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Analysis and control of nonlinear oscillators

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M.Sc. Report

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Preface

By finishing my Masters project my time as a student in Enschede has come to an end. In eight years time I completed the programs of both Electrical Engineering and Computer Science. I can look back on those eight years as being an inspiring and wonderful time. Not only did I learn a lot from actual studying, also being part of the student community was a great experience that enriched my live. Taking part in organizations as Twente SummerCampus, the study tour project to Japan, being a teachers assistant and going abroad for a half year internship where all activities that contribute to this.

First of all I would like to thank prof. Stramigioli for the support he gave to me in the last few years. From college to helping with the arrangement of the internship to finally giving me the opportunity to work on a very challenging thesis project. His enthusiasm, dedication and everlasting motivation is something that I really appreciated. Special thanks also to Gijs van Oort for providing feedback, help on problems with 20-sim and proofreading the papers.

I also would like to thank the other students working on their thesis projects in the group for the positive working atmosphere and of course the mostly off-topic, but nonetheless inspiring discussions during the coffee breaks. Last but certainly not least I want to thank my friends and family for their support during the last years. Not being a student anymore marks the start of a new life and starting a new job. I now feel ready and am looking forward to start working on this new challenge.

Michel van Dijk
Enschede, September 2007

Summary

Design of walking robots is a challenge as walking is an inherently unstable process. The stable gait of walking robot can be interpreted as form of oscillation as the states of the system behave in a periodic way. From mathematics, nonlinear oscillators are known that show a stable form of oscillation known as limit cycle oscillation. This research project focuses on the analysis and control of this type of oscillators that may be used in future designs of walking robots to obtain stable and robust behavior.

The Van der Pol oscillator is a nonlinear oscillator that has a globally attractive limit cycle. This system consists of a harmonic oscillator with an addition nonlinear damping term. This term behaves as an ordinary damping for high deflections, but it becomes a negative damping for small deflections. This results in oscillations of small amplitude being pumped up, while high amplitude oscillation are damped down leading to a globally attractive limit cycle.

Based on the same principle as the Van der Pol oscillator, systems that show oscillatory behavior can be brought in limit cycle oscillation by adding this type of nonlinear feedback. This feedback can be implemented by buffer element as a spring and a modulated transformer such as a continuous variable transmission (CVT). This leads to energy efficient limit cycle oscillations as energy that otherwise would be dissipated now is stored and can be fed back to the system when the feedback behaves generatively.

The second part of the research focused on the design of a feedback controller in such a way that exactly the desired periodic behavior could be obtained. This can be achieved using a Lagrange multiplier based approach, calculation of energy difference or parameterizing the limit cycle in time. The latter two can be implemented by an algorithm which is simple considering the amount and complexity of calculations and therefore this can be well implemented in a real-time controller. Parameterizing the limit cycle in time has the additional benefit that the phase of the system is explicitly known which makes synchronization between subsystems a straightforward task. This approach and the Lagrange multiplier approach is also extensible for use in higher order systems.

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CHAPTER 1

Introduction

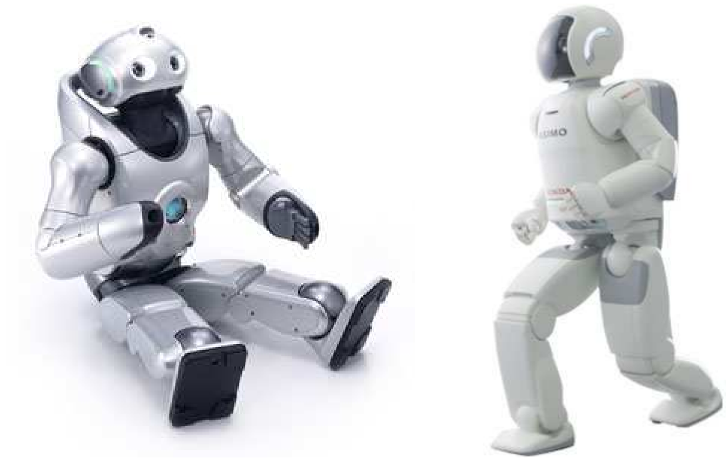
The motivation for this Master's assignment comes from the research on walking robots. One import research question in this field is how to design a walking robot that is able to walk stable, is robust for disturbances from its environment and at the same time is energy efficient. Involving nonlinear oscillator theory is a new approach that may provide a base for development of future walking robots. In order to better understand the goals of this project and the implication of its results, this report is started with an introduction on walking robots. This introduction is finished by a presentation of the goals of the project and the outline of this report.

1.1 Walking robots

Robots have been in use in industrial production environments for decades. They have replaced humans working on repetitive, dangerous or heavy tasks on assembly lines and this resulted in production processes that are more efficient, more consistent and have a higher throughput. Over the last few years interest in robots is also increasing for more domestic appliances such as for example vacuum cleaning, lawn mowing or floor washing. These are relative simple tasks, but it is expected that with the development of technology, robots will become more and more integrated in our daily life as well as being able to perform increasingly complex tasks.

Humanoid Robots

An interesting subject in the field of walking robots is that of humanoid robots. A humanoid is a type of robot which structurally resembles the human body. It has a torso supported by legs, possibly a head and usually is equipped with arms. Research on humanoid robots is still in an early stage but industry has already built some working examples such as Honda's Asimo and Sony's QRIO that are shown in figure 1.1. These have mainly been built for entertainment or proof-of-technology, but it is expected that in the future humanoids will be built that can do useful tasks. The main reason for building humanoids or walking robots in general is that the most suitable way of locomotion in the



(a) QRIO (b) Asimo

Figure 1.1: Industry built humanoid robots

human environment is walking. Designs based on wheels or tracks will have difficulties on coping with sills, stairs and doors for example. Walking on the contrary is very flexible and that makes it suitable for more irregular shaped environments.

Research on walking robots

The main problem in the design of a walking robot is to prevent instability. Walking inherently comes with the risk that the robot can fall over. In other forms of locomotion this is not so much an issue as the contact with the ground is always present. With walking, the feet of the robot alternatively lift from the ground and instability is more likely to occur. In humanoid robots there are only two legs to support the robot which makes ensuring stability even more difficult.

The common approach in the design for stability is to use a form of so-called static walking. With static walking it is ensured that the center of mass of the robot is always within the foot print of the foot that is currently on the ground. As long as the robot is moving slowly, inertial effects will be negligible and the robot will not fall over. Ensuring that the center of mass is over the foot on the ground can be done by means of actively controlling the joints of the robot. An extension of this approach is to use the Zero Moment Point (ZMP) approach. In this approach the inertial effects that occur when the robot is moving are incorporated to calculated the center of pressure (COP). As longs as the COP is withing the foot print of the foot on the ground the robot will be stable. This approach is used both in Asimo and QRIO. Although static walking offers great flexibility since each joint can be controlled individually a serious drawback is that of energy consumption. This is a result from actively controlling all the joints but mainly from not incorporation the robot's natural dynamics in calculating the motion profiles.

A complete different approach initiated by McGeer [McG90] in the early nineties is that of passive dynamic walking. He showed how a simple frame without any actuation was capable of walking down a shallow slope. All energy needed to overcome friction and losses due to impacts of the feet with ground is supplied by gravity. The motion of the system results as an ensemble of the natural dynamics of the system and its environment consisting of the slope and gravity. Inspired by the ideas of McGeer, researchers around the world have started to work on the design of robots that are based on passive dynamic walking of which examples can be found [WvF03], [CR05] and [Wis04]. Actuators are added to be able to walk also on flat terrain and accounting for the natural dynamics of the system resulted in walkers that are very energy efficient. Compared with static walking this is a great improvement but also this approach comes with a drawback. Although researchers succeeded in building stable walking robots, the designs are usually not very robust. Small disturbances such a an irregularity in the floor can destroy the stable gait of the robot and cause it to fall.

Research at the Control Engineering group

The Control Engineering group has been involved in the research on walking robots for about five years. The research started with the work of Vincent van Duindam [Dui06] who worked as a PhD-student on modeling and control of bipedal walking robots. He supervised several Master students that did an assignment that contributed to his research as listed below:

- Niels Beekman, Analysis and development of a 2D walking machine [Bee04]
- Edwin Dertien, Realization of an energy-efficient walking robot [Der05]
- Gijs van Oort, Strategies for stabilizing a 3D Dynamically Walking Robot[vO05]
- Eddy Veltman, Foot shapes and ankle actuation for a walking robot [Vel06]
- Yanzhen Xie, Dynamic effects of an upper body on a 2D bipedal robot [Xie06]
- Michel Franken, Ankle actuation for planar bipedal robots [Fra07]

The collaborative work of these persons has resulted in the biped walking robot 'Dribbel' shown in figure 1.2. Based on passive dynamic walking, this design can walk stable while consuming only a small amount of power. Nowadays Vincent has finished his PhD-research and has left the group. Gijs van Oort has become a PhD-student working on the stabilization of three dimensional dynamic walkers. The ultimate goal of the research in the group is to build a highly energy efficient and fully functional humanoid prototype.

