Evolution of the Modal Densities of a Class of Stochastic Hybrid Systems

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June 2001

...to my parents and Ilan.

Contents

Preface

This report is based on the work I did at the Nationaal Lucht-en Ruimtevaartlaboratorium (NLR) during the period December 2000 - May 2001. The work is done as a final project, a part of an international master program in Engineering Mathematics at Twente University, the Netherlands.

In this text, a study on evolution of modal density functions of certain classes of stochastic hybrid systems is conducted. The motivation behind the study is based on the possible applications in the field of air traffic management.

I would like to thank the following people for their support and contribution during the work and the writing of this report. Prof. Dr. Arjan van der Schaft (UT), Dr. Ir. Henk Blom (NLR), and Ir. Jelmer Scholte (NLR) are my final project supervisors. Ir. Bert Bakker (NLR), although not officially a supervisor, has been actively involved in supervising the work. Prof. Dr. Arun Bagchi, Dr. Michel Vellekoop, and Dr. Suresh Kumar, all from the Faculty of Applied Mathematics, UT, also contributed with valuable suggestions. Emad Imreizeeq and Arianto Wibowo, both Ph.D. students at the same faculty are always enthusiastic discussion partners and have given valuable inputs. I owe my gratitude to the people mentioned above. I would also like to express my gratitude to Cut Zidatul Fazla for constantly giving support during the work. Last but not least, I would like to thank my friends and colleagues, especially those in the *Contact99* group for the togetherness I have enjoyed in the past two years.

Finally, the author would like to express his hope that the results presented here can be useful for the reader, especially for further research.

Amsterdam, June 2001 A. Agung Julius

List of Abbreviations

The following is the list of abbreviations used in this report.

Abbreviation	Meaning
ATM	Air Traffic Management
BHP	Boundary Hitting Process
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PPP	Poisson Point Process
SDE	Stochastic Differential Equation
SHS	Stochastic Hybrid System

Chapter 1

Introduction

This report is based on research done at the National Aerospace Lab (NLR). The work is motivated by the possibility of applications in the field of air traffic management. In this report, the evolution of modal density functions of some classes of Stochastic Hybrid Systems (SHS) is considered.

The stochastic hybrid systems discussed in this report represent the dynamics of an object. The object can be thought of as, for example, an aircraft. However, the systems are intended to be rather general, so that the application will not be restricted to the particular problem involving aircrafts. Let us first introduce some notations that will be used throughout the rest of the report.

$$\begin{aligned} x_t &\in \mathsf{R}^{N_d} \\ \theta_t &\in M = \{m_1, m_{2,\cdots}, m_\mu\} \subset \mathsf{Z} \\ \xi_t &\triangleq (x_t, \theta_t)^{\mathrm{T}} \\ t &\in \mathsf{R}^+, \end{aligned}$$

We refer to x_t, θ_t, and ξ_t respectively as the *continuous state*, *discrete state* (or *mode*), and *hybrid state* of the object.

We assume that the hybrid state evolves according to a set of Ito stochastic differential equations (SDE) with a linear drift term, possibly involving a Boundary Hitting Process (BHP) and/or a Poisson Point Process (PPP).

In this report, we limit our attention to the case with $N_d = 1$, i.e. we assume that the continuous state x_t is an element of R.

1.1 Motivational Background

In this section we will discuss the reasoning behind the choice to employ the linear drift model.

First of all, it is generally known that linear models are in some sense simpler than nonlinear ones. For deterministic systems, linear models in the form of

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

have been widely used, and their properties have been intensively studied.

One of the stochastic counterparts of the system described in (1.1) is the *Ornstein-Uhlenbeck* process

$$dx_t = Ax_t \cdot dt + B \cdot dW_t, \tag{1.2}$$

which is also well known. Hence, it is expected that we can exploit some known results from these systems.

The other reason follows from the analysis below. We are interested in the behavior of the system without the presence of the stochastic elements. Let us consider the case where the Brownian motion and the Poisson process are not present. We will then have a deterministic process given by the following differential equation.

$$\dot{x}_t = Ax_t + \alpha \tag{1.3}$$

If A is invertible, (1.3) can be written as

$$\dot{x}_t = A(x_t + A^{-1}\alpha), \tag{1.4}$$

which is a linear system with a shifted coordinate such that the equilibrium is at $x = -A^{-1}\alpha$. The behavior of such system resembles that of a usual linear system, so we will not discuss it further. However, it should be realized that such drift can be used, for example, to model trajectories converging to a specific point in the continuous state space.

If A is singular, some other behavior can be encountered. Recalling that $x_t \in \mathsf{R}^{N_d}$, we assume that A is an $N_d \times N_d$ real matrix. Let us denote the rank of $\mathrm{Im}(A)$ as $n_A < N_d$. There is a linear invertible coordinate transformation

$$v = Tx, \tag{1.5}$$

such that

$$TAT^{-1} = \begin{bmatrix} A_v & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \updownarrow n_A \\ \updownarrow N_d - n_A \end{matrix}, \tag{1.6}$$

where A_v is a nonsingular $n_A \times n_A$ real matrix. We can then have the following relation

$$\dot{v} = \begin{bmatrix} A_v & 0 \\ 0 & 0 \end{bmatrix} v + T\alpha,$$

which can be broken down into two parts

$$\begin{aligned} [\dot{v}_1 \cdots \dot{v}_{n_A}]^{\mathrm{T}} &= A_v[v_1 \cdots v_{n_A}]^{\mathrm{T}} + T_1\alpha \\ [\dot{v}_{n_A} \cdots \dot{v}_{N_d}]^{\mathrm{T}} &= T_2\alpha \end{aligned} \tag{1.7}$$

where T_1 consists of the first n_A rows of T and T_2 being the rest.

If A_v is a Hurwitz matrix, the trajectories of the state will converge asymptotically to the hyperplane characterized by $[v_1 \cdots v_{n_A}]^{\mathrm{T}} = -A_v^{-1}T_1\alpha$. There is also, however, a constant drift given by the second line of (1.7).

This kind of behavior can be quite useful in modeling the trajectory of an aircraft. Imagine an aircraft moving in a three dimensional space (hence we take $N_d = 3$). If $n_A = 2$, we can model the trajectory converging to a line of the space. In real life this could be thought as an aircraft following a specific predefined route. Using $n_A = 1$, we can, for example, model the trajectory converging to a plane, e.g. an aircraft maneuvering to obtain a specific altitude.

Other features in the models discussed in this report can also be related with certain phenomenons in the field of air traffic management. The presence of white noise can be thought of as a representation for inaccuracy in measurements in the real life and influence from the environment that is stochastic by nature.

Changes of modes can be considered as representations of changes in flying dynamics or flying directives. Changes that are related to the PPP can be used to model certain failures in the aircraft.

The BHP can be used to model the changes in the flying directives. The changes of flying directives is very common in real life. An aircraft flying from airport A to airport B, for example, will pass through several zones, each of which has a specific flying directive, e.g. climb to altitude X, fly toward direction Y, etc. This phenomenon can be represented by including mode changes due to BHP, i.e. we divide the state space such that in a specific subset of the space a certain dynamics is observed.

Another use of the BHP is for modeling crashes between aircrafts. Suppose that we have two or more aircrafts, we can extend the state space by combining all the states of each aircraft. We can then define a *safe set* in the state space, which is the collection of states where all the aircrafts are separated by more than a certain minimal distance. A crash between aircrafts can be defined as the event when the process hits the boundary of the safe set.

1.2 Organization of the Report

In the first chapter, an introduction to the report, including the motivational background for the research and the structure of the report, is given. In Chapter 2, we consider a class of stochastic hybrid systems where a Poisson point process is involved. In Chapter 3, we consider a class of diffusion processes with mode switchings regulated by a boundary hitting process. In Chapter 4, we combine the features of the hybrid systems described in the previous two chapters to obtain a larger class, i.e. systems with diffusion with mode switchings regulated by a boundary hitting process and a Poisson point process. The last chapter, Chapter 5 includes description of possible extensions for the research and some conclusions.

Chapter 2

A Model with a Poisson Point Process

2.1 Model Formulation

In this chapter we consider a model of linear systems with mode changes triggered by a PPP. The model we use is of the form

$$\left.\begin{array}{c} dx_t = [a(\theta_t)x_t + \alpha(\theta_t)]\,dt \\ d\theta_t = c(\theta_{t^-}) \cdot dP_t \end{array}\right\} \tag{2.1}$$

$$\begin{array}{rcl} x_t & \in & \mathsf{R} \\ \theta_t & \in & M = \{0, 1\} \\ \xi_t & , & (x_t, \theta_t)^{\mathrm{T}} \\ t & \in & \mathsf{R}^+, \end{array}$$

The symbol θ_{t^-} is defined as $\lim\limits_{\tau \uparrow t} \theta_\tau = 0$. The symbol P_t denotes a right continuous Poisson point process with constant rate λ. We define

$$c(\theta_t) \;,\; \left\{ \begin{array}{l} 1,\ \text{if } \theta_t = 0 \\ -1,\ \text{if } \theta_t = 1 \end{array} \right. . \tag{2.2}$$

Moreover, it is assumed that the initial values

$$\xi_0 \;,\; (x_0\ \theta_0)^{\mathrm{T}}$$

are distributed according to some known distributions. We assume that there exists a probability space $(\Omega, P, \mathcal{F})$ together with a filtration $\mathcal{F}_t$ such that ξ_t and P_t is adapted to it.

The interpretation for the model described above is as follows. We can think of the model as a representation of the dynamics of a system with two modes. Each mode has its own linear dynamics described by the first line of (2.1). The mode of the dynamics changes whenever P_t (a PPP with constant rate λ) generates a point.

The reader is referred to [Sie97] to get a brief information on PPP.

2.2 Transition Density and Modal Density

If X_t is a Markov process, we define the transition density function of X_t as follows

Definition 2.1 *The transition density function* $p(x, t, y)$ *is the density function such that*

$$P(X_{s+t} \in \Gamma | X_s = x) = \int_\Gamma p(x, t, y) dy$$

for all Γ *(measurable) subset of* R *and* $s \geq 0$.

Notation 2.2 *Let* X *be a random variable. In this report, we denote the distribution density function of* X *as* $p_X(\cdot)$.

Definition 2.3 *In case of SHS, in this report we introduce the notion of modal density,* $\Psi^\theta_{x_t}(x, t)$ *where*

$$P(x_t \in \Gamma, \theta_t = \theta) = \int_\Gamma \Psi^\theta_{x_t}(x, t) dx$$

for all Γ *(measurable) subset of* R *and* $t \geq 0$.

Notation 2.4 *Throughout the report, we will also use the notation* $\Psi^\theta_{x_s}(x)$ *to denote the function* $\Psi^\theta_{x_t}(x, t)\big|_{t=s}$.

Notice that the modal density can be thought as the transition density for SHS, since for SHS we have not only density for the continuous state but also for the modes. We are interested in studying the evolution of $\Psi^\theta_{x_t}(x, t)$, especially in obtaining the expression for $\frac{\partial}{\partial t}\Psi^\theta_{x_t}(x, t)$ in relation with $\Psi^\theta_{x_t}(x, t)$.

In this report, it is always assumed that we know the initial distributions of the state, i.e. the initial modal density.

$$\Psi^\theta_{x_t}(x, t) = \Psi^\theta_{x_0}(x), \forall \theta \in M$$

It is also possible that we need to assume that the initial modal densities satisfy some other conditions, e.g. see Chapter 3 and Chapter 4.

2.3 Evolution of the Modal Density

In order to obtain the expression for $\frac{\partial}{\partial t}\Psi^\theta_{x_t}(x, t)$, we shall make use of the following propositions.

Proposition 2.5 *The following relation holds,*

$$\lim_{\Delta \downarrow 0} \frac{P(\textit{at least 1 point is generated by } P_t \textit{ in } [s, s+\Delta])}{\Delta} > 0 \tag{2.3}$$

for all $s \geq 0$.

Proof. By the definition of the Poisson point process,

$$P(\text{at least 1 point is generated by } P_t \text{ in } [s, s+\Delta]) = 1 - e^{-\lambda\Delta}. \tag{2.4}$$

Hence

$$\begin{aligned}\lim_{\Delta\downarrow 0}\frac{P(\text{at least 1 point is generated by } P_t \text{ in } [s,s+\Delta])}{\Delta} &= \lim_{\Delta\downarrow 0}\frac{1-e^{-\lambda\Delta}}{\Delta}\\ &= \lambda. \end{aligned}\tag{2.5}$$

■

The consequence of this proposition is that we cannot completely neglect the probability of a point generated by the Poisson process when we are doing the analysis over an infinitesimal time interval. However, the following proposition is also useful.

Proposition 2.6 *The following relation holds,*

$$\lim_{\Delta\downarrow 0}\frac{P(\text{at least 2 points are generated by } P_t \text{ in } [s,s+\Delta])}{\Delta} = 0 \tag{2.6}$$

for all $s \geq 0$.

Proof. By the definition of the Poisson point process,

$$P(\text{at least 2 points are generated by } P_t \text{ in } [s,s+\Delta]) = 1-e^{-\lambda\Delta}-\Delta\lambda e^{-\lambda\Delta}. \tag{2.7}$$

Hence

$$\begin{aligned}\lim_{\Delta\downarrow 0}\frac{P(\text{at least 2 points are generated by } P_t \text{ in } [s,s+\Delta])}{\Delta} &= \lim_{\Delta\downarrow 0}\frac{1-e^{-\lambda\Delta}-\Delta\lambda e^{-\lambda\Delta}}{\Delta}\\ &= 0. \end{aligned}\tag{2.8}$$

■

Consequently, it is sufficient if we take into account the case where at most one point is generated in the infinitesimal time interval. We can then obtain the following theorem

Definition 2.7 *Let* $\tau_{s<}$ *be defined as the first time after* s *when* P_t *generates a point, i.e.* $\tau_{s<}$, $\min_{\tau>s}(\tau - s : P_t \text{ generates a point at } t=\tau)$.

Definition 2.8 *The function* $p_{\tau_{s<}}(\cdot)$ *is the distribution density* $\tau_{s<}$ *such that*

$$P(P_t \text{ generates the first point after } s \text{ in } [s+a,s+b]) = \int_a^b p_{\tau_{s<}}(r)\;dr \tag{2.9}$$

for $b > a \geq 0$.

Theorem 2.9 *Assuming that at most one point is generated by* P_t *in* $[s,s+\Delta]$ *for all* $s \geq 0$, *the following relations hold true.*

$$\Psi^0_{x_t}(x,s+\Delta) = e^{-\lambda\Delta}\psi^{0,0}(x,s+\Delta) + \int_0^\Delta p_{\tau_{s<}}(r)\psi^{0,1}(x,s+\Delta,r)dr \tag{2.10a}$$

$$\Psi^1_{x_t}(x,s+\Delta) = e^{-\lambda\Delta}\psi^{1,1}(x,s+\Delta) + \int_0^\Delta p_{\tau_{s<}}(r)\psi^{1,0}(x,s+\Delta,r)dr \tag{2.10b}$$

where $\psi^{0,0}(x,t)$ *is the solution of the PDE*

$$\left.\begin{array}{l}\frac{\partial\psi^{0,0}(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[(a(0)x+\alpha(0))\,\psi^{0,0}(x,t)\right] = 0\\ \psi^{0,0}(x,s) = \Psi^0_{x_t}(x,s)\\ x\in\mathsf{R}, t>s\end{array}\right\}\tag{2.11}$$

and $\psi^{1,1}(x,t)$ is the solution of the PDE

$$\left.\begin{array}{l}\frac{\partial\psi^{1,1}(x,t)}{\partial t}+\frac{\partial}{\partial x}\left[\left(a(1)x+\alpha(1)\right)\psi^{1,1}(x,t)\right]=0\\ \psi^{1,1}(x,s)=\Psi^{1}_{x_t}(x,s)\\ x\in\mathsf{R},t>s\end{array}\right\} \quad (2.12)$$

The function $\psi^{0,1}(x,t,r)$ satisfies the following system of PDE

$$\left.\begin{array}{l}\frac{\partial\psi^{0,1}(x,t)}{\partial t}+\frac{\partial}{\partial x}\left[\left(a(1)x+\alpha(1)\right)\psi^{0,1}(x,t)\right]=0\\ \psi^{0,1}(x,s)=\Psi^{1}_{x_t}(x,s)\\ x\in\mathsf{R},s+r\geq t>s,r\leq\Delta\end{array}\right\} \quad (2.13a)$$

$$\left.\begin{array}{l}\frac{\partial\psi^{0,1}(x,t)}{\partial t}+\frac{\partial}{\partial x}\left[\left(a(0)x+\alpha(0)\right)\psi^{0,1}(x,t)\right]=0\\ x\in\mathsf{R},t>s+r,r\leq\Delta\end{array}\right\} \quad (2.13b)$$

The interpretation for the PDE system above is that we solve (2.13a) first to get $\psi^{0,1}(x,s+r,r)$ and use this function as a boundary condition for (2.13b). Analogously, the function $\psi^{1,0}(x,t,r)$ satisfies the following system of PDE

$$\left.\begin{array}{l}\frac{\partial\psi^{1,0}(x,t)}{\partial t}+\frac{\partial}{\partial x}\left[\left(a(0)x+\alpha(0)\right)\psi^{1,0}(x,t)\right]=0\\ \psi^{1,0}(x,s)=\Psi^{0}_{x_t}(x,s)\\ x\in\mathsf{R},s+r\geq t>s,r\leq\Delta\end{array}\right\} \quad (2.14)$$

$$\left.\begin{array}{l}\frac{\partial\psi^{1,0}(x,t)}{\partial t}+\frac{\partial}{\partial x}\left[\left(a(1)x+\alpha(1)\right)\psi^{1,0}(x,t)\right]=0\\ x\in\mathsf{R},t>s+r,r\leq\Delta\end{array}\right\} \quad (2.15)$$

Proof. Due to symmetry, the proof will be given for half of the proposition i.e. the part related to (2.10a), (2.11), (2.13a), and (2.13b). The other half will follow the same line.

