Assessment of model risk through hedging simulations:

Valuation of Bermudan swaptions with a one-factor Hull-White model

A thesis presented

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Abstract

In times of globalized financial markets, where the complexity of derivative contracts is severely increasing, the use of advanced models for pricing and hedging is required. Under this situation, the financial institutions desire to quantify the risk and the P&L of their trading portfolios.

Part of these portfolios consists of contracts with model dependent values. The models for pricing and risk management are used as an imitation of the actual market evolution and they are based on various assumptions. The risk management practice considers as a key factor the quantification of the risk due to the use of a particular model.

This motivates us to investigate the model risk of a popular model within the interest rate markets, namely the Hull and White short rate model. In this project we will show how the calibrated Hull and White model affects the hedging outcomes of discrete replicating strategies on Bermudan swaptions.

Our analysis will allow drawing several conclusions upon the behavior of the model under mean reversion uncertainty. Finally, we are able to give an estimate of mean reversion risk using an alternative approach of measurement based on the risk level of the derivative contract.

Keywords: Model risk, market risk, model reserves, model misspecification, parameter uncertainty, model risk measures, model error, error decomposition, discretization, incomplete markets, hedging, sensitivities, interest rate exotics, swap, swaptions, Bermudan.

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I hope to have the chance to pay you back in the future.

Sincerely, Panos

Contents

Preface

This document has been designed to offer a useful insight on a very important aspect of quantitative finance and risk management, namely the quantification of model risk. The text targets on final year master students of financial mathematics and market practitioners with some knowledge of derivatives pricing.

Chapter 1 starts with an overview of the problem and states the main objectives of this project. In chapter 2 the formal definition of model risk follows. Chapter 3 is a brief survey of previous researches on the topic starting from 1989 up to now. This chapter aims to give a general picture about how other people interpret model risk and how they have tried to extract it from the given information without getting into much details.

A full documentation is provided for the implementation of the examined short rate model and how the replicating strategies are defined on the hedge test module on chapters 4 and 5 respectively. After a significant empirical exposure with the subject we find the need of describing model risk as it is observed in the experimental outcomes of the hedge tests. For that reason, chapter 6 presents our interpretation on model risk and its related errors. Our understanding has been influenced by ideas from the literature and from our experimental knowledge. This description will help the reader to understand the results that follow next.

The first part of the results, chapter 7, is dedicated on the validation of our hedger module by testing its performance on different interest rate vanilla payoffs. Taking advantage of the simplicity of these payoffs several tasks like hedging frequency, type of sensitivities, decomposition of errors and other are being investigated. Chapter 8 presents the main results of this project, the hedge tests of Bermudan swaptions. Finally we conclude on the experimental evidences and the results of their analysis. At the end we summarize and we address potential directions for the quantification of model risk.

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Chapter 1
Introduction

1.1 Motivation

1.1.1 The problem

Nowadays, we witness an increasing complexity of the financial derivatives being traded in the market. For this reason it is essential to use sophisticated models for pricing and risk management. The subject of model risk is related to the inaccurate valuation and hedging by a certain model. For liquid instruments this risk can be obtained from the difference between the market and model price. Nevertheless in this text, our interest is focused on more exotic trades where no market price is available and the hedging portfolio plays an important role on the product value. This is because the fair derivative price should represent the total cost of its replicating strategy.

So far, it is well understood that perfect replication by any self-financing strategy is only possible in complete markets with no transaction costs and continuous hedging. In case of incomplete markets our replicating portfolio is no longer risk-free and is subjected to market risk. Market risk appears as an extra cost on the hedging portfolio due to changes of market factors like interest rates, volatilities etc. In order to allocate the amount of capital for the exposure of issuing new products one needs to calculate some risk measures using a variety of models and techniques. These measures can be computed as soon as we choose a model to describe the evolution of the underlying factors and for this, simulation is necessary.

Historically, model simulations have been used for testing trading strategies since 1977 starting with Galai and followed by Merton & Scholes [1978] and Gladstein [1982]. With simulations we can imitate the behavior, for instance, of the term structure of interest rates or the volatility surface of options prices or any other stochastic market variable. In this way, it is possible to generate a class of scenarios from which a Profit-Loss (P&L) distribution is created for the trading portfolio. In risk management practice, people are mainly interested in the probability of extreme losses and this can be quantified by using several market risk measures like VaR or Expected Shortfall (see Basel Committee regulations [1999]). Unfortunately, all models are based on assumptions and they are simply approximations of the actual dynamics (see Derman [1996] and Rebonato [2001]). In reality we are not able to capture the "real" generating process and that makes both pricing and hedging model sensitive. In order to illustrate this in more detail we will try to describe it through three simple examples.

1.1.2 Examples

Consider the well-known pricing model of Black-Scholes (BS) [1973]. The BS model assumes that the underlying asset follows a lognormal diffusion process. However, this model underestimates the probability of high increments, while the practice shows that the tails of the empirical distribution can be more extreme than those of a lognormal distribution. In other words, a standard valuation model may be misspecified when it is based on specific distributional assumptions which are not supported by the actual financial markets.

Moreover, consider a European call on a stock while the underlying follows a lognormal process. Then, a model like Black-Scholes would be a reasonable choice for pricing. However, the model assumes constant volatility which in reality is not correct. Traders try to correct this assumption by calibrating the model to input data in order to imply the volatility for the current day. Calibration is a numerical method which depends on the quality of the market data. Due to this fact, the implied parameter can be higher or lower than the real volatility of the call option. This inevitably will lead to pricing and hedging errors. The parameter specification therefore will be an additional source of risk, even if the model satisfies the distributional behavior of the underlying.

Another example which is more related to the current financial situation is the introduction of counterparty risk value adjustment (CVA) to pricing since the beginning of the recent crisis. Suppose that before that period the pricing of a financial derivative would require the use of one interest rate curve. After the crisis, the establishment of CVA on pricing requires at least two interest rate curves, one for discounting and one for the calculation of the forward rates. That means that if the market prices of those products were driven by a one-factor model before the summer of 2007, the current market prices might behave at least as a two-factor model.

Taking into account the previous example, assume that we are able to successfully approximate the real model process with a highly consistent model to market prices. Suppose that the model shows remarkable performance for the last ten years. Next, imagine that today a product, maturing at three years from now, is priced with this model. However, six months after the issuing date something happens and the market prices start behaving as a different process. That means that the product price may be completely different than what was initially expected from the issuer. If this new market process belongs to the set of our known models then the risk can be seen within the range of all model prices. If the opposite happens, meaning that the new market price is out of the known price range, we are facing a high risk. This is because our current market knowledge does not allow us to identify such a process much earlier. Therefore, model risk can be completely unexpected even when one uses the most accurate models of its time.

In a competitive market, related situations of model uncertainty can bring huge losses to the option issuer. Because of similar reasons the world of finance has experienced several embarrassing incidents due to wrong or inadequate models for the last twenty years.

1.1.3 Lessons from the past

In the past we have seen cases of big losses due to the use of derivative products even for simple underlyings such as bonds or stocks. Some well known examples are those of Barings, Metallgesellschaft, Procter & Gamble, Orange County, Showa Oil, Gibson Greetings or Long Term Capital Management. More specifically, in 1997, the Bank of Tokyo-Mitsubishi announced that its New York-based derivatives unit was taking an $83 million after-tax write-off because a computer model overvalued a portfolio of swaps and options on USD interest rates.

The same year, many derivatives traders noted that NatWest Markets, one of the largest banks in the UK, was aggressively pricing interest rate options and swaptions. It is believed that their valuation was ignoring the effect of volatility smile in its OTC swaptions prices with different moneyness on Sterling/German mark. The failure of the bank's pricing and risk management models to incorporate the "volatility smile" effect led to a significant over-valuation of the portfolio. Losses from these trades eventually totaled £90 million. People speculate that this may have occurred for a total period of three years.

These are only few examples, where model risk was the reason of big losses. Cases like those attract our interest to study a series of different deals and models in a wide range of historical market scenarios.

1.2 Objective

With this in mind, our project aims to assess the exposure of pricing and hedging interest rate exotics by the one-factor Hull and White short rate model. Our experiments will focus on Bermudan swaptions for which very limited information has been reported in the literature (see chapter 3). The assessment of model risk will be based on the study of hedging simulations and the analysis of their final hedging outcomes.

The first task of our analysis is to describe model risk and its related errors on pricing and hedging (see chapter 6). The description will be influenced from ideas in the literature and our experimental experience. In order to gain hedging experience we will need to create a big variety of hedge tests, for several interest rate vanilla products (see chapter 7), and analyze the related hedging errors.

The last part of this research will concentrate on the extensive analysis of the hedging results. Our interest is to learn how and which values are affected from the mean reversion risk. The final goal of the project is to estimate model risk (see chapter 8) based on a set of artificial experiments and identify a possible direction that can lead to a sound model risk quantification in the future.

Chapter 2
Model risk

The point of option pricing theory is usually the specification of a stochastic model and a set of future scenarios $(\Omega, \mathcal{F})$ with a probability measure $\mathbb{P}$ defined on these outcomes. There are many cases in financial decision making where the decision maker is not able to assign an exact probability to the future outcomes. Such measures describe the odds of a financial pricing rule (or model). The difficulty of defining a risk-averse pricing rule attracts the interest of both practitioners and academics and it is the main problem of model risk.

The first section of this chapter provides a philosophical description of model risk using definitions from the literature. The second section defines three model risk measures that have been proposed for model risk quantification in previous researches.

2.1 Definition

Consider a sample space Ω of all possible market scenarios. $\mathcal{F}_t$ is a collection of subsets of Ω which represents all the market information until time t. Because there is no reference probability measure on Ω, we define an objective probability measure $\mathbb{P}$ on the set of all market scenarios $(\Omega, \mathcal{F}_t)$. That means $\mathbb{P}$ describes the objective probabilities of market evolution. We also define an $\mathcal{F}_t$-measurable process $\{S_t\}_{t=0}^{T}$ which is a mapping of the form $S_t : \Omega \to \mathbb{R}$. Moreover, if there exists another probability measure $\mathbb{Q}_m$ on $(\Omega, \mathcal{F}_t)$ such that S_t is a martingale under $\mathbb{Q}_m$, we can create an arbitrage-free pricing rule on that space. In other words, $\mathbb{Q}_m$, describes the risk neutral probabilities of m pricing model and it is uniquely defined in a complete market (see Bjork [2004]).

However, there is always an uncertainty on defining such measures. In complete markets the uncertainty on $\mathbb{Q}_m$ depends only on the lack of identification (read p.11 for more details) of the measure $\mathbb{P}$ due the limitation to historical data. On the other hand, in incomplete markets even if we are certain about $\mathbb{P}$, $\mathbb{Q}_m$ is no longer unique. Hence, there is an ambiguity of choosing a risk neutral probability measure to describe the future outcomes.

Definition 1 *Risk is the uncertainty over the future outcomes, while we are able to specify a unique probability measure.*

Definition 2 *Ambiguity (Knightian uncertainty) is the uncertainty over the choice of the right probability measure.*

The message from this setting is that under ambiguity we cannot condition neither the hedging strategy nor the derivative price on a fixed model. In that sense we do not know what is the unique fair price or what is the unique distribution of the hedging errors. Below, we present model risk as it is appeared in the literature and several methodologies to quantify it. For that reason, we will need to consider that there exists a class of models $\mathcal{K}$, such that a risk-neutral model $m \in \mathcal{K}$ is defined on a probability space $(\Omega_m, \mathcal{F}_m, \mathbb{Q}_m)$.

Model risk: general definition

"The risk associated to the mismatch between the model dynamics and the actual dynamics is called model risk", Kerkhof [2002]. "Model risk is the risk of occurrence of a significant difference between the mark-to-model value of a complex and/or illiquid instrument, and the price at which the same instrument is revealed to have traded in the market.” Rebonato [2004].

Model risk is the exposure to financial losses due to misspecified or incorrectly applied model. In case of risk management, we can interpret model risk as the cost of our hedging strategy additionally to the cost of market risk which is represented on our P&L distribution. The cost of model risk arises due to the inaccurate generation of the P&L distribution. Under this framework of pricing models, the model risk can be decomposed to the following categories (see Kerkhof [2010]).

Estimation risk

Suppose the true model $m(\vartheta) \in \mathcal{K}$ is known, with ϑ a structural parameter of m. If $\hat{\vartheta}_0$ is the optimal estimate of ϑ for the current market data, the risk of calculating a price using $m(\hat{\vartheta}_i)$ with $i \neq 0$ is called estimation risk. This source of risk depends on the quality and liquidity of the input data and the robustness of the optimization process.

Misspecification risk

Consider a class of models $\mathcal{K}$. Assume that the price of the true model is within the range of all model prices of class $\mathcal{K}$, nevertheless the real model may not be available in class $\mathcal{K}$. The valuation by any model $m \in \mathcal{K}$ will probably deviate around the real price. The uncertainty of pricing or hedging with a wrong model m is usually called misspecification risk.

Identification risk

In order to model a physical or a financial process first we need to observe its behavior. For instance, we can identify the drift, the density of its increments, mean reversion characteristics or whatsoever. Then we can try to replicate this behavior by defining a stochastic process with the same (or similar) characteristics. In practice, we are only able to observe the past. As a result theoretically we can only define a highly specified[1] model for our historical data. In finance, nevertheless, the market in driven by the human factor and not from physical laws. As a consequence the market process may change its behavior (model) at a certain point in the future. Since the future data are not known, it is impossible to identify the real process, hence it is impossible to replicate its behavior. This is called, from Kerkhof [2010], identification risk. This is the risk of identifing the market process based only on previous market data. Under this situation it is not easy to measure model risk by any of the available methods. However, the literature proposes some measurements methods regarding the first two categories which we are going to analyze on the next section.

2.2 Model risk measures

Suppose that we have a T-claim $\mathcal{X}$ and the price of its replicating portfolio is given by $\Pi(T;\mathcal{X})$ at time T. Then we denote as $D = \mathcal{X} - \Pi(T;\mathcal{X})$ as the premium of market and model risk that need to be added to the price of our portfolio to meet the final condition of claim $\mathcal{X}$. This can be interpreted as the capital that somebody would need to keep as a protection from model uncertainty.

[1] Specified model is a highly accurate stochastic process which successfully can replicate the observed data.

Worst-case measure

Giboa&Schmeidler [1989] started the foundation of this approach and several strategies have been presented on the papers of Kirch [2002], Föllmer&Schied [2002] and Kerkhof [2002]. This approach gives an upper bound for the model risk of the T-claim $\mathcal{X}$. The formal definition is given by Kerkhof [2002].

Consider a class $\mathcal{C}$ of all possible models and a model m which is an element of $\mathcal{C}$. Let also $\mathcal{K} \subset \mathcal{C}$ a subset of the initial class with $m \in \mathcal{K}$. We will call $\mathcal{K}$ a tolerance set and m nominal model. Suppose that we have a product Π which is defined on the class $\mathcal{C}$ and the model risk measure is defined on class $\mathcal{K}$. Then the worst-scenario measure for model risk is defined under some probability measure $\mathbb{P}$, while $\mathbb{P}$ is based on the current statistical knowledge of the market process:

$$\rho_{\mathbb{P}}(\Pi, m, \mathcal{K}) = \sup_{k \in \mathcal{K}} U_k(\Pi_k) - U_m(\Pi_m)$$

where Π_k is the price of the portfolio with respect to k and U_k is the risk management method to assess the profit or loss of the hedging portfolio.

MaxMin measure

An alternative method to that of the worst-case approach is to calculate a "MaxMin" measure which is the range of all model prices within the set $\mathcal{K}$. This measure has been proposed by Cont [2006] and does not depend on the choice of the nominal model. This measure is defined as follows

$$\rho_{\mathbb{P}}(\Pi, m, l, \mathcal{K}) = \sup_{k \in \mathcal{K}} U_k(\Pi_k) - \inf_{l \in \mathcal{K}} U_l(\Pi_l)$$

Bayesian measure

This approach was first introduced by Hoeting [1999]. The model risk measure is a weighted average of risk measures. A financial institution can give less weight to those models when it is believed that they are more risky than others.

Consider again the set $\mathcal{K} = \{m_1, m_2, ..., m_N\}$ with the candidates models. Let us suppose that these models have parameters $\vartheta_i \in E_i$ for $i = \{1, 2, ..., N\}$.Then, the density that represents our view about the model parameters ϑ_i conditionally that the model m_i holds can be denoted as $p(\vartheta_i|m_i)$. Moreover, the financial institution may set the model probabilities $\mathbb{P}(m_i)$ according to its experience for the actual model.

The weights that a financial institution assigns to each model can be expressed as probabilities, based on the Bayes rule for a set of historical observations B. Then, $p(B|m_i)$ is a likelihood integral of the data under the model m_k,

$$p(B|m_k) = \int_{E_i} p(\vartheta_k|m_k)\mathbb{P}(B|\vartheta_k, m_k)d\vartheta_k$$

and the probability for a model $m_{k,}$, given a set of observations B, is

$$\mathbb{P}(m_k|B) = \frac{\mathbb{P}(m_k \cap B)}{\mathbb{P}(B)} = \frac{p(B|m_k)\mathbb{P}(m_k)}{\sum_{i=1}^{N} p(B|m_i)\mathbb{P}(m_i)}$$

The main idea of this approach is that if we want to compute the moment of a random variable X under model uncertainty, we can use the set of observations B and the conditional probabilities that this set indicates. The approximation of the moments of X is a weighted average of all the moments with respect to a certain model. For example the first moment of X given by B is

$$E[X|B] = \sum_{i=1}^{N} E[X|m_i, B]\mathbb{P}(m_i|B)$$

The expectation, finally, will represent the nominal price of variable X. Once, the nominal price is defined measurements can be applied.

Bayesian vs. Worst-case approach

A known disadvantage of the worst-case measure is the difficulty to create the set of the candidate models. An even more difficult task, though, is to define the nominal model $m \in \mathcal{K}$. The MaxMin measure overcomes this obstacle but it becomes more conservative to quantify model risk and more expensive in a competitive market. The choice of nominal model is also not a problem for the Bayesian approach (see Bragner&Schlag [2004]). Nonetheless, the Bayesian measure does not assume that the actual model belongs to the set of candidate models. The main idea of this method is to assign small probability to unfavorable models without making implicit assumption on the true model. In that sense, it is hard and sort of arbitrary to decide which model is more significant than the other. This makes the Bayesian approach difficult to implement from a practical point of view (see Cont [2006] and Kerkhof [2010]). In practice we need to have a strong knowledge about the models and big experience according to previous market data in order to assign reasonable weights to each model. Limited data or the possibility of poor understanding of the real data process could be too risky for the Bayesian measure.

On the following chapter we provide a quick summary of twenty years of research on the topic. In this summary, several trials of extracting model risk out of the observed prices are listed.

Chapter 3
Literature survey

In this chapter we present a brief overview of previous researches on model risk and we discuss their impact on derivatives pricing and hedging. The purpose of this survey is to get a general idea of how people tried to extract or study model risk without getting into unnecessary details. In this chapter we identify three different patterns of available information. The first group is application-oriented papers, the second is about more general and axiomatic approaches and the last group focuses on the model uncertainty in incomplete and especially in interest rate markets.

3.1 First attempt

Looking back retrospectively at 1983 we found the first attempt of investigating model risk by Galai. In his paper the author tries to decompose the returns of the hedging strategies due to discretization and due to the model choice. The decomposition of model error is based on nothing else than **comparing actual and model prices derived from plain vanilla equity options**. The difference between the current and previous prices is considered to be the discretization effect of the strategy. In order to get the relevant error terms, the differences are applied either on the derivative or on the hedging portfolio price. For the experimental scenarios historical data are used. A daily hedging based on Black-Scholes (BS) model is implemented for European equity call options traded on CBOE. The results indicate that the contribution of the model error plays an important role on determining the prices.

