other cases, it is unlikely that the different objectives would be optimized by the same alternative parameter choice. This means that some trade-offs are needed between criteria to ensure a satisfactory model. Solving a multiobjective problem often results in a multitude of solutions and not all these solutions are of interest. For a solution to be interesting, there must exist a dominance relation between the solution considered and the other solutions. We say that a solution vector $\bar{v}$ dominates another solution vector v^* if:

$\bar{v}$ is at least as good as v^* for all objectives, and

$\bar{v}$ is strictly better than v^* for at least one objective.

In general, a multiobjective problem has no optimal solution that could optimize all objectives simultaneously. But there exists a set of non-dominant or non-inferior or equally efficient alternative solutions, known as the Pareto optimal set. In other words, solutions which dominate others but do not dominate themselves are called optimal solutions in the Pareto sense. A Pareto optimal solution has the property that it is not possible to reduce any of the objective functions without increasing at least one of the other objective functions. Let us formally give the definition of Pareto optimal solution.

Definition 3.2.1a

If $C_k(v)$ denotes the objective function for objective $k \; \varepsilon \; K$, a solution vector $\bar{v} \; \varepsilon \; V$ is Pareto optimal if and only if there does not exist any other solution vector $v \; \varepsilon \; V$ such that the following holds:

$$C_k(v) \leq C_k(\bar{v}) \qquad \forall k \; \varepsilon \; K \qquad and$$

$$C_j(v) < C_j(\bar{v}) \qquad \text{for at least one } j \; \varepsilon \; K$$

Set of all Pareto points (sometimes called efficient points) to multiobjective optimization problem is called the Pareto or efficient frontier (20). Solution methods for multiobjective problems are given in appendix B. We now focus on the solution method for the objectives of interest in this thesis. The next section discusses the objective functions and solution method we employ in attempt to optimize all objective simultaneously.

3.3 Multiobjective in transportation: formulation and solution method

Relevant objectives

In our model, we seek an optimal flow pattern given that all other network attributes such as links' freeflow speeds and capacities, average number of household around a given road, emission factors, value of time among others are given. Thus the only variable we consider in our optimization is the traffic flows. Hence, we define the following relevant objectives:

Let

$t(v)$ be the vector of link travel time functions whose element,
$t_a(v)$, is the travel time function (in minutes) for link a.

$e(v)$ be the vector of link emission functions whose element,
$e_a(v)$ is the emission function (in grammes) for link a.

$n(v)$ be the vector of link noise functions whose element,
$n_a(v)$ is the noise function (in decibels(dB)) for link a.

$i(v)$ be the vector of infrastructure-damage cost functions whose element,
$i_a(v)$ is the infrastructure-damage cost function (in monetary value) for link a.

$s(v)$ be the vector of safety functions whose element,
$s_a(v)$ is the safety function (in number of injury crashes) for link a.

K be the set of objective indices, i.e. $K = \{t, e, n, i, s\}$, where t, e, n, i *and* s stand for travel time, emission, noise,
infrastructure and safety respectively.

See page (41 *and* 42) for concrete functions we use in this thesis.

Since we want to know when a trade-off is beneficial or not, it is necessary that all objectives be defined in one and the same unit. By such definition, we can easily shrink the Pareto set of solutions. For the mean time, we consider only one user and one vehicle class. Thus, we define the following monetary values for the above objectives:

Let

β be the monetary value of time per minute (VOT).

α be the vector of monetary value of emission per gramme (VOE) whose element,
α_a is the monetary value of emission per gramme (VOE) which depends on the urbanization (among other factors)
of link a.

γ be the monetary equivalent of $1dB(A)$ (VON) defined for a certain noise level.

ϱ be the 'average' monetary cost of an injury crash (VOS).

C be the vector of total network cost functions whose element,
$C_k(v)$ is the total network cost function for the k^{th} objective under the flow vector v.
E.g. $C_t(v) = \sum_{a \varepsilon A} \beta v_a t_a(v)$

$T(v)$ be the total network cost under the flow vector v.

$T(v) = \sum_{k \epsilon K} C_k(v)$

3.3.1 Model theory, formulation and solution method

As we mentioned earlier, there does not exist a vector v that optimizes all objectives simultaneously, otherwise, there is no need to solve the problem with multiple objectives. Our aim therefore is to find a satisfactory trade off between objectives. In the real sense, we intend to get as close as we can to the

optimal value of each objective.
Define v_k^* to be the optimal flow vector for objective k. This means that $C_k(v_k^*)$ is the optimal value (in the absolute sense) for objective k. We will call v_k^* the ideal vector point for objective k, and $C_k(v_k^*)$, the ideal objective value for objective k, and C^*, the ideal objective vector containing the ideal objective values. Observe that for any objective $k \varepsilon K$, $v_k^* \neq v_j^*$ for at least one objective $j \varepsilon K$, where $k \neq j$, otherwise, the objectives would not be in conflict with one another. $C_k(v_k^*)$ will serve as the lower bound for objective k. Since all the ideal points can not be reached simultaneously, we find a solution vector $\bar{v}$ that will make all objectives as close as possible to their respective ideal points. We have mentioned earlier that a multitude of feasible solutions are obtained in a multiobjective optimization problem. The question is, which of these solutions will be chosen? This decision depends solely on the choice of the decision maker (dm). As some objectives may be more important to the dm than others, solutions more favourable to less important objectives are discarded and solutions favourable to more important objectives are selected for further analysis. The judgement of which solution to choose among the remaining or equally important objectives is always a challenging task since no solution is better than the other in Pareto sense. It may be possible for the decision maker to define a utility or value function to enable him evaluate or quantify the trade offs or the objective values. If an explicit utility function can be constructed, then, the objectives can be aggregated into one criterion, and in this way, the multiobjective problem reduces to single objective problem. Definition of a good value or utility function is very difficult in practice though. No matter the valuation of the dm, the final accepted solution should be a point on the Pareto frontier.
We now assume that we have only one decision maker, say the government. Without loss of generality, we assume that the dm wants to keep as low as possible the costs of total system travel time, emission, noise, safety, and road damage. Since it is not possible to find a feasible flow vector v^* that results in ideal objective value for all objectives simultaneously, we then seek a feasible flow pattern (or vector) $\bar{v}$ that is as close as possible to the interest of the dm. The question now is; what is the interest of the dm? As we said, he wants to keep as low as possible the cost of individual mentioned objectives. In other words, he wants to find a flow vector $\bar{v}$ such that the resulting objective values are as close as possible to their respective ideal values while making useful trade offs between objectives. Mathematically, he intends to find $\bar{v}$ such that the trade off $C_k(\bar{v}_k) - C_k(v_k^*) = \epsilon_k$, is small $\forall k \varepsilon K$, and the sum of the trade offs $\sum_{k \varepsilon K} \epsilon_k$ is as small as possible. In this way, we have defined in literal terms the utility or value function for the decision maker. Since all our objectives are monetized, and for the fact that the dm's sole interest is to minimize the entire network cost, give no objective function preference over the other. Since the efficient solution points for the multiobjective problem are all on Pareto frontier, any choice for one solution point that favours one objective may lead to increase in the cost of another objective value. This move may or may not decrease the entire network cost. We then seek a flow vector $\bar{v}$ on the Pareto frontier that minimizes the total network cost.
The dm's objective to keep the trade off ϵ_k as small as possible for each $k \varepsilon K$, and his intention to minimize the total trade off translate to one aim; minimizing the entire network costs. This leads to the formulation of the following objective:

$$\min \sum_{k \varepsilon K} (C_k(v) - C_k(v_k^*)) \qquad (3.1)$$

$$s.t$$

$$v \varepsilon V$$

where $C_k(v_k^*)$ is the ideal objective value for objective $k \varepsilon K$.
Observation: The objective tends to minimize the gap between objective value and the ideal objective value for all objectives. By the summation, equation (3.1) minimizes the sum of the trade off which ensures that the trade offs made between objectives result in a minimum system cost. Another look at

(3.1) reveals that the formulation is simply minimizing the entire network costs. Since all dominant solutions lie on the Pareto frontier, we now prove that solution to the above problem is Pareto optimal .

Observation 1

The multiobjective formulation (3.1) is a special case of the standard multiobjective solution method, *weighted p – Norm method* (*or method of weighted different metrics*)

$$\min \ Z_p = \left[\sum_{k \epsilon K} [w_k(C_k(v) - C_k(v_k^*))]^p \right]^{1/p} \qquad (3.2)$$

$$s.t$$

$$v \, \varepsilon \, V$$

Where $w_k \geq 0 \ \forall k \epsilon K$

See that if $w_k = 1 \ \forall k \epsilon K \ and \ p = 1,$ then, (3.2) reduces to (3.1). This choice of equal weigh in (3.1) implies that all are objectives are equally important for the reasons given earlier. Next we proof that any flow vector that solves (3.2) is a Pareto optimal flow vector (25)

Theorem 3.1

The solution of the multiobjective problem (3.2) (*when* $1 \leq p \leq \infty$) is Pareto optimal if either

i) the solution is unique or

ii) all the weighting coefficients are positive.

Proof

i) Let $\bar{v} \, \varepsilon \, V$ be a unique solution to system (3.2). Suppose $\bar{v}$ is not Pareto optimal, then there exist $v \, \varepsilon \, V$ such that

$$C_k(v) \leq C_k(\bar{v}) \qquad \forall k \, \varepsilon \, K \qquad and$$

$$C_l(v) < C_l(\bar{v}) \qquad \text{for at least one } l \, \varepsilon \, K$$

and since $w_k \geq 0 \ \forall k \epsilon K,$ we have that

$$\Rightarrow \quad [w_k(C_k(v) - C_k(v_k^*))]^p \leq [w_k(C_k(\bar{v}) - C_k(v_k^*))]^p \qquad \forall k \, \varepsilon \, K$$

$$\Rightarrow \sum_{k \, \varepsilon \, K} [w_k(C_k(v) - C_k(v_k^*))]^p \leq \sum_{k \, \varepsilon \, K} [w_k(C_k(\bar{v}) - C_k(v_k^*))]^p$$

$$OR \left[\sum_{k \, \varepsilon \, K} [w_k(C_k(v) - C_k(v_k^*))]^p \right]^{1/p} \leq \left[\sum_{k \, \varepsilon \, K} [w_k(C_k(\bar{v}) - C_k(v_k^*))]^p \right]^{1/p}$$

But, uniqueness of $\bar{v}$ means that

$$\left[\sum_{k \, \varepsilon \, K} [w_k(C_k(\bar{v}) - C_k(v_k^*))]^p \right]^{1/p} < \left[\sum_{k \, \varepsilon \, K} [w_k(C_k(v) - C_k(v_k^*))]^p \right]^{1/p}$$

The last inequalities contradicts themselves and this concludes the proof of *i*).

ii) Let $\bar{v} \, \varepsilon \, V$ be a solution to system (3.2) with $w_k > 0 \ \forall k \epsilon K.$ Suppose $\bar{v}$ is not Pareto optimal, then, the rest of the proof follows from *i*) and results to the following contradiction:

$$\left[\sum_{k \, \varepsilon \, K} [w_k(C_k(v) - C_k(v_k^*))]^p \right]^{1/p} < \left[\sum_{k \, \varepsilon \, K} [w_k(C_k(\bar{v}) - C_k(v_k^*))]^p \right]^{1/p}$$

Since the second term of system (3.1) is a constant, we are left with the following optimization problem:

$$\min \sum_{k \varepsilon K} C_k(v) \qquad (3.3)$$

$$s.t$$

$$v \varepsilon V$$

The weighted methods has a shortcoming of not been able to find all the Pareto optimal solutions of nonconvex problems. The objective functions studied in this thesis is either convex or linear in v, thus, the feasible objective space is convex. An important feature of a multiobjective problem is the connectedness of the sets of Pareto optimal and weakly Pareto optimal solutions. It is often useful to know how well one can move continuously from one Pareto optimal solution to another. Steuer (31), proves that the Pareto optimal set of a multiobjective optimization is connected when the objectives are linear. Warburton (37), proves the connectedness of Pareto optimal set for convex case.
See appendix B for note on multiobjective solution methods.

Observation 2
It follows from theorem 3.1 that the solution of system (3.3) is Pareto optimal.

Observation 3
The social or system cost is lower when the objectives are optimized in an aggregated form as in (3.3) than when the objectives are singly optimized.

-This follows from the following arguement, assume $\bar{v} \varepsilon V$ is a link flow vector that solves problem (3.3), and suppose that there exist a single objective ideal vector point v^* such that

$$\sum_{k \varepsilon K} C_k(v^*) < \sum_{k \varepsilon K} C_k(\bar{v})$$

then, it means that the vector $\bar{v} \varepsilon V$ does not solve problem (3.2), contradicting our initial assumption that $\bar{v}$ solves problem (3.2).
Since fixed demand is a special case of elastic demand, we discus only when demand is elastic, the case for fixed demand follows directly.

3.4 Multiobjective Road Pricing Problem for Elastic Demand (MORPP_ED)

We kick off by defining the intention of the system controller or decision maker through the so called system problem. We assume we have only one system controller. His intention, is to maximize the entire system's economic benefit. Here we use directly the cost formulations described above.

3.4.1 System Problem for Multiobjective_Elastic Demand (SPM_ED)

The system problem can be seen as the problem of the decision maker or the system controller. It is the problem statement that describes the objective of the system controller. For our multiobjective problem, the aim or objective of the system controller can be stated as follows:

max *Social Welfare* (*or Economic Benefit* (*EB*))
$s.t$
flow & *environmental feasibility conditions*.

Recall from section 2.4 that the Social Welfare or EB is given by

$$EB = UB - SC$$

where UB is the User Benefit, given by

$$UB = \sum_{w \varepsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

and SC is the social cost, now given by

$$SC = \sum_{k \varepsilon K} C_k(v)$$

$B_w(d_w)$ is the inverse demand or benefit function for the OD pair $w \varepsilon W$, and $C_k(v)$ is the cost function for objective $k \varepsilon K$.

