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Modified potential approach to efficient, linear and symmetric values for TU-games

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Abstract

The potential approach is a useful tool in physics, and the successful treatment of such concept turned out to be reproducible, in the late eighties, in the field of cooperative game theory. This report is devoted to the modified potential approach for values on cooperative games, particularly for the Shapley value, the Solidarity value, and the class of values satisfying efficiency, linearity and symmetry (ELS values).

Chapter 1 introduces the basic concepts in the field of game theory, especially in the cooperative part. Besides the set-valued solutions, such as the imputation set and the stable set, we primarily discuss the Shapley value and the Solidarity value, both of which are single-valued solutions. After presenting several properties of solutions using in axiomatization, two equivalent forms of ELS values are studied in particular. Concerning the noncooperative part, games in normal form and the famous solution concept–Nash equilibrium are discussed.

In Chapter 2, the classical potential approach, which depict the equivalence between the classical gradient and the Shapley value, is introduced. In view of the classical potential, we consider a more general concept called the modified potential, which also satisfies the 0-normalized property, but with a modified gradient in the efficiency condition. Concerning this new concept, a value possesses a modified potential representation only when it equals to the modified gradient. For the ELS value, in which the Shapley value and the Solidarity value are two special cases, we discuss its sufficient and necessary condition when it admits a modified potential representation, especially in the separable case.

In Chapter 3, the Shapley value can be written as a linear combination of the corresponding coordinates based on the unanimity game. In order to simplify such expression, we define another basis analogously to the collection of unanimity games, associated with the Shapley value. Similarly, two new basis of the game space, with respect to the Solidarity value and ELS values respectively, are defined, such that these two values admit a simple sum expression concerning their coordinates. According to these basis, the modified potential admits a new expression, thus the modified potential representation of different values can be verified. In the last section, the concept of the potential game, which belongs to the field of noncooperative game theory, is introduced. We investigate the Solidarity value and the ELS value representations of potential games, which are closely related to its Shapley value representation.

Chapter 4 presents the reduced game and the corresponding reduced game property, which says the payoffs of players in a subset should not change or they should have no reason to renegotiate, if they apply the same solution rule among themselves as in the original game. For the Shapley value, Sobolev defined a special reduced game, such that the Shapley value in such $(n-1)$-person reduced game equals to that in the original n-person game. By the modified potential approach discussed in Chapter 2, we find the

same reduced game for the Shapley value, and further, new reduced games corresponding to the Solidarity value as well as ELS values.

Chapter 5 summarizes all the results we got in previous chapters, and gives an example concerning the Shapley value, in which the classical potential approach is extended to the Abelian group structure.

Notation

$\mathbb{N} = \{0, 1, 2, \ldots\}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$N = \{1, 2, \ldots, n\}$	the player set
(N, v)	the cooperative game with player set N
(N, D, u)	the noncooperative game with player set N
$S(S \subseteq N)$	the subset of N
$s, \|S\|$	the cardinality of the set S
$\mathcal{G}^N$	the space of cooperative games with player set N
$\mathcal{G}$	the universe of all game spaces
$\mathcal{G}^{N,D}$	the space of cooperative games with action choices
$\mathbb{R}^N$	the vector space with coordinates indexed by N
$\mathbb{R}^n$	the n-dimensional vector space
π	the permutation of N
$v = (v_S)_{S \subseteq N}$	the worth vector of cooperative games
$(v_d)_{d \in D}$	the worth vector of cooperative games with action choices
$u = (u_i)_{i \in N}$	the worth vector of noncooperative games
$D = D_1 \times \ldots \times D_n$	the strategy space of noncooperative games
P	the classical potential
P'	the modified potential
∇P	the classical gradient
$\nabla' P'$	the modified gradient
$Sh = (Sh_i)_{i \in N}$	the Shapley value of cooperative games

$ENSC = (ENSC_i)_{i \in N}$	the egalitarian non-separable contribution value of cooperative games
$Sol = (Sol_i)_{i \in N}$	the Solidarity value of cooperative games
$\Phi = (\Phi_i)_{i \in N}$	the ELS value of cooperative games
V	the potential function of potential games

Contents

1 Introduction **1**

1.1 Game theory . . . 1

1.2 Cooperative games . . . 2

1.2.1 Preliminaries . . . 2

1.2.2 Solution concepts . . . 3

1.2.3 Properties of solutions . . . 7

1.2.4 ELS values . . . 8

1.3 Noncooperative games . . . 11

1.4 Overview . . . 12

2 Modified potential representation **14**

2.1 Classical potential representation . . . 14

2.2 Modified potential representation . . . 16

2.2.1 General representation . . . 16

2.2.2 Representation for ELS values . . . 20

3 Representations of the potential game **24**

3.1 New basis associated to values . . . 24

3.1.1 Shapley value . . . 24

3.1.2 Solidarity value . . . 26

3.1.3 ELS values . . . 28

3.2 Modified potential representation on new basis . . . 33

3.3 ELS values representation of Potential games . . . 36

3.3.1 Potential games . . . 36

3.3.2 Representation of Potential games . . . 38

4 Reduced game property for ELS values **41**

4.1 Reduced game property . . . 41

4.2 RGP by the modified potential approach . . . 44

4.2.1 Shapley value . . . 45

4.2.2 Solidarity value . . . 46

4.2.3 ELS values . . . 50

5 Conclusions **56**
5.1 Overall results 56
5.2 Extension to the group structure 56

Bibliography **59**

Index **61**

Acknowledgements **63**

Chapter 1

Introduction

In this chapter, we introduce the basic concepts in the field of game theory, especially in the cooperative game theory.

In the cooperative part, some special game concepts, for instance, unanimity game, monotone game, and convex game, are presented. Concerning the solution part, a list of famous solution concepts is given, including core, kernel, stable set and nucleolus. After introducing the well-known single-valued solution called Shapley value, we discuss some frequently-used properties of solutions. For a special class of values satisfying efficiency, linearity and symmetry, we study its equivalent forms, and write the Solidarity value in agreement with the latter form.

In the noncooperative part, the concept of n-person game's normal form and the famous Nash equilibrium are introduced. Particularly, for the mixed extension of any finite bimatrix game, we present the Equilibrium point theorem and the Minimax theorem after defining the common value of such games and optimal strategy sets of both players.

1.1 Game theory

Game theory is a formally mathematical field which studies situations of competition and/or cooperation among involved parties. It provides general mathematical techniques for analyzing situations in which two or more agents make decisions that will influence one another's benefit. Game theory is a very dynamic and expanding field with large number of applications, which ranges from strategic questions in warfare to analogizing economic competition and cooperation, from fair distribution in social problems to behavior of animals in competitive situations, and certainly much more other aspects closely related to our common life.

Game theoretical approaches are classified into two branches: cooperative and non-cooperative game theory. The usual distinction between these two classes is, players can form coalitions and make binding agreements on how to distribute the payoffs of these coalitions in cooperative games, whereas in noncooperative games players have explicit strategies and can not make binding agreements. This distinction is not sharp in some case however, for instance, the theory of implementation is concerned with representing outcomes from cooperative solutions as equilibrium outcomes of specific noncooperative solutions.

1.2 Cooperative games

Cooperative games describe what each possible coalition can earn by cooperation. It is more abstract than noncooperative games, because strategies are not explicitly modeled.

1.2.1 Preliminaries

In the cooperative game with transferable utility (TU-game), the earning of a coalition can be expressed by a single number, whereas in the cooperative game with nontransferable utility (NTU-game), the possibilities from cooperation for each coalition are described by a set. More generally, the single number in the TU-game is an amount of money and the implicit assumption is that, it makes sense to transfer this utility among the players.

Definition 1.2.1. *A **cooperative game with transferable utility** or **TU-game** is an ordered pair (N, v), where N is a finite set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$.*

In the following chapters, the cooperative game we considered is always with transferable utility, i.e., TU-game.

Definition 1.2.2. *A nonempty subset S of N, is called a **coalition**, and the associated real number $v(S)$ is called the **worth** of coalition S.*

The size of coalition S is denoted by $|S|$ or, if no ambiguity arises, by s. Particularly, n denotes the size of the player set N. The coalition N is called the *grand coalition*. Given a transferable utility game (N, v) and a coalition S, we denote by (S, v) the *subgame* obtained by restricting v to subsets of S only.

Example 1.2.3. (Glove game)[19] *Let $N = \{1, 2, \ldots, n\}$ be divided into two disjoint subsets L and R. Members of L possess a left hand glove, members of R a right hand glove. A single glove is worth nothing, a right-left pair of gloves has value of one euro. This situation can be modeled as a TU-game (N, v), where for each $S \in 2^N$, we have $v(S) := min\{|L \cap S|, |R \cap S|\}$, and particularly $v(N) := min\{|L|, |R|\}$.*

Let $\mathcal{G}^N$ denote the set of all cooperative TU-games with player set N. Then the set $\mathcal{G}^N$ of characteristic functions of coalitional games forms a $(2^n - 1)$-dimensional linear space, with the usual operations of addition and scalar multiplication of functions. Denote by $\mathcal{G}$ the universe of all game spaces.

Definition 1.2.4. *A basis of space $\mathcal{G}^N$ is supplied by the collection of the* ***unanimity game*** $\{(N, u_T) \mid T \in 2^N \setminus \{\emptyset\}\}$, *that are defined by,*

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

The interpretation of the unanimity game u_T is that a unitary gain (or cost savings) can be achieved if and only if all players in coalition T are involved in cooperation.

Theorem 1.2.5. *For each $v \in \mathcal{G}^N$, we have,*

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^v u_T \quad \text{with} \quad c_T^v = \sum_{R \subseteq T} (-1)^{t-r} v(R)$$

If $v(S) \in \{0, 1\}$ for all $S \subseteq N$, and $v(\emptyset) = 0$, $v(N) = 1$, then game (N, v) is *simple*. Note that the unanimity game u_T, $T \in 2^N \setminus \{\emptyset\}$ is a special simple game. We call a game (N, v) *monotone* if $v(S) \leq v(T)$, for all $S, T \subseteq N$ and $S \subseteq T$. Game (N, v) is called *zero-normalized* if for all $i \in N$, $v(\{i\}) = 0$. A *superadditive* game (N, v) satisfies $v(S) + v(T) \leq v(S \cup T)$, for all $S, T \subseteq N$ and $S \cap T = \emptyset$. Particularly, it is called *inessential* or *additive* if the equality holds, or equivalently, $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. A game is called *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, for all $S, T \subseteq N$. Conversely, (N, v) is a *concave* game, if and only if $(N, -v)$ is convex.

Consider a game (N, v), for any $i \in N$, the *utopia vector* $\mu(v) = (\mu_i(v))_{i \in N} \in \mathbb{R}^N$ is defined by $\mu_i(v) := v(N) - v(N \setminus \{i\})$, and the vector $a(v) = (a_i(v))_{i \in N} \in \mathbb{R}^N$ satisfying $a_i(v) = max_{S \ni i} \left[v(S) - \sum_{j \in S \setminus \{i\}} \mu_j(v) \right]$ is called the *minimal right vector*. A game (N, v) is *quasi-balanced* if for all $i \in N$, $a_i(v) \leq \mu_i(v)$ and $\sum_{i \in N} a_i(v) \leq v(N) \leq \sum_{i \in N} \mu_i(v)$.

Definition 1.2.6. *Let $v \in \mathcal{G}^N$, for each $i \in N$ and $S \subseteq N$, the* ***marginal contribution*** *of player i to the coalition S is,*

$$\begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S \end{cases}$$

1.2.2 Solution concepts

Two main problems appear when researching on a game, one is which coalitions can form, the other one is how to distribute payoffs among all players. In cooperative games, one only need to consider the second problem, because players will always choose to join the grand coalition due to superadditivity.

Definition 1.2.7. *A* ***value*** $\psi = (\psi_i)_{i \in N}$ *is a mapping which assigns to every cooperative game $v \in \mathcal{G}^N$ exactly one element $\psi(N, v) \in \mathbb{R}^n$.*

Thus, a value ψ that associated a payoff vector $\psi(v) = (\psi_i(v))_{i \in N} \in \mathbb{R}^n$ with every game (N, v), assigns a payoff profile to every cooperative game. The value $\psi_i(v)$ of player

i in the game (N, v) represents an assessment by i of his gains from participating in the game. For a payoff vector $x \in \mathbb{R}^n$ and a coalition $S \subseteq N$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff to the members of the coalition S. Note that only payoff vectors $x \in \mathbb{R}^n$ satisfying $\sum_{i \in N} x_i \leq v(N)$ are reachable in the game (N, v).

Consider a game (N, v), a payoff vector x on $\mathbb{R}^n$ is said to be *efficient* if, for all $v \in \mathcal{G}^N$,

$$\sum_{i \in N} x_i = v(N)$$

The *preimputation set* $I^*(v)$ denotes the set of all efficient payoff vectors in the game (N, v), i.e.,

$$I^*(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \right\}$$

Definition 1.2.8. *A payoff vector $x \in \mathbb{R}^n$ is in the* ***imputation set*** *$I(v)$ for the game (N, v) if it is efficient and individually rational, i.e.,*

$$I(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{j \in N} x_j = v(N) \quad \textit{and} \quad x_i \geq v(\{i\}) \quad \textit{for all } i \in N \right\}$$

Note that if the proposed allocation $x \in I(v)$ is such that there is at least one player $i \in N$ whose payoff x_i satisfies $x_i < v(\{i\})$, the grand coalition would never form, because such a player would prefer not to cooperate since acting on his own can obtain more. Hence, the *individual rationality* condition should hold, i.e., for all $i \in N$, $x_i \geq v(\{i\})$.

If the criterion of individual rationality is strengthened by demanding not only one player, but every coalition $S \subseteq N$, which means the coalition S should receive at least the worth it can obtain by operating on his own, i.e., $\sum_{i \in S} x_i \geq v(S)$, then we have got another set-valued solution concept, which can be depicted as a multifunction from $\mathcal{G}^N$ to $\mathbb{R}^n$.

Definition 1.2.9. *The* ***core*** [9] *$C(v)$ of a game (N, v) is the set,*

$$C(v) := \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \quad \textit{for all} \quad S \subseteq N \right\}$$

Although the core is empty in some cases, it is still one of the most important set-valued solutions. If $C(v) \neq \emptyset$, then elements of $C(v)$ can be easily obtained, because the core is defined by a finite system of linear inequalities.

For a game (N, v), let x, y be two imputations, then x *dominates* y if there exists a coalition $S \subseteq N$, $S \neq \emptyset$, such that $\sum_{j \in S} x_j \leq v(S)$ and $x_i > y_i$ for all $i \in S$. In other words, players in S strictly prefer the payoff from x to those from y, and they can threaten to leave the grand coalition if y is used, because the payoff they obtain on their own is at least as large as the allocation they receive under x.

Definition 1.2.10. *Given a game $v \in \mathcal{G}^N$, a set $A(v) \subseteq I(v)$ is a* ***stable set*** [28] *for the game v if it satisfies,*

- *If $x, y \in A(v)$, then not x dominates y.*
- *If $x \notin A(v)$, then there is some $y \in A(v)$ such that y dominates x.*

Thus, a stable set satisfies the two conditions of *internal stability* (no payoff vector in $A(v)$ is dominated by another) and *external stability* (any payoff vector outside $A(v)$ is dominated by at least one vector in $A(v)$).

Consider a game (N, v), let $x \in \mathbb{R}^n$ be an efficient payoff vector, then the *excess* of S with respect to x in the game v is $e^v(S, x) := v(S) - \sum_{j \in S} x_j$. Note that the negative(positive) excess $e^v(S, x)$ can be regarded as a measure of the (dis)satisfaction by coalition S if payoff vector x was suggested as the final payoff. The greater $e^v(S, x)$, the more ill-treated S would feel.

Definition 1.2.11. *Let $s^v_{ij}(x)$ denote the* ***maximum surplus*** *of player i over player j different from i with respect to x in the game $v \in \mathcal{G}^N$, i.e.,*

$$s^v_{ij}(x) := max \left\{ v(S) - \sum_{k \in S} x_k \mid S \subseteq N \setminus \{j\}, S \ni i \right\}$$

This surplus at the payoff vector x is the maximal amount player i can gain (or lose, if negative) without player j by withdrawing under the payoff vector x, assuming that the other players of the formed coalition are satisfied with their payoffs under x.

Definition 1.2.12. *The* ***kernel*** [4] *$\chi(v)$ of a game (N, v) is the set of imputations x, such that, for every pair of players i, j,*

$$\begin{cases} \left(s^v_{ij}(x) - s^v_{ji}(x)\right)(x_j - v(\{j\})) \leq 0 \\ \left(s^v_{ji}(x) - s^v_{ij}(x)\right)(x_i - v(\{i\})) \leq 0 \end{cases}$$

The kernel is defined as the set of all imputations where no player has his bargaining power over another. If $s^v_{ij}(x) > s^v_{ji}(x)$, player i has more bargaining power than j with respect to x, but player j is immune to i's threats if $x_j = v(\{j\})$, because he can obtain this payoff on his own.

For a game (N, v) and a payoff vector $x \in \mathbb{R}^n$, let $\theta(x)$ be the 2^n-tuple, whose components are the excesses $e^v(S, x)$, $S \subseteq N$, arranged in nonincreasing order, i.e.,

$$\theta_i(x) \geq \theta_j(x) \quad \text{whenever } 1 \leq i \leq j \leq 2^n$$

Consider two payoff vectors x, y, we say $\theta(x)$ is *lexicographically* smaller that $\theta(y)$, if there exists an integer $1 \leq k \leq 2^n$, such that $\theta_i(x) = \theta_i(y)$ for $1 \leq i < k$, and $\theta_k(x) < \theta_k(y)$.

Definition 1.2.13. *For a game (N, v), the* ***nucleolus*** [21] *is the lexicographically minimal imputation, based on this ordering.*

Note that the nucleolus is always unique. When the core is non-empty, the nucleolus is in the core, and it is always in the kernel.

One of the most well-known single-valued solution concepts is the Shapley value, which is quite different from the core, because it assigns a unique payoff distribution to the players in the the grand coalition to every TU-game.

Definition 1.2.14. *For any $v \in \mathcal{G}^N$, the **Shapley value** [22] $Sh = (Sh_i)_{i\in N}$ is given by,*

$$Sh_i(N,v) := \sum_{S\subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}\left[v(S\cup\{i\}) - v(S)\right] \quad \text{for all } i \in N \tag{1.2}$$

In the following chapters, we will always let $h(n,s) := \frac{(s-1)!(n-s)!}{n!}$, in order to simplify the expression. For instance, the Shapley value of player i in the game v can be represented as,

$$Sh_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\left[v(S\cup\{i\}) - v(S)\right]$$

Definition 1.2.15. *Let (N,v) be a game. For each coalition T, the **dividend** $\triangle_v(T)$ [10] is defined, recursively, as follows,*

$$\begin{cases} \triangle_v(\emptyset) := 0 \\ \triangle_v(T) := v(T) - \sum_{S\subsetneq T} \triangle_v(S) \quad \text{if } |T| \geq 1 \end{cases}$$

The relationship between the Shapley value and dividends is that, the Shapley value of player i equals to the sum of all equally distributed dividends of coalitions to which player i belongs.

Theorem 1.2.16. [10] *Let $v = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T^v u_T$ as in (1.1). Then, it holds,*

- $\triangle_v(T) = c_T^v$ *for all* $T \neq \emptyset$
- $Sh_i(N,v) = \sum_{\substack{T\subseteq N\\ T\ni i}} \frac{\triangle_v(T)}{|T|}$ *for all* $i \in N$

Definition 1.2.17. *For a game (N,v), the **egalitarian non-separable contribution value** $ENSC = (ENSC_i)_{i\in N}$ [13] is defined by,*

$$ENSC_i(N,v) := \mu_i(v) + \frac{1}{n}\left[v(N) - \sum_{j\in N}\mu_j(v)\right] \quad \text{for all } i \in N \tag{1.3}$$

The $ENSC$ value arises from both the separable contribution principle, applied to the the game itself, and the egalitarianism. That is, the resulting allocation is determined by the egalitarian division of what is left of the total savings $v(N)$ after any player i is conceded to get his separable contribution $\mu_i(v)$.

Definition 1.2.18. *For a game* (N, v)*, the* ***Solidarity value*** [15] $Sol = (Sol_i)_{i\in N}$ *is defined by,*

$$Sol_i(N,v) = \sum_{\substack{S\subseteq N\\ S\ni i}} \frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{s}\sum_{j\in S}[v(S) - v(S\setminus\{j\})] \quad \textit{for all } i \in N \tag{1.4}$$

The Solidarity value is based on the assumption that if a coalition S forms, then players who contribute to S more than the average marginal contribution of a member of S support in some sense their weaker partners in S. Sometimes, it happens that the Solidarity value belongs to the core of a game while the Shapley value does not.