Notice that under the assumption that at most one point is generated in $[s.s+\Delta]$ we have for $\Gamma\subset\mathsf{R}$

$$\begin{aligned}&P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0)=\\ &=P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0,\theta_s=0)+P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0,\theta_s=1).\end{aligned} \quad (2.16)$$

Let us define density functions for each term of the right hand side of (2.16), such that

$$P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0,\theta_s=0) \triangleq \int_\Gamma\tilde{\psi}^{0,0}(x;s,\Delta)dx \quad (2.17)$$

$$P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0,\theta_s=1) \triangleq \int_\Gamma\tilde{\psi}^{0,1}(x;s,\Delta)dx \quad (2.18)$$

$$\Psi^0_{x_t}(x,s+\Delta)=\tilde{\psi}^{0,0}(x;s,\Delta)+\tilde{\psi}^{0,1}(x;s,\Delta) \quad (2.19)$$

From Chapman - Kolmogorov equation, we have the following relations

$$\begin{aligned}P(x_{s+\Delta} &\in \Gamma,\theta_{s+\Delta}=0,\theta_s=0)=\\ &= \int_{-\infty}^{\infty}\Psi^0_{x_t}(y,s)\cdot P(x_{s+\Delta}\in\Gamma,\theta_{s+\Delta}=0|\theta_s=0,x_s=y)dy\end{aligned} \quad (2.20)$$

Now, we also have that

$$\begin{aligned} P(x_{s+\Delta} &\in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 0, x_s = y) = \\ &= P(x_{s+\Delta} \in \Gamma | \theta_{s+\Delta} = 0, \theta_s = 0, x_s = y) \cdot P(\theta_{s+\Delta} = 0 | \theta_s = 0, x_s = y) \end{aligned} \tag{2.21}$$

Because of the assumption, we also have that

$$\begin{aligned} P(\theta_{s+\Delta} = 0 | \theta_s = 0, x_s = y) &= P(\theta_{s+t} = 0, \forall t \in [0, \Delta] | \theta_s = 0, x_s = y) \\ &= e^{-\lambda \Delta}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} P(x_{s+\Delta} &\in \Gamma | \theta_{s+\Delta} = 0, \theta_s = 0, x_s = y) = \\ &= P(x_{s+\Delta} \in \Gamma | x_s = y, \ \theta_{s+t} = 0, \forall t \in [0, \Delta]) \end{aligned} \tag{2.23}$$

Let us introduce the density $\psi^0(x; y, s, \Delta)$ such that

$$P(x_{s+\Delta} \in \Gamma | x_s = y, \ \theta_{s+t} = 0, \forall t \in [0, \Delta]) \triangleq \int_\Gamma \psi^0(x; y, s, \Delta) dx$$

In [Dav93] and [Dav84], it is given that $\psi^0(x; y, s, \Delta) \equiv \tilde{p}^0(x, t; y)\big|_{t=s+\Delta}$ is the solution of the PDE

$$\left. \begin{array}{l} \frac{\partial \tilde{p}^0}{\partial t} + \frac{\partial}{\partial x} \left[(a(0)x + \alpha(0)) \, \tilde{p}^0 \right] = 0 \\ \tilde{p}^0(x, s; y) = \delta(x - y) \\ x > 0, t > s \end{array} \right\}, \tag{2.24}$$

Combining (2.20) - (2.24), we will get

$$\begin{aligned} P(x_{s+\Delta} &\in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 0) = \\ &= \int_{-\infty}^{\infty} \Psi^0_{x_t}(y, s) \cdot e^{-\lambda \Delta} \cdot \int_\Gamma \psi^0(x; y, s, \Delta) dx dy \end{aligned}$$

or equivalently

$$\begin{aligned} \int_\Gamma \tilde{\psi}^{0,0}(x; s, \Delta) dx &= \int_{-\infty}^{\infty} \Psi^0_{x_t}(y, s) \cdot e^{-\lambda \Delta} \cdot \int_\Gamma \psi^0(x; y, s, \Delta) dx dy \\ &= \int_\Gamma e^{-\lambda \Delta} \int_{-\infty}^{\infty} \Psi^0_{x_t}(y, s) \cdot \psi^0(x; y, s, \Delta) dy dx \end{aligned} \tag{2.25}$$

If we denote

$$\psi^{0,0}(x, s + \Delta) \triangleq \int_{-\infty}^{\infty} \Psi^0_{x_t}(y, s) \psi^0(x; y, s, \Delta) dy,$$

it can be derived that $\psi^{0,0}(x, s + \Delta)$ satisfies (2.11) since $\psi^0(x; y, s, \Delta)$ is the Green's function of the PDE. We then have the following relation

$$\tilde{\psi}^{0,0}(x; s, \Delta) = e^{-\lambda \Delta} \psi^{0,0}(x, s + \Delta) \tag{2.26}$$

We turn our attention to $\tilde{\psi}^{0,1}(x, s+\Delta)$. We start with the following relation

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1) =$$
$$= \int_0^\Delta P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1, \tau_{s<} \in [r, r+dr])dr$$
$$= \int_0^\Delta P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1 | \tau_{s<} \in [r, r+dr]) \times P(\tau_{s<} \in [r, r+dr])dr$$
$$= \int_0^\Delta P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1 | \tau_{s<} \in [r, r+dr]) \times p_{\tau_{s<}}(r)dr \tag{2.27}$$

Note that since there can be at most one jump in $[s, s+\Delta]$ we have that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1 | \tau_{s<} \in [r, r+dr]) =$$
$$= P(x_{s+\Delta} \in \Gamma, \theta_s = 1, \theta_{s+t} = 0, \forall t \in [r, \Delta] | \tau_{s<} \in [r, r+dr]) \tag{2.28}$$

Let us define a density $\psi^{0,1}(x, s+\Delta)$ such that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1 | \tau_{s<} \in [r, r+dr]) \triangleq \int_\Gamma \psi^{0,1}(x, s+\Delta)dx. \tag{2.29}$$

Now, combining (2.27) and (2.29), we get

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1) = \int_0^\Delta p_{\tau_{s<}}(r) \int_\Gamma \psi^{0,1}(x, s+\Delta)dxdr,$$

or equivalently

$$\int_\Gamma \tilde{\psi}^{0,1}(x, s+\Delta)dx = \int_\Gamma \int_0^\Delta p_{\tau_{s<}}(r)\psi^{0,1}(x, s+\Delta)drdx$$
$$\tilde{\psi}^{0,1}(x, s+\Delta) = \int_0^\Delta p_{\tau_{s<}}(r)\psi^{0,1}(x, s+\Delta)dr \tag{2.30}$$

To show that $\psi^{0,1}(x,t)$ satisfies (2.13a) and (2.13b) can be done by following the way of showing $\psi^{0,0}(x,t)$ satisfies (2.11). We know that in the interval $s < t \leq s+r$ the dynamics is given by

$$dx_t = [a(1)x_t + \alpha(1)]\, dt$$
$$\theta_t = 1,$$

hence (2.13a), and in $s+r < t \leq s+\Delta$ the dynamics is given by

$$dx_t = [a(0)x_t + \alpha(0)]\, dt$$
$$\theta_t = 0,$$

which is represented by (2.13b). ■

Based on Theorem 2.9 above we can derive expressions for the evolution of the modal densities.

Theorem 2.10 *The modal densities $\Psi^0_{x_t}(x,t)$ and $\Psi^1_{x_t}(x,t)$ satisfy the following PDE*

$$\left.\begin{array}{c} \frac{\partial \Psi^0_{x_t}}{\partial t} = -\frac{\partial}{\partial x}\left[(a(0)x + \alpha(0))\,\Psi^0_{x_t}\right] + \lambda(\Psi^1_{x_t} - \Psi^0_{x_t}) \\ \frac{\partial \Psi^1_{x_t}}{\partial t} = -\frac{\partial}{\partial x}\left[(a(1)x + \alpha(1))\,\Psi^1_{x_t}\right] + \lambda(\Psi^0_{x_t} - \Psi^1_{x_t}) \\ \Psi^0(x,0) = \Psi^0_{x_0}(x) \\ \Psi^1_{x_t}(x,0) = \Psi^1_{x_0}(x) \\ t \geq 0, x \in \mathsf{R} \end{array}\right\} \tag{2.31}$$

Proof. From (2.10a) and (2.10b) we can derive the following relations for any $s \geq 0$

$$\frac{\Psi^0_{x_t}(x,s+\Delta)-\Psi^0_{x_t}(x,s)}{\Delta} = \frac{e^{-\lambda\Delta}\psi^{0,0}(x,s+\Delta)-\Psi^0_{x_t}(x,s)}{\Delta} + \\ + \frac{\int_0^\Delta p_{\tau_{s<}}(r)\psi^{0,1}(x,s+\Delta,r)dr}{\Delta} \tag{2.32}$$

$$\frac{\Psi^1_{x_t}(x,s+\Delta)-\Psi^1_{x_t}(x,s)}{\Delta} = \frac{e^{-\lambda\Delta}\psi^{1,1}(x,s+\Delta)-\Psi^1_{x_t}(x,s)}{\Delta} + \\ + \frac{\int_0^\Delta p_{\tau_{s<}}(r)\psi^{1,0}(x,s+\Delta,r)dr}{\Delta} \tag{2.33}$$

If we take the limit of $\Delta \downarrow 0$, we would obtain

$$\frac{\partial}{\partial t}\Psi^0_{x_t}(x,t) = \lim_{\Delta\downarrow 0}\frac{e^{-\lambda\Delta}\psi^{0,0}(x,s+\Delta)-\Psi^0_{x_t}(x,s)}{\Delta} + \\ + \lim_{\Delta\downarrow 0}\frac{\int_0^\Delta p_{\tau_{s<}}(r)\psi^{0,1}(x,s+\Delta,r)dr}{\Delta} \tag{2.34}$$

$$\frac{\partial}{\partial t}\Psi^1_{x_t}(x,t) = \lim_{\Delta\downarrow 0}\frac{e^{-\lambda\Delta}\psi^{1,1}(x,s+\Delta)-\Psi^1_{x_t}(x,s)}{\Delta} + \\ + \lim_{\Delta\downarrow 0}\frac{\int_0^\Delta p_{\tau_{s<}}(r)\psi^{1,0}(x,s+\Delta,r)dr}{\Delta} \tag{2.35}$$

Referring to (2.11), it can be derived that $f(x,\Delta) \triangleq e^{-\lambda\Delta}\psi^{0,0}(x,s+\Delta)$ satisfies the following PDE

$$\left.\begin{array}{l} \frac{\partial f}{\partial \Delta} + \frac{\partial}{\partial x}\left[\left(a(0)x+\alpha(0)\right)f\right] + \lambda f = 0 \\ f(x,0) = \Psi^0_{x_t}(x,s) \\ x \in \mathsf{R}, t > s \end{array}\right\} \tag{2.36}$$

Hence

$$\lim_{\Delta\downarrow 0}\frac{e^{-\lambda\Delta}\psi^{0,0}(x,s+\Delta)-\Psi^0_{x_t}(x,s)}{\Delta} = -\frac{\partial}{\partial x}\left[\left(a(0)x+\alpha(0)\right)\Psi^0_{x_t}(x,s)\right] - \lambda\Psi^0_{x_t}(x,s). \tag{2.37}$$

Similarly, by referring to (2.12), it can be derived that $g(x,\Delta) \triangleq e^{-\lambda\Delta}\psi^{1,1}(x,s+\Delta)$ satisfies the following PDE

$$\left.\begin{array}{l} \frac{\partial g}{\partial \Delta} + \frac{\partial}{\partial x}\left[\left(a(1)x+\alpha(1)\right)g\right] + \lambda g = 0 \\ g(x,0) = \Psi^1_{x_t}(x,s) \\ x \in \mathsf{R}, t > s \end{array}\right\} \tag{2.38}$$

Hence

$$\lim_{\Delta\downarrow 0}\frac{e^{-\lambda\Delta}\psi^{1,1}(x,s+\Delta)-\Psi^1_{x_t}(x,s)}{\Delta} = -\frac{\partial}{\partial x}\left[\left(a(1)x+\alpha(1)\right)\Psi^1_{x_t}(x,s)\right] - \lambda\Psi^1_{x_t}(x,s). \tag{2.39}$$

Now, we also have that

$$\begin{aligned}\lim_{\Delta\downarrow 0}\frac{\int_0^{\Delta} p_{\tau_{s<}}(r)\psi^{0,1}(x,s+\Delta,r)dr}{\Delta} &= p_{\tau_{s<}}(0)\psi^{0,1}(x,s,0)\\ &= p_{\tau_{s<}}(0)\Psi^{1}_{x_t}(x,s)\end{aligned} \tag{2.40}$$

$$\begin{aligned}\lim_{\Delta\downarrow 0}\frac{\int_0^{\Delta} p_{\tau_{s<}}(r)\psi^{1,0}(x,s+\Delta,r)dr}{\Delta} &= p_{\tau_{s<}}(0)\psi^{1,0}(x,s,0)\\ &= p_{\tau_{s<}}(0)\Psi^{0}_{x_t}(x,s)\end{aligned} \tag{2.41}$$

In addition, the following relation also holds true

$$\begin{aligned}p_{\tau_{s<}}(t) &= \frac{\partial}{\partial t}P(\tau < t)\\ &= \frac{\partial}{\partial t}\left(1-e^{-\lambda t}\right) = \lambda e^{-\lambda t}.\end{aligned} \tag{2.42}$$

Hence

$$p_{\tau_{s<}}(0) = \lambda. \tag{2.43}$$

Substituting (2.43) to (2.40) and (2.41), we yield

$$\lim_{\Delta\downarrow 0}\frac{\int_0^{\Delta} p_{\tau_{s<}}(r)\psi^{0,1}(x,s+\Delta,r)dr}{\Delta} = \lambda\Psi^{1}_{x_t}(x,s) \tag{2.44}$$

$$\lim_{\Delta\downarrow 0}\frac{\int_0^{\Delta} p_{\tau_{s<}}(r)\psi^{1,0}(x,s+\Delta,r)dr}{\Delta} = \lambda\Psi^{0}_{x_t}(x,s). \tag{2.45}$$

If we substitute (2.37) and (2.44) to (2.34), and (2.39) and (2.45) to (2.35), we will obtain the first two lines of (2.31).

The boundary conditions follow from the assumption that the initial modal densities are known (see Section 2.2). ■

2.4 Example

2.4.1 Theoretical Results

In this section we will present an implementation of the theoretical results derived in the previous section. Suppose we have a process described by the following equations.

$$\left.\begin{aligned} dx_t &= [a(\theta_t)x_t + \alpha(\theta_t)]\,dt\\ d\theta^i_t &= c(\theta_{t^-})\cdot dP_t\end{aligned}\right\} \tag{2.46}$$

$$\begin{aligned} x_t &\in \mathsf{R}\\ \theta_t &\in M = \{0,1\}\\ \xi_t &\triangleq (x_t,\theta_t)^{\mathrm{T}}\\ t &\in \mathsf{R}^+,\end{aligned}$$

The symbol P_t denotes a Poisson point process with constant rate λ. We define

$$[a(0)x_t + \alpha(0)] \quad , \quad 1 \tag{2.47a}$$
$$[a(1)x_t + \alpha(1)] \quad , \quad -1 \tag{2.47b}$$

$$c(\theta_t) \ , \ \begin{cases} 1, \text{ if } \theta_t = 0 \\ -1, \text{ if } \theta_t = 1 \end{cases} . \tag{2.48}$$

Moreover, it is assumed that the initial state is deterministic

$$\xi_0 \ , \ (x_0 \, \theta_0)^{\mathrm{T}} = (0\, 0)^{\mathrm{T}}. \tag{2.49}$$

Now, according to Theorem 2.10 the following PDE describes the evolution of $\Psi^0_{x_t}(x,t)$ and $\Psi^1_{x_t}(x,t)$.

$$\left.\begin{array}{c} \frac{\partial \Psi^0_{x_t}}{\partial t} = -\frac{\partial \Psi^0_{x_t}}{\partial x} + \lambda(\Psi^1_{x_t} - \Psi^0_{x_t}) \\ \frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{\partial \Psi^1_{x_t}}{\partial x} + \lambda(\Psi^0_{x_t} - \Psi^1_{x_t}) \\ \Psi^0(x,0) = \delta(x) \\ \Psi^1_{x_t}(x,0) = 0 \\ t \geq 0, x \in \mathsf{R} \end{array}\right\} \tag{2.50}$$

We will proceed with solving (2.50). First, we mention two properties of the process we will exploit in finding the solution.

- $P(x_t = t, \theta_t = 0) \neq 0$ for any finite $t \in \mathsf{R}^+$, since $P(x_t = t, \theta_t = 0)$ is equal to $P(P_t$ generates no points in $[0,t])$.
- $P(|x_t| > t) = 0$ for any $t \in \mathsf{R}^+$.

Considering these two properties, we propose a solution in the form of

$$\Psi^0_{x_t}(x,t) = \phi(t)\delta(x-t) + f_1(x,t)(\mathbf{1}(x+t) - \mathbf{1}(x-t)), \tag{2.51}$$

$$\Psi^1_{x_t}(x,t) = f_2(x,t)(\mathbf{1}(x+t) - \mathbf{1}(x-t)), \tag{2.52}$$

where $\phi(t)$, $f_1(x,t)$ and $f_2(x,t)$ are smooth functions and $\mathbf{1}(\cdot)$ denotes the unit step function.