3.2 Application oriented research

Bakshi, Cao and Chen [1997] **compare the pricing and hedging performance**, of vanilla equity call options from S&P500. In this paper it is examined how the Black-Scholes model performs against models that combine stochastic volatility(SV), stochastic interest(SI) and random jumps(J). The assessment of model misspecification is based on comparing implied volatility graphs of different models, across different moneyness and time to maturity (see Rubinstein [1985]), and by checking the consistency of implied volatility given from one model (see Bates [1996]) compared to this observed from the market data. The experiment involves estimating the model parameters from sample data and then testing the process out of sample. The results indicate better performance for SV model, however SV, SVJ, SVSI models are significantly misspecified in terms of internal consistency. Random jumps seem to improve the pricing of short-term options; whereas modeling stochastic interest rates, the random jumps can enhance the fit of long-term options.

Green&Figlewski [1999] have shown how to minimize the risk of derivative's position by dynamic hedging strategies in presence of model risk for equity, FX, interest rate vanilla products. The experiments are using historical scenarios taken from S&P500 and other important markets. The research considers hedging through diversification across different markets, cash flow matching and delta hedging. The risk of these trading strategies is examined for standard European puts and calls that are valued with some appropriate form of Black-Scholes model. The volatility input for the BS model is forecasted from the historical data either as an unconditional standard deviation or as an exponential weighted deviation. **The impact on the deviation of the prices of these trading strategies is examined for non-optimal volatility parameters and different money-**

ness. The results conclude that delta hedging is the most reasonable hedging strategy when the volatility is optimally estimated from the historical data.

Jiang&Oomen [2001], two years later, have discussed the impact of model misspecification on hedging. The authors claim that the magnitude of model risk can be isolated by **comparing the performance of hedging based on a wrong or misspecified model to those based on correctly specified models**. This argument is based on the idea of Bakshi, Cao & Chen [1997] to answer the question "Which is the least misspecified model?". For instance the SVSI-J (stochastic volatility, stochastic interest, stochastic jump) model which combines all the extensions of BS model outperforms the models which are missing one of these characteristics with respect to pricing error and hedging. In this spirit, Jiang & Oomen examine the performance of different hedging strategies varies across different pricing models, option's moneyness, option's maturity and by performing delta-vega-(rho) hedging strategies on European call options. They apply their method to examine the performance of volatility options and how the risk factors (interest, volatility, asset returns, etc.) can be hedged with certain strategies. Regarding the input values for the interest rate, the simulation results show that the choice between the market implied and model implied values has a very small impact on the hedging performance. However, when the stochastic interest rate is considered as constant the impact on the hedging effectiveness of ATM options, especially for medium- and long-term options is significant. Finally the research shows that the most important factor which affects the hedging performance under model misspecification is the input for the volatility parameter.

One year later, Hull&Suo [2002] investigate a method to quantify the exposure due to model risk for illiquid exotics. They propose the following approach: Assume that prices in the market are governed by a plausible **multi-factor no-arbitrage model, the so called "true model"** (see Bakshi, Cao & Chen [1997]), that is a lot more different and complex than the model being tested. In this paper the model risk of CR-IVF model (continuously recalibrated - implied volatility model) is examined by continuously fitting the "true model", a two-factor stochastic volatility model to market data (for more details see Schöbel and Zhu [1998]). Thereafter, the prices of compound and barrier options, given from the stochastic model, are compared to those that are priced under the Black-Scholes assumptions using the implied volatility taken from the CR-IVF formula. Hull&Suo find that the CR-IVF model gives reasonably good results for compound options while the results for barrier options are much less satisfactory. This indicates that barrier option more sensitive to path dependence than the compound options and that the size of model error is positively correlated to the number of payoff dates.

In the same period, Kerkhof, Melenberg & Schumacher [2002], have tried to assess the total hedging error instead of simply calculating hedging ratios. Since the actual dynamics are not available historical data simulation is applied, and following the steps proposed by Hull&Suo [2002] a benchmark model is defined. They investigate OTM, ATM, ITM European call options according to dividend paying Black-Scholes model for FX and S&P500 options. **The total model risk is decomposed in model risk due to estimation error and model risk due to misspecification**. The estimation risk is calculated by testing one model for different input parameters and the misspecification risk for a set of different models. The research applies the worst-case measure for the quantification and it stresses that the misspecification risk dominates the estimation risk. It is also shown that the nonparametric estimates of the P&L distribution, under the model assumptions, are more or less symmetric while the empirical density is skewed to the left. This explains why it happens more often than expected that the actual cost of hedging exceeds the option premium by a substantial amount.

3.3 More axiomatic approaches

After a couple of years we find researchers trying to move from application-oriented papers to more general and axiomatic approaches for model risk. Branger&Schlag [2004] emphasize that before doing hedging simulations it is essential to choose a robust market risk measure and an optimally minimized hedging strategy which should be model insensitive in order to come up with a solid ground for model risk assessment. They examine a naive, the worst case and the Bayesian approach for model risk quantification. The authors stress that the aggregated risk is very important for the optimization of the hedging strategy and they believe that the Bayesian method is more appropriate to aggregate the market and model risk (see chapter 2 for the definition and the difficulties of this approach).

As model risk attracts more attention from the research community, in 2006, an interesting work by Cont comes to light. In this paper the author tries to distinguish the uncertainty of econometric estimation from the uncertainty of risk neutral models and he presents a number of properties[2] for model risk assessment. According to these properties a coherent[3] measure is defined for model risk, as **the range of all candidate model prices**, while models are marked-to-market. The applicability of the measure is studied for European options priced with the Black-Scholes

[2] Properties that have already been addressed on previous papers like liquidity of benchmark instruments, model-free replicating strategies, the risk of static hedging, the risk of market prices, etc.

[3] A measure defined by Artzner [1999], which satisfies the properties of monotonicity, sub-additivity, translation invariance and positive homogenuity (see also Frey [2005])

(BS) model under local volatility uncertainty and for static hedging strategies. In one of the examples where a barrier option is priced with a local volatility model and a BS model with jumps the measurement shows that model risk composes the 40% of the derivatives selling price. In addition to that, an alternative measure is proposed which does not require model calibration. In order to achieve this Cont introduces a convex[4] measure which takes as an adjustment the difference norm between the model and market price along the set of benchmark instruments. Then the model risk is defined as **the range of all these convex measures** that correspond to the set of all candidate models. For more details we refer to [53].

In the beginning of the current year, Kerkhof, Melenberg&Schumacher [2010] publish a paper which incorporates model risk into risk measure calculations using classes of models based on standard econometric methods. This work continues on the same spirit as the paper of [2002], while it tries to incorporate the new concepts of ambiguity and lack of identification (see chapter [2] for more details). In that sense an new type of uncertainty is defined, the identification risk. The quantification of model risk, is achieved on top of market risk, by using the worst-case approach to measure the aggregated total risk. By using this measurement the nominal risk measure, such as VaR or Expected Shortfall, is adjusted accordingly. In this way **market risk is estimated using a class of models and not only one particular model**. The method is applied on S&P500 and FX data and the analysis is done by using a rolling window of two years for a range of historical data.

[4] A measure defined by Föllmer and Schied [2002], which relaxes the property of positive homogenuity to $\rho(\lambda X + (1-\lambda)Y) = \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\forall\lambda \in [0,1]$. Where X and Y represent the payoff of an option or portfolio. This measure can be defined as $\rho(X) = \sup_{\mathbb{Q}\in\mathcal{K}} \left\{E^{\mathbb{Q}}[X] - \alpha(\mathbb{Q})\right\}$ instead of the classic worst-case form $\rho(X) = \sup_{\mathbb{Q}\in\mathcal{K}} \left\{E^{\mathbb{Q}}[X]\right\}$. $\mathbb{Q}$ is the risk neutral measure, $\mathcal{K}$ is the set of candidate models and $\alpha(\mathbb{Q})$ is a penalty function (see Cont [2006]), which is the measure's adjustment to the systematic mispricing of the benchmark instruments by the corresponding model.

3.4 Interest rate markets

Bossy, Gibson, Lhabitant, Pistre &Talay [2007], in the same spirit as in [2000] and [2001], have developed a conceptual framework for **decomposing the P&L related to model misspecification for interest rate claims**. For this, an analytical way is proposed to decompose P&L to initial pricing error, model pricing error at any time and the cumulative risk due to hedging error. They investigate the sensitivity of forward P&L with respect to volatility, forward yield curve and frequency of rebalancing for bond options. The research is restricted to model risk assessment of Markovian univariate HJM and short rate models (Ho-Lee, Vasicek, CIR) but the underlying methodology can be applied to a larger class of hedging strategies with univariate Markov models. The results, like those of Figlewski&Green [1999], suggest that the discreteness of the replicating portfolio magnifies model risk, even for short rebalancing time intervals. The paper leaves to a future work the assessment of the hedging strategy when higher order multi-factor models are used.

Furthermore, there are few other noticeable papers related to the interest rate market which we will mention below without going much into detail. There are two similar approaches like that of Hull&Suo [2002] namely the work of Longstaff, Santa-Clara & Schwartz [2001] and Andersen&Andreasen [2001]. These papers test the performance of one-factor short rate model in a market where the term structure behaves as a multifactor model The first paper examines the cost of exercising American swaptions. The research shows **significant losses** for the one-factor model. which implied that the **model risk is more important than the optimal early exercise strategy** of the option. This is because the American swaption prices directly depend on the autocorrelation between interest rates of different maturities, while one-factor models imply perfect correlation between interest rates of different maturities. On the other hand, the second paper **tests the ef-**

fectiveness of the one-factor short rate model for pricing Bermudan swaptions. The model is recalibrated daily to caps or European swaptions or both. The results support the use of continuously re-calibrated one-factor models to price Bermudan swaptions, as long as the calibration is sufficiently comprehensive (calibration to a wide range of market instruments).

A similar work from Driessen [2003] examined both caps and swap options. The research gives an evidence that the difference of using one-factor and two-factor models for hedging disappears when the **set of hedging instruments maturities cover all payoff dates**. Additionally, Gupta&Subrahmanyam [2005] test several interest rate models for the pricing and hedging of caps and floors with multiple strike prices. They show that **one-factor models are adequate for pricing** and **two-factor models are more adequate for the hedging of caps and floors** and probably for other interest rate products in general. According to the authors opinion there are maybe significant benefits of using higher order multi-factor models for hedging more complicated payoffs like swaptions and spread options.

To summarize, we studied a long period of research, on model risk, started from 1989 until now. The main objective of researchers was always to extract (decompose) and quantify the model risk. Initially, we have seen the self-evident method of comparing model prices to market prices for plain vanilla products. Next, we meet the popular idea of comparing a very good benchmark model with other candidate models. The model risk in that case is always the difference between the "good" and the candidate model prices. Finally, we also find a conservative approach which considers as model risk the price range of all candidate models prices. This method targets model risk without the requirement of any analysis or decomposition.

As we initially stated, due to the limited information of model risk quantification on interest rate exotics, our objective is to study through hedging simulations the risk exposure of using the one-factor Hull-White model on Bermudan swaptions. In this project we decide to restrict our research on the mean reversion risk which in practice depends on the traders' choice. This is because the calibration of the mean reversion parameter to vanilla swaption prices does not fully incorporate the autocorrelation structure of the swap rates for the Bermudan swaptions.

On the next two chapters we provide the setup of our experimental infrastructure. Chapter 4 explains the basic implementation of the pricing procedure. Chapter 5 gives information about the hedging setup of our experiments. In chapter 6 we present the last part of our theoretical approach. In this chapter we will try to explain in a general and intuitive way the expected errors of the experimenter's results due to model, hedging and market risk.

Chapter 4
Valuation framework

For pricing interest rate options we usually need to know how the term structure of interest rates will evolve through the time. The zero rates of the yield curve can be described using an indirect method by modeling the instantaneous short rate[5] $r(t)$. In this way the prices of bonds or any other interest rate product depend only on the risk neutral dynamics of r. The main task of our project is to investigate the uncertainty of using one of the most popular short rate models, which has been introduced by John Hull and Alan White in 1990. We will base the complete investigation on a specific product namely Bermudan swaption. The purpose of this chapter is to provide the reader with all the relevant information regarding our implementation of the model, the calibration to market data, and the valuation of callable swaps with this model.

4.1 One factor Hull-White model

The Hull-White (or Extended Vašíček) model is a no-arbitrage short rate model which corrects the inability of Vašíček model to fit the initial term structure, by allowing to take it as an input. Practically, this can be obtained by introducing a function of time in the drift term of the model. The risk neutral dynamics of the one-factor Hull-White model are given by the affine diffusion

$$dr(t) = (\vartheta(t) + \alpha r(t))\, dt + \sigma dW(t) \tag{4.1}$$

[5] This is a variable which is not observed directly from the market.

where α and σ are constant. The model is mean reverting to $\frac{\vartheta(t)}{\alpha}$ with a rate of α

$$dr(t) = \alpha \left(\frac{\vartheta(t)}{\alpha} + r(t) \right) dt + \sigma dW(t)$$

The property of mean reversion ensures that the model is consistent with the empirical observation that long rates are less volatile than short rates. In addition, $\vartheta(t)$ is the parameter which is responsible for fitting the current zero rates while α and σ are chosen[6] appropriately to provide a good volatility structure.

Under the assumptions of affine modeling (see Appendix (A)) we can replace the structural parameters of the Hull-White model to the equations (A.3) and (A.4) of the proposition 6 in Appendix (A), we get the solution for the parameters of the affine term structure (A.6) and (A.5)

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) 1 - e^{-\alpha(T-t)} \tag{4.2}$$

$$A(t,T) = \int_t^T \left\{ \frac{1}{2}\sigma^2 B^2(s,T) - \vartheta(t) B(s,T) \right\} ds \tag{4.3}$$

4.1.1 Term structure

After this step we need to fit the observed bond prices $P^*(t,T)$ to the theoretical ones. As we can leverage from the one to one relationship of bond prices and forward rates we get

$$f^*(t,T) = \frac{\partial}{\partial T} \log P^*(t,T) \quad , \quad \forall T > 0$$

6 The choice of α, the speed of mean reversion, is usually left to the user and σ is determined from the calibration process.

which gives us

$$\begin{aligned} f^*(t,T) &= \frac{\partial}{\partial T}B(t,T)r(t) - \frac{\partial}{\partial T}A(t,T) \\ &= e^{-\alpha T} + \int_0^T e^{-\alpha(T-s)}\vartheta(s)ds - \frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha T}\right)^2 \end{aligned} \tag{4.4}$$

Given the observed $f^*(0,T)$ which can be derived from the market prices of zero coupon bonds we want to solve $\vartheta(t)$ such that we fit the current zero rates. One way of solving that is separate to $f^*(0,T) = x(T) - g(T)$, where $g(t)$ is the deterministic part of equation (4.4)

$$\begin{aligned} dx(t) &= -\alpha x(t)dt + \sigma dW(t) \\ x(0) &= r(0) \end{aligned}$$

and

$$g(t) = \frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha T}\right)^2 = \frac{\sigma^2}{2}B^2(0,t)$$

That gives us

$$\begin{aligned} \frac{\partial}{\partial T}f^*(0,T) &= \frac{\partial}{\partial T}x(T) - \frac{\partial}{\partial T}g(T) \\ &= -\alpha x(T) + \vartheta(T) - \frac{d}{dT}g(T) \end{aligned}$$

which solves $\vartheta(t)$ as follows

$$\vartheta(t) = \frac{\partial}{\partial t}f^*(0,t) + \alpha x(t) + \frac{d}{dt}g(t)$$

After having $\vartheta(t)$ we can obtain $r(t)$ by solving a mean reverting Ornstein-Ulhenbeck process, then its is easy to get

$$r(t) = r(s)e^{-a(t-s)} + c(t) - c(t)e^{-a(t-s)} + \sigma\int_s^t e^{-a(t-u)}dW(u)$$

where $c(t) = f^*(0,t) + g(t)$. Moreover by replacing $\vartheta(t)$ to equation (4.3) we get the affine term structure of equation (A.2) for the Hull-White model

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\left\{B(t,T)f^*(0,T) - \frac{\sigma^2}{4a}B^2(0,t)(1-e^{-2\alpha t}) - B(t,T)r(t)\right\}$$

For practical reasons, we can look back at equation (4.1) and imagine a short rate process reverting around zero

$$\begin{aligned} dx(t) &= -\alpha x(t)dt + \sigma dW(t) \\ x(0) &= 0 \end{aligned} \tag{4.5}$$

which takes the integral form of

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

while $r(t) = x(t) + g(t)$. Then, the term structure for Hull-White model is given by

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\left\{-G(t,T)\left(1-e^{-\alpha t}\right) - B(t,T)\, x(t)\right\} \tag{4.6}$$

where

$$G(t,T) = \frac{\sigma^2}{2a}B^2(0,t)(1-e^{-\alpha t})\left\{\frac{B(t,T)}{2}(1+e^{-\alpha t}) + \frac{(1-e^{-\alpha t})}{2}\right\}$$

The above expression offers to practitioners a clear formula for implementation.

4.1.2 Volatility structure and mean reversion

The volatility structure of Hull-White model is determined by the structural parameters α and σ. The volatility of a zero coupon bond at time t with maturity at T is

$$\frac{\sigma}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) \tag{4.7}$$

The volatility of a spot rate that corresponds to a zero-coupon bond of maturity T at time t is

$$\frac{\sigma}{\alpha\,(T-t)}\left(1-e^{-\alpha(T-t)}\right) \tag{4.8}$$

and the volatility of the forward rate at t that corresponds to a forward starting zero-coupon bond is $\sigma e^{-\alpha(T-t)}$. The diffusion parameter σ will determine the volatility of the instantaneous short rate and the speed of mean reversion will affect the shape of the volatility smile.

For a product like Bermudan swaption the derivative's price is highly correlated to the forward swap rates. By looking at equations (4.7) and (4.8), the previous statement implies that the speed of mean reversion α can play an important role on the calibration of the short rate model and consequently on the pricing afterwards. The calibration of the mean reversion parameter to non-callable instruments will not incorporate information related to the autocorrelation structure of the swap rates which is important for callable products. As a result, we will restrict the choice of α within a certain range.

The range of α depends on the currency and the different maturities that a product is traded. For example, for a Bermudan swaption for less than 10 years maturity, traded in Euro, a reasonable[7] range will be [0-5%] while a for higher than 10 year tenors can be [0-3%]. Nevertheless, these ranges are not small to be ignored. For that reason, in this project we decide to dedicate our investigation on the model uncertainty due to the speed of mean reversion parameter. As we can see on chapter 8 the impact of different α on the option's replicating portfolio price is far from being negligible.

[7] The data are based on recommendations given from the Market Risk Management department of ING Bank.

4.1.3 Calibration

Calibration is the learning process where a model iteratively tries to adjust its structural parameters until the output of the model matches the values of the training data. The training data must be market prices of highly traded products. In such a way the model is getting up-to-date[8] with the most recent behavior of the market. A reasonable choice for the calibration of a short rate model is to use liquid vanilla caps/floors and swaptions (see Andersen & Andreasen [2001] and Gupta & Subrahmanyam [2005]). For this project we will calibrate the one factor Hull-White model only to a set[9] of vanilla swaption prices.

The value of a payer[10] swaption at time $t < T_0$ priced with the Hull-White model, with tenor $\mathcal{T}_n = \{T_0, T_1, ..., T_n\}$, strike (fixed rate) K and notional N is

$$PSO(t, \mathcal{T}_n, N, K) = N\sum_{i=1}^{n} c_i \left[P(t_i, T_0)P\left(T_0, T_i, x(T_0)\right)N(d_1) - P(t, T_i)N(d_2)\right]$$

where $c_i = \tau_i K$, for $i = 1, 2, ..., n-1$ and $c_m = 1 + \tau_n K$. Moreover, $x(T_0)$ satisfies

$$\sum_{i=1}^{n} c_i P\left(T_0, T_i, x(T_0)\right) = 1$$

$P\left(T_0, T_i, x(T_0)\right)$ is the price of a zero coupon maturing at T_i that depends on process $x(t)$. Finally, the Hull and White model is calibrated to piecewise constant volatility of which the representation is given in Appendix (B).