1.2.3 Properties of solutions

After Shapley [22] introduced the Shapley value in 1953, the axiomatization approach to values has been widely used. The purpose is to pose a minimal list of desirable properties that fully characterize the solution. Let ψ be a value on $\mathcal{G}^N$, we mention now some desirable properties for single-valued solution concepts. Extensions of these properties to set-valued solution concepts are straightforward.

- **Individual rationality**: $\psi_i(v) \geq v(\{i\})$, for all $v \in \mathcal{G}^N$, $i \in N$.
- **Efficiency**: $\sum_{j\in N}\psi_j(v) = v(N)$, for all $v \in \mathcal{G}^N$.
- **Additivity**: $\psi(v) + \psi(w) = \psi(v + w)$, for all v, $w \in \mathcal{G}^N$.
- **Linearity**: $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$, for all α, $\beta \in \mathbb{R}$, v, $w \in \mathcal{G}^N$.
- **Symmetry**: $\psi_{\pi(i)}(\pi v) = \psi_i(v)$, for all $i \in N$, $v \in \mathcal{G}^N$, and every permutation π on N. The game $(N, \pi v)$ is given by $(\pi v)(S) = v(\pi^{-1}(S))$ for all $S \subseteq N$.
- **Substitution property**: $\psi_i(v) = \psi_j(v)$, for substitutes i and j in any game (N, v). Players i and j are called **substitutes** if both of them are more desirable, or equivalently, the equality for their marginal contributions $v(S \cup \{i\}) = v(S \cup \{j\})$, for all $S \subseteq N \setminus \{i, j\}$.
- **Dummy player property**: $\psi_i(v) = v(\{i\})$, for all $v \in \mathcal{G}^N$ and for all **dummy players** i in (N, v), i.e., players $i \in N$ such that $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$.
- **Null player property**: $\psi_i(v) = 0$, for all $v \in \mathcal{G}^N$ and for all **null players** i in (N, v), i.e., players $i \in N$ such that $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$.
- **A-null player property**: $\psi_i(v) = 0$, for all $v \in \mathcal{G}^N$ and for all **A-null players** i in (N, v), i.e., for every coalition T containing i, $A^v(T) = \frac{1}{t}\sum_{j\in T}(v(T) - v(T\setminus\{j\})) = 0$.
- **Covariance (strategic equivalence)**: $\psi(\alpha v + \beta) = \alpha\psi(v) + \beta$, for all $v \in \mathcal{G}^N$, $\alpha > 0$, and $\beta \in \mathbb{R}^n$. Here the game $(N, \alpha v + \beta)$ is given by $(\alpha v + \beta)(S) := \alpha v(S) + \sum_{j\in S}\beta_j$ for all $S \subseteq N$, $S \neq \emptyset$.

- **Continuity**: if for every pointwise convergent sequence of games $\{(N, v_k)\}_{k=0}^{\infty}$, the limit of which is the game $(N, \overline{v})$, the corresponding sequence of values $\{\psi(v_k)\}_{k=0}^{\infty}$ converges to the value $\psi(\overline{v})$.

The first axiomatization concerning the Shapley value given by L.S Shapley [22] states that, a solution satisfies additivity, symmetry, efficiency and the dummy player property if and only if it is the Shapley value. Afterwards, there appeared many other axiomatization methods using different lists of properties for the Shapley value. For other solution concepts, there also exists various kinds of axiomatizations. For instance, the Solidarity value [15] is characterized by additivity, symmetry, efficiency and A-null player property.
Remark: Note that, both the Shapley value and the Solidarity value are efficient, linear and symmetric, because a value satisfies additivity if and only if it is linear. Obviously, additivity can be deduced from linearity. Next we show the other direction is also true. Let ψ be a value of game $v \in \mathcal{G}^N$ that satisfies additivity, then $\psi(-v) + \psi(v) = 0$, that is, $\psi(-v) = -\psi(v)$. Thus, for every $n \in \mathbb{Z}$,

$$\psi(nv) = n\psi(v)$$

Furthermore, for every $q \in \mathbb{Z}$, $q \neq 0$,

$$\psi(v) = \psi(q\frac{1}{q}v) = q\psi(\frac{1}{q}v)$$

Therefore,

$$\psi(\frac{1}{q}v) = \frac{1}{q}\psi(v)$$

Consider an arbitrary rational number $\frac{p}{q}$, then for every $v \in \mathcal{G}^N$,

$$\psi(\frac{p}{q}v) = p\psi(\frac{1}{q}v) = \frac{p}{q}\psi(v)$$

Since rational numbers are dense in real numbers and ψ is continuous, we have $\psi(kv) = k\psi(v)$ for every $k \in \mathbb{R}$. Therefore ψ is linear.

1.2.4 ELS values

For the class of values satisfying efficiency, linearity and symmetry, in which the Shapley value and the Solidarity value are two special cases, there exists an explicit formula.

Theorem 1.2.19. [20] *A value $\Phi : \mathcal{G}^N \to \mathbb{R}^n$ verifies efficiency, linearity and symmetry if and only if there exists $\rho_s (s = 1, 2, \ldots, n-1)$ such that, for any $v \in \mathcal{G}^N$,*

$$\Phi_i(v) := \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N \\ S \ni i}} \rho_s \frac{v(S)}{s} - \sum_{\substack{S \subseteq N \\ S \not\ni i}} \rho_s \frac{v(S)}{n-s} \quad \textit{for all } i \in N \tag{1.5}$$

Proof. It can be checked easily that a value defined by (1.5) satisfies the three properties. Next we show the other direction holds.

Suppose a value Φ on $\mathcal{G}^N$ satisfies efficiency, linearity and symmetry. Consider a basis of the game space $\mathcal{G}^N$, which is a collection of games $\{(N, e_T) \mid T \in 2^N \setminus \{\emptyset\}\}$, defined by,

$$e_T(S) = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

By linearity, for any $v \in \mathcal{G}^N$, all $i \in N$,

$$\Phi_i(v) = \Phi_i\left(\sum_{\substack{T \subseteq N \\ T \neq \emptyset}} v(T) e_T\right) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} v(T)\Phi_i(e_T) \tag{1.6}$$

By symmetry, for any $T \subsetneq N$, there exist some λ_T and τ_T, such that,

$$\Phi_i(e_T) = \begin{cases} \lambda_T & \text{if } i \in T \\ \tau_T & \text{if } i \notin T \end{cases}$$

Note that, λ_T and τ_T only depend on the size t. In fact, for λ_T, consider a coalition $T \subsetneq N$, two players $i, j \notin T$, then i and j are substitutes in the game $(N, e_{T\cup\{i\}} + e_{T\cup\{j\}})$. By symmetry, we have $\Phi_i(e_{T\cup\{i\}} + e_{T\cup\{j\}}) = \Phi_j(e_{T\cup\{i\}} + e_{T\cup\{j\}})$, which means, $\lambda_{T\cup\{i\}} = \lambda_{T\cup\{j\}}$. Therefore, we can use λ_t instead of λ_T, so do τ_T.

By efficiency, for $T = N$, we have $\Phi_i(e_N) = \frac{1}{n}$ for any $i \in N$; for $T \subsetneq N$, it holds $t\lambda_t + (n-t)\tau_t = 0$, therefore by (1.6), for all $i \in N$,

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{T \subsetneq N \\ T \ni i}} \lambda_t v(T) - \sum_{\substack{T \subseteq N \\ T \not\ni i}} \frac{t}{n-t}\lambda_t v(T)$$

Let $\rho_t := t\lambda_t$, then we can obtain (1.5). □

We denote by ELS the class of values satisfying these three properties. Consider another coefficient $\alpha_{n,s}$ instead of ρ_s, there exists another equivalent form for this class of values.

Corollary 1.2.20. [7] *A value $\Phi : \mathcal{G}^N \to \mathbb{R}^n$ verifies efficiency, linearity and symmetry if and only if there exists a fixed collection of constants $A_n = \{\alpha_{n,s} | 1 \leq s \leq n-1\}$ such that, for any $v \in \mathcal{G}^N$,*

$$\Phi_i(v) := \frac{v(N)}{n} + \sum_{S \subsetneq N\setminus\{i\}} h(n, s+1)\alpha_{n,s+1} v(S\cup\{i\}) - \sum_{S \subseteq N\setminus\{i\}} h(n, s+1)\alpha_{n,s} v(S) \quad \text{for all } i \in N \tag{1.7}$$

Remark: For all $1 \leq s \leq n-1$, let $\rho_s := sh(n,s)\alpha_{n,s}$ in (1.5), then we can obtain (1.7) directly.

Example 1.2.21. *Remind the Solidarity value (1.4), which belongs to the ELS class. We can rewrite it as follows,*

$$
\begin{aligned}
& Sol_i(N,v) \\
=& \sum_{\substack{S\subseteq N\\ S\ni i}} h(n,s)\cdot \frac{1}{s}\sum_{j\in S}[v(S)-v(S\setminus\{j\})] \\
=& \sum_{\substack{S\subseteq N\\ S\ni i}} h(n,s)v(S) - \sum_{\substack{S\subseteq N\\ S\ni i}} h(n,s)\cdot\frac{1}{s}\left[v(S\setminus\{i\}) + \sum_{j\in S\setminus\{i\}} v(S\setminus\{j\})\right] \\
=& \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)v(S\cup\{i\}) - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} - \sum_{\substack{S\subseteq N\\ S\ni i}} h(n,s)\sum_{j\in S\setminus\{i\}}\frac{v(S\setminus\{j\})}{s}
\end{aligned}
\tag{1.8}
$$

Consider the last part in (1.8), by changing the order of summations, we have,

$$
\begin{aligned}
\sum_{\substack{S\subseteq N\\ S\ni i}} h(n,s)\sum_{j\in S\setminus\{i\}}\frac{v(S\setminus\{j\})}{s} &= \sum_{j\in N\setminus\{i\}}\sum_{\substack{S\subseteq N\\ S\ni i,j}} h(n,s)\frac{v(S\setminus\{j\})}{s} \\
&= \sum_{j\in N\setminus\{i\}}\sum_{\substack{S\subseteq N\setminus\{j\}\\ S\ni i}} h(n,s+1)\frac{v(S)}{s+1} \\
&= \sum_{j\in N\setminus\{i\}}\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+2)\frac{v(S\cup\{i\})}{s+2} \\
&= \sum_{S\subsetneq N\setminus\{i\}}\sum_{j\in N\setminus(S\cup\{i\})} h(n,s+2)\frac{v(S\cup\{i\})}{s+2} \\
&= \sum_{S\subsetneq N\setminus\{i\}} (n-s-1)h(n,s+2)\frac{v(S\cup\{i\})}{s+2}
\end{aligned}
$$

Submit the equation above into (1.8), the Solidarity value equals to,

$$
\begin{aligned}
& Sol_i(N,v) \\
=& \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)v(S\cup\{i\}) - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} - \sum_{S\subsetneq N\setminus\{i\}} (n-s-1)h(n,s+2)\frac{v(S\cup\{i\})}{s+2} \\
=& \frac{v(N)}{n} + \sum_{S\subsetneq N\setminus\{i\}}\left[h(n,s+1) - \frac{n-s-1}{s+2}h(n,s+2)\right]v(S\cup\{i\}) - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} \\
=& \frac{v(N)}{n} + \sum_{S\subsetneq N\setminus\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1}
\end{aligned}
\tag{1.9}
$$

In the following chapters, we will always use (1.9) when concerning the Solidarity value.

1.3 Noncooperative games

In the field of noncooperative game theory, players make decisions independently. Although they may be able to cooperate, any cooperation must be self-enforcing. For a noncooperative game, we always consider it in normal form when players independently choose one of their strategies, and payoffs take place depending on the chosen strategies. In such case, each player wants to maximize his own payoff.

Definition 1.3.1. *An n-person game in **normal form** is described by the $2n$-tuple,*

$$(D_1, D_2, \ldots, D_n; u_1, u_2, \ldots, u_n)$$

*where **strategy set** D_i contains the pure strategies of player i, and $u_i : D_1 \times D_2 \times \ldots \times D_n \to \mathbb{R}$ is the **payoff function** of player i.*

Example 1.3.2. (Oligopoly model) [25] *Cournot introduced the model in 1838, which contains n products of mineral water, the costs for producer i at quantity d_i is $c(d_i)$ and the sales price per unit is $B\left(\sum_{j=1}^{n} d_j\right)$. Then the game in normal form is $D_i = [0, \infty)$ and*

$$u_i(d_1, \ldots, d_n) = max\left\{0, d_i B\left(\sum_{j=1}^{n} d_j\right) - c(d_i)\right\}$$

Consider the two-person game in normal form $(D_1, D_2; u_1, u_2)$, then this game is called a *zero-sum game* if $u_2 = -u_1$. When D_1 and D_2 are finite sets, it is a *finite game*. We call it a *bimatrix game* (E, F) with $m \times n$-payoff matrices $E = [e_{ij}]_{m\times n}$ and $F = [f_{ij}]_{m\times n}$ if $D_1 = \{1, 2, \ldots, m\}$, $D_2 = \{1, 2, \ldots, n\}$ (possibly $m, n = \infty$), $u_1(i, j) = e_{ij}$, $u_2(i, j) = f_{ij}$. If $F = -E$ in the bimatrix game, it is a *matrix game* with $m \times n$-payoff matrix E.

In noncooperative games, the most well-known solution concept is the Nash equilibrium, named after John Forbes Nash. We consider it in the two-person game, and the n-person game case can be deduced similarly.

Definition 1.3.3. (d_1^*, d_2^*) *is a **Nash equilibrium** of the game $(D_1, D_2; u_1, u_2)$ if,*

$$u_1(d_1^*, d_2^*) \geq u_1(d_1, d_2^*) \quad \text{for all } d_1 \in D_1$$
$$u_2(d_1^*, d_2^*) \geq u_2(d_1^*, d_2) \quad \text{for all } d_2 \in D_2$$

The Nash equilibrium is a solution concept in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his own strategy unilaterally.

Definition 1.3.4. *The **mixed extension of an $m \times n$-bimatrix game** (E, F) is given by $(S^m, S^n; H_1, H_2)$ where for all $p \in S^m, q \in S^n$,*

$$S^m = \left\{p \in \mathbb{R}^m \mid p \geq 0, \sum_{i=1}^{m} p_i = 1\right\}; \; S^n = \left\{q \in \mathbb{R}^n \mid q \geq 0, \sum_{j=1}^{n} q_j = 1\right\}$$

$$H_1(p, q) = pEq^T = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i e_{ij} q_j; \qquad H_2(p, q) = pFq^T = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i f_{ij} q_j$$

The Nash equilibrium in such bimatrix game can be extended from its common definition. For this mixed strategy, Nash introduced a well-known theorem which can be proved by the L.Brouwer fixed point theorem.

Theorem 1.3.5. (Equilibrium point theorem) [25] *The mixed extension of each finite bimatrix game possesses a Nash equilibrium.*

Definition 1.3.6. *For the mixed extension of a finite matrix game, i.e., $F = -E$, the* upper and lower value *is,*

$$\underline{v}(E) := sup_{p \in S^m} inf_{q \in S^n} pEq^T$$
$$\overline{v}(E) := inf_{q \in S^n} sup_{p \in S^m} pEq^T$$

Definition 1.3.7. *The **value** of the mixed extension of a finite matrix game are $v(E) = \underline{v}(E) = \overline{v}(E)$. The **optimal strategy sets** of the players are*

$$O_1(E) := \left\{p^* \in S^m \mid p^* E e_j^T \geq v(E) \text{ for all } j = 1, \ldots, n\right\}$$
$$O_2(E) := \left\{q^* \in S^n \mid e_i E q^{*T} \leq v(E) \text{ for all } i = 1, \ldots, m\right\}$$

von Neumann introduced the famous Minimax theorem concerning the finite matrix game, and the minimax solution is just the same as the Nash equilibrium in such games.

Theorem 1.3.8. (Minimax theorem) [25] *For each finite matrix game E : $\underline{v}(E) = \overline{v}(E)$, $O_1(E) \neq \emptyset$ and $O_2(E) \neq \emptyset$.*

1.4 Overview

In this section we give an overview of our contributions to the study of game theory. This report makes use of the modified potential approach, to study the Shapley value, the Solidarity value and the class of values satisfying efficiency, linearity and symmetry.

Chapter 1 introduces basic concepts and notations in the field of game theory, especially in the cooperative game theory. A list of set-valued and single-valued solutions, and their properties used in axiomatization are presented. For the noncooperative game, its normal form and the Nash equilibrium are introduced.

In Chapter 2, based on the definition of the potential given by Hart and Mas-Colell [11], who proved the equivalence between the gradient of potential and the Shapley value, we define another type of the potential called modified potential, which is closely related to the weighted pseudo-potential presented by Driessen and Radzik [7]. Consider a value of the cooperative game, it admits a modified potential representation if and only if it belongs to ELS values and satisfys two more conditions concerning the coefficient of such values.

Chapter 3 considers the collection of unanimity games, which is a basis of the game space. We define a new basis with respect to the Shapley value, Solidarity value, and all ELS values, respectively, such that all these values can be represented as a simple sum of the corresponding coordinates. The modified potential of ELS values can be expressed by coordinates of the corresponding new basis, thus its modified potential representation can be verified. In view of these new basis, we define the Solidarity value and ELS values representations of potential games on the basis of the Shapley value representation given by Takashi [27].

In Chapter 4, by the potential approach, we obtain the reduced game corresponding to the Shapley value, which is the same as the Sobolev's reduced game. Make use of the similar modified potential approach, reduced games with respect to the Solidarity value and all ELS values are defined, respectively, such that these values satisfy the reduced game property with respect to their own reduced games.

Finally in Chapter 5 we give an example concerning the Shapley value, in which the classical potential approach is extended to the Abilian group structure.

Chapter 2

Modified potential representation

The concept of potential was once frequently used in Physics, and was brought into the field of mathematics successfully in the late eighties. Based on the definition of potential given by Hart and Mas-Colell [11] who proved the equivalence between the gradient of potential and the Shapley value, we aim to define another type of the potential called the modified potential. According to the 0-normalized condition and efficiency, the modified potential can be calculated based on the definition of the modified gradient. For the ELS value, we find it admits a modified potential representation if it satisfies another two conditions. Furthermore, when a value admits such modified potential representation, there exists special relationships among the coefficients of the modified gradient, which can be simplified if separable.

2.1 Classical potential representation

The potential approach is a successful tool in physics. Daniel Bernoulli (1738) was the first to introduce the idea that a conservative force can be derived by a potential in Hydrodynamics.

An illustrative example is the gravitational vector field, which says that the gravitational force acting on a particle is a function of its position in the space, i.e., $f = f(\overrightarrow{r}) = f(x, y, z)$. The work done by moving a particle continuously from position A to B through the path σ is the integrate of $f(\overrightarrow{r})$ on σ, i.e., $W = \int_\sigma f(\overrightarrow{r})d\overrightarrow{r}$. The gravitational field is *conservative* in the sense that it is *path independent*. But a field is conservative if there exists a continuous differentiable function P, such that $W = -\int_\sigma \nabla P d\overrightarrow{r}$. Therefore, $-\nabla P(\overrightarrow{r}) = f(\overrightarrow{r})$.

There exist several characterizations of conservative vector field, e.g., every contour integral with respect to the vector field is zero. Surprisingly, the successful treatment of the potential in physics turned out to be reproducible, in the late eighties, in cooperative game theory.

Concerning the solution part for TU-games, Hart and Mas-Colell [11] was the first to introduce the potential approach to values on cooperative TU-games. They proved that the Shapley value [22] can result as the vector of marginal contributions of a particular potential, the uniqueness of which is implied by the efficiency. Nevertheless, Dubey et al. [8] showed that semivalues, which does not satisfy the efficiency, also can be obtained by an associated potential. For the class of values which satisfies efficiency, linearity and symmetry, Driessen and Radzik [7] proved such value admit a pseudo-potential representation. Ortmann [16] clarified several analogues between the potential concepts in the cooperative game theory (without the efficiency constraint) and physics. In addition, Calvo and Santos [3] established that, any value that admits a potential representation is equivalent to the Shapley value in the sense that the value of any games is equal to the Shapley value of a strongly adapted game arising from initial games as well as the value itself.

In summary, the concept of the potential is a powerful tool within the solution part for TU-games.

Definition 2.1.1. *A* ***potential*** [11] *is a function* $P : \mathcal{G} \to \mathbb{R}$, $P(\emptyset, v) = 0$, *satisfying, for every* $v \in \mathcal{G}$,

$$\sum_{i \in N} \nabla_i P(N, v) = v(N) \tag{2.1}$$

Here the i's component of the gradient $\nabla_i P(N, v) := P(N, v) - P(N \setminus \{i\}, v)$ with v the restriction to $N \setminus \{i\}$ in the last expression.

Definition 2.1.2. *A value* ψ *is said to possess a* ***potential representation*** *if it is the discrete gradient of a real-valued potential* P, *i.e.* $\psi = \nabla P$.