We will then have the following relations.

$$\begin{aligned} \frac{\partial \Psi^0_{x_t}}{\partial t} \quad &= \quad \frac{d\phi}{dt}\delta(x-t) - \phi(t)\delta'(x-t) + \frac{\partial f_1}{\partial t}(\mathbf{1}(x+t) - \mathbf{1}(x-t)) + \\ &\quad +f_1(x,t)(\delta(x+t) + \boldsymbol{\delta}(x-t)) \end{aligned} \tag{2.53}$$

$$\frac{\partial \Psi^0_{x_t}}{\partial x} = \phi(t)\delta'(x-t) + \frac{\partial f_1}{\partial x}(\mathbf{1}(x+t) - \mathbf{1}(x-t)) + f_1(x,t)(\delta(x+t) - \delta(x-t)) \tag{2.54}$$

$$\frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{\partial f_2}{\partial t}(\mathbf{1}(x+t) - \mathbf{1}(x-t)) + f_2(x,t)(\delta(x+t) + \boldsymbol{\delta}(x-t)) \tag{2.55}$$

$$\frac{\partial \Psi^1_{x_t}}{\partial x} = \frac{\partial f_2}{\partial x}(\mathbf{1}(x+t) - \mathbf{1}(x-t)) + f_2(x,t)(\delta(x+t) - \boldsymbol{\delta}(x-t)) \tag{2.56}$$

By using Eqs.(2.51) - (2.56), we can rewrite the first line of Eq.(2.50) as

$$\begin{aligned} &\frac{d\phi}{dt}\delta(x-t) + \left[\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x}\right](\mathbf{1}(x+t) - \mathbf{1}(x-t)) + 2f_1(x,t)\delta(x+t) = \\ &\quad = \lambda\left[f_2(x,t) - f_1(x,t)\right](\mathbf{1}(x+t) - \mathbf{1}(x-t)) - \lambda\phi(t)\delta(x-t). \end{aligned} \tag{2.57}$$

While the second line becomes

$$\left[\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x}\right](\mathbf{1}(x+t) - \mathbf{1}(x-t)) + 2f_2(x,t)\delta(x-t) =$$
$$= \lambda\left[f_1(x,t) - f_2(x,t)\right](\mathbf{1}(x+t) - \mathbf{1}(x-t)) + \lambda\phi(t)\delta(x-t) \tag{2.58}$$

From Eq.(2.57), assuming that $\phi(t)$, $f_1(x,t)$ and $f_2(x,t)$ are smooth, we can infer the following relations.

$$\frac{d\phi}{dt} = -\lambda\phi(t) \tag{2.59}$$
$$f_1(-t,t) = 0 \tag{2.60}$$
$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = \lambda\left[f_2(x,t) - f_1(x,t)\right], \text{ for } |x| \le t \tag{2.61}$$

Moreover, considering the initial modal densities and (2.51), the following initial condition can be deduced.

$$\phi(0) = 1 \tag{2.62}$$

From Eq.(2.58), we can obtain the following relations.

$$f_2(t,t) = \frac{\lambda}{2}\phi(t) \tag{2.63}$$
$$\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} = \lambda\left[f_1(x,t) - f_2(x,t)\right], \text{ for } |x| \le t \tag{2.64}$$

By solving Eq.(2.59) with initial condition (2.62), we get

$$\phi(t) = e^{-\lambda t}. \tag{2.65}$$

Substituting Eq.(2.65) to (2.63), we obtain

$$f_2(t,t) = \frac{\lambda}{2}e^{-\lambda t}. \tag{2.66}$$

We now have a coupled PDE defined on the quarter plane $|x| \le t$,

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = \lambda\left[f_2(x,t) - f_1(x,t)\right] \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} = \lambda\left[f_1(x,t) - f_2(x,t)\right] \end{array}\right\} \tag{2.67}$$

with boundary conditions

$$\left.\begin{array}{c} f_1(-t,t) = 0 \\ f_2(t,t) = \frac{\lambda}{2}e^{-\lambda t} \end{array}\right\} \tag{2.68}$$

Note that although we still have a coupled PDE to solve, the boundary conditions are now smooth. Now, let us introduce a coordinate transformation as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} \;,\; \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix}$$
$$\begin{bmatrix} x \\ t \end{bmatrix} \;,\; \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

The coupled PDE (2.67) and the boundary conditions (2.68) are transformed into

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial u} = \lambda(f_2(u,v) - f_1(u,v)) \\ \frac{\partial f_2}{\partial v} = \lambda(f_1(u,v) - f_2(u,v)) \end{array}\right\} \tag{2.69}$$

$$\left.\begin{array}{r} f_1(0,v) = 0 \\ f_2(u,0) = \frac{\lambda}{2}e^{-\lambda u} \end{array}\right\} \tag{2.70}$$

From the first line of Eq.(2.69) we obtain

$$f_2(u,v) = \frac{1}{\lambda}\frac{\partial f_1}{\partial u} + f_1(u,v). \tag{2.71}$$

By substituting Eq.(2.71) to the second line, we can decouple the PDE into

$$\frac{1}{\lambda}\frac{\partial^2 f_1}{\partial u \partial v} + \frac{\partial f_1}{\partial v} + \frac{\partial f_1}{\partial u} = 0, \tag{2.72}$$

while the boundary conditions (2.70) become

$$\left.\begin{array}{r} f_1(0,v) = 0 \\ \left(\frac{1}{\lambda}\frac{\partial f_1}{\partial u} + f_1(u,v)\right)_{v=0} = \frac{\lambda}{2}e^{-\lambda u} \end{array}\right\}. \tag{2.73}$$

We will then use the method of Laplace transform as follows.

$$F_1(v;s) \triangleq \int_0^\infty f_1(u,v)\cdot e^{-su}\cdot du. \tag{2.74}$$

Applying the transform to (2.72), we get

$$\begin{aligned} \frac{s}{\lambda}\frac{d}{dv}F_1(v;s) + \frac{d}{dv}F_1(v;s) + sF_1(v;s) &= 0 \\ \left(\frac{s}{\lambda}+1\right)\frac{d}{dv}F_1(v;s) + sF_1(v;s) &= 0. \end{aligned} \tag{2.75}$$

From the boundary conditions, we get

$$\frac{s}{\lambda}F_1(0;s) + F_1(0;s) = \frac{\lambda}{2(s+\lambda)}, \tag{2.76}$$

or equivalently

$$F_1(0;s) = \frac{\lambda^2}{2(s+\lambda)^2}. \tag{2.77}$$

Solving the ordinary differential equation (2.75) with initial condition (2.77), we obtain the following relation

$$F_1(v;s) = \frac{\lambda^2 e^{\frac{-s\lambda v}{s+\lambda}}}{2(s+\lambda)^2}. \tag{2.78}$$

To obtain the inverse transform of Eq.(2.78) we start with rewriting it as

$$F_1(v;s) = \frac{\lambda^2 e^{-\lambda v}}{2}\cdot\frac{e^{\frac{\lambda^2 v}{s+\lambda}}}{(s+\lambda)^2}. \tag{2.79}$$

Now, it can be obtained that

$$\mathcal{L}^{-1}\left(\frac{e^{\frac{\lambda^2 v}{s}}}{s^2}\right) = \frac{1}{\lambda}\sqrt{\frac{u}{v}}I_1\left(2\lambda\sqrt{uv}\right), \tag{2.80}$$

The symbol $\mathcal{L}^{-1}$ denotes the inverse of the Laplace transform defined in Eq.(2.74), and $I_1(\cdot)$ denotes the second order modified Bessel function of the first kind. Using the frequency shifting and linearity properties of the Laplace transform, it is relatively clear that we can get the following relation.

$$f_1(u,v) = \frac{\lambda e^{-\lambda(v+u)}}{2}\sqrt{\frac{u}{v}} I_1\left(2\lambda\sqrt{uv}\right) \tag{2.81}$$

By transforming back the coordinate, we obtain

$$f_1(x,t) = \frac{\lambda e^{-\lambda t}}{2}\sqrt{\frac{t+x}{t-x}} I_1\left(\lambda\sqrt{t^2-x^2}\right). \tag{2.82}$$

From Eqs.(2.71) and (2.72) we can obtain

$$f_2(u,v) = \frac{\lambda e^{-\lambda(v+u)}}{2} I_0\left(2\lambda\sqrt{uv}\right), \tag{2.83}$$

where $I_0(\cdot)$ denotes the first order modified Bessel function of the first kind. By transforming the coordinate in Eq.(2.83), we can obtain the following solution.

$$f_2(x,t) = \frac{\lambda e^{-\lambda t}}{2} I_0\left(\lambda\sqrt{t^2-x^2}\right) \tag{2.84}$$

Finally, we have the solution of Eqs.(2.50), and the solution is as follows.

$$\begin{aligned}\Psi^0_{x_t}(x,t) &= \frac{\lambda e^{-\lambda t}}{2}\sqrt{\frac{t+x}{t-x}} I_1\left(\lambda\sqrt{t^2-x^2}\right)\cdot(\mathbf{1}(x+t)-\mathbf{1}(x-t))+ \\ &+e^{-\lambda t}\delta(x-t)\end{aligned} \tag{2.85}$$

$$\Psi^1_{x_t}(x,t) = \frac{\lambda e^{-\lambda t}}{2} I_0\left(\lambda\sqrt{t^2-x^2}\right)\cdot(\mathbf{1}(x+t)-\mathbf{1}(x-t)) \tag{2.86}$$

2.4.2 Comparison with Simulation Results

Here some results from simulation will be presented to be compared with the theoretical results (2.85) and (2.86). We run four simulations in order to get estimations of $\Psi^0_{x_t}(x,t)$ and $\Psi^1_{x_t}(x,t)$ at time $t = 1$, $2, 5$ and 10 seconds respectively. Each simulation generates 10000 samples of (x_t, θ_t). Eight histograms are made from the data as the results of the simulations.

In the simulation, we divide time into intervals of length Δ. We compute the probability of changing mode during the time interval as

$$P(\theta_{(k+1)\Delta} \neq \theta_{k\Delta}) = 1 - e^{-\lambda\Delta}. \tag{2.87}$$

We then use the following approximation

$$\begin{aligned} x_{(k+1)\Delta} &\simeq x_{k\Delta} + \Delta\cdot\alpha(\theta_{k\Delta}) \\ \theta_{(k+1)\Delta} &\simeq \theta_{k\Delta} + N_k\cdot c(\theta_{k\Delta}) \end{aligned} \tag{2.88}$$

where $c(\cdot)$ is defined as in Eq.(2.48) and N_k is a sequence of independent identically distributed random variables, such that for any k, the following relation holds.

$$\left.\begin{array}{r} P(N_k = 1) = 1 - e^{-\lambda\Delta} \\ P(N_k = 0) = e^{-\lambda\Delta} \end{array}\right\}. \tag{2.89}$$

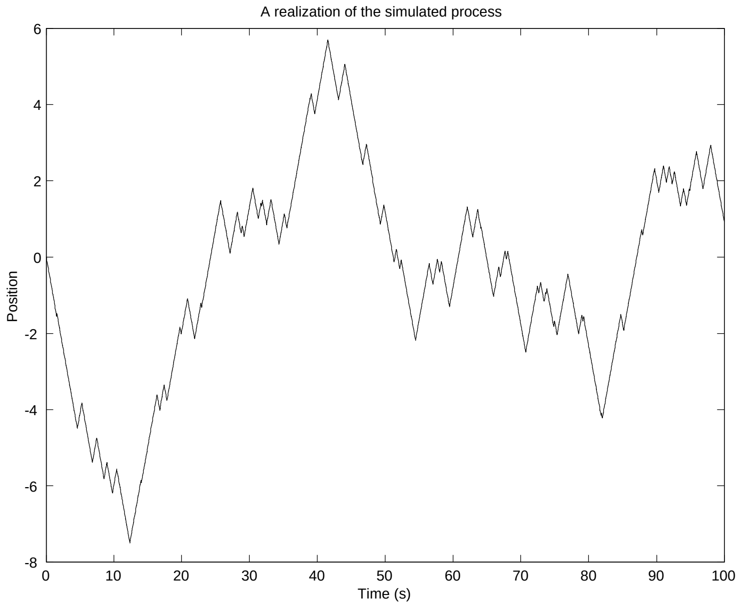

Figure 2.1: A realization of the simulated process

In Figure 2.1 we can see a realization of the simulated process.

In Figure 2.2, a pair of histograms as approximation of $\Psi^0_{x_t}(x,1)$ and $\Psi^1_{x_t}(x,1)$ is shown. There are 20000 data in total used in both histograms. Curves indicating the scaled theoretical density function are also shown. The theoretical density functions are scaled to match the sample size of the simulation.

In Figure 2.3, a pair of histograms as approximation of $\Psi^0_{x_t}(x,2)$ and $\Psi^1_{x_t}(x,2)$ is shown. There are 20000 data in total used in both histograms.

In Figure 2.4, a pair of histograms as approximation of $\Psi^0_{x_t}(x,5)$ and $\Psi^1_{x_t}(x,5)$ is shown. The number of data used is reduced to 10000 in order to reduce the computation time.

In Figure 2.5, a pair of histograms as approximation of $\Psi^0_{x_t}(x,10)$ and $\Psi^1_{x_t}(x,10)$ is shown. The number of data used is also 10000 in total for both histograms.

We can observe that after 1 second, the density $\Psi^0_{x_t}(x,1)$ is still dominated by the $\delta-$component, while $\Psi^1_{x_t}(x,1)$ does not exhibit any specific pattern. However, as time passes, both densities turn to a shape resembling the normal curve. It can also be observed that the agreement between the simulation results and the theoretical ones is quite good, in particular when more samples are used.

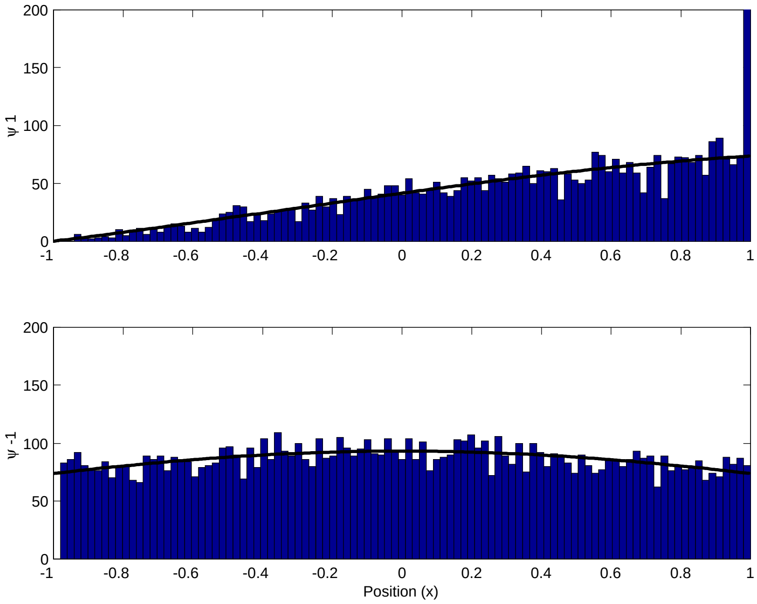

Figure 2.2: Histograms of the state at $t = 1$. Above : $\theta_1 = 0$. Below : $\theta_1 = 1$. The curves represent the scaled theoretical density functions.

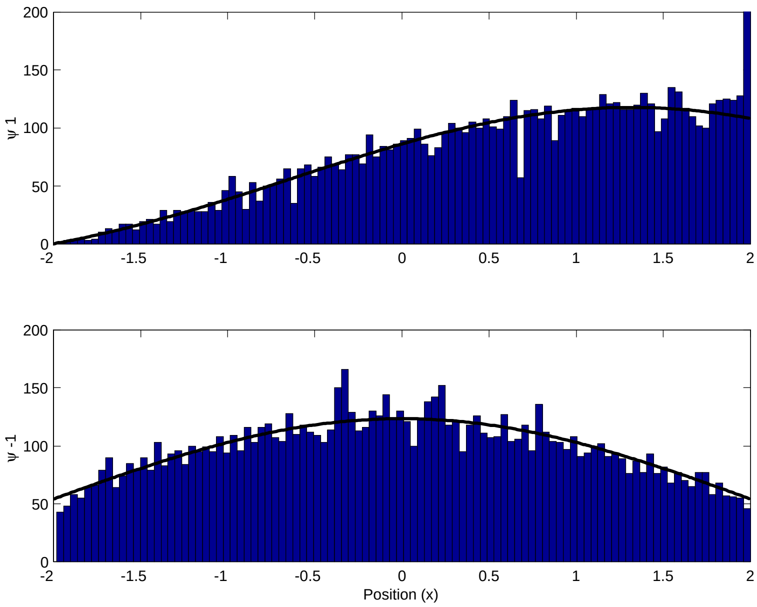

Figure 2.3: Histograms of the state at $t = 2$. Above : $\theta_2 = 0$. Below : $\theta_2 = 1$. The curves represent the scaled theoretical density functions.

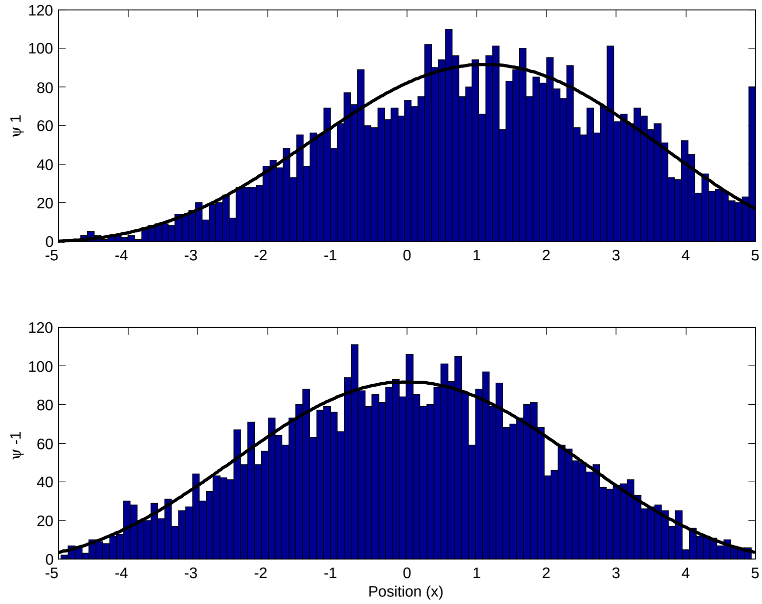

Figure 2.4: Histograms of the state at $t = 5$. Above : $\theta_5 = 0$. Below : $\theta_5 = 1$. The curves represent the scaled theoretical density functions.

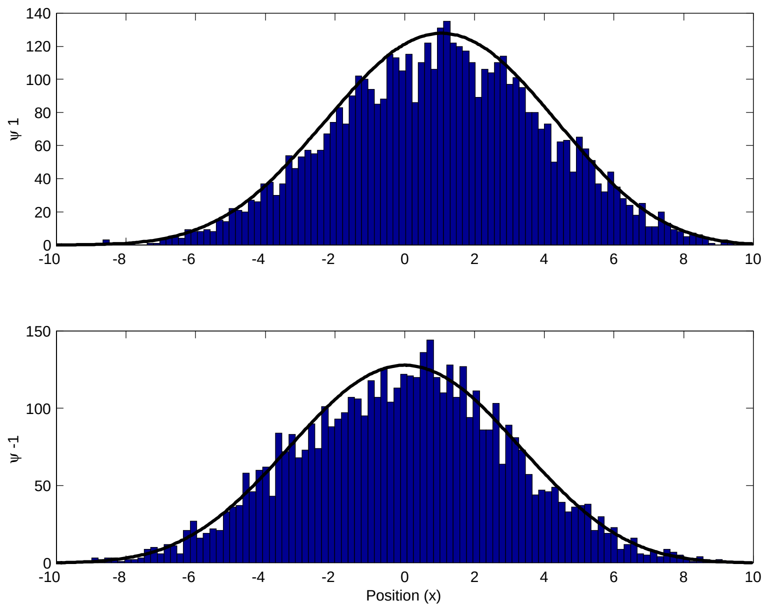

Figure 2.5: Histograms of the state at $t = 10$. Above : $\theta_{10} = 0$. Below : $\theta_{10} = 1$. The curves represent the scaled theoretical density functions.

Chapter 3

A Model with Diffusion and Boundary Hitting Process

3.1 Model Formulation

In this chapter we consider a model of linear systems with mode changes triggered by a BHP. The model we use is of the form described below.