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8 The calibration adjusts the model parameters until the match satisfies a threshold of certain accuracy. This method, though, does not take into account the pricing imperfection of the model. In that sense one can claim that calibration will give wrong model parameters because the model systematically misprices the benchmark instruments (see Cont [2006]). On the other hand, one can see the calibration as the method of model correction (see Andrersen&Andreassen [2001]) that transforms the model in such a way that the mispricing errors are minimal for a wide range of benchmark instruments. Nevertheless, there are additional issues that can cause mistakes on the estimation process related to the quality of market data and other reasons (see estimation risk at chapter [3]).

9 The available market data contain only quotes for at-the-money vanilla swaptions.

10 The owner pays fixed interest rate K.

4.2 Longstaff-Schwartz method

As we stated in chapter (1.2) our interest is focused on the valuation of Bermudan swaptions. A (payer) Bermudan swaption offers the option to the owner to enter in a swap agreement with fixed rate K with first reset date T_l and last pay date T_n, where $T_l \in [T_k, ..., T_h]$ a range of discrete initial reset dates and $T_k < T_h < T_n$. The pricing of Bermuda style payoffs reduces to an optimal stopping problem. Thus, the only thing we need to know is the conditional expectation of the derivative price $F(x_i)$ with underlying x_i at $t = t_i$,

$$\mathbb{E}\left[F(x_{i+1})|x_i\right] \tag{4.9}$$

The problem can be solved with classical PDE methods associated with the variational inequalities (see Wilmott, Howison & Dewynne [1995]). However, for high dimensional problems pricing with Monte-Carlo simulations is required. Then the solution to the optimal stopping problem is not straightforward anymore. Carrière [1996], Tsitsiklis & Van Roy [1999], [2001] and Longstaff & Schwartz [2001] proposed methods which give a proxy for the conditional expectation based on dynamic programming and regression. The basic idea of this is that the option price can be approximated as a linear combination of basis functions,

$$\mathbb{E}\left[F(x_{i+1})|x_i\right] \approx \sum_{k=0}^{K} a_k g_k(x_i) \tag{4.10}$$

Where, the weights of these functions are products of regression.

Our implementation realizes the Longstaff-Schwartz method, a technique very popular among practitioners. As stated in Clément, Lamberton & Protter [2002] and Glasserman & Yu [2005] the approximation of (4.9) is an orthogonal projection on a complete space of linearly independent basis functions. A possible choice for this span is the set of polynomials $g_n(x) = x^n$.

4.2.1 Valuation algorithm: Least-Square-Method

To solve (4.10) with respect to a_k we need to find the solution for the least squared error

$$\mathbb{E}\left[\left(\mathbb{E}\left[F(x_{i+1})|x_i\right] - \sum_{k=0}^{M} a_k g_k(x_i)\right)^2\right] \to 0$$

this gives

$$\mathbb{E}\left[\mathbb{E}\left[F(x_{i+1})|x_i\right] g_k(x_i)\right] = \sum_{k=0}^{M} a_k \mathbb{E}\left[g_k(x_i) g_l(x_i)\right]$$

Lets denote

$$A_{k,l} = \mathbb{E}\left[g_k(x_i) g_l(x_i)\right]$$

and

$$\begin{aligned} B_k &= \mathbb{E}\left[\mathbb{E}\left[F(x_{i+1})|x_i\right] g_k(x_i)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[F(x_{i+1}) g_k(x_i)|x_i\right]\right] \\ &= \mathbb{E}\left[F(x_{i+1}) g_k(x_i)\right] \end{aligned}$$

where the above equality comes from the fact that $g_k(x_i)$ is x_i measurable and the property of tower rule. Then we find the weights by inverting $A_{k,l}$

$$a = A_{k,l}^{-1} B_k$$

The calculation of these coefficients requires the Monte-Carlo simulation of the underlying for the set of exercise dates $T_1, T_2, ..., T_M$. Therefore for N paths we get

$$\hat{A}_{k,l} = \frac{1}{N} \sum_{n=0}^{N} g_k\left(x_i^{(n)}\right) g_l\left(x_i^{(n)}\right)$$

and

$$\hat{B}_k = \frac{1}{N} \sum_{n=0}^{N} F\left(x_{i+1}^{(n)}\right) g_k\left(x_i^{(n)}\right)$$

The **regression algorithm** then goes as follows:

- Simulate for N paths for the set of exercise dates $T_1, T_2, ..., T_M$
- The final condition of each path will be $F(x_M)$
- After having the final step of the simulation we can go backwards following the discrete time setting
  - We calculate $\hat{A}_{k,l}$ and $\hat{B}_k$
  - We find the inverse matrix $\hat{A}_{k,l}^{-1}$
  - Next, calculate $\hat{a}_k = \hat{A}_{k,l}^{-1} \hat{B}_k$

As it is stated in Glasserman & Yu [2005] for the valuation we can use $M \neq N$ paths in order to get a low biased estimate. This procedure is separate from the previous steps. Usually it is more efficient to proceed a forward calculation since their is always the possibility that the option is exercised in one of the first maturity dates. Then the **valuation algorithm** goes as follows:

- $\mathbb{E}\left[F(x_{i+1})|x_i\right] \approx \sum_{k=0}^{K} \hat{a}_k g_k(x_i)$
- Next compare the expected future value $\mathbb{E}\left[F(x_{i+1})|x_i\right]$ and the immediate exercise $F(x_i)$ at $t = T_i$ and according to the type of the payoff to set the value of the derivative at the current time step.

The convergence of the least square estimator increases as the number of paths and the number of polynomials functions increase. On Clément, Lamberton & Protter [2002] and Glasserman & Yu [2005] the reader can find additional information regarding the quality of the estimators, the convergence and the rate of convergence of this approximation.

On the next chapter we define our replicating strategy for hedging interest rate claims based on the current valuation framework. This will be our benchmark for studying the impact of model risk on Callable swaps like Bermudan swaptions.

Chapter 5
Replication

The goal of this project is to assess model risk through hedging simulations for Bermudan swaptions under the valuation framework of the previous chapter. For this reason we have developed a hedge test module that is able to perform Δ, V and ΔV hedging. The module in principal is able to replicate several interest rate claims using different short rate models for valuation. In this chapter, we want to give an extensive description of the replication process which is applied on our C++ framework.

Consider a T-claim $\mathcal{X}$ in a market which is consisted of n risky underlyings given a priori

$$S_1, S_2, ..., S_n$$

with $\mathbb{P}$-dynamics ($\mathbb{P}$ is an objective measure),

$$dS_i(t) = a_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{n}\sigma_{ij}d\bar{W}_j(t)$$

where W_j can be correlated $\mathbb{P}$-Brownian motions with correlation matrix ρ and $Cov(d\bar{W}_i, d\bar{W}_j) = \rho_{ij}dt$. In this market we also have a standard risk free asset

$$dB(t) = r(t)B(t)dt$$

where the instantaneous short rate $r(t)$ is an adapted stochastic process.

Definition 3 *The **wealth process** of a spot trading strategy $h(t)$ is*

$$\mathcal{V}^h(t) = h(t) \cdot S(t) = \sum_{i=1}^{n} h_i(t)S_i(t) \qquad , \qquad \forall t \in (0, T]$$

where the dot "$\cdot$" stands for inner product and S is a vector of assets from the existing market. The initial wealth of this portfolio is $\mathcal{V}^h(0) = h(0) \cdot S(0)$ and $\mathcal{V}^h(0) \in \mathbb{R}$.

5.1 Self-financing portfolio

A self-financing portfolio at any instant time $t = 1, 2, ..., T$ is solely financed by selling assets already in the portfolio.

Definition 4 *A spot trading strategy is said to be **self-financing** if it satisfies the budget equation*

$$\mathcal{V}^h(t) = h(t - dt) \cdot S(t) = h(t) \cdot S(t) \quad , \quad \forall t \in (0, T] \tag{5.11}$$

*where **capital gains** at time t are modeled as a backward differential equation.*

$$\mathcal{V}^h(t) - \mathcal{V}^h(t - dt) = h(t - dt) \cdot (S(t) - S(t - dt)) \quad , \quad \forall t \in (0, T] \tag{5.12}$$

*Moreover, in continuous time, taking the limit $dt \to 0$, the **gain process** is defined as follows*

$$d\mathcal{V}^h(t) = h(t) \cdot dS(t) \quad , \quad \forall t \in (0, T] \tag{5.13}$$

with $h(t)$ an $\mathcal{F}_{t-dt} - measurable$ process (predictable/non-anticipating at time t).

The equation (5.11) implies that at the beginning of period t our wealth equals what we get if we sell the old portfolio at today's prices.

Lemma 1 *A spot trading strategy $h(t)$ is self-financing iff $\mathcal{V}^h(t) = \mathcal{V}^h(0) + \Delta\mathcal{V}^h(t)$, $\forall t \in (0, T]$.*

Proof. Assume that $h(t)$ is self-financing. Then taking into account equations (5.11) and (5.12)

$$\begin{aligned} \mathcal{V}^h(t) &= h(0)\cdot S(0) + \sum_{u=1}^{t} \left(h(u)\cdot S(u) - h(u)\cdot S(u-1)\right) \\ &= h(0)\cdot S(0) + \sum_{u=1}^{t-1} h(u)\cdot \left(S(u) - S(u-1)\right) \\ &= \mathcal{V}^h(0) + \Delta\mathcal{V}^h(t) \end{aligned}$$

The inverse can be established in a similar way. ■

Definition 5 *A self-financing trading strategy* h *is called an* ***arbitrage*** *opportunity if* $P(\mathcal{V}^h(t) = 0) = 1$ *while the final wealth is*

$$\begin{aligned} P(\mathcal{V}^h(T) &\geq 0) = 1 \\ P(\mathcal{V}^h(T) &> 0) > 0 \end{aligned}$$

Intuitively arbitrage is a self financing trading strategy which generates profit with positive probability while it cannot generate loss. In addition, we say that spot market $\mathcal{M} = (S, H)$ is arbitrage-free if there are no arbitrage opportunities in the class H of all self-financing strategies.

Definition 6 *A* T*-claim* $\mathcal{X}$ *is* ***reachable*** *(can be replicated) iff there exists a self-financing portfolio* $h(t)$ *such that*

$$P\left(\mathcal{X} = \mathcal{V}^h(T)\right) = 1$$

$h(t)$ *is the hedge (replicating portfolio) against* $\mathcal{X}$.

Proposition 2 *Suppose there exists a self-financing portfolio such that* $\mathcal{V}^h(t)$ *has dynamics*

$$d\mathcal{V}^h(t) = k(t)\mathcal{V}^h(t)dt$$

then $\mathcal{V}^h(t)$ *is* ***arbitrage-free*** *if and only if* $k(t) = r(t)$*, where* $r(t)$ *is the instantaneous short rate.*

The interpretation of this proposition is that dynamics with no source of uncertainty, "locally riskless", should earn a return equal to the short rate of interest. The general solution of $\mathcal{V}^h$ is $\mathcal{V}^h(t) = \mathcal{V}^h(0)e^{\int_0^t r(s)ds}$.

5.2 Constructing a hedge

According to the previous definition we will attempt to construct a self-financing portfolio [11] to replicate a T-claim $\mathcal{X}$ using the priori market. In that sense we set

$$\mathcal{V}^h(t) = h_B(t)B(t) + h_1(t)S_1(t) + ... + h_n(t)S_n(t)$$

Bjork [2004] shows that the weights of such portfolio are

$$h_i(t) = \frac{\partial F}{\partial S_i} \tag{5.14}$$

$$h_B(t) = \frac{\mathcal{V}^h(t) - \sum_{i=1}^n S_i \frac{\partial F}{\partial S_i}}{B(t)} \tag{5.15}$$

where F satisfies the partial differential equation

$$rF = \frac{\partial F}{\partial t} + r\sum_{i=1}^n S_i \frac{\partial F}{\partial S_i} + \frac{1}{2}\sum_{i=1}^n \sigma_i S_i \sigma_j S_j \rho_{ij} \frac{\partial^2 F}{\partial S_i \partial S_j} \tag{5.16}$$

with final condition $F(T, S(T)) = \mathcal{X}$. Moreover the replicating portfolio $\mathcal{V}^h(t)$ is the same diffusion process as the derivative price $F(t, S(t))$.

Alternatively, we can see equation (5.16) from a more financial point of view as

$$rF = \Theta + r\sum_{i=1}^n S_i \Delta_i + \frac{1}{2}\sum_{i=1}^n \sigma_i S_i \sigma_j S_j \rho_{ij} \Gamma_{ij} \tag{5.17}$$

[11] Assume constant volatility for the option's price process.

Remark 1 *Consider t, in discrete time, defined for a given partition $\tau_n : 0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$. To simplify things, suppose we have available one asset $S(t_i)$ in our economy then the equation (5.17) becomes*

$$rF = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma$$

The partial derivative of function F at point $(t_i, S(t_i))$ with respect to t is

$$\begin{aligned} \frac{\partial}{\partial t} F(t_i, S(t_i)) &= \frac{F(t_i + dt, S(t_i + dt)) - F(t_i, S(t_i))}{dt} \quad , \quad \forall dt > 0 \\ &= \Theta \end{aligned} \tag{5.18}$$

Using the chain rule we can derive the partial derivatives with respect the rest of the variables

$$\begin{aligned} \frac{\partial F}{\partial S} &= \frac{\partial F}{\partial t}\frac{\partial t}{\partial S} \\ &= \frac{F(t_i + dt, S(t_i + dt)) - F(t_i, S(t))}{dt} \frac{dt}{S(t_i + dt) - S(t_i)} \\ &= \frac{F(t_i + dt, S(t_i + dt)) - F(t_i, S(t))}{S(t_i + dt) - S(t_i)} \\ &= \Delta \end{aligned} \tag{5.19}$$

Then in order Δ to be well-defined we must have $S(t_i + dt) - S(t_i) \neq 0$. Since the calculation of Δ is performed at t_i, $S(t_i)$ is deterministic, hence the only possible assumption we can take is that $S(t_i + dt) = S(t_i) + c$ with $c \neq 0$. In the same way[12] *we can derive $\frac{\partial}{\partial S}\left(\frac{\partial F}{\partial S}\right) = \Gamma$ as well.*

[12] This type of calculation it is sometimes called "forward" sensitivity in the word of quantitative finance. Many trading platforms though, in ING and elsewhere, still use the so called "spot" sensitivities (see section (5.4)). For our calculations we use the "forward" sensitivities and the practical reason of doing that is explained in section (5.3). In section (7.4.2) we provide evidence for this choice.

5.3 The idea of replication

This section will try to offer a financial interpretation of equation (5.19) using a simple example from risk neutral pricing in discrete time. For that reason, we will consider the binomial model to describe the dynamics of derivative F with underlying asset S and tenor T. In discrete time, t is defined for a given partition $\tau_n : 0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$ and $t \in [t_{i-1}, t_i]$.

Now, suppose that we issue a derivative F at $t = t_i$. Taking the assumption of no-arbitrage opportunities, we want to construct a self-financing portfolio $\mathcal{V}(t) = \Delta(t)S(t)$ at $t = t_i$ such that there is no uncertainty about its value at $t = t_{i+1}$. Then, we would like to know how much $\Delta(t_i)$ we have to choose, at $t = t_i$, in order to create a locally riskless portfolio until time $t = t_{i+1}$. $\Delta(t_i)$ is an $\mathcal{F}_{t_i}$-measurable process, where $\mathcal{F}_{t_i}$ represents the available market information until $t = t_i$.

In the simple case of binomial tree, at $t = t_i$, it is assumed that the value of the underlying will go either up or down by some amount $u, d \in \mathbb{R}$. Therefore, on the next discrete time step we potentially observe either $S(t_{i+1})^+ = S(t_i) \cdot u$ (for **state 1**) or $S(t_{i+1})^- = S(t_i) \cdot d$ (for **state 2**) as the price of the underlying, where $S(t_{i+1})^+$ and $S(t_{i+1})^-$ are both $\mathcal{F}_{t_i}$-measurable. The situation is described in figure (5.3). Under this hypothetical world, we would like to have the value of the portfolio $\mathcal{V}$ equal for both states, thus $\mathcal{V}_{state1} = \mathcal{V}_{state2}$, in order to remain riskless until time $t = t_{i+1}$. Obviously, we can calculate the hypothetical price[13] of the derivative $F(t_{i+1}, S(t_{i+1}))$ while we are still on step $t = t_i$ for both states **1** and **2.** For this calculation we only need to use $T - t_{i+1}$ (the time to maturity) and $S(t_{i+1})^+$ or $S(t_{i+1})^-$ respectively.

[13] We mention that the functions of the form $F(t, S(t)^+)^+$ or $F(t, S(t)^-)^-$ represent the bumped prices of a derivative with function of the form $F(t, S(t))$ due to bumped underlying $S(t)^+$ or $S(t)^-$ respectively.

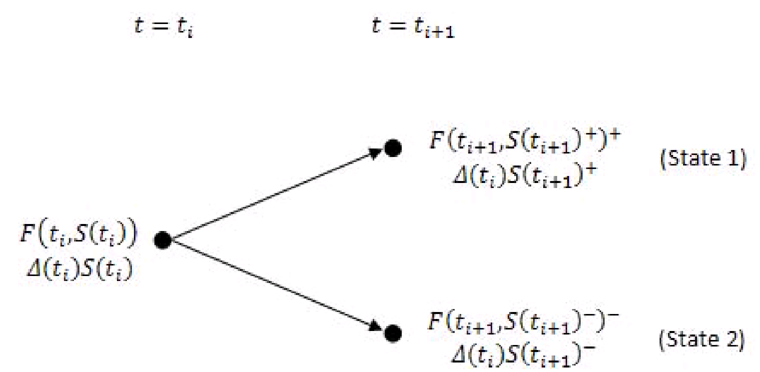

Figure (5.3). The binomial model is a simple example where we can easily interpert the idea of risk neutral pricing and replication.

The solution for $\Delta(t_i)$ finally is the sensitivity of $F(t_i, S(t_i))$ to the movement of the underlying asset

$$\Delta(t_i) = \frac{F(t_{i+1}, S(t_{i+1})^+)^+ - F(t_{i+1}, S(t_{i+1})^-)^-}{S(t_{i+1})^+ - S(t_{i+1})^-}$$

such that

$$\begin{aligned} F(t_i, S(t_i)) - \Delta(t_i)S(t_i) &= DF(t_i, t_{i+1})\mathcal{V}_{state1} \\ &= DF(t_i, t_{i+1})\mathcal{V}_{state2} \end{aligned}$$

where $DF(t_i, t_{i+1})$ stands for the discount factor for the time range $t \in [t_i, t_{i+1}]$ and $F(t_i, S(t_i))$ the risk-neutral price of the derivative at $t = t_i$. This example seems to be enough to grasp the philosophy of replication and the spirit that follows on the next paragraphs.

5.4 Why forward sensitivities

The reason that forward sensitivities are preferred against the spot counterparts is explained below. Equations (5.18) and (5.19) show that theoretically there is only one way of calculating sensitivities,

$$\frac{\partial F}{\partial S} = \frac{\partial F}{\partial t}\frac{\partial t}{\partial S} = \frac{F(t_i + dt, S(t_i + dt)) - F(t_i, S(t))}{S(t_i + dt) - S(t_i)}$$

While the spot calculation is given by,

$$\frac{\partial F^{(Spot)}}{\partial S} = \frac{F(t_i, S(t_i) + h) - F(t_i, S(t))}{S(t_i) + h - S(t_i)} \tag{5.20}$$

By applying the chain rule to the “spot” ratio we must be able to derive equation (5.18) which represents Θ.

$$\begin{aligned} \frac{\partial F^{(Spot)}}{\partial t} &= \frac{\partial F^{(Spot)}}{\partial S}\frac{\partial S}{\partial t} = \\ &= \frac{F(t_i, S(t_i) + h) - F(t_i, S(t))}{dt} \\ &\neq \frac{F(t_i + dt, S(t_i + dt)) - F(t_i, S(t_i))}{dt} \end{aligned} \tag{5.21}$$

Hence, the Θ calculation is not consistent if Δ is derived according to (5.20). The $\frac{\partial F}{\partial t}^{(Spot)}$ ratio is not well defined anymore. As we can see on the figure below the price of the bumped $F(t_i, S(t_i) + h)$ will lie on a vertical slope (see red dotted line in figure (5.21)).