If P is a potential then (2.1) says that the 'gradient' $\nabla P(N, v) := (\nabla_i P(N, v))_{i \in N}$ is an efficient payoff vector for the game (N, v), and in particular, note that,

$$P(\{i\}, v) = v(\{i\})$$
$$P(\{i, j\}, v) = \frac{1}{2}[v(\{i, j\}) + v(\{i\}) + v(\{j\})]$$

More generally, it follows from (2.1) that,

$$P(N, v) = \frac{1}{n}\left[v(N) + \sum_{i \in N} P(N \setminus \{i\}, v)\right] \tag{2.2}$$

so the potential of game (N, v) is uniquely determined by the potential of subgames of (N, v).

Example 2.1.3. *Hart and Mas-Colell* [11] *showed that, there exists a unique potential* $P : \mathcal{G} \to \mathbb{R}$, *such that, for every game* (N, v), *the resulting payoff vector* $(\nabla_i P(N, v))_{i \in N}$ *coincides with the Shapley value (1.2) of the game. Moreover, the potential of any game* (N, v) *is uniquely determined by efficiency applied only to the game and its subgames (i.e., to* (S, v) *for all* $S \subseteq N$*).*

This classical potential P *is given by, for any* $v \in \mathcal{G}$,

$$P(N, v) = \sum_{S \subseteq N} h(n, s) v(S) \tag{2.3}$$

Using (2.2), one can prove the existence and uniqueness of $P(N,v)$ recursively starting with $P(\emptyset,v)=0$.

By the definition of the gradient, for all $i \in N$,

$$\begin{aligned}
\nabla_i P(N,v) &= P(N,v) - P(N\setminus\{i\},v) \\
&= \sum_{S\subseteq N} h(n,s)v(S) - \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)v(S) \\
&= \sum_{S\subseteq N\setminus\{i\}} [h(n,s+1)v(S\cup\{i\}) + h(n,s)v(S)] - \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)v(S) \\
&= \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)v(S\cup\{i\}) - \sum_{S\subseteq N\setminus\{i\}} [h(n-1,s)-h(n,s)]v(S) \\
&= \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)[v(S\cup\{i\}) - v(S)] \\
&= Sh_i(N,v)
\end{aligned}$$

Therefore, the Shapley value is the unique value that admits a classical potential representation.

Example 2.1.4. *For any game (N,v), consider another value also satisfying the efficiency principle, the so-called egalitarian non-separable contribution value $ENSC$ (1.3).*

If we simply let $P(N,v)=v(N)$, then the $ENSC$ value can be represented as, for all $i \in N$,

$$ENSC_i(N,v) = \frac{1}{n}P(N,v) - P(N\setminus\{i\},v) + \frac{1}{n}\sum_{l\in N} P(N\setminus\{l\},v)$$

Note that, the classical gradient with respect to the Shapley value is a linear combination which contains two different items, $P(N,v)$ and $P(N\setminus\{i\},v)$. Whereas in the $ENSC$ value, we can represent it by a linear combination of three items, one of which is different from that of Shapley value, say $\frac{1}{n}\sum_{l\in N} P(N\setminus\{l\},v)$.

2.2 Modified potential representation

Driessen and Radzik [7] introduced a special definition called weighted pseudo-potential representation, with respect to a specific kind of values. Based on their results, and the potential representation for Shapley value as well as $ENSC$ value discussed in the previous section, we aim to give a general potential representation for values on TU-games.

2.2.1 General representation

In this new potential representation, we use three items to prescribe a player's gain from participating in a game (N,v). Firstly, a player receives some share of the solution $P'(N,v)$ from the game; secondly, players different from i will contribute some efforts to the game according to $P'(N,v)$, so we remove what other players would gain. In this way, we adopt

$(-P'(N \setminus \{i\}, v))$ as well as the average sum $(-\frac{1}{n}\sum_{l\in N} P'(N \setminus \{l\}, v))$, and distinguish each part by taking into account different shares assuming symmetry with respect to the size of the player set. Thus, we have the following definition.

Definition 2.2.1. *Consider three sequences $a \equiv (a_k)_{k\in\mathbb{N}}$, $b \equiv (b_k)_{k\in\mathbb{N}}$, $c \equiv (c_k)_{k\in\mathbb{N}}$ of real numbers and a function $P' : \mathcal{G} \to \mathbb{R}$, $P'(\emptyset, v) = 0$.*

- *The* ***modified gradient*** *$\nabla' P' = (\nabla'_i P')_{i\in N}$ is defined to be, for any n-person game (N, v), $i \in N$,*

$$\nabla'_i P'(N, v) = a_n P'(N, v) - b_n P'(N \setminus \{i\}, v) - \frac{1}{n} c_n \sum_{l\in N} P'(N \setminus \{l\}, v) \tag{2.4}$$

- *Function $P' : \mathcal{G} \to \mathbb{R}$ is called a* ***modified potential****, if it satisfies, for any n-person game (N, v),*

$$\sum_{i\in N} \nabla'_i P'(N, v) = v(N) \tag{2.5}$$

This modified potential P' satisfies the 0-normalized condition and its modified gradient is efficient according to (2.5), which is similar to the potential defined by Hart and Mas-Colell [11]. The only difference is that, we use the modified gradient $\nabla' P'$ instead of ∇P in the efficiency condition (2.1). Evidently, if $a_n = b_n = 1, c_n = 0$, the modified gradient would equal to the classical one, i.e., $\nabla P = \nabla' P'$.

Remark: Consider the 1-person game, the efficiency condition (2.5) would reduce to $a_1 P'(N, v) = v(N)$. In order to achieve $P'(N, v) = v(N)$ in 1-person game, we restrict that $a_1 = 1$. Concerning such coefficient, in the following chapters, we will always respect $a_1 = 1$.

Definition 2.2.2. *A value $\psi = (\psi_i)_{i\in N}$ on $\mathcal{G}$ has a* ***modified potential representation****, if there exists three sequences of real numbers $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ with $a_1 = 1$ and $a_k \neq 0$ for all $k > 1$, and a modified potential $P' : \mathcal{G} \to \mathbb{R}$, such that, its associated modified gradient satisfies, for any n-person game (N, v), all $i \in N$,*

$$\psi_i(N, v) = \nabla'_i P'(N, v) = a_n P'(N, v) - b_n P'(N \setminus \{i\}, v) - \frac{1}{n} c_n \sum_{l\in N} P'(N \setminus \{l\}, v) \tag{2.6}$$

In this definition, we restrict $a_k \neq 0$ for all $k > 1$, to make sure that the totalitarian fraction of the potential $P'(N, v)$ for any n-person game (N, v) does not vanish. Therefore we can guarantee the uniqueness of the modified potential P'.

Remind in Example 2.1.3, there exists a unique classical potential P with respect to the Shapley value. In fact, based on the efficiency principle (2.5) and the modified potential representation (2.6), there exists a general expression for the modified potential which can be calculated recursively.

Theorem 2.2.3. *Fix three sequences of real numbers $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ with $a_1 = 1$ and $a_k \neq 0$ for all $k > 1$.*

- *For any $v \in \mathcal{G}^N$, the modified potential P' is uniquely given by,*

$$P'(N,v) = \sum_{S \subseteq N} h(n,s) q_{n,s} v(S) \tag{2.7}$$

- *For any $v \in \mathcal{G}^N$ with $n > 1$, the associated modified gradient $\nabla' P'$ is, for all $i \in N$,*

$$\nabla'_i P'(N,v) = \frac{v(N)}{n} + b_n \sum_{S \subsetneq N \setminus \{i\}} h(n,s+1) q_{n-1,s+1} v(S \cup \{i\}) - b_n \sum_{S \subseteq N \setminus \{i\}} h(n,s+1) q_{n-1,s} v(S) \tag{2.8}$$

where $q_{n,s}$ is defined to be,

$$\begin{cases} q_{n,s} = \dfrac{\prod_{j=s+1}^{n} (b_j + c_j)}{\prod_{j=s}^{n} a_j} & \text{for all } 1 \le s \le n-1 \\ q_{n,n} = \dfrac{1}{a_n} \end{cases} \tag{2.9}$$

Remark: During the proof, we will use the following relationship, for all $1 \le s \le n-1$,

$$q_{n,s} = q_{n-1,s} \cdot q_{n,n} \cdot (b_n + c_n) \tag{2.10}$$

$$q_{n,s} = q_{n,s+1} \cdot q_{s,s} \cdot (b_{s+1} + c_{s+1}) \tag{2.11}$$

Proof. Fix three sequences of real numbers $a = (a_k)_{k \in \mathbb{N}}, b = (b_k)_{k \in \mathbb{N}}, c = (c_k)_{k \in \mathbb{N}}$ with $a_1 = 1$ and $a_k \neq 0$ for all $k > 1$. For any game (N, v), we have,

$$\begin{aligned} v(N) &\overset{(2.5)}{=} \sum_{i \in N} \nabla'_i P'(N,v) \\ &\overset{(2.4)}{=} \sum_{i \in N} \left[a_n P'(N,v) - b_n P'(N \setminus \{i\}, v) - \frac{1}{n} c_n \sum_{l \in N} P'(N \setminus \{l\}, v) \right] \\ &= n a_n P'(N,v) - (b_n + c_n) \sum_{l \in N} P'(N \setminus \{l\}, v) \end{aligned}$$

which is equivalent to,

$$P'(N,v) = \frac{1}{n a_n} v(N) + \frac{b_n + c_n}{n a_n} \sum_{l \in N} P'(N \setminus \{l\}, v) \tag{2.12}$$

Uniqueness is obvious according to this recursive formula, next we show the existence of this modified potential P' recursively, starting from the 0-normalized condition $P'(\emptyset, v) = 0$.

For 1-person game, i.e. $n = 1$, because $P'(\emptyset, v) = 0$ and $a_1 = 1$, the efficiency condition is reduced to $P'(\{i\}, v) = v(\{i\})$ for all $i \in N$.

Suppose (2.7) holds for all $(n-1)$-person game, we have,

$$\begin{aligned}\sum_{l\in N} P'(N\setminus\{l\},v) &= \sum_{l\in N}\sum_{S\subseteq N\setminus\{l\}} h(n-1,s)q_{n-1,s}v(S)\\ &= \sum_{S\subsetneqq N}(n-s)h(n-1,s)q_{n-1,s}v(S)\\ &= n\sum_{S\subsetneqq N} h(n,s)q_{n-1,s}v(S)\end{aligned}\tag{2.13}$$

Substituting the expression above into (2.12), one can get,

$$\begin{aligned}P'(N,v) &= \frac{1}{na_n}v(N)+\frac{b_n+c_n}{na_n}\sum_{l\in N}P'(N\setminus\{l\},v)\\ &= \frac{1}{na_n}v(N)+\frac{b_n+c_n}{a_n}\sum_{S\subsetneqq N}h(n,s)q_{n-1,s}v(S)\end{aligned}$$

By (2.10), the equation above changes to,

$$\begin{aligned}P'(N,v) &= \frac{q_{n,n}}{n}v(N)+\sum_{S\subsetneqq N}h(n,s)q_{n,s}v(S)\\ &= \sum_{S\subseteq N}h(n,s)q_{n,s}v(S)\end{aligned}$$

Thus, (2.7) holds for n-person game. By this inductive proof, the modified potential P' is of form (2.7).

For the i's component of the modified gradient $\nabla' P'$, we have,

$$\begin{aligned}&\nabla'_i P'(N,v)\\ &\overset{(2.4)}{=} a_nP'(N,v)-b_nP'(N\setminus\{i\},v)-\frac{1}{n}c_n\sum_{l\in N}P'(N\setminus\{l\},v)\\ &\overset{(2.7)}{=} a_n\sum_{S\subseteq N}h(n,s)q_{n,s}v(S)-b_n\sum_{S\subseteq N\setminus\{i\}}h(n-1,s)q_{n-1,s}v(S)-c_n\sum_{S\subsetneqq N}h(n,s)q_{n-1,s}v(S)\end{aligned}$$

By the relation (2.10), the equation above can be simplified as follows,

$$\begin{aligned}&\nabla'_i P'(N,v)\\ =\ &\frac{v(N)}{n}+b_n\sum_{S\subsetneqq N}h(n,s)q_{n-1,s}v(S)-b_n\sum_{S\subseteq N\setminus\{i\}}h(n-1,s)q_{n-1,s}v(S)\\ \overset{(2.11)}{=}&\frac{v(N)}{n}+b_n\sum_{S\subsetneqq N\setminus\{i\}}h(n,s+1)q_{n-1,s+1}v(S\cup\{i\})+b_n\sum_{S\subseteq N\setminus\{i\}}(h(n,s)-h(n-1,s))q_{n-1,s}v(S)\\ =\ &\frac{v(N)}{n}+b_n\sum_{S\subsetneqq N\setminus\{i\}}h(n,s+1)q_{n-1,s+1}v(S\cup\{i\})-b_n\sum_{S\subseteq N\setminus\{i\}}h(n,s+1)q_{n-1,s}v(S)\end{aligned}$$

□

Example 2.2.4. *Let the three sequences $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ of real numbers with $a_1 = 1$ and $a_k \neq 0$ for all $k > 1$ be arbitrary. Consider the Shapley value (1.2), for any game (N, v), we have $b_n q_{n-1,s} = 1$ when $1 \leq s \leq n-1$.*

If we let s be $(n-1)$ and $(n-2)$ respectively, the relationship between b_n and $q_{n-1,s}$ would lead to $b_n = a_{n-1}$ and $c_{n-1} = 0$.

Therefore, the Shapley value has various modified potential representations, given by,

$$Sh_i(N, v) = \nabla_i' P'(N, v) = a_n P'(N, v) - a_{n-1} P'(N \setminus \{i\}, v) \tag{2.14}$$

and the modified potential is,

$$P'(N, v) = \frac{1}{a_n} \sum_{S \subseteq N} h(n, s) v(S) \tag{2.15}$$

Evidently, if we choose the unitary sequence $a = (a_k)_{k\in\mathbb{N}} = 1$, then the modified potential gradient is reduced to the classical one given by Hart and Mas-Colell [11].

2.2.2 Representation for ELS values

Besides the Shapley value, Driessen and Radzik [7] showed that, there exists a special class of values satisfying efficiency, linearity and symmetry (ELS values (1.7)), which admits a weighted pseudo-potential representation. This conclusion is also suitable for our modified potential gradient.

Theorem 2.2.5. *Let Φ be a value on $\mathcal{G}$, then Φ admits a modified potential representation (cf. Definition 2.2.2), if and only if Φ is the ELS value (1.7), in which the collection of constants $A_n = \{\alpha_{n,s} | 1 \leq s \leq n-1\}$ with $n > 1$ satisfies two conditions:*

- *For all $1 \leq s \leq n-1$, $\frac{\alpha_{n,s-1}}{\alpha_{n,s}}$ is independent of n provided $\alpha_{n,s} \neq 0$;*
- *If there exist some s, $1 \leq s \leq n-1$, s.t. $\alpha_{n,s} = 0$, then $\alpha_{n,k} = 0$ for all $k \leq s$.*

Proof. ($\Rightarrow$) Suppose Φ has a modified potential representation with three sequences $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ of real numbers satisfying $a_1 = 1$ and $a_k \neq 0$ for all $k > 1$. Then Φ agrees with (2.8).

If $b_n = 0$, then for all $i \in N$, it holds $\Phi_i(N, v) = \frac{v(N)}{n}$. If $b_n \neq 0$, we consider a collection of constants $A_n = \{\alpha_{n,s} \mid 1 \leq s \leq n-1\}$ with $n > 1$, such that,

$$b_n q_{n-1,s} = \alpha_{n,s} \quad \text{for all } n > 1 \text{ and } 1 \leq s \leq n-1 \tag{2.16}$$

Thus, it holds that, for all $n > 1$ and $1 \leq s \leq n-1$,

$$\frac{\alpha_{n,s-1}}{\alpha_{n,s}} \overset{(2.16)}{=} \frac{b_n q_{n-1,s-1}}{b_n q_{n-1,s}} \overset{(2.11)}{=} \frac{b_s + c_s}{a_{s-1}} \tag{2.17}$$

$$\alpha_{n,s-1} \overset{(2.16)}{=} b_n q_{n-1,s-1} \overset{(2.11)}{=} \alpha_{n,s} q_{s-1,s-1}(b_s + c_s) \tag{2.18}$$

So $\alpha_{n,s}$ is uniquely determined by (2.16) based on three sequences a, b, c, and the two conditions in the theorem is equivalent to the relation (2.17) and (2.18), respectively.
($\Leftarrow$) Suppose the value Φ is the ELS value with form (1.7) and satisfies the two conditions in the theorem, then the collection of constants $A_n = \{\alpha_{n,s} \mid 1 \leq s \leq n-1\}$ with $n > 1$ is well defined.

For all $n > 1$ and $1 \leq s \leq n$, consider three sequences a, b, c, where a_s can be choosen arbitrarily with $a_s \neq 0$, $a_1 = 1$, b_s and c_s are defined as follows,

$$b_s = a_{s-1}\alpha_{s,s-1} \qquad \text{for all } 1 < s \leq n, n \geq s+1 \tag{2.19}$$

$$c_s = a_{s-1}\left(\frac{\alpha_{n,s-1}}{\alpha_{n,s}} - \alpha_{s,s-1}\right) \qquad \text{for all } 1 < s \leq n, n \geq s+1 \tag{2.20}$$

Note that, the two conditions in the theorem guarantee the correctness of form (2.20).

Assume for all $i \in N$, $\Phi_i(N, v) \neq \frac{v(N)}{n}$, we claim $\alpha_{n,n-1} \neq 0$. In fact, if $\alpha_{n,n-1} = 0$, then by the second condition in the theorem, for all $1 \leq t \leq n-1$, it holds $\alpha_{n,t} = 0$. Thus $\Phi_i(N, v) = \frac{v(N)}{n}$, which contradicts the assumption.

Therefore, $b_n \neq 0$, and there exists at least one k, $k \geq n$, such that, $\alpha_{k,n-1} \neq 0$. By (2.19) and (2.20), for all $1 < s \leq n, n \geq s+1$,

$$b_s + c_s = a_{s-1}\frac{\alpha_{k,s-1}}{\alpha_{k,s}} = a_{s-1}\frac{\alpha_{n,s-1}}{\alpha_{n,s}} \tag{2.21}$$

Make use of the equation above, then,

$$\begin{aligned} q_{n-1,s} &\overset{(2.9)}{=} \frac{\prod_{j=s+1}^{n-1}(b_j + c_j)}{\prod_{j=s}^{n-1} a_j} \\ &\overset{(2.21)}{=} \frac{1}{a_{n-1}}\frac{\alpha_{n,s}}{\alpha_{n,n-1}} \\ &\overset{(2.19)}{=} \frac{1}{b_n}\alpha_{n,s} \end{aligned}$$

Therefore, we have,

$$\alpha_{n,s} = b_n q_{n-1,s} \quad \text{for all } n > 1 \text{ and } 1 \leq s \leq n-1$$

Thus, the value Φ has a modified potential representation which arises from the three constructed sequences a, b, c. □

During the above proof, we obtain the relationship among the three sequences a, b, c which is depicted in the following corollary.

Corollary 2.2.6. *Let Φ be the ELS value (1.7) satisfying the two conditions in Theorem 2.2.5. Then for any n-person game (N, v) with $n > 1$,*

$$\Phi_i(N, v) = \nabla'_i P'(N, v) = a_n P'(N, v) - b_n P'(N\backslash\{i\}, v) - \frac{1}{n}c_n \sum_{l \in N} P'(N\backslash\{l\}, v) \quad \text{for all } i \in N$$

where the three sequences $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ of real numbers are defined recursively to be, for all $1 < s \leq n$,

$$\begin{cases} a_s \text{ is chosen arbitrarily with } a_s \neq 0 \text{ and } a_1 = 1; \\ b_s := a_{s-1}\alpha_{s,s-1}; \\ c_s := a_{s-1}\left(\dfrac{\alpha_{n,s-1}}{\alpha_{n,s}} - \alpha_{s,s-1}\right) \end{cases} \tag{2.22}$$

Note that by the relation (2.22), the modified potential (2.7) can be written as the following expression which only contains one undecided sequence $a = (a_k)_{k\in\mathbb{N}}$,

$$P'(N,v) = \frac{1}{a_n \cdot \alpha_{n,n}} \sum_{S\subseteq N} h(n,s)\alpha_{n,s}v(S)$$

For the collection of constants $A_n = \{\alpha_{n,s} | 1 \leq s \leq n-1\}$ with $n > 1$, we restrict them to satisfy two conditions, one of which is the quotient of $\alpha_{n,s-1}$ and $\alpha_{n,s}$ should be independent of n provided $\alpha_{n,s} \neq 0$. In order to avoid this assumption, we suppose the coefficient $\alpha_{n,s}$ is *separable*, i.e., $\alpha_{n,s} = \mu_n\nu_s$, which means $\alpha_{n,s}$ result from a product of two independent sequences, one refers to n, and the other one to s. In this way, the quotient of $\alpha_{n,s-1}$ and $\alpha_{n,s}$ only depends on s. By (2.22), we can get the following corollary.