The **SPM_ED**, can then be stated mathematically as follows:

$$\min Z = \sum_{k \varepsilon K} C_k(v) - \sum_{w \varepsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$\begin{aligned} & s.t \\ v &= \Lambda f && \psi \\ \Gamma f &= d && \lambda \\ g(v) &\leq 0 && \xi \\ f &\geq 0 && \rho \\ d &\geq 0 && \vartheta \end{aligned} \qquad (3.4)$$

The flow feasibility conditions are as described in the previous chapter. $g(v) \leq 0$ $(where\ g(v)\ \varepsilon\ \mathbb{R}^{|A| \times |K|})$ are the possible side constraints on the link flow vector v. These constraints are likely to be environmental quality standard constraints, which may require:

1) that total emission on certain links should not exceed the stipulated emission standard.
2) that total noise level on certain links should not exceed the standard allowed dB(A) level.
3) that total number of cars using certain roads should not exceed a given number so as to preserve the pavement and check accidents.

$(\psi, \lambda, \xi, \rho, \vartheta)$ are the *KKT* multipliers associated with the constraints. Without loss of generality, we assume link separable cost functions for all objectives, and absence of the side constraints $g(v)$. The remaining constraints we again call the *feasibility conditions for elastic demand* denoted by *FeC_ED*. We thus define the following cost functions for the relevant objectives:

$C_t(v) = \sum_{a \varepsilon A} \beta v_a t_a(v_a) = \beta T_a^{ff}\left(1 + \eta\left(\frac{v_a}{\hat{C}_a}\right)^{\phi}\right) = Bureau\ for\ Public\ Roads(BPR)\ function$; where

T_a^{ff} is the free flow travel time on link a,

v_a is the total flow on link a,

$\hat{C}_a$ is the practical capacity of link a, and

η *and* ϕ are *BPR* scaling parameters.

$C_e(v) = \sum_{a \varepsilon A} \alpha_a e_a(v_a) = \sum_{a \varepsilon A} \alpha_a \chi_a v_a l_a$; where

χ_a is the emission factor for link a (depending on the emission type and the vehicle speed of link a) in g/vehicle-kilometre.

l_a is the length of link a

$C_n(v) = \sum_{a \varepsilon A} \gamma n_a(v_a) = \sum_{a \varepsilon A} \gamma[A + B\log(\frac{\upsilon_a}{\upsilon_0}) + 10\log(\frac{v_a}{\upsilon_a})]H_a$; where

A and B are vehicle specific constants as given in "*Rekenen Meet Voorschrift*" (*RMVI*) calculation. They are measured in $dB(A)$

υ_a *and* υ_0 are respectively the average and reference speed of vehicles on link a.

H_a is the number of households along link a.

$C_i(v) = \sum_{a \varepsilon A} i_a(v_a) = \sum_{a \varepsilon A} \tau_a v_a (\frac{H_a}{J_a}) l_a$; where

τ_a is the load equivalence factor (LEF) that measures the amount of pavement deterioration produced by each vehicle on link a.

H_a is the initial cost for the infrastructure per kilometre.

J_a is the design standard of link a. It is measured by the design number of equivalent axle load (ESAL) repetitions.

$\frac{H_a}{J_a}$ is the unit investment cost per ESAL-kilometre. Observe that, the higher the design standard of an infrastructure, the smaller the factor ($\frac{H_a}{J_a}$), meaning that infrastructure with a high design standard are the most cost-effective way to handle high traffic volumes (17). (see "Road damage" of section 3. 1).

$C_s(v) = \sum_{a \varepsilon A} \varrho s_a(v_a) = \sum_{a \varepsilon A} \varrho \tau_a E_a$; where

τ_a is the risk factor for link a. It is measured in number of injury crashes/vehicle-kilometre

$E_a = l_a * v_a$ is the measure of level of exposure on link a.

$(\beta, \alpha, \gamma, \varrho)$ are the monetary value vectors as given in section 3.1.3.

We now look at the *KKT* optimality conditions of system (3. 4). Let L be the Lagrangian, and $\bar{v}$, $\bar{d}$ be the solution of the above program, then, there exists $(\psi, \lambda, \rho, \vartheta)$ such that the following hold:

$$L = \sum_{k \varepsilon K} C_k(v) - \sum_{w \varepsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma + (\Lambda f - v)^T \psi + (d - \Gamma f)^T \lambda - f^T \rho - d^T \vartheta$$

$$\frac{dL}{dv_a} = \beta\left(t_a(\bar{v}_a) + v_a \frac{d(t_a(\bar{v}_a))}{dv_a}\right) + \alpha_a \frac{d(e_a(\bar{v}_a))}{dv_a} + \gamma \frac{d(n_a(\bar{v}_a))}{dv_a} + \frac{d(i_a(\bar{v}_a))}{dv_a} + \varrho \frac{d(s_a(\bar{v}_a))}{dv_a} - \psi_a = 0 \; ; \forall a \, \varepsilon \, A$$

$$\frac{dL}{df} = \Lambda^T \psi - \Gamma^T \lambda - \rho = 0$$

$$\Rightarrow \sum_{a \varepsilon A} \psi_a \, \delta_{ar} - \lambda^w - \rho^r = 0 \qquad \forall r \, \varepsilon \, R_w, \; \forall w \, \varepsilon \, W$$

$$\frac{dL}{d(d)} = \lambda - B(\bar{d}) - \vartheta = 0$$

$$\Rightarrow \lambda^w - B(\bar{d}_w) - \vartheta^w = 0 \qquad \forall w \, \varepsilon \, W$$

$$f^T \rho = 0 \qquad \forall w \, \varepsilon \, W \qquad (3.5)$$

$$d^T \vartheta = 0 \qquad \forall w \, \varepsilon \, W \qquad (3.6)$$

$$\rho, \; \vartheta \geq 0, \qquad OR \qquad \rho^r, \vartheta^w \geq 0 \qquad \forall a \, \varepsilon \, A, \forall k \, \varepsilon \, K, \; \forall r \, \varepsilon \, R, \; \forall w \, \varepsilon \, W$$

$$\Rightarrow \rho^r f_r = 0 \qquad \forall r \, \varepsilon \, R$$

$$\Rightarrow d_w \vartheta^w = 0 \qquad \forall w \, \varepsilon \, W$$

Equations (3. 5)and (3. 6) are complementarity constraints.

Let Q be a vector whose element Q_a is given by

$$Q_a = \left[\beta v_a \frac{d(t_a(\bar{v}_a))}{dv_a} + \alpha_a \frac{d(e_a(\bar{v}_a))}{dv_a} + \gamma \frac{d(n_a(\bar{v}_a))}{dv_a} + \frac{d(i_a(\bar{v}_a))}{dv_a} + \varrho \frac{d(s_a(\bar{v}_a))}{dv_a} \right] \qquad (3.7)$$

$\Rightarrow$

$$\beta t_a(\bar{v}_a) + Q_a = \psi_a \qquad \forall a \,\varepsilon\, A$$

$$\sum_{a \,\varepsilon\, A} \psi_a \, \delta_{ar} = \lambda^w + \rho^r \qquad \forall r \,\varepsilon\, R_w,\ \forall w \,\varepsilon\, W$$

$$\Rightarrow \sum_{a \,\varepsilon\, A} (\beta t_a(\bar{v}_a) + Q_a)\, \delta_{ar} = (B(\bar{d}_w) + \vartheta^w) + \rho^r \ \geq B(\bar{d}_w) \qquad \forall r \,\varepsilon\, R_w,\ \forall w \,\varepsilon\, W \qquad (3.8)$$

since $\vartheta^w,\ \rho^r \geq 0,\ \forall r \,\varepsilon\, R_w,\ \forall w \,\varepsilon\, W.$
Recall from equation (2.7) of the previuos chapter that

$$\sum_{a \,\varepsilon\, A} (\beta t_a(\bar{v}_a) + Q_a)\bar{v}_a = \sum_{w \,\varepsilon\, W} \lambda^w d_w$$

$\Rightarrow$

$$\sum_{a \,\varepsilon\, A} (\beta t_a(\bar{v}_a) + Q_a)\bar{v}_a = \sum_{w \,\varepsilon\, W} (B(\bar{d}_w) + \vartheta^w) d_w = \sum_{w \,\varepsilon\, W} B(\bar{d}_w) d_w + \sum_{w \,\varepsilon\, W} \vartheta^w d_w = \sum_{w \,\varepsilon\, W} B(\bar{d}_w) d_w$$

$$\text{since } \ \vartheta^w d_w = 0; \forall w \,\varepsilon\, W$$

thus we summarize our observations as follows:

$$\boxed{\begin{array}{ll} \displaystyle\sum_{a \,\varepsilon\, A} (\beta t_a(\bar{v}_a) + Q_a)\, \delta_{ar} \geq B(\bar{d}_w) & \forall r \,\varepsilon\, R_w,\ \forall w \,\varepsilon\, W \\ \displaystyle\sum_{a \,\varepsilon\, A} (\beta t_a(\bar{v}_a) + Q_a)\bar{v}_a = \sum_{w \,\varepsilon\, W} B(\bar{d}_w) d_w & \end{array}} \qquad (3.9)$$

3.4.2 User Problem for Multiobjective_Elastic Demand (UPM_ED)

Again, without loss of generality, we assume that the user only considers the costs and the benefits he enjoys in making a trip. In this way, the only determinant of user's route choice behaviour is the travel costs and benefits of a trip. Notice that the problem statement for UPM_ED is exactly the same as that described in the previous chapter for UP_ED in section $2.4.2$. We only need to replace the variational inequality with the following convex programme since the travel time cost functions are assumed to be separable and monotonic, and the benefit function, linear.

$$\min \sum_{a \,\varepsilon\, A} \int_0^{v_a} \beta t_a(u)du - \sum_{w \,\varepsilon\, W} \int_0^{d_w} B_w(\varsigma)d\varsigma$$

$$s.t$$

flow and demand feasibility conditions

Since this does not change the results of UP_ED in section $2.4.2$, we thus, borrow results of that section, and the following holds for UPM_ED: given that (v^*, d^*) solves the UPM_ED, then from equation (2.19), we have

$$\boxed{\begin{array}{ll} \displaystyle\sum_{a \,\varepsilon\, A} \beta t_a(v_a^*)\delta_{ar} \geq B(d_w^*) & \forall r \,\varepsilon\, R_w,\ \forall w \,\varepsilon\, W \\ \displaystyle\sum_{a \,\varepsilon\, A} \beta t_a(v_a^*)v_a^* = \sum_{w \,\varepsilon\, W} B(d_w^*)d_w^* & \end{array}} \qquad (3.10a)$$

OR

$$\boxed{\begin{aligned}&\sum_{a\,\varepsilon\,A}(\beta t_a(v_a^*)+\varrho s_a(v_a^*))\delta_{ar}\geq B(d_w^*) && \forall r\,\varepsilon\,R_w,\ \forall w\,\varepsilon\,W\\ &\sum_{a\,\varepsilon\,A}(\beta t_a(v_a^*)+\varrho s_a(v_a^*))v_a^*=\sum_{w\,\varepsilon\,W}B(d_w^*)d_w^*\end{aligned}} \quad (3.10b)$$

if users also consider the safety cost before embarking on a trip.

Without loss of generality, let us assume that the number of users who considers road safety before embarking on a trip is negligible, then, equation $(3.10a)$ holds.

The first line of equation (3.10) confirms the Wardrop's first principle (see UP_ED-section $2.4.2$), and the second line is the travel cost-benefit balance equation.

Notice that the only difference between the optimality condition (3.9) for SPM_ED and that of user problem UPM_ED $(3.10a)$, is the absence of the term Q_a in (3.10). Observe that, by perturbing the link cost function $\beta t_a(v_a^*)$ by $Q_a|_{v_a^*}$ $;\forall a\ \varepsilon\ A,$ the optimality conditions in equation $(3.10a)$ for the user problem become exactly the same as those in equation (3.9). So, by charging each user of link a, a toll equal to Q_a, for all links in the network, the users' route choice behavior will result in an optimal (in Pareto sense) flow pattern. We now define the terms in Q_a.

$\beta v_a \frac{d(t_a(\bar{v}_a))}{dv_a}$ is the additional travel cost imposed on all existing users by an additional user of link a.

$\alpha_a \frac{d(e_a(\bar{v}_a))}{dv_a}$ is the additional cost to the environment due to emission caused by an additional user of link a.

$\gamma \frac{d(n_a(\bar{v}_a))}{dv_a}$ is the additional cost to the society due to increase in noise level caused by an additional user of link a.

$\frac{d(i_a(\bar{v}_a))}{dv_a}$ is the cost of the road damaged caused by a single car using link a.

$\varrho \frac{d(s_a(\bar{v}_a))}{dv_a}$ is the cost of the increase in risk level on link a due to an additional user of this link.

Therefore, by charging each user of link a a toll equal to Q_a, all the costs imposed on other users, on the environment, and on the society by a single user are now internalized. In this way, every user pays 'completely' for the externalities he is causing to the society. Charging each user of link a a toll equal to Q_a does not only make sure that every user is responsible for his action, it also ensures that users' route choice behaviour will follow a flow pattern that agrees with the dm's flow pattern and stipulated environmental quality constraints. We call

$\beta v_a \frac{d(t_a(\bar{v}_a))}{dv_a}$ the congestion toll on link a for a single user,

$\alpha_a \frac{d(e_a(\bar{v}_a))}{dv_a}$ the emission toll on link a for a single user,

$\gamma \frac{d(n_a(\bar{v}_a))}{dv_a}$ the noise toll on link a for a single user,

$\frac{d(i_a(\bar{v}_a))}{dv_a}$ the toll for infrastructure damage on link a for a single user,

$\varrho \frac{d(s_a(\bar{v}_a))}{dv_a}$ the safety toll on link a for a single user, and

We call (3.7) the Marginal Social Cost Toll for elastic demand. It is the bases of our Marginal Social Cost Pricing (*MSCP*) where every user is 'fully' responsible for any cost he is causing to the society. By perturbing the link travel cost function by Q_a, the optimality conditions of the user problem are exactly the same with those of system problem. This means that by tolling Q_a, on link a $\forall a\ \varepsilon\ A,$ the users will be forced to use the network in a way desired by the system controller.

