MASTER THESIS

DECOMPOSITIONS FOR STOCHASTIC PRODUCT FORM PETRI NETS

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Summary

Stochastic Petri nets are used to model many systems in which competition over resources plays an important part, such as computer systems, telecommunication networks or hospitals. Composition and decomposition of these stochastic Petri nets allows for a more efficient analysis and a better understanding of network behavior and performance. In this work we extended existing decomposition results, which allows us to decompose any $S\Pi^2$-net into separate common input bag classes. Secondly these decomposition results are used to formulate an algorithm to find the normalising constant and first order performance measures efficiently.

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Chapter 1

Introduction

Competition over resources is an important issue in many practical systems. For example, computer systems, telecommunication networks and hospitals fall in this category. These systems typically have different flows of items, such as data packages, customers or patients, that move along stochastic paths through the system where they require acces to shared resources in order to move along. Typical questions arising are identification of bottlenecks, achievable throughput and maximization of resource utilization. Therefore, performance analysis is an important issue in the design and implementation of such real life systems.

One of the models that can be used to analyze such systems is the stochastic Petri net. The concept of Petri nets was first published by Carl Adam Petri in 1962 [12] and is able to explicitly incorporate concurrency and competition in the system. However, just as for queueing networks, the huge state space required to describe these systems often makes computing exact solutions for these models infeasable. Different techniques to overcome these problems have been developped such as discrete-event simulation, approximate methods or exact analytical results. The latter approach aims at describing the steady-state probabilities and other performance measures as functions of a fixed set of parameters of the states, derived from the model structure. Models for which such solutions may be developed are said to be product form models, since the structure of the functions are a product of elementary terms corresponding to the parameters.

From different angles by several authors ([2, 3, 5, 6, 7]) a subclass of Petri nets has been identified that allows such a product form solution for the equilibrium distribution. This means that the equilibrium distribution can be written as a product of terms, where each term is a function of the marking of one of the places in the Petri net. This closed form for the equilibrium distribution completely defines it up to a normalising constant. So in order to completely find the equilibrium distribution the normalising constant still has to be determined, which is in general a difficult task. Somewhat efficient algorithms by Coleman [3] and Coyle et al. [4] are known for a specific subset of product form Petri nets.

Composition and decomposition of closed form results contribute to less computational effort requirements and greater understanding of network behavior and performance. They allow for studying a system by analyzing the characteristics of separate components. The first decomposition results of stochastic Petri nets into seperate common input bag classes were obtained by Kortbeek and Boucherie, for a more detailed overview of the literature available on product form stochastic Petri nets the reader is referred to their work [9].

The contribution of this research is twofold. First, we extend the decomposition results of Kortbeek and Boucherie [9] in such a way that every product form Petri net can be decomposed into seperate common input bags. Secondly we use this new decomposition to obtain an algorithm to find the normalising constant and the performance measures of a subset of product form Petri nets in an efficient way.

The rest of this thesis is organised as follows: in chapter 2 we will give an overview of the definitions and known results that we will need in order to establish our own results and we will give an algorithm to find all possible decompositions that could be obtained from the results of Kortbeek and Boucherie [9]. Then, in chapter 3 we will give our own decomposition result. Followed by several examples to illustrate this result in chapter 4. In chapter 5 we will show our method to obtain the normalising constant and the performance measures of a stochastic product form Petri net. Finally, in the last chapter we will give a discussion of our results and some possible directions for future research.

Chapter 2

Definitions and preliminaries

2.1 Definitions

In this section we give an overview of the definitions and results known from literature, that we will use in the subsequent sections. For a more extensive overview of the Petri net concept the reader is referred to the survey of Murata [11].

First we will introduce the concept of a Petri-net and some of its basic properties.

Definition 2.1 (Petri net)**.** *A Petri net is a 4-tuple* (P, T, I, O)*, where*

- $P = \{p_1, \ldots, p_{n_p}\}$ *is a finite set of places,*
- $T = \{t_1, \ldots, t_{n_t}\}$ *is a finite set of transitions,*
- $I, O : P \times T \to \mathbb{N}$ *are the input and output functions identifying the relation between the places and the transitions.*

Definition 2.2 (Marking)**.** *A marking* $\boldsymbol{m} = (m(i), i = 1, \ldots, n_p)$ *of a Petri net is a vector in* $\mathbb{N}_0^{n_p}$*, where* $m(i)$ *represents the number of tokens at place* p_i*.*

Definition 2.3 (Marked Petri net)**.** *A marked Petri net is a combination of a Petri net and an initial marking* $(P, T, I, O, \boldsymbol{m}_0)$

Definition 2.4 (Input bag and Output bag)**.** *The input bag and the output bag of transition* $t \in T$ *are defined as the vectors* $\boldsymbol{I}(t) = (I(p_1, t), \ldots, I(p_{n_p}, t))$ *and* $\boldsymbol{O}(t) = (O(p_1, t), \ldots, O(p_{n_p}, t))$*. They represent the number of tokens consumed at the places to fire transition* t *and the number of tokens released to the places after firing transition* t *respectively.*

Definition 2.5 (Transition enabling and firing)**.** *A necessary and sufficient condition for transition t to be enabled in marking $\boldsymbol{m}$ is that $m(i) \geq I(p_i, t)$, $i = 1, \ldots, n_p$. When transition t fires, then the next state of the Petri net is $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Symbolically this is denoted as $\boldsymbol{m}\,|t\!> \boldsymbol{m}'$.*

Definition 2.6 (Stochastic Petri net)**.** *A stochastic Petri net is a 5-tuple $\mathcal{SPN}$ $= (P, T, I, O, Q)$ where (P, T, I, O) is a Petri net and $Q = (q(t_1) \ldots q(t_{n_t}))$ is a set of exponential firing rates of the transitions $\{t_1 \ldots t_{n_t}\}$.*

Definition 2.7 (Firing sequence)**.** *A finite sequence of transitions $\sigma = t_{\sigma_1}, t_{\sigma_2}, \ldots, t_{\sigma_k}$ is a finite firing sequence of the Petri net if there exists a sequence of markings $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{k+1}$ for which $\boldsymbol{m}_i\,|t_{\sigma_i} > \boldsymbol{m}_{i+1}$, $i = 1, \ldots, k$. Symbolically this will be denoted as $\boldsymbol{m}_1\,|\sigma\!> \boldsymbol{m}_{k+1}$*

Definition 2.8 (Firing count vector)**.** *A vector $\bar{\sigma}$ is the firing count vector of the firing sequence σ if $\bar{\sigma}(t)$ equals the number of times transition t occurs in the firing sequence σ.*

Definition 2.9 (Incidence matrix)**.** *The incidence matrix $\boldsymbol{A}$ with entries $A(i, j)$ $= O(p_i, t_j) - I(p_i, t_j)$ describes the change in the number of tokens in place p_i when transition t_j fires.*

Definition 2.10 (State equation)**.** *If $\boldsymbol{m}_0\,|\sigma > \boldsymbol{m}$, then $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{A}\bar{\sigma}$. This equation is referred to as the state equation for the Petri net.*

Definition 2.11 (Reachable marking)**.** *A marking $\boldsymbol{m}'$ is reachable from marking $\boldsymbol{m}_0$ if there exists a firing sequence σ such that $\boldsymbol{m}_0\,|\sigma\!> \boldsymbol{m}'$.*

Definition 2.12 (Reachability set)**.** *The reachability set $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$ is the set of all markings of $\mathcal{PN}$ reachable from $\boldsymbol{m}_0$.*

Now that we have defined the basic concept of the Petri-net, we will look at two central concepts of Petri-nets called P-invariants and T-invariants. The structure of the T-invariants will allow us to identify the subclass of Petri-nets that allow a product form equilibrium distribution. The T-invariants will also allow us to identify seperate parts of the Petri-net which will form the basis of our decomposition results of Section 3.

Definition 2.13 (T-invariant)**.** *A vector $\boldsymbol{x} \in \mathbb{N}_0^{n_t}$ is a T-invariant if $\boldsymbol{x} \neq 0$ and $\boldsymbol{A}\boldsymbol{x} = 0$. From the state equation we obtain that a T-invariant is the firing count vector of a potential firing sequence that brings a marking back to itself.*

Definition 2.14 (P-invariant)**.** *A vector $\boldsymbol{y} \in \mathbb{N}_0^{n_p}$ is a P-invariant if $\boldsymbol{x} \neq 0$ and $\boldsymbol{y}\boldsymbol{A} = 0$. A P-invariant represents a weighted sum of the tokens present at the places that is constant for all reachable markings.*

Definition 2.15 (Support)**.** *The support of a T-invariant $\boldsymbol{x}$ or a P-invariant $\boldsymbol{y}$ is the set of transitions or places respectively corresponding to the non-zero entries of $\boldsymbol{x}$ and $\boldsymbol{y}$ and are denoted by $||\boldsymbol{x}||$ and $||\boldsymbol{y}||$.*

Definition 2.16 (Minimal invariant)**.** *A T-invariant $\boldsymbol{x}$ is minimal if there is no other T-invariant $\boldsymbol{x}'$ such that $x'(t) \leq x(t)$ for all $t \in T$. A minimal P-invariant is defined analogously.*

Definition 2.17 (Minimal support invariant)**.** *A support is minimal if no proper subset of the support is the support of another invariant. A minimal support invariant is a minimal invariant with a minimal support. I.e. $\boldsymbol{x}$ is a minimal support invariant if there is no invariant $\boldsymbol{x}'$ such that $||\boldsymbol{x}'|| \subset ||\boldsymbol{x}||$ or $x'(t) \leq x(t)$ for all $t \in T$.*

Definition 2.18 (Closed set)**.** *For $\mathcal{T} \subseteq T$ define $\mathcal{R}(\mathcal{T})$, the set of input and output bags for the transitions in $\mathcal{T}$, as $\mathcal{R}(\mathcal{T}) = \bigcup_{t \in \mathcal{T}} \{\boldsymbol{I}(t) \cup \boldsymbol{O}(t)\}$. $\mathcal{T}$ is a closed set if for all $g \in \mathcal{R}(\mathcal{T})$ there exist $t, t' \in \mathcal{T}$ such that $g = \boldsymbol{I}(t)$ and $g = \boldsymbol{O}(t')$, that is if each output bag is also an input bag and each input bag is also an output bag for a transition in $\mathcal{T}$.*

Definition 2.19 (Minimal closed support T-invariant)**.** *A minimal closed support T-invariant is a minimal support T-invariant whose support is a closed set. Let $ClT = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ be the set of minimal closed support T-invariants.*

Result 2.20 (Memmi and Roucairol [10])**.** *Every T-invariant can be written as a linear combination of minimal support T-invariants.*

$$\boldsymbol{x} = \sum_{i=1}^{p} \lambda_i \boldsymbol{x}^i$$

where $\lambda_i \in \mathbb{Q}, i = 1 \ldots p$. The equivalent result holds for P-invariants.

Result 2.21 (Boucherie and Sereno [1])**.** *A T-invariant $\boldsymbol{x}$ is a minimal closed support T-invariant if the firing sequence of $\boldsymbol{x}$ is linear, that is for each $t \in ||\boldsymbol{x}||$ there is a unique $t' \in ||\boldsymbol{x}||$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. Conversely, if the firing sequence of a T-invariant $\boldsymbol{x}$ is linear, then $\boldsymbol{x}$ is a closed support T-invariant.*

Definition 2.22 (Common input bag relation)**.** *Let $\boldsymbol{x}, \boldsymbol{x}' \in ClT$. We say that $\boldsymbol{x}, \boldsymbol{x}'$ are in common input bag relation (notation: $\boldsymbol{x}\ CI\ \boldsymbol{x}'$ if there exist $t \in ||\boldsymbol{x}||, t' \in ||\boldsymbol{x}'||$ such that $I(t) = I(t')$. The relation CI^* is the transitive closure of CI.*

Definition 2.23 (Common input bag classes)**.** *The common input bag class $CI(\boldsymbol{x})$ is the equivalence class of $\boldsymbol{x} \in ClT$, that is $CI(\boldsymbol{x}) = \{\boldsymbol{x}' | \boldsymbol{x}\ CI^*\ \boldsymbol{x}'\}$. Let $\{CI^1, \ldots, CI^k\}$ be the set of all common input bag classes and let $K = \{1, \ldots, k\}$ be the set of all indices of the common input bag classes.*

Definition 2.24 (Transition and place set of a CI class)**.** *The transition set of common input bag class CI^i, $\mathcal{T}(CI^i)$, is the set of all transitions belonging to common input bag class CI^i, i.e. $\mathcal{T}(CI^i) = \{t \in T | \exists \boldsymbol{x} \in CI^i : t \in ||\boldsymbol{x}||\}$.*

The place set of common input bag class CI^i, $\mathcal{P}(CI^i)$, is the set of all places belonging to common input bag class CI^i, i.e. $\mathcal{P}(CI^i) = \{p \in P | \exists t \in \mathcal{T}(CI^i) : I(p,t) > 0\}$.

The T-invariants allow us to define two subclasses of Petri-nets, the first are the $S\Pi$-nets which are nets that allow a product form equilibrium distribution over the markings of places if the firing rates satisfy an extra condition [6]. Second are the $S\Pi^2$-nets which allow a product form equilibrium distribution for all firing rates [6].

Definition 2.25 ($S\Pi$-net)**.** *A Π-net is a Petri net for which every transition $t \in T$ is covered by a minimal closed support T-invariant. That is for every $t \in T$ there is a minimum closed support T-invariant $\boldsymbol{x}^i$ such that $t \in ||\boldsymbol{x}^i||$. A $S\Pi$-net is a stochastic Π-net.*

Definition 2.26 ($S\Pi^2$-net)**.** *A Π^2-net is a Π-net such that for every $r \in \mathcal{R}(T)$, there is an $a_r \in \mathbb{Q}^{n_p}$ such that*

$$a_r \boldsymbol{A} = b_r$$

in which for $t = 1, \ldots, n_t$

$$b_r(t) = \begin{cases} -1 & \text{if } r = I(t), \\ 1 & \text{if } r = O(t), \\ 0 & \text{otherwise} \end{cases}$$

A $S\Pi^2$-net is a stochastic Π^2-net.