Corollary 2.2.7. *Let Φ be the ELS value (1.7) on $\mathcal{G}$, suppose $\alpha_{n,s} = \mu_n\nu_s$ for all $1 \leq s \leq n-1$, where μ_n and ν_s are two independent sequences refers to n and s, respectively. For any n-person game (N,v) with $n > 1$,*

$$\Phi_i(N,v) = \nabla_i' P'(N,v) = a_n P'(N,v) - b_n P'(N\backslash\{i\},v) - \frac{1}{n}c_n \sum_{l\in N} P'(N\backslash\{l\},v) \quad \text{for all } i \in N$$

where the three sequences $a = (a_k)_{k\in\mathbb{N}}, b = (b_k)_{k\in\mathbb{N}}, c = (c_k)_{k\in\mathbb{N}}$ of real numbers are defined recursively to be, for all $1 < s \leq n$,

$$\begin{cases} a_s \text{ is chosen arbitrarily with } a_s \neq 0 \text{ and } a_1 = 1; \\ b_s := a_{s-1}\mu_s\nu_{s-1}; \\ c_s := a_{s-1}\nu_{s-1}\left(\frac{1}{\nu_s} - \mu_s\right) \end{cases} \tag{2.23}$$

in addition, the collection of constants $A_n = \{\alpha_{n,s} | 1 \leq s \leq n-1\}$ should satisfy that, if there exist some s, $1 \leq s \leq n-1$, s.t. $\alpha_{n,s} = 0$, then $\alpha_{n,k} = 0$ for all $k \leq s$.

In this circumstance, the number $q_{n,s}$ in the modified potential is just $\frac{1}{a_n}\frac{\nu_s}{\nu_n}$, therefore (2.7) can be reduced to,

$$P'(N,v) = \frac{1}{a_n\nu_n} \sum_{S\subseteq N} h(n,s)\nu_s v(S) \tag{2.24}$$

Example 2.2.8. *Consider the Solidarity value (1.4), $\alpha_{n,s}$ here is separable with $\mu_n = 1$, $\nu_s = \frac{1}{s+1}$ for all $1 \leq s \leq n-1$. By (2.23), we have $b_n = \frac{1}{n}a_{n-1}$, $c_n = a_{n-1}$, a_n is chosen arbitrarily with $a_n \neq 0$ and $a_1 = 1$. The Solidarity value admits a modified potential representation as follows:*

$$Sol_i(N,v) = \nabla_i' P'(N,v) = a_n P'(N,v) - \frac{1}{n}a_{n-1}P'(N\setminus\{i\},v) - \frac{1}{n}a_{n-1}\sum_{l\in N} P'(N\setminus\{l\},v) \tag{2.25}$$

where the modified potential is,

$$P'(N,v) = \frac{n+1}{a_n}\sum_{S\subseteq N} h(n,s)\frac{v(S)}{s+1} \tag{2.26}$$

Evidently, the assumption that $\alpha_{n,s}(1 < s \leq n-1)$ is separable is helpful for the calculation. Next we verify the correctness of the modified potential representation for the Solidarity value.

In view of the modified potential P' for the Solidarity value, we have,

$$\begin{aligned} P'(N\setminus\{i\},v) &= \frac{n}{a_{n-1}}\sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\frac{v(S)}{s+1} \\ \sum_{l\in N} P'(N\setminus\{l\},v) &\overset{(2.13)}{=} \frac{n^2}{a_{n-1}}\sum_{S\subsetneq N} h(n,s)\frac{v(S)}{s+1} \end{aligned}$$

Substituting expressions above into (2.25), then the modified gradient $\nabla_i' P'$ equals to,

$$\begin{aligned} \nabla_i' P'(N,v) &= (n+1)\sum_{S\subseteq N} h(n,s)\frac{v(S)}{s+1} - \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\frac{v(S)}{s+1} - n\sum_{S\subsetneq N} h(n,s)\frac{v(S)}{s+1} \\ &= \frac{v(N)}{n} + \sum_{S\subsetneq N} h(n,s)\frac{v(S)}{s+1} - \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\frac{v(S)}{s+1} \\ &= \frac{v(N)}{n} + \sum_{S\subsetneq N\setminus\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\setminus\{i\}} (h(n-1,s)-h(n,s))\frac{v(S)}{s+1} \\ &= \frac{v(N)}{n} + \sum_{S\subsetneq N\setminus\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} \\ &\overset{(1.9)}{=} Sol_i(N,v) \end{aligned}$$

Therefore, the modified gradient $\nabla_i' P'$ in (2.25) equals to the Solidarity value, which means (2.25) is a proper modified potential representation for the Solidarity value (1.4).

Chapter 3

Representations of the potential game

Based on the collection of unanimity games, which is a well-known basis of $\mathcal{G}^N$, we aim to define a new basis with respect to the Shapley value, the Solidarity value, and all ELS values, respectively, such that all these values can be represented as a simple sum of the corresponding coordinates. The modified potential of the ELS value can be represented as a new linear combination with respect to the new basis, thus its modified gradient can be verified. Making use of this new basis, we consider the potential game, which is a special noncooperative game concept that admits a potential function. Inspired by the Shapley value representation of the potential game discussed by Takashi [27], we present its Solidarity value and ELS values representations, respectively.

3.1 New basis associated to values

Remind in Theorem 1.2.5, the characteristic function of any game can be represented as a linear combination of unanimity games u_T with coordinates c_T^v for all $T \in 2^N \setminus \{\emptyset\}$. Making use of this basis, we can represent the Shapley value in terms of c_T^v.

3.1.1 Shapley value

The Shapley value (1.2) in the unanimity game u_T with form (1.1) is, for any $T \in 2^N \setminus \{\emptyset\}$,

$$Sh_i(N, u_T) = \sum_{S \subseteq N \setminus \{i\}} h(n, s+1)[u_T(S \cup \{i\}) - u_T(S)] \quad \text{for all } i \in N \tag{3.1}$$

- $i \notin T$. For all $S \subseteq N \setminus \{i\}$, it holds $u_T(S \cup \{i\}) = u_T(S)$, thus $Sh_i(N, u_T) = 0$.

- $i \in T$. For all $S \subseteq N \setminus \{i\}$, it holds $u_T(S) = 0$, thus by (3.1),

$$\begin{aligned}
Sh_i(N, u_T) &= \sum_{S \subseteq N \setminus \{i\}} h(n, s+1) u_T(S \cup \{i\}) \\
&\overset{(1.1)}{=} \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \supseteq T \setminus \{i\}}} h(n, s+1) \\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} h(n, s+1) \\
&= \frac{1}{t\binom{n}{t}} \sum_{s=t-1}^{n-1} \binom{s}{t-1} \\
&= \frac{1}{t}
\end{aligned}$$

Remark: During the derivation above, we use the combinatorial result,

$$\sum_{s=t-1}^{n-1} \binom{s}{t-1} = \binom{n}{t} \quad \text{for all } 1 \leq t \leq n \tag{3.2}$$

In fact, suppose the equation holds when $n = k-1$, then for $n = k$,

$$\begin{aligned}
\sum_{s=t-1}^{k-1} \binom{s}{t-1} &= \sum_{s=t-1}^{k-2} \binom{s}{t-1} + \binom{k-1}{t-1} \\
&= \binom{k-1}{t} + \binom{k-1}{t-1} = \binom{k}{t}
\end{aligned}$$

Therefore, $Sh_i(N, u_T) = \frac{1}{t}$ when $i \in T$, and 0 otherwise. By the linearity of the Shapley value, we derive that, for all $i \in N$,

$$Sh_i(N, v) = Sh_i\left(N, \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^v u_T\right) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^v Sh_i(N, u_T) = \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{c_T^v}{t} \tag{3.3}$$

Next we consider another basis of the space $\mathcal{G}^N$, which is associated with the Shapley value.

Definition 3.1.1. *A basis of the space $\mathcal{G}^N$ concerning the Shapley value is supplied by the collection $B^{Sh} = \left\{(N, u_T^{Sh}) \mid T \in 2^N \setminus \{\emptyset\}\right\}$, defined by,*

$$u_T^{Sh}(S) = \begin{cases} t & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Theorem 3.1.2. *For each $v \in \mathcal{G}^N$, we have,*

- $v = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T^{v,Sh} u_T^{Sh}$ *with* $c_T^{v,Sh} = \frac{1}{t}\sum_{R\subseteq T}(-1)^{t-r}v(R)$
- $Sh_i(N,v) = \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,Sh}$ *for all* $i \in N$

Proof. For any $S \subseteq N$,

$$\begin{aligned}\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,Sh}u_T^{Sh}(S) &= \sum_{\substack{T\subseteq S\\ T\neq\emptyset}} tc_T^{v,Sh}\\ &= \sum_{\substack{T\subseteq S\\ T\neq\emptyset}}\sum_{R\subseteq T}(-1)^{t-r}v(R)\\ &= \sum_{R\subseteq S}\sum_{t=r}^{s}\binom{s-r}{t-r}(-1)^{t-r}v(R)\\ &= \sum_{R\subseteq S}(1-1)^{s-r}v(R)\\ &= v(S)\end{aligned}$$

By the similar derivation as in the unanimity game u_T, one can derive that $Sh_i(N, u_T^{Sh}) = 1$ if $i \in T$, and 0 otherwise, which means, for all $i \in N$,

$$Sh_i(N,v) = Sh_i(N, \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,Sh}u_T^{Sh}) = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,Sh} Sh_i(N,u_T^{Sh}) = \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,Sh}$$

□

3.1.2 Solidarity value

Next we consider the Solidarity value (1.4), which is also efficient, linear, and symmetric. We already derived in Example 1.2.21 that, the Solidarity value (1.4) can be written as,

$$Sol_i(N,v) = \frac{v(N)}{n} + \sum_{S\subsetneqq N\setminus\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} \tag{3.5}$$

Consider two characteristic functions $v, w \in \mathcal{G}$, then the relationship between the Solidarity value and the Shapley value is, $Sol_i(N,v) = Sh_i(N,w)$ when,

$$\begin{cases} v(S) = w(S)(s+1) & \text{where } S \subsetneqq N\\ v(N) = w(N) \end{cases}$$

Based on u_T^{Sh} and the relation above, we define another basis associated to the Solidarity value.

Definition 3.1.3. *A basis of the space $\mathcal{G}^N$ concerning the Solidarity value is supplied by the collection $B^{Sol} = \{(N, u_T^{Sol}) \mid T \in 2^N \setminus \{\emptyset\}\}$, defined by,*

$$u_T^{Sol}(S) = \begin{cases} t & \text{if } S = N\\ t(s+1) & \text{if } T \subseteq S \text{ and } S \subsetneqq N\\ 0 & \text{otherwise}\end{cases} \tag{3.6}$$

Theorem 3.1.4. *For any $v \in \mathcal{G}^N$, we have,*

- $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^{v,Sol} u_T^{Sol}$ *with* $c_T^{v,Sol} = \begin{cases} \frac{1}{t} \sum_{R \subseteq T} (-1)^{t-r} \frac{1}{r+1} v(R) & \text{if } T \subsetneq N \\ \frac{1}{n} \left[v(N) + \sum_{R \subsetneq N} (-1)^{n-r} \frac{1}{r+1} v(R) \right] & \text{if } T = N \end{cases}$
- $Sol_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,Sol}$ *for all $i \in N$*

Proof. For any $S \subsetneq N$, one can prove $\sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,Sol} u_T^{Sol}(S) = v(S)$ by a similar derivation as for the Shapley value; consider the case $S = N$, we have,

$$\begin{aligned}
\sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,Sol} u_T^{Sol}(N) &= \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} t c_T^{v,Sol} + n c_N^{v,Sol} \\
&= \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} \sum_{R \subseteq T} (-1)^{t-r} \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= \sum_{R \subsetneq N} \sum_{\substack{T \subsetneq N, T \neq \emptyset \\ T \supseteq R}} (-1)^{t-r} \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= \sum_{R \subsetneq N} \sum_{t=r}^{n-1} \binom{n-r}{t-r} (-1)^{t-r} \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= \sum_{R \subsetneq N} \left[\sum_{t=r}^{n} \binom{n-r}{t-r} (-1)^{t-r} - \binom{n-r}{n-r} (-1)^{n-r} \right] \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= \sum_{R \subsetneq N} \left[(1-1)^{n-r} - (-1)^{n-r} \right] \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= \sum_{R \subsetneq N} (-1)^{n-r+1} \frac{1}{r+1} v(R) + n c_N^{v,Sol} \\
&= v(N)
\end{aligned}$$

Next we calculate the Solidarity value on this new basis u_T^{Sol} via (3.5),

- $i \notin T$. For all $S \subsetneq N \setminus \{i\}$, $T \subseteq S \cup \{i\}$ is equivalent to $T \subseteq S$, thus,

$$Sol_i(N, u_T^{Sol}) = \frac{u_T^{Sol}(N)}{n} - h(n,n) \frac{u_T^{Sol}(N \setminus \{i\})}{n} = \frac{t}{n} - \frac{t}{n} = 0$$

- $i \in T$. For all $S \subseteq N \setminus \{i\}$, it holds $u_T(S) = 0$, thus,

$$\begin{aligned}
Sol_i(N, u_T^{Sol}) &= \frac{1}{n} u_T^{Sol}(N) + \sum_{S \subsetneq N \setminus \{i\}} h(n, s+1) \frac{u_T^{Sol}(S \cup \{i\})}{s+2} \\
&\overset{(3.6)}{=} \frac{1}{n} t + t \sum_{\substack{S \subsetneq N \setminus \{i\} \\ S \supseteq T \setminus \{i\}}} h(n, s+1) \\
&= t \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} h(n, s+1) \\
&= \frac{t}{t \binom{n}{t}} \sum_{s=t-1}^{n-1} \binom{s}{t-1} \\
&\overset{(3.2)}{=} 1
\end{aligned}$$

Due to the linearity of the Solidarity value, for all $i \in N$,

$$Sol_i(N, v) = Sol_i \left(N, \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,Sol} u_T^{Sol} \right) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,Sol} Sol_i(N, u_T^{Sol}) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,Sol} \tag{3.7}$$

□

3.1.3 ELS values

For the Shapley value and the Solidarity value, we have already found new basis associated with them. Based on these two special cases, we derive a general basis, which concerns all the values satisfying efficiency, linearity and symmetry (ELS values).

Definition 3.1.5. *A basis of the space $\mathcal{G}^N$ concerning the ELS value (1.7) is supplied by the collection $B^\Phi = \{(N, u_T^\Phi) \mid T \in 2^N \setminus \{\emptyset\}\}$, defined by,*

$$u_T^\Phi(S) = \begin{cases} t & \text{if } S = N \\ \frac{t}{\alpha_{n,s}} & \text{if } T \subseteq S \text{ and } S \subsetneq N \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

Theorem 3.1.6. *For any $v \in \mathcal{G}^N$, we have,*

- $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^{v,\Phi} u_T^\Phi$ *with* $c_T^{v,\Phi} = \begin{cases} \frac{1}{t} \sum_{R \subseteq T} (-1)^{t-r} \alpha_{n,r} v(R) & \text{if } T \subsetneq N \\ \frac{1}{n} \left[v(N) + \sum_{R \subsetneq N} (-1)^{n-r} \alpha_{n,r} v(R) \right] & \text{if } T = N \end{cases}$
- $\Phi_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,\Phi}$ *for all* $i \in N$

Proof. First we verify $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^{v,\Phi} u_T^\Phi$.

- $S \subsetneq N$

$$\begin{aligned}
\sum_{\substack{T\subseteq N\\T\neq\emptyset}} c_T^{v,\Phi} u_T^{\Phi}(S) &= \frac{1}{\alpha_{n,s}} \sum_{\substack{T\subseteq S\\T\neq\emptyset}} c_T^{v,\Phi} t \\
&= \frac{1}{\alpha_{n,s}} \sum_{\substack{T\subseteq S\\T\neq\emptyset}} \sum_{R\subseteq T} (-1)^{t-r} \alpha_{n,r} v(R) \\
&= \frac{1}{\alpha_{n,s}} \sum_{R\subseteq S} \sum_{\substack{T\subseteq S, T\neq\emptyset\\T\supseteq R}} (-1)^{t-r} \alpha_{n,r} v(R) \\
&= \frac{1}{\alpha_{n,s}} \sum_{R\subseteq S} \alpha_{n,r} v(R) \sum_{t=r}^{s} \binom{s-r}{t-r} (-1)^{t-r} \\
&= \frac{1}{\alpha_{n,s}} \sum_{R\subseteq S} \alpha_{n,r} v(R) (1-1)^{s-r} \\
&= v(S)
\end{aligned}$$

- $S = N$

$$\begin{aligned}
\sum_{\substack{T\subseteq N\\T\neq\emptyset}} c_T^{v,\Phi} u_T^{\Phi}(N) &= \sum_{\substack{T\subsetneq N\\T\neq\emptyset}} t c_T^{v,\Phi} + n c_N^{v,\Phi} \\
&= \sum_{\substack{T\subsetneq N\\T\neq\emptyset}} \sum_{R\subseteq T} (-1)^{t-r} \alpha_{n,r} v(R) + n c_N^{v,\Phi} \\
&= \sum_{R\subsetneq N} \alpha_{n,r} v(R) \sum_{\substack{T\subsetneq N, T\neq\emptyset\\T\supseteq R}} (-1)^{t-r} + n c_N^{v,\Phi} \\
&= \sum_{R\subsetneq N} \alpha_{n,r} v(R) \sum_{t=r}^{n-1} \binom{n-r}{t-r} (-1)^{t-r} + n c_N^{v,\Phi} \\
&= \sum_{R\subsetneq N} \alpha_{n,r} v(R) \left[\sum_{t=r}^{n} \binom{n-r}{t-r} (-1)^{t-r} - \binom{n-r}{n-r} (-1)^{n-r} \right] + n c_N^{v,\Phi} \\
&= \sum_{R\subsetneq N} \alpha_{n,r} v(R) [(1-1)^{n-r} - (-1)^{n-r}] + n c_N^{v,\Phi} \\
&= \sum_{R\subsetneq N} (-1)^{n-r+1} \alpha_{n,r} v(R) + n c_N^{v,\Phi} \\
&= v(N)
\end{aligned}$$

According to Corollary 1.2.20, Φ can be represented as, for all $i \in N$,

$$\Phi_i(N,v) = \frac{v(N)}{n} + \sum_{S\subsetneq N\setminus\{i\}} h(n,s+1)\alpha_{n,s+1} v(S\cup\{i\}) - \sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\alpha_{n,s} v(S)$$

Using the formula above, we calculate $\Phi_i(N, u_T^{\Phi})$,

- $i \notin T$. For any $S \subsetneq N \setminus \{i\}$, $T \subseteq S \cup \{i\}$ is equivalent to $T \subseteq S$, thus,

$$\Phi_i(N, u_T^\Phi) = \frac{u_T^\Phi(N)}{n} - h(n,n)\alpha_{n,n-1}u_T^\Phi(N \setminus \{i\}) = \frac{t}{n} - \frac{t}{n} = 0$$

- $i \in T$. For any $S \subseteq N \setminus \{i\}$, it holds $u_T^\Phi(S) = 0$. Thus,

$$\begin{aligned}
\Phi_i(N, u_T^\Phi) &= \frac{u_T^\Phi(N)}{n} + \sum_{S \subsetneq N\setminus\{i\}} h(n, s+1)\alpha_{n,s+1}u_T^\Phi(S \cup \{i\}) \\
&\overset{(3.8)}{=} \frac{t}{n} + t \sum_{\substack{S \subsetneq N\setminus\{i\} \\ S \supseteq T\setminus\{i\}}} h(n, s+1) \\
&= t \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} h(n, s+1) \\
&= \frac{1}{\binom{n}{t}} \sum_{s=t-1}^{n-1} \binom{s}{t-1} \\
&\overset{(3.2)}{=} 1
\end{aligned}$$

By the linearity of the value Φ, we have, for any $i \in N$,

$$\Phi_i(N, v) = \Phi_i\left(N, \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,\Phi} u_T^\Phi\right) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} c_T^{v,\Phi} \Phi_i\left(N, u_T^\Phi\right) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,\Phi} \tag{3.9}$$

□

Suppose the coefficient $\alpha_{n,s}$ in the ELS value is separable, i.e., $\alpha_{n,s} = \mu_n \nu_s$ for all n and $1 \leq s \leq n-1$, we have the following conclusion.