By utilizing the toll pricing framework described in the previous chapter, we can find other toll vectors that can be used to achieve the optimal (in Pareto sense) network flow pattern. Suppose $(\bar{v},\bar{d})$ is the system optimal (in Pareto sense) flow pattern for the SPM_ED, then, by utilizing equation (2.20), any social toll vector θ, whose element θ_a is the social toll on link a, satisfying the following set of linear conditions will also yield a system optimal (in Pareto sense) flow:

$$\boxed{\begin{aligned} &\sum_{a \varepsilon A} (\beta t_a(\bar{v}_a) + \theta_a)\delta_{ar} \geq B(\bar{d}_w) && \forall r \varepsilon R_w,\ \forall w \varepsilon W \\ &\sum_{a \varepsilon A} (\beta t_a(\bar{v}_a) + \theta_a)\bar{v}_a = \sum_{w \varepsilon W} B(\bar{d}_w)\bar{d}_w \end{aligned}} \qquad (3.11a)$$

which we can condense in matrix form as

$$\boxed{\begin{aligned} \Lambda^T(\beta t(\bar{v}) + \theta) &\geq \Gamma^T B(\bar{d}) \\ (\beta t(\bar{v}) + \theta)^T \bar{v} &= B(\bar{d})^T(\bar{d}) \end{aligned}} \qquad EqC\_ED \qquad (3.11b)$$

Where we use EqC_ED to mean equilibrium constraint for elastic demand.
Observe from the last equation that

$$\boxed{\theta^T \bar{v} = B(\bar{d})^T(\bar{d}) - \beta t(\bar{v})^T \bar{v}} \qquad (3.12)$$

The *RHS* of equation (3.12) is a constant. This shows that, when demand is elastic, then, the total toll revenue collected in a network is constant no matter the toll vector used.
Note that the *MSCP* toll vector Q whose element $Q_a = MSCP$ toll for link a is a valid social toll vector for (3.11).
When demand is fixed, then, (3.11) becomes

$$\boxed{\begin{aligned} \Lambda^T(\beta t(\bar{v}) + \theta) &\geq \Gamma^T \lambda \\ (\beta t(\bar{v}) + \theta)^T \bar{v} &= (d^*)^T \lambda \end{aligned}} \qquad EqC\_FD \qquad (3.13)$$

where λ is a free scalar.

3.5 First Best Pricing Algorithms for the Multiobjective Problem

3.5.1 Algorithm for First Best Pricing scheme for Multiobjective Road Pricing Problem_Fixed Demand

Steps involve in designing the first best optimal tolls for the multiobjective problem when demand is fixed are given below:
1) Solve the system problem **SPM_FD** to get the desired Pareto optimal link flow vector $\bar{v}$
2) Find any social toll vector θ satisfying EqC_FD (equation (3.13))
3) Update the travel cost on link a to include θ_a, $\forall a \varepsilon A$.
Notice that $\theta_a = Q_a = Marginal\ Social\ Cost\ Toll;\ \forall a \varepsilon A$ is a valid solution.
By utilizing equation (3.13), secondary objectives on the tolls can be achieved. The formulations for secondary objectives on tolls described in sub-section 2.3.4 remain valid with condition (3.13) replacing condition $(2.12b)$.

3.5.2 Algorithm for First Best Pricing scheme for Multiobjective Road Pricing Problem_Elastic Demand

Steps involve in designing the first best optimal tolls for the multiobjective problem when demand is elastic is given below:
1) Solve the system problem **SPM_ED** to get the desired Pareto optimal link flow vector $\bar{v}$ and corresponding OD demand vector $\bar{d}$
2) Find any social toll vector θ satisfying EqC_ED (equation (3.11))

3) Update the travel cost on link a to include θ_a, $\forall a \,\varepsilon\, A$.
Notice that $\theta_a = Q_a = Marginal\ Social\ Cost\ Toll$; $\forall a \,\varepsilon\, A$ is a valid solution.
Again, by utilizing equation (3.11), secondary objectives on the tolls can be achieved. The the formulations for secondary objectives on tolls described in sub-section 2.3.4 remain valid with equation (3.11b) replacing equation (2.20).

3.5.3 Numerical Examples First Best Pricing for the Multiobjective Problem

3.5.3.1 Five-Node Network Example for First Best Pricing_FD

We now present a Five-Node network example to demonstrate the strengths of the tools developed in the preceding sections. We only consider two emission types; NO_x and PM_{10}. Additional emission types can easily be incorporated in a similar fashion.

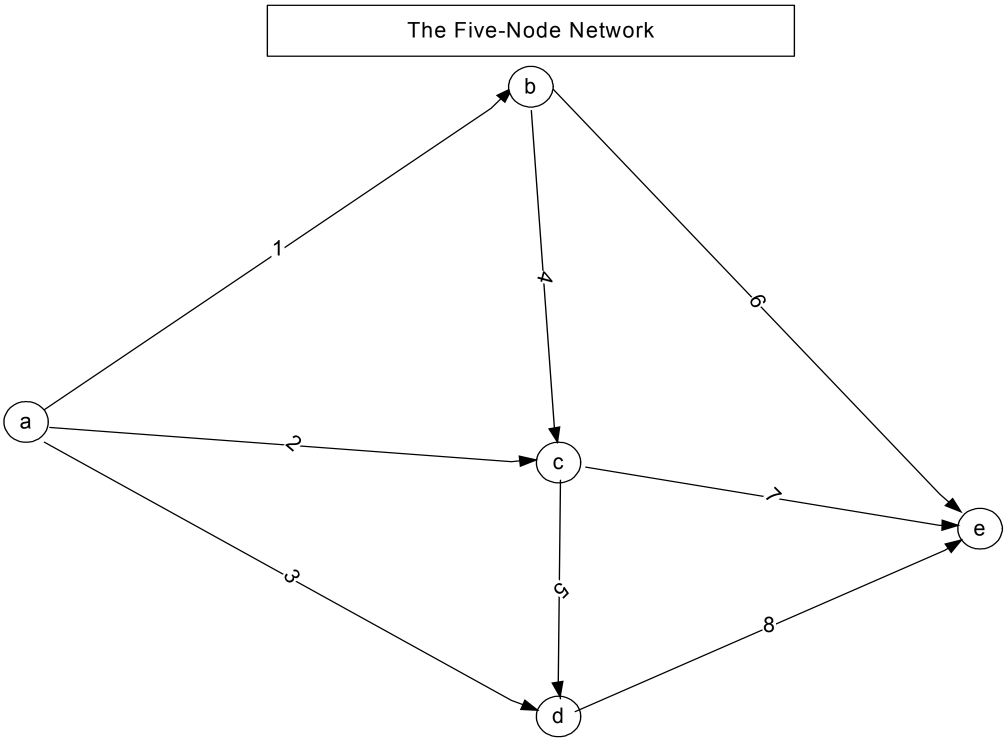


Figure 3.1

Link attributes and inputs

Emission factors, emission costs, cost for 1$dB(A)$ and safety costs used here revolves within the neighbourhood of the values stated in (1). The value of time (VOT) used is from (3). The injury costs used are arbitrary. An optimization software AIMMS is used for all the numerical examples in this thesis.

Table 3.1

Link Attributes	Vehicle Class =	Personenauto's						
Link	Length(km)	Free Speed(km/hr)	Capacity	# of HouseHolds	Air Emission Cost for NOx (€/gram)	Emission factor PM_{10}(g/km)	Damage to infrastru-cture (€/veh-km)	Safety factor (Injury per Veh-km)
1	10	100	400	7	0.008	0.043	0.0024	0.00000027
2	7	70	300	200	0.0119	0.035	0.0024	0.000001
3	10.5	100	350	8	0.009	0.043	0.0024	0.0000003
4	5	70	200	200	0.0119	0.043	0.0024	0.000001
5	4	70	250	200	0.0126	0.043	0.0024	0.000001
6	10	90	250	9	0.0075	0.043	0.0024	0.00000027
7	5	80	250	200	0.012	0.035	0.0024	0.0000001
8	8.5	90	300	10	0.0088	0.043	0.0024	0.00000027

Table 3.2

Injury Cost (€/Injury)	Value of Time (VOT) (€/min)	Noise	
		A (dB(A))	69.4
600	0.1671667	B (dB(A))	27.6
		Vref(km/hr)	80

Table 3.3

Cost of noise per household as measured from road traffic (Euro per year in 2007 price scale)

dB(A)	< 55	55 - 65	66 - 75	> 75
Euro per dB(A)	0	27	40	45.4

	Emission Factor (g/km/veh)	
Speed (km/hr)	NOx	PM10
< 15	0.702	0.061
<= 30	0.456	0.059
<= 45	0.48	0.059
< 65	0.227	0.035
>= 65	0.236	0.043

Table 3.4

	a--e
OD-Demand	1000

Table 3.5

BPR scaling parameters	
BPR (η)	0.15
BPR (ϕ)	4

Results

Table 3.6

Link flows							
Links/Objectives	UE	Travel Time	Emission	Noise	Safety	Infrastructure	Social/System
1	281.05	319.86	140.79	994.73	1000.00	0.00	320.03
2	369.07	324.05	377.25	3.58	0.00	1000.00	329.49
3	349.89	356.09	481.96	1.69	0.00	0.00	350.48
4	0.00	15.67	0.00	0.00	0.00	0.00	3.67
5	0.00	0.00	0.00	0.67	0.00	0.00	0.00
6	281.05	304.19	140.79	994.73	1000.00	0.00	316.35
7	369.07	339.72	377.25	2.90	0.00	1000.00	333.17
8	349.89	356.09	481.96	2.36	0.00	0.00	350.48

Table 3.7

Path flows(f_r)							
Paths (r) /Objectives	UE	Travel Time	Emission	Noise	Safety	Infrastructure	Social/System
a--b--e	281.05	304.19	140.79	994.73	1000.00	0.00	316.35
a--b--c--e	0.00	15.67	0.00	0.00	0.00	0.00	3.67
a--b--c--d--e	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a--c--e	369.07	324.05	377.25	2.90	0.00	1000.00	329.49
a--c--d--e	0.00	0.00	0.00	0.67	0.00	0.00	0.00
a--d--e	349.89	356.09	481.96	1.69	0.00	0.00	350.48
Total OD demand	**1000.00**	**1000.00**	**1000.00**	**1000.00**	**1000.00**	**1000.00**	**1000.00**

Cost Table

Table 3.8 All costs and tolls in €

Costs						
Objectives/Costs	Travel Time	Emission	Noise	Safety	Infrastructure	Social/System
UE	2421.10	143.64	340.16	3.71	40.07	**2948.69**
Travel Time ($C_t(v)$)	2388.50	145.76	395.54	3.68	40.92	**2974.41**
Emission ($C_e(v)$)	2935.63	130.20	338.87	3.73	39.60	**3448.02**
Noise ($C_n(v)$)	49519.85	291.10	13.53	3.25	47.94	**49875.67**
Safety ($C_s(v)$)	50789.07	291.81	13.54	3.24	48.00	**51145.66**
Infrastructure ($C_i(v)$)	44275.96	274.45	331.10	4.50	28.80	**44914.81**
Social/System ($T(v)$)	2391.34	141.88	319.93	3.67	40.83	**2897.64**