1.2 Assignment goals

A new way to improve the stability of walking robots may be to use nonlinear oscillatory theory. The stable gait of a walking robot can be interpreted as a form of stable oscillation. The robot itself is a high order dynamical system of which the states are made up of the velocities and positions of the rigid bodies that the robot consists of. When the robot is walking in a stable gait, the states will behave periodically. This can be depicted as a closed trajectory in the shape space of the robot.

From mathematics, nonlinear oscillators are known that show a very stable form of oscillation. Independent on the initial condition, such an oscillator always converges to the same periodic behavior. In other words, the trajectory in the state space that the oscillator converges to is unique and this is known as a limit cycle. If a walking robot

Figure 1.2: Dribbel

can be designed in a way that it behaves as a such a nonlinear oscillator this would mean stable walking behavior is obtained automatically.

Before this can be done a better understanding of nonlinear oscillators is needed and that is exactly what is the goal of this assignment. Starting from the well known Van der Pol oscillator it needs to be understand how the mechanism works that ensures convergence to the limit cycle. From there the question will be how an oscillator can be designed that has exactly the periodic behavior that is needed. The ultimate goal will be to become capable of designing oscillators for any possible periodic behavior.

1.3 Report outline

The results of the assignment are presented in two papers that make up the subsequent two chapters of this report. The papers are written in such a way that they can be read separately. The first papers deals mainly with the analysis part of the project. It shows how nonlinear oscillators work and how stable limit cycles can be obtained in oscillating systems. The paper was written halfway the project and has been submitted to the IEEE conference on Intelligent Robots and Systems, but was unfortunately not selected for publication. The second paper deals with control on nonlinear oscillators. It describes how a system can be designed that behaves as a nonlinear oscillator with a stable limit cycle of predefined periodic behavior. It is planned to combine the results of the two papers into one paper and submit it for IEEE Transactions on Robotics and Automation.

CHAPTER 2

Energy Conservative Limit Cycle Oscillations

Energy Conservative Limit Cycle Oscillations

Michel van Dijk, Stefano Stramigioli

***Abstract*— This paper shows how globally attractive limit cycle oscillations can be induced in a system with a nonlinear feedback element. Based on the same principle as the Van der Pol oscillator, the feedback behaves as a negative damping for low velocities but as an ordinary damper for high velocities. This nonlinear damper can be physically implemented with a continuous variable transmission and a spring, storing energy in the spring when the damping is positive and reusing it when the damping is negative. The resulting mechanism has a natural limit cycle oscillation that is energy conservative and can be used for the development of robust, dynamic walking robots.**

I. Introduction

THE results described in this paper are motivated by the search for robust, energy efficient walking robots. Over the last decade, researchers around the world in both industry and universities have been working on walking robots and succeeded in building numerous working examples [1], [2]. While industry mainly focuses on so-called static walkers such as Honda's Asimo and Sony's QRIO, some universities based their research on dynamic walking of which examples can be found in [3], [4], [5]. The benefit of dynamic walking is that it exploits the natural dynamics of the mechanics of the walker, which results in highly energy efficient and natural looking locomotion.

Research on dynamic walking was initiated by McGeer [6] in the early nineties. Originally inspired by toys, he developed several passive walking mechanisms that could walk down a shallow slope only powered by gravity. From his results the view emerged that dynamic walkers could be created based on the same principle, with the addition of actuators to provide energy instead of using gravity.

The stable gait of a dynamic walker can be interpreted as a stable limit cycle of the system [2]. Once the walker has converged to the stable gait it keeps repeating the same pattern over and over again. Unfortunately the dynamic walkers that have been built so far suffer from a lack of robustness. The stability of the gate is easily destroyed by even relatively small disturbances, usually resulting in the robot falling down. Apparently the limit cycle of the system, although being stable, has only a narrow area of attraction. Current research is focused on improving this shortcoming and is expected to yield more robust behavior.

The dynamics of a walking robot are generally nonlinear and on top of that the regular impacts with the ground causes a switching behavior that makes it hard to understand the dynamics of these systems in an analytical way. This explains why the current generation of dynamic walkers is more often a result of trial and error and parameter optimization rather than a thorough analysis of the dynamic behavior that is responsible for the stable limit cycle oscillation. Although also this paper does not give a full analysis of the nonlinear dynamics, a new approach is taken in the design of dynamic walkers that focuses on generation of stable limit cycles. Inspired by nonlinear oscillators famous for their globally attractive limit cycles such as the Van der Pol oscillator, a new way is proposed for inducing limit cycle oscillation in mechanisms based on energy feedback. The result described in this paper is a mechanism that has a natural limit cycle oscillation, is energy efficient and on top of that fairly easy to implement. It is expected that this concept will enable us to build robust, dynamic walkers that excel in a combination of simplicity and performance.

The remainder of this paper is organized as follows. Section II introduces some necessary background information on nonlinear oscillators that exhibit stable limit cycle oscillations. In section III an implementation is proposed that is energy conservative, based on the theory of port-Hamiltonian systems. In section IV an example is given of how the proposed implementation can be used in a physical system. Finally in section V this paper is concluded and future research on this subject is discussed.

II. Limit cycles and nonlinear oscillators

A limit cycle is a periodic solution of a differential equation with the additional property that it is isolated. In the phase space of the system a periodic solution is a trajectory that is a closed orbit. Isolated means that any neighboring trajectory of the limit cycle is not closed, they spiral either towards or away from the limit cycle. Mathematically it could also be said that there exists an open neighborhood that contains only one periodic solution. If all neighboring trajectories spiral towards the limit cycle it is stable or attractive, otherwise it is unstable or half-stable in some exceptional cases. For the design of robust walking robots it is interesting to look at stable limit cycles, with a basin of attraction that is as large as possible. The possibility of a limit cycle solution is restricted to nonlinear systems. In a linear system, if $x(t)$ is a solution then because of linearity also $c{\cdot}x(t)$ is a solution for any constant c. In the phase space this can be seen as an infinite number of closed trajectories encircling the single equilibrium point in the origin, however non of these trajectories is isolated.

A. Lienard systems

There exist nonlinear systems which are known to have a globally attractive limit cycle. An example is the famous Van der Pol oscillator that is described by:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + Kx = 0. \qquad (1)$$

It was discovered by the Dutch scientist Balthasar van der Pol during the early development of radio technology in which vacuum tubes were used. The equation is similar to the damped harmonic oscillator but with a nonlinear damping term $\mu(x^2-1)\dot{x}$. For positive μ the damping term is negative for $|x| < 1$ and positive for $|x| > 1$. This results in small amplitude oscillations being pumped up, while large amplitude oscillations are damped down. Intuitively it is understandable that this must lead to a stable oscillation of intermediate amplitude. An oscillator closely related to the Van der Pol oscillator is the somewhat less famous Rayleigh oscillators that is based on the same principle. In this oscillator the damping term only depends on the derivative $\dot{x}$, which is also what is used in the examples further on in this paper. The equation describing the Rayleigh oscillator is:

$$\ddot{x} + \mu(\dot{x}^2 - 1)\dot{x} + Kx = 0. \tag{2}$$

The relation between the two oscillators described above can be found by first differentiating (2) with respect to time and then replacing $\dot{x}$ with y.

The Van der Pol equation is a specific case of a Lienard system as described by the equation below:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3}$$

Here $f(x)$ and $g(x)$ may be nonlinear functions. Lienards theorem [7] states that (3) has a unique, stable limit cycle surrounding the origin of the phase space if the following conditions are satisfied:

1) $f(x)$ and $g(x)$ are continuously differentiable for all x
2) $g(-x) = -g(x)$ for all x ($g(x)$ is *odd*)
3) $g(x) > 0$ for $x > 0$
4) $f(-x) = f(x)$ for all x ($f(x)$ is *even*)
5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$

The conditions on $g(x)$ ensure that its behavior is like that of a restoring force like a spring and the conditions on $f(x)$ ensure a damping behavior that is amplifies small amplitude oscillations, but damp down large amplitude oscillations. More information on nonlinear oscillators and nonlinear dynamics in general can be found in books as for example [7], [8] or [9].