Result 2.27 (Kortbeek en Boucherie [9])**.** *A $S\Pi$-net is a $S\Pi^2$-net if and only if all minimal support T-invariants $\boldsymbol{x}$ are minimal closed support T-invariants.*

Definition 2.28 (Routing chain [8])**.** *The routing chain of a stochastic Petri-net $\mathcal{SPN} = (P, T, I, O, Q)$ was defined by Henderson et al. as the markov chain $\boldsymbol{Y} = (Y(t), t \geq 0)$ at finite state space $S = \{\boldsymbol{I}(t) | t \in T\}$ with transition rates*

$$q_Y(\boldsymbol{I}(t), \boldsymbol{I}(t')) = \sum_{\{t'' \in T | \boldsymbol{I}(t) = \boldsymbol{I}(t''), \boldsymbol{I}(t') = \boldsymbol{O}(t'')\}} q(t'').$$

The global balance equations for $\boldsymbol{Y}$ are, for all $t \in T$,

$$\sum_{I(t') \in \mathcal{R}(T)} y(\boldsymbol{I}(t)) q_Y(\boldsymbol{I}(t), \boldsymbol{I}(t')) - y(\boldsymbol{I}(t')) q_Y(\boldsymbol{I}(t'), \boldsymbol{I}(t)) = 0. \tag{2.1}$$

Result 2.29 (Kortbeek and Boucherie [9])**.** *The equilibrium distribution of a $S\Pi^2$-net $\mathcal{SPN}$ is given by*

$$\pi(\boldsymbol{m}) = B^{-1} \prod_{p=1}^{n_p} f_p^{m_p}, \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$$

where $f_p = e^{-z_p}$ for $\boldsymbol{z}$ a solution of $\boldsymbol{z}A = \boldsymbol{C}$ with $\boldsymbol{C} \in \mathbb{R}^{n_t}$ is a row vector defined as $C_t = \log(y(\boldsymbol{I}(t))/y(\boldsymbol{O}(t)))$, where $y(\cdot)$ is the solution of the routing chain (2.1) and B is a normalising constant such that

$$B = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \prod_{p=1}^{n_p} f_p^{m_p}.$$

Finally, for the SΠ^2-nets Kortbeek and Boucherie [9] provided a decomposition result in terms of T-invariants by identifying so-called surplus place sets and conflict places by showing that removing these from the net results in a decomposition.

Definition 2.30 (Sufficient place set and Surplus place set)**.** *A subset of places $\mathcal{P}^{suf} \subseteq P$ is a sufficient place set if for each initial marking $\boldsymbol{m}_0$, the marking of the places $p \in \mathcal{P}^{suf}$ combined with $\boldsymbol{m}_0$ provides sufficient information to uniquely define the marking of all places. A subset of places $\mathcal{P}^{sur} \subseteq P$ is a surplus place set if the subset op places $P \backslash \mathcal{P}^{sur}$ is a sufficient place set.*

Definition 2.31 (Conflict place)**.** *Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be minimal closed support T-invariants such that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, i.e. $CI(\boldsymbol{x}^1) \neq CI(\boldsymbol{x}^2)$. Let p be a place that is an element of both $\mathcal{P}(CI(\boldsymbol{x}^1))$ and $\mathcal{P}(CI(\boldsymbol{x}^2))$. Then p is called a conflict place of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$.*

Result 2.32 (Kortbeek and Boucherie [9])**.** *Consider a SΠ^2-net and a surplus place set P^{sur} with corresponding sufficient place set P^{suf}. If there is no transition for which the complete input bag is contained in the intersection of the surplus place set and the conflict place set, i.e. $\mathcal{P} = \{p \in P | p \in (\mathcal{P}^{con} \cap P^{sur})\}$ and $\nexists t \in T$ for which $\{p \in P | I(p,t) \geq 0\} \subseteq \mathcal{P}$, then*

- *removing all places $p \in \mathcal{P}$ and all arcs incident to these places yields s product form SΠ-nets: $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^s$; each $\mathcal{SPN}^i$ corresponding to one or more connected common input bag classes.*

- *the equilibrium distribution π of $\mathcal{SPN}$ is a product over the invariant measures of the subnets:*

$$\pi(\boldsymbol{m}) = B \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where $\boldsymbol{m}^i$ is the submarking in places that belong to $\mathcal{SPN}^i$ and $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ is the invariant measure of subnet $\mathcal{SPN}^i$ with

$$\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i) = \prod_{\{p \in \bigcup_{j \in I^i} \mathcal{P}(CI^j) \backslash \mathcal{P}\}} f_p^{m(p)},$$

where CI^j, $j \in I^i \subset K$, denote the common input bag classes contained in subnet $\mathcal{SPN}^i$, and B is normalising constant such that

$$B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$$

.

2.2 Finding all decompositions

Since a surplus place set is in general not unique, the decomposition according to result 2.32 is not unique either. Every choice of a surplus place set will remove a complete input bag, result in a decomposition or keep the net in one piece. In order to find all possible decompositions, all possible surplus place sets will have to be identified. To this end lemma 2.33 is presented.

Lemma 2.33. *A set of places $\mathcal{P} \subseteq P$ from a $S\Pi$-net is a sufficient place set with corresponding surplus place set $\bar{\mathcal{P}} = P \backslash \mathcal{P}$ if and only if all the rows of A can be written as linear combinations of the rows of A corresponding to places in $\mathcal{P}$, i.e.*

$$\alpha_j = \sum_{i \in \mathcal{P}} \lambda_{ij} \alpha_i, \forall j \in P \tag{2.2}$$

where α_p is the row of A corresponding to place p.

Proof. First we show that this is a sufficient condition. For every reachable marking m there is a firing sequence σ such that $m_0|\sigma > m$ which means that $m = m_0 + A\bar{\sigma}$. So if condition (2.2) on the rows of A is met then we know

$$m(j) = m_0(j) + \alpha_j \bar{\sigma} = m_0(j) + \sum_{i \in \mathcal{P}} \lambda_{ij} \alpha_i \bar{\sigma} = m_0(j) + \sum_{i \in \mathcal{P}} \lambda_{ij} m(i)$$

therefore the marking of any place j can be uniquely determined from the markings of the places in $\mathcal{P}$.

In order to show necessity we first assume that α_j, $j \in \bar{P}$ can not be written as a linear combination of the rows of A corresponding to places in $\mathcal{P}$. Then it is possible to find a solution $x \in \mathbb{Q}^{n_t}$ to the system of equations

$$\begin{aligned} \alpha_i x &= 0, \forall i \in \mathcal{P} \\ \alpha_j x &= 1 \end{aligned}$$

Because $x \in \mathbb{Q}^{n_t}$ it is possible to find a $c \in \mathbb{Z}/0$ such that $cx \in \mathbb{Z}^{n_t}$. Furthermore because of the definition of a $S\Pi$-net we know that every transition t is covered by a minimal closed support T-invariant x_i, so it is possible to find c_i such that $cx + \sum_i c_i x_i \in \mathbb{N}_+^{n_t}$. Therefore it is possible to construct a firing sequence σ with a firing count vector $\bar{\sigma} = cx + \sum_i c_i x_i$. For any such firing sequence σ it is possible to find an initial marking m_0^σ from which σ can be fired to get marking m^σ, i.e. $m_0^\sigma |\sigma > m^\sigma$. The two markings m^σ and m_0^σ are different because $m^\sigma(j) = m_0^\sigma(j) + \alpha_j \bar{\sigma} = m_0^\sigma(j) + \alpha_j(cx + \sum_i c_i x_i) = m_0^\sigma(j) + c$. However for these two markings we know that $m^\sigma(p) = m_0^\sigma(p) + \alpha_p \bar{\sigma} = m_0^\sigma(p) + \alpha_p(cx + \sum_i c_i x_i) = m_0^\sigma(p)$ for all $p \in \mathcal{P}$. Therefore, from places $p \in \mathcal{P}$ it is impossible to tell whether m_0^σ or m^σ is observed and therefore $\mathcal{P}$ can not be a a sufficient place set. This contradicts the initial assumption that $\mathcal{P}$ is a sufficient place set. So α_j has to be a linear combination of the rows of A corresponding to the places in $\mathcal{P}$. □

Next we will present an algorithm to check whether or not a give set is a surplus place set. After a short example to clarify the given algorithm a lemma will be formulated to show that this algorithm is indeed correct. The algorithm will form the basis of the algorithm to generate all possible decompositions resulting from Theorem 2.32.

Algorithm 2.34 (Checking a surplus place set of a bounded $S\Pi$-net)**.**

Step 1: *Consider a structurally bounded $S\Pi$-net $\mathcal{SPN}$ and a potential surplus place set $\mathcal{P} \in P$.*

Step 2: *Obtain the set of minimal closed support P-invariants $\{y^1 \dots y^p\}$ and a basis $\{\bar{y}^1 \dots \bar{y}^r\}$ composed of elements from $\{y^1 \dots y^p\}$. Define the matrix $\boldsymbol{Y}$ consisting of rows $\{y^1 \dots y^p\}$.*

Step 3: *Swap the columns of $\boldsymbol{Y}$ such that the columns corresponding to places $p \in \mathcal{P}$ are in the front. Denote the obtained matrix by $\tilde{\boldsymbol{Y}}$.*

Step 4: *Apply Gauss-Jordan elimination on $\tilde{\boldsymbol{Y}}$ to obtain its reduced row echelon form $rref(\tilde{\boldsymbol{Y}})$.*

Step 5: *If and only if $rref(\tilde{\boldsymbol{Y}})$ contains leading ones in each of the first $|\mathcal{P}|$ columns, then $\mathcal{P}$ is a surplus place set.*

Before we show that Algorithm 2.34 is correct, we will first give a small example to illustrate how the algorithm works.

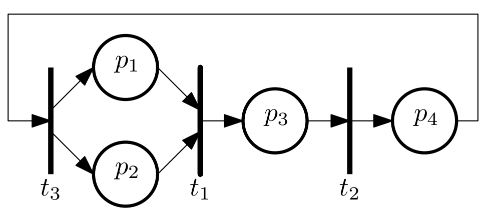


Figure 2.1: Petri net of Example 2.34

Example 2.35. *Consider the Petri-net shown in figure 2.1. We will now use Algorithm 2.34 to find the surplus place set $\mathcal{P}^{sur} = \{p_1, p_3\}$.*

Step 2: *From the incidence matrix*

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

we obtain the minimal closed support P-invariants $y^1 = [1011]$ and $y^2 = [0111]$. These are linearly independent so $\bar{y}^i = y^i$ for $i = 1, 2$. This gives us the matrix

$$\boldsymbol{Y} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step 3: *We want to check potential surplus place set* $\mathcal{P} = \{p_1, p_3\}$ *so we swap column 2 and 3 to obtain*

$$\tilde{\boldsymbol{Y}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step 4: *Applying Gauss-Jordan elimination to* $\tilde{\boldsymbol{Y}}$ *we obtain*

$$rref(\tilde{\boldsymbol{Y}}) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step 5: *We see that* $rref(\tilde{\boldsymbol{Y}})$ *contains leading ones in the first two rows, so* $\mathcal{P}^{sur}$ *is indeed a surplus place set.*

Observe that $\alpha_{p_1} = \alpha_{p_2}$ *and* $\alpha_{p_3} = -\alpha_{p_2} - \alpha_{p_4}$. *So according to Lemma 2.33,* $\{p_1, p_3\}$ *is indeed a surplus place set.*

Lemma 2.36. *Algorithm 2.34 concludes that* $\mathcal{P}$ *is a surplus place set if and only if* $\mathcal{P}$ *is a surplus place set.*

Proof. The theorem is equivalent to the statement that $rref(\tilde{\mathbf{Y}})$ contains leading ones in the first $|\mathcal{P}|$ columns if and only if $\mathcal{P}$ is a surplus place set. First we will show the only if part.

Let $\tilde{\mathbf{A}}$ be the permutation of $\mathbf{A}$ corresponding to the permutation used to obtain $\tilde{\mathbf{Y}}$. By definition of the P-invariants we know $\mathbf{YA} = 0$, so also $\tilde{\mathbf{Y}}\tilde{\mathbf{A}} = 0$ and $rref(\tilde{\mathbf{Y}})\tilde{\mathbf{A}} = 0$. Let $\tilde{y}^i$ be the ith row of $rref(\tilde{\mathbf{Y}})$. Now we know that $rref(\tilde{\mathbf{Y}})$ has leading ones in the first $|\mathcal{P}|$ columns so for every place $j \in \mathcal{P}$ there is an i such that $\tilde{y}^i(j) = 1$ and $\tilde{y}^i(p) = 0$ for all $p \in \mathcal{P}\backslash j$. Furthermore $\tilde{y}^i\tilde{A} = 0$ so $\alpha_j = \sum_{p \in P\backslash\mathcal{P}} \tilde{y}^i(p)\alpha_p$ and according to Theorem 2.33 $\mathcal{P}$ is a surplus place set.

For the second part, if $\mathcal{P}$ is a surplus place set then from Lemma 2.33 and equation 2.2 we can find a vector $z_i \in \mathbb{Q}^{n_p}$ for each $i \in \mathcal{P}$ such that $z_i\tilde{\mathbf{A}} = 0$ by taking $z_i(i) = 1$, $z_i(p) = 0$ for all $p \in \mathcal{P}\backslash i$ and $z_i(p) = -\lambda_{pi}$ for all $p \in P\backslash\mathcal{P}$. $\mathcal{SPN}$ is bounded so we know that it is covered by P-invariants. This means that for every $i \in \mathcal{P}$ there is a P-invariant $\mathbf{y}$ and a scalar $c \in \mathbb{N}^+$ such that $cz_i + \mathbf{y}$ is a P-invariant. This means that from Result 2.20 we know that $z_i \in span(\{y^1 \dots y^p\})$ and so $z_i \in rowspan\mathbf{Y}) = rowspan(\tilde{\mathbf{Y}}) = rowspan(rref(\tilde{\mathbf{Y}}))$. Now assume that $rref(\tilde{\mathbf{Y}})$ does not have leading ones in the first $|\mathcal{P}|$ columns. Let j be the first column that does not contain a leading one. Then by showing that the equation

$$z_j = \lambda rref(\tilde{\mathbf{Y}})$$

has no solution $\lambda \in \mathbb{R}^r$, we obtain the contradiction $z_j \notin rowspan(rref(\tilde{\mathbf{Y}}))$, from which we can conclude that $rref(\tilde{\mathbf{Y}})$ must have leading ones in the first $|\mathcal{P}|$ columns. j was the first column that did not have a leading one so $z_j(i) = 0$ for $i < j$ implies $\lambda(i) = 0$. However $z_j(j) = 1$ implies that there must be an $i < j$ such that $lambda(i) \neq 0$, because $rref(\tilde{\mathbf{Y}})$ contains only zeros at and below row j, otherwise it could have been a pivot column during the Gauss-Jordan elimination. So equation (2.2) does not have a solution. □

Using Algorithm 2.34 we can make the following algortihm that finds all possible decompositions of $S\Pi^2$-net $\mathcal{SPN}$ following from Result 2.32.