Cost comparison Charts

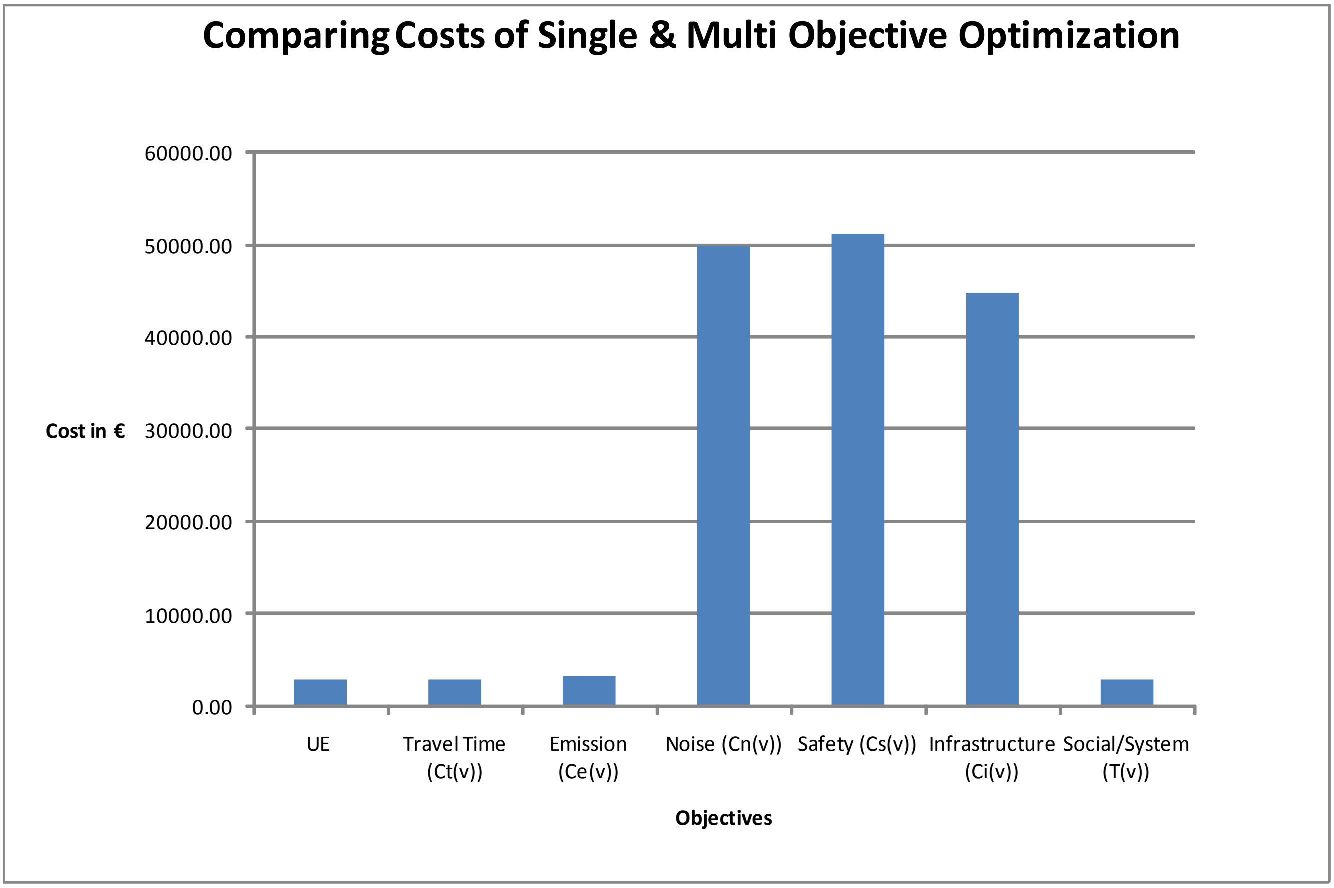


Figure 3.2

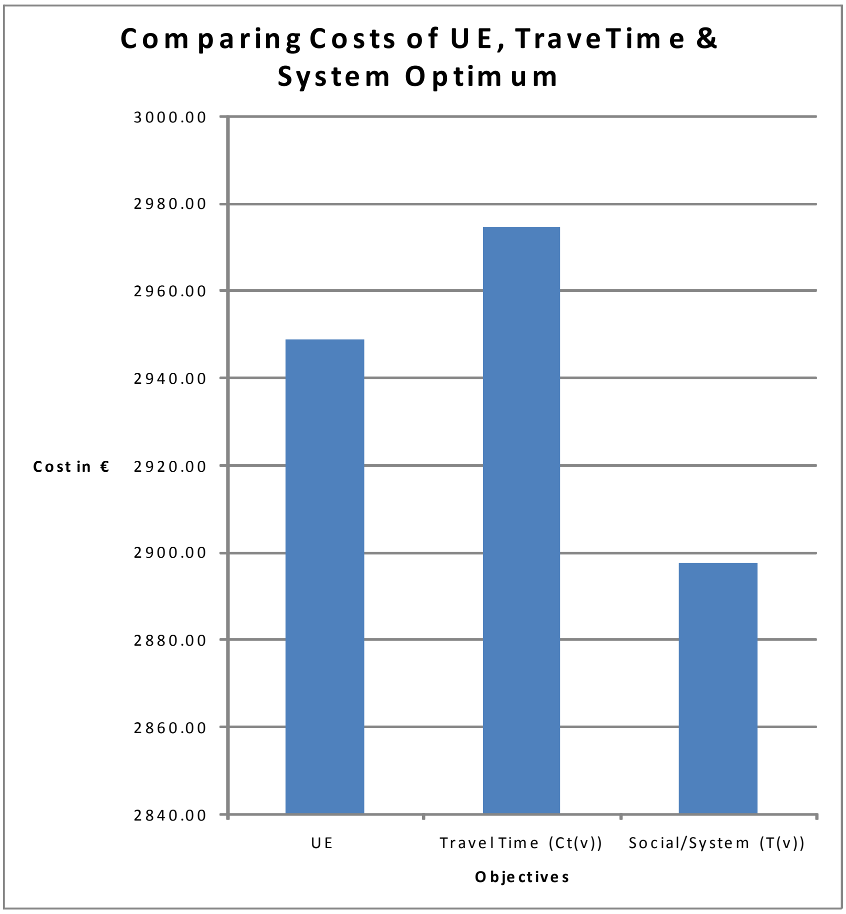


Figure 3.3

Optimal Toll Pricing Framework for the Pareto Optimal Flow Pattern
Table 3.9

Objectives on Toll						
Links/Toll Objective	**Link Flows (v)**	**θ_{MSCP}**	**$\theta_{MinRevenue}$**	**$\theta_{FixedRevenue}$**	**$\theta_{MinTollBooth}$**	**$\theta_{MinMaxToll}$**
1	320.03	0.37	0.00	0.09	0.00	0.00
2	329.49	1.01	0.56	0.65	0.56	0.56
3	350.48	1.22	0.19	0.38	0.28	0.56
4	3.67	0.08	0.00	0.00	0.00	0.00
5	0.00	0.07	0.00	0.00	0.00	0.56
6	316.35	1.85	0.10	0.19	0.10	0.56
7	333.17	1.67	0.00	0.09	0.00	0.46
8	350.48	1.18	0.09	0.09	0.00	0.18
Total Toll Revenue ($\theta^T v$)		**2431.82**	**313.21230**	**500.00**	**313.21330**	**775.54**
Toll Booths		**8**	**4**	**6**	**3**	**6**

Interpretation:
The cost table shows that, the cost for each externality is lowest when the corresponding objective is optimized. This is seen on the diagonal cost entries. This diagonal cost vector is what we called the ideal cost vector. Also notice that the system cost is minimal when we optimize the aggregated objective function (*see equation* (3.3)), and this reasserts observation 3. Observation from the second cost chart shows that the Pareto optimal cost for the *system cost* objective is clearly less than that of *travel cost* objective, which in turn, is less than that of *user equilibrium*. This result confirms our claim, that the most efficient, and hence, the optimal (in Pareto sense) use of transportation network, is not reached in general by just optimizing the travel cost. The toll pricing framework table for the Pareto optimal flow pattern displays some of the valid toll vectors that can be used to achieve the Pareto optimal flow pattern. The marginal cost social pricing (*MCSP*) scheme charges toll on all the links with very high toll revenue with respect to other toll objectives. The minimum revenue toll vector maintains minimum toll revenue among all other toll objectives. Minimum toll booth objective yields a valid toll vector by charging tolls only on three links, making it the smallest number of toll booths among other toll vectors shown. Minimax objective maintained lowest link tolls among other toll objectives.
Remarks: The entries in the attribute table for infrastructure cost assumed that all roads have the same unit investment cost per ESAL-kilometre. Different unit investment cost per ESAL-kilometre can be easily incorporated when these values are known.
The path flow pattern given in the path flow table is not unique in general.
It is by coincidence that the highest link toll on the minimax corresponds to the highest link tolls on the other toll objectives except for *MSCP*. In general, minimax objective has lower link tolls when compared to other valid toll vectors.
The fixed revenue of €500 is arbitrary for the fixed revenue objective.
The results in the tables above assumed there are no environmental quality standard constraints. If we assume that there are such constraints, then, we need to solve the system problem together with condition (3.13) to get the values of $\bar{\xi}_a^k$; $\forall a \; \varepsilon \; A$, $\forall k \; \varepsilon \; K$. These factors are needed to compute the MSCP toll vector Q. In that case, an extra term $\sum_{k \, \varepsilon \, K} \bar{\xi}_a^k \frac{d(g_a^k(\bar{v}_a))}{dv_a}$ is addedd to Q (3.7). Note that the matrix $\bar{\xi}$ is not needed to compute equivalent toll vectors.
Using these valid toll vectors, users' route choice behaviour will result in a flow pattern that is Pareto optimal and at the same time satisfying the environmental quality standard conditions.

3.5.3.2 Five-Node Network Example for First Best Pricing_ED

Using the same Five-Node Network with link attributes as stated in **Five-Node Network Example_Fixed Demand**, we define the following inverse demand (benefit) function for the OD pair $(a-e)$:

$$B(d_w) = 800 - \frac{d_w}{2}$$

where d_w is the variable OD demand for the OD pair $w = (a-e)$. By utilizing the solution methods together with the toll pricing framework described in this section, the following results were found: (All costs, benefits or welfare and tolls are in €)

NB: For single objective optimisation, we miximised the economic benefit of each objective with the following formulation:

$$\min Z_k = C_k(v) - \sum_{w \varepsilon W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$feasibility\ conditions$$

Table 3.10

Link flows							
Links/Objectives	**UE**	**Travel Time**	**Emission**	**Noise**	**Safety**	**Infrastructure**	**Social/System**
1	524.50	574.39	639.48	1594.75	1599.99	0.00	563.15
2	500.05	454.40	377.25	3.67	0.00	1599.94	449.46
3	565.69	540.21	582.93	1.56	0.00	0.00	555.74
4	42.35	107.31	0.00	0.03	0.00	0.00	92.39
5	0.00	26.03	0.00	0.80	0.00	0.00	3.67
6	482.15	467.08	639.48	1594.72	1599.99	0.00	470.76
7	542.41	535.68	377.25	2.90	0.00	1599.94	538.18
8	565.69	566.24	582.93	2.36	0.00	0.00	559.41

Table 3.11

Path flows							
Paths (r) /f_r	**UE**	**Travel Time**	**Emission**	**Noise**	**Safety**	**Infrastructure**	**Social/System**
a--b--e	482.15	467.08	639.48	1594.72	1599.99	0.00	470.76
a--b--c--e	42.35	107.31	0.00	0.03	0.00	0.00	90.67
a--b--c--d--e	0.00	0.00	0.00	0.00	0.00	0.00	1.72
a--c--e	500.05	428.37	377.25	2.88	0.00	1599.94	447.51
a--c--d--e	0.00	26.03	0.00	0.80	0.00	0.00	1.95
a--d--e	565.69	540.21	582.93	1.56	0.00	0.00	555.74
Total OD demand	**1590.25**	**1569.00**	**1599.67**	**1599.98**	**1599.99**	**1599.94**	**1568.35**

Table 3.12

Objectives --->	UE	Travel Time	Emission	Noise	Safety	Infrastructure	Total Social Welfare
Inverse Demand Function	4.875	15.500	0.167	0.009	0.003	0.029	15.825
User Benefit	639976.23	639759.74	639999.97	640000.00	640000.00	640000.00	639749.57

Welfare Table

Table 3. 13

Welfare table							
Objectives/Welfare	**UE**	**Travel Time**	**Emission**	**Noise**	**Safety**	**Infrastructure**	**Total Social Welfare**
UE	635972.22	632223.96	639659.05	639509.22	639970.36	639910.86	631368.53
Travel Time ($B_t(v)$)	635825.05	632427.59	639450.15	639208.04	639753.80	639693.98	631494.60
Emission ($B_e(v)$)	635380.87	629444.69	639724.51	639658.41	639994.30	639931.83	628753.86
Noise ($B_n(v)$)	536214.45	134618.71	639533.81	639986.13	639994.81	639923.26	134056.72
Safety ($B_s(v)$)	534542.36	126263.41	639533.11	639986.12	639994.82	639923.20	125700.66
Infrastructure ($B_i(v)$)	547973.07	190296.20	639560.90	639660.43	639992.80	639953.92	189464.25
Total Social Welfare(TB(v))	635834.56	632419.11	639442.75	639277.40	639743.74	639684.19	631568.91

Table 3. 14

Costs						
Objectives/Costs	Travel Time	Emission	Noise	Safety	Infrastructure	Total Social Welfare
UE	7752.27	317.18	467.01	5.87	65.37	**631368.53**
Travel Time ($B_t(v)$)	7332.15	309.59	551.70	5.94	65.76	**631494.60**
Emission ($B_e(v)$)	10555.28	275.46	341.56	5.67	68.14	**628753.86**
Noise ($B_n(v)$)	505381.29	466.19	13.87	5.19	76.74	**134056.72**
Safety ($B_s(v)$)	513736.59	466.89	13.88	5.18	76.80	**125700.66**
Infrastructure ($B_i(v)$)	449703.80	439.10	339.57	7.20	46.08	**189464.25**
Total Social Welfare(TB(v))	7330.46	306.82	472.17	5.83	65.38	**631568.91**

where $B_k(v)$ is the Economic Benefit with respect to objective k.