B. Passivity based oscillators

Another approach to the analysis of limit cycle oscillations is taken in [10]. Here the authors use dissipativity theory to characterize oscillators as open systems. This makes it possible to interconnect a network of oscillators and analyze their common behavior. For the purposes of this paper focus is on isolated oscillators only, but the passivity approach is useful because it allows looking at system connections from an energy based point of view. A system is passive with respect to its input $u(t)$ and output $y(t)$ if there exists as storage function $S(x(t))$, $S(0) = 0$ such that:

$$S(x(t)) \geq 0 \quad \text{and} \quad \dot{S}(x(t)) \leq u(t) \cdot y(t). \tag{4}$$

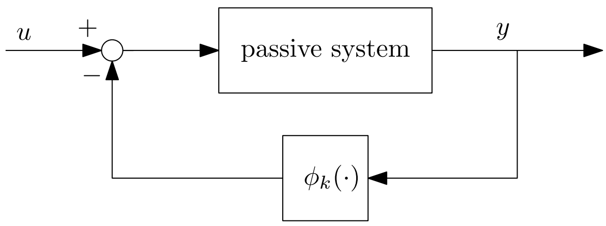


Fig. 1. Passive system with nonlinear feedback

Starting point for the analysis in [10] is the above described Van der Pol oscillator and the Fitzhugh-Nagumo oscillator, which is a simplified model of spike generation in neurons. The author generalizes these two types of oscillators to a form as shown in Fig. 1. The forward path consists of a passive system and the negative feedback is formed by a nonlinearity $\phi_k(y)$ that is the sum of a passive part $\phi(y)$ and an anti-passive or active part $-ky$.

$$\phi_k(y) = \phi(y) - ky \tag{5}$$

Here $\phi(y)$ is a smooth, static nonlinearity in the sector (0,∞) (thus passive), and moreover $\phi(y)$ is a stiffening nonlinearity, i.e. $\lim_{|y|\to\infty} \frac{\phi(y)}{y} = \infty$. Now take $G_k(s)$ the system formed by the linearized passive system with negative feedback $-ky$. Increasing k will eventually lead to instability of $G_k(s)$ as poles cross the imaginary axis and move onto the right half s-plane. Define k^* the smallest $k > 0$ for which $G_k(s)$ has a pole on the imaginary axis. Under the assumption of absolute stability of the system shown in Fig. 1 for $k = k^*$ two scenarios are possible:

1) Scenario 1, Van der Pol type: For $k = k^*$ a pair of complex conjugate poles cross the imaginary axis at non-zero speed causing a supercritical Hopf bifurcation. In this bifurcation the stable origin becomes unstable and a stable limit cycle emerges from the origin.

2) Scenario 2, Fitzhugh-Nagumo type: For $k = k^*$ a single pole crosses the imaginary axis causing a pitchfork bifurcation that results in a bistable system. Extending the negative feedback with a slow adaptation mechanism $\frac{1}{\tau s+1}$ transforms the bistable system into a system with a globally stable limit cycle.

In the isolated case where the system is not connected ($u = 0$), the system will exhibit a self-sustained stable limit cycle oscillation for $k \gtrsim k^*$. The existence of this limit cycle is not guaranteed for all $k > k^*$ since further bifurcations may occur that alter the system behavior.

C. Port Hamiltonian Systems

Key point for the limit cycle oscillations is the nonlinear element that is locally generative, but globally dissipative. The interaction of the passive part of the system with the nonlinearity can be described by an exchange of energy through a power port connecting the two parts. The amount of power $P(t)$ equals the product $u(t) \cdot y(t)$ and the total exchange of energy is the time integral of the power $E(t) = \int_0^t P(u)\,du$. A powerful framework for modeling dynamical systems that are described by an energy function and connections through powerports is that of port-Hamiltonian systems given by the following set of differential equations:

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial H(x)}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H(x)}{\partial x} + (K(x) - S(x))u \end{aligned} \tag{6}$$

In these equations x represents the state, $H(x)$ is the energy function or Hamiltonian, $J(x)$ and $K(x)$ are skew-symmetric matrices that model powercontinuous elements, $R(x)$ and $S(x)$ are positive semi-definite matrices that model dissipative elements and (u, y) is the port through which the system can interact with the outer world. These systems have the property that $\dot{H}(x) \leq u^T y$, so these systems are passive with storage function $H(x)$.

III. Energy conservative implementation

Nonlinear oscillators as the Van der Pol oscillator are generally considered nonconservative since energy dissipation takes place in the nonlinear element. When the nonlinearity behaves generatively, energy has to be supplied from an external source. However it is easy to see that for every periodic solution the change in energy of the system must be zero because energy can be expressed as a function of the state.

$$\left.\begin{aligned} \Delta E &= E(x(t+T)) - E(x(t)) \\ x(t+T) &= x(t) \end{aligned}\right\} \Rightarrow \quad \Delta E = 0 \tag{7}$$

Therefore, instead of using a nonlinearity that dissipates energy it would be useful to have an element that buffers energy so that it can be reused later. In this section it is described how to model an element that has the same characteristics as the nonlinearity, but buffers energy instead of dissipating it. The usage of this element results in oscillators that do not dissipate any energy once converged to the stable limit cycle.

Although not commonly known, bondgraphs as introduced by Paynter [11] can be very useful in the analysis of systems that are connected with powerports. The following analysis is based on bondgraph terminology, but is presented in a general form so that no bondgraph knowledge is required to understand the ideas presented.

A. Power continuous transmissions

To be able to shape the characteristic of the buffer a power continuous transmission (PCT) is used with transmission ratio n as described by the constitutive relations below and of which a graphical representation is shown in Fig. 2.

$$\begin{aligned} out_1 &= n \cdot in_2 \\ out_2 &= n \cdot in_1 \end{aligned} \tag{8}$$

The transmission is power continuous in the sense that the power that flows into the system at one port, flows out at the other port in the same amount. No energy is stored

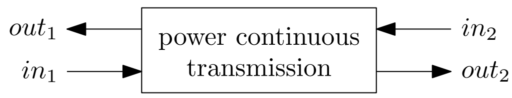


Fig. 2. Power continuous transmission

or dissipated by the transmission. This property is easily deduced with the constitutive relations of (8):

$$\begin{aligned} P_1 &= in_1 \cdot out_1 \\ &= (\frac{1}{n} \cdot out_2) \cdot (n \cdot in_2) \\ &= out_2 \cdot in_2 \\ &= P_2. \end{aligned} \tag{9}$$

It is good to realize that this power continuous property still holds if the transmission factor n is not constant. Such a transmission is referred to as being modulated by a factor $n(\cdot)$. In bondgraphs, this element is known as a (modulated) transformer. Physical examples of power continuous transmissions with a constant transmission ratio are an ideal electric transformer or a set of frictionless gears.

It can be shown that the modulation factor $n(\cdot)$ can be chosen such that the power flow is always in the same direction, that is choosing $n(\cdot)$ such that the PCT becomes a one way device. The positive direction of power flow is defined as the direction from (in_1, out_1) to (in_2, out_2). In perspective of Fig. 2, power flows from left to right if $in_1 \cdot out_1 > 0$ and consequently from right to left if $in_1 \cdot out_1 < 0$. Suppose $n = in_1 \cdot in_2$ is taken as modulation factor, resulting in:

$$\begin{aligned} P_1 = in_1 \cdot out_1 &= in_1 \cdot n \cdot in_2 \\ &= in_1^2 \cdot in_2^2 \\ &\geq 0. \end{aligned} \tag{10}$$

As can be seen the flow of power is always in the positive direction with this modulation factor. Similarly, the flow is always in the negative direction if $n = -in_1 \cdot in_2$ is taken. The structure of the PCT with this modulation is depicted in Fig. 3.