Definition 3.1.7. *A basis of the space $\mathcal{G}^N$ concerning the ELS value (1.7) is supplied by the collection $B^\Phi = \{(N, u_T^\Phi) \mid T \in 2^N \setminus \{\emptyset\}\}$, defined by,*

$$u_T^\Phi(S) = \begin{cases} t & \text{if } S = N \\ \frac{t}{\mu_n \nu_s} & \text{if } T \subseteq S \text{ and } S \subsetneq N \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

Corollary 3.1.8. *For any $v \in \mathcal{G}^N$, we have,*

- $v = \sum_{T \in 2^N\setminus\{\emptyset\}} c_T^{v,\Phi} u_T^\Phi$ *with* $c_T^{v,\Phi} = \begin{cases} \frac{\mu_n}{t} \sum_{R \subseteq T} (-1)^{t-r} \nu_r v(R) & \text{if } T \subsetneq N \\ \frac{1}{n}\left[v(N) + \mu_n \sum_{R \subsetneq N} (-1)^{n-r} \nu_r v(R)\right] & \text{if } T = N \end{cases}$

- $\Phi_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,\Phi}$ *for all* $i \in N$

Remark: When there exist some $\alpha_{n,s} = 0$, for $n > 1$ and $1 \leq s \leq n-1$, or equivalently $\mu_n \nu_s = 0$ in the separable case, the definition of the basis in (3.8) and (3.10) do not hold any more. However, the conclusion in the Theorem 3.1.6 and Corollary 3.1.8 are still true. We take the $ENSC$ value with form (1.3) as an example.

Remind that, the $ENSC$ value is defined by, for any $v \in \mathcal{G}^N$, all $i \in N$,

$$\begin{aligned} ENSC_i(N,v) &= v(N) - v(N \setminus \{i\}) + \frac{1}{n}\left[v(N) - \sum_{j\in N}(v(N) - v(N\setminus\{j\}))\right] \\ &= \frac{v(N)}{n} - v(N\setminus\{i\}) + \frac{1}{n}\sum_{j\in N} v(N\setminus\{j\}) \\ &= \frac{v(N)}{n} - \frac{n-1}{n}v(N\setminus\{i\}) + \frac{1}{n}\sum_{j\in N\setminus\{i\}} v(N\setminus\{j\}) \end{aligned} \tag{3.11}$$

Comparing the equation above to the ELS value (1.7), we have,

$$\begin{cases} h(n,n)\alpha_{n,n-1} = \frac{n-1}{n} \\ h(n,n-1)\alpha_{n,n-1} = \frac{1}{n} \end{cases}$$

Therefore, $\alpha_{n,n-1} = n-1$, and for all s, $1 \leq s < n-1$, it holds $\alpha_{n,s} = 0$. Next we verify the conclusion in the Theorem 3.1.6 and Corollary 3.1.8. By the definition of $c_T^{v,\Phi}$, for $T = N$,

$$\begin{aligned} c_N^{v,\Phi} &= \frac{1}{n}\left[v(N) + \sum_{R \subsetneq N}(-1)^{n-r}\alpha_{n,r}v(R)\right] \\ &= \frac{1}{n}\left[v(N) - \sum_{i\in N}\alpha_{n,n-1}v(N\setminus\{i\})\right] \\ &= \frac{1}{n}\left[v(N) - \sum_{i\in N}(n-1)v(N\setminus\{i\})\right] \end{aligned}$$

For any $i \in N$, $T = N \setminus \{i\}$, we have,

$$\begin{aligned} c_{N\setminus\{i\}}^{v,\Phi} &= \frac{1}{n-1}\sum_{R\subseteq N\setminus\{i\}}(-1)^{n-r-1}\alpha_{n,r}v(R) \\ &= \frac{1}{n-1}\alpha_{n,n-1}v(N\setminus\{i\}) \\ &= v(N\setminus\{i\}) \end{aligned}$$

For any other T with size $t < n-1$, it holds $c_T^{v,\Phi} = 0$. Thus,

$$\begin{aligned}
\sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,\Phi} &= c_N^{v,\Phi} + \sum_{j\in N\setminus\{i\}} c_{N\setminus\{j\}}^{v,\Phi} \\
&= \frac{1}{n}\left[v(N) - \sum_{i\in N}(n-1)v(N\setminus\{i\})\right] + \sum_{j\in N\setminus\{i\}} v(N\setminus\{j\}) \\
&= \frac{1}{n}v(N) - \frac{n-1}{n}v(N\setminus\{i\}) + \frac{1}{n}\sum_{j\in N\setminus\{i\}} v(N\setminus\{j\}) \\
&\overset{(3.11)}{=} ENSC_i(N,v)
\end{aligned}$$

In fact, in this case, we can define the basis in the Theorem 3.1.6 and Corollary 3.1.8 in the following way,

$$u_T^{\Phi}(S) = \begin{cases} t & \text{if } S = N, \text{ or } s < n-1 \\ \frac{t}{n-1} & \text{if } T \subseteq S \text{ and } s = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Then, it holds $v = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T^{v,\Phi} u_T^{\Phi}$. Here we only have to consider the two cases $T = N$ and $T = N\setminus\{i\}$ for any $i \in N$,

- $T = N$

$$\begin{aligned}
\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,\Phi} u_T^{\Phi}(N) &= \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} t c_T^{v,\Phi} \\
&= n c_N^{v,\Phi} + (n-1)\sum_{i\in N} c_{N\setminus\{i\}}^{v,\Phi} \\
&= v(N)
\end{aligned}$$

- $T = N\setminus\{i\}$

$$\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,\Phi} u_T^{\Phi}(N\setminus\{i\}) = c_{N\setminus\{i\}}^{v,\Phi} = v(N\setminus\{i\})$$

In the above example, we discuss the case when there exist some k, such that, $\alpha_{n,k} \neq 0$, and for all $s < k$, $\alpha_{n,s} = 0$. For these special ELS values, we can define u_T^{Φ} as follows,

$$u_T^{\Phi}(S) = \begin{cases} t & \text{if } S = N, \text{ or } s < k \\ \frac{t}{\alpha_{n,s}} & \text{if } T \subseteq S \text{ and } k \leq s < n \\ 0 & \text{otherwise} \end{cases}$$

Then the conclusion in the Theorem 3.1.6 and Corollary 3.1.8 holds.

3.2 Modified potential representation on new basis

For every efficient, linear and symmetric value Φ, we already found a basis associated with it, such that, player i's payoff $\Phi_i(N,v)$ can be represented as the total sum of $c_T^{v,\Phi}$ for all coalitions containing i. We can obtain a new expression for the modified potential for value Φ in terms of such coordinates $c_T^{v,\Phi}$, $T \in 2^N \setminus \{\emptyset\}$.

Example 3.2.1. *Consider the Shapley value (1.2), which can be represented as* $Sh_i(N,v) = \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,Sh}$ *for all* $i \in N$*, where* $c_T^{v,Sh}$ *is defined in Theorem 3.1.2.*

According to Example 2.2.4, consider the modified potential (2.15) for the Shapley value on the basis u_T^{Sh}*,*

$$\begin{aligned} P'(N,u_T^{Sh}) &\overset{(2.15)}{=} \frac{1}{a_n}\sum_{S\subseteq N} h(n,s)u_T^{Sh}(S) \\ &\overset{(3.4)}{=} \frac{t}{a_n}\sum_{\substack{S\subseteq N\\ S\supseteq T}} h(n,s) \\ &= \frac{1}{a_n}\frac{1}{\binom{n}{t}}\sum_{s=t}^{n}\binom{s-1}{t-1} \\ &\overset{(3.2)}{=} \frac{1}{a_n} \end{aligned}$$

Therefore by the linearity of P'*, we have* $P'(N,v) = \frac{1}{a_n}\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,Sh}$*. Similarly for the* $(n-1)$ *person game,* $P'(N\setminus\{i\},v) = \frac{1}{a_{n-1}}\sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} c_T^{v,Sh}$*, thus the modified potential representation for the Shapley value is, for all* $i \in N$*,*

$$\begin{aligned} \nabla_i' P'(N,v) &\overset{(2.14)}{=} a_n P'(N,v) - a_{n-1}P'(N\setminus\{i\},v) \\ &= \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,Sh} - \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} c_T^{v,Sh} \\ &= \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,Sh} \\ &\overset{(3.3)}{=} Sh_i(N,v) \end{aligned}$$

Based on the example above which is associated with the Shapley value, we want to find the general modified potential for all ELS values by means of coordinates $c_T^{v,\Phi}$.

Theorem 3.2.2. *For all efficient, linear, and symmetric value* Φ *(1.7), concerning the new basis* u_T^{Φ} *with form (3.8), the modified potential can be represented as,*

$$P'(N,v) = \frac{1}{na_n\alpha_{n,n}}\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} [n - t(1-\alpha_{n,n})]c_T^{v,\Phi} \tag{3.12}$$

and the modified gradient $\nabla' P'$ is equivalent to value Φ, i.e., for every game (N, v), $i \in N$,

$$\nabla'_i P'(N, v) = \Phi_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,\Phi} \tag{3.13}$$

In fact, if $\alpha_{n,s}$ is separable, we can get a similar theorem as follows, and a proof will be given in this case. The inseparable case can be proved in the same way.

Theorem 3.2.3. *For all efficient, linear, and symmetric value Φ (1.7), suppose the coefficient $\alpha_{n,s}$ is separable (cf. Corollary 2.2.7), concerning the new basis u_T^Φ with form (3.8), the modified potential can be represented as,*

$$P'(N, v) = \frac{1}{na_n\mu_n\nu_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} [n - t(1 - \mu_n\nu_n)] c_T^{v,\Phi} \tag{3.14}$$

and the modified gradient $\nabla' P'$ is equivalent to value Φ, i.e., for every game (N, v), $i \in N$,

$$\nabla'_i P'(N, v) = \Phi_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,\Phi} \tag{3.15}$$

Proof. On the basis u_T^Φ, the modified potential P' on $\mathcal{G}$ with form (2.24) can be rewritten as,

$$\begin{aligned}
P'(N, v) &\overset{(2.24)}{=} \frac{1}{a_n\nu_n} \sum_{S \subseteq N} h(n, s)\nu_s v(S) \\
&= \frac{1}{a_n\nu_n} \sum_{S \subsetneq N} h(n, s)\nu_s \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} \frac{t}{\mu_n\nu_s} c_T^{v,\Phi} + \frac{1}{na_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} tc_T^{v,\Phi} \\
&= \frac{1}{a_n\mu_n\nu_n} \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} tc_T^{v,\Phi} \sum_{s=t}^{n-1} \binom{n-t}{s-t} h(n, s) + \frac{1}{na_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} tc_T^{v,\Phi} \\
&\overset{(3.2)}{=} \frac{1}{a_n\mu_n\nu_n} \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} c_T^{v,\Phi} \frac{1}{\binom{n}{t}} \binom{n-1}{t} + \frac{1}{na_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} tc_T^{v,\Phi} \\
&= \frac{1}{a_n\mu_n\nu_n} \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} \frac{n-t}{n} c_T^{v,\Phi} + \frac{1}{na_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} tc_T^{v,\Phi} \\
&= \frac{1}{na_n\mu_n\nu_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} [n - t(1 - \mu_n\nu_n)] c_T^{v,\Phi}
\end{aligned}$$

Similarly, for the $(n-1)$ person game,

$$\begin{aligned}
P'(N\setminus\{i\},v) &\overset{(2.24)}{=} \frac{1}{a_{n-1}\nu_{n-1}} \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\nu_s v(S) \\
&= \frac{1}{a_{n-1}\nu_{n-1}} \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\nu_s \sum_{\substack{T\subseteq S\\ T\neq\emptyset}} \frac{t}{\mu_n\nu_s} c_T^{v,\Phi} \\
&= \frac{1}{a_{n-1}\mu_n\nu_{n-1}} \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} t c_T^{v,\Phi} \sum_{s=t}^{n-1} \binom{n-t-1}{s-t} h(n-1,s) \\
&= \frac{1}{a_{n-1}\mu_n\nu_{n-1}} \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} t c_T^{v,\Phi} \frac{1}{\binom{n-1}{t}} \sum_{s=t}^{n-1} \binom{s-1}{t-1} \\
&\overset{(3.2)}{=} \frac{1}{a_{n-1}\mu_n\nu_{n-1}} \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} c_T^{v,\Phi}
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\sum_{l\in N} P'(N\setminus\{l\},v) &= \sum_{l\in N} \frac{1}{a_{n-1}\mu_n\nu_{n-1}} \sum_{\substack{T\subseteq N\setminus\{l\}\\ T\neq\emptyset}} c_T^{v,\Phi} \\
&= \frac{1}{a_{n-1}\mu_n\nu_{n-1}} \sum_{\substack{T\subsetneqq N\\ T\neq\emptyset}} (n-t) c_T^{v,\Phi}
\end{aligned}$$

Therefore, the modified gradient is,

$$\begin{aligned}
\nabla_i' P'(N,v) &\overset{(2.6)}{=} a_n P'(N,v) - b_n P'(N\setminus\{i\},v) - \frac{1}{n} c_n \sum_{l\in N} P'(N\setminus\{l\},v) \\
&\overset{(2.23)}{=} a_n P'(N,v) - a_{n-1}\mu_n\nu_{n-1} P'(N\setminus\{i\},v) - \frac{1}{n} a_{n-1}\nu_{n-1}\left(\frac{1}{\nu_r} - \mu_n\right) \sum_{l\in N} P'(N\setminus\{l\},v) \\
&= \frac{1}{n\mu_n\nu_n} \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} [n - t(1-\mu_n\nu_n)] c_T^{v,\Phi} - \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} c_T^{v,\Phi} - \frac{1-\mu_n\nu_n}{n\mu_n\nu_n} \sum_{\substack{T\subsetneqq N\\ T\neq\emptyset}} (n-t) c_T^{v,\Phi} \\
&= \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} c_T^{v,\Phi} - \sum_{\substack{T\subseteq N\setminus\{i\}\\ T\neq\emptyset}} c_T^{v,\Phi} \\
&= \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,\Phi}
\end{aligned}$$

Until here we get the same form for Φ as in Theorem 3.1.6. □

Remark: For the $(n-1)$-person game, the modified potential $P'(N\setminus\{i\},v)$ is not just a simple generalization of $P'(N,v)$, because the basis u_T^Φ (3.8) is a sectional function, which is different on $S=N$ and $S\subsetneqq N$.

Example 3.2.4. *Remind the Example 2.2.8 concerning the Solidarity value (1.4), we have* $\mu_n = 1, \nu_s = \frac{1}{s+1}$ *for all* n *and* $1 \leq s \leq n$. *Using (3.14), the associated modified potential is,*

$$P'(N,v) = \frac{1}{a_n} \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} (n-t+1) c_T^{v,Sol}$$

For the $(n-1)$*-person game,*

$$P'(N \setminus \{i\}, v) = \frac{n}{a_{n-1}} \sum_{\substack{T \subseteq N \setminus \{i\} \\ T \neq \emptyset}} c_T^{v,Sol}$$

$$\sum_{l \in N} P'(N \setminus \{l\}, v) = \frac{n}{a_{n-1}} \sum_{\substack{T \subsetneq N \\ T \neq \emptyset}} (n-t) c_T^{v,Sol}$$

Therefore, it is easy to derive that, for every game (N,v), $i \in N$,

$$\nabla_i' P'(N,v) = Sol_i(N,v) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v,Sol}$$

3.3 ELS values representation of Potential games

The concept of potential games was proposed by Monderer and Shapley [12], which means games with potential functions. Note that, this potential function is different from the modified potential we discussed in Chapter 2.

3.3.1 Potential games

The potential function is defined as a function of strategy profiles such that the change of strategy concerned one player can be expressed in one global function. Potential games can be either ordinal or cardinal. Here we mainly talk about the *cardinal potential game*, which means the difference in individual payoffs for each player from individually changing one's strategy and other remains, have the same value as the difference in values for the potential function, whereas in the *ordinal potential game*, only the signs of differences have to be the same.

In such potential games, one has to consider different strategies for all the players participated in the game. Thus, instead of the cooperative game (N,v), we consider the noncooperative game (N,D,u) in this section, where N is a finite set of players, $D = (D_i)_{i \in N}$ is the finite strategy space, and $u = (u_i)_{i \in N}$ is the payoff function.

Consider a strategy d in the strategy space D, we write $d = (d_1, d_2, \ldots, d_n)$, in which d_i is the strategy of player i. For a strategy containing all players but i, we denote by $d_{-i} = (d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)$ and $D_{-i} = D_{N \setminus \{i\}}$. If the strategy of player i changes from d_i to d_i' while that of other players remain, we use $d \backslash d_i'$, which is just $(d_1, d_2, \ldots, d_{i-1}, d_i', d_{i+1}, \ldots, d_n)$. For a coalition S, $S \subseteq N$, we denote by $D_S = (D_i)_{i \in S}$

the subspace of the strategy space concerning players in coalition S. Similarly, we define $d_S = (d_i)_{i \in S} \in D_S$.

Formally, the potential game is defined as follows,

Definition 3.3.1. [12] *(N,D,u) is called a* ***potential game*** *if there exists a* ***potential function*** $V : D \to \mathbb{R}$*, such that for any* $i \in N$*,* $d'_i \in D_i$*,* $d \in D$*,*

$$u_i(d \backslash d'_i) - u_i(d) = V(d \backslash d'_i) - V(d)$$

Slade [23] introduced a necessary and sufficient condition for potential games as follows.

Theorem 3.3.2. [23] (N, D, u) *is a potential game if only if there exist functions* $V : D \to \mathbb{R}$ *and* $Q_i : D_{-i} \to \mathbb{R}$*, such that for any* $i \in N$*,* $d \in D$*,*

$$u_i(d) = V(d) + Q_i(d_{-i})$$

Note that the potential function is not unique. In fact, Monderer and Shapley [12] proved following lemma.

Lemma 3.3.3. [12] *Let* (N, D, u) *be a potential game,* V *and* V' *be potential functions. Then there exists a constant* c *such that* $V(d) = V'(d) + c$ *for any* $d \in D$*.*

In a potential game, a single potential function can be used to find all Nash equilibria due to the following result.

Lemma 3.3.4. [12] *Let* (N, D, u) *be a potential game with a potential function* V*. Let* $(N, D, (V)_{i \in N})$ *be a game in which every player's payoff function is* V*. Then the set of Nash equilibria of game* (N, D, u) *coincides with that of* $(N, D, (V)_{i \in N})$*.*

Assume strategy sets D are intervals of real numbers, Monderer and Shapley [12] discussed how to verify whether a game has a potential function.

Lemma 3.3.5. [12] *Suppose the payoff functions* $u_i : D_i \to \mathbb{R}$ *for player* $i \in N$ *are continuously differentiable, and let* $V : D \to \mathbb{R}$*. Then* V *is a potential function for game* (N, D, u) *if and only if* V *is continuously differentiable, and*

$$\frac{\partial u_i}{\partial d_i} = \frac{\partial V}{\partial d_i} \quad \text{for any } i \in N$$

Theorem 3.3.6. [12] *Suppose the payoff functions* $u_i : D_i \to \mathbb{R}$ *for player* $i \in N$ *are twice continuously differentiable, then* (N, D, u) *is a potential game if and only if,*

$$\frac{\partial^2 u_i}{\partial d_i \partial d_j} = \frac{\partial^2 u_j}{\partial d_i \partial d_j} \quad \text{for any } i, j \in N$$

Moreover, if d' *is an arbitrary (but fixed) strategy profile in* D*, then a potential function is given by,*

$$V(d) = \sum_{i \in N} \int_0^1 \frac{\partial u_i}{\partial d_i}(x(t))(x_i)'(t)dt$$

where $x : [0, 1] \to D$ *is a piecewise continuously differentiable path in* D *that connects* d' *to* d *(i.e.,* $x(0) = d'$ *and* $x(1) = d$*).*

3.3.2 Representation of Potential games

Takashi [27] proved that a game admits a potential function if and only if the payoff function coincide with the Shapley value of a particular class of cooperative games indexed by the set of strategy profiles. In addition, a potential function of the game coincides with a potential of the class of cooperative games. This particular class of cooperative games is called the TU-games with action choices defined as follows.

Definition 3.3.7. [27] *Given a set of players N and a strategy set D, the collection of TU-games $\{v_d\}_{d\in D}$ is called the **TU-game with action choices**, if for all $d, d' \in D$, any $T \subseteq N$,*

$$v_d(T) = v_{d'}(T) \quad \text{if } d_T = d'_T$$

Note that in such games, the value of a coalition is determined by its members and the strategies of its members, but not by strategies of players outside the coalition. Denote by $\mathcal{G}^{N,D}$ the set of all TU-games with action choices.

Theorem 3.3.8. [27] *(N, D, u) is a potential game if and only if there exists a collection $\{\Psi_T \mid \Psi_T : D_T \to \mathbb{R}, T \subseteq N\}$, such that, for any $d \in D$,*

$$u_i(d) = \sum_{\substack{T\subseteq N\\ i\ni T}} \Psi_T(d_T) \quad \text{for all } i \in N \tag{3.16}$$

Moreover, the potential function is given by,

$$V(d) = \sum_{T\subseteq N} \Psi_T(d_T) \tag{3.17}$$

By defining $\Psi_T(d_T) := \frac{c_T^{v_d}}{|T|}$ (cf. Theorem 1.2.5), for all $T \subseteq N$, $d_T \in D_T$, Takashi [27] proved the following theorem concerning the Shapley value, is equivalent to Theorem 3.3.8.