Consider a stochastic hybrid system characterized by the following (Ito) SDE.

$$\begin{aligned} x_t &\in \mathsf{R} \\ \theta_t &\in M = \{0, 1\} \\ \xi_t &\triangleq (x_t, \theta_t)^{\mathrm{T}} \\ t &\in \mathsf{R}^+, \end{aligned}$$

$$\begin{aligned} dx_t &= [a(\theta_t)x_t + \alpha(\theta_t)]\, dt + dW_t + b^0(\theta_{t^-})dk_t^0 + b^1(\theta_{t^-})dk_t^1 \\ d\theta_t &= c^0(\theta_{t^-})dk_t^0 + c^1(\theta_{t^-})dk_t^1 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} k_t^0 &= L_t(\mathsf{R}^+) \\ k_t^1 &= L_t(\mathsf{R}^-) \end{aligned} \tag{3.2}$$

and

$$\theta_{t^-} = \lim_{\tau \uparrow t} \theta_\tau$$

We assume that there exist a probability space $(\Omega, P, \mathcal{F})$ and a filtration $\mathcal{F}_t$ such that the process ξ_t is adapted to it.

As usual, W_t denotes a standard Brownian motion. Moreover, we also define

$$\begin{aligned} L_t(A) &: \Omega \to \{0, 1\} \\ L_t &= \begin{cases} 1, \text{ if } \lim_{s \uparrow t} x_s \in A \\ 0, \text{ otherwise} \end{cases} \end{aligned} \tag{3.3}$$

$$a(\theta) \leq 0, \forall \theta \in M \tag{3.4}$$

$$b^0(\theta_{t^-}) = \begin{cases} 0, & \theta_{t^-} = 0 \\ 1, & \theta_{t^-} = 1 \end{cases} \tag{3.5}$$

$$b^1(\theta_{t^-}) = \begin{cases} -1, & \theta_{t^-} = 0 \\ 0, & \theta_{t^-} = 1 \end{cases} \tag{3.6}$$

$$c^0(\theta_{t^-}) = \begin{cases} 0, & \theta_{t^-} = 0 \\ -1, & \theta_{t^-} = 1 \end{cases} \tag{3.7}$$

$$c^1(\theta_{t^-}) = \begin{cases} 1, & \theta_{t^-} = 0 \\ 0, & \theta_{t^-} = 1 \end{cases} \tag{3.8}$$

Throughout this text we will refer to x_t, θ_t, and ξ_t as the continuous state, discrete state (mode), and hybrid state resp. For the initial condition ξ_t, we assume that the following relation holds.

$$(x_0, \theta_0) \in ((\mathsf{R}^+ - \{0\}) \times \{0\}) \cup ((\mathsf{R}^- - \{0\}) \times \{1\}) \tag{3.9}$$

Definition 3.1 *As for the previous model, we define the modal density $\Psi^\theta_{x_t}(x,t)$ such that for every set Γ in the Borel field of R we have the following relation.*

$$P(x_t \in \Gamma, \theta_t = \theta) = \int_\Gamma \Psi^\theta_{x_t}(x,t) \cdot dx \tag{3.10}$$

We are particularly interested in describing and solving the evolution of $\Psi^0_{x_t}(x,t)$ and $\Psi^1_{x_t}(x,t)$.

Example 3.2 *As an example of a process described by (3.1) - (3.9), we define the drift term as in (3.11) below.*

$$\begin{array}{lcl} [a(0)x_t + \alpha(0)] & , & -1 \\ [a(1)x_t + \alpha(1)] & , & -x_t - 1. \end{array} \tag{3.11}$$

Later in this report, this process will be used as an example for the discussion. A realization of this process is plotted in Figure 3.1. The process depicted here starts at $(x_0, \theta_0) = (1, 0)$. When the process is about to hit the boundary (around $t = 0.3$ s), it jumps such that it starts at $x = -1$ and the mode switches to 1. Again, when the boundary is about to be hit (around $t = 1.5$ s) the process jumps back to the mode 0 and $x = 1$. In between the jumps, the process follows the diffusion equation described for the currently valid mode.

3.2 Diffusion with an Absorbing Boundary

Before we proceed with solving (3.1), we will devote this section for discussion of diffusion with an absorbing boundary. Later it will become clear that the discussion in this section is closely related to the solution of (3.1).

Let x_t be the solution of

$$dx_t = a(x,t)dt + b(x,t)dW_t,$$

in an open proper domain $G \subset \mathsf{R}$ with a smooth boundary ∂G. We assume that $a(\cdot)$ and $b(\cdot)$ satisfy all conditions that guarantee that there exists a unique solution to the

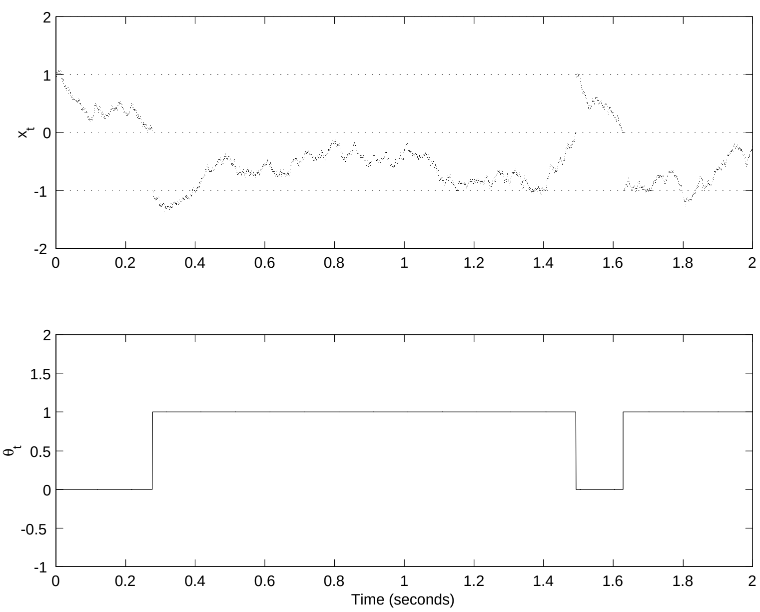

Figure 3.1: A realization of a process described by Example 3.2. Top : The continuous state. Bottom : The mode

SDE (refer to e.g. [GS72] and [WH85]). Considering (3.1), for example, we can take $G = \{x \in \mathsf{R} \mid x > 0)$ or $G = \{x \in \mathsf{R} \mid x < 0)$ and $\partial G = \{0\}$. We can describe a diffusion process with absorbing boundary as a phenomenon where upon arrival at a boundary point (state) the process is to remain in that state for all times. In order to give a mathematical representation of such process, let us first consider the following definitions.

Definition 3.3 *The first passage time τ_x is defined as*

$$\tau_x = \inf\{t \geq 0 : x_t = 0\}$$

Using the definition of τ_x, the absorbed process can be defined as follows.

Definition 3.4 *We define the absorbed process $\tilde{x}_t$ as the stochastic process satisfying*

$$\tilde{x}_t = \begin{cases} x_t, & \textit{if } t < \tau_x \\ 0, & \textit{if } t \geq \tau_x \end{cases} \tag{3.12}$$

with initial condition $\tilde{x}_0 = x_0$.

An example of the realization of an absorbed process is shown in Figure 3.2.

Consider the following definition of a *stopping time.*

Definition 3.5 *A stopping time τ for a process x_t is a random variable mapping $\Omega \rightarrow (\mathsf{R}^+ \cup \infty)$ such that the event $(\tau \leq t)$ is measurable under $\mathcal{F}_t$.*

We can notice that τ_x is a stopping time since the event $(\tau_x \leq t) = (\exists s \leq t : x_s = 0) \in \mathcal{F}_t$. The expectation and density of the stopping time τ_x will be given in the following two theorems.

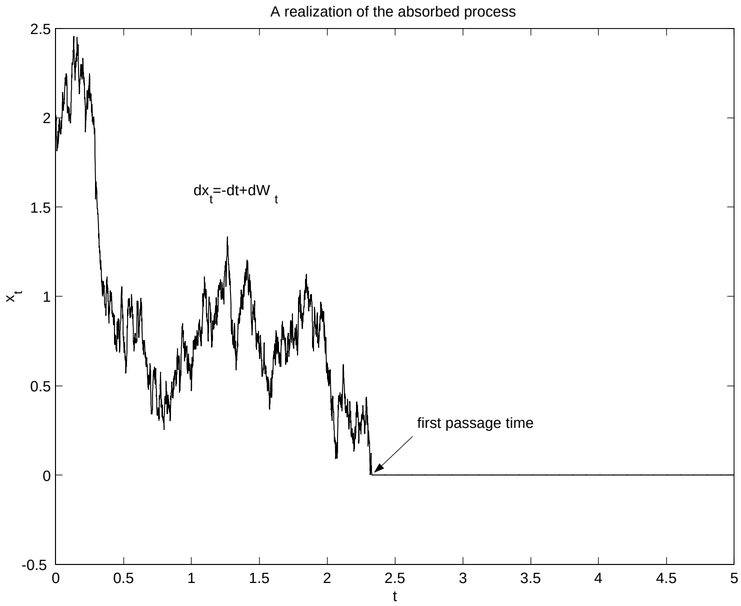

Figure 3.2: A realization of an absorbed process

Theorem 3.6 *Let x_t be the solution of*

$$dx_t = a(x,t)dt + b(x,t)dW_t,$$

in an open proper domain $G \subset \mathrm{R}$ with a smooth boundary ∂G. Define

$$\tau_x(y,s) \triangleq \inf(t \geq s : x(t) \in \partial G, x(s) = y).$$

Then the following relation holds true

$$E\tau_x(y,s) = s + u(y,t)|_{t=s}, \tag{3.13}$$

where $u(y,t)$ is the solution of the following PDE.

$$\frac{\partial u}{\partial t} + a(y,t)\frac{\partial u}{\partial y} + \frac{1}{2}b^2(y,t)\frac{\partial^2 u}{\partial y^2} = -1,\ t \geq s, y \in G \tag{3.14}$$

$$u(y,t) = 0,\ y \in \partial G. \tag{3.15}$$

Proof. The derivation is given in [Sch80] for the multi dimensional case. It will not be rewritten here. ∎

Theorem 3.7 *Let $\tau_x(y,s)$ be defined as in Theorem 3.6, the following relation is also valid.*

$$P(\tau_x(y,s) < T) = q(y,t;T)|_{t=s},$$

where $q(y,t,T)$ is the solution of

$$\frac{\partial q}{\partial t} + a(y,t)\frac{\partial q}{\partial y} + \frac{1}{2}b^2(y,t)\frac{\partial^2 q}{\partial y^2} = 0, T > t \geq s, y \in G \tag{3.16}$$

$$q(y,t;T) = 1,\ y \in \partial G, T \geq t \geq s \tag{3.17}$$

$$q(y,T;T) = 0,\ y \in G \tag{3.18}$$

Proof. The derivation of this theorem is also given in [Sch80]. ■

We will now implement Theorem 3.7 to obtain the density of the first passage time for each mode of the process in Example 3.2. We assume that $x = 0$ is an absorbing boundary. Implementing the theorem to the process described in Example 3.2, we obtain the following relations.

- If $x_0 = y > 0$,

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial y} + \frac{1}{2}\frac{\partial^2 q}{\partial y^2} = 0, T > t \geq 0, y > 0 \tag{3.19}$$

$$q(y,t;T) = 1,\ y = 0, T \geq t \geq 0 \tag{3.20}$$

$$q(y,T;T) = 0,\ y > 0 \tag{3.21}$$

- If $x_0 = y < 0$

$$\frac{\partial q}{\partial t} - (1+y)\frac{\partial q}{\partial y} + \frac{1}{2}\frac{\partial^2 q}{\partial y^2} = 0, T > t \geq 0, y < 0 \tag{3.22}$$

$$q(y,t;T) = 1,\ y = 0, T \geq t \geq 0 \tag{3.23}$$

$$q(y,T;T) = 0,\ y < 0 \tag{3.24}$$

Let us solve the PDE described by (3.19)-(3.21). First, we do the following substitution of independent variable.

$$\tilde{t} \ , \ T - t \tag{3.25}$$

The problem is then transformed into

$$-\frac{\partial q}{\partial \tilde{t}} - \frac{\partial q}{\partial y} + \frac{1}{2}\frac{\partial^2 q}{\partial y^2} = 0, T \geq \tilde{t} > 0, y > 0 \tag{3.26}$$

$$q(y,\tilde{t}) = 1,\ y = 0, T \geq \tilde{t} \geq 0 \tag{3.27}$$

$$q(y,\tilde{t}) = 0,\ y > 0, \tilde{t} = 0 \tag{3.28}$$

Let us define the Laplace transformation of $q(y,\tilde{t};T)$ in the $\tilde{t}$ direction,

$$Q(y;s) \ , \ \int_0^\infty q(y,\tilde{t}) \cdot e^{-s\tilde{t}} d\tilde{t}, \tag{3.29}$$

then using the initial condition (3.28), (3.26) and (3.27) are transformed into the following ODE.

$$-sQ - \frac{dQ}{dy} + \frac{1}{2}\frac{d^2Q}{dy^2} = 0 \tag{3.30}$$

$$Q(0;s) = \frac{1}{s} \tag{3.31}$$

Another extra condition can be added, i.e.

$$Q(\infty;s) = 0 \tag{3.32}$$

The interpretation of (3.32) is clear i.e. the probability of hitting the boundary within the time frame is zero if we start from a point with infinite distance from the boundary. The solution of (3.30) is in the form of

$$Q(y; s) = C_1(s)e^{(1+\sqrt{1+2s})y} + C_2(s)e^{(1-\sqrt{1+2s})y}. \tag{3.33}$$

Imposing the conditions (3.31) and (3.32), we yield

$$Q(y; s) = \frac{e^{(1-\sqrt{1+2s})y}}{s}. \tag{3.34}$$

After doing inverse transform on (3.34), we will obtain

$$q(y, \tilde{t}) = \int_0^{\tilde{t}} \frac{y}{t\sqrt{2\pi t}} \exp\left(-\frac{(y-t)^2}{2t}\right) dt. \tag{3.35}$$

Now, regarding Theorem 3.7 and (3.25), we can interpret $q(y, \tilde{t})$ as the probability that the process hits the boundary $\tilde{t}$ time unit after it starts at $x_{t_0} = y$ for any $t_0 \geq 0$. Without losing generality, we can set $t_0 = 0$. Then the following relation holds true.

$$P(\tau_x(y, 0) < \tilde{t}) = q(y, \tilde{t}) \tag{3.36}$$

The density function $p_{\tau_x(y,0)}(t)$ can then be given as

$$p_{\tau_x(y,0)}(t) = \frac{y}{t\sqrt{2\pi t}} \exp\left(-\frac{(y-t)^2}{2t}\right) \tag{3.37}$$

Solving the PDE system (3.22) - (3.24) is more complicated. The results provided below are obtained from modifying the results presented in [Lef96]. The moment generating function of the difference between the first passage time $\tau_x(y, s)$ and the starting time s is given by the following relation

$$\phi(\sigma, y) \;,\; E(e^{-\sigma(\tau_x(y,s)-s)})$$

This expression is independent of s. For $y < 0$, the solution $\phi(\sigma, y)$ is given by

$$\phi(\sigma, y) = e^{\frac{(y+1)^2-1}{2}} \cdot \frac{D_{-\sigma}\left(-\sqrt{2}(y+1)\right)}{D_{-\sigma}\left(-\sqrt{2}\right)} \tag{3.38}$$

where $D_{-\sigma}(\cdot)$ is a parabolic cylinder function called the Weber function. See [MI72] for more information on this function.

3.3 Solution of Equation (3.1)

We will propose a way to construct the solution ξ_t. For $i = 1, 2, 3, \cdots$, let $\check{\xi}_t^i \;,\; (\check{x}_t^i, \check{\theta}_t^i)^{\mathrm{T}}$ be the solution of

$$\begin{aligned} d\check{x}_t^i &= \left[a(\check{\theta}_t^i)x_t + \alpha(\check{\theta}_t^i)\right] dt + dW_t \\ d\check{\theta}_t^i &= 0 \end{aligned} \tag{3.39}$$

with initial condition at time $t = \tau_{i-1}$ as follows $\check{\xi}^i_{\tau_{i-1}} \triangleq (\check{x}^i_0, \check{\theta}^i_0)^{\mathrm{T}}$. We define

$$\tau_i \triangleq \inf(t > \tau_{i-1} : \check{x}^i_t = 0) \tag{3.40}$$

with $\tau_0 = 0$. Notice that τ_i is a stopping time.

Let $T \in \mathsf{R} > 0$. Define a stochastic process $\check{\xi}_t$ whose sample paths in $[0, T]$ are constructed as follows.

1. Let $\check{\xi}^1_0 = \xi_0$.
2. Obtain a sample path of $\check{\xi}^1_t$ until the stopping time τ_1.
3. For $i = 1, 2, 3, \cdots$, let the initial condition $\check{\xi}^{i+1}_{\tau_i}$ be determined from $\check{\xi}^i_{\tau_i}$. That is, $\check{\xi}^{i+1}_{\tau_i} = (-1, 1)^{\mathrm{T}}$ if $\check{\theta}^i_{\tau_i} = 0$, and $\check{\xi}^{i+1}_{\tau_i} = (1, 0)^{\mathrm{T}}$ if $\check{\theta}^i_{\tau_i} = 1$.
4. Start the process $\check{\xi}^{i+1}_t$ from the stopping time τ_i until the stopping time τ_{i+1}.
5. Repeat step 3 and 4 until a sample path of $\check{\xi}^N_t$ is obtained where $\tau_N > T$.
6. A sample path of $\check{\xi}_t$ is obtained by concatenating those of $\check{\xi}^i_t$, i.e.

$$\check{\xi}_t = \check{\xi}^i_t, \text{ for } t \in [\tau_{i-1}, \tau_i) \tag{3.41}$$

Definition 3.8 *We define the set* $J_T \triangleq \{\tau_1, \tau_2, \tau_3, \cdots\}$ *with* $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 \cdots \leq T$.

Remark 3.9 *Let* $t \in [0, T]$, *and* ξ_t *be the solution of (3.1) with initial condition* ξ_0. *Let* $J_T = \{\tau_1, \tau_2, \tau_3, \cdots\}$ *with* $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 \cdots \leq T$. *The state at the jump times* ξ_{τ_i}, $i \in \mathsf{N}$ *are actually* $\mathcal{F}_0 - measurable$.

The following theorem provides us with the way to characterize a solution of (3.1).

Theorem 3.10 *Let* $t \in [0, T]$, *and* $\check{\xi}_t$ *be the stochastic process as described above. The process* $\check{\xi}_t$ *is a solution of (3.1) with initial condition* ξ_0.