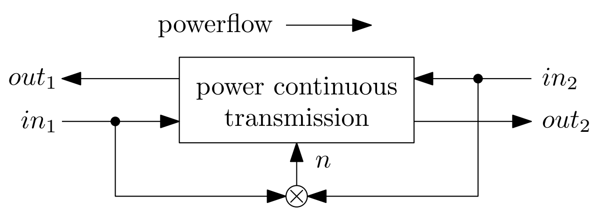


Fig. 3. Modulated PCT, with powerflow from port 1 to port 2

B. Storage element

Besides the power continuous transmission to guide the flow of power also a storage element is needed to store and supply the energy associated with the power flows, which in bondgraph terms is implemented by I- or C-type buffers. Such a storage element can be modeled by a simple integrator as shown in Fig. 4. In the storage element the input is

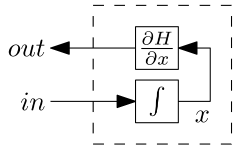


Fig. 4. Storage element

integrated to obtain a state x and the output equals the partial derivative of the stored energy with respect to the state. In the linear case this simplifies to $\frac{x}{C}$, where the constant C represents the capacity of the storage element. This describes for example a spring with the velocity as input, length as state, capacity $\frac{1}{K}$ and output the force, or a mass with the force as input, impulse as state, mass as capacity and velocity as output. The amount of energy stored in the element is found by calculating the integral over the product of the input and output.

$$\int in \cdot out \, dt = \int \frac{\partial H}{\partial x} \dot{x} \, dt = H(x) \tag{11}$$

In the linear case this equals:

$$H(x) = \int \frac{x}{C} \dot{x} \, dt = \frac{1}{2C} x^2. \tag{12}$$

C. Replacing the nonlinear element

It is now possible to show how the combination of a modulated power continuous transmission and a storage element can be used as a substitution for the nonlinear element. Starting from the nonlinearity $\phi(\cdot)$ where the output is a nonlinear function of the input, that is:

$$out = \phi(in) \tag{13}$$

and using a structure as shown in Fig. 5, the following relation can be deduced:

$$\left.\begin{aligned} out &= n \cdot b_{out} = n \cdot \frac{\partial H}{\partial x} \\ out &= \phi(in) \end{aligned}\right\} \quad n = \frac{\phi(in)}{\frac{\partial H}{\partial x}} \tag{14}$$

That is the structure of Fig. 5 where modulation factor n is according to (14) will have the same input-output characteristic as nonlinearity $\phi(\cdot)$. When the power inflow is positive and $\phi(\cdot)$ behaves as a dissipative element, now energy is stored in the buffer instead of being dissipated. When $\phi(\cdot)$ would have a generative characteristic, the buffer supplies the previously stored energy. It can do so as long as there is energy stored in to buffer, that is as long as $\frac{\partial H}{\partial x} > 0$. At the same time it is clear that $\phi(\cdot)$ can be freely chosen, so it is possible to replace any nonlinearity with this system. In section IV an example and possible physical implementation of this system is given.

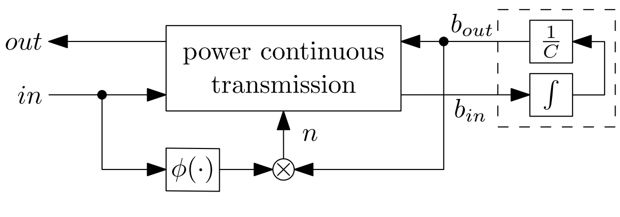


Fig. 5. Modulated PCT with storage element

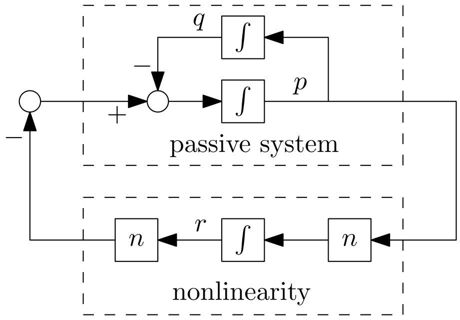


Fig. 6. Van der Pol oscillator in conservative form

D. Conservative Van der Pol oscillator

Using the buffer as sketched above to replace the nonlinear damping term in the Van der Pol equation, it can be written in a form that is conservative. First the Van der Pol oscillator is rewritten to equal the form of Fig. 1.

$$\ddot{x} + \phi(x, \dot{x}) + x, \quad \phi(x, \dot{x}) = \mu(x^2 - 1)\dot{x} \tag{15}$$

A block diagram of the system is shown in Fig. 6 where the following variables are used: $q = x$, $p = \dot{x}$, r the state of the storage element with capacity $C = 1$ and $n = \phi(\cdot)/r$. The equations describing the system in matrix form are:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -n \\ 0 & n & 0 \end{bmatrix} \begin{bmatrix} q \\ p \\ r \end{bmatrix}. \tag{16}$$

Which is a port-Hamiltonian system with state vector x and skew-symmetric matrix J denoted by:

$$x = \begin{bmatrix} q & p & r \end{bmatrix}^T \quad J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -n \\ 0 & n & 0 \end{bmatrix}. \tag{17}$$

The Hamiltonian of the system is:

$$H(x) = \frac{1}{2} x^T \cdot x = \frac{1}{2} q^2 + \frac{1}{2} p^2 + \frac{1}{2} r^2. \tag{18}$$

This is a conservative system as can be seen by calculating the time derivative of $H(x)$:

$$\begin{aligned} \dot{H}(x) &= \frac{\partial H}{\partial x} \dot{x} \\ &= x^T J x \\ &= 0 \end{aligned} \tag{19}$$

where equality to zero follows from the skew-symmetric property of J.

It can be verified that the behavior of the system in conservative form is the same as that of the normal Van der Pol oscillator by a numerical simulation. In Fig. 7 a solution of the system for $\mu = 1$ is shown with initial conditions $x = [2, 2, 10]$. The figure shows convergence to a limit cycle in the (q,p,r)-space, and the projection of the trajectory on the (q,p)-surface shows the limit cycle associated with the

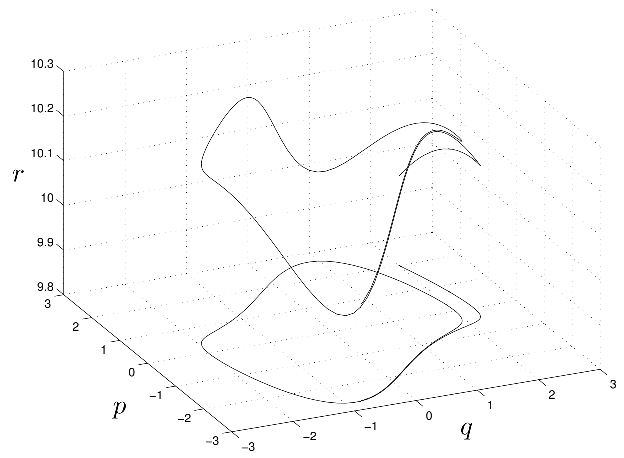


Fig. 7. Simulation of the conservative Van der Pol implementation

normal Van der Pol oscillator, thereby verifying that the behavior of the conservative system is indeed the same.

E. Extension to systems with dissipation

In the previous example the passive part consists of conservative elements only. If the passive part contains also dissipative elements then the total energy flow into the nonlinearity will be smaller than the amount of energy it has to supply. As a result the stored energy in the buffer will decrease as time evolves until the buffer becomes empty. To overcome such problems energy has to be injected into the system that compensates the dissipation of energy in the passive part. Connecting an actuator directly to the system however will generally influence the dynamics of the system and thereby possibly disturb the stability properties of the limit cycle. With the use of the storage element it is possible to circumvent this problem in an elegant way. Instead of injecting energy in the system, it is possible to directly inject energy into the storage element. As the modulation factor of the power continuous transmission compensates for any variation of energy storage this can be done without influencing the system behavior. Energy injection can be done by extending the storage element to a two port system as shown in Fig. 8 where port 1 connects to the system

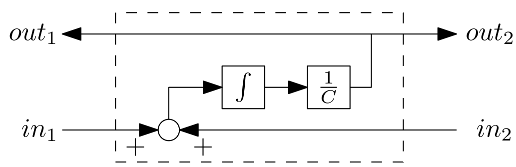


Fig. 8. Two port storage element

through the power continuous transmission as before and port 2 is used to supply the required energy. If the storage element is implemented as a torsional spring for example, one side of the spring connects to the system and the other side to a motor which winds up the spring similarly like what is done in an analogue wristwatch.

IV. Applications and implementation

The last part of this paper discusses an application and possible physical implementation of the system described so far. As the motivation for this paper comes from research on walking robots the application will be in that field. The leg of a walking robot can be roughly interpreted as an inverse pendulum, or double inverse pendulum in case of legs with knees and therefore it is chosen to look at how limit cycle oscillations can be induced in a pendulum.