Algorithm 2.37. *Finding all decompositions*

Step 1: *Consider a structurally bounded $S\Pi^2$-net $\mathcal{SPN}$. Determine from the set of common input bag classes the set of conflict places $\mathcal{P}^{con}$.*

Step 2: *Obtain the powerset $P^{con}_{all} = \mathcal{P}(\mathcal{P}^{con})$. Remove from P^{con}_{all} all sets that contain a complete input bag. Start with an empty set of surplus place sets that generate a decomposition $P^{sur}_{all} = \emptyset$.*

Step 3: *Take an element $\mathcal{P} \in P^{con}_{all}$ and apply Algorithm 2.34 to check whether or not $\mathcal{P}$ is a surplus place set. If it is then go to step 4 else go to step 5.*

Step 4: *Remove $\mathcal{P}$ and all its subsets from P^{con}_{all}, $P^{con}_{all} := P^{con}_{all}/\mathcal{P}(\mathcal{P})$ and add them to the set of surplus place sets $P^{sur}_{all} := P^{sur}_{all} \cup \mathcal{P}(\mathcal{P})$. Go to step 6.*

Step 5: *Remove $\mathcal{P}$ and all its supersets from P^{con}_{all}, i.e. $P^{con}_{all} := P^{con}_{all}/\{\tilde{\mathcal{P}}|\tilde{\mathcal{P}} \in P^{con}_{all}, \mathcal{P} \subseteq \tilde{\mathcal{P}}\}$.*

Step 6: *If $P^{con}_{all} \neq \emptyset$ then return to step 3 else continue to step 7.*

Step 7: *Each possible surplus place set $\mathcal{P}^{sur} \in P^{sur}_{all}$ results in a possible decomposition according to Result 2.32*

It should be noted that the efficiency of Algorithm 2.37 heavily depends on the order in which the potential surplus place sets are tried during step 3. It may also be possible to save a lot of time by memoization of the intermediate results of Algorithm 2.34, because if the first few points chosen during two different runs of Algorithm 2.34 are the same, then the first few steps of the Gauss Jordan elimination will be exactly the same, so they could be skipped if the intermediate results were still available. We did not investigate the efficiency of Algorithm 2.37 in detail, since it is not our main focus. We merely wanted to show that it is possible to find all possible decompositions.

Example 2.38. *In order to illustrate the application of Algorithm 2.37 we will give an example of a $S\Pi^2$-net that can be decomposed in two distinct ways. Consider the Petri-net in Figure 2.2. From the incidence matrix:*

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

we obtain the three minimal support T-invariants $\boldsymbol{x}^1 = [110000]$, $\boldsymbol{x}^2 = [001100]$ and $\boldsymbol{x}^3 = [000011]$ and four minimal support P-invariants $\boldsymbol{y}^1 = [1101000]$, $\boldsymbol{y}^2 =$

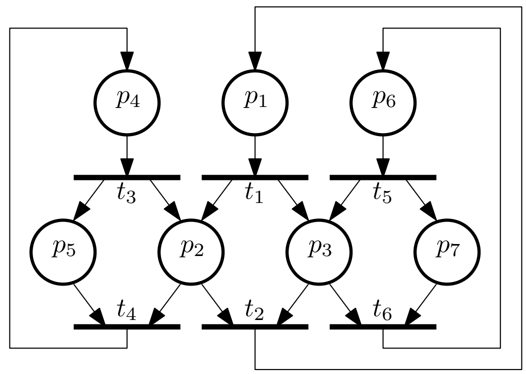


Figure 2.2: Petri net of Example 2.38

[1010010], $\boldsymbol{y}^3 = [0001100]$ *and* $\boldsymbol{y}^4 = [0000011]$, *which are linearly independent. As we can see the three minimal support T-invariants are all closed, so the Petri-net is a* $S\Pi^2$*-net. Furthermore* $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ *and* $\boldsymbol{x}^3$ *are not in common input bag relation so they result in three CI classes,* $CI^1 = \{\boldsymbol{x}^1\}$, $CI^2 = \{\boldsymbol{x}^2\}$ *and* $CI^3 = \{\boldsymbol{x}^3\}$. *This results in the conflict place set* $\mathcal{P}^{con} = \{p_2, p_3\}$. *This gives us enough information to perform Algorithm 2.37.*

Step 1: *The conflict place set is* $\mathcal{P}^{con} = \{p_2, p_3\}$.

Step 2: *Therefore* $\mathcal{P}^{con}_{all} = \{\{p_2\}, \{p_3\}, \{p_2, p_3\}\}$ *from which* $\{p_2, p_3\}$ *is removed because it contains the whole input bag of transition* t_2.

Step 3-6: *Using Algorithm 2.34 we find that both* $\{p_2\}$ *and* $\{p_3\}$ *are surplus place sets.*

Step 7: *This gives us two possible decompositions, both of these options decompose the net in two pieces such that*

$$\pi(\boldsymbol{m}) = B\pi_y^{SPN^1}(\boldsymbol{m}^1)\pi_y^{SPN^2}(\boldsymbol{m}^2), m \in \mathcal{M}(SPN, m_0),$$

where if we use the first surplus place set $\{p_2\}$ *we get* SPN^1 *containing* CI^1 *and* CI^3 *while* SPN^2 *contains* CI^2 *and this results in the following equilibrium distributions*

$$\begin{aligned}
\pi_y^{SPN^1}(\boldsymbol{m}^1) &= \left(\frac{\mu_2}{\mu_1}\right)^{m^1(p_1)} \left(\frac{\mu_6}{\mu_5}\right)^{m^1(p_6)} \\
\pi_y^{SPN^2}(\boldsymbol{m}^2) &= \left(\frac{\mu_4}{\mu_3}\right)^{m^2(p_4)} \\
\pi(\boldsymbol{m}) &= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)} \left(\frac{\mu_6}{\mu_5}\right)^{m(p_6)} \left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)}.
\end{aligned}$$

When we use the second surplus place set $\{p_3\}$ *to obtain the decomposition we get* SPN^1 *containing* CI^1 *and* CI^2 *while* SPN^2 *contains* CI^3*. This results in the following equilibrium distributions*

$$\begin{aligned}
\pi_y^{SPN^1}(\boldsymbol{m}^1) &= \left(\frac{\mu_2}{\mu_1}\right)^{m^1(p_1)} \left(\frac{\mu_4}{\mu_3}\right)^{m^1(p_4)} \\
\pi_y^{SPN^2}(\boldsymbol{m}^2) &= \left(\frac{\mu_6}{\mu_5}\right)^{m^2(p_6)} \\
\pi(\boldsymbol{m}) &= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)} \left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)} \left(\frac{\mu_6}{\mu_5}\right)^{m(p_6)}.
\end{aligned}$$

Chapter 3

Decomposition by adding bag count places

This section will introduce the Bag Count Place Extended Petri-net of a bounded $S\Pi^2$-net (BCPE-$S\Pi^2$-net) by defining a set of bag count places and adding these to the net. The definition of these places is such that the marking of these bag count places has a one-to-one correspondence to the marking of the original places of the $S\Pi^2$-net, once a choice for the initial marking of these places is made. This will enable us to decompose the extended $S\Pi^2$-net in seperate components for each common input bag class. The equilibrium distribution of the bag count places will provide an equilibrium distribution of the original places, because of the one-to-one correspondence between the marking of the original places and the bag count places.

Definition 3.1. *BCPE-$S\Pi^2$-net*

Given a bounded $S\Pi^2$-net $\mathcal{SPN} = (P, T, I, O, Q)$ define for every bag $r \in \mathcal{R}(T)$ a bag count place $\tilde{p}_r$. Let $\tilde{P} = P \cup \{\tilde{p}_r | r \in \mathcal{R}(T)\}$ and $\tilde{I}, \tilde{O} : \tilde{P} \times T \to \mathbb{N}$ where

$$\tilde{I}(p,t) = \begin{cases} I(p,t) & \text{if } p \in P \\ 1 & \text{if } p = \tilde{p}_r, r = I(t) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{O}(p,t) = \begin{cases} O(p,t) & \text{if } p \in P \\ 1 & \text{if } p = \tilde{p}_r, r = O(t) \\ 0 & \text{otherwise} \end{cases}$$

Now let $\widetilde{\mathcal{SPN}} = (\tilde{P}, T, \tilde{I}, \tilde{O}, Q)$ be the BCPE-$S\Pi^2$-net of $\mathcal{SPN}$.

Definition 3.1 defines the structure of the BCPE-$S\Pi^2$-net. This net will be used to analyse the behaviour of the original net. The bag count places, in particular, will be used to monitor the behaviour of the rest of the net. In order for this to be possible we first have to show that the addition of these places to form the BCPE-$S\Pi^2$-net does not influence the behaviour of the net on the original places. First, we will show that the BCPE-$S\Pi^2$-net is still a $S\Pi^2$-net and second we give two conditions for the initial marking of the BCPE-$S\Pi^2$-net that guarantee that a firing sequence σ can be fired in the original net if and only if σ can be fired in the BCPE-$S\Pi^2$-net. This will guarantee that the behaviour of both nets on the original places will be exactly the same. In Lemma 3.6 it will be shown that it is always possible to find an initial marking that satisfies these conditions.

Lemma 3.2. *The BCPE-$S\Pi^2$-net $\widetilde{\mathcal{SPN}}$ of a $S\Pi^2$-net $\mathcal{SPN}$ is a $S\Pi^2$-net.*

Proof. In order to show that $\widetilde{\mathcal{SPN}}$ is a $S\Pi^2$-net we will show that every T-invariant of $\widetilde{\mathcal{SPN}}$ is also a T-invariant of $\mathcal{SPN}$ and every minimal closed support T-invariant of $\mathcal{SPN}$ is a closed support T-invariant of $\widetilde{\mathcal{SPN}}$. These two statements combined show that every minimal support T-invariant of $\widetilde{\mathcal{SPN}}$ is a minimal closed support T-invariant. By Result 2.27 this means that $\widetilde{\mathcal{SPN}}$ is a $S\Pi^2$-net.

Any T-invariant $\mathbf{x}$ of $\widetilde{\mathcal{SPN}}$ is a T-invariant of $\mathcal{SPN}$, because by construction the first $|P|$ rows of the incidence matrix $\tilde{\mathbf{A}}$ are equal to the rows of $\mathbf{A}$. This means that $\tilde{\mathbf{A}}\mathbf{x} = 0$ implies $\mathbf{A}\mathbf{x}$.

Every minimal closed support T-invariant $\mathbf{x}$ of $\mathcal{SPN}$ is also a T-invariant of $\widetilde{\mathcal{SPN}}$, because by Result 2.21 the firing sequence of $\mathbf{x}$ is linear. This means that for any transition $t \in ||\mathbf{x}||$ there is a unique $t' \in ||\mathbf{x}||$ such that $\mathbf{I}(t) = \mathbf{O}(t)$ and by construction of the BCPE-$S\Pi^2$-net this means that $\tilde{\alpha}_{\tilde{p}_{I(t)}}\mathbf{x} = 0$. For each input bag r for which there is no transition $t \in ||\mathbf{x}||$ such that $\mathbf{I}(t) = r$ there is also no transition $t' \in ||\mathbf{x}||$ such that $\mathbf{O}(t) = r$. So $\tilde{\alpha}_{\tilde{p}_r}\mathbf{x} = 0$ for all $r \in \mathcal{R}(T)$. Finally, $\mathbf{x}$ is also closed in $\widetilde{\mathcal{SPN}}$, because by construction if $\mathbf{I}(t) = \mathbf{O}(t)$ then also $\tilde{\mathbf{I}}(t) = \tilde{\mathbf{O}}(t)$ □

Lemma 3.3. *If the initial marking, $\tilde{m}_0$, of the BCPE-$S\Pi^2$-net $\widetilde{\mathcal{SPN}}$, belonging to the marked $S\Pi^2$-net $(\mathcal{SPN}, m_0)$, satisfies the following two conditions:*

1. *$\tilde{m}_0(p) = m_0(p)$ for all $p \in P$*
2. *$\tilde{m}(p) > 0$, for all $p \in \tilde{P} \backslash P$ and for each $\tilde{m} \in \mathcal{M}(\widetilde{\mathcal{SPN}}, \tilde{m}_0)$,*

then any firing sequence σ can be fired in $\mathcal{SPN}$ from m_0 if and only if it can be fired in $\widetilde{\mathcal{SPN}}$ from $\tilde{m}_0$.

Proof. Every firing sequence σ that can be fired from $\tilde{m}_0$ in $\widetilde{\mathcal{SPN}}$ can also be fired from m_0 in $\mathcal{SPN}$, because from condition 1 we know that both initial markings are equal on the original places and any transition $t \in T$ consumes and deposits the same number of tokens from the same original places in both nets. This means that during any step of the firing sequence the marking on the original places of both nets will be equal and in $\mathcal{SPN}$ there are no other places to put restrictions on the enabling of any transition, so if σ can be fired in $\widetilde{\mathcal{SPN}}$

it can also be fired in $\mathcal{SPN}$. Conversely, the original places of $\widetilde{\mathcal{SPN}}$ will never disable a firing sequence that can be fired in $\mathcal{SPN}$, because the marking of the original places will always be equal in both nets. The bag count places will never disable a transition, because by construction every transition consumes at most 1 token from a bag count place. So condition 2 ensures that a transition will never be disabled because of a shortage of tokens on the bag count places. □

Now that we know that the behaviour of the $S\Pi^2$-net and its BCPE-$S\Pi^2$-net are equal, we will show that there is a one-to-one correspondence between the marking of the original places and the marking of the bag count places. This means that the bag count places are a sufficient place set and this will allow us to apply Result 2.32 to get a new decomposition result in Theorem 3.7.