Welfare comparison Charts

Figure 3.4

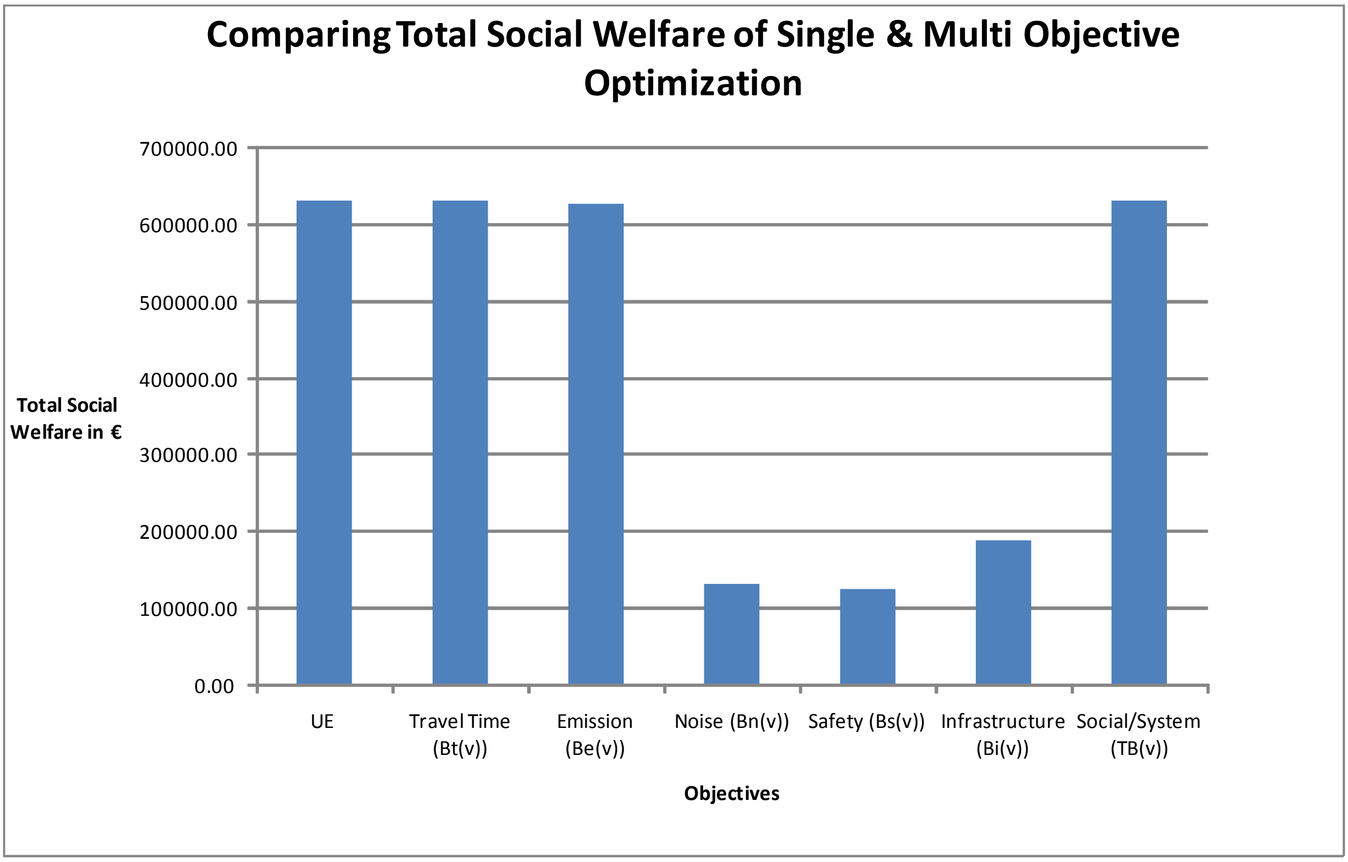
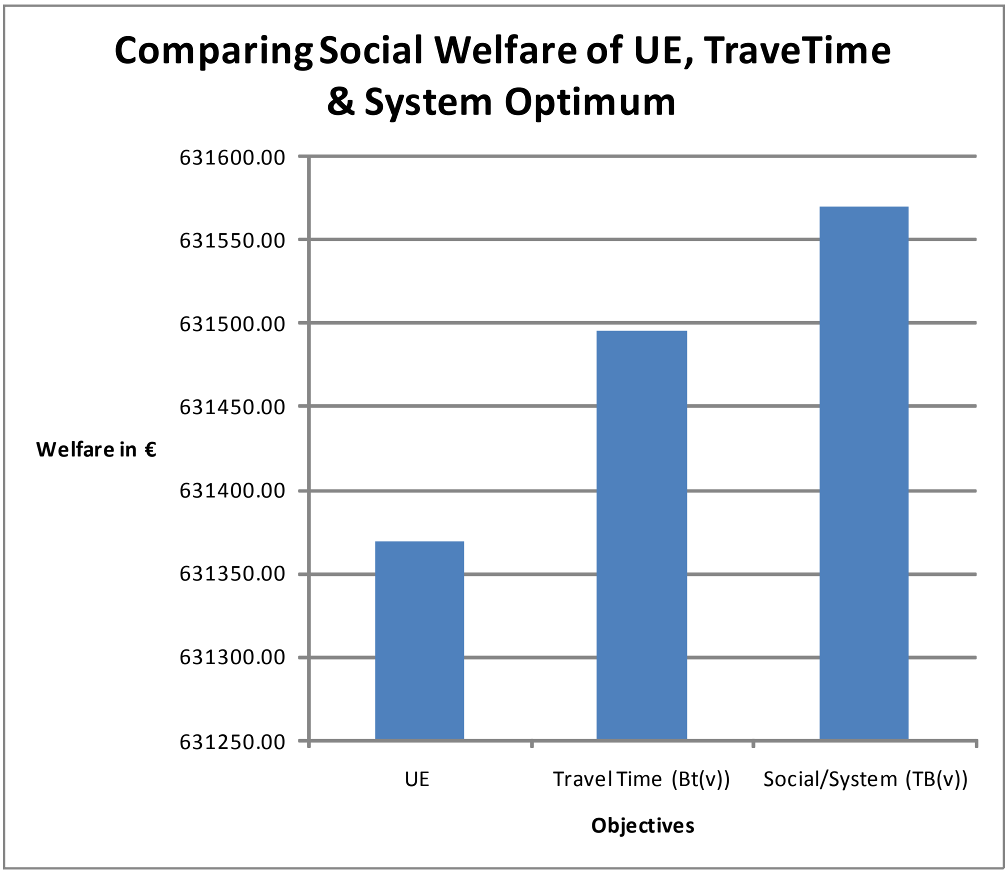


Figure 3.5



Optimal Toll Pricing Framework for the Pareto Optimal Flow Pattern

Table 3.15

Objectives on Toll						
Links/Toll Objective	**Link Flows (v)**	θ_{MSCP}	$\theta_{MinRevenue}$	$\theta_{MaxRevenue}$	$\theta_{MinTollBooth}$	$\theta_{MinMaxToll}$
1	563.15	2.49	4.77	4.77	0.00	5.09
2	449.46	3.19	5.32	5.32	0.00	5.65
3	555.74	4.11	5.75	5.75	0.00	5.92
4	92.39	0.21	0.00	0.00	0.00	0.00
5	3.67	2.79	0.15	0.15	4759.93	0.00
6	470.76	8.54	6.25	6.25	0.00	5.92
7	538.18	8.19	6.10	6.10	0.00	5.77
8	559.41	6.98	5.35	5.35	0.00	5.18
Total Toll Revenue ($\theta^T v$)		**17488.60**	**17488.60**	**17488.60**	**17488.60**	**17488.60**
Toll Booths		**8**	**7**	**7**	**1**	**6**

Interpretation:

The path-flow table shows different demands when the objectives are optimized singly. Again, even though the OD demand shown in the path flow table is unique, the path flow pattern is not unique in general. The welfare table shows that the ideal welfare (diagonal entries) remain the Pareto optimal for single objective optimization. The social welfare or benefit is maximum when the objectives are optimized in an aggregated form (*see equation* (3.3)). The second chart conspicuously shows that, by optimizing the cost of travel time, the most efficient use of the transportation network and thus, the optimal social welfare can not be reached in general with such approach. This result seals up our original claim and confirms observation 3. From the toll pricing framework table, see that the total toll revenue generated by each of the toll objectives is the same even when we try to maximize the total toll revenue (*see* $\theta_{MaxRevenue}$). This confirms our mathematical result (*see equation* 3.12) of constant toll revenue for elastic demand given any valid toll vector. The minimax objective clearly maintains the lowest maximum link tolls over all other toll objectives. By minimizing the number of toll booths, a huge amount of toll is charged on link 5, making it the only tolled link for this toll objective. It makes sense to say that, this high toll causes only very few people to use that link (see the corresponding link flow). As expected, the MSCP charges toll on all the links. Since all total toll revenues are the same, it is up to the decision maker or the system controller to know which of the toll vectors to implement. When the cost of toll booths is very high, the toll vector for $\theta_{MinTollBooth}$ may be used. $\theta_{MinMaxToll}$ toll vector is used when the decision maker wants to maintain a low link tolls owing to users acceptability. It is more likely that users will not complain so much when the tolls are spread evenly in the network than when they are concentrated in some parts of the network. It is interesting to note that, the toll collected from MSCP scheme can be easily differentiated according to each externality. This is also true for fixed demand. In this way, one can know exactly how much toll has been paid by the users for the congestion they cause other users, the cost of damages they cause to the society due to air and noise pollution, accident risk and pavement maintenance.

3.6 Second Best Pricing Algorithms for the Multiobjective Problem

3.6.1 Algorithm for Second Best Pricing scheme for Multiobjective Road Pricing Problem_Fixed Demand

The algorithm is very similar to the one given in sub section 2.5.1 and the one discussed below.

3.6.2 Algorithm for Second Best Pricing scheme for Multiobjective Road Pricing Problem_Elastic Demand

We use a formulation similar to the one used in second best pricing for congestion pricing discussed in sub section 2.6.1. For our multiobjective problem, the second best pricing formulation can be stated in words as follows:

max *The Social Welfare or Economic Benefit*

s.t

*The flows and demands are feasible

*The flows and demands are in constrained tolled user equilibrium

*Certain links cannot be tolled.

*Certain environmental standard quality constraints are satisfied.

Given that Y is the set of links in the network that cannot be tolled, the following describes formally the steps involve in computing the second best tolls for the multiobjective road pricing problem when demand is elastic.

1) Solve the system problem **SPM_ED** to obtain the Pareto optimal flow pattern $(\bar{v}, \bar{d})$

2) Find solution set $\mathcal{F}$ containing any social toll vector θ satisfying the equilibrium condition (3.11) and $\theta_a = 0 \,; \forall a \,\varepsilon\, Y$

3) Check if $\mathcal{F}$ is empty, if NO, GOTO step 4 to compute the First Best social tolls, else GOTO step 5 to compute the Second Best social tolls

4) The vector θ is the first best social toll vector. Update the link cost function to $\beta t_a(v_a) + \theta_a \,; \ \forall a \,\varepsilon\, A$ where $\theta_a = 0 \,; \forall a \,\varepsilon\, Y$ STOP.

5) Using $\bar{v}, \bar{d}$ as the initial flow vectors, solve the following bi-level toll pricing problem

$$\min Z = \sum_{k \,\varepsilon\, K} C_k(v) - \sum_{w \,\varepsilon\, W} \int_0^{d_w} B_w(\varsigma) d\varsigma$$

$$\begin{aligned} & s.t \\ & EqC\_ED \\ & FeC\_ED \\ & \theta_a = 0 \qquad \forall a \,\varepsilon\, Y \end{aligned}$$

The first two constraints ensure that the flow resulting from the above system is a feasible user equilibrium flow pattern.The last constraint ensures the feasibility of toll pattern. As discussed in SBP_FD, MPECs are hard to solve due to non linearity and non convexity of the second constraint. The algorithm above has a unique solution if the system (SPM_ED) and the user (UPM_ED) problems have unique solutions. It may be difficult to achieve by non linear optimization tools though. We again state the following lemma:

Lemma 4

If SPM_ED and UPM_ED have Pareto solutions $(\bar{v}, \bar{d})$ and (v^*, d^*) respectively and the functions $(t(v), e(v), n(v), i(v), s(v) \text{ and } B(d))$ are continuous and monotonic or linear in v and d , then, the second best algorithm in step1-5 above has a unique solution in the interval $\left[\sum_{k \,\varepsilon\, K} C_k(v^*) - \sum_{w \,\varepsilon\, W} \int_0^{d_w^*} B_w(\varsigma) d\varsigma \ , \ \sum_{k \,\varepsilon\, K} C_k(\bar{v}) - \sum_{w \,\varepsilon\, W} \int_0^{\bar{d}_w} B_w(\varsigma) d\varsigma \right]$ provided the derivatives $(\nabla t(v), \nabla e(v), \nabla n(v), \nabla i(v), \nabla s(v), \nabla B(d))$ exist and the environmental side constraints are not strict.

Proof:

The proof is very similar to that given in lemma 2. □

3.6.3 Five-Node Network Example-Second Best Pricing_ED

We illustrate the second best pricing scheme for multiobjective problem only for elastic demand case since fixed demand is a special case of elastic demand. The second best pricing scheme when demand is fixed follows the same pattern.

We retain the links' attributes of section 3.6.1 and the inverse demand function of the Five-Node Network Example for First Best Pricing_ED given in section 3.6.2 as the input variables. Let Y be the set of links that cannot be tolled. If on different occasions we are not allowed to toll the following links:

a) $\{4\}\ \varepsilon\ Y$

b) $\{1,2,3,4,6,7,8\}\ \varepsilon\ Y$

c) $\{4,5\}\ \varepsilon\ Y$

then, the second best algorithm for multiobjective-elastic demand (*section* 3.6.2) terminates in step 4, thus yielding the same flow pattern as the system problem. In other words, the flow pattern resulting from the algorithm is Pareto optimal flow pattern. The optimal toll vectors that can be used to achieve this Pareto optimal flows are as shown below:

a) see the toll vector $\theta_{MinRevenue}$ of Five-Node Network Example for First Best Pricing_ED (*Table* 3.15)

b) see the toll vector $\theta_{MinTollBooth}$ of Five-Node Network Example for First Best Pricing_ED (*Table* 3.15)

c) see the toll vector $\theta_{MinMaxToll}$ of Five-Node Network Example for First Best Pricing_ED (*Table* 3.15)

Since these toll vectors are not unique, it is interesting to note that we can still find other toll vectors that can achieve the Pareto optimal flow pattern when the set of links in Y are different from the ones given above (a,b,c). Again, the toll vectors given in *Table* 3.15 are not unique in general, this means that other toll vectors may exist which can achive the Pareto optimal flow with the Y restriction (a, b & c).