Theorem 3.3.9. [27] *(N, D, u) is a potential game if and only if there exists $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$, such that, for any $d \in D$,*

$$u_i(d) = Sh_i(N, v_d) \quad \text{for all } i \in N$$

Moreover, the potential function is given by,

$$V(d) = \sum_{T\subseteq N} h(n,t)v_d(T) \overset{(2.3)}{=} P(N, v_d)$$

Based on the theorem above, we want to define new representations of potential games with respect to the Solidarity value and the ELS value, respectively.

Remind that we already find a basis u_T^{Sol} with form (3.6) of the space $\mathcal{G}^N$. By using the corresponding $c_T^{v,Sol}$ (cf. Theorem 3.7), we can rewrite the Solidarity value as a simple sum, i.e., for all $i \in N$,

$$Sol_i(N, v) = \sum_{\substack{T\subseteq N\\ T\ni i}} c_T^{v,Sol} \tag{3.18}$$

For any $T \subseteq N$, define,

$$B^{Sol}(N, v) = \sum_{T \subseteq N} c_T^{v,Sol} \tag{3.19}$$

On the restrict space $\mathcal{G}^{N,D}$, the Solidarity value can also be written as the sum but with another $c_T^{v_d,Sol}$ which is restricted to the game $\{v_d\}_{d\in D}$ defined by,

$$v_d = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^{v_d,Sol} u_T^{Sol} \quad \text{with} \quad c_T^{v_d,Sol} = \begin{cases} \frac{1}{t} \sum_{R \subseteq T} (-1)^{t-r} \frac{1}{r+1} v_d(R) & \text{if } T \subsetneqq N \\ \frac{1}{n} \left[v_d(N) + \sum_{R \subsetneqq N} (-1)^{n-r} \frac{1}{r+1} v_d(R) \right] & \text{if } T = N \end{cases} \tag{3.20}$$

Lemma 3.3.10. *$\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$ if and only if $d_T = d'_T$ implies $c_T^{v_d,Sol} = c_T^{v_{d'},Sol}$ for any $T \subseteq N$, $d, d' \in D$.*

Proof. ($\Rightarrow$) Suppose $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$, then for all $R \subseteq T$, $v_d(R) = v_{d'}(R)$ if $d_T = d'(T)$. Thus by (3.20), $d_T = d'_T$ implies $c_T^{v_d,Sol} = c_T^{v_{d'},Sol}$.

($\Leftarrow$) Suppose that $d_T = d'_T$ implies $c_T^{v_d,Sol} = c_T^{v_{d'},Sol}$ for any $T \subseteq N$, $d, d' \in D$. By Theorem 3.1.4, for any $T \subsetneqq N$,

$$v_d(T) = \sum_{R \subseteq N} c_R^{v_d,Sol} u_R^{Sol}(T) = (s+1) \sum_{R \subseteq S} r c_R^{v_d,Sol}$$

and particularly, $v_d(N) = \sum_{R \subseteq N} r c_R^{v_d,Sol}$. Thus $d_T = d'_T$ also implies $v_d(T) = v_{d'}(T)$. Therefore, $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$. □

Based on $c_T^{v_d,Sol}$ and the above lemma, we have the following theorem.

Theorem 3.3.11. *(N, D, u) is a potential game if and only if there exists $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$, such that, for any $d \in D$,*

$$u_i(d) = Sol_i(N, v_d) \quad \textit{for all } i \in N \tag{3.21}$$

Moreover, the potential function is given by,

$$V(d) = B^{Sol}(N, v_d) \tag{3.22}$$

Proof. ($\Rightarrow$) Suppose (N, D, u) is a potential game. By Theorem 3.3.8, there exists a collection $\{\Psi_T \mid \Psi_T : D_T \to \mathbb{R}, T \subseteq N\}$, such that,

$$u_i(d) = \sum_{\substack{T \subseteq N \\ T \ni i}} \Psi_T(d_T)$$

Let $d \in D$, consider the game $\{v_d\}_{d\in D}$. Define $v_d = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T^{v_d,Sol} u_T^{Sol}$, where $c_T^{v_d,Sol} = \Psi_T(d_T)$. Then,

$$u_i(d) = \sum_{\substack{T \subseteq N \\ T \ni i}} \Psi_T(d_T) = \sum_{\substack{T \subseteq N \\ T \ni i}} c_T^{v_d,Sol} \overset{(3.18)}{=} Sol_i(N, v_d)$$

$$V(d) = \sum_{T \subseteq N} \Psi_T(d_T) = \sum_{T \subseteq N} c_T^{v_d,Sol} \overset{(3.19)}{=} B^{Sol}(N, v_d)$$

($\Leftarrow$) Suppose there exists $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$, such that, for all $i \in N$, $d \in D$,

$$u_i(d) = Sol_i(N, v_d)$$

By Theorem 3.3.8, define $\Psi_T(d_T) = c_T^{v_d,Sol}$, and the potential function $V(d) = B^{Sol}(N, v_d) = \sum_{T\subseteq N} c_T^{v_d,Sol}$, then,

$$\sum_{\substack{T\subseteq N \\ T\ni i}} \Psi_T(d_T) = \sum_{\substack{T\subseteq N \\ T\ni i}} c_T^{v_d,Sol} = Sol_i(N, v_d) = u_i(d)$$

Thus by Theorem 3.3.8, (N, D, u) is a potential game. □

Similar to the Solidarity value, we can define $c_T^{v_d,\Phi}$ for ELS values as follows,

$$v_d = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T^{v_d,\Phi} u_T^{\Phi} \quad \text{with} \quad c_T^{v_d,\Phi} = \begin{cases} \frac{1}{t}\sum_{R\subseteq T}(-1)^{t-r}\alpha_{n,r}v_d(R) & \text{if } T \subsetneq N \\ \frac{1}{n}\left[v_d(N) + \sum_{R\subsetneq N}(-1)^{n-r}\alpha_{n,r}v_d(R)\right] & \text{if } T = N \end{cases}$$

and the function B^{Φ} on $\mathcal{G}^{N,D}$ is given by,

$$B^{\Phi}(N, v_d) = \sum_{T\subseteq N} c_T^{v_d,\Phi}$$

Thus, there exists another representation for potential games.

Theorem 3.3.12. *(N, D, u) is a potential game if and only if there exists $\{v_d\}_{d\in D} \in \mathcal{G}^{N,D}$, such that, for any $d \in D$,*

$$u_i(d) = \Phi_i(N, v_d) \quad \textit{for all } i \in N$$

Moreover, the potential function is given by,

$$V(d) = B^{\Phi}(N, v_d)$$

Chapter 4

Reduced game property for ELS values

The Shapley value satisfies the reduced game property with respect to the Sobolev's reduced game, which is a special game concept we aim to derive by the modified potential approach introduced in Chapter 2. By the similar approach, there exist special reduced games with respect to the Solidarity value and the ELS value, respectively. In this chapter, we will always suppose that $A_n = \{\alpha_{n,s} \mid 1 \leq s \leq n-1\}$ concerning the ELS value (cf. Corollary 1.2.20) is a nonzero sequence.

4.1 Reduced game property

To introduce the concept of a reduced game and the reduced game property, we first look at an example.

Example 4.1.1. [17] *Consider a three-person game (N, v) given in the following table, the dividends (cf. Definition 1.2.15) of coalitions and the potential (cf. Definition 2.1.1) of subgames are given in lines 3 and 4 of this table, respectively. It follows that,*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1	2	3	5	6	9	15
$\triangle_v(S)$	0	1	2	3	2	2	4	1
$P(S,v)$	0	1	2	3	4	5	7	$10\frac{1}{3}$

$$\begin{aligned}
Sh(N,v) &= (Sh_1(N,v), Sh_2(N,v), Sh_3(N,v)) \\
&= (\bigtriangledown_1 P(N,v), \bigtriangledown_2 P(N,v), \bigtriangledown_3 P(N,v)) \\
&= \left(10\frac{1}{3} - 7, 10\frac{1}{3} - 5, 10\frac{1}{3} - 4\right) = \left(3\frac{1}{3}, 5\frac{1}{3}, 6\frac{1}{3}\right) \\
Sh(\{1,2\},v) &= (Sh_1(\{1,2\},v), Sh_2(\{1,2\},v)) = (4-2, 4-1) = (2,3) \\
Sh(\{2,3\},v) &= (Sh_2(\{2,3\},v), Sh_3(\{2,3\},v)) = (7-3, 7-2) = (4,5)
\end{aligned}$$

Suppose that all players in this game agree on using the Shapley value, and consider one possible coalition $\{1,3\}$. Players 1 and 3 will have $3\frac{1}{3} + 6\frac{1}{3} = 9\frac{2}{3}$ if they pool their Shapley value payoffs together. Another way to obtain this amount is to take the worth of the grand coalition, 15, and to subtract player 2's payoff, $5\frac{1}{3}$.

Consider $\{1\}$ as a subcoalition of $\{1,3\}$. Player 1 could form a coalition with player 2 and obtain the worth 5, but he would have to pay player 2 according to the Shapley value of game $(\{1,2\},v)$, which is the vector $(2,3)$. So player 1 is left with $5-3=2$. Similarly, player 3 could form a coalition with player 2 and obtain $v(\{2,3\})=9$ minus the Shapley value payoff for player 2 in the game $(\{2,3\},v)$, which is 4. So player 3 is left with $9-4=5$.

Thus, a 'reduced game' $(\{1,3\},\widetilde{v})$ has been constructed with $\widetilde{v}(\{1\})=2$, $\widetilde{v}(\{3\})=5$, and $\widetilde{v}(\{1,3\})=9\frac{2}{3}$. The Shapley value of this game is $\left(3\frac{1}{3},6\frac{1}{3}\right)$. Note that these payoffs are equal to the Shapley value payoffs in the original game. This is not a coincidence; the particular way of constructing a reduced game as illustrated here leaves the Shapley value invariant.

For any game (N,v), a subset of players, say T, $T\subseteq N$, consider the game arising among the players in T. The reduced game property means, in general, the payoff of players in T should not change or they should have no reason to renegotiate, if they apply the same 'solution rule' in the reduced game $(T,\widetilde{v})$ as in the original game (N,v). There are many different ways to define the reduced game. Here we discuss one of the reduced games, say the Sobolev's reduced game [24].

Definition 4.1.2. [24] *Given any n-person game with $n\geq 2$, player $i\in N$, and payoff vector $x\in\mathbb{R}^n$, the corresponding **reduced game** $(N\setminus\{i\},v^x)$ with respect to x is as follows,*

$$v^x(S) := \frac{s}{n-1}(v(S\cup\{i\})-x_i)+\frac{n-1-s}{n-1}v(S) \quad \text{for all } S\subseteq N\setminus\{i\} \tag{4.1}$$

Note that the worth of any non-empty coalition in the above reduced game is obtained as a convex combination of the worth of the coalition in the original game and the original worth of the coalition together with the single player minus the payoff x_i to the single player i for his participation.

Definition 4.1.3. *The solution ψ on $\mathcal{G}$ possesses the **reduced game property (RGP)**, if for any n-person game with $n\geq 2$, $\emptyset\neq T\subseteq N$ and $x\in\psi(N,v)$, it holds $x^T\in\psi(T,v^x)$, where $x^T\in\mathbb{R}^T$ denotes the restriction of $x\in\mathbb{R}^n$ to $T\subseteq N$.*

In order to axiomatize the Shapley value on $\mathcal{G}$, Sobolev [24] introduced the following theorem concerning four different properties.

Theorem 4.1.4. [24] *The Shapley value is the unique value on $\mathcal{G}$ which possesses the Substitution property, Covariance, Efficiency and the reduced game property with respect to the reduced game (4.1).*

Corollary 4.1.5. *The Shapley value (1.2) satisfies the reduced game property with respect to the reduced game (4.1), i.e., for any game (N, v), $i \in N$, if $x = Sh(N, v)$, then,*

$$Sh_j(N \setminus \{i\}, v^x) = Sh_j(N, v) \quad \text{for all } j \in N \setminus \{i\}$$

Proof. We prove this corollary in the following five steps.

- Step 1: Substituting the reduced game (4.1) into the Shapley value (1.2) in the $(n-1)$-person reduced game $(N \setminus \{i\}, v^x)$;
- Step 2: Calculate the coefficient of x_i by combinatorial counting;
- Step 3: Rewrite x_i which is just the Shapley value for player i in the game (N, v), by distinguishing coalitions with or without player j;
- Step 4: Substituting the result of Step 2 and Step 3 into that of Step 1 to simplify the Shapley value in the $(n-1)$-person reduced game;
- Step 5: Compare the result of Step 4 with the Shapley value (1.2) for player j in the game (N, v).

Step 1: By the definition of the Shapley value (1.2) and the reduced game (4.1), fix $i \in N$, for all $j \in N \setminus \{i\}$,

$$\begin{aligned} Sh_j(N \setminus \{i\}, v^x) &\overset{(1.2)}{=} \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1)[v^x(S \cup \{j\}) - v^x(S)] \\ &\overset{(4.1)}{=} \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1)\left[\frac{s+1}{n-1}(v(S \cup \{i,j\}) - x_i) + \frac{n-s-2}{n-1}v(S \cup \{j\})\right] \\ &\quad - \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1)\left[\frac{s}{n-1}(v(S \cup \{i\}) - x_i) + \frac{n-s-1}{n-1}v(S)\right] \end{aligned} \tag{4.2}$$

Step 2: Consider the coefficient of x_i in the equation above,

$$\begin{aligned} &- \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1)\frac{s+1}{n-1}x_i + \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1)\frac{s}{n-1}x_i \\ &= -\frac{x_i}{n-1}\sum_{s=0}^{n-2}\binom{n-2}{s} h(n-1, s+1) \\ &= -\frac{x_i}{n-1} \end{aligned} \tag{4.3}$$

Step 3: x here is the Shapley value in the n-person game, thus,

$$\begin{aligned} x_i &= Sh_i(N, v) \\ &= \sum_{S \subseteq N \setminus \{i\}} h(n, s+1)[v(S \cup \{i\}) - v(S)] \\ &= \sum_{S \subseteq N \setminus \{i,j\}} h(n, s+2)[v(S \cup \{i,j\}) - v(S \cup \{j\})] + \sum_{S \subseteq N \setminus \{i,j\}} h(n, s+1)[v(S \cup \{i\}) - v(S)] \end{aligned} \tag{4.4}$$

Step 4: By (4.3), (4.2) is equivalent to,

$$Sh_j(N \setminus \{i\}, v^x) = \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1) \left[\frac{s+1}{n-1} v(S \cup \{i,j\}) + \frac{n-s-2}{n-1} v(S \cup \{j\}) \right]$$
$$- \sum_{S \subseteq N \setminus \{i,j\}} h(n-1, s+1) \left[\frac{s}{n-1} v(S \cup \{i\}) + \frac{n-s-1}{n-1} v(S) \right] - \frac{x_i}{n-1}$$

Substituting (4.4) into the equation above, we have,

$$Sh_j(N \setminus \{i\}, v^x) = \sum_{S \subseteq N \setminus \{i,j\}} \left[\frac{h(n-1, s+1)}{n-1}(s+1) - \frac{h(n, s+2)}{n-1} \right] v(S \cup \{i,j\})$$
$$+ \sum_{S \subseteq N \setminus \{i,j\}} \left[\frac{h(n-1, s+1)}{n-1}(n-s-2) + \frac{h(n, s+2)}{n-1} \right] v(S \cup \{j\})$$
$$- \sum_{S \subseteq N \setminus \{i,j\}} \left[\frac{h(n-1, s+1)}{n-1} s + \frac{h(n, s+1)}{n-1} \right] v(S \cup \{i\})$$
$$- \sum_{S \subseteq N \setminus \{i,j\}} \left[\frac{h(n-1, s+1)}{n-1}(n-s-1) + \frac{h(n, s+1)}{n-1} \right] v(S)$$

After simplifying, the equation above changes to,

$$Sh_j(N \setminus \{i\}, v^x)$$
$$= \sum_{S \subseteq N \setminus \{i,j\}} h(n, s+2)[v(S \cup \{i,j\}) - v(S \cup \{i\})] + \sum_{S \subseteq N \setminus \{i,j\}} h(n, s+1)[v(S \cup \{j\}) - v(S)]$$
$$= \sum_{\substack{S \subseteq N \setminus \{j\} \\ S \ni i}} h(n, s+1)[v(S \cup \{j\}) - v(S)] + \sum_{\substack{S \subseteq N \setminus \{j\} \\ S \not\ni i}} h(n, s+1)[v(S \cup \{j\}) - v(S)]$$
$$= \sum_{S \subseteq N \setminus \{j\}} h(n, s+1)[v(S \cup \{j\}) - v(S)]$$

Step 5: Note that the equation above is just $Sh_j(N, v)$, therefore,

$$Sh_j(N \setminus \{i\}, v^x) = Sh_j(N, v)$$

Thus, the Shapley value satisfies the reduced game property with respect to the reduced game (4.1). □

4.2 RGP by the modified potential approach

We already proved that the Shapley value satisfies the reduced game property (RGP) with respect to the Sobolev's reduced game. Next we use the modified potential approach introduced in Chapter 2, to find the reduced game not only with respect to the Shapley value, but also to the Solidarity value and the ELS value.

4.2.1 Shapley value

Remind that in Example 2.1.3, we discussed the relationship between the classical potential P and the Shapley value Sh, that is, for any game (N, v) with $n \geq 2$, all $i \in N$,

$$Sh_i(N, v) = P(N, v) - P(N \setminus \{i\}, v) \tag{4.5}$$

where $P(N, v) = \sum_{S \subseteq N} h(n, s)v(S)$.

Next we use the potential approach to derive the same reduced game (4.1) associated with the Shapley value in the following five steps.

- Step 1: Substituting the potential P into the potential representation (4.5) for the Shapley value;
- Step 2: Simplify the result of Step 1 by distinguishing coalitions with or without player i;
- Step 3: Rewrite the result of Step 2, in order to make $v(S \cup \{i\})$ and $v(S)$ have the same coefficient in different summations, respectively;
- Step 4: Recognize the Shapley value (1.2) from the result of Step 3, and split it among the two other summations by combinatorial counting;
- Step 5: Repeat Step 1 with respect to the $(n-1)$-person reduced game, and compare it to the result of Step 4.

Step 1: For any n-person game (N, v) with $n \geq 2$, for all $j \in N$,

$$\begin{aligned} Sh_j(N, v) &\overset{(4.5)}{=} P(N, v) - P(N \setminus \{j\}, v) \\ &= \sum_{S \subseteq N} h(n, s)v(S) - \sum_{S \subseteq N \setminus \{j\}} h(n-1, s)v(S) \end{aligned}$$

Step 2: For any $i \in N \setminus \{j\}$, distinguishing coalitions S with or without player i, then,

$$\begin{aligned} Sh_j(N, v) = &\sum_{S \subseteq N \setminus \{i\}} [h(n, s+1)v(S \cup \{i\}) + h(n, s)v(S)] \\ &- \sum_{S \subseteq N \setminus \{i,j\}} [h(n-1, s+1)v(S \cup \{i\}) + h(n-1, s)v(S)] \\ = &\sum_{S \subseteq N \setminus \{i\}} h(n-1, s) \left[\frac{s}{n} v(S \cup \{i\}) + \frac{n-s}{n} v(S) \right] \quad (4.6) \\ &- \sum_{S \subseteq N \setminus \{i,j\}} h(n-2, s) \left[\frac{s}{n-1} v(S \cup \{i\}) + \frac{n-s-1}{n-1} v(S) \right] \quad (4.7) \end{aligned}$$

Step 3: In order to make the coefficient of $v(S \cup \{i\})$ and $v(S)$ respectively be identical in (4.6) and (4.7), we split (4.6) into two summations,

$$\sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\left[\frac{s}{n}v(S\cup\{i\})+\frac{n-s}{n}v(S)\right]$$

$$=\sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\left[\left(\frac{s}{n-1}-\frac{s}{n(n-1)}\right)v(S\cup\{i\})+\left(\frac{n-s-1}{n-1}+\frac{s}{n(n-1)}\right)v(S)\right]$$

$$=\sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\left[\frac{s}{n-1}v(S\cup\{i\})+\frac{n-s-1}{n-1}v(S)\right]$$

$$-\frac{1}{n-1}\sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\left[v(S\cup\{i\})-v(S)\right] \tag{4.8}$$

Step 4: Note that (4.8) is just the Shapley value (1.2) with coefficient $(-\frac{1}{n-1})$, thus,

$$\begin{aligned}
Sh_j(N,v) &= \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\left[\frac{s}{n-1}v(S\cup\{i\})+\frac{n-s-1}{n-1}v(S)\right] \\
&- \sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)\left[\frac{s}{n-1}v(S\cup\{i\})+\frac{n-s-1}{n-1}v(S)\right]-\frac{Sh_i(N,v)}{n-1} \\
&= \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)\left[\frac{s}{n-1}v(S\cup\{i\})+\frac{n-s-1}{n-1}v(S)-\frac{s}{n-1}Sh_i(N,v)\right] \\
&- \sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)\left[\frac{s}{n-1}v(S\cup\{i\})+\frac{n-s-1}{n-1}v(S)-\frac{s}{n-1}Sh_i(N,v)\right] \\
&= \sum_{S\subseteq N\setminus\{i\}} h(n-1,s)v^x(S) - \sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)v^x(S)
\end{aligned}$$

Step 5: Compared to the result of Step 1, the equation above is just $Sh_j(N\setminus\{i\},v^x)$. Thus, we have obtained the Sobolev's reduced game (4.1) by the potential approach.