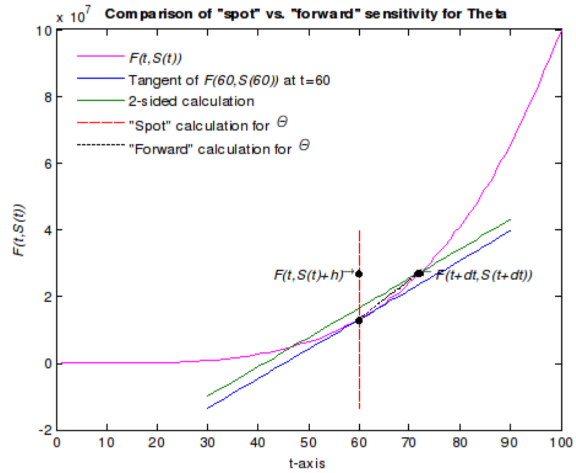

Figure (5.21)**.** The figure offers a visual illustration of the main difference between“spot” and “forward” hedging ratios.

An evidence for the robustness of the forward sensitivities is provided on section (7.4.2).

5.5 A $\triangle$-neutral portfolio

To neutralize the Δ-sensitivity of interest rate claim, such as Bermudan swaption (or any other claim sensitive to swap rates), we choose to use n liquid par swaps. Lets denote the price of the hedgable claim as PSO and the hedging instruments as PS_i. Next, we need to calculate the weights h_i such that both sides $PSO = h_1 PS_1 + ... + h_n PS_n$ should have same Δ-sensitivities. Hence we get a system of linear equations that need to be solved.

$$
\begin{aligned}
\frac{\partial PSO(t_{i+1})}{\partial r_1} &= h_1(t)\frac{\partial PS_1(t_{i+1})}{\partial r_1} + ... + h_n(t)\frac{\partial PS_n(t_{i+1})}{\partial r_1} \\
\frac{\partial PSOt_{i+1})}{\partial r_2} &= h_1(t)\frac{\partial PS_1(t_{i+1})}{\partial r_2} + ... + h_n(t)\frac{\partial PS_n(t_{i+1})}{\partial r_2} \\
&\vdots \\
\frac{\partial PSO(t_{i+1})}{\partial r_n} &= h_1(t)\frac{\partial PS_1(t_{i+1})}{\partial r_n} + ... + h_n(t)\frac{\partial PS_n(t_{i+1})}{\partial r_n}
\end{aligned}
$$

where r_i the market swap rates. The problem, because of the nature of swap rates and their relation with the zero rates, the above linear equations reduce to the following system

$$
\begin{pmatrix}
\frac{\partial PS_1(t_{i+1})}{\partial r_1} & \frac{\partial PS_2(t_{i+1})}{\partial r_1} & \cdots & \frac{\partial PS_{n-1}(t_{i+1})}{\partial r_1} & \frac{\partial PS_n(t_{i+1})}{\partial r_1} \\
 & \frac{\partial PS_2(t_{i+1})}{\partial r_2} & & \frac{\partial PS_{n-1}(t_{i+1})}{\partial r_2} & \frac{\partial PS_n(t_{i+1})}{\partial r_2} \\
 & & \ddots & & \vdots \\
 & & & \frac{\partial PS_{n-1}(t_{i+1})}{\partial r_{n-1}} & \frac{\partial PS_n(t_{i+1})}{\partial r_{n-1}} \\
0 & & & & \frac{\partial PS_n(t_{i+1})}{\partial r_n}
\end{pmatrix}
\begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix}
=
\begin{pmatrix} \frac{\partial PSO(t_{i+1})}{\partial r_1} \\ \frac{\partial PSO(t_{i+1})}{\partial r_2} \\ \vdots \\ \frac{\partial PSO(t_{i+1})}{\partial r_n} \end{pmatrix}
$$

5.5.1 A self-financed $\triangle$-hedging portfolio

In order to replicate a derivative PSO which is producing a T-claim $\mathcal{X}$ we construct a self-financed portfolio using par swaps $\{PS_1, ..., PS_n\}$ to make our position locally riskless. Then, we create a replicating portfolio $\mathcal{V}^h$ to hedge the claim $\mathcal{X} = PSO(T)$. The replicating portfolio is a stochastic process $\left\{V^h(t)\right\}_{t\in[0,T]}$, where t is defined on a discrete partition, as in the previous section. In terms of clarity we will follow the notation $V^h(i)$ instead of $V^h(t_i)$.

Initially at $t = 0$, we issue a PSO, we put the premium amount to a risk-free investment like a deposit account or risk-free bonds(non-defaultable bonds). We make the portfolio Δ-neutral by going long on a portfolio of par swaps (zero premium is required).

$$PSO(0) = V^h(0) = \left\{\sum_{\mathbf{i}=1}^{n} h_i(0)PS_i(0)\right\}_{=0} + \left[h_B(0)\right]_{cash}$$

Note that $\sum_{i=1}^{n} h_i(0)PS_i(0) = 0$, hence $V^h(0) = h_B(0)$.

At $t = 1$, we construct a new portfolio $\sum_{\mathbf{j}=1}^{n} h_j(\mathbf{1})PS_j(\mathbf{1}) = 0$ (where $PS_j(\mathbf{1})$ are the par swaps at $t = 1$) which makes the total portfolio Δ-neutral again. Moreover, we sell $\sum_{\mathbf{i}=1}^{n} h_i(\mathbf{0})PS_i(\mathbf{1}) \neq 0$ and we invest this money to a risk-free investment as well. Notice that the financing of the portfolio is not based on any exogenous infusion of money.

$$\begin{aligned}
PSO(1) &\approx V^h(1) = \left\{\sum_{\mathbf{j}=1}^{n} h_j(1)PS_j(1)\right\}_{=0} + \left[\sum_{\mathbf{i}=1}^{n} h_i(0)PS_i(\mathbf{1}) + V^h(0)AF(0,1)\right]_{cash} \\
PSO(2) &\approx V^h(2) = \left\{\sum_{\mathbf{k}=1}^{n} h_k(2)PS_k(2)\right\}_{=0} + \left[\sum_{\mathbf{j}=1}^{n} h_j(1)PS_j(\mathbf{2}) + V^h(1)AF(1,2)\right]_{cash} \\
PSO(3) &\approx V^h(3) = \left\{\sum_{\mathbf{l}=1}^{n} h_l(3)PS_l(3)\right\}_{=0} + \left[\sum_{\mathbf{k}=1}^{n} h_k(2)PS_k(\mathbf{3}) + V^h(2)AF(2,3)\right]_{cash} \\
&\vdots
\end{aligned}$$

where $AF(i-1,i) = e^{\int_{t_{i-1}}^{t_i} r(s)ds}$.

5.6 A ΔV-neutral portfolio

In order to immunize a claim which is sensitive to volatility we need to make the portfolio V-neutral. A derivative like Bermudan swaption can be seen as a stream of vanilla swaptions. This derivative will be sensitive to the vanilla swaption volatilities. For this reason we need to use additional hedging instruments on our portfolio sensitive to these volatilities. We denote the price of these new hedging instruments as pso_i. Now the portfolio consists of $\{PS_1, ..., PS_n, pso_1, pso_2, ..., pso_m\}$, which is a combination of two portfolios in order to replicate our claim. In our world we have only available at-the-money swaptions with Black volatilities σ_i. The derivative after that should equal $PSO = h_1 PS_1 + ... + h_n PS_n + w_1 pso_1 + ... + w_m pso_m$, where w_i are the portfolio position on the vanilla swaption instruments. Given the fact that the swaps are not sensitive to volatility we can easily solve the following system

$$\underbrace{\begin{pmatrix} \frac{\partial PS_1(t_{i+1})}{\partial \sigma_1} & \cdots & \frac{\partial PS_n(t_{i+1})}{\partial \sigma_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial PS_1(t_{i+1})}{\partial \sigma_m} & \cdots & \frac{\partial PS_{10}(t_{i+1})}{\partial \sigma_m} \end{pmatrix}}_{=0} \vec{h}(t) + \begin{pmatrix} \frac{\partial pso_1(t_{i+1})}{\partial \sigma_1} & \cdots & \frac{\partial pso_{10}(t_{i+1})}{\partial \sigma_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial pso_1(t_{i+1})}{\partial \sigma_m} & \cdots & \frac{\partial pso_m(t_{i+1})}{\partial \sigma_m} \end{pmatrix} \vec{w}(t) = \begin{pmatrix} \frac{\partial PSO(t_{i+1})}{\partial \sigma_1} \\ \frac{\partial PSO(t_{i+1})}{\partial \sigma_2} \\ \vdots \\ \frac{\partial PSO(t_{i+1})}{\partial \sigma_m} \end{pmatrix}$$

after solving this system $\vec{w}$ is known and then easily we solve for $\vec{h}$ as well,

$$\begin{pmatrix} \frac{\partial PS_1(t_{i+1})}{\partial r_1} & \cdots & \frac{\partial PS_{10}(t_{i+1})}{\partial r_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial PS_1(t_{i+1})}{\partial r_n} & \cdots & \frac{\partial PS_n(t_{i+1})}{\partial r_n} \end{pmatrix} \vec{h}(t) + \begin{pmatrix} \frac{\partial pso_1(t_{i+1})}{\partial r_1} & \cdots & \frac{\partial pso_m(t_{i+1})}{\partial r_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial pso_1(t_{i+1})}{\partial r_n} & \cdots & \frac{\partial pso_m(t_{i+1})}{\partial r_n} \end{pmatrix} \vec{w}(t) = \begin{pmatrix} \frac{\partial PSO(t_{i+1})}{\partial r_1} \\ \frac{\partial PSO(t_{i+1})}{\partial r_2} \\ \vdots \\ \frac{\partial PSO(t_{i+1})}{\partial r_{10}} \end{pmatrix}$$

5.6.1 A self-financed ΔV-hedging portfolio

As we defined above, we attempt to construct a replicating portfolio which is solely financed from the wealth generated from its assets.

$$PSO(0) = V^h(0) = \left\{\sum_{\mathbf{i}=1}^{n} h_i(0)PS_i(0)\right\}_{=0} + \left\{\sum_{\mathbf{i}=1}^{m} w_i(0)pso_i(0)\right\}_{\neq 0} + h_B(0)$$

On the next hedging step we liquidate the portfolio of swaps and swaptions. According to profit or loss from the current positions we take a relevant cash position in order to go long on a portfolio of new swaptions to make the total portfolio ΔV-neutral and with zero cost we buy a portfolio of par swaps. Like that, the next rebalancing step is as follows

$$\begin{aligned} V^h(1) &= \left\{\sum_{\mathbf{j}=1}^{n} h_j(1)PS_j(1)\right\}_{=0} + \left\{\sum_{\mathbf{j}=1}^{m} w_j(1)pso_j(1)\right\}_{\neq 0} + h_B(1) \\ V^h(2) &= \left\{\sum_{\mathbf{k}=1}^{n} h_k(2)PS_k(2)\right\}_{=0} + \left\{\sum_{\mathbf{k}=1}^{m} w_k(2)pso_k(2)\right\}_{\neq 0} + h_B(2) \\ &\vdots \end{aligned}$$

where $h_B(t)$ is the cash adjustment, either long or short position on cash equivalent with the available cash from previous step and the additional cash generated from selling the previous swaps and swaptions minus the cost of buying $\{w_1(t), ..., w_m(t)\}$ swaptions. Like that, the value the portfolio at time t worths

$$V^h(t) = \sum_{\mathbf{j}=1}^{n} h_j(t-1)PS_j(t) + \sum_{\mathbf{j}=1}^{m} w_j(t-1)pso_j(t) + h_B(t-1)AF(t-1,t)$$

then we liquidate and immediately at the same time we construct a new portfolio with new instruments

$$V^h(t) = \left\{\sum_{\mathbf{k}=1}^{n} h_k(t)PS_k(t)\right\}_{=0} + \sum_{\mathbf{k}=1}^{m} w_k(t)pso_k(t) + h_B(t)$$

with cash position

$$h_B(t) = V^h(t) - \left\{ \sum_{\mathbf{k=1}}^{m} w_k(t) pso_k(t) \right\}$$

Chapter 6
Model risk and related errors

The principle of derivatives pricing indicates that the cost of replicating a new product should equal its initial price. Hence, we are both interested to know how much our model values (initial price and final hedging error) differ from the "real" ones, whenever model valuation is applied. The quantification of model risk in practice is more difficult than the methods presented in chapter 2. The reason is that model risk cannot be observed directly from the given prices.

To be able to study model risk extensively, first we need to identify the errors due to pricing, hedging and market risk as separate[14] random sources. With this chapter we attempt to identify the sources of model risk as they might appear in practice. In this way we aim to help the readers to create the first intuition about the problem and boost their understanding for the upcoming results in chapters 7 and 8. The theory presented on the following sections will be the base of our explanations for a wide variety of experimental results.

6.1 Errors on pricing

For the list below it holds that all random variables are defined on the same sample space $(\Omega, \mathcal{F}, \mathbb{Q})$ as we defined in chapter 2.

- Let $F(t)$ a random variable which describes the **"real" market price** of product F at t.

[14] For simplicity we assume independence between the random sources of risk. Any possible correlation structure (if there exists) between the random sources will require additional research which is not a part of the current project.

- Let $F(t)^M$ a random variable which describes the **model price** of product F at t, while no parameter[15] uncertainty is assumed.
- Let $\epsilon(t)^M = F(t)^M - F(t)$ a random variable which describes the **pricing error due to model** M, while no parameter uncertainty is assumed.
- Let $F(t)^M_{calib}$ a random variable which describes the **model price** of product F at t, while the model is **calibrated** to liquid market instruments. At this stage we face parameter uncertainty because of the estimation.
- Let $\epsilon(t)^M_{calib} = F(t)^M_{calib} - F^M(t)$ a random variable which describes the **pricing error due to calibration** of model M to liquid market instruments. This error depends on the quality and liquidity of the market prices.

Proposition 3 *The **model price** at t, when the model is marked-to-market, equals the "real" market price plus the additional model errors,*

$$F(t)^M_{calib} = \epsilon(t)^M_{calib} + \epsilon(t)^M + F(t) \tag{6.22}$$

*The **total pricing error** at t due to misspecified model M equals*

$$\epsilon(t) = \epsilon(t)^M_{calib} + \epsilon(t)^M \tag{6.23}$$

[15] Assume that we use the optimal model parameters according to the current market data.

6.2 Errors on hedging

The pricing error of equation (6.23) will play a role on the value of the hedging ratios. For a discrete hedging portfolio this will cause an additional uncertainty on top of market risk[16].

- Let $\mathcal{V}(t)$ a random variable which describes the price of the hedging **portfolio** at t while we apply continuous rebalancing and the prices are driven by the true model.

- Let $\Pi(t)_D$ a random variable which describes the price of hedging **portfolio** at t with **discrete** rebalancing and the prices are driven by the true model.

- Let $\mathcal{E}(t)_D = \Pi(t)_D - \mathcal{V}(t)$ a random variable which describes the portfolio's **error** due to **discrete rebalancing**. This error represents market risk. $\mathcal{E}_D$ depends on the ability of numerical hedging ratios to approximate sufficiently the sensitivity of the claim to market changes.

- Let $\Pi(t)_D^M$ a random variable which describes the price of the hedging **portfolio** at t with discrete rebalancing, when **model** M is used for the approximation of the hedging ratios and no parameter uncertainty is assumed for that model.

- Let $\mathcal{E}(t)_D^M = \Pi(t)_D^M - \Pi(t)_D$ a random variable which describes the discrete portfolio **error** due to **misspecified model** M for discrete rebalancing while no parameter uncertainty is assumed for that model.

[16] Market risk is the risk due to the changes on market variables such as interest rates, swap rates, volatility, inflation, etc.

- Let $\Pi(t)^M_{D,calib}$ is a random variable which describes the model price of discretely rebalanced **portfolio** at t when the misspecified model M is **calibrated** to liquid market instruments.

- Let $\mathcal{E}(t)^M_{D,calib} = \Pi(t)^M_{D,calib} - \Pi(t)^M_D$ a random variable which describes the discrete portfolio **error** due to **calibration** of model M to the current market data.

Proposition 4 *The discrete model price of **portfolio** at time t while the model is marked-to-market equals the price of the "real" portfolio plus the discretization error and model errors. Then the price is*

$$\Pi(t)^M_{D,calib} = \mathcal{E}(t)^M_{D,calib} + \mathcal{E}(t)^M_D + \mathcal{E}(t)_D + \mathcal{V}(t) \tag{6.24}$$

*The **total hedging error** for one discrete hedging step is*

$$\mathcal{E}(t) = \mathcal{E}(t)^M_{D,calib} + \mathcal{E}(t)^M_D + \mathcal{E}(t)_D \tag{6.25}$$

and

$$\mathcal{E}(t)^M = \mathcal{E}(t)^M_{D,calib} + \mathcal{E}(t)^M_D \tag{6.26}$$

*is the additional **error due to model** M when discrete rebalancing is applied.*

6.3 Errors on a $\triangle$-hedging portfolio

After keeping in mind proposition (1) and (2) we are able to apply the same logic on a Δ-hedging portfolio. Suppose that we have a market which is consisted of the risky underlying $S(t)$ given a priori and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Then, the $\mathbb{P}$-dynamics of the underlying are given by

$$dS(t) = a(t)S(t)dt + \sigma(t)S(t)d\bar{W}(t) \tag{6.27}$$

Moreover, the market consists of a risk-free asset $B(t)$ with dynamics

$$dB(t) = r(t)B(t)dt$$

$\bar{W}(t)$ is a $\mathbb{P}$-Wiener process and $r(t)$ the instantaneous short rate.