Let us assume that we can only toll links 2 and 4, that is $\{1,3,5,6,7,8\}\ \varepsilon\ Y$. We assume that node c is a city center and paths $\overline{abe}$ and $\overline{ade}$ are highways or freeways. Links 2 and 4 are the only entrances into the city c. We want to maintain the system cost as low as possible and at the same time discouraging people from going to e through c. This is a typical example of the so called cordon pricing scheme or cordon charging. To solve this second best pricing problem, we employ the second best solution algorithm described in section 3.6.2.

Results:

There does not exist any toll vector θ such that the set of conditions in step 2 of the algorithm are satisfied. This means that the set $\mathcal{F}$ is empty. We thus proceed to step 5. The second best solution satisfying the toll constraint is as follows (all costs, tolls, benefits, and welfare are in €):

Link flows and Tolls - *Table* 3.16

Resulting Flows/Tolls		
Links(a)	Link Flows (v_a)	Link Toll per vehicle (θa)
1	565.85	0.00
2	453.06	0.54
3	571.12	0.00
4	86.11	0.00
5	0.00	0.00
6	479.74	0.00
7	539.17	0.00
8	571.12	0.00
Total Toll revenue ($\theta^T v$)		244.66
Toll Booths		1

Cost and Social Benefit - *Table* 3.17

Costs						
Objectives/Costs	Travel Time	Emission	Noise	Safety	Infrastructure	Total Social Welfare
Total Social Welfare(TB(v))	7684.40	317.81	471.73	5.88	66.25	631429.06

Comparison Table - *Table* 3.18

	Path Flows	
Paths [j]	First Best Pricing	Second Best Pricing
a--b--e [i]	470.76	479.74
a--b--c--e [ii]	90.67	86.11
a--b--c--d--e [iii]	1.72	0.00
a--c--e [iv]	447.51	453.06
a--c--d--e [v]	1.95	0.00
a--d--e [vi]	555.74	571.12

	UE	First Best Pricing	Second Best Pricing
Demand for OD pair (a,e)	1590.25	1568.35	1590.03
Inverse Demand Function	4.88	15.82	4.99
User Benefit	639976.23	639749.63	639975.13
Total Social Welfare	631368.53	631568.91	631429.06

Observation:

The total social welfare has been decreased by €139.85 (with reference to the first best pricing scheme). This shows that this toll vector does not result in the most efficient (in Pareto sense) use of the simple network. This is as expected since the algorithm does not terminate in step 4. Only one of the tollable links (*link* 2) is tolled as seen in the link flows/tolls table. Notice that the total traffic that passes through the city center c has been decreased and the freeway (or highway) traffic increased (with reference to first best pricing). Though the user benefit from the second best pricing (SBP) scheme exceeds that of first best pricing (FBP) scheme, the social cost generated by the increased number of users in the SBP by far exceeds that of FBP. This change in cost swallows the change in the user benefit, thus leaving the total social welfare of FBP better than that of SBP scheme. The same applies to UE and FBP schemes. It is also interesting to note that, though the SBP scheme generates a social benefit or welfare which is lower than that generated by FBP scheme, its (i.e. SBP) social welfare is higher than that generated by user equilibrium (UE). This reasserts our claim in **Lemma 4**, which states that if our system problem (SO) (which can always be reached through first best pricing scheme), and the user problem (UE) have unique solutions, then, the second best pricing will also have a solution in the closed interval $\left[TotalSocialWelfare_{[UE]} \text{ , } TotalSocialWelfare_{[SO]} \right]$.

In this chapter, we have assumed that we have only one system controller who has one interest; to minimize the total system cost or maximize the societal welfare, which he can achieve by using pricing approach. Sometimes, this assumption may not be obtainable in practice if stake holders affected by users actions in the transportation sector are allowed to toll road networks. These stake holders may include the insurance companies who pay hospital bills for injuries or death resulting from road accidents, and ill health as a result of inhalation of poisonous gases and exposure to high level of noise, ministry of economics with interest in the nation's productivity which may be adversely affected by man-hours lost in congestion, environmental agencies that are only concerned about the cleanness of air, government or the private companies responsible for road maintenance. Clearly, these agencies have conflicting objectives. If allowed to toll the network individually, then, the Pareto optimal flow pattern and the pricing schemes described in this section have to be modified to incorporate this new setup of multiple system controllers. This is the subject of the next chapter.

Chapter 4
Multi-Leader-Multi-Follower Problem in Road Pricing

4.1 Introduction

When one decision maker (dm), specifically the government, absolutely controls all the affairs (traffic flow) of a transportation system say through road pricing, then, it is likely that some other stakeholders affected by activities of transportation may not be happy with the decisions made by this central government. The reason for this is very simple: when a central body models the road pricing, all traffic externalities are simultaneously considered with or without preference for any externality. When preference is given, say, to congestion, then the effect of the preferred externality subdues the effect of other externalities. Such model may result in frequent accidents occurrences and continuous exposure to high noise level and/or emission level. The end result of this is readily seen. The injury costs together with the costs of ill health resulting from the aforementioned, translate to a low profit level for the insurance companies which do not care about how fast users get to their destinations, but how safe the routes they are using are. Companies responsible for road or infrastructure maintenance may as well be spending much due to the fact that government or the ministry that tries to reduce man-hour loss (in congestion) has done so by channeling traffic to pavements not meant for them. It may be that heavy vehicles now use small pavements not meant for them, or that the traffic volume now exceeds the level which such infrastructure was originally designed to support. All these point to one fact: agitation of unfavoured stakholders. Even when no preference is given to any of the traffic externalities, it is natural that stakeholders still will prefer to impose tolls individually since they will only consider the externality which affects them. In other words, every stakeholder will want his objective to be optimized. No ideal flow vector exists which can achieve these ideal objectives as we saw in the previous chapter. The reason for this lies in the fact that the stakeholders have conflicting objectives. The questions that comes in mind at this point should be; what happens when you allow these stakeholders to optimize their individual objectives simultaneously? That is, what happens when one stakeholder optimizes his objective given that other stakeholders are doing the same? This question is the subject of this chapter, and we proceed by formally introducing the mathematical and economic theory behind it.

4.2 Mathematical and Economic Theory

4.2.1 Problem description and model formulation

In the previous chapter, we assumed we had only one decision maker who sets tolls on the links in a way that achieve a flow pattern which is at equilibrium and at the same time optimal for his objective. This we called mathematical problem with equilibrium constraint (*MPEC*). *MPEC* as described in that chapter is analogous to a Stackelberg game where the system controller is the leader and the network users are the followers. Sometimes, a Stackelberg game is referred to as leader-follower game. In this thesis, we call it single-leader-multi-follower game or single-leader-multi-follower problem. If we assume that different stakeholders are allowed to toll the network, then, users are influenced not only by just one leader, but by more than one decision maker or leader. From now on we will call the stakeholders the "leaders". Hence, when leaders do not cooperate, we refer to it as a multi-leader-multi-follower game or problem. By cooperation we mean that leaders pursue one objective, which in our case is to keep the system cost as low as possible. The multiobjective problem described in the previous chapter can be seen as cooperative game among leaders. In multi-leader-multi-follower problem, the leaders take decisions at the upper level which influence the followers at the lower level. When the followers perceive these (upper level) decisions, they react accordingly, and this reaction may cause the leaders to update their individual decisions which also results in lower level players updating their reactions also. These updates continue until a stable situation is reached. A stable state is reached when the lower level users can no longer improve on their current decisions (or strategies) by unilaterally switching to other decisions (or strategies), and the same applies to upper level players or the leaders. In our road pricing problem, the lower level users are in equilibrium or steady state when no user can decrease his cost (or increase his benefit) further by unilaterally switching to another route or decide whether to travel or not (when demand is elastic). In the same way, the upper level players are in equilibrium state when no leader can further improve his objective by unilaterally switching to another toll vector.

In the above scenario, each leader continuously looks for a toll vector that optimizes his objective and at the same time resulting in an equilibrated flow pattern while taking into account other leaders' toll vectors. In other words, each leader is continuously solving an *MPEC* which is influenced by other leaders' *MPECs*. These continuous processes terminate when each leader can no longer improve his objective by unilaterally changing his current toll vector. A closer look reveals that we are already confronted with an equilibrium problem subject to equilibrium constraints (*EPEC*). In game theory, a Nash equilibrium is defined as a state where no player can improve his or her outcome by altering his or her decision unilaterally. Hence, in classical game theory, our problem translates to solving two Nash problems, namely; Nash game (or equilibrium problem) in the upper level and Nash game (or equilibrium problem) in the lower level, and these result to Nash toll vectors and Nash flow vectors respectively. Our aim therefore is to find a toll vector for all leaders such that the upper level is at Nash equilibrium as well as the lower level (see *figure* 4. 1).

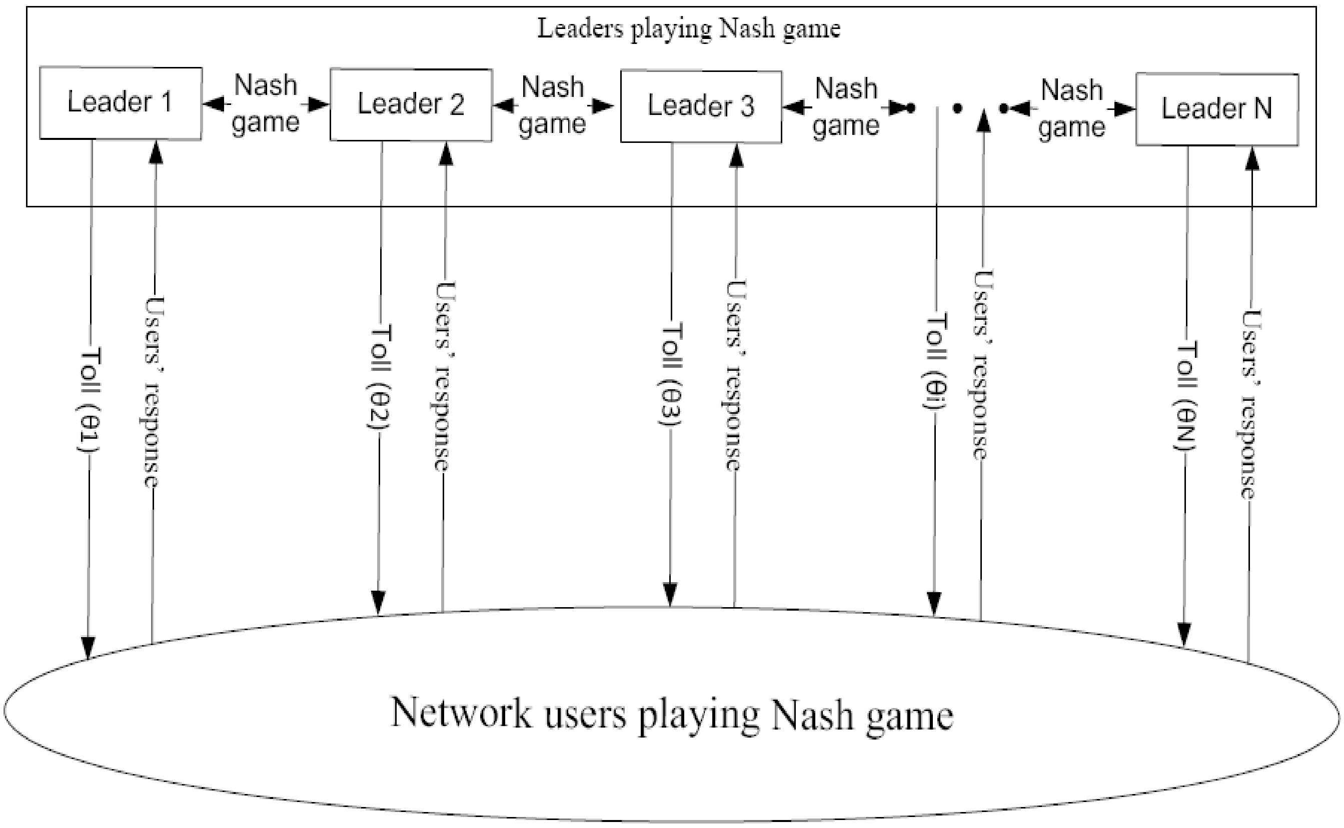


Figure 4.1 Multi-leader-multi-follower Nash representation

4.2.2 Mathematical models for Nash Equilibrium

Assumptions: We assume

i) that the inverse demand functions used here are separable and strictly monotonic and

ii) that the cost satisfy the following

$$(t_k(v^k) - t_k(v^j))^T(v^k - v^j) \geq 0 \qquad v^k \neq v^j$$

The assumptions ensure that the Wadrop's equilibrium (WE) (v, d) is unique.
Again we will consider only the case when demand is elastic since fixed demand is just a special case of elastic demand models. Given that $C_k(v^k)$ is the cost objective function for leader k, then, we want to find a toll vector $\bar{\theta}^k$ and flow vector $\bar{v}^k$, $\forall k \in K$ such that

$$\Psi_k(\bar{\theta}^k, \bar{\theta}^{-k}, \bar{v}^k, \bar{d}^k) \leq \Psi_k(\theta^k, \bar{\theta}^{-k}, v^k, d^k) \qquad \forall k \in K$$

where Ψ_k is the *MPEC* program (see program (4.1)) for leader k. We have used θ^{-k} to mean all toll vectors in $K \backslash k$. We call the vector $(\bar{\theta}, \bar{v}, \bar{d})$, an upper level Nash equilibrium solution vector. This means that each leader k is solving the following *MPEC* problem:

maximize economic benefit with respect to k's system cost

s.t

** flows are in quilibrium*

** feasibility conditions are satisfied*

which can be mathematically stated as;

$$\Psi_k(\bar{\theta}^k, \bar{\theta}^{-k}, \bar{v}^k, \bar{d}^k) = \min_{v^k, d^k_w} Z = C_k(v^k) - \sum_{w \in W} \int_0^{d^k_w} B_w(\varsigma) d\varsigma$$

$$s.t$$

$$\begin{aligned} \Lambda^T\left(\beta t(v^k) + \theta^k + \sum_{j \in K \backslash k} \bar{\theta}^j\right) &\geq \Gamma^T B(d^k) \\ \left(\beta t(v^k) + \theta^k + \sum_{j \in K \backslash k} \bar{\theta}^j\right)^T v^k &= B(d^k)^T (d^k) \\ v^k &= \Lambda f^k \\ \Gamma f^k &= d^k \\ f^k &\geq 0 \\ \theta^k &\geq 0 \end{aligned} \tag{4.1}$$

All notations remain as previously defined. The first two constraints are equilibrium conditions (*EqC_ED*), and the last four are the feasibility conditions (*FeC_ED*). Observe that in the optimization problem above, each leader k can only change his own link toll vector θ^k, $\bar{\theta}^j$ are other leaders strategy which leader k can not change. The bar sign "$^-$" indicates that these strategies are fixed for the above optimization problem. We have also assumed that all links can be tolled.
Suppose $(\bar{v}^k, \bar{d}^k, \bar{\theta}^k)$ is a solution vector for leader k that solves (4.1). Then the following corollary holds:

Corollary 1
Since $\bar{v}^k, \bar{d}^k$ is *Wadrop's equilibrium* (*WE*) with respect to $\bar{\theta} = \sum_{k \in K} \bar{\theta}^k$, then, it means that $\bar{v}^k, \bar{d}^k$ is the same $\forall k \in K$.