4.2.2 Solidarity value

Next we consider another efficient, linear, and symmetric value, the Solidarity value (1.9). By using the similar modified potential approach as the one for the Shapley value, we want to find a reduced game with respect to the Solidarity value, such that, this value satisfies the reduced game property.

Remind that in Example 2.2.8, we already obtained the modified potential representation for the Solidarity value. Let the nonzero sequence $(a_k)_{k\in\mathbb{N}}$ be arbitrary with $a_1=1$, for any n-person game (N,v) with $n\geq 2$, all $j\in N$,

$$Sol_j(N,v)=a_nP'(N,v)-\frac{1}{n}a_{n-1}P'(N\setminus\{j\},v)-\frac{1}{n}a_{n-1}\sum_{l\in N}P'(N\setminus\{l\},v) \tag{4.9}$$

where the modified potential P' on $\mathcal{G}$ is,

$$P'(N,v)=\frac{n+1}{a_n}\sum_{S\subseteq N}h(n,s)\frac{v(S)}{s+1}$$

We will split the total derivation in five steps similar to the one described in the previous section, but to replace the Shapley value by the Solidarity value.

Step 1: Substituting the modified potential into (4.9), for all $j \in N$, we have,

$$\begin{aligned} Sol_j(N,v) =&(n+1)\sum_{S\subseteq N} h(n,s)\frac{v(S)}{s+1} - \sum_{S\subseteq N\backslash\{j\}} h(n-1,s)\frac{v(S)}{s+1} - n\sum_{S\subsetneq N} h(n,s)\frac{v(S)}{s+1} \\ =&\sum_{S\subsetneq N} h(n,s)\frac{v(S)}{s+1} - \sum_{S\subseteq N\backslash\{j\}} h(n-1,s)\frac{v(S)}{s+1} + \frac{v(N)}{n} \end{aligned}$$

Step 2: Distinguishing coalitions S with or without player i, then,

$$\begin{aligned} Sol_j(N,v) =& \sum_{S\subsetneq N\backslash\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} + \sum_{S\subseteq N\backslash\{i\}} h(n,s)\frac{v(S)}{s+1} \\ &- \sum_{S\subseteq N\backslash\{i,j\}} h(n-1,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\backslash\{i,j\}} h(n-1,s)\frac{v(S)}{s+1} + \frac{v(N)}{n} \\ =& \sum_{S\subsetneq N\backslash\{i\}} \frac{h(n-1,s)}{s+1}\left[\frac{s}{n}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s}{n}v(S)\right] \qquad (4.10) \\ &- \sum_{S\subseteq N\backslash\{i,j\}} \frac{h(n-2,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S)\right] \qquad (4.11) \\ &+\frac{1}{n}v(N) + \frac{1}{n^2(n-1)}v(N\setminus\{i\}) \end{aligned}$$

Step 3: Split (4.10) into two summations,

$$\begin{aligned} &\sum_{S\subsetneq N\backslash\{i\}} \frac{h(n-1,s)}{s+1}\left[\frac{s}{n}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s}{n}v(S)\right] \\ =& \sum_{S\subsetneq N\backslash\{i\}} \frac{h(n-1,s)}{s+1}\left[\left(\frac{s}{n-1} - \frac{s}{n(n-1)}\right)\frac{s+1}{s+2}v(S\cup\{i\}) + \left(\frac{n-s-1}{n-1} + \frac{s}{n(n-1)}\right)v(S)\right] \\ =& \sum_{S\subsetneq N\backslash\{i\}} \frac{h(n-1,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S)\right] \\ &-\frac{1}{n-1}\sum_{S\subsetneq N\backslash\{i\}} h(n,s+1)\left[\frac{1}{s+2}v(S\cup\{i\}) - \frac{1}{s+1}v(S)\right] \qquad (4.12) \end{aligned}$$

Step 4: Compare (4.12) to the Solidarity value (1.9), then we have,

$$\begin{aligned} Sol_j(N,v) =& \sum_{S\subsetneq N\backslash\{i\}} \frac{h(n-1,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S)\right] \\ &- \sum_{S\subseteq N\backslash\{i,j\}} \frac{h(n-2,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S)\right] \\ &+\frac{v(N)}{n} + \frac{v(N\setminus\{i\})}{n^2(n-1)} - \frac{1}{n-1}\left[Sol_i(N,v) - \frac{v(N)}{n} + \frac{v(N\setminus\{i\})}{n^2}\right] \end{aligned}$$

Split $Sol_i(N,v)$ into the two summations, then,

$$\begin{aligned}Sol_j(N,v) = &\sum_{S\subsetneqq N\setminus\{i\}} \frac{h(n-1,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S) - \frac{s(s+1)}{n-1}Sol_i(N,v)\right]\\ &- \sum_{S\subseteq N\setminus\{i,j\}} \frac{h(n-2,s)}{s+1}\left[\frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S) - \frac{s(s+1)}{n-1}Sol_i(N,v)\right]\\ &+\frac{1}{n-1}[v(N) - Sol_i(N,v)]\end{aligned}$$

For all $S \subsetneqq N \setminus \{i\}$, let,

$$v^x_{Sol}(S) := \frac{s}{n-1}\frac{s+1}{s+2}v(S\cup\{i\}) + \frac{n-s-1}{n-1}v(S) - \frac{s(s+1)}{n-1}Sol_i(N,v)$$

and in particular, $v^x_{Sol}(N\setminus\{i\}) := v(N) - Sol_i(N,v)$, thus,

$$Sol_j(N,v) = \sum_{S\subsetneqq N\setminus\{i\}} \frac{h(n-1,s)}{s+1}v^x_{Sol}(S) - \sum_{S\subseteq N\setminus\{i,j\}} \frac{h(n-2,s)}{s+1}v^x_{Sol}(S) + \frac{1}{n-1}v^x_{Sol}(N\setminus\{i\}) \tag{4.13}$$

Step 5: Compare the equation above with the result of Step 1, one can find (4.13) is just the Solidarity value (1.9) in the $(n-1)$-person reduced game $(N\setminus\{i\}, v^x_{Sol})$, thus,

$$Sol_j(N,v) = Sol_j(N\setminus\{i\}, v^x_{Sol})$$

From the derivation above, we obtain the following theorem.

Theorem 4.2.1. *The Solidarity value (1.4) on $\mathcal{G}$ possesses the reduced game property with respect to the following reduced game,*

$$v^x_{Sol}(S) := \begin{cases} \frac{n-s-1}{n-1}v(S) + \frac{s(s+1)}{n-1}\left(\frac{v(S\cup\{i\})}{s+2} - x_i\right) & \text{if } S \subsetneqq N\setminus\{i\} \\ v(N) - x_i & \text{if } S = N\setminus\{i\} \end{cases} \tag{4.14}$$

that is, for all $i \in N$, when $x = Sol(N,v)$,

$$Sol_j(N\setminus\{i\}, v^x_{Sol}) = Sol_j(N,v) \quad \text{for all } j \in N\setminus\{i\}$$

Proof. We will use the five steps described in the proof for Corollary 4.1.5, but to replace the Shapley value by the Solidarity value.
Step 1: Consider the Solidarity value (1.9) in the $(n-1)$-person reduced game $(N\setminus\{i\}, v^x_{Sol})$, for all $i\in N$, $j \in N\setminus\{i\}$,

$$\begin{aligned}&Sol_j(N\setminus\{i\}, v^x_{Sol})\\ =&\sum_{S\subsetneqq N\setminus\{i,j\}} h(n-1,s+1)\frac{v^x_{Sol}(S\cup\{j\})}{s+2} - \sum_{S\subseteq N\setminus\{i,j\}} h(n-1,s+1)\frac{v^x_{Sol}(S)}{s+1} + \frac{v^x_{Sol}(N\setminus\{i\})}{n-1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S \subsetneqq N\setminus\{i,j\}} \frac{h(n-1,s+1)}{s+2}\left[\frac{n-s-2}{n-1}v(S\cup\{j\}) + \frac{s+2}{s+3}\frac{s+1}{n-1}v(S\cup\{i,j\}) - (s+2)\frac{s+1}{n-1}x_i\right] \\
&- \sum_{S \subseteq N\setminus\{i,j\}} \frac{h(n-1,s+1)}{s+1}\left[\frac{n-s-1}{n-1}v(S) + \frac{s+1}{s+2}\frac{s}{n-1}v(S\cup\{i\}) - (s+1)\frac{s}{n-1}x_i\right] + \frac{v(N)-x_i}{n-1}
\end{aligned} \tag{4.15}
$$

Step 2: Calculate the coefficient of x_i in (4.15),

$$
\begin{aligned}
&-\frac{1}{n-1}x_i - \sum_{s=0}^{n-3}\binom{n-2}{s}h(n-1,s+1)\frac{s+1}{n-1}x_i + \sum_{s=0}^{n-2}\binom{n-2}{s}h(n-1,s+1)\frac{s}{n-1}x_i \\
&= -\frac{1}{n-1}x_i - \sum_{s=0}^{n-3}\frac{s+1}{(n-1)^2} + \sum_{s=0}^{n-2}\frac{s}{(n-1)^2} \\
&= -\frac{1}{n-1}x_i
\end{aligned} \tag{4.16}
$$

Step 3: Note that x here is just the Solidarity value (1.9) in the n-person game, thus,

$$
\begin{aligned}
x_i &= Sol_i(N,v) \\
&= \sum_{S \subsetneqq N\setminus\{i\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} - \sum_{S \subseteq N\setminus\{i\}} h(n,s+1)\frac{v(S)}{s+1} + \frac{v(N)}{n} \\
&= \sum_{S \subsetneqq N\setminus\{i,j\}} h(n,s+2)\frac{v(S\cup\{i,j\})}{s+3} + \sum_{S \subseteq N\setminus\{i,j\}} h(n,s+1)\frac{v(S\cup\{i\})}{s+2} \\
&- \sum_{S \subsetneqq N\setminus\{i,j\}} h(n,s+2)\frac{v(S\cup\{j\})}{s+2} - \sum_{S \subseteq N\setminus\{i,j\}} h(n,s+1)\frac{v(S)}{s+1} + \frac{v(N)}{n} - \frac{v(N\setminus\{i\})}{n^2}
\end{aligned} \tag{4.17}
$$

Step 4: By (4.16) and (4.17), (4.15) equals to,

$$
\begin{aligned}
&Sol_j(N\setminus\{i\}, v^x_{Sol}) \\
&= \sum_{S \subsetneqq N\setminus\{i,j\}} h(n-1,s+1)\frac{s+1}{n-1}\frac{v(S\cup\{i,j\})}{s+3} - \sum_{S \subseteq N\setminus\{i,j\}} h(n-1,s+1)\frac{s}{n-1}\frac{v(S\cup\{i\})}{s+2} \\
&+ \sum_{S \subsetneqq N\setminus\{i,j\}} h(n-1,s+1)\frac{n-s-2}{n-1}\frac{v(S\cup\{j\})}{s+2} - \sum_{S \subseteq N\setminus\{i,j\}} h(n-1,s+1)\frac{n-s-1}{n-1}\frac{v(S)}{s+1} + \frac{v(N)-x_i}{n-1}
\end{aligned} \tag{4.18}
$$

Substituting (4.17) into the equation above, then,

$$\begin{aligned}
Sol_j(N\setminus\{i\},v^x_{Sol}) = &\sum_{S\subsetneqq N\setminus\{i,j\}}\left[\frac{h(n-1,s+1)}{n-1}(s+1)-\frac{h(n,s+2)}{n-1}\right]\frac{v(S\cup\{i,j\})}{s+3}\\
&+\sum_{S\subsetneqq N\setminus\{i,j\}}\left[\frac{h(n-1,s+1)}{n-1}(n-s-2)+\frac{h(n,s+2)}{n-1}\right]\frac{v(S\cup\{j\})}{s+2}\\
&-\sum_{S\subseteq N\setminus\{i,j\}}\left[\frac{h(n-1,s+1)}{n-1}s+\frac{h(n,s+1)}{n-1}\right]\frac{v(S\cup\{i\})}{s+2}\\
&-\sum_{S\subseteq N\setminus\{i,j\}}\left[\frac{h(n-1,s+1)}{n-1}(n-s-1)-\frac{h(n,s+1)}{n-1}\right]\frac{v(S)}{s+1}+\frac{v(N)}{n}+\frac{v(N\setminus\{i\})}{n^2(n-1)}
\end{aligned} \tag{4.19}$$

After simplifying, the equation above changes to,

$$\begin{aligned}
&Sol_j(N\setminus\{i\},v^x_{Sol})\\
=&\sum_{S\subsetneqq N\setminus\{i,j\}} h(n,s+2)\frac{v(S\cup\{i,j\})}{s+3}+\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+1)\frac{v(S\cup\{j\})}{s+2}\\
-&\sum_{S\subsetneqq N\setminus\{i,j\}} h(n,s+2)\frac{v(S\cup\{i\})}{s+2}-\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+1)\frac{v(S)}{s+1}+\frac{v(N)}{n}-\frac{v(N\setminus\{j\})}{n^2}\\
=&\sum_{S\subsetneqq N\setminus\{j\}} h(n,s+1)[v(S\cup\{j\})-v(S)]+\frac{v(N)}{n}+\frac{v(N\setminus\{i\})}{n^2(n-1)}
\end{aligned} \tag{4.20}$$

Step 5: Note that the equation above is just $Sol_j(N,v)$. Therefore,

$$Sol_j(N\setminus\{i\},v^x_{Sol}) = Sol_j(N,v)$$

Thus, the Solidarity value satisfies the reduced game property associated to the reduced game v^x_{Sol} defined by (4.14). □

4.2.3 ELS values

Using the modified potential approach, we already find reduced games v^x and v^x_{Sol} associated with the Shapley value and the Solidarity value, respectively. Next we want to use the same method to find a general expression for all values satisfying efficiency, linearity and symmetry.

In order to simplify the derivation, we suppose the coefficient $\alpha_{n,s}$ of ELS values (1.7) is separable, i.e., $\alpha_{n,s} = \mu_n \nu_s$ for any n and $1 \le s \le n-1$. The calculation process is almost the same but to use $\alpha_{n,s}$ instead of the product of μ_n and ν_s, if $\alpha_{n,s}$ is not separable.

By Corollary 2.2.7, let the nonzero sequence $(a_k)_{k\in\mathcal{G}}$ be arbitrary with $a_1 = 1$, then the modified potential representation for ELS values is, for any n-person game (N,v) with

$n \geq 2$, all $j \in N$,

$$\Phi_j(N,v) = a_n P'(N,v) - a_{n-1}\mu_n\nu_{n-1}P'(N\setminus\{j\},v) - \frac{a_{n-1}\nu_{n-1}}{n\nu_n}(1-\mu_n\nu_n)\sum_{l\in N}P'(N\setminus\{l\},v)$$

where the modified potential P' on $\mathcal{G}$ is,

$$P'(N,v) = \frac{1}{a_n\nu_n}\sum_{S\subseteq N}h(n,s)\nu_s v(S)$$

Remind the five steps we discussed in section 4.2.1. Next we will use the similar five steps but to replace the Shapley value by the ELS value.

Step 1: Substituting the modified potential into the modified gradient, then for all $j \in N$,

$$\begin{aligned}\Phi_j(N,v) =& \frac{1}{\nu_n}\sum_{S\subseteq N}h(n,s)\nu_s v(S) - \mu_n\sum_{S\subseteq N\setminus\{j\}}h(n-1,s)\nu_s v(S) - \left(\frac{1}{\nu_n}-\mu_n\right)\sum_{S\subsetneqq N}h(n,s)\nu_s v(S)\\ =&\mu_n\sum_{S\subsetneqq N}h(n,s)\nu_s v(S) - \mu_n\sum_{S\subseteq N\setminus\{j\}}h(n-1,s)\nu_s v(S) + \frac{1}{n}v(N)\end{aligned}$$

Step 2: Distinguishing coalitions S with or without player i, the equation above changes to,

$$\begin{aligned}\Phi_j(N,v) =&\mu_n\sum_{S\subsetneqq N\setminus\{i\}}h(n,s+1)\nu_{s+1}v(S\cup\{i\}) + \mu_n\sum_{S\subseteq N\setminus\{i\}}h(n,s)\nu_s v(S)\\ &-\mu_n\sum_{S\subseteq N\setminus\{i,j\}}h(n-1,s+1)\nu_{s+1}v(S\cup\{i\}) - \mu_n\sum_{S\subseteq N\setminus\{i,j\}}h(n-1,s)\nu_s v(S) + \frac{v(N)}{n}\\ =&\mu_{n-1}\sum_{S\subsetneqq N\setminus\{i\}}h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n}v(S\cup\{i\}) + \frac{\mu_n}{\mu_{n-1}}\frac{n-s}{n}v(S)\right] \qquad (4.21)\\ &-\mu_{n-1}\sum_{S\subseteq N\setminus\{i,j\}}h(n-2,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\}) + \frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)\right]\\ &+\frac{1}{n}v(N) + \frac{\mu_n\nu_{n-1}}{n(n-1)}v(N\setminus\{i\})\end{aligned}$$

Step 3: Split (4.21) into two summations, then,

$$\begin{aligned}&\mu_{n-1}\sum_{S\subsetneqq N\setminus\{i\}}h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n}v(S\cup\{i\}) + \frac{\mu_n}{\mu_{n-1}}\frac{n-s}{n}v(S)\right]\\ =&\mu_{n-1}\sum_{S\subsetneqq N\setminus\{i\}}h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\left(\frac{s}{n-1}-\frac{s}{n(n-1)}\right)v(S\cup\{i\})\right]\\ &+\mu_{n-1}\sum_{S\subsetneqq N\setminus\{i\}}h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\left(\frac{n-s-1}{n-1}v(S)+\frac{s}{n(n-1)}\right)v(S)\right]\end{aligned}$$

$$
\begin{aligned}
=&\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)\right]\\
&-\frac{\mu_n}{n-1}\sum_{S\subsetneq N\setminus\{i\}} h(n,s+1)[\nu_{s+1}v(S\cup\{i\})-\nu_s v(S)] \qquad (4.22)
\end{aligned}
$$

Step 4: Compare (4.22) to the ELS value (1.7), then the equation above is equivalent to,

$$
\begin{aligned}
&\Phi_j(N,v)\\
=&\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)\right]\\
&-\mu_{n-1}\sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)\right]\\
&+\frac{v(N)}{n}+\frac{\mu_n\nu_{n-1}}{n(n-1)}v(N\setminus\{i\})-\frac{1}{n-1}\left[\Phi_i(N,v)-\frac{v(N)}{n}+\frac{\mu_n\nu_{n-1}}{n}v(N\setminus\{i\})\right]\\
=&\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} h(n-1,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)-\frac{1}{\mu_{n-1}}\frac{1}{\nu_s}\frac{s}{n-1}\Phi_i(N,v)\right] \qquad (4.23)\\
&-\mu_{n-1}\sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)\nu_s\left[\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)-\frac{1}{\mu_{n-1}}\frac{1}{\nu_s}\frac{s}{n-1}\Phi_i(N,v)\right] \qquad (4.24)\\
&+\frac{1}{n-1}[v(N)-\Phi_i(N,v)]
\end{aligned}
$$

Denote by $v^x_\Phi(S)$ the expression inside the bracket in (4.23) and (4.24), and let $v^x_\Phi(N\setminus\{i\}):=v(N)-\Phi_i(N,v)$, then the equation above equals to,

$$
\Phi_j(N,v)=\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} h(n-1,s)\nu_s v^x_\Phi(S)-\mu_{n-1}\sum_{S\subseteq N\setminus\{i,j\}} h(n-2,s)\nu_s v^x_\Phi(S)-\frac{1}{n-1}v^x_\Phi(N\setminus\{i\})
$$

Step 5: Note that, compared to the result of Step 1, the equation above is just the ELS value (1.7) of player j in the $(n-1)$-person reduced game $(N\setminus\{i\},v^x_\Phi)$. Therefore we have,

$$
\Phi_j(N,v)=\Phi_j(N\setminus\{i\},v^x_\Phi)
$$

From the derivation above, we get the following theorem.