Proof. We will show that $\check{\xi}_s$, $s \in [0, T]$ satisfies

$$\check{x}_s = x_0 + \int_0^s \left[a(\check{\theta}_t)\check{x}_t + \alpha(\check{\theta}_t)\right] dt + \int_0^s dW_t + \int_0^s b^0(\check{\theta}_{t-})dk^0_t + \\ + \int_0^s b^1(\check{\theta}^-_t)dk^1_t \tag{3.42}$$

$$\check{\theta}_t = \theta_0 + \int_0^s c^0(\check{\theta}_{t-})dk^0_t + \int_0^s c^1(\check{\theta}_{t-})dk^1_t \tag{3.43}$$

By the nature of the construction of $\check{\xi}_t$ we will evaluate the integrals in (3.42) and (3.43) segment by segment. Equivalently, for $s \in [\tau_i, \tau_{i+1}], \forall i \in \{i \mid i \geq 0, \tau_{i+1} \in J_T\}$ the relations (3.42) and (3.43) can also be written as

$$\check{x}_s = \check{x}_{\tau_i} + \int_{\tau_i}^s \left[a(\check{\theta}_t)\check{x}_t + \alpha(\check{\theta}_t)\right] dt + \int_{\tau_i}^s dW_t + \int_{\tau_i}^s b^0(\check{\theta}_{t-})dk^0_t + \\ + \int_{\tau_i}^s b^1(\check{\theta}_{t-})dk^1_t \tag{3.44}$$

$$\check{\theta}_t = \check{\theta}_{\tau_i} + \int_{\tau_i}^{s} c^0(\check{\theta}_{t-})dk_t^0 + \int_{\tau_i}^{s} c^1(\check{\theta}_{t-})dk_t^1 \tag{3.45}$$

It is trivial to verify that the initial condition is satisfied. Moreover, if $s \in [\tau_i, \tau_{i+1})$ the last two terms of (3.44) and (3.45) evaluate to zero. Hence for $s \in [\tau_i, \tau_{i+1})$ the relations (3.44) and (3.45) are satisfied by the construction of $\check{\xi}_t^{i+1}$. If $s = \tau_{i+1}$, it can also be verified that the jump (i.e. $\check{\xi}_{\tau_{i+1}^-}$ to $\check{\xi}_{\tau_{i+1}}$) is consistent with (3.44) and (3.45), as the integral terms involving dk_t^0 and dk_t^1 will cover the gap caused by the jump.

The verification will be done as follows :

- For the jumps from $\check{\xi}_{\tau_{i+1}^-} = (0, 0)$ to $\check{\xi}_{\tau_{i+1}} = (-1, 1)$

$$\begin{aligned} \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^0(\check{\theta}_{t-})dk_t^0 + \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^1(\check{\theta}_{t-})dk_t^1 &= 0 - 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^0(\check{\theta}_{t-})dk_t^0 + \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^1(\check{\theta}_{t-})dk_t^1 &= 0 + 1 \\ &= 1 \end{aligned}$$

- For the jumps from $\check{\xi}_{\tau_{i+1}^-} = (0, 1)$ to $\check{\xi}_{\tau_{i+1}} = (1, 0)$

$$\begin{aligned} \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^0(\check{\theta}_{t-})dk_t^0 + \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^1(\check{\theta}_{t-})dk_t^1 &= 1 + 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^0(\check{\theta}_{t-})dk_t^0 + \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^1(\check{\theta}_{t-})dk_t^1 &= -1 + 0 \\ &= -1 \end{aligned}$$

The reader is suggested to refer to Figure 3.1.

■

Proposition 3.11 *Almost every sample path of $\check{\xi}_t^i$ is uniformly continuous in $[0, T]$.*

Proof. The proof is based on [GS72][1]. It is sufficient if we can show that there exists a constant $K > 0$ such that

$$\left[a(\check{\theta}_t^i)x + \alpha(\check{\theta}_t^i)\right]^2 + 1 \leq K^2(1 + x^2) \tag{3.46}$$

$$\left|a(\check{\theta}_t^i)\right| \leq K \tag{3.47}$$

For the ease of writing the proof, $a(\check{\theta}_t^i)$ and $\alpha(\check{\theta}_t^i)$ will be simply written as a and α respectively.

[1] see page 40 of the reference.

Manipulating (3.46), we obtain

$$(K^2 - a^2)x^2 - 2a\alpha x + K^2 - 1 - \alpha^2 \geq 0 \tag{3.48}$$

We then have to find K such that the discriminant of (3.48) is less than zero. In addition, (3.47) is also has to be satisfied. We then get the following relation

$$\begin{aligned} 4a^2\alpha^2 - 4(K^2 - a^2)(K^2 - 1 - \alpha^2) &< 0 \\ K^4 - (a^2 + \alpha^2 + 1)K^2 + a^2 &> 0 \end{aligned} \tag{3.49}$$

Notice that the discriminant of (3.49) is

$$\begin{aligned} (a^2 + \alpha^2 + 1)^2 - 4a^2 &= (a^2 + \alpha^2 + 1 - 2a)(a^2 + \alpha^2 + 1 + 2a) \\ &= ((a-1)^2 + \alpha^2) \cdot ((a+1)^2 + \alpha^2) \\ &\geq 0 \end{aligned}$$

which guarantees that (3.49) has real solutions. Let q be the largest root of (3.49) then choosing

$$K > \max(|a|\,, \sqrt{\max(0, q)})$$

will satisfy (3.46) and (3.47), thus completing the proof

Theorem 3.12 *Equation (3.1) admits a unique right continuous solution for every initial condition satisfying (3.9).*

■

Proof. It has been shown that the solution can be constructed as piecewise diffusions. By construction of the solution and the path continuity of $\breve{\xi}^i_t$, right continuity is established (see Theorem 3.10 and Proposition 3.11). It can be proven that the solutions for the diffusions are unique for each eligible initial condition. In [Oks98] and [GS72] it is stated that if the conditions in the proof of Proposition 3.11 are satisfied then the diffusion admits unique solution.

To show the uniqueness of the whole solution we need to establish the uniqueness of the jumps. But this is more or less trivial since the jumps are constructed such that the states where the process jumps from and where it jumps into are unique for each piece of diffusion. Hence the uniqueness of the solution is established. ■

3.4 Evolution of Modal Density

Let us introduce the modal transition density.

Definition 3.13 *The modal transition density, denoted as $\Psi^{\theta}_{x_t|\xi_s}(x,t)$, with $s \leq t$, is the density function such that for any set Γ in the Borel field of R we have the following relation*

$$P(x_t \in \Gamma, \theta_t = \theta \mid \xi_s) = \int_{\Gamma} \Psi^{\theta}_{x_t|\xi_s}(x,t) \cdot dx$$

We will use the model of diffusion with absorbing boundary to assist us in formulating the evolution of the transition density. First, consider the shifted process $(\xi^+_s)_t$.

Notation 3.14 *The shifted process $(\xi^+_s)_t$, ξ_{s+t}. This notation follows the one in [BW90]. The continuous and discrete part of the shifted process are then denoted as $(x^+_s)_t$ and $(\theta^+_s)_t$.*

Next, consider the following relations

$$P(x_t \in \Gamma, \theta_t = \theta \mid \xi_s) = P((x_s^+)_{t-s} \in \Gamma, (\theta_s^+)_{t-s} = \theta \mid (\xi_s^+)_0) \tag{3.50}$$

$$\begin{aligned} P((x_s^+)_{t-s} \in \Gamma, (\theta_s^+)_{t-s} = \theta \mid (\xi_s^+)_0) &= P((x_s^+)_{t-s} \in \Gamma, (\theta_s^+)_{t-s} = \theta, \tau_{(x_s^+)} < t-s \mid (\xi_s^+)_0) \\ &+ P((x_s^+)_{t-s} \in \Gamma, (\theta_s^+)_{t-s} = \theta, \tau_{(x_s^+)} \geq t-s \mid (\xi_s^+)_0) \end{aligned} \tag{3.51}$$

Notice that the process definition (3.1) and the limitations on the initial density (3.9) imply that

$$\begin{aligned} P(x_t &\leq 0, \theta_t = 0) = 0 \\ P(x_t &\geq 0, \theta_t = 1) = 0. \end{aligned} \tag{3.52}$$

It is then natural to assume that Γ is strictly contained in either half of R. In the following discussion we will evaluate the first term of the right hand side of (3.51).

Proposition 3.15 *Given the process described by (3.1), the probability of* re-entrance *within a small time interval Δ is $o(\Delta)$, i.e. for all $t \geq 0$, the following relation holds.*

$$P(\exists s : t < s < t + \Delta,\ \theta_t = \theta_{t+\Delta} \neq \theta_s | \xi_t) = o(\Delta)$$

Proof. The proof is based on the fact that for the re-entrance to occur, the process have to hit the boundary twice. Once the process hit the boundary it will jump to a state away from the boundary so it will have to traverse along a certain distance before hitting the boundary again.

Suppose that

$$\left.\begin{array}{c} x_t = y \neq 0 \\ a(\theta_t) = a \leq 0 \\ \alpha(\theta_t) = \alpha \end{array}\right\} \tag{3.53}$$

It has been proved (Theorem 3.10) that the solution of (3.1) can be decomposed such that there is a $\delta > 0$, such that if $h < \delta$ the following relation is true.

$$\begin{aligned} x_{t+h} = x_t + \int_t^{t+h} [a x_\tau + \alpha]\, d\tau + W_{t+h} - W_t \\ \theta_{t+h} = \theta_t \end{aligned} \tag{3.54}$$

Consider the process given by (3.54) for all $h > 0$. Now, it can be derived from (3.54) that

$$\begin{aligned} x_{t+h} - x_t = \left(y + \frac{\alpha}{a}\right)(e^{ah} - 1) + e^{a(t+h)} \int_t^{t+h} e^{-a\tau} dW_\tau,\ a \neq 0 \\ x_{t+h} - x_t = h\alpha + \int_t^{t+h} dW_\tau,\ a = 0 \end{aligned} \tag{3.55}$$

We can then have the following relations

$$E(x_{t+h} - x_t | \xi_t) = \begin{cases} \left(y + \frac{\alpha}{a}\right)(e^{ah} - 1),\ a \neq 0 \\ h\alpha,\ a = 0 \end{cases} \tag{3.56}$$

since $a(\theta_t) \leq 0$, we have that

$$\left|e^{ah} - 1\right| < |ah|\,. \tag{3.57}$$

We can infer that

$$\begin{aligned} |E(x_{t+h} - x_t|\xi_t)| &\leq (|ay| + |\alpha|)\, h \\ &= (-a\,|y| + |\alpha|)\, h \end{aligned} \tag{3.58}$$

Equation (3.58) implies that for any y we can choose a limit $l(y)$ such that

$$h \leq l(y) \Longrightarrow |E(x_{t+h} - x_t|\xi_t)| \leq \frac{y}{2} \tag{3.59}$$

From (3.55) and (3.56) we can obtain

$$\Delta_h \;, \; x_{t+h} - x_t - E(x_{t+h} - x_t|\xi_t) = \begin{cases} \int_t^{t+h} e^{-a(\tau - t - h)} dW_\tau, \; a \neq 0 \\ \int_t^{t+h} dW_\tau, \; a = 0 \end{cases}$$

Now, Δ_h is a zero mean normal variable, and through the Ito isometry we know that

$$E(\Delta_h^2) \leq E(W_{t+h} - W_t)^2 = h, \forall h > 0 \tag{3.60}$$

Since the solution of (3.54) is continuous (and hence separable), by symmetry we have that

$$P\left(\sup_{0\leq s\leq h} \Delta_s > \frac{|y|}{2}\right) = P\left(\inf_{0\leq s\leq h} \Delta_s < \frac{-|y|}{2}\right) \leq P\left(\sup_{0\leq s\leq h} (W_{t+s} - W_t) > \frac{|y|}{2}\right) \tag{3.61}$$

It is known that (see e.g. [GS72]) the following relation holds true

$$\begin{aligned} P\left(\sup_{0\leq s\leq h} (W_{t+s} - W_t) > \frac{|y|}{2}\right) &= 2P\left((W_{t+h} - W_t) > \frac{|y|}{2}\right) \\ &\leq 2h\left(\frac{2}{|y|}\right)^2 = \frac{8h}{y^2} \end{aligned} \tag{3.62}$$

The inequality above is a Markov inequality.

We have the following relation

$$\begin{aligned} \sup_{0\leq\Delta\leq h} (|x_{t+\Delta} - x_t|) \leq & \sup_{0\leq\Delta\leq h} (|x_{t+\Delta} - x_t| - E(|x_{t+\Delta} - x_t|)) + \\ & + \sup_{0\leq\Delta\leq h} (E(|x_{t+\Delta} - x_t|)). \end{aligned} \tag{3.63}$$

Now, if we take $h \leq l(y)$ then (3.59), (3.61),(3.62) and (3.63) imply that

$$P(\sup_{0\leq\Delta\leq h} |x_{t+\Delta} - x_t| \geq y|\xi_t) \leq \frac{8h}{y^2}, \tag{3.64}$$

and hence

$$P(\text{the process hits the boundary in } [t, t+h]|\xi_t) \leq \frac{8h}{y^2} \tag{3.65}$$

In order to prove the proposition, it is sufficient if we can show that the following relations hold true.

$$\begin{aligned} &P(\text{the process hits the boundary in } [t, t+\Delta]|x_t = y > 0, \theta_t = 0) \times \\ &\times P(\text{the process hits the boundary in } [t, t+\Delta]|x_t = -1, \theta_t = 1) = o(\Delta) \end{aligned} \tag{3.66}$$

$$P(\text{the process hits the boundary in } [t, t+\Delta] | x_t = y < 0, \theta_t = 1) \times$$
$$\times P(\text{the process hits the boundary in } [t, t+\Delta] | x_t = 1, \theta_t = 0) = o(\Delta) \qquad (3.67)$$

Notice that (3.66) and (3.67) are upper bounds for the re-entrance probability for $x_t > 0$ and $x_t < 0$ respectively.

For simplicity, let us denote the left hand sides of (3.66) and (3.67) as $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then the following relations can be derived through (3.65).

$$\lim_{\Delta \downarrow 0} \frac{\mathcal{X}}{\Delta} \leq \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \frac{8\Delta}{y^2} \frac{8\Delta}{1} = 0 \qquad (3.68)$$

$$\lim_{\Delta \downarrow 0} \frac{\mathcal{Y}}{\Delta} \leq \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \frac{8\Delta}{y^2} \frac{8\Delta}{1} = 0 \qquad (3.69)$$

This completes the proof. ■

Proposition 3.15 indicates that if we want to analyze the variation of the modal density in a small time interval, we can assume that there is only one boundary hitting event in such an interval.

Notation 3.16 *Recall the definition of the stopping times τ_i in 3.3. We introduce the following notation,*

$$\tau_{s<} \triangleq \tau_\nu - s, \text{ where } \nu \triangleq \min_{i \geq 0}(i \mid \tau_i > s).$$

In other words, $\tau_{s<}$ denotes the time difference between s and the first jump time after s.

Proposition 3.17 *Consider the process described by (3.1). Assume that $s \geq 0$ and $\Delta > 0$. Given $\Psi^0_{x_t}(x, s)$ and $\Psi^1_{x_t}(x, s)$, under assumption that at most one jump occurs in $[s, s+\Delta]$, the following relations hold true.*

$$\Psi^0_{x_t}(x, s+\Delta) = \psi^{0,0}(x; s, \Delta) + \psi^{0,1}(x; s, \Delta) \qquad (3.70a)$$
$$\Psi^1_{x_t}(x, s+\Delta) = \psi^{1,1}(x; s, \Delta) + \psi^{1,0}(x; s, \Delta) \qquad (3.70b)$$

where $\psi^{0,0}(x; s, \Delta) \equiv p^0(x,t)\big|_{t=s+\Delta}$ is the solution of the PDE

$$\left.\begin{array}{l} \frac{\partial p^0}{\partial t} - \frac{1}{2}\frac{\partial^2 p^0}{\partial x^2} + \frac{\partial}{\partial x}\left([a(0)x + \alpha(0)]\, p^0\right) = 0 \\ p^0(x,s) = \Psi^0_{x_t}(x,s) \\ p^0(0,t) = 0 \\ x > 0, t > s \end{array}\right\}, \qquad (3.71)$$

and $\psi^{1,1}(x; s, \Delta) \equiv p^1(x,t)\big|_{t=s+\Delta}$ is the solution of the PDE

$$\left.\begin{array}{l} \frac{\partial p^1}{\partial t} - \frac{1}{2}\frac{\partial^2 p^1}{\partial x^2} + \frac{\partial}{\partial x}\left([a(1)x + \alpha(1)]\, p^1\right) = 0 \\ p^1(x,s) = \Psi^1_{x_t}(x,s) \\ p^1(0,t) = 0 \\ x < 0, t > s \end{array}\right\}. \qquad (3.72)$$

The other two terms are defined as

$$\psi^{0,1}(x; s, \Delta) = \int_0^\Delta q^0(x, \Delta - r) \cdot p_{\tau^1_{s<}}(r) dr \qquad (3.73a)$$

$$\psi^{1,0}(x; s, \Delta) = \int_0^\Delta q^1(x, \Delta - r) \cdot p_{\tau^0_{s<}}(r) dr \qquad (3.73b)$$

The densities $p_{\tau^0_{s<}}(\cdot)$ *and* $p_{\tau^1_{s<}}(\cdot)$ *are defined as follows. For any* $\tau > 0$,

$$P(\tau_{s<} \leq \tau, \theta_s = 0\ ) \triangleq \int_0^{\tau} p_{\tau^0_{s<}}(r)dr \tag{3.74a}$$

$$P(\tau_{s<} \leq \tau, \theta_s = 1\ ) \triangleq \int_0^{\tau} p_{\tau^1_{s<}}(r)dr \tag{3.74b}$$

The functions $q^0(x,t)$ *and* $q^1(x,t)$ *are defined as the solutions of the following PDEs*

$$\left.\begin{array}{l} \frac{\partial q^0}{\partial t} - \frac{1}{2}\frac{\partial^2 q^0}{\partial x^2} + \frac{\partial}{\partial x}\left(\left[a(0)x + \alpha(0)\right] q^0\right) = 0 \\ q^0(x,0) = \delta(x-1) \\ q^0(0,t) = 0 \\ x > 0, t > 0 \end{array}\right\} \tag{3.75}$$

$$\left.\begin{array}{l} \frac{\partial q^1}{\partial t} - \frac{1}{2}\frac{\partial^2 q^1}{\partial x^2} + \frac{\partial}{\partial x}\left(\left[a(1)x + \alpha(1)\right] p^1\right) = 0 \\ q^1(x,0) = \delta(x+1) \\ q^1(0,t) = 0 \\ x < 0, t > 0 \end{array}\right\}. \tag{3.76}$$

Proof. Due to symmetry, the proof will be given for half of the proposition i.e. the part related to (3.70a), (3.71), (3.73a), (3.96), and (3.75). The other half will follow the same line.