A. Pendulum

The pendulum is a classic physical example of a nonlinear differential equation. The differential equation describing the damped pendulum of Fig. 9 with pointmass m, length l, damping d and input torque T is:

$$\ddot{\theta} + \frac{d}{m \cdot l}\dot{\theta} + g \sin\theta - \frac{T}{m \cdot l} = 0. \quad (20)$$

The system is passive with respect to input T and output $\dot{\theta}$, so limit cycle oscillation is expected if a negative feedback of the form $-T = \dot{\theta}^3 - k\dot{\theta}$ is used. Although not strictly a Lienard system, the system with this feedback is similar to the Rayleigh oscillator and the nonlinearity fulfills the conditions described in section II. It can be seen that the term $k\dot{\theta}$ will compensate the damping term $\frac{d}{m \cdot l}\dot{\theta}$ and thus limit cycle oscillation may be expected for $k > \frac{d}{m \cdot l}$. That is, the system is stable for small k, increasing k results in a Hopf-bifurcation in which the stable equilibrium becomes unstable and a globally stable limit cycle surrounding the origin appears.

These expectations are verified by numerical simulation of which the results are shown in Fig. 10 and 11. The parameters used in the simulation are: $m = 1$, $l = 1$, $d = 2$ and $g = 9.81$. In Fig. 10 $k = 1$, which results in a damped, stable system as the damping of the pendulum is stronger than the active part of the nonlinear feedback. Increasing k results in a limit cycle oscillation as is shown in Fig. 11 for a value $k = 4$.

B. Implementation

In order to build the pendulum with nonlinear feedback in a power continuous way, a physical implementation for the modulated power continuous transmission and storage element have to be found. The (input, output) combinations of the transmission are of the form (rotational velocity,

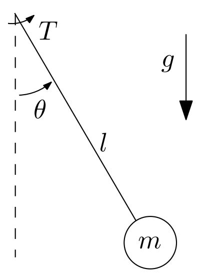


Fig. 9. Classic pendulum

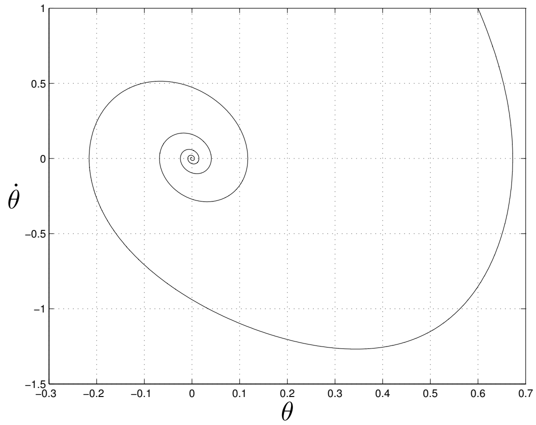


Fig. 10. Damped pendulum

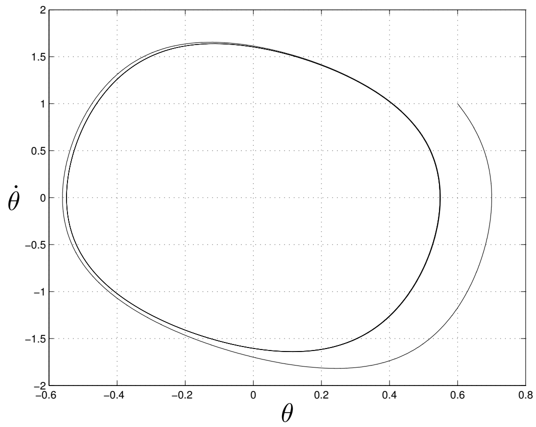


Fig. 11. Pendulum in limit cycle

torque) so the device must transform rotation to rotation. A set of gears would suffice if the transmission ratio would be constant, but in this case the ratio must be variable. A physical device with a variable transmission ratio is called a continuous variable transmission (CVT). There are several possible implementations to create a physical CVT. In the automobile industry it is most common to use two adjustable pulleys connected by a steel belt. The pulleys are created in such a way that their radius can be increased by compressing the pulley. The change of the radius results in a different transmission ratio. Another possible implementation is schematically drawn in Fig. 12. This implementation consists of a pair of lined up conic cylinders that are connected by a belt. By adjusting the position of the belt on the cylinders with active control, a continuous range of transmission ratios can be selected. The implementation with conic cylinders

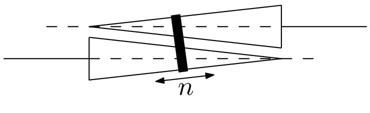


Fig. 12. CVT implementation with conic cylinders

seems more suitable for walking robots as it can be made long and thin, in order to fit in for example a leg or hip joint. Besides the variable transmission ratio the device will also have to be fitted with a reversing mechanism so that negative transmission ratios can be accomplished as well. The storage element can simply consist of a toroidal spring as described in the previous section. If one end of the spring is connected to a motor and a controller that ensures the spring is always under tension this will provide energy injection to overcome damping in the pendulum.

V. Conclusions and future work

It was shown how stable limit cycle oscillations can be induced in a passive system with the use of a nonlinear feedback. Similarly to the Van der Pol oscillator, this feedback pumps up small amplitude oscillation, but damps down large amplitude oscillation thus resulting in a stable oscillation of intermediate amplitude. The passive system and the nonlinear feedback continuously exchange energy. Whereas in the Van der Pol oscillator energy that flows into the nonlinear feedback is dissipated, it was shown how to convert this into a conservative system. With the combination of a modulated power continuous transmission and a storage element any nonlinear characteristic can be implemented by choosing the appropriate modulation factor. Energy that otherwise would be dissipated can now be reused and fed back to the system. It was also shown how the concept can be extended with a two port storage element to compensate for energy losses in the passive part of the system.

Systems that exhibit stable limit cycle oscillation are interesting for the development of robust, dynamic walking robots. Future research will focus on how the concept described in the paper can be implemented in a walking robot. It would be interesting to analyze what the exact influence of the nonlinear feedback is on the shape of the resulting limit cycle. Other points of interest are how the system can be generalized to higher dimensions and how the oscillation can be synchronized with impacts of the feet with the ground.

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CHAPTER 3

Design of Nonlinear Oscillators with Specified Periodic Behavior

Design of Nonlinear Oscillators with Specified Periodic Behavior

Michel van Dijk, Stefano Stramigioli

***Abstract*— Nonlinear oscillators can exhibit stable, self-induced oscillations known as limit cycle oscillations. Systems that need to perform a periodic repeating behavior can also be seen as oscillating systems. From this point of view, nonlinear oscillators are interesting in the control of such systems. This paper discusses how nonlinear oscillators can be designed with exactly the desired periodic behavior. Furthermore it is discussed how this can be applied to higher dimensional systems such as walking robots and how several oscillators can be synchronized.**

I. Introduction

ROBOTS have been in use in industrial production environments for decades. They have replaced human workers which resulted in production processes that are more efficient, more consistent and of higher throughput. Over the last few years interest in robots is also increasing for domestic appliances such as for example vacuum cleaning or floor washing [1]. These are relatively simple tasks, but it is expected that with the development of technology, robots will become more and more integrated in our daily life and being able to perform increasingly complex tasks.

Wheeled locomotion is very suitable on solid, flat surfaces, but is not very convenient in an uneven terrain. Walking is much more flexible and therefore more suitable for the environment in which we live in. Furthermore, walking can in fact be a very energy efficient way of locomotion if the underlying dynamics of the mechanism are optimally used [2]. There tend to be two different approaches in the control of walking robots. Most designs, such as Honda's Asimo [3], use control techniques involving Center of Pressure (CoP) and Zero Moment Point (ZMP) to calculate stable motion profiles for each actuator in the robot. Traditional control engineering then ensures that the robot follows exactly the desired trajectory. While this technique offers great flexibility as each joint can be individually controlled, a serious drawback is the energy consumption of this kind of systems. A complete different approach is that of passive dynamic walking as initiated by McGeer [4]. He was the first to show that it is possible to build a mechanism that can walk down a shallow slope without any actuation at all. All energy needed to compensate for friction and losses that result from impacts with the ground is supplied by gravity. The combination of the dynamics of the system, the slope and gravity results in a natural walking motion. This motion not only looks very natural and smooth, it clearly is also very energy efficient.

Inspired by the ideas of McGeer, researchers have been working on walking robots that are based on passive dynamic walking but are equipped with actuators so that they can walk on level ground. Examples of these types of robots can be found in [5], [6], [7]. In our own laboratory of the Control Engineering group of the University of Twente the robot Dribbel has been built [8]. The main problem with these walking robots is that it is hard to ensure robustness. Relatively small disturbances can cause the walking pattern to become unstable and causing to robot to fall down. As the walking can be interpreted as an oscillation in the shape space, our approach is to use nonlinear oscillators for developing robust walking robots. In previous work we showed how stable limit cycle oscillations can be induced in systems in an energy efficient way using a nonlinear feedback [9]. In this paper we outline how to design the feedback system to get exactly the periodic motion that is needed.