Lemma 3.4. *The marking of the bag count places in the BCPE-$S\Pi^2$-net can be expressed in the marking of the original places as follows*

$$\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r) = a_r(m - m_0),$$

where a_r is any vector as given in definition 2.26.

Proof. For every reachable marking $\tilde{m}$ there is a firing sequence σ such that $\tilde{m}_0|\sigma > \tilde{m}$. This means that $\tilde{m} - \tilde{m}_0 = \tilde{A}\bar{\sigma}$. For the row belonging to $\tilde{p}_r$ this means that $\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r) = \tilde{\alpha}_{\tilde{p}_r}\bar{\sigma}$. From Definition 3.1 it follows that the row of the incidence matrix belonging to $\tilde{p}_r$ is equal to the vector b_r, from Definition 2.26, so $\tilde{\alpha}_{\tilde{p}_r} = b_r = a_r A$. Combining these results gives the following expression:

$$\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r) = \tilde{\alpha}_{\tilde{p}_r}\bar{\sigma} = a_r A\bar{\sigma} = a_r(m - m_0).$$

It should be noted that neither a_r nor σ is uniquely defined, however for all a_r^1, a_r^2 satisfying the conditions in definition 2.26 and all σ_i such that $m_0|\sigma_i > m$ for $i \in \{1, 2\}$ we know that

$$a_r^1 A\bar{\sigma}_1 = b_r\bar{\sigma}_1 = a_r^2 A\bar{\sigma}_1 = a_r^2(m - m_0) = a_r^2 A\bar{\sigma}_2.$$

This means that the marking of the bag count places can be uniquely determined from the marking of the original places, independent of the choice of a_r and firing sequence σ. □

Before stating the next theorem it should be noted that the marking of a bag count place $\tilde{p}_r$ changes if and only if a transition fires that either uses r as its input bag, in this case the marking of $\tilde{p}_r$ decreases by one, or creates r as its output bag, in this case the marking of $\tilde{p}_r$ increases by one. So the marking of $\tilde{p}_r$ indicates the number of times bag r is created minus the number of times bag r is used. This insight immediatly shows how to obtain the marking of the original places from the marking of the bag count places.

Lemma 3.5. *The marking of the original places of the $S\Pi^2$-net can be expressed in the marking of the bag count places of the BCPE Petri net as follows*

$$m - m_0 = \sum_{r \in \mathcal{R}(T)} (\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r))r.$$

Proof. As stated above for every bag r the marking $\tilde{m}(\tilde{p}_r)$ indicates exactly how many times bag r is created minus the number of times bag r is used. This means that the current marking of the petri net can be found by starting from the initial marking m_0 and adding $\tilde{m}(\tilde{p}_r)$ times bag r to it for every bag $r \in \mathcal{R}(T)$. □

Combining these two results we see that there is a one-to-one correspondence between the marking of the original places and the marking of the bag count places. Moreover, the marking of the bag count places is a linear combination of the marking of the original places. This means that if the original $S\Pi^2$-net was bounded, then the corresponding marking of the bag count places is bounded and an initial marking $\tilde{m}_0$ for the bag count places can be chosen such that the marking of the bag count places never drops below 1 as is required by the conditions of Lemma 3.3.

Lemma 3.6. *For a structurally bounded $S\Pi^2$-net $\mathcal{SPN}$ and for every initial marking m_0 an initial marking $\tilde{m}_0$ can be chosen such that for any reachable marking $\tilde{m}$, $\tilde{m}(\tilde{p}_r) > 0$.*

Proof. From theorem 3.4 we know that $\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r) = a_r(m - m_0)$ and since $\mathcal{SPN}$ is bounded we know that there is a constant C_p such that $0 \le m(p) < C_p$ for all $p \in P$. Therefore

$$C_1 = \sum_{p \in P} \min(0, a_r(p)C_p) \le a_r m \le \sum_{p \in P} \max(0, a_r(p)C_p) = C_2,$$

so taking initial marking $\tilde{m}_0(\tilde{p}_r) = 1 - C_1 + a_r \cdot m_0$, we get

$$\tilde{m}(\tilde{p}_r) = \tilde{m}_0(\tilde{p}_r) + a_r(m - m_0) = 1 - C_1 + a_r m \ge 1 > 0.$$

□

Lemma 3.6 shows that a transition in $\widetilde{\mathcal{SPN}}$ is never disabled due to a shortage of tokens in one of the bag count places. This means that any firing sequence that can fire in the original $\mathcal{SPN}$ can also fire in the BCPE-$S\Pi^2$-net $\widetilde{\mathcal{SPN}}$ and because $\mathcal{SPN}$ and $\widetilde{\mathcal{SPN}}$ behave exactly the same on the original places, any firing sequence that can not fire in $\mathcal{SPN}$ can not fire in $\widetilde{\mathcal{SPN}}$ either. This means that the two Petri nets behave similar if we only observe the original places in $\widetilde{\mathcal{SPN}}$. Moreover by construction none of the bag count places are a conflict place and from theorem 3.4 we know that the marking of the original places can be uniquely determined from the bag count places. So the set of bag count places $\{p_r | r \in \mathcal{R}(T)\}$ forms a sufficient place set and the set of original places P a surplus place set. So by Result 2.32 we know that $\widetilde{\mathcal{SPN}}$ can be decomposed in seperate components for every common input bag class.

Theorem 3.7. *Consider a $S\Pi^2$-net $\mathcal{SPN} = (P, T, I, O, Q)$, its BCPE-$S\Pi^2$-net $\widetilde{\mathcal{SPN}} = (\tilde{P}, T, \tilde{I}, \tilde{O}, Q)$, a set of vectors a_r satisfying the conditions of Definition 2.26 and an initial marking $\tilde{m}_0$ satisfying the conditions of Definition 3.3. Then,*

1. *removing all original places $p \in P$ from $\widetilde{\mathcal{SPN}}$ yields k state machines: $\mathcal{SM}^1, \ldots, \mathcal{SM}^k$; each $\mathcal{SM}^i$ corresponding to exactly one common input bag class.*

2. *The equilibrium distribution π of $\mathcal{SPN}$ is equal to the equilibrium distribution $\tilde{\pi}$ of $\widetilde{\mathcal{SPN}}$ which is a product of the invariant measures of the state machines:*

$$\pi(m) = \tilde{\pi}(\tilde{m}) \quad = \quad B \prod_{i=1}^{s} \pi_y^{\mathcal{SM}^i}(\tilde{m}^i), m \in \mathcal{M}(\mathcal{SPN}, m_0) \qquad (3.1)$$

   where $\tilde{m}^i$ is the submarking of the bag count places that belong to $\mathcal{SM}^i$ and $\pi^{\mathcal{SM}^i}$ is the invariant measure of subnet $\mathcal{SM}^i$ with

$$\pi_y^{\mathcal{SM}^i}(\tilde{m}^i) = \prod_r y(r)^{\tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r)} = \prod_r y(r)^{a_r(m-m_0)}$$

   where $y(\cdot)$ is the solution of the routing chain of state machine $\mathcal{SM}^i$.

Proof. Statement 1 is true by construction of the BCPE-$S\Pi^2$-net. Every transition has exactly one bag count place in its input bag and exactly one bag count place in its output bag. This means that removing all original places from the net will yield a state machine. This state machine will consist of k separate components, because two bag count places $\tilde{p}_1$ and $\tilde{p}_2$ are connected in this state machine if and only if there is a CI class CI^i such that $\tilde{p}_1, \tilde{p}_2 \in \mathcal{P}(CI^i)$.

For statement 2: from Lemma 3.5 we know that $\tilde{P} \backslash P$ is a surplus place set and all conflict places are original places by construction. So from Result 2.32 it follows that the equilibrium distribution can be decomposed as shown. □

Remark 3.8. *It can be noted that a state machine Petri net is equivalent to a Jackson network. So the routing chain is equivalent to the well-known traffic equations from queueing theory.*

Chapter 4

Decomposition examples

This section will illustrate the similarities and differences between Result 2.32 and Theorem 3.7 by giving three examples. The first example is a Petri-net consisting of two CI classes linked by a single conflict place. This conflict place will form a surplus place set by itself which means that Result 2.32 already gives us the means to decompose it into two seperate CI classes. This example shows that both methods result in the same decomposition, however they follow a different path to get there. For the second example we will revisit Example 2.38. This example was of a Petri-net which has three CI classes and could be decomposed in two ways in two parts using Result 2.32 and we will show that using Theorem 3.7 it will decompose in three parts, exactly one for each CI class. The third and last example have a Petri-net that has three CI classes, however all its places will be conflict places. This example will show that even though the CI classes are tangled very closely together and the product form over the places does not seem to be able to be decomposed it is still possible to seperate the different CI classes and look at their behaviour seperately.

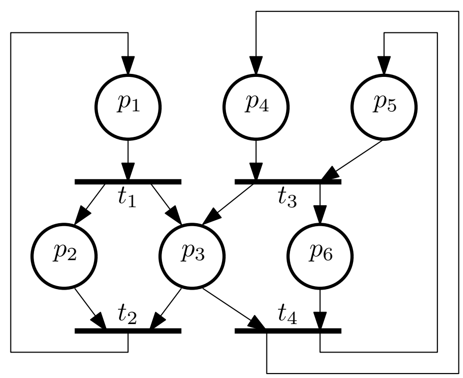
Example 4.1. *Consider the Petri net shown in Figure 4.1a. From the incidence matrix*

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

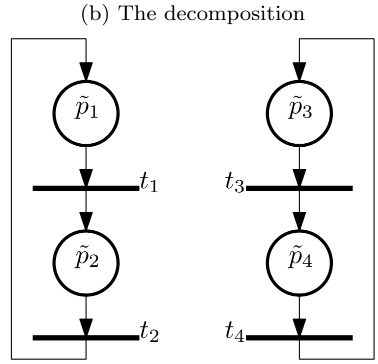
we obtain two minimal support T-invariants $\boldsymbol{x}^1 = [1100]$ *and* $\boldsymbol{x}^2 = [0011]$ *and five minimal support P-invariants* $\boldsymbol{y}^1 = [110000]$, $\boldsymbol{y}^2 = [101100]$, $\boldsymbol{y}^3 = [101010]$, $\boldsymbol{y}^4 = [000101]$ *and* $\boldsymbol{y}^5 = [000011]$ *of which the first four are linearly independent. The two T-invariants are both closed so the net is indeed a* $S\Pi^2$*-net and the T-invariants are not in common input bag relation so it has two common input bag classes* $CI^1 = \{\boldsymbol{x}^1\}$ *and* $CI^2 = \{\boldsymbol{x}^2\}$. *This gives us one conflict place* p_3 *and using Algorithm 2.34 we find that* $\{p_3\}$ *is a surplus place set. This means that*

Figure 4.1: The Petri nets of Example 4.1

(a) The original net



(b) The decomposition



using $\{p_3\}$ as the surplus place set in Result 2.32, we obtain a decomposition into SPN^1 consisting of places p_1 and p_2 and transitions t_1 and t_2 and SPN^2 consisting of places p_4, p_5 and p_6 and transitions t_3 and t_4. This means that

$$\begin{aligned}\pi(m) &= B\pi_y^{SPN^1}(m^1)\pi_y^{SPN^2}(m^2) \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)}\left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)},\end{aligned}$$

where μ_i is the firing rate of transition t_i.

In order to use Theorem 3.7 we need to add the four bag count places, $\tilde{p}_1,\ldots,\tilde{p}_4$, to the net and then remove the original places, $p_1,\ldots,p_6$. This gives us the net shown in Figure 4.1b. Which results in the following equilibrium distribution:

$$\begin{aligned}\pi(m) &= B\pi_y^{SM^1}(\tilde{m}^1)\pi_y^{SM^2}(\tilde{m}^2) \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{\tilde{m}(\tilde{p}_1)}\left(\frac{\mu_4}{\mu_3}\right)^{\tilde{m}(\tilde{p}_3)} \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{a_1 m}\left(\frac{\mu_4}{\mu_3}\right)^{a_3 m}.\end{aligned}$$

A possible choice for the vectors a_r is $a_{I(t_1)} = [100000]$ and $a_{I(t_3)} = [000100]$. Using this choice we see that both decompositions result in the same equilibrium

We can see that this way the net also decomposes in two pieces and the pieces correspond to the same part of the net as the pieces of the previous decomposition. However the structure of the pieces is not necessarily the same for both decompositions. We can see that the part corresponding to CI^1 is the same in both cases, however the part corresponding to CI^2 has a different structure.

Example 4.2. *Example 2.38 revisited*

To illustrate the extra strength of Theorem 3.7 over Result 2.32 we take another look at the Petri-net of Example 2.38. From Example 2.38 we know that the net could be decomposed in two parts, either keeping CI^1 and CI^2 connected or keeping CI^1 and CI^3 connected. By adding the six bag count places, $\tilde{p}_1,\ldots,\tilde{p}_6$ to the net and then removing all original places, $p_1,\ldots,p_7$ we obtain the Petri-net shown in Figure 4.2.