We start with the following relation for $\Gamma \subset \mathsf{R}^+$.

$$\begin{gathered} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0) = \\ = P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 0) + P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1). \end{gathered} \tag{3.77}$$

Let us define density functions for each term of the right hand side of (3.77), such that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 0) \equiv \int_{\Gamma} \psi^{0,0}(x; s, \Delta)dx \tag{3.78}$$

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1) \equiv \int_{\Gamma} \psi^{0,1}(x; s, \Delta)dx \tag{3.79}$$

Then (3.77) can be written as

$$\int_{\Gamma} \Psi^0_{x_t}(x, s+\Delta)dx = \int_{\Gamma} \psi^{0,0}(x; s, \Delta)dx + \int_{\Gamma} \psi^{0,1}(x; s, \Delta)dx. \tag{3.80}$$

Since (3.80) is valid for all $\Gamma \subset \mathsf{R}^+$, then we have (3.70a). Notice that since at most one jump can occur in $[s, s+\Delta]$, we have the following relation

$$\begin{gathered} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 0, x_s = y > 0) = \\ = P(x_{s+\Delta} \in \Gamma, \theta_{s+t} = 0, \forall t \in [0, \Delta] | \theta_s = 0, x_s = y > 0) \end{gathered} \tag{3.81}$$

To prove that $\psi^{0,0}(x; s, \Delta)$ satisfies (3.71), we introduce another density $\psi^0(x; x_s, s, \Delta)$ such that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+t} = 0, \forall t \in [0, \Delta] | \theta_s = 0, x_s = y > 0) \equiv \int_{\Gamma} \psi^0(x; y, s, \Delta)dx \tag{3.82}$$

In [BW90], it is given that $\psi^0(x; y, s, \Delta) \equiv \tilde{p}^0(x, t; y)\big|_{t=s+\Delta}$ is the solution of the PDE

$$\left.\begin{array}{l} \frac{\partial \tilde{p}^0}{\partial t} - \frac{1}{2}\frac{\partial^2 \tilde{p}^0}{\partial x^2} + \frac{\partial}{\partial x}\left([a(0)x + \alpha(0)]\, p^0\right) = 0 \\ \tilde{p}^0(x, s; x_s) = \delta(x - y) \\ \tilde{p}^0(0, t; y) = 0 \\ x > 0, t > s \end{array}\right\}, \tag{3.83}$$

Now, we also have that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 0) =$$
$$= \int_0^\infty \Psi^0_{x_t}(y, s) \cdot P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 0, x_s = y) dy. \tag{3.84}$$

Using (3.78) and (3.82) we can obtain

$$\int_\Gamma \psi^{0,0}(x; s, \Delta) dx = \int_0^\infty \Psi^0_{x_t}(y, s) \left(\int_\Gamma \psi^0(x; y, s, \Delta) dx \right) dy, \tag{3.85}$$

which is a Chapman-Kolmogorov equation. By exchanging the order of integration on the right hand side, we obtain

$$\int_\Gamma \psi^{0,0}(x; s, \Delta) dx = \int_\Gamma \int_0^\infty \Psi^0_{x_t}(y, s) \psi^0(x; y, s, \Delta) dy dx$$

hence

$$\psi^{0,0}(x; s, \Delta) = \int_0^\infty \Psi^0_{x_t}(y, s) \cdot \psi^0(x; y, s, \Delta) dy. \tag{3.86}$$

Defining that $\psi^{0,0}(x; s, \Delta) \equiv p^0(x, t)\big|_{t=s+\Delta}$, we get

$$p^0(x, t) = \int_0^\infty \Psi^0_{x_t}(y, s) \cdot \tilde{p}^0(x, t; y) dy, \ s + \Delta \geq t > s \tag{3.87}$$

Now, notice that $\tilde{p}^0(x, t; x_s)$ is the Green's function of the PDE (3.83), hence (3.87) implies that $p^0(x, t)$ is the solution of the same PDE with $\Psi^0_{x_t}(x, s)$ as a boundary condition. This proves (3.71).

To prove that $\psi^{0,1}(x; s, \Delta)$ is characterized by (3.73a), (3.74a), and (3.75) we use the following relation

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 1, x_s = y < 0) =$$
$$= \int_{r=0}^{r=\Delta} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \tau_{s<} \in [r, r + dr] | \theta_s = 1, x_s = y < 0). \tag{3.88}$$

Using the Chapman-Kolmogorov equation, we can get (see (3.84))

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \theta_s = 1) =$$
$$= \int_{-\infty}^0 \Psi^1_{x_t}(y, s) \cdot P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 1, x_s = y < 0) dy \tag{3.89}$$

We also have that

$$\begin{aligned} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0, \tau_{s<} \in [r, r + dr] | \theta_s = 1, x_s = y < 0) = \\ = P(\tau_{s<} \in [r, r + dr] | \theta_s = 1, x_s = y < 0) \times \\ \times P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 1, x_s = y < 0, \tau_{s<} \in [r, r + dr]) \end{aligned} \tag{3.90}$$

Notice that by the assumption that there can be at most one jump in $[s, s+\Delta]$ the following relation holds true

$$\begin{aligned} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = 0 | \theta_s = 1, x_s = y < 0, \tau_{s<} \in [r, r+dr]) = \\ = P(x_{s+\Delta} \in \Gamma, \theta_{s+t} = 0, \forall t \in [r, \Delta] | \theta_s = 1, x_s = y < 0, \tau_{s<} \in [r, r+dr]) \end{aligned} \tag{3.91}$$

We introduce the following compact notation

$$p_{\tau_{s<}|\xi_s}(y, r; 1) \ , \ p_{\tau_{s<}|x_s=y,\theta_s=1}(r)$$

It is clear by the definition of $p_{\tau_{s<}|\xi_s}(y, r; 1)$ that

$$P(\tau_{s<} \in [r, r+dr] | \theta_s = 1, x_s = y < 0) = p_{\tau_{s<}|\xi_s}(y, r; 1) dr \tag{3.92}$$

Moreover, if we define a function density $q^0(x, t)$ such that

$$\begin{aligned} P(x_{s+\Delta} \in \Gamma, \theta_{s+t} = 0, \forall t \in [r, \Delta] | \theta_s = 1, x_s = y < 0, \tau_{s<} \in [r, r+dr]) = \\ = \int_\Gamma q^0(x, \Delta - r) dx, \end{aligned} \tag{3.93}$$

it is known from [BW90] that $q^0(x, t)$ will satisfy (3.75). Now (3.89) can be written as

$$\int_\Gamma \psi^{0,1}(x; s, \Delta) dx = \int_{-\infty}^0 \Psi^1_{x_t}(y, s) \int_0^\Delta \left(\int_\Gamma q^0(x, \Delta - r) dx \right) \cdot p_{\tau_{s<}|\xi_s}(y, r; 1) dr dy. \tag{3.94}$$

Exchanging the order of integration, we can obtain

$$\begin{aligned} \int_\Gamma \psi^{0,1}(x; s, \Delta) dx &= \int_\Gamma \int_0^\Delta \int_{-\infty}^0 q^0(x, \Delta - r) \cdot \Psi^1_{x_t}(y, s) \cdot p_{\tau_{s<}|\xi_s}(y, r; 1) dy dr dx \\ &= \int_\Gamma \int_0^\Delta q^0(x, \Delta - r) \cdot p_{\tau^1_{s<}}(r) dr dx \end{aligned} \tag{3.95}$$

We use Lemma 3.18 to derive the second line of (3.95). Since (3.95) is valid for all $\Gamma \subset \mathsf{R}^+$, we obtain (3.73a). ■

Lemma 3.18 *Let $p_{\tau_{s<}|\xi_s}(x, t; \theta)$ be the distribution density of $\tau_{s<}$ given $(x_s = x, \theta_s = \theta)$, such that for any $\tau > 0$*

$$P(\tau_{s<} \leq \tau | x_s = x, \theta_s = \theta) \equiv \int_0^\tau p_{\tau_{s<}|\xi_s}(x, t; \theta) dt$$

then, by referring to (3.74a) and (3.74b), the following relations hold true,

$$p_{\tau^0_{s<}}(r) = \int_0^\infty \Psi^0_{x_t}(x, s) \cdot p_{\tau_{s<}|\xi_s}(x, r; 0) dx \tag{3.96}$$

$$p_{\tau^1_{s<}}(r) = \int_{-\infty}^0 \Psi^1_{x_t}(x, s) \cdot p_{\tau_{s<}|\xi_s}(x, r; 1) dx \tag{3.97}$$

Proof. By symmetry, only (3.96) will be proven. Referring to (3.74a), we have that for any $\tau > 0$,

$$P(\tau_{s<} \leq \tau, \theta_s = 0) \ , \ \int_0^\tau p_{\tau^0_{s<}}(r) dr.$$

But we also have that

$$P(\tau_{s<} \leq \tau, \theta_s = 0) = \int_{y \in \mathbb{R}^+} P(\tau_{s<} \leq \tau, x_s \in [y, y+dy], \theta_s = 0)$$
$$= \int_{y \in \mathbb{R}^+} P(\tau_{s<} \leq \tau | x_s \in [y, y+dy], \theta_s = 0) \times P(x_s \in [y, y+dy], \theta_s = 0)$$
$$= \int_{y \in \mathbb{R}^+} \int_0^\tau p_{\tau_{s<}|\xi_s}(y, r; 0) dr \cdot \Psi^0_{x_s}(y) dy \tag{3.99}$$

By exchanging the order of integration, we obtain

$$P(\tau_{s<} \leq \tau, \theta_s = 0) = \int_0^\tau \int_{y \in \mathbb{R}^+} p_{\tau_{s<}|\xi_s}(y, r; 0) \Psi^0_{x_s}(y) dy dr. \tag{3.100}$$

Finally, by comparing (3.100) and (3.74a) we obtain (3.96). ∎

Theorem 3.19 *The densities $\Psi^0_{x_t}(x,t)$ and $\Psi^1_{x_t}(x,t)$ satisfy the following PDEs*

$$\left.\begin{array}{l} \frac{\partial \Psi^0_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} - \frac{\partial}{\partial x}\left([a(0)x + \alpha(0)]\, \Psi^0_{x_t}\right) + p_{\tau^1_{t<}}(0) \cdot \delta(x-1) \\ \Psi^0_{x_t}(x,0) = \Psi^0_{x_0}(x) \\ \Psi^0_{x_t}(0,t) = 0 \\ x > 0, t \geq 0 \end{array}\right\} \tag{3.101}$$

$$\left.\begin{array}{l} \frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^1_{x_t}}{\partial x^2} - \frac{\partial}{\partial x}\left([a(1)x + \alpha(1)]\, \Psi^1_{x_t}\right) + p_{\tau^0_{t<}}(0) \cdot \delta(x+1) \\ \Psi^1_{x_t}(x,0) = \Psi^1_{x_0}(x) \\ \Psi^1_{x_t}(0,t) = 0 \\ x < 0, t \geq 0 \end{array}\right\} \tag{3.102}$$

where $\Psi^0_{x_0}(x)$ and $\Psi^1_{x_0}(x)$ are the densities of the initial condition.

Proof. Refer to Proposition 3.17. For any $s \geq 0$, we have that

$$\frac{\Psi^0_{x_t}(x, s+\Delta) - \Psi^0_{x_t}(x,s)}{\Delta} = \frac{\psi^{0,0}(x; s, \Delta) - \Psi^0_{x_t}(x,s)}{\Delta} + \frac{\psi^{0,1}(x; s, \Delta)}{\Delta}. \tag{3.103}$$

By taking the limit of $\Delta \downarrow 0$, we obtain

$$\left.\frac{\partial \Psi^0_{x_t}}{\partial t}\right|_{t=s} = \lim_{\Delta \downarrow 0} \frac{\psi^{0,0}(x; s, \Delta) - \Psi^0_{x_t}(x,s)}{\Delta} + \lim_{\Delta \downarrow 0} \frac{\psi^{0,1}(x; s, \Delta)}{\Delta} \tag{3.104}$$

Referring to (3.71), we see that

$$\lim_{\Delta \downarrow 0} \frac{\psi^{0,0}(x; s, \Delta) - \Psi^0_{x_t}(x,s)}{\Delta} = \left.\frac{\partial \psi^{0,0}(x; s, \Delta)}{\partial \Delta}\right|_{\Delta=0} =$$
$$= \left.\frac{\partial p^0}{\partial t}\right|_{t=s} = \frac{1}{2}\frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} + \frac{\partial}{\partial x}\left([a(0)x + \alpha(0)]\, \Psi^0_{x_t}\right). \tag{3.105}$$

We also have that (see (3.73a))

$$\lim_{\Delta \downarrow 0} \frac{\psi^{0,1}(x; s, \Delta)}{\Delta} = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \int_0^\Delta q^0(x, s+\Delta-r) \cdot p_{\tau^1_{s<}}(r) dr$$
$$= q^0(x,s) \cdot p_{\tau^1_{s<}}(0)$$
$$= \delta(x-1) \cdot p_{\tau^1_{s<}}(0) \tag{3.106}$$

Proving the boundary conditions in (3.101) is more or less trivial since they can be derived easily from those of (3.71) and (3.75). Hence, from (3.105) and (3.106) we can obtain (3.101).

Following the same line we can derive for any $s \geq 0$

$$\frac{\partial \Psi^1_{x_t}}{\partial t} = \lim_{\Delta \downarrow 0} \frac{\psi^{1,1}(x; s, \Delta) - \Psi^1_{x_t}(x, s)}{\Delta} + \lim_{\Delta \downarrow 0} \frac{\psi^{1,0}(x; s, \Delta)}{\Delta}. \tag{3.107}$$

From (3.72), the following relation can be derived.

$$\begin{aligned} \lim_{\Delta \downarrow 0} \frac{\psi^{1,1}(x; s, \Delta) - \Psi^1_{x_t}(x, s)}{\Delta} &= \left. \frac{\partial \psi^{1,1}(x; s, \Delta)}{\partial \Delta} \right|_{\Delta=0} = \\ &= \frac{1}{2} \frac{\partial^2 \Psi^1_{x_t}}{\partial x^2} + \frac{\partial}{\partial x} \left([a(1)x + \alpha(1)] \, \Psi^1_{x_t} \right) \end{aligned} \tag{3.108}$$

We also have that (see (3.73b))

$$\begin{aligned} \lim_{\Delta \downarrow 0} \frac{\psi^{1,0}(x; s, \Delta)}{\Delta} &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \int_0^\Delta q^1(x, s + \Delta - r) \cdot p_{\tau^0_{s<}}(r) dr \\ &= q^1(x, s) \cdot p_{\tau^0_{s<}}(0) \\ &= \delta(x + 1) \cdot p_{\tau^0_{s<}}(0) \end{aligned} \tag{3.109}$$

Since the boundary conditions in (3.102) can also be derived easily from those of (3.72) and (3.76). Hence, from (3.108) and (3.109) we can obtain (3.102). ■

In addition to the results above, there are something more about the density of $\tau_{t<}$.

Proposition 3.20 *For the process described by (3.1), the following relations hold true*

$$p_{\tau^1_{t<}}(0) = -\frac{1}{2} \left. \frac{\partial \Psi^1_{x_t}(x, t)}{\partial x} \right|_{x=0} \tag{3.110}$$

$$p_{\tau^0_{t<}}(0) = \frac{1}{2} \left. \frac{\partial \Psi^0_{x_t}(x, t)}{\partial x} \right|_{x=0} \tag{3.111}$$

Proof. We start with the following relations

$$P(\theta_t = 0) = \int_0^\infty \Psi^0_{x_t}(x, t) dx \tag{3.112}$$

$$P(\theta_t = 1) = \int_{-\infty}^0 \Psi^1_{x_t}(x, t) dx. \tag{3.113}$$

We then have that

$$\frac{\partial}{\partial t} P(\theta_t = 0) = \int_0^\infty \frac{\partial}{\partial t} \Psi^0_{x_t}(x, t) dx \tag{3.114a}$$

$$\frac{\partial}{\partial t} P(\theta_t = 1) = \int_{-\infty}^0 \frac{\partial}{\partial t} \Psi^1_{x_t}(x, t) dx. \tag{3.114b}$$

Substituting (3.101) to (3.114a) yields

$$\begin{aligned} \frac{\partial}{\partial t} P(\theta_t = 0) = \int_0^\infty & \frac{1}{2} \frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} - \frac{\partial}{\partial x} \left([a(0)x + \alpha(0)] \, \Psi^0_{x_t} \right) dx + \\ & + \int_0^\infty p_{\tau^1_{t<}}(0) \cdot \delta(x - 1) dx \end{aligned} \tag{3.115}$$

Hence

$$\frac{\partial}{\partial t}P(\theta_t = 0) = \frac{1}{2}\left.\frac{\partial \Psi^0_{x_t}(x,t)}{\partial x}\right|_{x=\infty} - \frac{1}{2}\left.\frac{\partial \Psi^0_{x_t}(x,t)}{\partial x}\right|_{x=0} +$$
$$+ \left([a(0)x + \alpha(0)]\,\Psi^0_{x_t}(0,t)\right) - \lim_{x\to\infty}\left([a(0)x + \alpha(0)]\,\Psi^0_{x_t}(x,t)\right) + p_{\tau^1_{t<}}(0). \quad (3.116)$$

Since $\Psi^0_{x_t}(0,t) = 0$,

$$\left([a(0)x + \alpha(0)]\,\Psi^0_{x_t}\right) = 0. \quad (3.117)$$

We also have the following relations

$$\lim_{x\to\infty} \Psi^0_{x_t}(x,t) = 0 \quad (3.118a)$$

$$\int_0^\infty \Psi^0_{x_t}(x,t)dx < \infty. \quad (3.118b)$$

Assuming that the following limit exists, (3.118a) implies that

$$\left.\frac{\partial \Psi^0_{x_t}(x,t)}{\partial x}\right|_{x=\infty} = 0, \quad (3.119)$$

and (3.118b) implies that

$$\lim_{x\to\infty}\left([a(0)x + \alpha(0)]\,\Psi^0_{x_t}\right) = 0 \quad (3.120)$$

since the tail of $\Psi^0_{x_t}$ has to decay faster than $1/x$. Hence we have

$$\frac{\partial}{\partial t}P(\theta_t = 0) = -\frac{1}{2}\left.\frac{\partial \Psi^0_{x_t}(x,t)}{\partial x}\right|_{x=0} + p_{\tau^1_{t<}}(0) \quad (3.121)$$

Since the change in $P(\theta_t = 0)$ is determined by the probability of the process entering and exiting mode 0, we also have

$$\frac{\partial}{\partial t}P(\theta_t = 0) = p_{\tau^1_{t<}}(0) - p_{\tau^0_{t<}}(0) \quad (3.122)$$

Hence we obtain (3.110). Following the similar steps, from (3.114b) we can obtain

$$\frac{\partial}{\partial t}P(\theta_t = 1) = \frac{1}{2}\left.\frac{\partial \Psi^1_{x_t}(x,t)}{\partial x}\right|_{x=0} + p_{\tau^0_{t<}}(0). \quad (3.123)$$

Since we also have that

$$\frac{\partial}{\partial t}P(\theta_t = 1) = p_{\tau^0_{t<}}(0) - p_{\tau^1_{t<}}(0), \quad (3.124)$$

we obtain (3.111). ■

We conclude this chapter with the following example.