The remainder of this paper is organized as follows. In section II the basics of periodic systems and nonlinear oscillators are analyzed. Sections III and IV contain the main part of this paper. Section III discusses how oscillatory systems can be turned into controlled oscillators and section IV discusses how any predetermined periodic movement can be induced in a system. In section V we discuss extensions to higher dimensional oscillators and the synchronization between oscillators. Finally in section VI this paper is concluded and future research is discussed.

II. Analysis of periodic systems

In this section it is presented how periodic behavior is obtained in systems by looking both at controlled systems and at systems that behave as an oscillator. Also the basic concepts of limit cycle oscillation are presented, as this is widely used throughout this paper.

A. Periodicity in controlled systems

The most straightforward way to get a system to move periodically is to use a standard control system setup with a periodic reference signal as is shown in Figure 1. If the employed position controller is stiff enough this will result in the system following the reference signal closely. This is the approach that is commonly used in robots that use a Zero Moment Point type of control strategy. Motion profiles for each joint are calculated beforehand and the control system in each joint ensures that this profile is followed accurately.

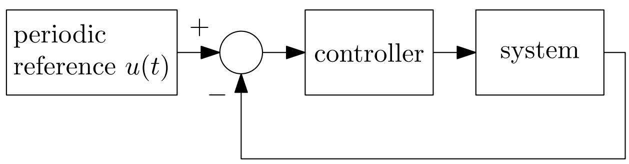

Fig. 1. Control system with periodic reference

There are however two main disadvantages of this approach compared to oscillators:

1) Absolute time dependency: The reference signal is a function of the absolute time and as a consequence the motion of the system becomes dependent on absolute time. Absolute time however is generally not relevant in the case of a walking robot. It does not matter if the phase of the movements is synchronized with absolute time, only the relevant position of each joint with respect to each other is of importance.

2) Energy consumption: By using a stiff controller the system is forced into a motion without accounting for the natural dynamics of the system. If the periodic reference does not match with the natural dynamics of the system, then the controller needs to supply energy to counteract the forces resulting from these dynamics. In many cases this is not necessary. Swinging a pendulum for example will consume a minimum amount of energy if it swings in its natural motion resulting from gravitational forces.

B. Periodicity in oscillators

An oscillator is a system that shows periodic behavior without the need for an external reference signal. Well-known examples are the harmonic oscillator described by $\ddot{x} + x = 0$ or the undamped pendulum $\ddot{x} + g \cdot \sin(x) = 0$ with unit length and mass and gravitational constant g. The motion of a system can be represented as a trajectory in the state space, which is $[x, \dot{x}]$ for these two second order systems. Periodic motion or oscillation is a closed orbit in the state space. In case of a linear system the number of possible closed orbits is infinite; The state space is filled with a continuum of concentric orbits and it dependents on the initial condition on what orbit the system settles. In case of disturbances the system changes from one orbit to another.

Nonlinear systems can have the attractive property of limit cycle oscillation. A limit cycle is a closed orbit in the state space with the additional property that it is isolated. This means there is no continuum of closed orbits surrounding the limit cycle and as a consequence trajectories must either spiral into or away from the limit cycle. In the first case the limit cycle is stable or attractive, in the other case it is an unstable limit cycle. Oscillators that show stable limit cycle oscillation are of interest because for a certain range of initial conditions known as the basin of attraction, they always converge to the same periodic movement. Since there is no external reference signal needed, this is also known as self sustained oscillation.

There exist nonlinear oscillators which have a stable limit cycle that is also globally attractive. By globally attractiveness it is meant that the system basin of attraction equals the entire state space. A well known example is the famous Van der Pol oscillator [10]:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + Kx = 0 \qquad (1)$$

and the closely related, but somewhat less famous Rayleigh oscillator:

$$\ddot{x} + \mu(\dot{x}^2 - 1)\dot{x} + Kx = 0. \qquad (2)$$

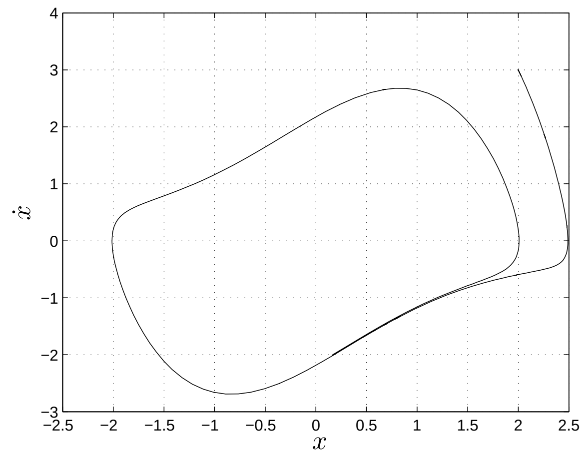

Fig. 2. Van der Pol oscillator

The working of these oscillators is easy to understand intuitively. The systems consist of a harmonic oscillator with a nonlinear damping term. In the Van der Pol case this term behaves as an ordinary damping for $x^2 > 1$ but it becomes a negative damping for $x^2 < 1$. For the Rayleigh oscillator this is the same except with x replaced by $\dot{x}$. As a result these systems behave as a damped oscillator for states far away from the origin with trajectories spiraling towards the origin. On the contrary, for states close to the origin the system becomes an unstable harmonic oscillator and trajectories spiral away from the origin. As trajectories in the state space cannot intersect this must result in an attractive limit cycle surrounding the origin. The origin itself is an unstable equilibrium in both systems. Figure 2 shows the limit cycle of the Van der Pol oscillator with parameters $\mu = 1$ and $K = 1$, converging from initial conditions $(2, 3)$. Detailed information on the Van der Pol oscillator can be found in [10], [11].

III. Control of oscillators

Similarly as in the Van der Pol and Rayleigh oscillators, limit cycles can be induced in an oscillatory system by means of a nonlinear feedback that is locally generative but globally dissipative as was explained in detail in [9]. The resulting system consisting of an oscillating system with a feedback controller is shown in Figure 3. The key question is what this feedback should look like in order to obtain exactly the limit cycle that is desired. Obviously shaping this feedback

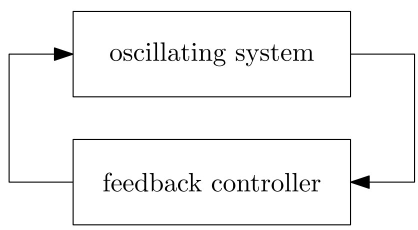

Fig. 3. Oscillating system with feedback

will affect the obtained limit cycle but when looking closer to the Van der Pol and Rayleigh oscillator it can be seen that we are limited in what resulting limit cycles we can get. The Van der Pol oscillator divides the state space in three regions along the x-axis whereas in case of the Rayleigh oscillator there are three regions along the $\dot{x}$-axis. The problem is that the feedback is always actively steering the system under control except when the system is on the border that separates the regions. Therefore the natural motion of the oscillatory system under control is always disturbed by the feedback. In case of the harmonic oscillator for example, it is impossible to get a pure sinusoidal oscillation.

The conclusion to be drawn is that in order not to disturb the natural oscillation, the feedback should do nothing as long as the system is on the limit cycle. So instead of dividing the state space into regions along the axes, it should be divided into a region inside and outside the limit cycle. If the current state is inside the limit cycle the energy in the system is too low and the feedback must be generative. If outside the limit cycle, there is too much energy in the system and the feedback must be dissipative so that the motion is forced back to the limit cycle. This concept is basically the same as both the Van der Pol and the Rayleigh oscillator, but by choosing the regions in this way any desired limit cycle can be obtained. The feedback law that realizes this concept is:

$$T = -\kappa \cdot D \cdot \dot{x}. \qquad (3)$$

Here D is a metric for the distance between the current state and the limit cycle and κ a proportional gain that determines the strength of the feedback. If the current state is outside the limit cycle D is positive, inside the limit cycle it is negative.