This net allows a simple choice of the a_r vectors similar to the previous example: $a_{I(t_1)} = [100000]$, $a_{I(t_2)} = [-100000]$, $a_{I(t_3)} = [0001000]$, $a_{I(t_4)} = [000-1000]$, $a_{I(t_5)} = [0000010]$ and $a_{I(t_6)} = [00000-10]$. Which gives us the following equilibrium distribution

$$\begin{aligned}\pi(m) &= B\pi_y^{SM^1}(\tilde{m}^1)\pi_y^{SM^2}(\tilde{m}^2)\pi_y^{SM^3}(\tilde{m}^3) \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{\tilde{m}(\tilde{p}_1)}\left(\frac{\mu_4}{\mu_3}\right)^{\tilde{m}(\tilde{p}_3)}\left(\frac{\mu_6}{\mu_5}\right)^{\tilde{m}(\tilde{p}_5)} \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)}\left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)}\left(\frac{\mu_6}{\mu_5}\right)^{m(p_6)}\end{aligned}$$

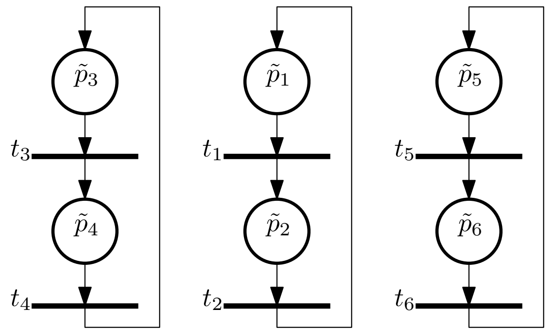


Figure 4.2: Bag count places of the Petri net of Example 4.2

Example 4.3. *In this last example we look at the Petri-net with the following incidence matrix*

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

There are three minimal support T-invariants $\boldsymbol{x}^1 = [110000]$, $\boldsymbol{x}^2 = [001100]$ *and* $\boldsymbol{x}^3 = [000011]$ *and one minimal support P-invariant* $\boldsymbol{y}^1 = [1111]$. *All the T-invariants are closed so it is a* $S\Pi^2$*-net and none of the T-invariants are in common input bag relation, so there are three CI classes,* $CI^1 = \{\boldsymbol{x}^1\}$, $CI^2 = \{\boldsymbol{x}^2\}$ *and* $CI^3 = \{\boldsymbol{x}^3\}$. *All places belong to each of the three CI classes so the set of conflict places is* $\{p_1, p_2, p_3, p_4\}$. *In order to decompose it we first need to add the six bag count places to the net to get the BCPE-*$S\Pi^2$*-net with incidence matrix:*

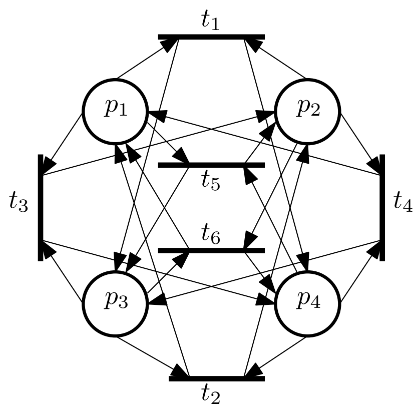
$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Next we require a set of vectors a_r, *a possible choice is :*

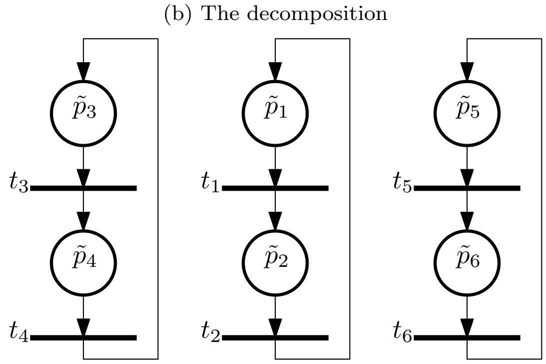
$$\begin{array}{rcl} a_{I(t_1)} & = & [\ \ \frac{1}{2} \ \ \frac{1}{2} \ \ 0 \ \ 0\ ] \\ a_{I(t_2)} & = & [\ -\frac{1}{2} \ -\frac{1}{2} \ \ 0 \ \ 0\ ] \\ a_{I(t_3)} & = & [\ \ \frac{1}{2} \ \ 0 \ \ \frac{1}{2} \ \ 0\ ] \\ a_{I(t_4)} & = & [\ -\frac{1}{2} \ \ 0 \ -\frac{1}{2} \ \ 0\ ] \\ a_{I(t_5)} & = & [\ \ 0 \ -\frac{1}{2} \ -\frac{1}{2} \ \ 0\ ] \\ a_{I(t_6)} & = & [\ \ 0 \ \ \frac{1}{2} \ \ \frac{1}{2} \ \ 0\ ] \end{array}$$

Figure 4.3: The Petri nets of Example 4.3

(a) The original net



(b) The decomposition



Removing the original places from the net we obtain the Petri-net shown in Figure 4.3b. Which is exactly the same as the net we obtained in Example 4.2. So we obtain the following equilibrium distribution:

$$
\begin{aligned}
\pi(m) &= B\pi_y^{SM^1}(\tilde{m}^1)\pi_y^{SM^2}(\tilde{m}^2)\pi_y^{SM^3}(\tilde{m}^3) \\
&= B\left(\frac{\mu_2}{\mu_1}\right)^{\tilde{m}(\tilde{p}_1)}\left(\frac{\mu_4}{\mu_3}\right)^{\tilde{m}(\tilde{p}_3)}\left(\frac{\mu_6}{\mu_5}\right)^{\tilde{m}(\tilde{p}_5)} \\
&= B\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}m(p_1)+\frac{1}{2}m(p_2)}\left(\frac{\mu_4}{\mu_3}\right)^{\frac{1}{2}m(p_1)+\frac{1}{2}m(p_3)}\left(\frac{\mu_5}{\mu_6}\right)^{\frac{1}{2}m(p_2)+\frac{1}{2}m(p_3)} \\
&= \left(\frac{\mu_2\mu_4}{\mu_1\mu_3}\right)^{\frac{1}{2}m(p_1)}\left(\frac{\mu_2\mu_5}{\mu_1\mu_6}\right)^{\frac{1}{2}m(p_2)}\left(\frac{\mu_4\mu_5}{\mu_3\mu_6}\right)^{\frac{1}{2}m(p_3)}
\end{aligned}
$$

Chapter 5

Normalising constant and performance measures

In this section, we will use the decomposition results from section 3 in order to formulate an algorithm that gives the marginal distribution of a number of CI classes, given the equilibrium distribution of all CI classes in isolation. This will give a method for two different objectives. First, the algorithm can be used to find the normalising constant of the Petri net. Second, it allows us to find the first order performance measures, like the average number of tokens present in a place or the probability that a transition is enabled.

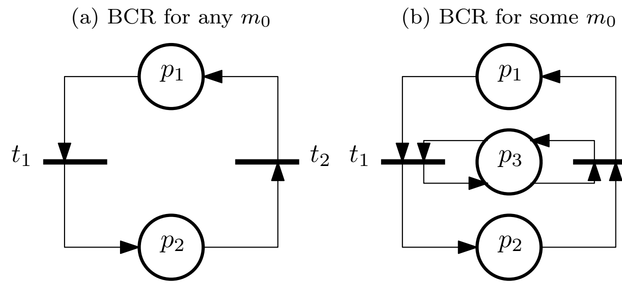
Definition 5.1 (First order performance measures of a set of CI classes)**.** *The first order performance measures of a set of CI classes* $\mathcal{C}(I) = \{CI^i | i \in I \subseteq K\}$ *are the performance measures that can be obtained if you know the marginal probabilities of finding any marking* m_1 *on the places of the set of common inputbag classes and the firing rates of all transitions belonging to these common input bag classes, i.e. if you know for any marking* m_1 *the probability*

$$Pr\left[m(p) = m_1(p), \forall p \in \bigcup_{i \in I} \mathcal{P}(CI^i)\right] = \sum_{m: m(p) = m_1(p), \forall p \in \bigcup_{i \in I} \mathcal{P}(CI^i)} \pi(m).$$

First, we will give the condition that defines the bag count reachable $S\Pi^2$-nets for which the algorithm can be used. Then, a couple of definitions will be given that we need for the algorithm. After this the algorithm itself will be presented and, finally, we will make a start with characterizing the defined subclass of $S\Pi^2$-nets by showing that all P-invariant reachable $S\Pi^2$-nets satisfy the given condition.

Definition 5.2 (Bag Count Reachable $S\Pi^2$-net)**.** *The Bag Count Reachable* $S\Pi^2$*-nets (BCR-*$S\Pi^2$*-nets) are all marked* $S\Pi^2$*-nets* $\mathcal{SPN} = (P, T, I, O, Q)$*,*

Figure 5.1: Two Petri nets to illustrate bag count reachability



with initial marking m_0, for which the following two statements are equivalent for every vector $m \in \mathbb{Z}^{n_p}$:

1. $m \in \mathcal{M}(\mathcal{SPN}, m_0)$

2. $m \geq 0$, $\sum_{r \in \mathcal{R}(\mathcal{T}(CI^i))} a_r(m - m_0) = 0$ *for all* $i \in K$

By recalling the bag count places and Lemma 3.4 we can see that statement 2 of Definition 5.2 is equivalent to $\sum_{r \in \mathcal{R}(\mathcal{T}(CI^i))} \tilde{m}(\tilde{p}_r) - \tilde{m}_0(\tilde{p}_r) = 0$. This means that any marking with the same total number of tokens on the bag count places of each common inputbag class CI^i satisfies this condition. In other words, there are no markings m that are unreachable, for which there is a positive integer valued vector $\bar{\sigma} \in \mathbb{N}_0^{n_t}$ such that $m = m_0 + \mathbf{A}\bar{\sigma}$.

Whether a $S\Pi^2$-net is a BCR-$S\Pi^2$-net depends on its initial marking, so it is a behavioural property. Some $S\Pi^2$-nets are bag count reachable for any initial marking, for example the Petri net in Figure 5.1a. However in some cases it depends on the chosen initial marking, for instance the Petri net of Figure 5.1b is bag count reachable with initial marking $m_0 = [101]$, however it is not bag count reachable for intial marking $m_0 = [100]$, because then marking $m = [010]$ is not reachable, however it does satisfy statement 2 of Definition 5.2.

Remark 5.3. *It can be noted that for any $S\Pi^2$-net statement 1 implies statement 2 of definition 5.2. This is true because, firstly, $m \geq 0$ is true for any marking m and secondly, for any reachable marking m there is a firing sequence from m_0 to m and firing any transition t reduces $\tilde{m}(\tilde{p}_{I(t)})$ by one and increases $\tilde{m}(\tilde{p}_{O(t)})$ by one. This means that the total number of tokens on the bag count places within each common inputbag class remains constant, because $\boldsymbol{I}(t)$ and $\boldsymbol{O}(t)$ belong to the same common inputbag class. However statement 2 does not imply statement 1 in general.*

In order to introduce the Algorithm of this section we first need to define three types of place sets concerned with sets of common input bag classes and a Petri net called the $CIclass$ in isolation.

Definition 5.4 (Interior place set)**.** *With $I \subset K$, a place $p \in P$ is called an interior place of $\mathcal{C}(I) = \{CI^i | i \in I\}$, if there is an $i \in I$ such that $p \in \mathcal{P}(CI^i)$ and there is no $j \in K \backslash I$ such that $p \in \mathcal{P}(CI^j)$.*

The interior place set of $\mathcal{C}(I)$ is the set of all places that are an interior place of $\mathcal{C}(I)$ and is denoted by P^I_{int}, i.e. $P^I_{int} = \bigcup_{i \in I} \mathcal{P}(CI^i) \backslash \bigcup_{j \in K \backslash I} \mathcal{P}(CI^j)$.

Definition 5.5 (Boundary place set)**.** *With $I \subset K$, a place $p \in P$ is called a boundary place of $\mathcal{C}(I)$, if there are an $i \in I$ and a $j \in K \backslash I$ such that $p \in \mathcal{P}(CI^i) \cap \mathcal{P}(CI^j)$.*

The boundary place set of $\mathcal{C}(I)$ is the set of all places that are a boundary place of $\mathcal{C}(I)$ and is denoted by P^I_{bnd}, i.e. $P^I_{bnd} = \bigcup_{i \in I} \mathcal{P}(CI^i) \cap \bigcup_{j \in K \backslash I} \mathcal{P}(CI^j)$.

Definition 5.6 (Exterior place set)**.** *With $I \subset K$, a place $p \in P$ is called an exterior place of $\mathcal{C}(I)$, if it is neither an interior place nor a boundary place of $\mathcal{C}(I)$.*

The exterior place set of $\mathcal{C}(I)$ is the set of all places that are an exterior place of $\mathcal{C}(I)$ and is denoted by P^I_{ext}, i.e. $P^I_{ext} = P \backslash (P^I_{int} \cup P^I_{bnd})$

Definition 5.7 (Common inputbag class in isolation)**.** *Common input bag class CI^i in isolation is a Petri net $\mathcal{SPN}^i = (P^i, T^i, I^i, O^i)$ where $T^i = \mathcal{T}(CI^i)$ is the set of all transitions in T that belong to CI^i. $P^i = \mathcal{P}(CI^i)$ is the set of all places from P that belong to CI^i. $I^i(p,t) = I(p,t)$ and $O^i(p,t) = O(p,t)$ for all $p \in P^i, t \in T^i$.*

The structure of $\mathcal{SPN}^i$ is chosen such that it consists of one common inputbag class. This means that if we look at the decomposition of this Petri net according to Theorem 3.7 we see that it remains in one piece consisting of one statemachine $\mathcal{SM}$ which is identical to the statemachine $\mathcal{SM}^i$ from the decomposition of the original net $\mathcal{SPN}$. So from Theorem 3.7 we get that the equilibrium distribution of $\mathcal{SPN}$ is equal, up to a constant factor, to the product of the equilibrium distributions of the state machines $\mathcal{SM}^i$, $i \in K$. I.e. $\pi(m) = B_1 \prod_i \pi^{SM^i}(\tilde{m}^i)$, for all $m \in \mathcal{M}(\mathcal{SPN}, m_0)$. In turn, the equilibrium distribution of these state machines is equal, up to a constant factor, to the equilibrium distribution of the common input bags in isolation. I.e. $\pi^i(m^i) = B_2 \pi^{SM^i}(\tilde{m}^i)$, for all $m^i \in \mathcal{M}(\mathcal{SPN}, m^i_0)$. Combining these two observations forms the basis for our algorithm, however in order for us to be able to use this observation to find the equilibrium distribution of the original net, we still need to define an initial markings m^i_0, for all $i \in K$, such that any reachable marking $m \in \mathcal{M}(\mathcal{SPN}, m_0)$ corresponds to reachable markings m^i in the common inputbag classes in isolation.