Example 3.21 *Let* ξ_t *where*

$$\begin{aligned} x_t &\in \mathsf{R} \\ \theta_t &\in M = \{0,1\} \\ \xi_t &\triangleq (x_t,\theta_t)^{\mathsf{T}} \\ t &\in \mathsf{R}^+, \end{aligned}$$

be a process described by (3.1) - (3.9), with

$$a(\theta) = \alpha(\theta) = 0,\ \theta = 0, 1.$$

This means that the diffusion for each mode is just a standard Brownian motion. We also assume that the initial modal densities are given as follows.

$$\Psi^0_{x_t}(x, 0) = \frac{1}{2}\delta(x-1) \tag{3.125a}$$

$$\Psi^1_{x_t}(x, 0) = \frac{1}{2}\delta(x+1) \tag{3.125b}$$

According to Theorem 3.19 we can describe the evolution of the modal densities as

$$\left.\begin{array}{l} \frac{\partial \Psi^0_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} + p_{\tau^1_{t<}}(0) \cdot \delta(x-1) \\ \Psi^0_{x_t}(x, 0) = \frac{1}{2}\delta(x-1) \\ \Psi^0_{x_t}(0, t) = 0 \\ x > 0, t \geq 0 \end{array}\right\} \tag{3.126}$$

$$\left.\begin{array}{l} \frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^1_{x_t}}{\partial x^2} + p_{\tau^0_{t<}}(0) \cdot \delta(x+1) \\ \Psi^1_{x_t}(x, 0) = \frac{1}{2}\delta(x+1) \\ \Psi^1_{x_t}(0, t) = 0 \\ x < 0, t \geq 0 \end{array}\right\} \tag{3.127}$$

Using Proposition 3.20, we can obtain

$$\left.\begin{array}{l} \frac{\partial \Psi^0_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} - \frac{1}{2}\left.\frac{\partial \Psi^1_{x_t}}{\partial x}\right|_{x=0} \cdot \delta(x-1) \\ \Psi^0_{x_t}(x, 0) = \frac{1}{2}\delta(x-1) \\ \Psi^0_{x_t}(0, t) = 0 \\ x > 0, t \geq 0 \end{array}\right\} \tag{3.128}$$

$$\left.\begin{array}{l} \frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^1_{x_t}}{\partial x^2} + \frac{1}{2}\left.\frac{\partial \Psi^0_{x_t}}{\partial x}\right|_{x=0} \cdot \delta(x+1) \\ \Psi^1_{x_t}(x, 0) = \frac{1}{2}\delta(x+1) \\ \Psi^1_{x_t}(0, t) = 0 \\ x < 0, t \geq 0 \end{array}\right\} \tag{3.129}$$

Here we have coupled PDEs, but by construction of the example, we introduce symmetry between the two modes, such that

$$\Psi^1_{x_t}(x, t) = \Psi^0_{x_t}(-x, t) \tag{3.130}$$

Consequently we will also have that

$$\left.\frac{\partial}{\partial x}\Psi^1_{x_t}(x, t)\right|_{x=0} = -\left.\frac{\partial}{\partial x}\Psi^0_{x_t}(x, t)\right|_{x=0} \tag{3.131}$$

Hence the PDEs can be decoupled into

$$\left.\begin{array}{l} \frac{\partial \Psi^0_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^0_{x_t}}{\partial x^2} + \frac{1}{2}\left.\frac{\partial \Psi^0_{x_t}}{\partial x}\right|_{x=0} \cdot \delta(x-1) \\ \Psi^0_{x_t}(x, 0) = \frac{1}{2}\delta(x-1) \\ \Psi^0_{x_t}(0, t) = 0 \\ x > 0, t \geq 0 \end{array}\right\} \tag{3.132}$$

$$\left.\begin{array}{l} \frac{\partial \Psi^1_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^1_{x_t}}{\partial x^2} - \frac{1}{2}\left.\frac{\partial \Psi^1_{x_t}}{\partial x}\right|_{x=0} \cdot \delta(x+1) \\ \Psi^1_{x_t}(x,0) = \frac{1}{2}\delta(x+1) \\ \Psi^1_{x_t}(0,t) = 0 \\ x < 0, t \geq 0 \end{array}\right\} \tag{3.133}$$

We can notice that (3.132) is (3.133) mirrored around the $t-$axis. Solving any of them, however, is beyond the scope of the report.

Chapter 4

A Model with Diffusion, Poisson Point Process and Boundary Hitting Process

In this chapter we combine the features of the stochastic hybrid systems discusses in the two earlier chapters. We will now consider a process whose mode switchings are triggered by both PPP and BHP.

In this case, we double the number of modes. There are then two pairs of modes. One mode in each pair can be referred to as *normal* and the other as *fault*. The motivation behind this, is that in the application in ATM field, we assume that the PPP represents some kind of faults occurring in the aircraft/pilot. We also assume that the time needed to fix the fault follows the exponential distribution. Hence switching from a fault mode to a normal one can be modeled as another PPP. In addition, since naturally the expected time needed to fix the fault and the expected time between faults (MTBF = Mean Time Between Failures) are different, we would also assume in the model that the rate of the PPP is a function of the mode.

The mathematical formulation of the process is given as follows.

4.1 Mathematical Formulation

Consider a stochastic hybrid system characterized by the following (Ito) SDE.

$$\begin{aligned} x_t &\in \mathsf{R} \\ \theta_t &\in M = \{0, 1, 2, 3\} \\ \xi_t &\triangleq (x_t, \theta_t)^{\mathrm{T}} \\ t &\in \mathsf{R}^+, \end{aligned}$$

$$\begin{aligned} dx_t &= [a(\theta_t)x_t + \alpha(\theta_t)]\, dt + dW_t + \sum_{i=0}^{1} b^i(\theta_{t^-}) dk_t^i \\ d\theta_t &= \sum_{i=0}^{1} c^i(\theta_{t^-}) dk_t^i + d(\theta_{t^-}) dP_t \end{aligned} \quad (4.1)$$

where P_t is a PPP with mode dependant rate $\lambda(\theta_t)$.As usual, W_t denotes a standard Brownian motion.

We assume that there exist a probability space $(\Omega, P, \mathcal{F})$ and a filtration $\mathcal{F}_t$ such that the process ξ_t is adapted to it. Moreover, we also define

$$\begin{aligned} k_t^0 &= L_t(\mathsf{R}^+) \\ k_t^1 &= L_t(\mathsf{R}^-) \end{aligned} \tag{4.2}$$

$$\begin{aligned} L_t(A) &: \Omega \to \{0,1\} \\ L_t &= \begin{cases} 1, \text{ if } \lim_{s\uparrow t} x_s \in A \\ 0, \text{ otherwise} \end{cases} \end{aligned} \tag{4.3}$$

$$a(\theta) \le 0, \forall \theta \in M \tag{4.4}$$

$$b^0(\theta_{t^-}) = \begin{cases} 0, \ \theta_{t^-} = 0,2 \\ 1, \ \theta_{t^-} = 1,3 \end{cases} \tag{4.5}$$

$$b^1(\theta_{t^-}) = \begin{cases} -1, \ \theta_{t^-} = 0,2 \\ 0, \ \theta_{t^-} = 1,3 \end{cases} \tag{4.6}$$

$$c^0(\theta_{t^-}) = \begin{cases} 0, \ \theta_{t^-} = 0,2 \\ -1, \ \theta_{t^-} = 1,3 \end{cases} \tag{4.7}$$

$$c^1(\theta_{t^-}) = \begin{cases} 1, \ \theta_{t^-} = 0,2 \\ 0, \ \theta_{t^-} = 1,3 \end{cases} \tag{4.8}$$

$$d(\theta_{t^-}) = \begin{cases} 2, \ \theta_{t^-} = 0,1 \\ -2, \ \theta_{t^-} = 2,3 \end{cases} \tag{4.9}$$

Let us assume that $\{0, 1)$ is the set of the normal modes, and $\{2, 3\}$ being the set of the fault mode. We assume that we start in a normal mode. Hence for the initial condition ξ_t, we assume that the following relation holds.

$$(x_0, \theta_0) \in ((\mathsf{R}^+ - \{0\}) \times \{0\}) \cup ((\mathsf{R}^- - \{0\}) \times \{1\}) \tag{4.10}$$

4.2 Solution of Equation (4.1)

We will propose a way to construct the solution ξ_t. We will use a construction analogous to the one in Section 3.3. For $i = 1, 2, 3, \cdots$, let $\check{\xi}_t^i$, $(\check{x}_t^i, \check{\theta}_t^i)^{\mathrm{T}}$be the solution of

$$\begin{aligned} d\check{x}_t^i &= \left[a(\check{\theta}_t^i)x_t + \alpha(\check{\theta}_t^i)\right] dt + dW_t \\ \check{\theta}_t^i &= \check{\theta}^i \end{aligned} \tag{4.11}$$

with initial condition $\check{\xi}_{\tau_{i-1}}^i$, $(\check{x}_0^i, \check{\theta}_0^i)^{\mathrm{T}}$. We define the stopping time

$$\tau_i^p = \inf(t > \tau_{i-1} : P_t \text{ generates a point at time } t) \tag{4.12}$$

which has an exponential distribution such that for every $x \ge 0$,

$$P(\tau_i^p \le x + \tau_{i-1}) = 1 - e^{-x\lambda(\check{\theta}^i)}. \tag{4.13}$$

Moreover, let

$$\tau_i^b = \inf(t > \tau_{i-1} : \check{x}_t^i = 0) \tag{4.14}$$

and

$$\tau_i = \min(\tau_i^b, \tau_i^p),\ \tau_0 = 0. \tag{4.15}$$

Both τ_i^b and τ_i are also stopping times.

Let $T \in \mathsf{R} > 0$. Define a stochastic process $\check{\xi}_t$ whose sample paths in $[0, T]$ are constructed as follows.

1. Let $\check{\xi}_0^1 = \xi_0$.
2. Obtain a sample path of $\check{\xi}_t^1$ until the stopping time τ_1.
3. For $i = 1, 2, 3, \cdots$, let the initial condition $\check{\xi}_{\tau_i}^{i+1}$ be determined as follows. If $\tau_i^p < \tau_i^b$ go to step 4. If $\tau_i^p > \tau_i^b$ go to step 5. If $\tau_i^p = \tau_i^b$ go to step 6.
4. $\check{\xi}_{\tau_i}^{i+1} = (\check{x}_{\tau_i}^i, \check{\theta}^i + d(\check{\theta}^i))^{\mathrm{T}}$. Go to step 7.
5. $\check{\xi}_{\tau_i}^{i+1} = (-\operatorname{sign}(\check{x}_{\tau_{i-1}}^i), \check{\theta}^i - \operatorname{sign}(\check{x}_{\tau_{i-1}}^i))^{\mathrm{T}}$. Go to step 7.
6. $\check{\xi}_{\tau_i}^{i+1} = (-\operatorname{sign}(\check{x}_{\tau_{i-1}}^i), \check{\theta}^i - \operatorname{sign}(\check{x}_{\tau_{i-1}}^i) + d(\check{\theta}^i))^{\mathrm{T}}$. Go to step 7.
7. Start the process $\check{\xi}_t^{i+1}$ from the stopping time τ_i until the stopping time τ_{i+1}.
8. Back to step 3 until a sample path of $\check{\xi}_t^N$ is obtained where $\tau_N > T$
9. A sample path of $\check{\xi}_t$ is obtained by concatenating those of $\check{\xi}_t^i$, i.e.

$$\check{\xi}_t = \check{\xi}_t^i,\ \text{for } t \in [\tau_{i-1}, \tau_i) \tag{4.16}$$

Definition 4.1 *We define the set* $J_T \triangleq \{\tau_1, \tau_2, \tau_3, \cdots\}$ *with* $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 \cdots \leq T$.

Remark 4.2 *After each stopping time* τ_i *we can consider started afresh due to the strong Markov property.*

The following theorem provides us with the way to characterize a solution of (4.1).

Theorem 4.3 *Let* $t \in [0, T]$*, and* $\check{\xi}_t$ *be the stochastic process as described above. The process* $\check{\xi}_t$ *is a solution of (4.1) with initial condition* ξ_0*.*

Proof. We will show that $\check{\xi}_s$, $s \in [0, T]$ satisfies

$$\check{x}_s = x_0 + \int_0^s \left[a(\check{\theta}_t)\check{x}_t + \alpha(\check{\theta}_t)\right] dt + \int_0^s dW_t + \sum_{i=0}^{i=1} \int_0^s b^i(\check{\theta}_{t-}) dk_t^i \tag{4.17}$$

$$\check{\theta}_t = \theta_0 + \sum_{i=0}^{i=1} \int_0^s c^i(\check{\theta}_{t-}) dk_t^i + \int_0^s d(\check{\theta}_{t-}) dP_t \tag{4.18}$$

By the nature of the construction of $\check{\xi}_t$ we will evaluate the integrals in (4.17) and (4.18) segment by segment. Equivalently, for $s \in [\tau_i, \tau_{i+1}], \forall i \in \{i \,|i \geq 0, \tau_{i+1} \in J_T\}$ the relations (4.17) and (4.18) can also be written as

$$\check{x}_s = \check{x}_{\tau_i} + \int_{\tau_i}^{s} \left[a(\check{\theta}_t)\check{x}_t + \alpha(\check{\theta}_t)\right] dt + \int_{\tau_i}^{s} dW_t + \sum_{i=0}^{i=1} \int_{\tau_i}^{s} b^i(\check{\theta}_{t-})dk_t^i \tag{4.19}$$

$$\check{\theta}_t = \check{\theta}_{\tau_i} + \sum_{i=0}^{i=1} \int_{\tau_i}^{s} c^i(\check{\theta}_{t-})dk_t^i + \int_{\tau_i}^{s} d(\check{\theta}_{t-})dP_t \tag{4.20}$$

It is trivial to verify that the initial condition is satisfied. Moreover, if $s \in [\tau_i, \tau_{i+1})$ the terms involving the PPP and the BHP of (4.19) and (4.20) evaluate to zero. Hence for $s \in [\tau_i, \tau_{i+1})$ the relations (4.19) and (4.20) are satisfied by the construction of $\check{\xi}_t^{i+1}$. If $s = \tau_{i+1}$, it can also be verified that the jump (i.e. $\check{\xi}_{\tau_{i+1}^-}$ to $\check{\xi}_{\tau_{i+1}}$) is consistent with (3.44) and (3.45), as the integral terms involving the PPP and BHP will cover the gap caused by the jump.

The verification will be done as follows :

- If the jump is caused by the PPP, i.e. $\tau_p^i < \tau_{\check{x}^i}$

$$x_{\tau_{i+1}} - x_{\tau_{i+1}^-} = 0$$

$$\theta_{\tau_{i+1}} - \theta_{\tau_{i+1}^-} = \int_{\tau_{i+1}^-}^{\tau_{i+1}} d(\check{\theta}_{t-})dP_t = d(\check{\theta}_{\tau_{i+1}^-})$$

- If the jump is caused by the BHP, i.e. $\tau_p^i < \tau_{\check{x}^i}$, with $x_{\tau_i} > 0$

$$x_{\tau_{i+1}} - x_{\tau_{i+1}^-} = \sum_{i=0}^{1} \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^i(\check{\theta}_{t-})dk_t^i = -1$$

$$\theta_{\tau_{i+1}} - \theta_{\tau_{i+1}^-} = \sum_{i=0}^{1} \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^i(\check{\theta}_{t-})dk_t^i = 1$$

- If the jump is caused by the BHP, i.e. $\tau_p^i < \tau_{\check{x}^i}$, with $x_{\tau_i} < 0$

$$x_{\tau_{i+1}} - x_{\tau_{i+1}^-} = \sum_{i=0}^{1} \int_{\tau_{i+1}^-}^{\tau_{i+1}} b^i(\check{\theta}_{t-})dk_t^i = 1$$

$$\theta_{\tau_{i+1}} - \theta_{\tau_{i+1}^-} = \sum_{i=0}^{1} \int_{\tau_{i+1}^-}^{\tau_{i+1}} c^i(\check{\theta}_{t-})dk_t^i = -1$$

- If the jump is caused by both the BHP and the PPP, i.e. $\tau_p^i = \tau_{\check{x}^i}$, the jump will be a superposition of those of the individual process. (See step 6 in the construction of $\check{\xi}_t$ above).

■

For the proof, the reader is referred to the analogous proof of Theorem 3.10. It is also interesting to notice that while the jump caused by both processes simultaneously is accommodated in the construction, the probability of having such jump is zero[1].

[1]By definition of the Poisson Point Process, the probability of having a point at a certain fixed time is zero.

Remark 4.4 *The solution described above is right continuous by construction (compare with the construction in Chapter 3). Uniqueness of solution will not be proven here. Instead, we assume that 4.1) admits a unique solution as described above.*

4.3 Evolution of the Modal Densities

In this section, we will discuss the evolution of the modal densities for the process described in (4.1). We start with the following proposition.

Proposition 4.5 *Given the process described by (4.1), the probability of having at least two jumps within a small time interval Δ is $o(\Delta)$, i.e. for all $t \geq 0$, the following relation holds.*

$$P(\exists i, j : i \neq j, \tau_i \in [t, t+\Delta], \tau_i \in [t, t+\Delta] | \xi_t) = o(\Delta)$$

Proof. It has been shown in Proposition 2.6 that the probability of having at least two PPP jumps in $[t, t+\Delta]$ is $o(\Delta)$. While in Proposition 3.15 it is shown that the probability of having at least two BHP jumps in $[t, t+\Delta]$ is also $o(\Delta)$. Hence it is sufficient now if we can show that the probability of having a PPP jump and a BHP jump in $[t, t+\Delta]$ is also $o(\Delta)$.

Now, let $\lambda_{\max}$ be the maximum rate of the PPP, i.e.

$$\lambda_{\max} \triangleq \max_{\theta \in \{0,1,2,3\}} \lambda(\theta)$$

and let $\mathcal{X}$ and $\mathcal{Y}$ be the events as follows.

$$\begin{aligned} \mathcal{X} &: \text{ the BHP generates a jump in } [t, t+\Delta] \\ \mathcal{Y} &: \text{ the PPP generates a jump in } [t, t+\Delta] \end{aligned}$$

Our goal is to show that

$$P(\mathcal{X} \cap \mathcal{Y} | \xi_t) = o(\Delta)$$

We have the following relation

$$P(\mathcal{X} \cap \mathcal{Y} | \xi_t) = P(\mathcal{Y} | \xi_t, \mathcal{X}) \cdot P(\mathcal{X} | \xi_t) \tag{4.21}$$

such that

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} P(\mathcal{X} \cap \mathcal{Y} | \xi_t) = \lim_{\Delta \downarrow 0} \frac{P(\mathcal{Y} | \xi_t, \mathcal{X})}{\Delta} P(\mathcal{X} | \xi_t).$$

But from Proposition 2.5 we know that

$$\lim_{\Delta \downarrow 0} \frac{P(\mathcal{Y} | \xi_t, \mathcal{X})}{\Delta} \leq \lambda_{\max}$$

hence

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} P(\mathcal{X} \cap \mathcal{Y} | \xi_t) \leq \lambda_{\max} \lim_{\Delta \downarrow 0} P(\mathcal{X} | \xi_t). \tag{4.22}$$

Since from (3.65) we know that

$$\lim_{\Delta \downarrow 0} P(\mathcal{X} | \xi_t) = 0, \tag{4.23}$$

the proof is completed. ■

Proposition 4.5 plays the same role as Proposition 2.6 and Proposition 3.15 in deriving the evolution of the modal densities for the previous models. We will proceed to derive a proposition which is analogous to Proposition 3.17. First we introduce the following notations.