Consider a derivative product $F(t, S(t))$ which produces T-claim $\mathcal{X} = F(T, S(T))$ at T. As it is shown in Bjork [2004] under the assumption of constant volatility the product F is tradable only if it satisfies the following partial derivative (in terms of clarity we temporarily omit to show the variables of the corresponding functions)

$$rF = \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}$$

with final condition $\mathcal{X} = F(T, S(T))$. The above equation can be also seen as

$$rF = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma \tag{6.28}$$

In order to hedge this claim we need to construct a **portfolio** $\mathcal{V}(t)$ which must be able to replicate the diffusion $F(t, S(t))$, $\forall t \in [0, T]$. Therefore, the diffusion $\mathcal{V}$ should have the same sensitivity to market movements as F. Then a continuously rebalanced hedging portfolio $\mathcal{V}$ should satisfy

$$r\mathcal{V} = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma \tag{6.29}$$

After this step, assume that $\Pi(t)$ represents a **Δ-hedging portfolio** in continuous time. For this portfolio Θ and Γ terms remain unhedged and the following equality holds

$$r\Pi = rS\Delta \tag{6.30}$$

while the portfolio error ${}_{\Delta}\mathcal{E}(t)$ for the Δ-hedging portfolio $\Pi(t)$ will be

$$
\begin{aligned}
{}_{\Delta}\mathcal{E} &= r\Pi - r\mathcal{V} \\
&= -\Theta - \frac{1}{2}\sigma^2 S^2 \Gamma
\end{aligned}
\tag{6.31}
$$

Next, consider $\Pi(t)$ and $\Delta(t)$ in **discrete** time. We will denote them as $\Pi(t)_D$ and $\Delta(t)_D$ respectively, such that the **discretization error** ${}_{\Delta}\mathcal{E}_D$ due to Δ-hedging in discrete time is

$$
\begin{aligned}
{}_{\Delta}\mathcal{E}_D &= r\Pi_D - r\Pi \\
&= rS(\Delta_D - \Delta)
\end{aligned}
\tag{6.32}
$$

Now suppose that we are in the inconvenient case that we have to use a **model** M for the calculation of the hedging ratio $\Delta(t)_D$ (no parameter uncertainty is assumed at this stage). We denote the new portfolio as $\Pi(t)_D^M$ and the hedging ratio as $\Delta(t)_D^M$. The use of model will introduce an additional error in comparison with the previous portfolio Π_D which equals

$$
\begin{aligned}
{}_{\Delta}\mathcal{E}_D^M &= r\Pi_D^M - r\Pi_D \\
&= rS(\Delta_D^M - \Delta_D)
\end{aligned}
\tag{6.33}
$$

Furthermore, if model M is calibrated to market data we denote $\Delta(t)_{D,calib}^M$ and $\Pi(t)_{D,calib}^M$ for the calibrated $\Delta(t)_D^M$ and $\Pi(t)_D^M$ respectively. Then the **calibration** contributes to an extra error term to the previous portfolio

$$
\begin{aligned}
{}_{\Delta}\mathcal{E}_{D,calib}^M &= r\Pi_{D,calib}^M - r\Pi_D^M \\
&= rS(\Delta_{D,calib}^M - \Delta_D^M)
\end{aligned}
\tag{6.34}
$$

Finally, the value of a discrete Δ-hedging portfolio is supported by a calibrated model M is[17]

$$
\begin{aligned}
r\Pi^M_{D,calib} &= {}_\Delta\mathcal{E}^M_{D,calib} + r\Pi^M_D \\
&= {}_\Delta\mathcal{E}^M_{D,calib} +_\Delta \mathcal{E}^M_D + r\Pi_D \\
&= {}_\Delta\mathcal{E}^M_{D,calib} +_\Delta \mathcal{E}^M_D +_\Delta \mathcal{E}_D + r\Pi \\
&= {}_\Delta\mathcal{E}^M_{D,calib} +_\Delta \mathcal{E}^M_D +_\Delta \mathcal{E}_D +_\Delta \mathcal{E} + r\mathcal{V}
\end{aligned}
\tag{6.35}
$$

After all this modification the **total hedging error** which is included on the final price of portfolio $\Pi^M_{D,calib}$ as a residual cost on excess of the "real" portfolio value $\mathcal{V}(t)$. Hence the total error equals

$$
\begin{aligned}
\mathcal{E} &= r\Pi^M_{D,calib} - r\mathcal{V} \\
&= [{}_\Delta\mathcal{E}]_{Unhedged} + \{{}_\Delta\mathcal{E}_D\}_{Market} + \left({}_\Delta\mathcal{E}^M_D +_\Delta \mathcal{E}^M_{D,calib}\right)_{Model}
\end{aligned}
\tag{6.36}
$$

6.4 Errors - Market risk - Model risk

$\mathcal{E}(t)$ is the random variable from which a P&L distribution is generated. The contribution of M model to the total error $\mathcal{E}(t)$ for a discrete Δ-hedging strategy is

$$
\mathcal{E}^M =_\Delta \mathcal{E}^M_D +_\Delta \mathcal{E}^M_{D,calib} \tag{6.37}
$$

- $\mathcal{E}^M$ is the random term that will contribute to the diversity of the final hedging outcomes along a set of different models.
- $\mathcal{E}_D$ is the error term that remains unaffected from the use of any model. This term is connected to the discreteness of the portfolio, the numerical approximation of the continuous

[17] For the derivation of (6.35) we use equations (6.31), (6.32), (6.33), (6.34).

partial derivative $\Delta \approx \frac{\partial F}{\partial S}$ and the change of the market variable S through the life of financial product F.

- ${}_{\triangle}\mathcal{E}$ which is composed from the remaining terms of the expansion (6.28) Θ and Γ. The ${}_{\triangle}\mathcal{E}$ will not be affected from the set of candidate models. Moreover, ${}_{\triangle}\mathcal{E}$ may have additional terms except from Θ and Γ if the Itô expansion (6.28) is applied on a different process $F(t, S(t))$, e.g. $F(t, \sigma(t), S(t, \sigma(t)))$.

$\Pi^{M}_{D,calib}$ will be the output of a usual hedge test. Therefore, $\mathcal{E}(t)$ can be derived by comparing $\Pi^{M}_{D,calib}(t)$ to the price of the financial derivative $F(t)$, where F moves with a different process than $\Pi^{M}_{D,calib}$. If a model-free $F(t)$ is available, $\mathcal{E}(t)$ can be extracted at any time. Otherwise this is only possible at time T where $F(T) = \mathcal{X}$.

After this description, we have a clear picture of the basic implementation and the risks that we expect to affect the price of our results. The next two chapters are dedicated to present the hedging outcomes based on the current infrastructure. On chapter 7 we present results from plain vanilla interest rate claims. Using such products we validate (without the presence of benchmark hedger) the accuracy of our results by identifying expected hedging errors. Any unexpected leakage on the portfolio would be considered as a bug. Taking advantage on the simplicity of these payoffs, we perform some additional experiments to improve our understanding on the hedging procedure and its related properties. For the experiments analytical (mark-to-market valuation) and model-based (Monte-Carlo) pricing is applied. Finally, chapter 8 is focused on the most interesting part of our research, which is the investigation of model risk on Callable swaps under the valuation setting of chapter 4.

Chapter 7
Results I: Hedge test

An important practical topic of model risk and risk management in general is the correct implementation of a model. For that reason, the first part of this chapter is devoted to the validation of our hedging module. The chapter is structured as follows.

The first section gives a brief introduction to the available market data. The second section describes our validation strategy and the set of the experiments that are going to be applied. After that the Δ, V and ΔV -hedging results of vanilla swaps and swaptions are presented.

The second part of this chapter is dedicated to provide additional explanations regarding the performance and the user's choices of the replication. Topics that are discussed are the use of forward and spot sensitivities, the hedging frequency and the bumping size of the hedging ratios.

Examples are also given as an evidence that the error decomposition is feasible when hedging simulations are performed under several assumptions (see table (7.1)). All the explanations that we currently provide are based on the error setting of chapter 6.

In all sections we apply mark-to-market (MtM) analytical pricing except from the last section (7.5) where both MtM and Monte-Carlo pricing are being used.

7.1 Historical data

The set of historical scenarios is based on market data of swap rates and at-the-money (ATM)[18] vanilla swaption volatilities. The graph below visualizes the quoted swap rates and few selected

[18] No smile is available for the Black swaption volatilities.

swaption volatilities that are mostly related to the deals that we are going to investigate later on. The data cover an 8 year calendar history that starts from 4-May-2001 until 2-June-2009. The corresponding market history contains a rich set of scenarios of high and low volatile periods such as the credit crunch of 2007 and the adjustment period right after that. All data are quoted in EURO currency.

To avoid any confusion, we choose to express all historical days on trading days starting from number 0 until 2107 which is the last historical day of our data. In order to make this mapping transparent the reader can observe two types of horizontal axis in figure (7.1). The first horizontal axis which corresponds to the swap rates is printed with respect to trading days, while for exactly the same market history the axis of volatilities is printed with the traditional calendar days.

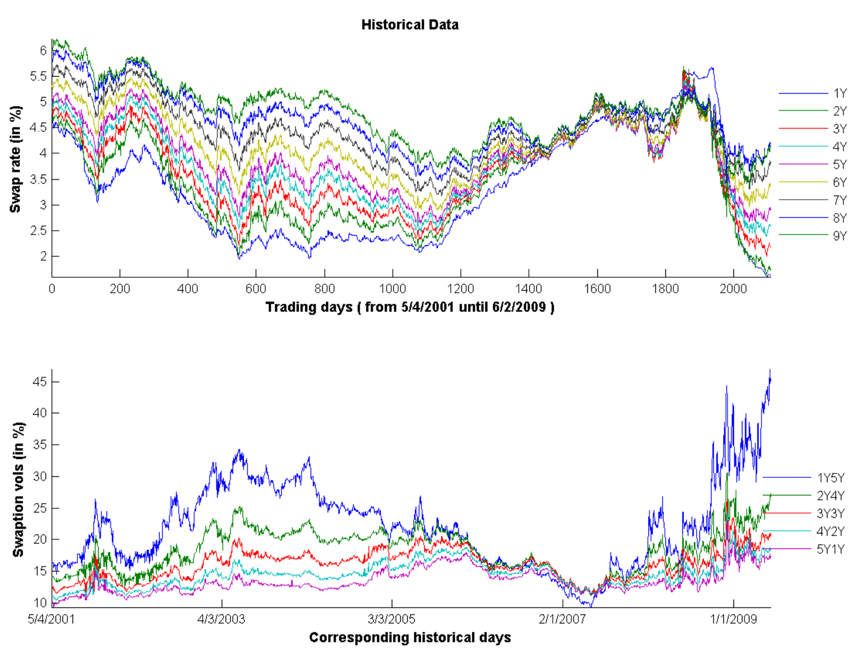

Figure (7.1). The figure shows data of 2107 trading days starting from 4-May-2001 until 2-June-2009. The data can can be downloaded from Bloomberg. The labels on the right side indicate the corresponding quotes. For example in case of swap rates 1Y stands for the 1 year swap rate. In case of swaption volatilities 2Y4Y stands for the volatility of 2 year swaption on a 4 year swap agreement. We emphasize that for the purpose of this project we map the corresponding calendar days to an equivalent trading day starting from 0 to 2107 which is the total number of the available history. All values are printed on percentage scale.

7.2 Validation strategy

For the validation of our hedging module we apply hedging on spot and forward starting vanilla swaps and vanilla swaptions. These are the closest interest rate products to our hedging instruments.

Experiment	**Hedging Type**	**Time**	**Rates**	**Vols**	**Unhedged Risks**
D1	Δ-hedging	-	change	-	Γ
D2	Δ-hedging	change	change	-	$\Gamma + \Theta$
D3	Δ-hedging	change	change	change	$\Gamma + \Theta + V + \mathbb{V}$
D4	Δ-hedging	-	change	change	$\Gamma + V + \mathbb{V}$
V1	V-hedging	-	-	change	$\mathbb{V}$
V2	V-hedging	change	-	change	$\Theta + \mathbb{V}$
V3	V-hedging	change	change	change	$\Delta + \Gamma + \Theta + \mathbb{V}$
V4	V-hedging	-	change	change	$\Delta + \Gamma + \mathbb{V}$
DV1	ΔV-hedging	-	change	change	$\Gamma + \mathbb{V}$
DV2	ΔV-hedging	change	change	change	$\Gamma + \Theta + \mathbb{V}$

Table 7.1. The table lists all the experiment types of the validation strategy. Each experiment implements different type of hedging under several assumptions. The marker "-" means that the factor specified in the column label does not change during hedging process, while the marker "change" stands for the opposite. The last column on the right hand side shows the risk factors which remain unhedged according to the type of hedging which is indicated on the second column. $\mathbb{V}$ stands for Volga.

To test extensively the performance of our module we perform hedging under different assumptions. The validation will be based on testing the performance of the hedger module in presence of different combinations of risk factors, like Θ, Γ, Δ and V. According to this idea we need to define a set of experiments that control the change of time (evaluation date), swap rates and swaption volatilities. The table (7.1) provides the 10 types of hedging experiments that we are going to use in this chapter.

In order to illustrate how these experiments work in practice consider that we want to examine the performance of Δ-hedging on vanilla swaptions. The first step to do that is to apply Δ-hedging in a world that the swap rates are changing[19] dynamically while Θ and V risk are not present. We can achieve this if we assume that the evaluation date and swaption volatilities never change. Then the hedging error will be composed from Γ plus a residual error ${}_{\Delta}\mathcal{E}_D$ due to the discrete Δ-hedging portfolio (see chapter 6 for the notation of hedging errors).

The dynamic change of time or of any market variable is considered to introduce a new source of risk to the replicating portfolio. By controlling the change of the market variables or of the evaluation date we can investigate the contribution of each risk, in many cases, separately.

7.3 Results

This section investigates the types of experiments as they are defined on table (7.1). On the first subsection we examine the Δ-hedging performance on spot and forward starting swaps which are priced analytically as discounted cash flows. V or ΔV-hedging are ignored since these payoffs are not sensitive to volatility (refer to the pricing formulas for that). The second subsection investigates cases of Δ, V and ΔV-hedging on plain vanilla swaptions. The swaptions prices are mark-to-market with the Blacks' formula. Refer to Brigo & Mercurio [2006] the analytical formulas.

[19] The way of keeping constant or dynamic the variables (time, rates or volatilities) on our experiments is simple. At the start day we use the current date, rates and volatilities as we find them on the dataset. Then, if for example the assumption sets the time (evaluation date) and volatilities constant on the next hedging step will only update on our code the rates. Evaluation date and volatilities will remain the same as "yesterday". This is the assumption D1. For the rest applies also the same.

We choose to perform hedging for almost 3 months period (61 trading days). This amount of hedging steps is enough to give us the evidence we need in order to illustrate the consistent behavior of the hedging module and verify the error description of chapter 6. On the results we associate positive errors to profit and negative errors to loss. In this section we apply hedging based on mark-to-market analytical formulas, hence there is no presence of model risk.

7.3.1 Vanilla swaps

Below we present D1 (only swap rates change) and D2 (swap rate and time change) type of Δ-hedging experiments. On ATM spot starting swap of 3 years and ATM 1 year forward starting swap of the same maturity are tested.

The first indication that the replication is working correctly is to use the fact that these instruments can be perfectly replicated on the start day of the deal by the equivalent hedging instruments. A short position on a 3 year swap can be replicated from a long 3 years swap hedging instrument. A short position on a 1Y3Y swap can be replicated with a long 4 years swap and a short 1 year swap.

The figures below include the Net Present Values (NPV) of the option and that of the replicating portfolio. The replication evolves along the axis of trading days. For hedging we use double-sided forward sensitivities with bumping size of 8bps (see section (7.4.2) and section (7.4.4) respectively).

The first hedge test shows almost perfect replication for the D1 case. Swaps are linear payoffs and theoretically there is no convexity. This can be observed on the D1 error which is zero apart from one day out of sixty days. The mismatch in this example might be either a discreteness or numerical error. The D2 case shows the effect of Θ risk. The Θ error term is added to the hedging portfolio after including the change on time (evaluation date).

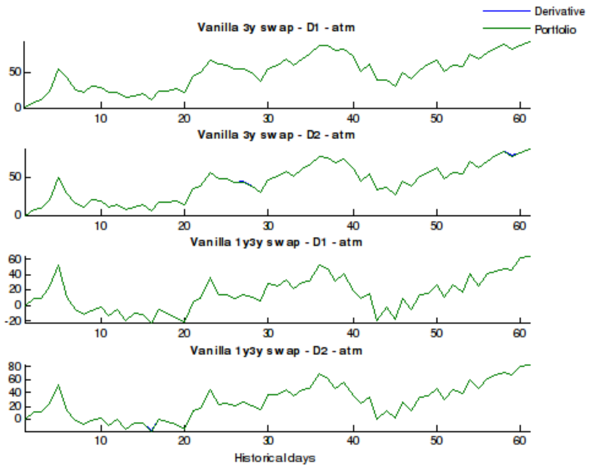

Figure (7.3.1.1). NPVs in bps of the interest rate receiver swaps and their hedging portfolio D1 and D2 are plotted. We remind that the numbering of the trading days is printed according to figure (7.1).

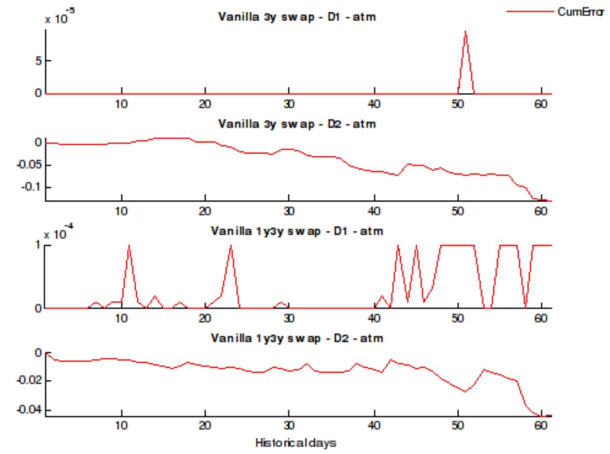

Figure (7.3.1.2). These are the corresponding NPVs (in bps) of the cumulative hedging errors. The errors are the difference between the price of option and hedging portfolio (see chapter (6) for detailed description).

7.3.2 Vanilla swaptions

On swaptions we can apply all types of Δ, V or ΔV-hedging experiments of table (7.1). The hedging is set up with double-sided forward sensitivities of 8bps bumping size for Δ and 10bps bumping size for V ratios. We test vanilla receivers[20] with fixed rate at 6%.

Δ-hedging

The results from D1 case show the Γ risk of vanilla swaptions. The D2 experiment is the case where both Θ and Γ remain unhedged. By looking figures (7.3.1.1) and (7.3.1.2) one may question, how is it possible to receive smaller error than in D1 case, since Θ risk is added on the

[20] To receive the fixed leg of the underlying swap.

portfolio? This example gives a good motivation that further analysis is required if we want to understand and finally measure the errors of our replicating strategy.

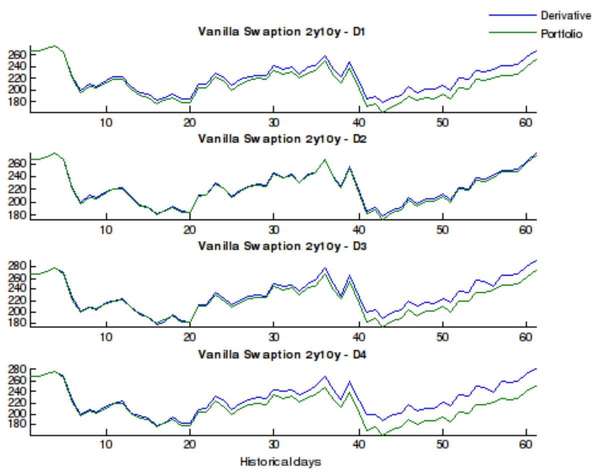

Figure (7.3.2.1). NPVs in bps of vanilla receiver swaptions and their hedging portfolio D1, D2, D3 and D4 are plotted. We remind that the numbering of the trading days is printed according to figure (7.1).

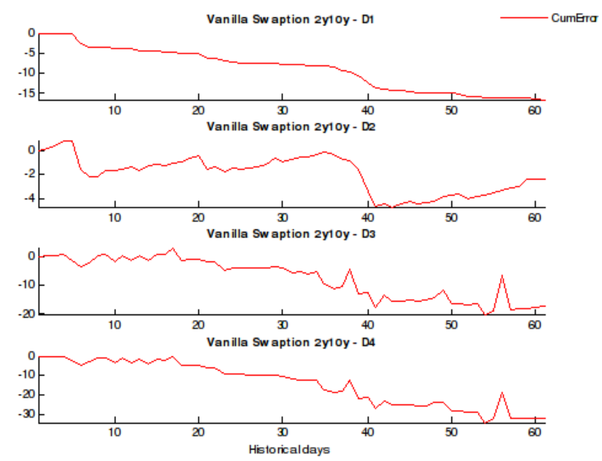

Figure (7.3.2.2). These are the corresponding NPVs (in bps) of the cumulative hedging errors. The errors are the difference between the price of option and hedging portfolio (see chapter (6) for detailed description).

The experiments emphasize that receiving small errors does not imply low risk or good hedging performance. As we will see on the next section (7.4.1) Γ and Θ risks partially cancel out with each other. Hence, we cannot relate the size of hedging error directly to performance without any further analysis. As a consequence, the impact of “hidden” risk factors will remain the main obstacle for the model risk quantification.