A. Implementation with Lagrange multipliers

To implement this feedback system it is necessary to calculate the value and sign of the distance to the limit cycle. Assuming there exists an equation $f(x, \dot{x}) = 0$ that defines the shape of the limit cycle, this can be done with the help of Lagrange multipliers. Given the harmonic oscillator with solution $x(t) = A \cdot \sin(\frac{t}{T})$ for example, the shape of the limit cycle is then given by:

$$\left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2 - 1 = 0. \qquad (4)$$

Where $y = \dot{x}$, $p = A$ and $q = \frac{A}{T}$. The situation is sketched in Figure 4 where $p = 1$, $q = 2$ and the pair (a, b) denotes the current state. The goal is now to find a pair (x, y) on the limit cycle such that the distance D between (a, b) and (x, y) is minimized. To simplify equations it is equally possible to minimize D^2 as this yields the same (x, y). D^2 is given by:

$$D^2 = (x - a)^2 + (y - b)^2. \qquad (5)$$

Equation (5) defines circles of radius D around (a, b). Starting from zero, increasing the radius D this will lead eventually to a situation where the circle around (a, b) is touching the limit cycle without intersecting it. In this case the limit cycle and the circle are tangent in (x, y) and

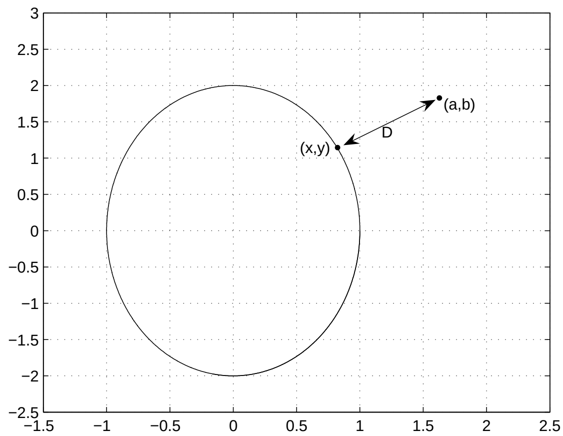

Fig. 4. Limit cycle of the harmonic oscillator

therefore the gradients of both must be equal apart from a scaling factor λ which is known as the Lagrange multiplier.

$$\begin{aligned} \nabla\left[(x-a)^2 + (y-b)^2\right] &= \lambda \cdot \nabla\left[\left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2\right] \\ \begin{bmatrix} 2(x-a) \\ 2(y-b) \end{bmatrix} &= \lambda \cdot \begin{bmatrix} 2\frac{x}{p} \\ 2\frac{y}{q} \end{bmatrix} \end{aligned} \qquad (6)$$

Combination of equations (4) and (6) results in the set of three nonlinear equations shown below.

$$\begin{aligned} 2(x-a) - 2\lambda\frac{x}{p} &= 0 \\ 2(y-b) - 2\lambda\frac{y}{q} &= 0 \\ \left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2 - 1 &= 0 \end{aligned} \qquad (7)$$

For the general case, the set of nonlinear equations is found by differentiating D_c, the distance constrained to the limit cycle with respect to x, y and λ, where D_c is defined as:

$$D_c(x, y, \lambda) = (x - a)^2 + (y - b)^2 + \lambda \cdot f(x, \dot{x}). \qquad (8)$$

The problem of minimizing the distance between the current state (a, b) and the limit cycle has thereby been reduced to solving this set of nonlinear equations and inserting the resulting (x, y) back in (5). Solving this set of equation however, may not be straightforward. In the example, substitution results in a fourth order polynomial which can be solved analytically, but this is not guaranteed for the general case. An approximation of the solution can be found using a root finding algorithm such as Newton-Raphson or a gradient search [12]. Care needs to be taken when implementing such an algorithm to prevent numeric instability. Also the complexity of the algorithm must be within certain limits to ensure that is applicable in a real-time controller. Instead of using Lagrange multipliers with the necessity of numerical algorithms a different approach can be used which simplifies the calculations considerably.

B. Implementation with energy control

For a certain class of systems the implementation can be simplified with the help of an energy function or Hamiltonian. If the system is conservative, the energy in the system will be constant and this energy level uniquely identifies the limit cycle of the system. This is generally the case for systems of the form $\ddot{x} + f(x) = 0$. By multiplying with $\dot{x}$ and then integrating over time we find the energy function of the state variables x and $\dot{x}$.

$$\begin{gathered} \ddot{x} \cdot \dot{x} + f(x) \cdot \dot{x} = 0 \\ \int \ddot{x} \cdot \dot{x} + f(x) \cdot \dot{x} \,\mathrm{d}t = E \\ \frac{1}{2}\dot{x}^2 + F(x) = E \end{gathered} \tag{9}$$

Thus, if the desired energy level E_d is known and the current energy level E_c is calculated as function of the current state, then D of equation (3) can be implemented as the difference of E_d and E_c. This gives the subsequent feedback rule:

$$T = -\kappa \cdot (E_c - E_d) \cdot \dot{x}. \tag{10}$$

When E_c is lower than E_d the feedback injects energy in the system while it damps down the system if E_c is too high. If E_c equals E_d, the system is exactly on the desired limit cycle and the feedback leaves the system untouched.

C. Change of dynamics with feedforward control

The feedback controller can also be used to impose an additional feedforward force on the system that is dependent on the state but not of the distance to the limit cycle. This relieves being restricted to the natural motion of the system and allows for inducing any motion of the form $\ddot{x} + g(x) = 0$. The feedforward term will consist of $g(x)$ as wel as a cancellation term for the natural dynamics of the system. In case of a system that is defined by $\ddot{x} + f(x) = T$, the feedback term becomes:

$$T = f(x) - g(x) - \kappa \cdot (E_d - E_c) \cdot \dot{x}. \tag{11}$$

The first term compensates the dynamics of the system, the second term induces the desired dynamics and the last term is the stabilizing term that ensures convergence to the limit cycle that is to be obtained.

In order to be able to implement this feedback, a model of the system is necessary to know how $f(x)$ looks like. In practice there will be differences between the model and the physical system. In a control setup where tracking of a reference signal is the goal, this may cause problems as unexpected dynamics are present. For the purpose of converging to a limit cycle however, small differences between the model and the system are acceptable. It will only result in a limit cycle that slight deviates from the desired one. Additionally, the stabilizing term will counteract deviations from the desired limit cycle which makes the effect of an inaccurate model smaller. The effects of differences between the model and the physical system can be even further reduced with help of a learning feedforward controller.

In the same way it is also possible to add a feedforward term that compensates for damping in the system. Without adding this term the damping in the system would have to be overcome by the stabilizing term of the controller. In that case the behavior would not follow the limit cycle but deviate to a trajectory where the stabilizing term and the damping cancel out. By adding damping compensation in the feedforward term this effect is reduced.

D. Example

To conclude this section an example is given that illustrates the proposed feedback system. Consider the harmonic oscillator to be swung up to an amplitude of one. The equation of motion is:

$$\ddot{x} + x = T, \tag{12}$$

and the energy function of the system is:

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2. \tag{13}$$

To obtain an amplitude of 1 this gives $E_d = 1$. The feedback law of (10) result in the trajectory of Figure 5 where we used $\kappa = 2$. As can be seen the oscillator is stabilized to the desired limit cycle with amplitude 1. An example including feedforward control to obtain different system behavior is given in the next section.

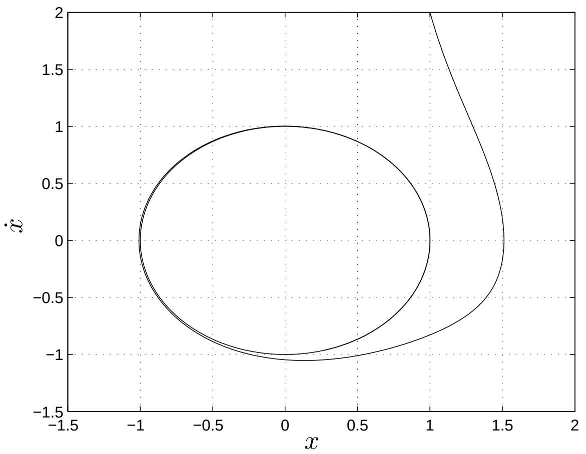

Fig. 5. Harmonic oscillator in limit cycle oscillation

IV. Control of periodic systems

In the previous section we looked at limit cycle oscillations where the shape of the oscillation emerges as a consequence of the system's natural dynamics. Using feedforward it is possible to change the dynamics, but one question not answered so far is how to induce a predetermined periodic movement $x(t)$ in a system. As explained before the absolute value of t is not relevant, it is only used to indicate the phase $\frac{t}{T}$ of the system. Instead of starting from the system dynamics it is now explained how to design a feedback controller that gives exactly $x(t)$ in the system under control.

A. Parameterization of the limit cycle

Given the desired motion $x(t)$ it is straightforward to calculate the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ by differentiation. The parameterized curve $C = (x(t), \dot{x}(t))$ then defines the trajectory we want to achieve. Projected on the state space, this trajectory may formally not be limit cycle as for certain $x(t)$ the trajectory could intersect itself. Considering the fact that the phase of the desired motion can be taken as extra state however, a non-intersecting trajectory in the state space is obtained and therefore it is always possible to indicate the desired trajectory as a limit cycle.