Notation 4.6 *Let θ be an element of M, we denote the mode that is related to θ through a PPP jump as θ^p and the mode that is related to θ through a BHP jump as θ^b. For example : $0^p = 2,\ 1^b = 0$, etc.*

Notation 4.7 *Referring to Notation 3.16,we introduce the following notation,*

$$\tau^b_{s<} \triangleq \tau_\nu - s,\ \text{where } \nu \triangleq \min_{i\geq 0}(i \mid \tau_i > s,\ \text{BHP jump }).$$

In other words, $\tau^b_{s<}$ denotes the time difference between s and the time of the first BHP jump after s. Analogously we can also introduce

$$\tau^p_{s<} \triangleq \tau_\nu - s,\ \text{where } \nu \triangleq \min_{i\geq 0}(i \mid \tau_i > s,\ \text{PPP jump })$$

as the time difference between s and the time of the first PPP jump after s.

Proposition 4.8 *Assume that $s \geq 0$ and $\Delta > 0$. Given $\Psi^\theta_{x_t}(x,s)$ for all $\theta \in M$, under assumption that at most one jump occurs in $[s, s+\Delta]$, the following relations hold true.*

$$\Psi^\theta_{x_t}(x, s+\Delta) = \psi^{\theta,\theta}(x; s,\Delta) + \psi^{\theta,\theta^b}(x; s,\Delta) + \psi^{\theta,\theta^p}(x; s,\Delta) \tag{4.24a}$$

where

$$\psi^{\theta,\theta}(x; s,\Delta) \equiv e^{-\lambda(\theta)\Delta}\ p^\theta(x,t)\big|_{t=s+\Delta} \tag{4.25}$$

and $p^\theta(x,t)$ is the solution of the PDE

$$\left.\begin{array}{l} \frac{\partial p^\theta}{\partial t} - \frac{1}{2}\frac{\partial^2 p^\theta}{\partial x^2} + \frac{\partial}{\partial x}\left([a(\theta)x + \alpha(\theta)]\, p^\theta\right) = 0 \\ p^\theta(x,s) = \Psi^\theta_{x_t}(x,s) \\ p^\theta(0,t) = 0 \\ x \in Inv(\theta), t > s \end{array}\right\}, \tag{4.26}$$

where $Inv(\theta)$[2] is defined as follows

$$Inv(\theta) = \begin{cases} \mathsf{R}^+, & \text{if } \theta \in \{0,2\} \\ \mathsf{R}^-, & \text{if } \theta \in \{1,3\} \end{cases} \tag{4.27}$$

The other terms are defined as

$$\psi^{\theta,\theta^b}(x; s,\Delta) = \int_0^\Delta q^\theta(x, s+\Delta-r)\cdot p_{\tau^{\theta^b}_{s<}}(r)dr \tag{4.28a}$$

The density $p_{\tau^b_{s<}}(\cdot)$ is defined as follows. For any $\Delta > \tau > 0$,

$$P(\tau^b_{s<} \leq \tau, \theta_s = \theta^b\ ) \triangleq \int_0^\tau p_{\tau^{\theta^b}_{s<}}(r)dr \tag{4.29a}$$

[2] $Inv(\theta)$ is read as 'invariant of θ'. This comes from terminology of hybrid systems. (See e.g. [VdSS00])

The functions $q^\theta(x,t)$ is defined as the solutions of the following PDE

$$\left.\begin{array}{l} \frac{\partial q^\theta}{\partial t} - \frac{1}{2}\frac{\partial^2 q^\theta}{\partial x^2} + \frac{\partial}{\partial x}\left(\left[a(\theta)x + \alpha(\theta)\right] q^\theta\right) = 0 \\ q^\theta(x,s) = \delta(x - j(\theta)) \\ q^\theta(0,t) = 0 \\ x \in Inv(\theta), t > s \end{array}\right\} \tag{4.30}$$

The symbol $j(\theta) \in Inv(\theta)$ here denotes the state into which the process goes after a BHP jump from mode θ^b, i.e.

$$j(\theta) = \begin{cases} 1, & \text{if } \theta \in \{0,2\} \\ -1, & \text{if } \theta \in \{1,3\} \end{cases}$$

The term $\psi^{\theta,\theta^p}(x;s,\Delta)$ is defined as

$$\psi^{\theta,\theta^p}(x;s,\Delta) = \int_0^\Delta \lambda(\theta^p) e^{-r\lambda(\theta^p)} f^\theta(x, s+\Delta, r)dr \tag{4.31}$$

where the function $f^\theta(x,t,r)$ satisfies the following system of PDE

$$\left.\begin{array}{l} \frac{\partial f^\theta(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\left(a(\theta^p)x + \alpha(\theta^p)\right) f^\theta(x,t)\right] = 0 \\ f^\theta(x,s) = \Psi^{\theta^p}_{x_t}(x,s) \\ x \in \mathsf{R}, s + r \geq t > s, r \leq \Delta \end{array}\right\} \tag{4.32}$$

$$\left.\begin{array}{l} \frac{\partial f^\theta(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\left(a(\theta)x + \alpha(\theta)\right) f^\theta(x,t)\right] = 0 \\ x \in \mathsf{R}, t > s + r, r \leq \Delta \end{array}\right\} \tag{4.33}$$

Proof. Under the assumption that at most one jump occur in $[s.s+\Delta]$ we have for $\Gamma \subset \mathsf{R}^+$

$$\begin{aligned} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta) &= P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta) + \\ +P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^p) &+ P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^b). \end{aligned} \tag{4.34}$$

Let us define density functions for each term of the right hand side of (4.34), such that

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta) \equiv \int_\Gamma \psi^{\theta,\theta}(x;s,\Delta)dx \tag{4.35}$$

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^p) \equiv \int_\Gamma \psi^{\theta,\theta^p}(x;s,\Delta)dx \tag{4.36}$$

$$P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^b) \equiv \int_\Gamma \psi^{\theta,\theta^b}(x;s,\Delta)dx \tag{4.37}$$

Hence we have (4.24a). Moreover, let $\mathcal{X}$ and $\mathcal{Y}$ be the events as follows.

$$\begin{array}{l} \mathcal{X} : \text{there is a BHP jump in } [s, s+\Delta] \\ \mathcal{Y} : \text{there is a PPP jump in } [s, s+\Delta] \end{array}$$

Using Chapman-Kolmogorov Equation we can obtain

$$
\begin{aligned}
&P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta) = \\
&= \int_{y \in Inv(\theta)} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta, x_s \in [y, y + dy]) \\
&= \int_{y \in Inv(\theta)} (P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta | \theta_s = \theta, x_s \in [y, y + dy]) \times \\
&\quad \times P(\theta_s = \theta, x_s \in [y, y + dy])) \\
&= \int_{y \in Inv(\theta)} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta | \theta_s = \theta, x_s \in [y, y + dy]) \cdot \Psi^{\theta}_{x_t}(x, s).
\end{aligned} \tag{4.38}
$$

and we also have that

$$
\begin{aligned}
&P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta | \theta_s = \theta, x_s \in [y, y + dy]) = \\
&= P(x_{s+\Delta} \in \Gamma, \mathcal{X}^c \cap \mathcal{Y}^c | \theta_s = \theta, x_s \in [y, y + dy]) = \\
&= P(x_{s+\Delta} \in \Gamma, \mathcal{X}^c | \mathcal{Y}^c, \theta_s = \theta, x_s \in [y, y + dy]) \times \\
&\quad \times P(\mathcal{Y}^c | \theta_s = \theta, x_s \in [y, y + dy]).
\end{aligned} \tag{4.39}
$$

Notice that to derive the second line of (4.39) we use the assumption that at most one jump is generated in $[s, s + \Delta]$. Now let there be a density $p^{\theta}(x, t)$ such that

$$
P(x_{s+\Delta} \in \Gamma, \mathcal{X}^c | \mathcal{Y}^c, \theta_s = \theta, x_s \in [y, y + dy]) = \int_{\Gamma} p^{\theta}(x, s + t) dx. \tag{4.40}
$$

It has been shown in the proof of Proposition 3.17 that $p^{\theta}(x, t)$ is described by (4.26). We also know that

$$
P(\mathcal{Y}^c | \theta_s = \theta, x_s \in [y, y + dy]) = e^{-\lambda(\theta)\Delta} \tag{4.41}
$$

Substituting (4.35), (4.40), and (4.41) to (4.38), we obtain

$$
\int_{\Gamma} \psi^{\theta,\theta}(x; s, \Delta) dx = \int_{\Gamma} e^{-\lambda(\theta)\Delta} p^{\theta}(x, s + t) dx \tag{4.42}
$$

which gives us (4.25).

Analogous to (4.38), we can obtain the following relations

$$
\begin{aligned}
&P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^p) = \\
&= \int_{y \in Inv(\theta^p)} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta | \theta_s = \theta^p, x_s \in [y, y + dy]) \cdot \Psi^{\theta^p}_{x_t}(x, s).
\end{aligned} \tag{4.43}
$$

$$
\begin{aligned}
&P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta, \theta_s = \theta^b) = \\
&= \int_{y \in Inv(\theta^b)} P(x_{s+\Delta} \in \Gamma, \theta_{s+\Delta} = \theta | \theta_s = \theta^b, x_s \in [y, y + dy]) \cdot \Psi^{\theta^b}_{x_t}(x, s).
\end{aligned} \tag{4.44}
$$

Let us focus on (4.43) first. Recall the density described in (4.36). Under the assumption that at most one jump occurs in $[s, s + \Delta]$, it has been shown in Theorem 2.9 that $\psi^{\theta,\theta^p}(x; s, \Delta)$ is characterized by (4.31) - (4.33)[3].

[3] compare (4.31) - (4.33) with (2.10a) and (2.13a) - (2.13b).

Analogously for (4.44), Proposition 3.17 implies that $\psi^{\theta,\theta^b}(x;s,\Delta)$ is characterized by (4.28a) - (4.30)[4]. ■

Given Proposition 4.8 above we are able to derive the evolution equation of the modal densities as in the previous chapters.

Theorem 4.9 *The density $\Psi^{\theta}_{x_t}(x,t)$ satisfies the following PDE*

$$\left.\begin{array}{l} \frac{\partial \Psi^{\theta}_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^{\theta}_{x_t}}{\partial x^2} - \frac{\partial}{\partial x}\left(\left[a(\theta)x+\alpha(\theta)\right]\Psi^{\theta}_{x_t}\right) - \lambda(\theta)\Psi^{\theta}_{x_t} + \\ +p_{\tau^{\theta^b}_{t<}}(0)\cdot\delta(x-j(\theta)) + \lambda(\theta^p)\Psi^{\theta^p}_{x_t} \\ \Psi^0_{x_t}(x,0) = \Psi^0_{x_0}(x) \\ \Psi^0_{x_t}(0,t) = 0 \\ x>0, t\geq 0 \end{array}\right\} \tag{4.45}$$

where $\Psi^{\theta}_{x_0}(x)$ is the density of the initial condition.

Proof. We start with the following relation.

$$\frac{\partial \Psi^{\theta}_{x_t}}{\partial t} = \lim_{\Delta\downarrow 0}\frac{\Psi^{\theta}_{x_t}(x,t+\Delta)-\Psi^{\theta}_{x_t}(x,t)}{\Delta} \tag{4.46}$$

From Proposition (4.8), we can obtain the following relation

$$\frac{\partial \Psi^{\theta}_{x_t}}{\partial t} = \lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta}(x;t,\Delta)-\Psi^{\theta}_{x_t}(x,t)}{\Delta} + \lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta^b}(x;t,\Delta)}{\Delta} + \lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta^p}(x;t,\Delta)}{\Delta} \tag{4.47}$$

Now, recalling (4.25) and (4.26), it can be shown that $\psi^{\theta,\theta}(x;t,\Delta)$ satisfies the following PDE

$$\left.\begin{array}{l} \frac{\partial \psi^{\theta,\theta}}{\partial \Delta} - \frac{1}{2}\frac{\partial^2 \psi^{\theta,\theta}}{\partial x^2} + \frac{\partial}{\partial x}\left(\left[a(\theta)x+\alpha(\theta)\right]\psi^{\theta,\theta}\right) + \lambda(\theta)\psi^{\theta,\theta} = 0 \\ \psi^{\theta,\theta}(x;t,0) = \Psi^{\theta}_{x_t}(x,t) \\ \psi^{\theta,\theta}(0;t,\Delta) = 0 \\ x\in Inv(\theta), t>s \end{array}\right\}. \tag{4.48}$$

We then obtain the following relation

$$\lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta}(x;t,\Delta)-\Psi^{\theta}_{x_t}(x,t)}{\Delta} = \frac{1}{2}\frac{\partial^2 \Psi^{\theta}_{x_t}}{\partial x^2} - \frac{\partial}{\partial x}\left(\left[a(\theta)x+\alpha(\theta)\right]\Psi^{\theta}_{x_t}\right) - \lambda(\theta)\Psi^{\theta}_{x_t}. \tag{4.49}$$

Following the proof of Theorem 3.19 it can be shown that

$$\lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta^b}(x;t,\Delta)}{\Delta} = p_{\tau^{\theta^b}_{t<}}(0)\cdot\delta(x-j(\theta)), \tag{4.50}$$

while from the proof of Theorem 2.10 the following relation can be derived.

$$\lim_{\Delta\downarrow 0}\frac{\psi^{\theta,\theta^p}(x;t,\Delta)}{\Delta} = \lambda(\theta^p)\Psi^{\theta^p}_{x_t}. \tag{4.51}$$

Combining (4.49) - (4.51), and substituting them to (4.47), we complete the proof. ■

[4] compare (4.28a) - (4.30) with (3.73a), (3.74a) and (3.75).

Chapter 5

Conclusions and Possible Extensions for the Research

5.1 Conclusions

In short we will try to summarize the results discussed so far in this report.

The research reported in this text is intended to study the evolution of the modal density functions of certain classes of hybrid systems. The modal density functions can be thought as the equivalence of probability density functions of the states in continuous variable stochastic processes.

There are three model discussed in the report. All models involve one dimensional continuous dynamics and discrete dynamics of the mode. In the first two models, we have jumps induced by Poisson Point Processes (PPP) and Boundary Hitting Processes (BHP). In the third model, we combine the features of the previous two models to devise a model with jumps induced by both the PPP and the BHP.

In deriving the equation of the evolution of the modal density functions we divide the paths of the processes into intervals separated by the jump times (which are stopping times) such that inside the intervals the solution satisfy usual Ornstein-Uhlenbeck diffusion equation with constant discrete state.

The evolution of the modal density of each mode is then analyzed as a Dirichlet problem of the Fokker-Planck equation of Forward Kolmogorov equation. We also established a limitation of the influence of certain modes to the others within an infinitesimal time interval. Basically, we have established that it is sufficient if we consider that at most one jump can occur within such interval, thus enabling us to interconnect the evolution of a modal density function only to those of the modes that can reach it by one jump.

5.2 Possible Extensions for the Research

The results discussed in this report are in some way limited. For example, so far we have only considered processes whose continuous dynamics is in a one dimensional space. Given the motivation of the research, i.e. implementation in ATM systems, we need to cope with dynamics in higher dimensional space.

We will therefore highlights some possible extensions for the research, where some aspects of the models can be altered in order to make it more capable in handling more complex phenomenons.

5.2.1 Employing nonlinear diffusion model

One of the limiting assumptions in this report is that the diffusion model is linear. In fact, it is also possible to extend the models so that nonlinear diffusion is used. For example, instead of (4.1), we might have the following SDE.

$$\begin{aligned} dx_t &= F(\xi_t, t)dt + G(\xi_t, t)dW_t + \sum_{i=0}^{1} b^i(\theta_{t-})dk_t^i \\ d\theta_t &= \sum_{i=0}^{1} c^i(\theta_{t-})dk_t^i + d(\theta_{t-})dP_t \end{aligned} \tag{5.1}$$

In this case, we still have the Dirichlet problem of the Fokker-Planck equation for each mode, with modification in the some of the terms. However, we should be careful when discussing the uniqueness and continuity of the sample paths of the diffusion. We have to pay more attention on assuring that the probability of having more than one jump in an interval of Δ is still $o(\Delta)$. Otherwise the modes will have to be interconnected in a different way.

5.2.2 Extending the state space

It is natural that we would want to extend the (hybrid) state space so that we can handle more complex dynamics. We can still follow the line of formulation in Chapter 4, with a major difference. In general, when the boundary set does not consists of only one point, or when there are more than one mode to where a mode can jump with the BHP, we have to consider the joint distribution of where and when the process hits the boundary instead of just when.

Another important consequence is that θ^p and θ^b will now be sets of mode, instead of just one mode. Hence, more one mode can be interconnected with more than just two modes as in (4.45).

5.2.3 Using random jumps

We can also modify the jumps such that the state to which the process jump is random. We can introduce a distribution density for $j(\theta)$ in (4.45) (some kind of transition measure in [Dav84]). Suppose that

$$p_{j(\theta)} = F^\theta(x), \ x \in Inv(\theta), \tag{5.2}$$

then provided that Proposition 4.5 is still valid, we will have the following relation instead of (4.45).

$$\left.\begin{array}{l} \frac{\partial \Psi^\theta_{x_t}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^\theta_{x_t}}{\partial x^2} - \frac{\partial}{\partial x}\left(\left[a(\theta)x + \alpha(\theta)\right]\Psi^\theta_{x_t}\right) - \lambda(\theta)\Psi^\theta_{x_t} + \\ + p_{\tau_{t<}^{\theta^b}}(0) \cdot F^\theta(x) + \lambda(\theta^p)\Psi^{\theta^p}_{x_t} \\ \Psi^0_{x_t}(x,0) = \Psi^0_{x_0}(x) \\ \Psi^0_{x_t}(0,t) = 0 \\ x > 0, t \geq 0 \end{array}\right\} \tag{5.3}$$

Notice that if $j(\theta)$ is deterministic, then $F^\theta(x)$ is just a Dirac measure, and (5.3) is reduced back to (4.45).

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