In order to show that such an initial marking can be found we first define a reduced firing sequence and then formally define the conditions on the initial marking and show that such an initial marking can always be found.

Definition 5.8 (Reduced firing sequence)**.** *The reduced firing sequence σ^i is obtained from a firing sequence σ by removing all transitions $t \notin \mathcal{T}(CI^i)$.*

Lemma 5.9. *Let $\mathcal{SPN}^i$ be the common inputbag class CI^i in isolation of a structurally bounded marked $S\Pi^2$-net $(\mathcal{SPN}, m_0)$. Choosing initial marking m_0^i as $m_0^i(p) = m_0(p)$ for all $p \in P_{int}^i$ and $m_0^i(p) = \sum_{t \in T^i} I(p,t)\tilde{m}_0(\tilde{p}_{\mathbf{I}(t)})$ for all $p \in P_{bnd}^i$ makes sure that: for any firing sequence σ that can be fired in the original net from m_0, the reduced firing sequence σ^i can be fired from m_0^i in $\mathcal{SPN}^i$.*

Proof. In order to prove this lemma we will show that there is no place $p \in P^i$ that will prohibit a transition from firing in $\mathcal{SPN}^i$ that was allowed to fire in $\mathcal{SPN}$. First, we look at the interior places of CI^i, by definition these places only belong to input bags of transitions $t \in \mathcal{T}(CI^i)$. That means that any transition in σ that uses these places will also be in σ^i. Therefore, the marking of these places will at any step during the firing sequences be equal and if a transition is disabled in $\mathcal{SPN}^i$ due to this place, then it is also disabled in $\mathcal{SPN}$.

The marking of the boundary places can be influenced by transitions from outside the common input bag class. However, we know that the statemachine $\mathcal{SM}^i$ is the decomposition of $\mathcal{SPN}^i$, so from Lemma 3.5 for every $p \in P^i$

$$(m^i(p) - m_0^i(p)) = \sum_{t \in T^i} I(p,t)(\tilde{m}(\tilde{p}_{\mathbf{I}(t)}) - \tilde{m}_0(\tilde{p}_{\mathbf{I}(t)})).$$

This means that for any $p \in P_{int}^i$, if we take m_0^i as given in this lemma, and recalling that $\tilde{m} \geq 1$ we get:

$$\begin{aligned} m^i(p) &= \sum_{t \in T^i} I(p,t)\tilde{m}(\tilde{p}_{\mathbf{I}(t)}) \\ &\geq \sum_{t \in T^i} I(p,t) \\ &\geq \max_{t \in T^i} I(p,t). \end{aligned}$$

Which means that no transition in σ^i is every disabled due to a boundary place. ☐

By now we have defined five different types of Petri-nets derived from a single $S\Pi^2$-net $\mathcal{SPN}$. All of these Petri nets are used in either Theorem 3.7 or Algorithm 5.15. These nets can differ in structure and/or initial marking. Table 5.1 gives an overview of all of these Petri nets and their notations.

Next we will give a short overview of these nets and some remarks about their initial marking and the relation between the markings of the different nets.

1. The $S\Pi^2$-net $\mathcal{SPN}$ is the original net that we would like to analyse. Its structure and initial marking are given.

2. The BCPE-net $\widetilde{\mathcal{SPN}}$ is obtained from $\mathcal{SPN}$ by adding a number of places called the bag count places as described in Definition 3.1. Its initial marking $\tilde{m}_0$ is equal to the initial marking of the original net on the original

Description	Petri net	equilibrium distribution	marking
The $S\Pi^2$-net	$\mathcal{SPN}, m_0$	π	m
BCPE-net	$\widetilde{\mathcal{SPN}}, \tilde{m}_0$	$\tilde{\pi}$	$\tilde{m}$
State machines	$SM^i, \tilde{m}_0^i$	π^{SM^i}	$\tilde{m}^i$
CI class in isolation	$\mathcal{SPN}^i, m_0^i$	π^i	m^i
SM of CI in isolation	$SM^i, \hat{m}_0^i$	$\hat{\pi}^i$	$\hat{m}^i$

Table 5.1: Notation for the different Petri-nets

places and is chosen on the new places in such a way that those places will never disable the firing of a transition that could have been fired in the original net, a possible choice for the initial marking is given in Lemma 3.6. There is a one to one correspondence between the markings m and $\tilde{m}$ such that m_0 and m_1 correspond to $\tilde{m}_0$ and $\tilde{m}_1$ respectively if and only if for any firing sequence σ, $m_0|\sigma > m_1$ iff $\tilde{m}_0|\sigma > \tilde{m}_1$, as can be seen in Lemmata 3.4 and 3.5. Moreover for any reachable marking m of the original net its corresponding marking $\tilde{m}$ of the BCPE-net is also reachable and the converse is also true, because of the choice of the conditions on the initial marking $\tilde{m}_0$.

3. The state machines SM^i, $i \in K$, are obtained from the decomposition of $\mathcal{SPN}$ as given in Theorem 3.7 or in other words by removing all original places from $\widetilde{\mathcal{SPN}}$. Their initial markings $\tilde{m}_0^i$ are equal to the markings of the corresponding places in $\tilde{m}_0$. There is a one to one correspondence between the marking of the BCPE-net and the marking of all state machines, because the marking $\tilde{m}$ consists of the marking of all state machines plus the marking of the original places and there is a one to one correspondence between these two parts. Here it should be noted that a reachable marking m always corresponds to a set of reachable markings $\tilde{m}^i$ however the converse is not guaranteed.

4. The CI classes in isolation $\mathcal{SPN}^i$ are obtained from the original net by removing all places and transitions that do not belong to CI^i as seen in Definition 5.7. The initial marking m_0^i is chosen such that for any firing sequence σ that can be fired from m_0 in $\mathcal{SPN}$ it is also possible to fire the reduced firing sequence σ^i from m_0^i, which can be done as shown in Lemma 5.9.

5. The state machine of the CI class in isolation SM^i is the state machine that is obtained by decomposing $\mathcal{SPN}^i$ by using Theorem 3.7. The structure of these state machines is equal to the state machines from point 3. The initial marking $\hat{m}_0^i$ of these state machines may differ from $\tilde{m}_0^i$. However there is a one to one correspondence: $\tilde{m}^i$ corresponds to $\hat{m}^i$ if $\tilde{m}^i - \tilde{m}_0^i = \hat{m}^i - \hat{m}_0^i$. And according to Lemmata 3.4 and 3.5 there is a one to one correspondence between the markings $\hat{m}^i$ and m^i.

The goal of our algorithm will be to calculate values $G^I(dm_c^I)$ for increasingly larger sets of indices I. These values G^I will represent all information about the common input bag classes CI^i, $i \in I$, that is needed in order to describe the behaviour of all other common inputbag classes CI^j, $j \in K/I$. Here dm_c^I is a vector of integers for every boundary place $p \in P_{bnd}^I$ which represents the difference from the initial marking resulting from transitions in $\mathcal{T}(CI^i)$, $i \in I$. $G^I(dm_c^I)$ can be thought of as the sum over all reachable markings, that differ $dm_c^I(p)$ from the initial marking $m_0(p)$ on the boundary places $p \in P_{bnd}^I$, of the invariant measure of the states of state machines SM^i, $i \in I$.

Definition 5.10. $G^I(dm_c^I)$

$G^I(dm_c^I)$ is defined as follows

$$G^I(dm_c^I) = \sum_{\substack{\tilde{m}^I: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i, \forall i \in I \\ m(p) \geq 0, \forall p \in P_{int}^I \\ m^I(p) - m_0^I(p) = dm_c^I(p), \forall p \in P_{bnd}^I}} \prod_{i \in I} \pi^{SM^i}(\tilde{m}^i)$$

Where $\tilde{m}^I$ is a vector that gives the marking of all bag count places from common inputbag classes CI^i, $i \in I$ and $m^I(p) - m_0^I(p)$ is the change in the marking of place p as a result of transitions from common inputbag classes CI^i, $i \in I$. I.e. $m^I(p) - m_0^I(p) = \sum_{t \in T^i}(\tilde{m}(\tilde{p}_{\boldsymbol{I}(t)}) - \tilde{m}_0(\tilde{p}_{\boldsymbol{I}(t)}))I(p,t)$.

Calculating these values $G^I(dm_c^I)$ will be the goal of our algorithm. First we will show why these values are interesting by showing how the normalising constant and first order performance measures can be found using the values $G^I(dm_c^I)$. Afterwards, we will show the algorithm itself.

Theorem 5.11. *For a BCR-$S\Pi^2$-net $\mathcal{SPN}$ the equilibrium distribution is given by equation (3.1) in Theorem 3.7. The normalisation constant B is given by $B^{-1} = G^K(\emptyset)$.*

Proof. B is the normalisation constant so it makes sure that the equilibrium distribution sums to one. Because $\mathcal{SPN}$ is a BCR-$S\Pi^2$-net we know that statement 1 and 2 from Definition 5.2 are equivalent, so:

$$\begin{aligned} 1 &= \sum_{m \in \mathcal{M}(\mathcal{SPN}, m_0)} \pi(m) \\ &= \sum_{m \in \mathcal{M}(\mathcal{SPN}, m_0)} B \prod_{i=1}^{s} \pi_y^{SM^i}(\tilde{m}^i) \\ &= B \sum_{\substack{\tilde{m}: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i, \forall i \in K \\ m(p) \geq 0, \forall p \in P}} \prod_{i \in K} \pi_y^{SM^i}(\tilde{m}^i) \\ &= BG^K(\emptyset) \end{aligned}$$

Where the last step can be taken because by definition $P^K_{bnd} = \emptyset$ and $P^K_{int} = P$. So $G^K(\emptyset) = B^{-1}$. □

In order to find the first order performance measures we need two things. We need the marginal distribution for the places and the firing rates of the transitions. The firing rates are straightforward, since we use state independent firing rates. So a transition is either enabled with the known firing rate or it is disabled and which of these is true in a given state is only dependent on the marking of the places belonging to the common inputbag class to which this transisiton belongs. So if we know the marginal distribution of a common inputbag class, then we also know everything about the firing rates of the transitions belonging to this common input bag class. Therefore we will now show how to find the marginal distributions from the values G^I.

Theorem 5.12. *The marginal probabilty of finding $m_1(p)$ tokens on the places $p \in \mathcal{P}(CI^i)$ is given by*

$$\begin{aligned} & Pr(m(p) = m_1(p), \forall p \in \mathcal{P}(CI^i)) \\ = \quad & B \sum_{\substack{m^i \in \mathcal{M}(\mathcal{SPN}^i, m^i_0): \\ m^i(p) - m^i_0(p) = \\ m_1(p) - m_0(p), \forall p \in P^i_{int}}} G^{K/i}(dm^i_c) \pi^{SM^i}_y(\tilde{m}^i), \end{aligned}$$

where $dm^i_c(p) = m_1(p) - m_0(p) - m^i(p) + m^i_0(p)$ for all $p \in P^i_{bnd}$.

Proof. The proof consists of four steps as shown below. The first step (1) rewrites the marginal probability by using its definition. In the second step (2) Theorem 3.7 and Definition 5.2 are used to rewrite the sum. In the third step (3) the sum is split in two parts and finally in the last step (4) Definition 5.10 is used to complete the proof.

$$\begin{aligned} & Pr\left[m(p) = m_1(p), \forall p \in \mathcal{P}(CI^i)\right] \\ \overset{(1)}{=} \quad & \sum_{\substack{m \in \mathcal{M}(\mathcal{SPN}, m_0): \\ m(p) = m_1(p), \forall p \in \mathcal{P}(CI^i)}} \pi(m) \\ \overset{(2)}{=} \quad & \sum_{\substack{\tilde{m}: \\ \sum \tilde{m}^j = \sum \tilde{m}^j_0, \forall j \in K \\ m(p) \geq 0, \forall p \notin \mathcal{P}(CI^i) \\ m(p) = m_1(p), \forall p \in \mathcal{P}(CI^i)}} B \prod_{j \in K} \pi^{SM^j}_y(\tilde{m}^j) \end{aligned}$$

$$\stackrel{(3)}{=} B \sum_{\substack{\tilde{m}^i: \\ \sum \tilde{m}^i = \sum \tilde{m}^i_0 \\ m^i(p) = m_1(p), \forall p \in P^i_{int} \\ (m^i(p) \geq 0, \forall p \in P^i_{bnd}}} \sum_{\substack{\tilde{m}^{K/i}: \\ \sum \tilde{m}^j = \sum \tilde{m}^j_0, \forall j \in K/i \\ m(p) \geq 0, \forall p \in P^i_{ext} \\ m^{K/i}(p) - m^{K/i}_0(p) = \\ (m_1(p) - m_0(p)) - \\ (m^i(p) - m^i_0(p)), \forall p \in P^i_{bnd}}} \pi^{SM^i}(\tilde{m}^i) \prod_{j \in K/i} \pi^{SM^j}_y(\tilde{m}^j)$$

$$\stackrel{(4)}{=} B \sum_{\substack{m^i \in \mathcal{M}(\mathcal{SPN}^i, m^i_0): \\ m^i(p) - m^i_0(p) = \\ m_1(p) - m_0(p), \forall p \in P^i_{int}}} G^{K/i}(dm^i_c) \pi^{SM^i}_y(\tilde{m}^i)$$

□

Now that we have established why we would like to calculate the values G^I, we will show how to find these values in a recursive manner. The algorithm consists of three steps. The first step is to determine the marginal distribution of the number of tokens present on the boundary places of all common inputbag classes in isolation. During the second step, the values $G^I(dm^I_c)$ will be calculated using the marginal probabilities, where I are sets consisting of only one index. The third step is to calculate $G^I(dm^I_c)$ for increasingly larger sets I by combining the values of two smaller sets.