D3 is the type of Δ-hedging where time, swap rates and swaption volatilities are changing during the life of the experiment. In this case we see that the impact of volatility risk amplifies the hedging error in comparison with the D2 case. The error of experiment D3 as it is stated on table (2) leaves unhedged Γ, Θ, V and $\mathbb{V}$. D4 experiment is the last of the Δ-hedge tests, in that case time is ignored. Θ risk is not anymore on the portfolio and this moves the hedging error to more loss. This is the second evidence together with the D2 errors which show that Θ risk has a positive sign for this portfolio.

V-hedging

The results for a vega hedged portfolio are also interesting. The first observation is that $\mathbb{V}$ is relatively smaller than the realized Γ errors (see D1 case on figure (7.3.2.2) for the size of Γ). This is not an occasional observation. We verify that the performance of a V-hedging portfolio is not significantly affected by using different bumping sizes for hedging. For a range [0.1-200]bps of several bumping sizes we tested the hedging performance with no important differences. The variability of the results was approximately a couple of basis points. The V2 hedging error shows the contribution Θ risk to the vega hedged portfolio. The V3 type of hedging, in addition, incorporates the unhedged Δ and Γ terms to the portfolio and V4 includes all risks except from Θ.

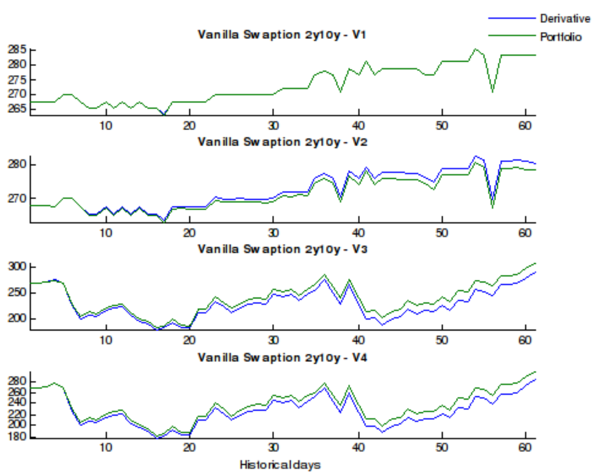

Figure (7.3.2.3). NPVs in bps of vanilla receiver swaptions and their hedging portfolio V1, V2, V3 and V4 are plotted. We remind that the numbering of the trading days is printed according to figure (7.1).

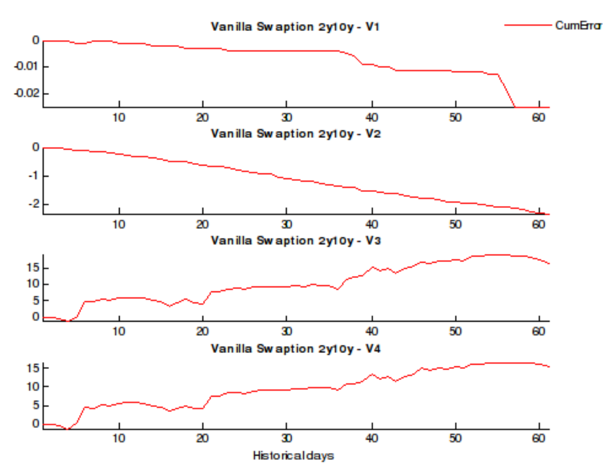

Figure (7.3.2.4). These are the corresponding NPVs (in bps) of the cumulative hedging errors. The errors are the difference between the price of option and hedging portfolio (see chapter (6) for detailed description).

ΔV-hedging

DV1 is the ΔV-hedging where the swap rates and volatilities change while time is assumed to be constant. DV2 is the ΔV-hedging experiment where the time change is also included. The errors of ΔV-portfolios are more difficult to understand from the first view. The hedging errors are a combination of risks that come from a portfolio of swaps and a portfolio of swaptions.

The amount of these two portfolios is not proportionally equal. The size of swaption positions depends on the total vega of the deal. Thus, the size of positions on swaps and swaptions will contribute different proportion of errors (e.g. Γ of swaps plus Γ of swaptions) to the hedging portfolio. The combinations that one can imagine regarding the error composition of the hedging portfolio in that case are quite a lot. The nature of errors of swaps and swaption are of different nature and this makes the explanation of these results not straightforward.

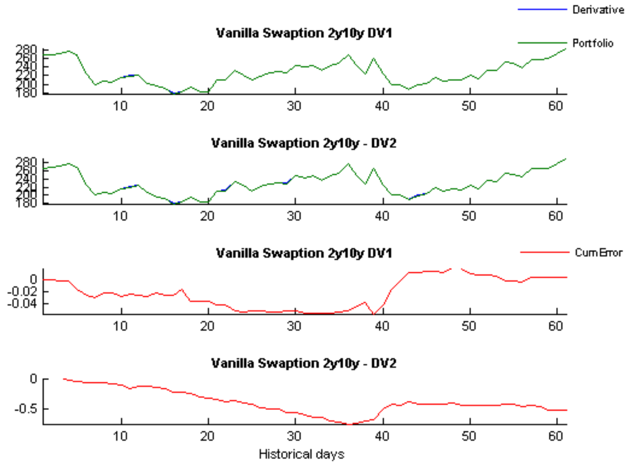

Figure (3). The figure shows on the upper part the NPVs in bps of the DV1 (no presence of Θ-risk) and DV2 hedging experiments (see table (7.2) for details). The lower part of the figure gives the corresponding cumulative errors (red color) of the experiments.

Since no benchmark hedger is available to compare our results, we will need to explain the reasons of having these sort of errors on our replicating portfolios. The main tool to do this is the use of the theoretical intuition of chapter (6). Next sections will provide answers upon several hedging details including the error decomposition.

7.4 Evaluation of hedging performance

In this section we give a quick illustration of error decomposition for the Δ-hedging experiments of figure (7.3.2.1). Next we provide evidence for the forward hedging sensitivities comparing their performance to the corresponding spot hedging sensitivities. Additionally, examples are presented regarding the hedging frequency and the second order effects due to bumped sensitivities.

7.4.1 Decomposition of hedging errors: Evidence

Here we want to show that the setting of chapter (6.3) can explain the errors of a previous experiment. To illustrate this in more detail we will try to decompose the hedging errors of figure (7.3.2.1). In order to verify that the decomposed errors are indeed the expected ones (according to our theoretical intuition) we will use them to derive errors that appeared on other experiments. The figure (7.4.1.2) provides the evidence to that.

The decomposition is applied as follows. From D1 error one can get Γ risk. The difference of D2 minus D1 error will give Θ. Moreover, Θ can be computed as the difference between D3 minus D4 hedging error. The basic idea is to use different experiments (D1, D2, D3, D4 and so on) and to combine the information from the last column of table (7.2) to add or subtract errors to get others. The same applies for V- hedging, however this naive decomposition it becomes more difficult for ΔV- hedging.

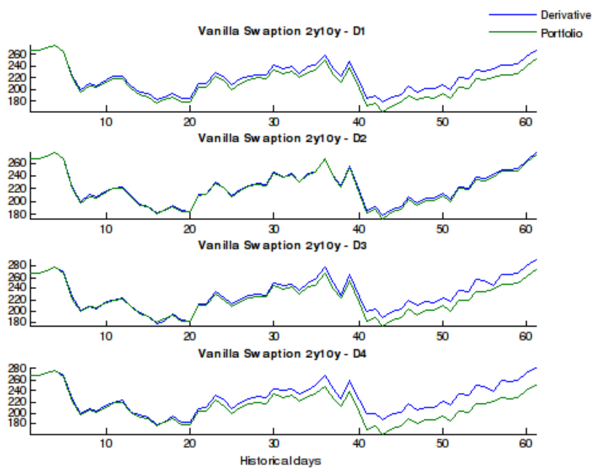

Figure (7.4.1.1). NPVs in bps of the vanilla swaptions and their hedging portfolio are plotted. This is the same as figure(7.3.2.5). It is given next to the decomposed examples to assist the visual inspection.

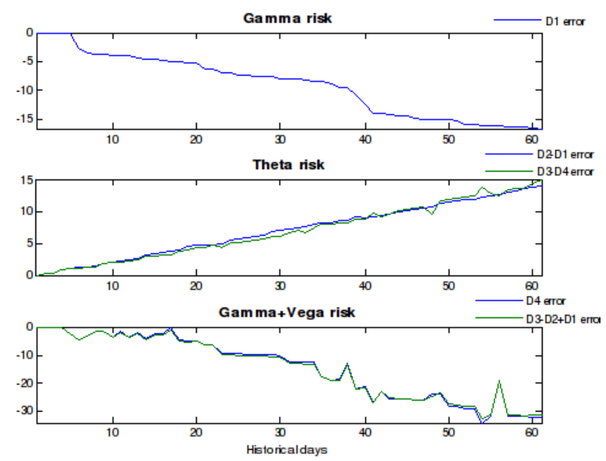

Figure (7.4.1.2). These are the corresponding NPVs of the cumulative hedging errors. The errors are analysed and reproduced by using the assumptions of table (7.2).

The promising message of this example is that by using a set independent experiments (hedge tests) we are able to reproduce errors or remove artifacts from our data. This fact will be used later on chapter (9) to recommend a possible solution for more complicated problems.

7.4.2 Forward vs Spot sensitivities: Robustness

On the following figures we will provide two examples as an evidence on the choice of using forward instead of spot sensitivities on our hedging. The first example considers a 3 and 10 years ATM vanilla swap that receives fixed. The second example examines the effect of forward and spot sensitivities on a non-linear deal. For this reason a vanilla swaption is replicated. Both examples are D2 type of hedging experiment, where Δ- hedging is applied while time (evaluation date) and swap rates are changing dynamically.

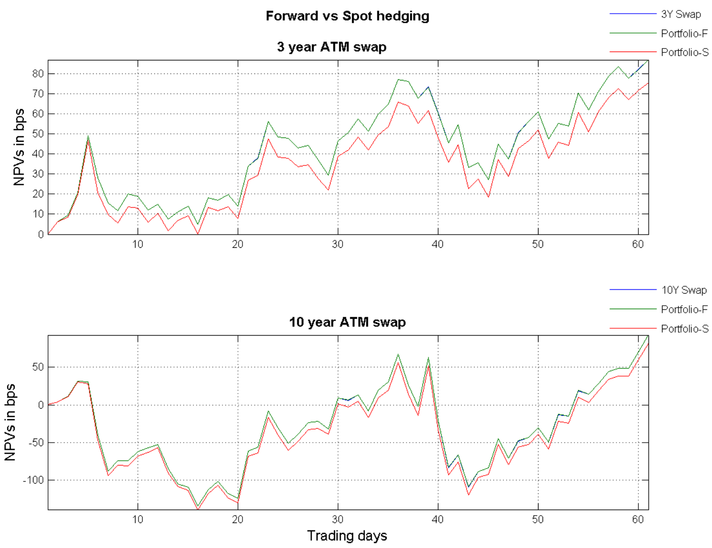

Figure (7.4.2.1). The figure shows the NPVs of 3 and 10 years ATM swaps and their hedging portfolios. The forward hedging portfolios replicate closely the option price process while the spot hedging portfolio is systematically losing. It is enough to notice the 3 year deal which is the same as in figure (7.3.1). The spot hedging error of this option is approximately five times bigger that the forward hedging error (2bps for forward, 10 bps for spot) only within 3 months period.

The forward hedged portfolio replicates closely the vanilla swap, while this is not possible with the spot hedged portfolio. The robustness of the spot sensitivities is directly depicted on the size of the hedging error. However, the difference on the hedging performance will be substantially smaller[21] for non-linear payoffs like vanilla or Bermudan swaptions. The next figure plots the NPVs of the analytically priced and replicated 2Y10Y vanilla swaption of figure (7.3.2.1).

The difference on performance is minimal. Although, the experience out of an extensive range of experiments suggests that the use of forward sensitivities is a more robust approximation of the actual sensitivities. The results of this section constitute a sound support for the theory of sections (5.3) and (5.4), where the idea of replication and the calculation of hedging ratios is presented.

[21] The reason is still under investigation. A possible explanation can be that the effect on swaps is immediately depicted on the price because of the linear nature of the payoff.

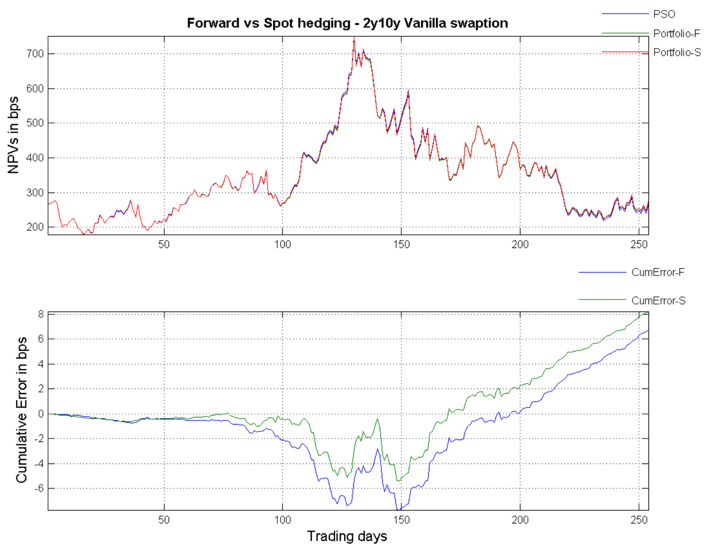

Figure (7.4.2.2). The figure shows the NPVs of the vanilla swaption price (PSO), the forward hedged portfolio (Portfolio - F) and the spot hedged portfolio (Portfolio - S) for a 1 year hedge test. The cumulative hedging error (CumError - F) of forward portfolio and the cumulative hedging error (CumError -S) of the spot portfolio are given on the lower plot.

7.4.3 Hedging frequency

The robustness of forward sensitivities can also be important when the frequency of rebalancing is taken into account. In this section we test the hedging performance for three different hedging frequencies. A 3 years ATM vanilla swap is replicated with both forward and spot hedging ratios. The test considers 1-day, 10-days and 20-days of rebalancing. These frequencies have 4 mutual steps on a period of 61 trading days. Due to this reason the graph shows only the portfolio price for the mutual hedging days.

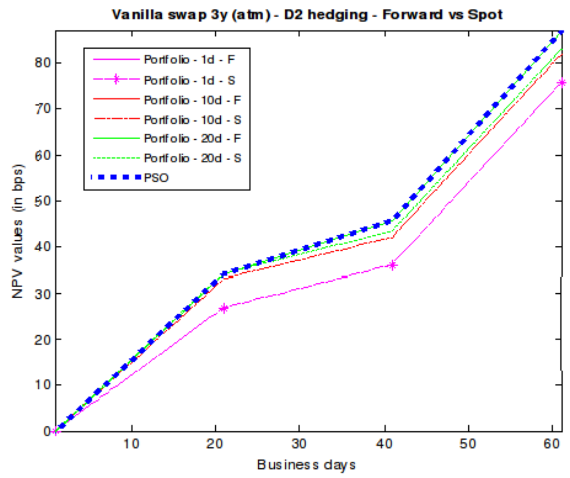

Figure (7.4.3). The figure shows spot (S) and forward (F) hedged portfolios. The D2 type of Δ-hedging is performed for 1, 10 and 20 days of rebalancing frequency. We plot NPVs of the derivative and its corresponding portfolios for only the 4 common steps of the different hedging frequencies. The hedging is applied on a 3 year ATM swap. PSO stands for the derivatives' price.

The forward sensitivities seem to be more robust for any of the hedge tests. The effect of spot sensitivities is quite more important on linear products and this is the reason that we have used swaps for our example.

In order not to create any confusion, we emphasize that the forward sensitivities **do not** hedge Θ (see the presence of Θ-risk on previous figures). The forward hedging portfolios on figure (7.4.3) are very close to the price of the vanilla swap (see the bold blue dashed line in the previous figure) while the errors are too small to be visible (see figure (7.3.1) for the magnitude of the hedging errors on swaps)

It is good to mention that the hedging frequency is not correlated with the hedging performance as we see on the figure above. This is because the hedging ratio will approximate the real sensitivity in presence of market risk. On the next rebalancing step the hedge will perform according to the movement of the market. The size of the market movement is not always dependent to the observation time (herein hedging frequency). Hence, the error of rebalancing in discrete time will not be directly connected to the size of the hedging frequency. Spot sensitivities show that are more vulnerable to market risk and hedging frequency than the forward sensitivities. This is the final evidence that we needed to support the consistency of equation (5.19).

7.4.4 Bumping size: The effect of second-order risks

One of the last things that we need to investigate, before we start the model risk experiments, is the choice of bumping sizes for the hedging ratios. As we have mentioned at the beginning of this chapter we used two sided bumped sensitivities. For the calculation of Δ ratios we use 8bps bumping size and for V ratios 10bps. The reason of choosing these bumping sizes is due to the fact that the standard deviation of the the daily changes of the quoted rates is around 5-6bps on a yearly periods. The standard deviation of the daily changes for swaption volatilities is around 10-15bps on a yearly bootstrapping window. The effect of second order risks of V is very small as we have already mentioned in section (7.3.2). On the other hand the effect of Γ risk is more important and the choice of the bumping size for Δ is crucial.

Piterbarg [2005], for a similar replication problem on Bermudan swaptions, remarks that a reasonable bumping size may be 10bps for Δ. Of course, this remark is based on different market data. Making several hedging experiments on our data we observed that 8bps is a fairly good choice. This bumping size is a bit higher[22] than the standard deviation of the daily increments of quoted swap rates.

[22] Piterbarg also mentions that a bigger bumping size may hedge a part of second order terms.

In the figure below we show the effect of Γ risk on the portfolio of the previous 2Y10Y vanilla swaption under the assumptions of D1 experiment where no Θ is taken into account. Then the observed errors associate to Γ risk only.

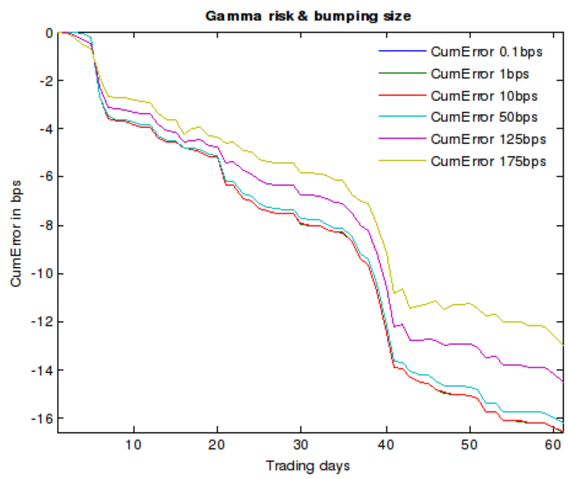

Figure (7.4.4). The figure shows the NPVs of the hedging errors of a D1 type of experiment on a 2y10y vanilla swaption. The errors in that case show only the Γ-risk.

The figure shows that different bumping sizes may perform better from one scenario to another. As an example we give figure (7.4.4). The bumped sensitivities of 125 and 175 bps result lower errors on the tested period. Although, on average bumping sizes higher that 15bps will give higher errors in a wide range of market scenarios. The choice of bumping size does not have a fixed rule and always depends on the market scenarios.

The size of 8 and 10 bps for Δ and V respectively, based on the standard deviation of the data increments, proves to be a good choice for the current dataset. The errors that we receive with these estimates do not exceed the absolute value of 45bps even on a 5 year horizon.

7.5 Model risk: The vanilla case

Before we continue with the most important part of this project, the experimental results of Bermudan swaptions, we will give the first feeling of model risk due to the Monte-Carlo simulation of the one-factor Hull-White model.

The next figure gives an example of the 2Y10Y vanilla swaption from section (7.3.2). On the graphs the NPVs of the derivative (namely "PSO" on the labels) and the hedging portfolio (namely "Portf" on the labels) are printed. Additionally, we provide the errors of the model based (namely "HW1" on the labels) and analytic portfolio ("Ana" on the labels).