Proof
In transportation models, it is generally assumed that user's route choice behaviour is only affected by the total travel cost which this user perceives. Hence, a user using a link a only perceives the travel time

cost $\beta t_a(v_a)$ and the tolls on this link $\bar{\theta}_a = \sum_{k \in K} \bar{\theta}_a^k$. Since the total toll imposed on every link by the leaders is the same for all users, and the travel time cost function also the same for all leaders as well as for all users, then, it means that the link flow vector $\bar{v}^k$ is the same $\forall k \in K$.
In a more simpler way, since $(\bar{v}^k, \bar{d}^k)$ are elastic WE to some $\bar{\theta}$, then by uniqueness, $(\bar{v}^k, \bar{d}^k)$ are the same. □

4.3 Coalition of leaders

When demand is elastic, leaders are only interested in maximizing the economic benefit with respect to their individual costs. Maximizing economic benefit translates to maximizing the consumer surplus, which is the amount users benefit by being able to travel for a cost less than they would be willing to pay (see *figure* 1.1). It turned out that leaders' interests are not conflicting in the sense that they all aim at maximizing consumer surplus, and this is best achieved when they cooperate. Grand cooperation among leaders means they are solving the following system:

$$\begin{aligned} \min Z = \sum_{k \in K} C_k(v) - \sum_{w \in W} \int_0^{d_w} B_w(\varsigma) d\varsigma \\ s.t \\ \Lambda^T(\beta t(v) + \theta) \geq \Gamma^T B(d) \\ (\beta t(v) + \theta)^T v = B(d)^T(d) \\ v \in V \end{aligned} \qquad (4.2)$$

When demand is fixed, then, grand coalition among leaders reduces to solving the following multiobjective problem:

$$\begin{aligned} \min Z = \sum_{k \in K} C_k(v) \\ s.t \\ \Lambda^T(\beta t(v) + \theta) \geq \Gamma^T \lambda \\ (\beta t(v) + \theta)^T v = (d^*)^T \lambda \\ v \in V \end{aligned} \qquad (4.3)$$

Where λ is a free vector variable.

Interpretation.
Systems 4.2 maximizes collective economic benefits for the leaders, whereas 4.3 minimizes the collective losses for the leaders, and can be interpreted as finding a particular solution to a multiobjective bilevel optimization problem (*MPEC*) by minimizing a convex combination of all leaders' objectives (recall the weighted sum multiobjective solution method described in the last chapter). Convex combination of all leaders' objectives can also be seen as leaders forming grand coalition.

Corollary 2
If demand is elastic, then, grand cooperation among leaders leads to higher average economic benefit among leaders as compared to the non cooperative scenario. Similarly, if demand is fixed, then, grand coalition among leaders leads to lower average cost among leaders than when leaders do not cooperate.

Proof
We only consider the fixed demand (FD) case, the proof for elastic case is very similar to that of FD. Let $(\bar{v},\bar{d},\bar{\theta}^k)$ be a solution vector for leader k that solves (4.1) and let $(\hat{v},\hat{d},\hat{\theta})$ be a global solution vector for (4.3). We want to prove that

$$\sum_{k \in K} C_k(\hat{v}) \leq \sum_{k \in K} C_k(\bar{v})$$

where $C_k(v)$ is the cost objective for leader k at link flow v.
Suppose there exist a toll vector $\bar{\theta}^k\ \forall k \in K$ such that

$$\sum_{k \in K} C_k(\bar{v}) \leq \sum_{k \in K} C_k(\hat{v})$$

then, this contradicts that $\hat{\theta}$ solves (4.3) since with $\hat{\theta}_a = \sum_{k \in K} \bar{\theta}_a^k$ a better solution for (4.3) is attained.

□

Corollary 3
If $(\bar{v},\bar{d},\bar{\theta}^k)$ is a solution vector for leader k that solves (4.1), then $(\bar{v},\bar{d},\bar{\theta})$ with $\bar{\theta} = \sum_{k \in K} \bar{\theta}^k$ is a Pareto (or local) solution to the multiobjective problem (4.2) *or* (4.3).

Proof
We want to prove that there does not exist another toll vector $\hat{\theta}$, such that

$$C_k(\hat{v},\hat{\theta}) \leq C_k(\bar{v},\bar{\theta}) \qquad \forall k \in K \quad and$$
$$C_j(\hat{v},\hat{\theta}) < C_j(\bar{v},\bar{\theta}) \ \ for\ at\ least\ one\ j \in K$$

Suppose such toll vector exists, it means that at least one leader j is better off with $\hat{\theta}$ with respect to Nash (w.r.t.N) and no other leader is worse off with $\hat{\theta}$, then,

$$define\ \mathring{\theta} = \left(\bar{\theta}^k,\ k \neq j,\ \ \mathring{\theta}^j = \hat{\theta} - \sum_{k \in K \backslash j} \bar{\theta}^k \quad with\ \ \mathring{\theta}_a^j = \hat{\theta}_a - \sum_{k \in K \backslash j} \bar{\theta}_a^k \quad \forall a \in A \right)$$

This means that leader j is better off with $\mathring{\theta}^j$ than $\bar{\theta}^j$ at Nash scenario. This contradicts the assumption that $(\bar{v},\bar{d},\bar{\theta}^j)$ is a solution vector for leader j that solves (4.1). □
Note that we assumed in corollary 3 that tolls are not restricted in sign.
The proof for elastic demand case is similar to the one given above.

4.3.1 Incentives to form grand coalition

We have seen that grand coalition among leaders inevitably leads to a lower average costs (when demand is fixed) or a higher average economic benefit (when demand is elastic) among leaders. This lower average cost is achieved by decreasing the costs of some leaders while increasing that of others w.r.t.N case. Observe that some leaders will be left worse than they are in Nash case if they buy the idea of grand coalition. With this, they will not have any incentive to form the grand coalition. On the other hand, some leaders will be better off than in Nash case when this grand coalition is formed. If the "gainers" are willing to offset the "losers'" losses (w.r.t.N), then, there will always be incentive to for grand coalition among leaders.
In a more analytical way, suppose

$$\sum_{k \in K} C_k(\hat{v}) \leq \sum_{k \in K} C_k(\bar{v}) \qquad (4.4)$$

where $\hat{v}$ and $\bar{v}$ are the equilibrium flow vectors when leaders do and do not cooperate respectively, then

$$\sum_{k\epsilon K} C_k(\hat{v}) = \sum_{k\epsilon K} C_k(\bar{v}) - \alpha \qquad where \quad \alpha \geq 0$$

$$= \sum_{k\epsilon K} (C_k(\bar{v}) - \alpha_k) \qquad where \ \ \alpha_k = \frac{\alpha}{|K|} \ \Rightarrow \ \sum_{k\epsilon K} \alpha_k = \alpha, \ and \ \alpha_k \geq 0 \quad \forall k \epsilon K$$

Now define

$$\begin{aligned} C_k(\hat{v}) &= (C_k(\bar{v}) - \alpha_k) && \forall k \epsilon K \\ &\leq C_k(\bar{v}) && \forall k \epsilon K \\ &< C_k(\bar{v}) && \forall k \epsilon K \quad if \quad \alpha > 0 \end{aligned}$$

We have assume an egalitarian rule for the excess profit sharing. Observe that for all leaders, the resulting cost after coalition is not worse than in Nash scenario. If the inequality in (4.4) is strict, then all leaders will be strictly better off on coalition (w.r.t.N). We used α to represent the total cost reduction or gain due to coalition.

Corollary 4

A tolling scheme that tolls each gaining leader a cost benefit which is equal to the excess of his Nash scenario, and setting off the losers losses, and finally sharing the remaining cost benefit using the egalitarian rule will make every leader happy, and will cause all leaders to agree to grand coalition.

Proof

Observe that in this way every leader is at least guaranteed of his Nash outcome which may be improved after the sharing of the remaining cost benefit. The mathematical proof follows from the argument above. □

4.4 Restriction on tolls

4.4.1 What happens when some links cannot be tolled?

Suppose now that a set of links Y cannot be tolled (the so called second best pricing), then, it means that we have to add this requirement to our Nash formulation (4.1):

$$\theta_a^k = 0 \quad \forall a \ \epsilon \ Y$$

4.4.2 What happen when leaders are allowed to place tolls on specific links?

Suppose that a set of links Y^k can not be tolled by leader k, then, we only need to add the following constraint to (4.1):

$$\theta_a^k = 0 \quad \forall a \ \epsilon \ Y^k \qquad \forall k \epsilon K$$

4.4.3 What happens when leaders are not allowed to toll a link beyond a certain amount?

Suppose on link a, leader k is not allowed to place a toll above a certain limit say $\varpi_a^k \epsilon \mathbb{R}$, then, to (4.1) we add the following constraint:

$$\theta_a^k \leq \varpi_a^k \qquad \forall a \ \epsilon A \, , \forall k \epsilon K$$

Any combination of the above conditions can easily be added to the problem.

4.5 Existence of Nash Equilibrium.

In game theory, existence of Nash has been a point of concern for game theorists as well as economists. In the bilevel game we have just described, though we have stated that there is every possibility that all leaders will cooperate, it is important if we know whether the toll setting game by leaders will ever terminate if they do not cooperate. Corollary 3 states that if Nash exists, then, the solution vector for problem (4.1) is a Pareto solution to the multiobjective problem (4.2) *or* (4.3). The reverse may not be the case.

Corollary 5
Assuming we can toll all links, and that link tolls are not restricted in any form, then, the non-cooperative version of the game described in this chapter has no Nash toll vectors for the leaders.

Proof
We only give proof for elastic demand since the proof is the same for fixed demand.
Assume a Nash vector $(\bar{v}, \bar{d}, \bar{\theta}^k)$ exist for all leaders. $\bar{\theta}^k$ is the Nash toll vector for leader k and, $\bar{v}$ *and* $\bar{d}$ represent the corresponding Nash link flow and demand vectors respectively. Now suppose $(\hat{v}, \hat{d})$ are respectively the ideal (optimal) link flow and demand vectors for travel time or cost of leader t, then, leader t will still be able to find a toll vector which will yield his ideal vectors $(\hat{v}, \hat{d})$. Recall from *equation* $3.11b$ that any toll vector $\hat{\theta}^t$ for leader t satisfying the following linear equation will yield the optimal vector $(\hat{v}, \hat{d})$ for leader t :

$$\Lambda^T\left(\beta t(\hat{v}) + \sum_{k \in K \backslash t} \bar{\theta}^k + \hat{\theta}^t\right) \geq \Gamma^T B(\hat{d})$$

$$\left(\beta t(\hat{v}) + \sum_{k \in K \backslash t} \bar{\theta}^k + \hat{\theta}^t\right)^T \hat{v} = B(\hat{d})^T(\hat{d})$$

Observe that leader t will always find a toll vector $\hat{\theta}^t$ satisfying the equation above. At least, leader t is sure of finding the following toll vector

$$\hat{\theta}^t = \hat{v}^T\left(\frac{d\beta t(v)}{dv}\bigg|_{v=\hat{v}}\right) - \sum_{k \in K \backslash t} \bar{\theta}^k$$

the *Marginal Congestion Cost toll vector* each time other leaders place their tolls $\bar{\theta}^k$. This contradicts that $(\bar{v}, \bar{d}, \bar{\theta}^k)$ is a Nash vector for all leaders since there exist a toll vector $\hat{\theta}^t$ different from $\bar{\theta}^t$ for leader t which yields a better flow vector $(\hat{v}, \hat{d})$ than $(\bar{v}, \bar{d})$. Since the ideal travel time vector $(\hat{v}, \hat{d})$ is not optimal for other objectives, other leaders will alter this flow vector accordingly by placing tolls. When this happens, leader t will alter his toll accordingly, returning the flow vector to $(\hat{v}, \hat{d})$. Since there is no retriction on tolls, this changing of toll vectors by the leaders does not terminate, thus no Nash equilibrium is reached. ☐

4.6 Numerical Example.

Again we use the five-node network described in the previous chapter retaining all link attributes mentioned in that chapter.

4.6.1 Cooperative and non-cooperative leaders game.

We present below the result of a sequential tolling of the five node network by three leaders.