First, we need to obtain the marginal distributions of the boundary places of the common input bag classes in isolation. These can be found using any known algorithm to find the equilibrium distribution of $S\Pi^2$-nets, for instance by enumerating the whole statespace or if the $CIclass$ in isolation is P-invariant reachable you could use one of the more efficient algorithms such as the one by Coleman [3] or Coyle et al. [4]. In general the size of the statespace of the common input bag classes in isolation will be orders of magnituted smaller than the statespace of the whole Petri net, since if there are n common input bag classes each with k possible states, then the size of the statespace of the whole net will be $O(k^n)$. So depending on the number of common input bag classes and how restrictive the conflict places are, finding the equilibrium distribution and the marginal distribution of the common input bag classes in isolation will be significantly easier than finding them for the whole net. The rest of the algorithm will recombine the common input bag classes by combining the parts one by one. During each intermediate step we only retain that part of the information that is relevant for the rest of the Petri-net in order to be as efficient as possible.

Next, we look at the second step of the algorithm and show how to obtain the initial values $G^i(dm^i_c)$ from the marginal probabilties.

Theorem 5.13. *For every $i \in K$ and every vector dm^i_c we can find $G^i(dm^i_c)$*

as follows:

$$G^i(dm_c^i) = \frac{\pi_y^{SM^i}(\tilde{m}_0^i)}{\pi^i(m_0^i)} \sum_{\substack{m^i \in \mathcal{M}(\mathcal{SPN}^i, m_0^i): \\ m^i(p) - m_0^i(p) = dm_c^i(p), \forall p \in P_{bnd}^i}} \pi^i(m^i).$$

Proof. We know that SM^i is the decomposition of $\mathcal{SPN}^i$. This means that from Theorem 3.7 we get

$$\pi^i(m^i) = B_1 \hat{\pi}_y(\hat{m}^i).$$

Because π^{SM^i} and $\hat{\pi}^i$ are the equilibrium distributions of two Petri nets that only differ in initial marking we know that $\pi_y^{SM^i} = \hat{\pi}_y^i$ and since they are in product form we know that $\pi_y^{SM^i}(\tilde{m}_1^i + \tilde{m}_2^i) = \pi_y^{SM^i}(\tilde{m}_1^i)\pi_y^{SM^i}(\tilde{m}_2^i)$. Furthermore, since we know that the two Petri nets only differ in intial marking we know that $\tilde{m}^i - \tilde{m}_0^i = \hat{m}^i - \hat{m}_0^i$. Combining these statements we get for any m^i and corresponding $\tilde{m}^i$ and $\hat{m}^i$,

$$\begin{aligned} \pi^i(m^i) &= B_1 \hat{\pi}_y(\hat{m}^i) \\ &= B_1 \pi_y^{SM^i}(\tilde{m}^i - \tilde{m}_0^i + \hat{m}_0^i) \\ &= B_1 \pi_y^{SM^i}(\hat{m}_0^i - \tilde{m}_0^i)\pi_y^{SM^i}(\tilde{m}^i) \\ &= B_2 \pi_y^{SM^i}(\tilde{m}^i). \end{aligned}$$

Here we can find B_2 by substituting any pair m^i and $\tilde{m}^i$ for instance the initial markings. Finally, using the definition of $G^i(dm_c^i)$ and the conditions for a BCR-$\mathcal{S}\Pi^2$-net we get:

$$\begin{aligned} G^i(dm_c^i) &= \sum_{\substack{\tilde{m}^i: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i \\ m^i(p) \geq 0, \forall p \in P^i \\ m^i(p) - m_0^i(p) = dm_c^i(p), \forall p \in P_{bnd}^i}} \pi_y^{SM^i}(\tilde{m}^i) \\ &= \frac{\pi_y^{SM^i}(\tilde{m}_0^i)}{\pi^i(m_0^i)} \sum_{\substack{m^i \in \mathcal{M}(\mathcal{SPN}^i, m_0^i): \\ m^i(p) - m_0^i(p) = dm_c^i(p), \forall p \in P_{bnd}^i}} \pi^i(m^i). \end{aligned}$$

□

Now that we have the values $G^i(dm_c^i)$ for single indices i, we will now show how to obtain the values $G^I(dm_c^I)$ for larger sets I recursively.

Theorem 5.14. *For $I \subset K$ and $J \subset K$ for which $I \cap J = \emptyset$ and $IJ = I \cup J$ and any vector dm_c^{IJ},*

$$G^{IJ}(dm_c^{IJ}) = \sum_{dm_c^I, dm_c^J} G^I(dm_c^I) G^J(dm_c^J)$$

where the sum is taken over all dm_c^I and dm_c^J such that:

1. *$dm_c^I(p) = dm_c^{IJ}(p)$ for all $p \in P_{bnd}^{IJ} \backslash P_{bnd}^J$*
2. *$dm_c^J(p) = dm_c^{IJ}(p)$ for all $p \in P_{bnd}^{IJ} \backslash P_{bnd}^I$*
3. *$dm_c^I(p) + dm_c^J(p) \geq -m_0(p)$ for all $p \in (P_{bnd}^I \cap P_{bnd}^J) \backslash P_{bnd}^{IJ}$*
4. *$dm_c^I(p) + dm_c^J(p) = dm_c^{IJ}(p)$ for all $p \in (P_{bnd}^I \cap P_{bnd}^J) \cap P_{bnd}^{IJ}$*

Proof. First let us name the place sets from the four statements above as follows and give a short description of where they come from,

1. $P^1 = P_{bnd}^{IJ} \backslash P_{bnd}^J$, this is the set of places that is a boundary place of $\mathcal{C}(I)$ and $\mathcal{C}(IJ)$ but not of $\mathcal{C}(J)$.
2. $P^2 = P_{bnd}^{IJ} \backslash P_{bnd}^I$, this is the set of places that is a boundary place of $\mathcal{C}(J)$ and $\mathcal{C}(IJ)$ but not of $\mathcal{C}(I)$.
3. $P^3 = (P_{bnd}^I \cap P_{bnd}^J) \backslash P_{bnd}^{IJ}$, this is the set of places that is a boundary place of $\mathcal{C}(I)$ and $\mathcal{C}(J)$ but not of $\mathcal{C}(IJ)$.
4. $P^4 = (P_{bnd}^I \cap P_{bnd}^J) \cap P_{bnd}^{IJ}$, this is the set of places that is a boundary place of $\mathcal{C}(I)$, $\mathcal{C}(J)$ and $\mathcal{C}(IJ)$.

By definition 5.10 we know

$$G^{IJ}(dm_c^{IJ}) = \sum_{\substack{\tilde{m}^{IJ}: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i, \forall i \in IJ \\ m(p) \geq 0, \forall p \in P_{int}^{IJ} \\ m^{IJ}(p) - m_0^{IJ}(p) = dm_c^{IJ}(p), \forall p \in P_{bnd}^{IJ}}} \prod_{i \in IJ} \pi^{SM^i}(\tilde{m}^i) \tag{5.1}$$

We will split this sum in three parts, one part will only depend on the common input bag classes $\mathcal{C}(I)$, one part will only depend on the common input bag classes $\mathcal{C}(J)$ and the third part sums over the rest.

First, we notice that $\tilde{m}^{IJ}$ consists of $\tilde{m}^I$ and $\tilde{m}^J$ so in order to split the sum in Equation 5.1 in two sums over these vectors we have to split the constraints on $\tilde{m}^{IJ}$ as well. The first set of constraints, $\sum m^i = \sum m_0^i$ for all $i \in IJ$, is easy to split, because every i belongs to either I or J it is a constraint for only one of the two sums.

For the second constraint, $m(p) \geq 0$ for all $p \in P_{int}^{IJ}$, we have $P_{int}^{IJ} = P_{int}^I \cup P_{int}^J \cup P_3$. For places from P_{int}^I, $m(p)$ is only dependant on $\tilde{m}^I$ and similarly for places from P_{int}^J, $m(p)$ is only dependant on $\tilde{m}^J$. This is true because from Lemma 3.5 we know that $m(p) = \sum_t I(t,p)\tilde{m}(\tilde{p}_{I(t)})$ and by definition of P_{int}^I we know that $I(t,p) = 0$ for all $p \in P_{int}^I$, $t \in \mathcal{T}(CI^j)$ for any $j \in K \backslash I$. For places $p \in P_3$ we have that $m(p) - m_0(p) = m^I(p) - m_0^I(p) + m^J(p) - m_0^J(p)$. So we can split the constraint $m(p) \geq 0$ into three parts: $dm_c^I(p) + dm_c^J(p) \geq -m_0(p)$, $m^I(p) - m_0^I(p) = dm_c^I(p)$ and $m^J(p) - m_0^J(p) = dm_c^J(p)$.

Finally, for the third constraints, $m^{IJ}(p) - m_0^{IJ}(p) = dm_c^{IJ}(p)$ for all $p \in P_{bnd}^{IJ}$, we have $P_{bnd}^{IJ} = P_1 \cup P_2 \cup P_4$. For $p \in P_1$ we know that $m(p)$ is independant of $\tilde{m}^J$ so $m^J(p) - m_0^J = 0$ and similarly for $p \in P_2$ we have $m^I(p) - m_0^I = 0$. For any place p $m^{IJ}(p) - m_0^{IJ} = m^I(p) - m_0^I(p) + m^J(p) - m_0^J(p)$. This means that the constraints for places in P_1 or P_2 belong to only one of the two sums, while the constraints for places from P_4 have to be split in three parts: $m^I(p) - m_0^I = dm_c^I(p)$, $m^J(p) - m_0^J = dm_c^J(p)$ and $dm_c^I(p) + dm_c^J(p) = dm_c^{IJ}(p)$. This results in the following way to split the sum

$$\begin{aligned}
& G^{IJ}(dm_c^{IJ}) \\
= & \sum_{\substack{dm_c^I, dm_c^J: \\ dm_c^I(p) = dm_c^{IJ}(p), \forall p \in P_1 \\ dm_c^J(p) = dm_c^{IJ}(p), \forall p \in P_2 \\ dm_c^I(p) + dm_c^J(p) \geq -m_0(p), \forall p \in P_3 \\ dm_c^I(p) + dm_c^J(p) = dm_c^{IJ}(p), \forall p \in P_4}} \sum_{\substack{\tilde{m}^I: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i, \forall i \in I \\ m(p) \geq 0, \forall p \in P_{int}^I \\ m^I(p) - m_0^I(p) = dm_c^I(p), \forall p \in P_{bnd}^I}} \\
& \sum_{\substack{\tilde{m}^J: \\ \sum \tilde{m}^i = \sum \tilde{m}_0^i, \forall i \in J \\ m(p) \geq 0, \forall p \in P_{int}^J \\ m^J(p) - m_0^J(p) = dm_c^J(p), \forall p \in P_{bnd}^J}} \prod_{i \in I} \pi^{SM^i}(\tilde{m}^i) \prod_{i \in J} \pi^{SM^i}(\tilde{m}^i) \\
= & \sum_{\substack{dm_c^I, dm_c^J: \\ dm_c^I(p) = dm_c^{IJ}(p), \forall p \in P_1 \\ dm_c^J(p) = dm_c^{IJ}(p), \forall p \in P_2 \\ dm_c^I(p) + dm_c^J(p) \geq -m_0(p), \forall p \in P_3 \\ dm_c^I(p) + dm_c^J(p) = dm_c^{IJ}(p), \forall p \in P_4}} G^I(dm_c^I) G^J(dm_c^J)
\end{aligned}$$

□

Now we have all the parts of the algorithm available and we can combine them to get the algorithm shown below.

Algorithm 5.15. *Calculating the values $G^I(dm_c^I)$ for $\mathcal{SPN}$*

Step 1: *Obtain the BCPE-net $\widetilde{\mathcal{SPN}}$ as given in Definition 3.1 and an initial marking $\tilde{m}_0$ satisfying the conditions in Lemma 3.3, for example by taking m_0 as given in Lemma 3.6.*

Step 2: *Obtain the common input bag classes in isolation $\mathcal{SPN}^i$ as given in Definition 5.7, their initial markings m^i as given in Lemma 5.9 and equilibrium distribution π^i in any way, for instance by enumerating the whole statespace.*

Step 3: *From the equilibrium distributions π^i obtain the values $G^i(dm_c^i)$ for all indices $i \in K$ and vectors dm_c^i as given in Theorem 5.13. Let S be the set containing of all sets of one index, $S = \bigcup_{i\in I}\{\{i\}\}$.*

Step 4: *If S contains at least two elements, continue with Step 5, otherwise you are done.*

Step 5: *Take two sets $I, J \in S$, $I \neq J$, and calculate $G^{IJ}(dm_c^{IJ})$ from $G^I(dm_c^I)$ and $G^J(dm_c^J)$ as given in Theorem 5.14. Remove I and J from S and add IJ to S, $S := (S \cup \{IJ\})\backslash\{I, J\}$. Go back to step 4.*

Example 5.16. *Consider the Petri-net shown in Figure 5.2a with initial marking $m_0 = [0101001000]^T$ and all firing rates are equal to 1.*

From the incidence matrix

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix},$$

we obtain three minimal support T-invariants:

$$x^1 = [111000000]^T$$
$$x^2 = [000111000]^T$$
$$x^3 = [000000111]^T,$$

all three have closed support so it is a $S\Pi^2$-net and they are not in common input bag relation so we have three common input bag classes.