We perform the hedging experiment DV2, where the evaluation date, swap rates and swaption volatilities are changing during the life of the experiment. The model is using mean reversion 3% and is calibrated to piecewise constant volatility (see Appendix (B)) and the number of paths is set to 10K.

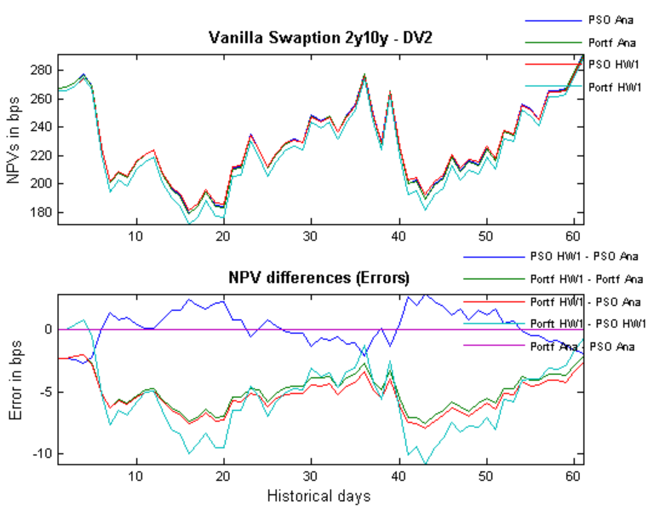

Figure (7.5). This is the first example of model risk. Here we compare the hedging performance of the one-factor Hull White model against the analytical hedging portfolio ("model-free" see footnote). In the second figure we plot in blue color the pricing error due to model risk. In green color we plot the hedging error due to model risk. The difference of "Portfolio HW1 - PSO HW1" is an approximation of the hedging error due to model risk in case when the market prices are not available.

	Difference in bps	Relative error
Pricing Model Error	-1.935	-0.663%
Portfolio HW1 - PSO Ana	-2.616	-0.896%
Portfolio HW1 - Portfolio Ana	-2.102	-0.720%
Portfolio HW1 - PSO Ana - Pricing Model Error	-0.681	-0.233%
Portfolio HW1 - PSO HW1	-0.681	-0.233%
Portfolio Ana - PSO Ana	0.004	0.001%

Table 7.2. This table shows the final error differences of the Monte-Carlo and analytical prices of figure (7.5). The relative errors are elements of the first column over the last MtM swaption price.

In case of liquid products the actual difference between the model and "model-free" [23] cumulative errors can be interpreted as the model risk of the hedging portfolio. The extraction of model risk for non liquid products will not be that easy. The study of model risk for exotic products does not have available the "model-free" derivative prices. The analysis of exotic portfolios requires a very good understanding of market and model risk impact on the final prices. This was exactly the scope of this chapter, to warm up the reader's understanding of the final experiments that follow in chapter 8. For the next of the experiments the difference "Portfolio HW1 - PSO Ana - Pricing Model Error" will be the approximation of the hedging error due to model risk.

[23] "Model-free portfolio": The mark-to-market prices of vanilla swaptions are quoted as volatilities that are used as input to Black's formula to get the market price. Hence the hedged portfolio which uses this analytical formula is considered to be model free.

Chapter 8
Results II: Model risk assessment

For the model risk assessment there are two options of testing the parameter and model uncertainty. In order to study the performance of different models it is required that models should have sufficiently specified their structural parameters to be consistent with the market prices (internally consistent model). The wrong use of internal parameters can push a model to substantial losses (take as an example the results of Longstaff, Santa-Clara & Schwartz [2001] ref.[22]). Hence, before the experimenter proceeds to the study of model uncertainty it is important to start from the internal consistency of his/her models. As a consequence on this project we will restrict on the mean reversion uncertainty of the one-factor Hull-White model on Bermudan swaptions.

The uncertainty on our model remains on the mean reversion parameter α since σ is calibrated as a piecewise constant parameter. The mean reversion parameter is responsible for the shape (curvature) of the volatility structure (see section (4.1.2)). The specification of this parameter for Bermudan swaptions is recommended for the range or [0-5]% for short and [0-3]% for long maturity deals (see validation report [66]). On our experiments the range of the final hedging error translates to a range of 15bps for short and 70bps for long maturity deals. The ranges of α are set to {-1,1,3,10}% and {-1,1,3,5}% respectively for a notional of 10000bps in EURO.

For the experimental part of this chapter we start with the introduction of our methodology including definition of experiments, collection of observations, data preprocessing and data analysis. We separate our experiments with respect to riskiness and we present the results according to that.

The analysis of our experimental observations includes methods to imply correlation structures within the data and the formation of distinct clusters due to the effect of mean reversion factor. The results of 5 and 20 years deals are presented separately and for their description an extensive variety of graphical tools is being used.

At the end of the chapter we are speculating upon the results and a possible practical way to estimate model risk and the mean reversion uncertainty. To illustrate this we give a short example applied to our data and we leave the suggestions of its use for further investigation.

8.1 Methodology

The first task of our research is to produce a significant amount of results that will cover a fairly wide range of scenarios. The market scenarios[24] we use are described in figure (7.1). The trivial idea to hedge the options up to maturity, in order to use the model free payoff price at expiry, cannot be applied at this stage. This would require a huge amount of market scenarios which is practically not feasible. The available market data cover a period of 8 trading years. One hedge test may be enough to use the whole period of our data. Then the next experiment will be highly correlated to the previous one.

Another issue which is related to callable options is that the contracts can be finalized before their maturity. Then we face the following problem. For instance, imagine that a 10 year Bermudan swaption annually exercised is called after 4 years under a certain scenario. For another scenario the same product is exercised in two years. Under this situation we cannot analyze the

[24] The real market scenarios are considered as "model-free" compared to artificially generated scenarios. Nonetheless, market scenarios are associated with the past, while models are supposed to forecast the future. Hence, the market scenarios of previous market history still suffer from identification risk.

hedging results as a set of identical observations. This is because, we cannot compare the hedging performance of portfolios that have different durations. For example, the 4 year hedge test corresponds 1012 accumulated errors while the second scenario corresponds to 506 errors (assume daily rebalancing).

8.1.1 Definition of experiments

Due to this reason we decide to create deals of Bermudan swaptions which are annually exercised. We divide the 8 years market data to 8 equal partitions. The hedge will be observed until the first exercise date. Each experiment will have a start date at the beginning of these periods. The first exercise date of each deal will be set on the same data partition with the start date. Thus, 8 non-overlapping scenarios, of one year trading history, are defined for our market data. On each data partition we create artificial ATM, OTM and ITM deals[25]. The hedging of these deals will be realized for 4 different mean reversion parameters. This will finally give us 106 hedge tests of one trading year period for one type of Bermudan.

Due to limited time we only perform 106 tests on 5 year Bermudan swaptions and 8 more tests on 20 year Bermudans. The options give the right to receive the fixed leg of the underlying swap. The spread on the floating leg is set to zero and the notional of the swap to 10000bps in EURO. All positions are long and the rebalancing of the portfolios is set on daily basis.

Remark 2 *For the rest of the chapter we will use the following terminology. The hedge test which is defined on a specific data partition, for a specific Bermudan option with fixed rate K,*

[25] The moneyness is defined according to the value of the first underlying of the Bermudan swaption. For a 5 year Bermudan swaption with first exercise date in 1 year the moneyness depends on the price of the 1Y5Y forward swap. The ITM rate is set 200bps higher than the ATM rate. The OTM fixed rate is set 150bps lower than the ATM rate.

will be called ***"experimental case"****. One experimental case will have 4 different "****realizations"*** *equal to the number of mean reversion parameters that are used. So, the four elements of one "experimental case" will be characterized from the same market risk and the same moneyness.*

8.1.2 Collection of results: Observed characteristics

The observed characteristics is vector of features which all together constitute the description of an instance (herein experiment). Our characteristics are defined from the collection of the NPVs of the derivative's price, the hedging portfolio and the daily sensitivities Δ and V. Furthermore, besed on these observations we define some additional characteristics:

- **Cumulative Hedging Error:** The difference between the NPVs of derivative's price and the price of replicating portfolio at each hedging step.
- **Daily Hedging Error:** The daily increments of the cumulative hedging error.
- **Total Daily Vega (Delta):** The total bucketed vega(delta) of each trading day.
- **Average Total Vega (Delta):** The average of the "total daily vegas (deltas)" on yearly basis.

Additionally we apply a simple statistical analysis on the daily hedging errors to derive the first four moments of the P&L distribution of the hedging experiment. The P&L of each hedge test is the empirical distribution of the daily hedging errors.

8.1.3 Data analysis

The scope of the data analysis is to identify which of the observed characteristics are systematically being affected from the mean reversion uncertainty.

The initial inspection of the hedge test results combines visualization of the NPVs including the portfolio's price, the hedging errors and the bucketed sensitivities. Additionally we provide the P&L histogram and the quantile-quantile[26] plot (Q-Q plot) of the daily hedging errors. This part of the analysis is to make us familiar with the observed data (results) and if possible to identify a sound behavior due to the different specifications of each experiment.

For the analysis of the collected data, with respect to mean reversion, we normalize the observed characteristics of each experiment. The characteristic values of each individual experiment then are transformed to standard normal variables on the following table.

MeanReversion	**Final error 1**	**Normalized Final error 1**
1%	1.1	-0.24019
-1%	1.4	1.200961
3%	0.9	-1.20096
5%	1.2	0.240192
Group Mean	1.15	0
Group St.Deviation	0.208167	1

Table 8.3. The table illustrates an example of data normalization. Consider an arbitrary experimental case of 4 mean reversions $\{1\%, -1\%, 3\%, 5\%\}$. From these mean reversions we get a group of 4 different errors x. The mean μ and standard deviation σ of the group is calculated and the values x are normalized as $z = \frac{x-\mu}{\sigma}$. On this table the values of the second column are the original observations. The normalized characteristics are given on the third column. The normalized values represent the effect of mean reversion on each characteristic value. The same applies to other characteristics.

[26] Quantile-Quantile plot is a graphical method for comparing two probability distributions by plotting their quantiles against each other. Usually, the comparison is between the empirical and a theoretical distribution.

The normalization is a necessary step once we cannot compare values of different data periods and of different moneyness on the same axis. This procedure is a naive way to eliminate market risk effects from our data.

After this step classification analysis is performed using scatter[27] and silhouette[28] diagrams. This procedure is a standard practice of data classification (see Duda, Hart & Stork [2001]). This analysis will show which characteristic values does the mean reversion parameter affect.

At the end of our analysis we apply a rough estimate of model risk for our deals. The estimate will be a combination of the deviation and the riskiness of each experimental case.

8.2 Hedge test results: 5 year Bermudan swaption

We perform 106 hedge tests according to section (8.1.1). The replicating portfolio is based on forward hedging ratios. For these hedge tests we define an "experimental case" for {-1,1,3,10}% mean reversions. -1% and 10% are extreme values while 1% and 3% are within the practice standards (see validation report [66]). The model is calibrated to 5 vanilla swaptions with maturities that cover all the exercise dates of the Bermudan swaption. Hence we will have in total 5 bucketed vegas. The number of Monte-Carlo paths is set to 25K.

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[27] A scatter plot is used when the observation parameter is under the control of the experimenter. The scope of scatter plot is to show if a parameter is systematically increased or decreased by another one. The parameter which is responsible for change of other parameters it is called the control parameter and is traditionally plotted along the horizontal axis. Scatter plots can suggest various kinds of correlations between variables with a certain confidence interval.

[28] Silhouettes use the internal point distance to assign values from -1 to 1 to cluster points (see appendix (C). The measure shows of how similar that point is to points in its own cluster in comparison with points in other clusters. Values close to -1 implies that the point should belong to another cluster, 1 implies prefect fit with its cluster and 0 that the point cannot be an element of any cluster according to its internal point distance.

The visual inspection of the hedge tests have shown that the market scenarios and the moneyness of the experiment affect significantly the performance of hedging and the P&L of the daily hedging errors. Meanwhile, the mean reversion does not contribute to big changes on the final values. The results at the first stage suggest three main types of hedging behaviors.

The first type of hedging behavior is observed when vega is high. In that case, the hedging error is very sensitive to market movements. The associated deals are non ITM during the life of the experiment. These types of experiments will show significant second order effects. Due to high vega the hedging portfolio, in that cases, will be mainly consisted of vanilla swaption instruments.

The second category of hedging outcomes is when vega is small then the effect of market risk is minimal. The Bermudan is ITM and converges to the underlying forward swap. Thus, the product is transformed to a linear product. The option is almost not sensitive to volatility anymore. The second order effects are dramatically reduced and the most important type of risk that affects the portfolio's price is Θ. Θ will be positive or negative depending on the movement of market. Hence, the process of the hedging error is not volatile anymore and it will evolve approximately with a linear trend.

The last type of portfolio behavior is a combination of the previous two cases. This happens when an ATM option is transformed to ITM during the life of the experiment. Then the daily P&L looks like the convolution of an ITM and OTM error distribution.

Due to this diversity of results it is difficult to study model risk from the first view. The deviation of the hedging error for one "experimental case" can be amplified or shrank according the distribution of the daily hedging errors. A worst-case (or MaxMin) measure in that stage could not be that useful. For that reason, before we continue on the model risk assessment is important to get familiar with the nature of the portfolios that we incorporate in our research. On the next two sections we show one representative example for each case. Below we provide information all the "experimental cases". In Appendix (H) the initial option price and the hedging error of each hedge test are listed in tables (8.4) and (8.5).

Moneyness\Scenario	0	255	510	805	1060	1315	1570	1825
	Mean of Initial Option Price							
ITM	799.502	774.2225	807.6913	794.6253	849.1878	830.2965	837.3078	857.6495
OTM	13.21623	14.39065	11.06878	12.38603	8.101808	13.40333	10.08809	12.59838
ATM	143.358	154.078	159.8128	162.7968	143.214	152.155	134.7385	144.6905
	Range of Initial Option Price							
ITM	0.393	0.786	0.858	1.057	0.236	0.508	0.153	0.347
OTM	1.0541	1.2066	1.0781	1.2171	0.75436	1.151	0.86392	1.2179
ATM	6.617	7.249	6.591	7.039	6.242	7.319	6.875	7.214
	Standard Deviation of Initial Option Price							
ITM	0.176218	0.351213	0.385438	0.471225	0.104197	0.22485	0.070722	0.157917
OTM	0.457303	0.530891	0.4682	0.528867	0.327948	0.497157	0.373597	0.524156
ATM	2.883848	3.150209	2.869884	3.069535	2.721363	3.186185	2.991195	3.137619

Table 8.4. The table shows the average derivative price at time 0, the range and the standard deviation for each "experimental case".

8.2.1 High vega risk deals

The portfolios of this group of deals are mainly driven by vega as we already mentioned. The fluctuation of the of the hedging error is quite volatile and is correlated to the derivative's process. The variability of the daily hedging error indicates the presence of second order terms.

The size of vegas and the volatile evolution of deltas and the size of hedging errors are an indication of riskiness for the deal. Under this situation, the P&L of the daily errors will look close to Gaussian distribution.

Moneyness\Scenario	0	255	510	805	1060	1315	1570	1825
	Mean of Final Hedging Error							
ITM	8.92925	9.69	14.9125	9.1925	3.147	-2.08825	-5.1385	-19.745
OTM	-41.7755	24.32743	2.810375	28.88273	-4.09656	-0.82278	-3.08286	28.49475
ATM	-18.4486	17.99775	7.789	31.035	-12.3241	1.522925	6.1745	26.34575
	Range of Final Hedging Error							
ITM	1.034	1.15	1.25	0.27	0.596	0.747	3.912	11.65
OTM	4.87683	9.5588	4.9349	1.8182	1.6557	1.379187	1.87152	11.2966
ATM	14.3036	7.099	8.961	5.531	3.999	3.4235	15.554	4.628
	Standard Deviation of Final Hedging Error							
ITM	0.42221	0.539073	0.53761	0.121758	0.269534	0.324118	1.605368	5.22979
OTM	2.092162	4.326795	2.31186	0.795372	0.688984	0.636098	0.891373	5.666015
ATM	6.064569	3.236054	3.735363	2.486481	1.690143	1.522249	7.317225	1.954511

Table 8.5. The table shows the average hedging error at the end of the first hedging year, the range and the standard deviation for each "experimental case".

The figures below provide self-explanatory information about the starting point of market scenario, the mean reversion parameter of Hull-White model and the fixed rate of the underlying swap. The NPVs of the hedging errors are plotted together with the NPVs of the portfolio.

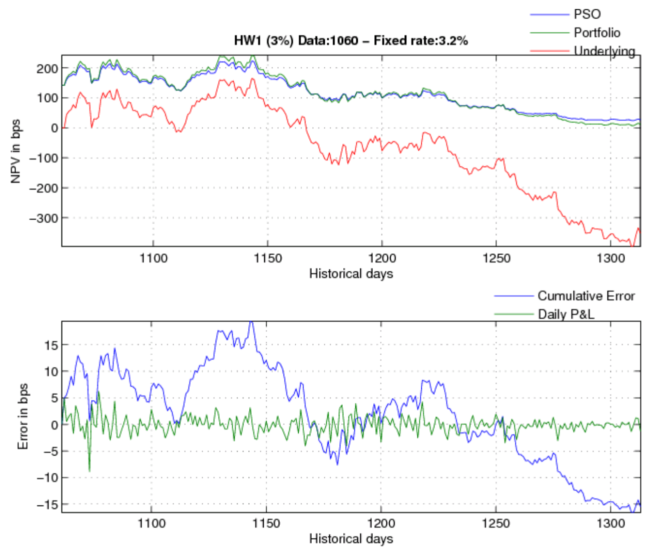

Figure (8.2.1.1). The figure gives the NPVs of the Bermudan swaption (namely "PSO"), the hedging potfolio (namely "Portfolio") and the first underlying swap (namely "Underlying"). The NPV of the swap is given by analytical formulas and constitutes the only model free visual tool to evaluate the evolution of the model-dependent NPVs.

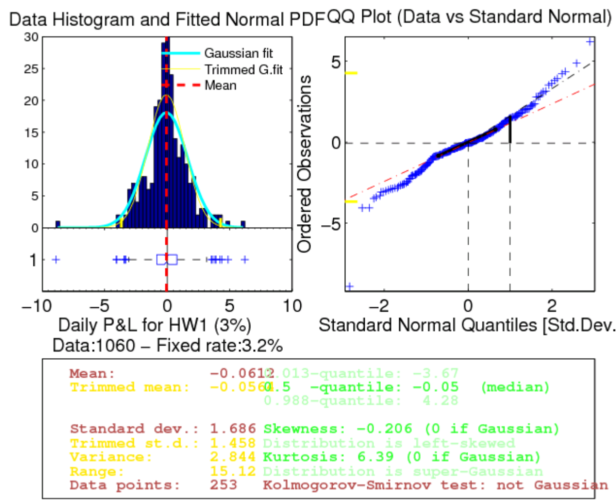

Figure (8.2.1.2). The histogram of the daily hedging errors (namely "P&L") is given along with the quantile-quantile diagram that compares the empirical distribution to the standard normal distribution. Additionally the statistical measures and the 2.5% quantiles of the distribution are provided. **Remark:** The kurtosis is corrected using the excess of kurtosis -3, such that 0 corresponds to Gaussian type.