Demand type: Fixed
Number of leaders (players): Three (leaders who minimize travel time cost ("t"), emission cost ("e") and safety cost ("s"))
Tollable links: All
Domain of tolls: Non negative
Maximum toll per link per leader: One Euro

Method: Sequential optimization of the leaders' objectives
Sequence: Travel time cost → Emission cost → Safety cost → Travel time cost → Emission cost → Safety cost → and so on (Note that this sequence is arbitrary)
Number of iteration: Eight

Results

1) Table 4.1 - Semi-ideal case (the best each leader can do given that no other leader is there)

Link Tolls

Links	Leader "t"	Leader "e"	Leader "s"
1		1.00	
2	0.57	1.00	1.00
3	0.05		
4			0.10
5			
6	0.18	0.38	
7		0.03	1.00
8	0.23		

Link Flows

Links	Leader "t"	Leader "e"	Leader "s"
1	319.86	140.97	420.04
2	324.05	376.97	88.91
3	356.09	482.06	491.05
4	15.67		
5			
6	304.19	140.97	420.04
7	339.72	376.97	88.91
8	356.09	482.06	491.05

Cost

Leaders	Semi-ideal Cost
Leader "t"	2388.50
Leader "e"	130.20
Leader "s"	3.37

Single objective optimization

Leaders	System Cost
Leader "t"	2974.41
Leader "e"	3448.02
Leader "s"	3636.67

2) Table 4.2 - Non-cooperate or Nash outcome

Link Tolls

Links	Leader "t"	Leader "e"	Leader "s"
1		1.00	
2	0.00		1.00
3	1.00		
4		0.12	
5			
6		1.00	
7	0.34		1.00
8	1.00		

Link Flows

Links	Leader "t"	Leader "e"	Leader "s"
1	299.14	299.14	299.14
2	334.31	334.31	334.31
3	366.56	366.56	366.56
4			
5			
6	299.14	299.14	299.14
7	334.31	334.31	334.31
8	366.56	366.56	366.56

Cost

Leaders	Cost
Leader "t"	2391.96
Leader "e"	144.34
Leader "s"	3.67
System Cost	**2539.97**

3) Table 4.3 - Grand coalition among leaders

Links	Link Tolls
1	0.44
2	1.00
3	0.53
4	
5	
6	1.00
7	0.83
8	1.00

Links	Link Flows
1	314.02
2	328.90
3	357.07
4	9.04
5	
6	304.98
7	337.94
8	357.07

Leaders	Cost
Leader "t"	2388.76
Leader "e"	145.24
Leader "s"	3.68
System Cost	**2537.68**

Observation

Though the leaders' costs are simultaneously smallest in the semi-ideal case, there is no single toll vector that can achieve that result as stated earlier (no flow pattern exists which achives the ideal objective vector - see chapter 3). The term semi-ideal is used since we have placed limits on the tolls. Observe that the system cost is higher when individual objective is optimized than in the Nash scenario, which in turn is higher than in grand coaliton. See from the Nash scenario that leader "s" placed high tolls on links 2 and 7 owing to the fact that less traffic on these links reduces the cost of safety. This is readily seen from the semi-ideal case. The same interpretation holds for the high tolls on links 1 and 6 by leader "e". Leaders "e" and "s" are better off when more traffic approximately 500 (see semi-ideal case) uses links 3 and 8. This why they do not impose any toll these links (see Nash scenario). On the other hand, such high traffic on links 3 and 8 is not beneficial to leader "t", and this justifies the high tolls placed on link 3 and 8 by leader "t".

In general, the average leaders' costs are minimum when they cooperate. If the cost benefit sharing rule described in section 4.3.1 is employed, then, the cost of each leader is as follows:

Table 4.4

Leaders	New cost after coalition
Leader "t"	2391.20
Leader "e"	143.57
Leader "s"	2.91
System Cost	**2537.68**

See that every leader is better off than in the Nash scenario. This is only possible if leader "t" that benefited from the coalition is willing to compensate other leaders up to what they can guarantee themselves in the Nash scenario. In addition that he will also give them something extra no matter how small (if he does not want to apply Egalitarian rule) in order to make 100% sure that they will cooperate. Whether he uses Egalitarian or not, all leaders will be better off than in Nash scenario, a drive to form grand coalition.

When other restrictions such as those described in section 4.4 are placed on the links, then, we only need to add the conditions and run the program again.

4.7 Related Discussion/Open questions

1) Can we always find a Nash toll vectors $\bar{\theta}_a^k$, $\forall a \in A$, $\forall k \in K$ that solve (4.1) when there is restriction on link tolls?
2) If Nash toll vector exists, is it unique?
2) Does the existence of Pareto or local solution to system (4.2) *or* (4.3) imply the existence of Nash toll vector solving (4.1)
4) Does there exist a core for the above leaders game?

Chapter 5
Concluding Remarks and Future Research

5.1 Summary of our findings

Owing to the fact that road pricing schemes centered around congestion alone may create negative benefits for society, thus, defeating its objective, we have designed a flexible tolling scheme that will help reduce the effect of most important traffic externalities. This flexible tolling schemes help to bridge the gap between the theories and practical feasibility. We showed that it is possible to attain a Pareto optimal flow pattern that minimizes the entire system cost when a single governing body is allowed to place tolls on the transportation network and when all links can be tolled. Since tolling all links of a network is not feasible in general, we also looked at the second best Pareto optimal solution when restrictions are placed on tolls. Although with this restriction, the first best Pareto optimal solution is not guaranteed in general, we were able to show that we may still achieve the first best solution with the flexibility of the tolling schemes designed.

On the other, it may be that numerous stakeholders have capabilities of tolling the transportation network to the best of their interests. It turned out that when there is no restrictions on tolls, stakeholders will continue to change their toll vectors perpetually with respect to other stakeholders' reaction. If "enough" restrictions are placed on link tolls, a Nash equilibrium may be reached. We showed that when stakeholders form grand coalition, there exist a tolling scheme and profit allocation rule that guaranty every stakeholder his outcome benefit in Nash equilibrium, and additional benefit as a great incentive for the stakeholders to form grand coalition. All our models were centered around classsical optimization formulations.

5.2 Research extensions and Recommendations

Since our models are centered around classical optimization formulations, the number of variables can grow uncontrollably big when the network is large. This calls for an efficient optimisation heuristic which can transform the analytical models into heuristic algorithms capable of handling large networks. It will be nice if the models used in this thesis is extended to time dependent models. It will also be nice if one investigates into the existence and uniqueness of Nash equilibrium when the stakeholders do not cooperate.

Appendix A
Bilevel Optimization

As the name implies, bilevel optimization problem involves solving a two-level optimization problem. In general terms, bilevel optimization problem can be mathematically stated as follows:

$$\min_{x \,\epsilon\, X,\, y} \; F(x,y)$$

$$s.t$$

$$G(x,y) \leq 0$$

$$\min_{y} \; f(x,y)$$

$$s.t$$

$$g(x,y) \leq 0$$

where $x \,\epsilon\, \mathbb{R}^{n1}$ are the upper level variables and $y \,\epsilon\, \mathbb{R}^{n2}$ are the lower level variables, $F : \mathbb{R}^{n1}$ X $\mathbb{R}^{n2} \to \mathbb{R}$ and $f : \mathbb{R}^{n1}$ X $\mathbb{R}^{n2} \to \mathbb{R}$ are respectively the upper and the lower level objective functions. $G(x,y) : \mathbb{R}^{n1}$ X $\mathbb{R}^{n2} \to \mathbb{R}^{m1}$ and $g(x,y) : \mathbb{R}^{n1}$ X $\mathbb{R}^{n2} \to \mathbb{R}^{m2}$ are the upper and the lower level constraints respectively. Observe that each level is a full optimization program. The upper level constraint $G(x,y)$ does not bind the lower level optimization problem.

Bilevel problem are generically non-convex and sometimes non-differentiable and thus, they are intrinsically hard. Disconnectedness of the solution space of bilevel problems also contributes to the hardness of the problem. Even for a linear case, it can be shown that the solution space is disconnected.

When the lower level problem is convex and regular, it can be replaced by its *KKT* conditions, yielding a aingle level optimization problem. Unfortunately, even under suitable convexity assumptions on the functions F, G, and the set X, the above mathematical program is still not easy to solve. This is due to the non convexities of the resulting complementarity and Lagerangean constraints. In literature, other solution methods exist. Colson, et al (9) discuss in detail the following solution methods:

a) Extreme point approaches for linear case
b) Branch and bound methods
c) Complementary pivoting
d) Descent methods
e) Penalty function methods
f) Trust region methods

A special case of a bilevel problem where the lower level problem involves equilibrium problem is the well known mathematical problem with equilibrium constraints (*MPECs*). This is the class of the bilevel problem examined in this thesis.

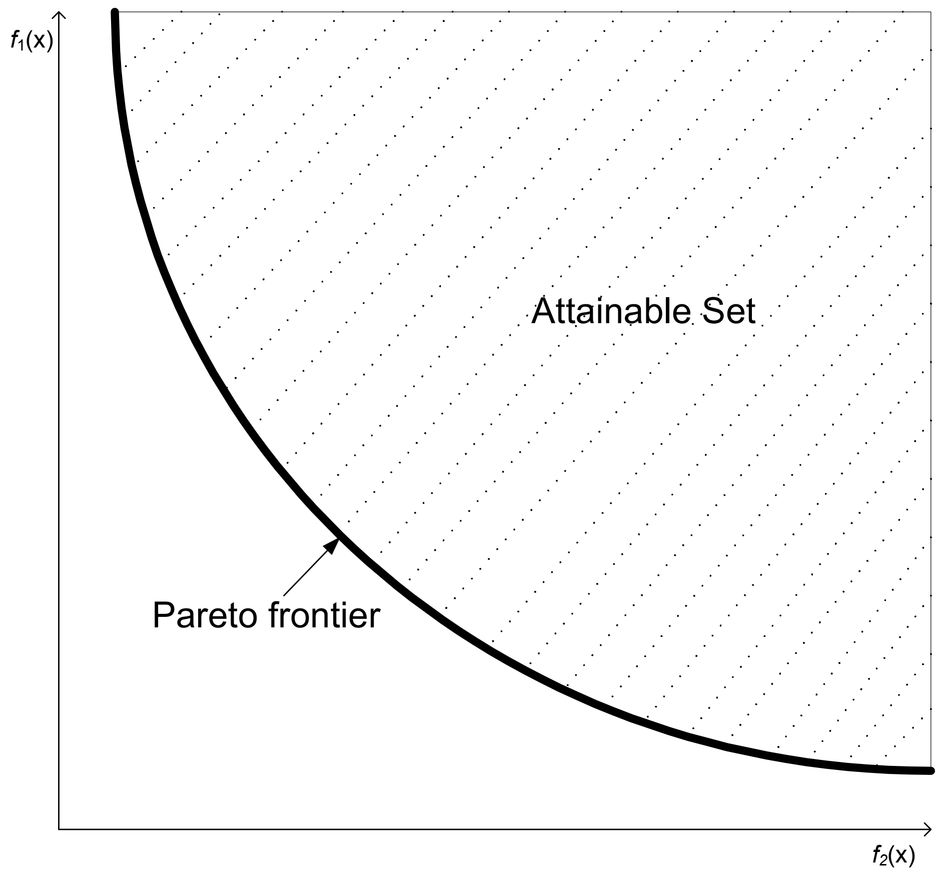
Appendix B
Multiobjective Optimization Problem

A number of practical optimization problems involve optimization of more than one objective simultaneously. Without loss of generality, the multiobjective optimization problem is the problem of minimizing the n objectives $f_i(x), k = 1,\ldots,n$ of a variable vector p in a universe X. Which can be mathematically be stated as follow:

$$\min F(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$$
$$s.t$$
$$x \in X$$
$$X = \left\{x : g_i(x) \leq 0,\ h_j(x) = 0, x = [x_1 \ldots x_n]^T,\ i = 1,\ldots,m_1;\ j = 1,\ldots,m_2\right\}$$

where x denotes the optimization decision variable , X represents the set of possible optimization variable x, $f_i(x)$ is a non linear objective function, and $g_i(x)$ *and* $h_j(x)$ are nonlinear inequality and equality constraint functions respectively. In general, the multiobjective problem has no optimal solution that could optimize all objectives simultaneously. But there exist a set of equally efficient, or non-inferior,alternative solutions, known as the Pareto optimal set or the Pareto frontier. A pareto optimal solution has the property that it is not possible to reduce any of the objective functions without increasing at least one of the other objective functions.

The figure below illustrates the Pareto optimal set for just two objectives $f_1(x)$ *and* $f_2(x)$, assuming convex (or linear) solution space.



A solution to the represented multiobjective optimization must lie on the Pareto frontier.

The multiobjective optimization problem is convex if all the objective functions and the feasible region are convex. Miettinen (25) shows that if the multiobjective optimization problem is convex, then every locally Pareto optimal solution is also globally Pareto optimal. When the solution space is not convex, then the problem becomes more difficult to solve. The connectedness of the solution space of multiobjective problems is also referred in (25).

In addition to the *weighted* $p-$*norm method* described in this thesis, other methods for solving multiobjective problem exist in literature. Detailed note on the following methods can be found in (20) *and* (25) :

a) ε –Constraint Method
b) Hybrib method
c) Method of weighted metrics
d) Value Function Method
e) Goal Programming
f) Goal Attainment Method
g) Iteractive Surrogate worth Trade-Off Method
h) Geoffrion-Dyer-Feinberg method
i) Sequential Proxy Optimization Technique
j) Tchebycheff method
k) Step Method (STEM)
l) GUESS Method
m) Satisficing Trade-Off Method
n) Light Beam Search
o) Reference Direction Approach (Method)
p) NIMBUS Method
q) Method of inequalities
r) Multiobjective Genetic Algorithm
s) Multiobjective Control Method
t) The minimax Reference Point Method
u) Iteractive Step Trade-Off Method
v) Gradient Projection Method

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