Step 1: *Using Theorem 3.7 we obtain the three statemachines shown in Figure 5.2b, where we take $\tilde{m}_0 = [010100100]^T$ as initial marking.*

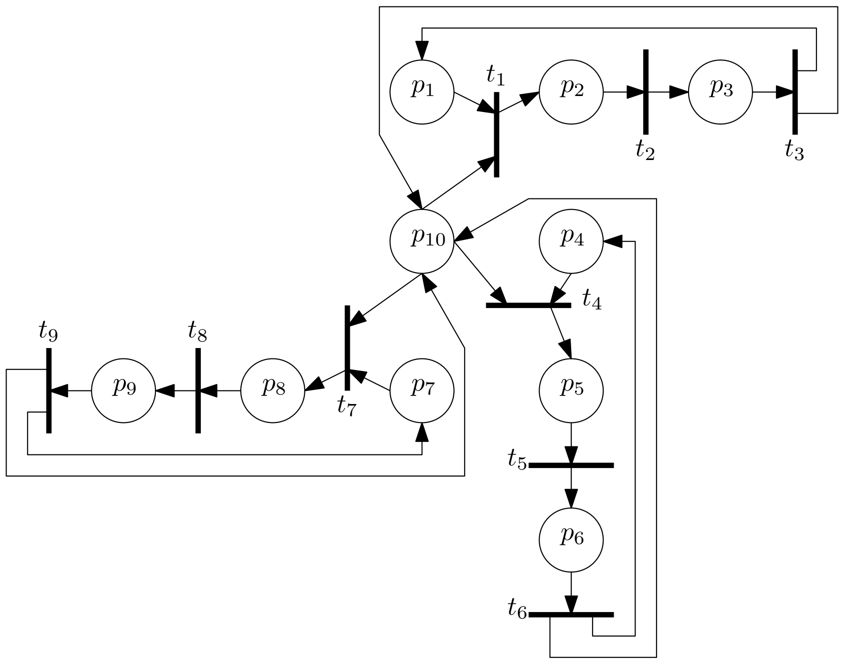
Step 2: *All three common inputbag classes in isolation have the same structure given by incidence matrix*

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

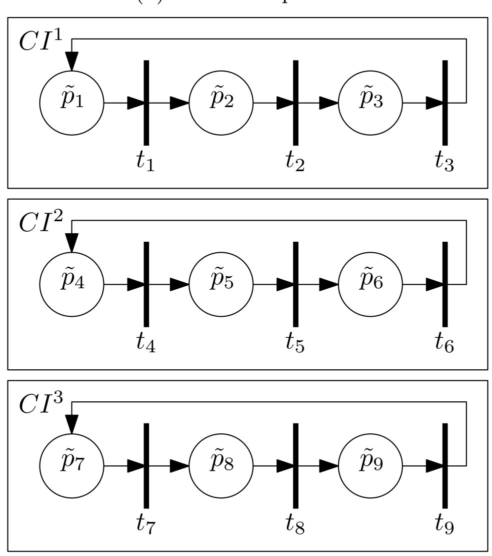
and the same solution to the traffic equations which is $v_r = 1$ for all r. Which gives an invariant measure $\pi_y^i(m^i) = 1$ for all m^i. From Lemma 5.9 and $\tilde{m}_0$ as defined in step 1 we obtain $m_0^1 = [0100]$ and $m_0^2 = m_0^3 = [1001]$. This gives three possible states in each of the nets, i.e. $\mathcal{M}(\mathcal{SPN}^i, m_0^i) = \{[1001], [0100], [0010]\}$. So we have $\pi^i(m^i) = \frac{1}{3}$ for all i and m^i.

Figure 5.2: Petri nets of Example 5.16

(a) The original net



(b) The decomposition



Step 3: *For G^1 we have two possible values for $dm_c^1(p_{10}) = m^1(p_{10}) - m_0^1(p_{10})$, i.e. 0 and 1. Which gives*

$$
\begin{aligned}
G^1([0]) &= \frac{\pi_y^{SM^1}(\tilde{m}_0^1)}{\pi^1(m_0^1)} \sum_{\substack{m^1 \in \mathcal{M}(\mathcal{SPN}^1, m_0^1) \\ m^1(p_{10})=0}} \pi^1(m^1) \\
&= \frac{1}{1/3}\left[\pi^1([0100]) + \pi^1([0010])\right] = 2 \\
G^1([1]) &= \frac{\pi_y^{SM^1}(\tilde{m}_0^1)}{\pi^1(m_0^1)} \sum_{\substack{m^1 \in \mathcal{M}(\mathcal{SPN}^1, m_0^1) \\ m^1(p_{10})=1}} \pi^1(m^1) \\
&= \frac{1}{1/3}\pi^1([1001]) = 1
\end{aligned}
$$

Similarly we find two possible values for dm_c^2 and dm_c^3, i.e. -1 and 0. This results in the following values of $G^i(dm_c^i)$:

i	$G^i(-1)$	$G^i(0)$	$G^i(1)$
1	*0*	*2*	*1*
2	*2*	*1*	*0*
3	*2*	*1*	*0*

Step 4&5: *From this we can calculate the values of G^{23} from G^2 and G^3. First we observe that $P_{bnd}^{23} = \{p_{10}\}$, so dm_c^{23} has one element, $dm_c^{23}(p_{10})$, which can take all values that $dm_c^2 + dm_c^3$ can take, i.e. -2, -1 and 0. So we get:*

$$
\begin{aligned}
G^{23}([-2]) &= \sum_{\substack{dm_c^2, dm_c^3: \\ dm_c^2(p_{10}) + dm_c^3(p_{10}) = -2}} G^2(dm_c^2) G^3(dm_c^3) \\
&= G^2([-1]) G^3([-1]) = 4 \\
G^{23}([-1]) &= \sum_{\substack{dm_c^2, dm_c^3: \\ dm_c^2(p_{10}) + dm_c^3(p_{10}) = -1}} G^2(dm_c^2) G^3(dm_c^3) \\
&= G^2([-1]) G^3([0]) + G^2([0]) G^3([-1]) = 4 \\
G^{23}([0]) &= \sum_{\substack{dm_c^2, dm_c^3: \\ dm_c^2(p_{10}) + dm_c^3(p_{10}) = 0}} G^2(dm_c^2) G^3(dm_c^3)
\end{aligned}
$$

$$= \quad G^2([0])G^3([0]) = 1$$

There are still two sets in S so we do step 4 and 5 once more.

Step4&5: *Now we will find G^{123} by combining G^1 and G^{23}. This time we see that $P^{123}_{bnd} = \emptyset$, so p_{10} belongs to P^3 in Theorem 5.14. This gives us:*

$$\begin{aligned} G^{123}(\emptyset) &= \sum_{\substack{dm_c^1, dm_c^{23}: \\ dm_c^1(p_{10}) + dm_c^{23}(p_{10}) \geq -m(p_{10})}} G^1(dm_c^1)G^{23}(dm_c^{23}) \\ &= G^1([0])G^{23}([0]) + G^1([1])G^{23}([-1]) + G^1([1])G^{23}([0]) \\ &= 7 \end{aligned}$$

Now that we have found these values G^I we will use them for two things. First, we calculate the normalising constant $B = G^{123}(\emptyset)^{-1} = \frac{1}{7}$. For this small instance we can easily check this to be correct, because there are only 7 possible states:

$$\begin{aligned} &[1001001001], \\ &[0101001000], \\ &[0011001000], \\ &[1000101000], \\ &[1000011000], \\ &[1001000100], \\ &[1001000010] \end{aligned}$$

and $\pi_y^{SM^i}(m^i) = 1$ for all i and m^i. Secondly, we can calculate a performance measure of CI^1 for instance the average number of tokens on place p_2. In order to do this we calculate the following marginal probabilities by using Theorem 5.12. In this case the sum for each marginal probability will have only one term, because the interior point of CI^1 uniquely define the state of common input bag class CI^1 in isolation. This gives us the following marginal probabilities, where $Pr([abcd]) = Pr(m|m(p_1) = a, m(p_2) = b, m(p_3) = c, m(p_{10}) = d)$:

$$\begin{aligned} Pr([1000]) &= B\pi_y^{SM^i}([100])G^{23}(-1) &= \tfrac{4}{7} \\ Pr([1001]) &= B\pi_y^{SM^i}([100])G^{23}(0) &= \tfrac{1}{7} \\ Pr([0100]) &= B\pi_y^{SM^i}([010])G^{23}(0) &= \tfrac{1}{7} \\ Pr([0010]) &= B\pi_y^{SM^i}([001])G^{23}(0) &= \tfrac{1}{7} \end{aligned}$$

This gives for the average number of tokens on place p_2:

$$0Pr([1000]) + 0Pr([1001]) + 0Pr([0010]) + 1Pr([0100]) = \frac{1}{7}$$

□

From Example 5.16 two important questions arise:

1. How efficient is our algorithm?

2. How do we check whether or not a $S\Pi^2$-net is a BCR-$S\Pi^2$-net, so we may apply our algorithm?

We did not analyse either question in detail, as this fell outside the scope of this research. Analysing the complexity of our algorithm and comparing it to other known algorithms as well as further characterising the class of BCR-$S\Pi^2$-nets would be an interesting topic for future research. However we did put some thought into these questions and below you can find our conclusions on them so far.

Let us look at the efficiency first. Example 5.16 was small enough to be analysed as a whole and we have seen that its reachable marking set had only seven elements, while each of the three common input bag classes in isolation had three which makes a total of nine states that had to be evaluated in the second step. In general the state space of the original net will be a subset of the carthesian product of state space of the common input bag classes in isolation. How large this subset is will depend on the conflict places and how restrictive they are in the original net. In Example 5.16 the conflict place allowed only one common input bag class to be active at the same time. This resulted in only seven reachable markings out of the potential $3^3 = 27$. In general we can say that step 2 will be significantly faster than analysing the net as a whole if there are many common input bag classes and the conflict places are not too restrictive in the original net.

The second time consuming step, is step 5 where the common input bag classes are recombined. The efficiency of these steps will heavily depend on the number of conflict places and the order in which the common input bag classes are recombined. The number of boundary places of $\mathcal{C}(I)$, $I \subset K$, will be the dimension of dm_c^I which means that the more boundary places a set I has the more values G^I you have to calculate and store. When combining I and J to find G^{IJ} the boundary places of IJ will consist of the boundary places of I and the boundary places of J, however if a place belongs to both I and J it will be counted only once and if it is not boundary place of $K \backslash IJ$ it is not counted at all. It will improve the efficiency greatly if the order of recombining is chosen such that I and J share many boundary places. So the efficiency of the algorithm is also dependent on the number of conflict places and whether an order of recombining can be found such that conflict places are eliminated as soon as possible.

Next, we look at characterising the class of BCR $S\Pi^2$-nets. We have not been able to find a conclusive way to check whether or not a given $S\Pi^2$-net is bag count reachable. However, we do have a sufficient condition, which is not necessary. First we define P-invariant reachable Petri nets as used by Coleman [3].

Definition 5.17 (P-invariant reachable Petri net)**.** *A stochastic Petri net $\mathcal{SPN}$ is P-invariant reachable if, for any two markings m_1 and m_2, the following two statements are equivalent:*

1. $m_2 \in \mathcal{M}(\mathcal{SPN}, m_1)$.
2. $\boldsymbol{Y}m_1 = \boldsymbol{Y}m_2$.

Where $\boldsymbol{Y}$ is the matrix that has all minimum support P-invariants of $\mathcal{SPN}$ as its rows.

Lemma 5.18. *Any P-invariant reachable $S\Pi^2$-net is a BCR $S\Pi^2$-net for any initial marking m_0.*

Proof. Three statements about any marking m are involved in the definitions of P-invariant reachable and BCR $S\Pi^2$-nets:

1. $m \in \mathcal{M}(\mathcal{SPN}, m_0)$
2. $m \geq 0$, $\sum_{r \in \mathcal{R}(\mathcal{T}(CI^i))} a_r(m - m_0) = 0$ for all $i \in K$
3. $m \geq 0$, $\mathbf{Y}m = \mathbf{Y}m_0$.

A Petri net is P-invariant reachable if statement 1 and 3 are equivalent for every initial marking m_0 and bag count reachable for an initial marking m_0 if 1 and 2 are equivalent for m_0. In Remark 5.3 we have shown that statement 1 implies statement 2 for any $S\Pi^2$-net. We will show that statement 2 implies statement 3 for any $S\Pi^2$-net to complete the proof.

From Lemma 3.4 we know $a_r(m-m_0) = \tilde{m}-\tilde{m}_0$. This means that statement 2 means that the total number of tokens on the bag count places for each common input bag class are equal in $\tilde{m}$ and $\tilde{m}_0$. By definition of the common input bag class, for any pair of input bags in the same common input bag class, r_1 and r_2, there is a sequence of transitions σ_{12} that transforms r_1 into r_2, which means that $r_1 + A\bar{\sigma}_{12} = r_2$. This means that since $\tilde{m}$ and $\tilde{m}_0$ have the same number of tokens on the bag count places in each common input bag class that by adding many sequences of transitions σ_{ij} together we can get a sequence of transitions σ such that $m = m_0 + \mathbf{A}\bar{\sigma}$ and by definition of the P-invariants we have that $YA = 0$. So statement 2 implies $\mathbf{Y}m = \mathbf{Y}m_0 + \mathbf{YA}\sigma = \mathbf{Y}m_0$. □

Finally we have one conjecture about a group of Petri nets that belong to the BCR $S\Pi^2$-nets. However we can not prove it at this point.

Conjecture 5.19. *For any $S\Pi^2$-net for which there is no place p that is part of at least two input bags in different quantities. I.e. there are no transitions t_1 and t_2 such that $I(p,t_1) \neq 0$, $I(p,t_2) \neq 0$ and $I(p,t+1) \neq I(p,t_2)$. Then there is a marking m_{min} such that for any initial marking $m_0 \geq m_{min}$ the marked $S\Pi^2$-net is bag count reachable.*

Chapter 6

Discussion

The results of this research are twofold. First, we have shown that each $S\Pi^2$-net can be decomposed into several smaller $S\Pi^2$-nets, where each smaller net represents one common input bag class of the original net and the equilibrium distribution of the original net is the product of the equilibrium distributions of the smaller nets. Second, we have shown an algorithm to find the normalising constant and first order performance measures of a bag count reachable Petri net. This algorithm can be applied to more Petri nets than other methods by Coleman [3] and Coyle et al. [4]. The efficiency of our algorithm has not been investigated, however for Petri nets with relatively few conflict places it is expected to perform well.

We see several promising areas for future research. First of all, the current decomposition results are formulated only for state independent firing rates. However we think it is possible to formulate these results for state dependent firing rates, similar to those used by Henderson et al. [7], Boucherie and Sereno [1] and Haddad et al. [6]. Secondly, we think that it is possible to decompose the Petri nets even further. We believe that if we replace the condition of sharing an input bag by sharing a transition in the definition of the common input bag class, then it is still possible to formulate the decomposition results.

For the results of section 5 there are some more posibilities for future research. Foremost is investigating the efficiency of the algorithm on its own and how it performs in comparison to other known algorithms. Furthermore, it would be interesting to see if the set of BCR-$S\Pi^2$ could be further characterised. By further investigating Conjecture 5.19. First by proving or disproving the conjecture and if it turns out to be true, then it would be interesting to see if a minimal marking m_{min} can be found.

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