The advantage of this approach is not only that it can be used to specify exactly the desired motion but it also reduces the complexity of calculating the minimum distance between the limit cycle and the current state. Compared with the Lagrange multiplier approach with three unknowns, the distance now is a function of t only.

$$D^2 = (x(t) - a)^2 + (\dot{x}(t) - b)^2 . \qquad (14)$$

Although still nonlinear, this is just one equation in one unknown. The Newton-Raphson method can be used to find $\hat{t}$ that minimizes D^2, but this method has a risk of numerical instability if the first guess t_0 is not close enough to $\hat{t}$. Since $\hat{t}$ is restricted between 0 and T instability is easy to prevent by using the proposed algorithm below.

B. Outline of the algorithm

We now propose an algorithm to find $\hat{t}$ that is guaranteed to give a solution without risking numerical instability. The idea is start from t_0 and than take steps in direction of decreasing distance until the minimum is found. This is a stable solution because $\hat{t}$ must be in the interval $[t_0 - T, t_0 + T]$.

Step 1: Determine t_0
The first step of the algorithm is to determine a starting point t_0. If this is the first iteration of the algorithm then no $\hat{t}$ was estimated before and an initial guess has to be determined based on the current state (a, b). This can be an arbitrary t in the interval $[0, T]$. If this is not the first iteration then $\hat{t}$ that was estimated in the previous iteration is a good choice since the system's current state will be more or less in the neighborhood of the previous $\hat{t}$.

Step 2: Step towards minimum distance
First calculate the time derivative of D^2. If it is positive, then decrease t stepwise until the derivative changes sign. If it is negative then t should be increased until a change of sign is detected. The value of $\hat{t}$ that gives the minimum distance is now in the interval of the found values of t indicated by t_{curr} and t_{curr-1}.

Step 3: Determine $\hat{t}$
Since the solution $\hat{t}$ is now bracketed between t_{curr} and t_{curr-1} a simple bracketed algorithm can be used to find $\hat{t}$. The false position method or bisection algorithm [12] can for example be used to do this.

Step 4: Apply feedback law
Substitution of $\hat{t}$ in the equation for the acceleration gives $\ddot{x}$. With the model of the system it can then be calculated how much the feedforward torque has to be applied to obtain this acceleration. The stabilizing term ensures convergence to the limit cycle in the same way as in the previous section.

C. Example

To illustrate the approach outlined above this section is concluded with an example. As system to be controlled we take the undamped pendulum described by $\ddot{x} + g \cdot \sin(x) = 0$, where g is the gravitational constant. Suppose the periodic motion to be induced is $x(t) = \sin(2t) + \sin(t)$. By differentiation the velocity and acceleration are found:

$$\begin{aligned} \dot{x}(t) &= 2 \cdot \cos(2t) + \cos(t) \\ \ddot{x}(t) &= -4 \cdot \sin(2t) - \sin(t). \end{aligned} \qquad (15)$$

The feedback controller implements the algorithm as described above to calculate $\hat{t}$ that gives the minimum distance D_{min} between the limit cycle and the current state. The feedback law that brings the pendulum in the desired motion now becomes:

$$T = g \cdot \sin(x) + \left(-4\sin(2\hat{t}) - \sin(\hat{t})\right) - D_{min} \cdot \dot{x}. \qquad (16)$$

Where the first term compensates for the pendulum's natural dynamics, the second term feedforwards the torque needed to get the desired acceleration and the third term stabilizes to the desired limit cycle. The behavior of the system was verified using numerical simulation. In Figure 6 the convergence of the system to the limit cycle can be seen for initial conditions $(1, 1)$. Figure 7 shows the corresponding time plot of $x(t)$ from which can be seen that the system converges to the periodic behavior $x(t) = \sin(2t) + \sin(t)$.

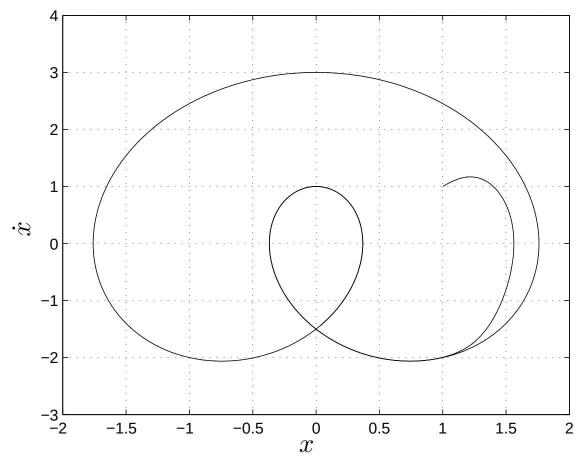

Fig. 6. Pendulum in intersecting limit cycle oscillation - state space

V. Extensions

In practical situations the systems to be controlled are usually not simple second order systems as discussed in

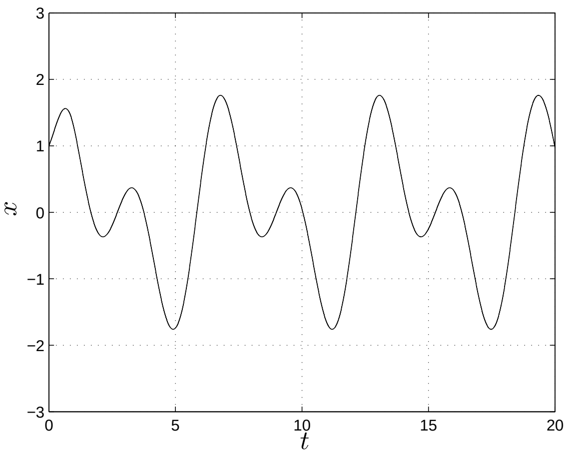

Fig. 7. Pendulum in intersecting limit cycle oscillation - time plot

this paper so far. Therefore this section explains how the introduced techniques can be applied for more advance applications. Two separated cases are discussed, synchronization of independent systems and a direct approach for higher order systems that are not independent.

A. Synchronization

In case of independent systems that are not mechanically coupled, a feedback controller that brings the system in limit cycle oscillation can be developed for each system separately. Synchronization between the system can be obtained be comparing the phase difference between the systems and change their relative speeds accordingly. The phase difference can be calculate by a separate phase controller connecting the systems as shown in Figure 8. The correction of the speeds of each system can be done by each feedback controller as this is achieved by changing the shape of the limit cycle. Increase of speeds is achieved by stretching the limit cycle along the $\dot{x}$-axis, while slowing down is achieved by shrinking the limit cycle.

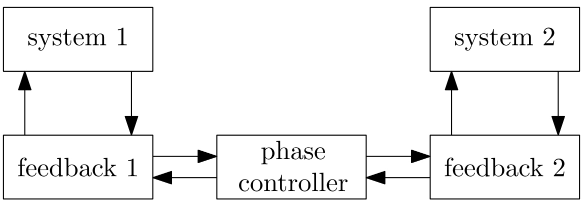

Fig. 8. Synchronization with phase controller

B. Higher order systems

In coupled mechanical systems such as for example a double pendulum,the second order subsystems cannot be controlled independently. This is because applying a torque to one system also influences the motion in the other system. For these kind of systems the limit cycle can be seen as a trajectory in a higher dimensional state space, for example fourth order for the double pendulum. This allows for direct application of the Lagrange multiplier approach or the parameterized limit cycle approach. In the last case the optimization is still one-dimensional as t is the only unknown variable; Only the equations for the distance become more complex. Unfortunately the energy based approach cannot be used for higher order system as it is not possible to express a trajectory in a space of three dimensions or higher by an energy function. Using an energy function in a three-dimensional space for example would define a sphere and not a trajectory.

VI. Conclusions

Inspired by the famous Van der Pol oscillator it was shown how limit cycle oscillations can be induced in a system using a nonlinear feedback. By proper design of a feedback controller the system can be stabilized on exactly the limit cycle that is desired. For simple systems this can be achieved with a straightforward energy based control. Using feedforward, the dynamics of the system can be changed and the shape of the limit cycle is altered. It is also possible to induce a specific periodic motion $x(t)$ with help of a somewhat more complicated control algorithm. This algorithm is computationally not very complex and therefore suitable for real-time applications. If the feedback controller is implemented with a CVT and buffer as in [9] this results in energy efficient limit cycle oscillations. Finally it was argued how these techniques can be extended for application on higher order system. Future research will focus on application of these concepts on robust, energy efficient walking robots. As a starting point a demonstration setup is going to be developed to proof this concept.

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