Theorem 4.2.2. *If $\alpha_{n,s}$ is separable, i.e., $\alpha_{n,s}=\mu_n\nu_s$, the ELS value (1.7) on $\mathcal{G}$ possesses the reduced game property with respect to the following reduced game,*

$$
v^x_\Phi(S):=\begin{cases}\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)+\frac{1}{\mu_{n-1}\nu_s}\frac{s}{n-1}\left[\mu_n\nu_{s+1}v(S\cup\{i\})-x_i\right] & \text{if } S\subsetneq N\setminus\{i\}\\ v(N)-x_i & \text{if } S=N\setminus\{i\}\end{cases}
$$

that is, for all $i \in N$, when $x = \Phi(N, v)$,

$$\Phi_j(N \setminus \{i\}, v^x_\Phi) = \Phi_j(N, v) \quad \text{for all } j \in N \setminus \{i\}$$

Example 4.2.3. *Consider the Solidarity value (1.4), which is a special case in the class of ELS values. We already got in Example 2.2.8 that, $\mu_n = 1,\ \nu_s = \frac{1}{s+1}$ for all $1 \le s \le n-1$. In view of Theorem 4.2.2, we can calculate the reduced game for the Solidarity value by substituting μ_n and ν_s into the general expression of reduced games for all ELS values. One can verify that, it is equivalent to (4.14).*

Until here we have discussed the separable case for the ELS value. In the general case, we have the following theorem.

Theorem 4.2.4. *The ELS value (1.7) on $\mathcal{G}$ possesses the reduced game property with respect to the following reduced game,*

$$v^x_\Phi(S) := \begin{cases} \frac{\alpha_{n,s}}{\alpha_{n-1,s}}\frac{n-s-1}{n-1}v(S) + \frac{1}{\alpha_{n-1,s}}\frac{s}{n-1}\left[\alpha_{n,s+1}v(S\cup\{i\}) - x_i\right] & \text{if } S \subsetneqq N\setminus\{i\} \\ v(N) - x_i & \text{if } S = N \setminus \{i\} \end{cases} \tag{4.25}$$

that is, for all $i \in N$, when $x = \Phi(N, v)$,

$$\Phi_j(N \setminus \{i\}, v^x_\Phi) = \Phi_j(N, v) \quad \text{for all } j \in N \setminus \{i\}$$

Proof. Make use of the similar five steps as the proof for Corollary 4.1.5, but to replace the Shapley value by the ELS value.

Step 1: Remind the expression for the value Φ (1.7) in the $(n-1)$-person reduced game $(N \setminus \{i\}, v^x_\Phi)$, fix $i \in N$, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}
&\Phi_j(N \setminus \{i\}, v^x_\Phi) \\
=& \sum_{S \subsetneqq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n-1,s+1} v^x_\Phi(S \cup \{j\}) \\
-& \sum_{S \subseteq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n-1,s} v^x_\Phi(S) + \frac{1}{n-1} v^x_\Phi(N \setminus \{i\}) \\
=& \sum_{S \subsetneqq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n,s+2}\frac{s+1}{n-1} v(S \cup \{i,j\}) - \sum_{S \subseteq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n,s+1}\frac{s}{n-1} v(S \cup \{i\}) \\
+& \sum_{S \subsetneqq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n,s+1}\frac{n-s-2}{n-1} v(S \cup \{j\}) - \sum_{S \subseteq N\setminus\{i,j\}} h(n-1, s+1)\alpha_{n,s}\frac{n-s-1}{n-1} v(S) \\
+& \frac{v(N) - x_i}{n-1} - \sum_{s=0}^{n-3}\binom{n-2}{s} h(n-1, s+1)\frac{s+1}{n-1}x_i + \sum_{s=0}^{n-2}\binom{n-2}{s} h(n-1, s+1)\frac{s}{n-1}x_i
\end{aligned} \tag{4.26}$$

Step 2: Consider the coefficient of x_i,

$$
\begin{aligned}
&-\frac{1}{n-1}x_i-\sum_{s=0}^{n-3}\binom{n-2}{s}h(n-1,s+1)\frac{s+1}{n-1}x_i+\sum_{s=0}^{n-2}\binom{n-2}{s}h(n-1,s+1)\frac{s}{n-1}x_i\\
=&-\frac{1}{n-1}x_i-\sum_{s=0}^{n-3}\frac{s+1}{(n-1)^2}x_i+\sum_{s=0}^{n-2}\frac{s}{(n-1)^2}x_i\\
=&-\frac{1}{n-1}x_i
\end{aligned}
$$

Step 3: Here x_i is just the ELS value (1.7) for player i in the game (N,v), thus,

$$
\begin{aligned}
x_i &= \Phi_i(N,v)\\
&=\sum_{S\subsetneqq N\setminus\{i\}} h(n,s+1)\alpha_{n,s+1}v(S\cup\{i\})-\sum_{S\subseteq N\setminus\{i\}} h(n,s+1)\alpha_{n,s}v(S)+\frac{v(N)}{n}\\
&=\sum_{S\subsetneqq N\setminus\{i,j\}} h(n,s+2)\alpha_{n,s+2}v(S\cup\{i,j\})+\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+1)\alpha_{n,s+1}v(S\cup\{i\})\\
&-\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+2)\alpha_{n,s+1}v(S\cup\{j\})-\sum_{S\subseteq N\setminus\{i,j\}} h(n,s+1)\alpha_{n,s}v(S)+\frac{v(N)}{n}-\frac{\alpha_{n,n-1}}{n}v(N\setminus\{i\})
\end{aligned}
\tag{4.27}
$$

Step 4: Concerning the result of Step 2, (4.26) is reduced to,

$$
\begin{aligned}
&\Phi_j(N\setminus\{i\},v^x_\Phi)\\
=&\sum_{S\subsetneqq N\setminus\{i,j\}} h(n-1,s+1)\alpha_{n,s+2}\frac{s+1}{n-1}v(S\cup\{i,j\})-\sum_{S\subseteq N\setminus\{i,j\}} h(n-1,s+1)\alpha_{n,s+1}\frac{s}{n-1}v(S\cup\{i\})\\
+&\sum_{S\subsetneqq N\setminus\{i,j\}} h(n-1,s+1)\alpha_{n,s+1}\frac{n-s-2}{n-1}v(S\cup\{j\})-\sum_{S\subseteq N\setminus\{i,j\}} h(n-1,s+1)\alpha_{n,s}\frac{n-s-1}{n-1}v(S)\\
+&\frac{1}{n-1}v(N)-\frac{1}{n-1}x_i
\end{aligned}
$$

Substituting (4.27) into the equation above, then,

$$
\begin{aligned}
&\Phi_j(N\setminus\{i\},v^x_\Phi)\\
=&\sum_{S\subsetneqq N\setminus\{i,j\}}\alpha_{n,s+2}\left[\frac{h(n-1,s+1)}{n-1}(s+1)-\frac{h(n,s+2)}{n-1}\right]v(S\cup\{i,j\})\\
+&\sum_{S\subsetneqq N\setminus\{i,j\}}\alpha_{n,s+1}\left[\frac{h(n-1,s+1)}{n-1}(n-s-2)+\frac{h(n,s+2)}{n-1}\right]v(S\cup\{j\})\\
-&\sum_{S\subseteq N\setminus\{i,j\}}\alpha_{n,s+1}\left[\frac{h(n-1,s+1)}{n-1}s+\frac{h(n,s+1)}{n-1}\right]v(S\cup\{i\})\\
-&\sum_{S\subseteq N\setminus\{i,j\}}\alpha_{n,s}\left[\frac{h(n-1,s+1)}{n-1}(n-s-1)+\frac{h(n,s+1)}{n-1}\right]v(S)+\frac{1}{n}v(N)+\frac{\alpha_{n,n-1}}{n(n-1)}v(N\setminus\{i\})
\end{aligned}
$$

After simplifying, we have,

$$\begin{aligned}
&\Phi_j(N \setminus \{i\}, v^x_\Phi) \\
=&\sum_{S \subsetneqq N\setminus\{i,j\}} \alpha_{n,s+2} h(n,s+2) v(S \cup \{i,j\}) + \sum_{S \subseteq N\setminus\{i,j\}} \alpha_{n,s+1} h(n,s+1) v(S \cup \{j\}) \\
-&\sum_{S \subsetneqq N\setminus\{i,j\}} \alpha_{n,s+1} h(n,s+2) v(S \cup \{i\}) + \sum_{S \subseteq N\setminus\{i,j\}} \alpha_{n,s} h(n,s+1) v(S) + \frac{1}{n} v(N) - \frac{\alpha_{n,n-1}}{n} v(N \setminus \{j\}) \\
=&\sum_{S \subsetneqq N\setminus\{j\}} h(n,s+1)[\alpha_{n,s+1} v(S \cup \{j\}) - \alpha_{n,s} v(S)] + \frac{1}{n} v(N) - \frac{\alpha_{n,n-1}}{n} v(N \setminus \{j\})
\end{aligned}$$

Step 5: Compared with (4.27), the equation above is just the ELS value $\Phi_j(N, v)$, therefore,

$$\Phi_j(N \setminus \{i\}, v^x_\Phi) = \Phi_j(N, v)$$

Thus, (4.25) is a proper reduced game with respected to ELS values.

□

Remark: Until here, we talked about the $(n-1)$-person reduced game $(N \setminus \{i\}, v^x_\Phi)$ associated to the ELS value. Next we consider a game with one more player deleted, i.e., the $(n-2)$-person reduced game $(N \setminus \{i,j\}, (v^x_\Phi)_{N\setminus\{i,j\}})$, in which $j \in N \setminus \{i\}$.

We can achieve such reduced game by deleting player j from the $(n-1)$-person reduced game $(N \setminus \{i\}, v^x_\Phi)$. Another way is to delete player i from the $(n-1)$-person reduced game $(N \setminus \{j\}, v^x_\Phi)$. In fact, both the two ways will arrive at the same result, which means, the reduced game is independent of the order of players deleting from the original game.

We consider the $(n-2)$-person reduced game $(N \setminus \{i,j\}, (v^x_\Phi)_{N\setminus\{i,j\}})$, obtained by deleting player j from the reduced game $(N \setminus \{i\}, v^x_\Phi)$, where $i \in N$, $j \in N \setminus \{i\}$.

For all $S \subsetneqq N \setminus \{i\}$, by the definition of the $(n-1)$-person reduced game (4.25),

$$\begin{aligned}
&(v^x_\Phi)_{N\setminus\{i,j\}}(S) \\
=&\frac{\alpha_{n-1,s}}{\alpha_{n-2,s}} \frac{n-s-2}{n-2} v^x_\Phi(S) + \frac{\alpha_{n-1,s+1}}{\alpha_{n-2,s}} \frac{s}{n-2} v^x_\Phi(S \cup \{j\}) - \frac{1}{\alpha_{n-2,s}} \frac{s}{n-2} x_j \\
=&\frac{\alpha_{n-1,s}}{\alpha_{n-2,s}} \frac{n-s-2}{n-2} \left[\frac{\alpha_{n,s}}{\alpha_{n-1,s}} \frac{n-s-1}{n-1} v(S) + \frac{\alpha_{n,s+1}}{\alpha_{n-1,s}} \frac{s}{n-1} v(S \cup \{i\}) - \frac{1}{\alpha_{n-1,s}} \frac{s}{n-1} x_i \right] - \frac{1}{\alpha_{n-2,s}} \frac{s}{n-2} x_j \\
+&\frac{\alpha_{n-1,s+1}}{\alpha_{n-2,s}} \frac{s}{n-2} \left[\frac{\alpha_{n,s+1}}{\alpha_{n-1,s+1}} \frac{n-s-2}{n-1} v(S \cup \{j\}) + \frac{\alpha_{n,s+2}}{\alpha_{n-1,s+1}} \frac{s+1}{n-1} v(S \cup \{i,j\}) - \frac{1}{\alpha_{n-1,s+1}} \frac{s+1}{n-1} x_i \right] \\
=&\frac{\alpha_{n,s+2}}{\alpha_{n-2,s}} \frac{s(s+1)}{(n-1)(n-2)} v(S \cup \{i,j\}) + \frac{\alpha_{n,s}}{\alpha_{n-2,s}} \frac{(n-s-1)(n-s-2)}{(n-1)(n-2)} v(S) \\
+&\frac{\alpha_{n,s+1}}{\alpha_{n-2,s}} \frac{s(n-s-2)}{(n-1)(n-2)} v(S \cup \{j\}) + \frac{\alpha_{n,s+1}}{\alpha_{n-2,s}} \frac{s(n-s-2)}{(n-1)(n-2)} v(S \cup \{i\}) \\
-&\frac{1}{\alpha_{n-2,s}} \frac{s}{n-2} x_j - \frac{1}{\alpha_{n-2,s}} \frac{s}{n-2} x_i
\end{aligned}$$

Note that, $v(S \cup \{i\})$ and $v(S \cup \{j\})$ share the same coefficient, so do x_i and x_j. Therefore for the class of values satisfying efficiency, linearity and symmetry, its reduced game has no relation on the order of players.

Chapter 5

Conclusions

In this chapter, we first summarize overall results in previous chapters, and then give an example which extend the classical potential approach to the Abelian group structure.

5.1 Overall results

In this monograph, the modified potential approach is applied to study the Shapley value, the Solidarity value, and particularly, the class of values satisfying efficiency, linearity and symmetry (ELS values). In terms of the weighted pseudo-potential presented by Driessen and Radzik [7], we consider a modified potential and the associated modified gradient. By these concepts, a value on $\mathcal{G}$ admits a modified potential representation, if and only if it belongs to ELS values and satisfies two more conditions. In order to express the Shapley value, Solidarity value and all ELS values respectively by a simple sum of special coordinates, we define the basis of $\mathcal{G}$ with respect to different values. By these new basis, the modified potential of these values own a new form, and the correctness of the corresponding modified potential representation can be verified consequently. Based on the Shapley value representation of the potential game, its Solidarity value and ELS values representations are defined. Making use of the potential approach, we obtain the reduced game with respect to the Shapley value, which is the same as the Sobolev's reduced game. Applying the similar modified potential approach, we derive reduced games corresponding to the Solidarity value and all ELS values, respectively, such that these values satisfy the reduced game property.

5.2 Extension to the group structure

We want to extend the results above to the group structure, in order to offer them a general meaning. The concept of a group is central to abstract algebra, because other algebraic structures, such as rings, fields, and vector spaces can all be seen as groups endowed with additional operations and axioms.

Concerning the operations needed in our calculations, we consider the *Abelian group*, also called the *commutative group*. It is a group in which the result of applying the group operation to two group elements does not depend on their order. Abelian groups generalize the arithmetic of addition of integers. They are named after Niels Henrik Abel.

Definition 5.2.1. [1] *An* ***Abelian group*** *is a set* $\mathcal{F}$*, together with an operation* $\bigoplus$ *that combines any two elements* a *and* b *to form another element denoted* $a \bigoplus b$*, where* $(\mathcal{F}, \bigoplus)$ *must satisfy five* ***Abelian group axioms****:*

- ***Closure****:* $a \bigoplus b$ *is in* $\mathcal{F}$*, for all* a*,* b *in* $\mathcal{F}$*.*
- ***Associativity****:* $(a \bigoplus b) \bigoplus c = a \bigoplus (b \bigoplus c)$ *holds, for all* a*,* b *and* c *in* $\mathcal{F}$*.*
- ***Identity element****: there exists an element* e *in* $\mathcal{F}$*, such that for all elements* a *in* $\mathcal{F}$*,* $e \bigoplus a = a \bigoplus e = a$ *holds.*
- ***Inverse element****: for each* a *in* $\mathcal{F}$*, there exists an element* b *in* $\mathcal{F}$*, such that* $a \bigoplus b = b \bigoplus a = e$*, where* e *is the identity element, we denote* $b = a^{-1}$*.*
- ***Commutativity****:* $a \bigoplus b = b \bigoplus a$ *holds, for all* a*,* b *in* $\mathcal{F}$*.*

In fact, whether the concept of the modified potential approach is possible to generalize to the Abelian group is still waiting for verifying. We give here a simple example, in which the classical potential approach is extended.

Consider a *group game* $(N, v, \mathcal{F})$, where N is a finite player set, $\mathcal{F}$ is an Abelian group with operation $\bigoplus$, $v : 2^N \to \mathcal{F}$ satisfying $v(\emptyset) = e$, where e is the unit of group $\mathcal{F}$. So the worth $v(S)$ of coalition S is an element of the group $\mathcal{F}$. For any group game $(N, v, \mathcal{F})$, the Shapley value can be defined as,

$$Sh_i(N, v, \mathcal{F}) = \bigoplus_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) \bigoplus v(S)^{-1}]^{h(n,s+1)} \quad \text{for all } i \in N$$

where $(v(T))^{\frac{a}{b}} = (v(T))^a \bigoplus (v(T)^{-1})^b$ for any $T \subseteq N$ and $a, b \in \mathbb{N}$.

The classical potential in the Abelian group changes to,

$$P(N, v, \mathcal{F}) = \bigoplus_{S \subseteq N} (v(S))^{h(n,s)}$$

consider the $(n-1)$-person game, we have,

$$P(N \setminus \{i\}, v, \mathcal{F}) = \bigoplus_{S \subseteq N \setminus \{i\}} (v(S))^{h(n-1,s)}$$

Note that the operation $\bigoplus$ is reduced to common $+$ if we consider the classical Shapley value and the corresponding potential.

Next we show the difference of $P(N, v, \mathcal{F})$ and $P(N \setminus \{i\}, v, \mathcal{F})$ is $Sh_i(N, v, \mathcal{F})$, that is, the gradient equals to the Shapley value holds even in the Abelian group structure. By definition, the classical gradient $\nabla_i P(N, v)$ equals to,

$$
\begin{aligned}
\nabla_i P(N,v) =& [P(N,v,\mathcal{F})] \bigoplus [P(N \setminus \{i\}, v, \mathcal{F})]^{-1} \\
=& \left[\bigoplus_{S \subseteq N} (v(S))^{h(n,s)}\right] \bigoplus \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S))^{h(n-1,s)}\right]^{-1} \\
=& \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}))^{h(n,s+1)}\right] \bigoplus \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S))^{h(n,s)}\right] \bigoplus \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S)^{-1})^{h(n-1,s)}\right] \\
=& \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}))^{h(n,s+1)}\right] \bigoplus \\
& \left[\bigoplus_{S \subseteq N \setminus \{i\}} \left(\underbrace{v(S) \bigoplus \ldots \bigoplus v(S)} \bigoplus \underbrace{v(S)^{-1} \bigoplus \ldots \bigoplus v(S)^{-1}}\right)\right] \qquad (5.1) \\
=& \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}))^{h(n,s+1)}\right] \bigoplus \left[\bigoplus_{S \subseteq N \setminus \{i\}} (v(S)^{-1})^{h(n,s+1)}\right] \\
=& \bigoplus_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) \bigoplus v(S)^{-1}]^{h(n,s+1)} \\
=& Sh_i(N, v, \mathcal{F})
\end{aligned}
$$

Note that in (5.1), the number of $v(S)$ and $v(S)^{-1}$ are $h(n,s)$ and $h(n-1,s)$, respectively. Therefore, by the Associativity and Inverse element property, we can delete the surplus and simplify it.

From this simple example, one can find the calculations and properties used in the derivation is more complex in operation $\bigoplus$ than in the common $+$. Hence it is still a question to verify whether all of our results can be extended to the group structure.

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Index

A-null player property, 7
A-null players, 7
additivity, 7

basis, 3

characteristic function, 2
coalition, 2, 4
  grand coalition, 2
  worth of coalition, 2
conservative, 13
continuity, 8
cooperative game, 2
core, 4, 6
covariance, 7

dividend, 6
dominate, 4
dummy player, 7
dummy player property, 7

efficiency, 4, 7, 13
egalitarian non-separable contribution value, 6, 15
ELS values, 8
excess, 5
external stability, 5

game, 2
  additive game, 3
  bimatrix game, 11
  concave game, 3
  convex game, 3
  finite game, 11
  inessential game, 3
  matrix game, 11
  monotone game, 3
  quasi-balanced game, 3
  simple game, 3
  superadditive, 3
  zero-normalized game, 3
  zero-sum game, 11
gradient, 13–15

imputation set, 4, 5
individual rationality, 4, 7
internal stability, 5

kernel, 5, 6

lexicographically, 5
lower value, 11

marginal contribution, 3
maximum surplus, 5
minimal right vector, 3
modified gradient, 16

Nash equilibrium, 11
noncooperative game, 10
normal form, 10
NTU-game, 2
nucleolus, 5
null player, 7
null player property, 7

optimal strategy set, 12

path independent, 13
payoff, 3, 5
players, 2
potential, 14
potential representation, 14

preimputation set, 4

set-valued solution, 4, 7
Shapley value, 6, 14
single-valued solution, 6
Solidarity value, 7, 9
strategic equivalence, 7
strategy set, 10
subgame, 2, 14
substitute, 7
substitution property, 7
symmetry, 7

TU-game, 2, 13

unanimity game, 3
upper value, 11
utopia vector, 3

value, 3, 12

worth, 2

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