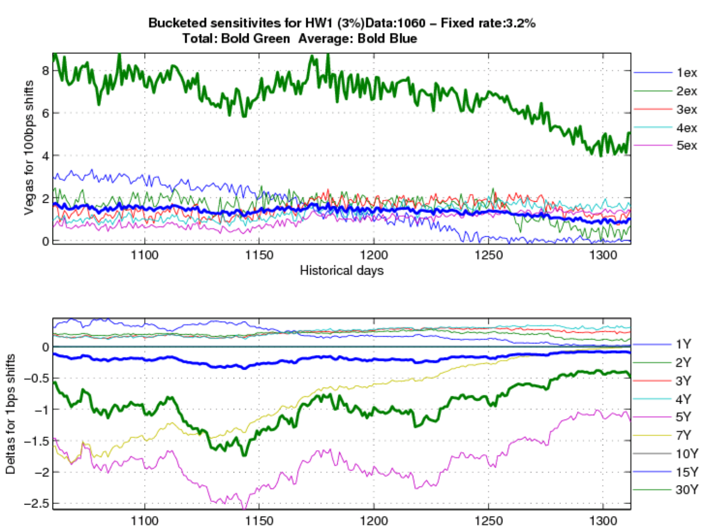

Figure (8.2.1.3). The figure gives on the upper gragh the bucketed vegas and on the lower graph the bucket deltas. The values represent the money value of 100bps shift on the quoted volatilities and 1bps shift on the quoted rates. The bold green lines correspond to the Total Bucketed vega (delta) and the bold blue lines correspond to the Average Bucketed vega (delta). On the right side the labels "1ex", "2ex", etc. correspond to the volatility of a vanilla swaption that matures on the 1st exercise date, 2nd exercise date, etc. respectively. The labels of "1Y", "2Y", etc. correspond the sensitivity of the 1 year, 2 years, etc. swap rates respectively. The minus sign on sesitivities stands for long positions.

8.2.2 Low vega risk deals

The low risk deals show a different behavior than this of the high risk deals. The reason is that ITM deals converge to the underlying asset, which is a forward swap. Then the payoff is linear and very easy to hedge (see figure (6.3.1)). As we mentioned at the beginning of the section the only risk that remains is Θ.

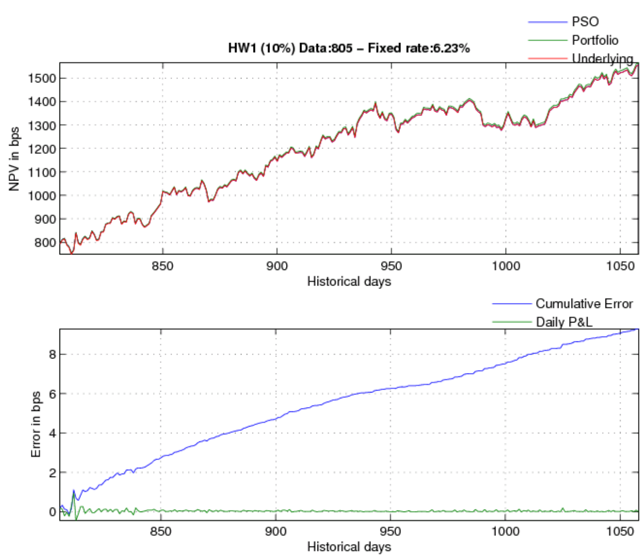

Figure (8.2.2.4). The presence of Θ is the most interesting detail of this example.

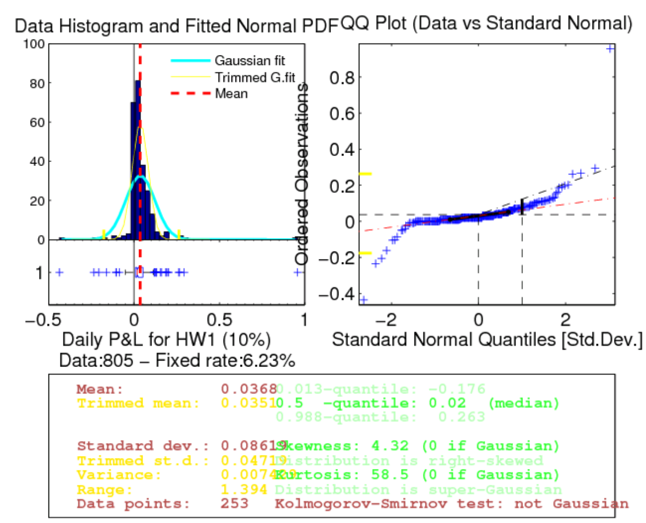

Figure (8.2.2.5). Observe the shape, the kurtosis and the skewness of the daily P&L. The daily errors are very small and this gives a leptokurtic empirical distribution.

When a deal is deep ITM at beginning of the option's life then everything is simple. The hedging errors are small and are skewed either to the left or to the right according to the sign of Θ. The vega of the option converges to zero while the amount of delta is linearly dependent to time. This implies that portfolio is mainly hedged by vanilla swap instruments. The second-order terms are minimal and the hedging errors are not volatile as in the previous example. This creates a leptokurtic P&L for the daily errors. The sign of Θ shifts the P&L either to the left or to the right.

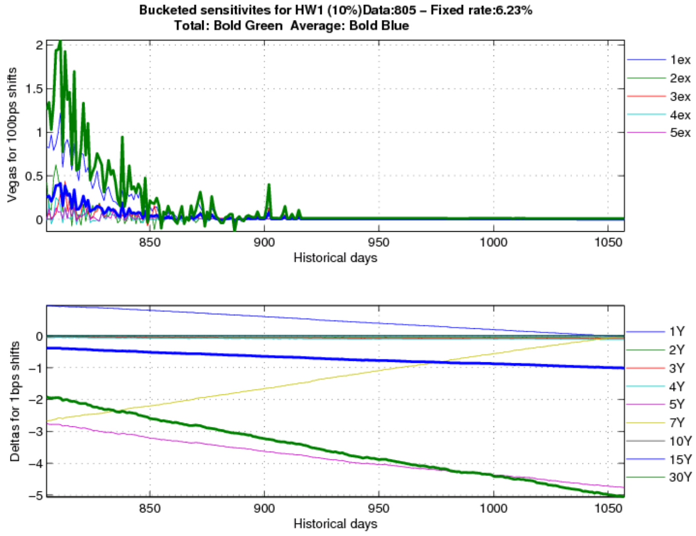

Figure (8.2.2.6). The sensitivities indicate the linear nature of the Bermudan swaption. The product is not sensitive anymore to swaption volatilities and the hedging turns to a linear combination of vanilla swaps during the life of the option. Therefore, delta is driving the process of the portfolio.

Another case of low risk experiment is when the ATM deals quickly turn to ITM. The P&L distribution of the daily errors combines the characteristics of ITM and OTM P&Ls. The fat tails of the P&L appear because the option initially is not ITM. Hence, initially the deviation of the daily errors is high. When the option will be transformed to an ITM deal the errors will be significantly smaller, close to zero and skewed to the right or to the left according to Θ. The figures below show a counter example of such hedge test.

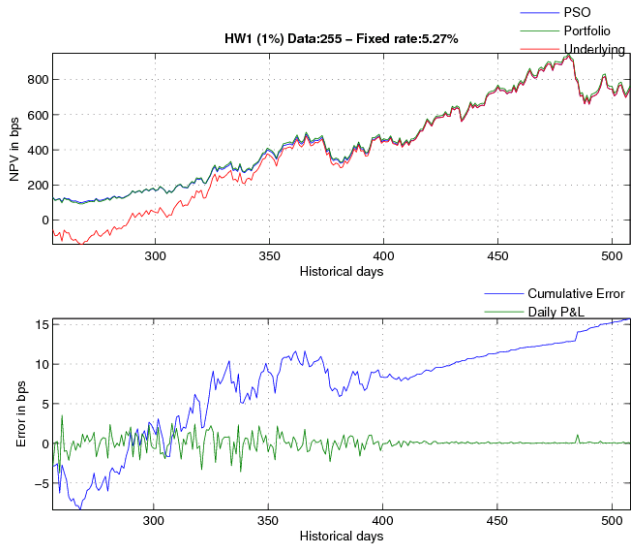

Figure (8.2.2.1). Observe that after trading day 400 the option turns to an ITM case the errors are almost zero. The contribution of Θ is present on the linear slope of the cumulative hedging error.

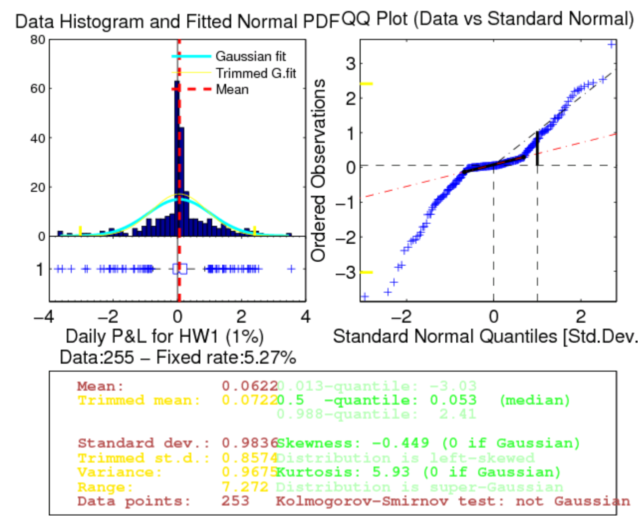

Figure (8.2.2.2). Observe the shape and the skewness on the daily P&L. The errors inherit characteristics of both high and low vega risk cases.

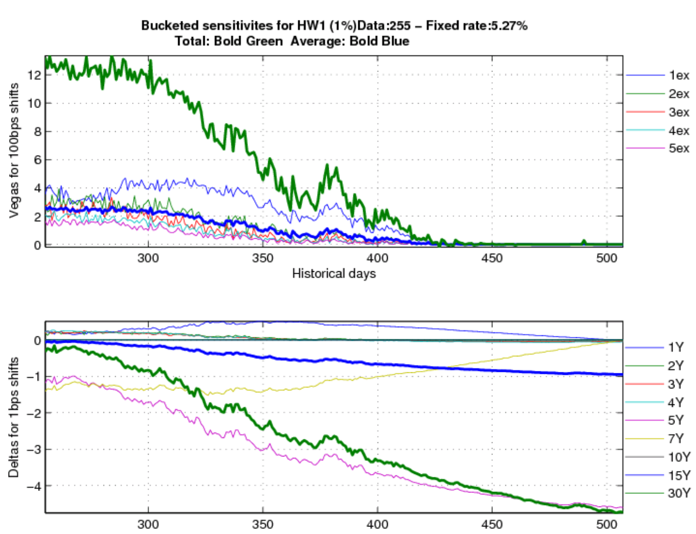

Figure (8.2.2.3). The magnitude of vegas gradually reduces to zero and the evolution of deltas is approximately linear after day 400.

8.2.3 Analysis of observed data

The first task of our analysis is to get a feeling of how the mean reversion parameter affects the experimental values. The normalization of the observed variables (see table (7.1.3)) facilitates a transparent comparison along the whole set of experiments and leaves the data dependent only to the mean reversion parameter.

The use of scatter plots indicates a positive correlation of the parameter α with the initial option prices while this is not straightforward for the final hedging error. Additionally to that the relation of mean reversion and the portfolio sensitivities is examined. The analysis takes the "average total bucketed sensitivities" into account. The scatter diagrams show significant correlation of vega sensitivity with the mean reversion parameter for ATM and OTM deals, while for delta this holds only for ITM deals. This result was expected according to our previous comments on what type of instruments contribute to the replication according to moneyness.

The following figures offer a visual evaluation for the effect of the mean reversion risk on different experimental variables.

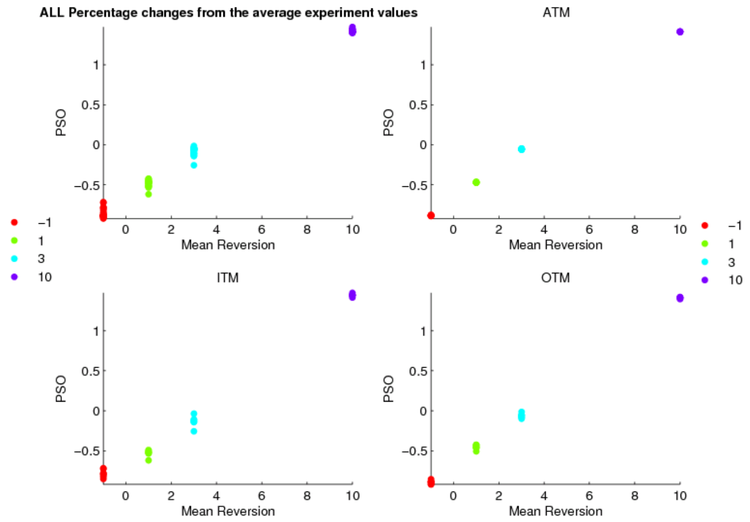

Figure (8.2.3.1). The upper-left diagram plots all the initial option prices "PSO". The plots that correspond to “ATM”, “OTM” and “ITM” titles show only the option prices that belong to ATM, OTM and ITM cases respetively. All diagrams suggest positive correlation between prices and mean reversion.

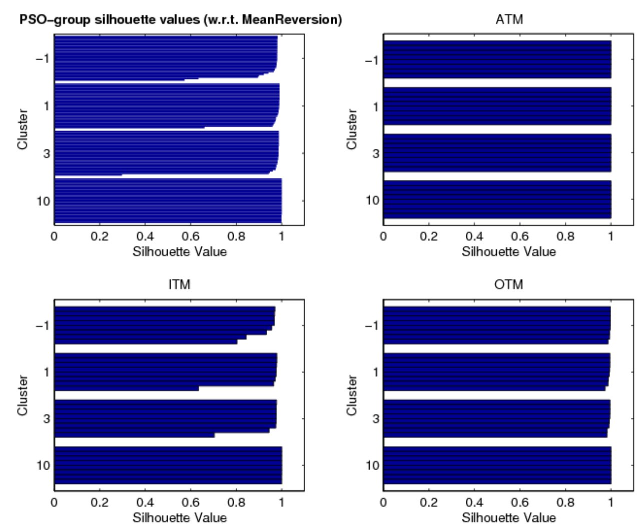

Figure (8.2.3.2). We evaluate the groups of the previous figure (8.2.3.1) w.r.t. α. The silhouettes indicate the formation of compact clusters with respect to the mean reversion. We observe that the ATM and ITM cases have a bit higher silhouettes than the ITM cases.

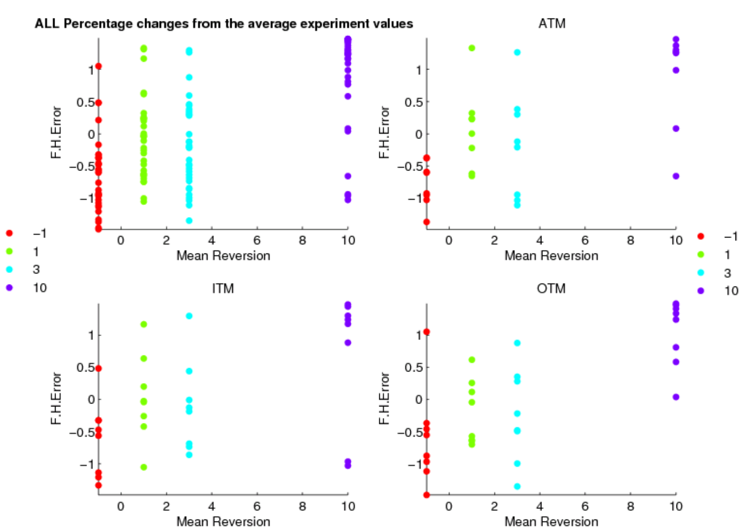

Figure (8.2.3.3). The scatter diagrams show that the final hedging error (herein F.H.Error) does not indicate any type of correlation to the mean reversion risk. Compare this fact the scatter diagram of the 20 year Bermudan on figure (8.3.2.1).

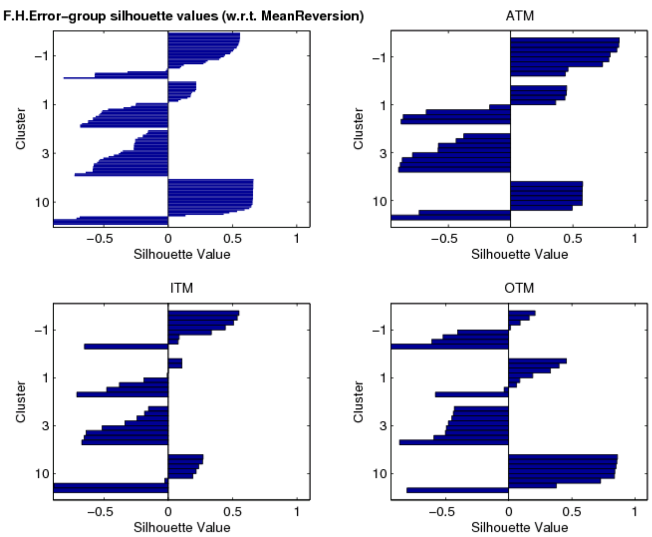

Figure (8.2.3.4). We evaluate the groups of the previous figure (8.2.3.3) w.r.t. α. The silhouettes do not indicate the formation of compact clusters with respect to the mean reversion.

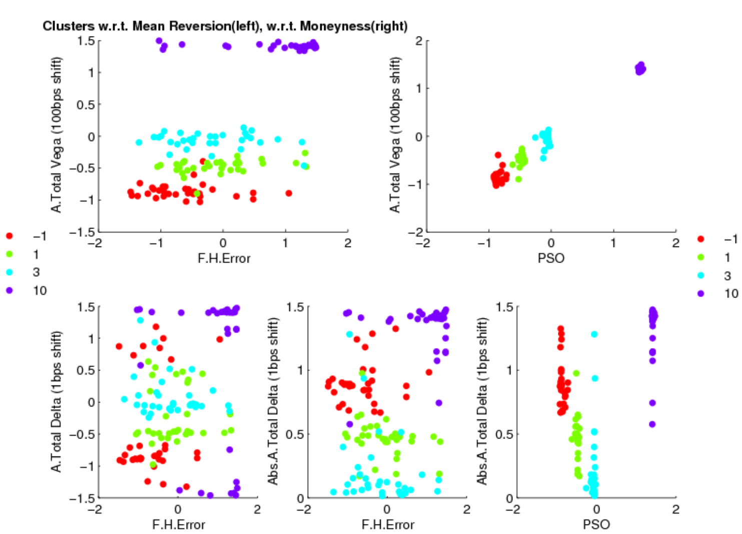

Figure (8.2.3.5). We investigate further the reasons of why mean reversion risk does not affect significantly the final hedging error. We use the classical scatter function for the following characteristics. The “A.Total Vega/Delta” is the Average(on year basis) Total Bucketed Vega/Delta. The “Abs.A.Total Delta” is the Absolute Average(on year basis) Total Bucketed Delta. The results for Delta show that is less sensitive to model risk than Vega which forms compact clusters and shows a positive correlation with the mean reversion parameter.

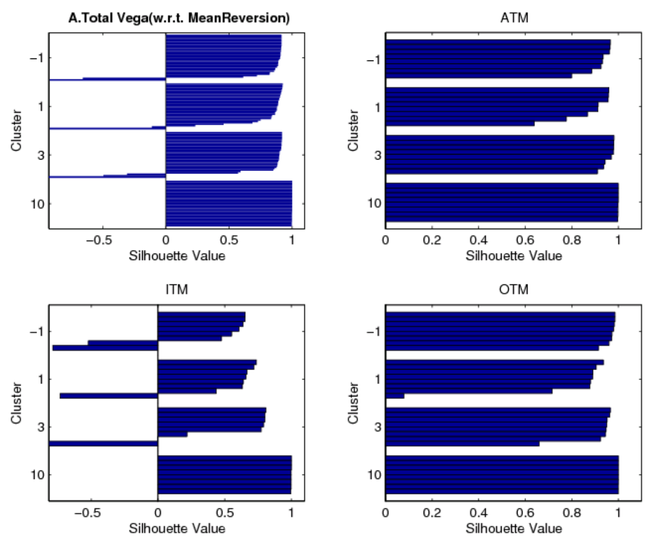

Figure (8.2.3.6). We evaluate then "A.Total Vega" groups of the previous figure (8.2.3.1) w.r.t. α. The figure shows only the silhouettes for the Average Total Bucketed Vega w.r.t. mean reversion. The ATM and OTM values form compact clusters w.r.t. α parameter. This is because the hedging positions are mainly taken on vanilla swaptions.

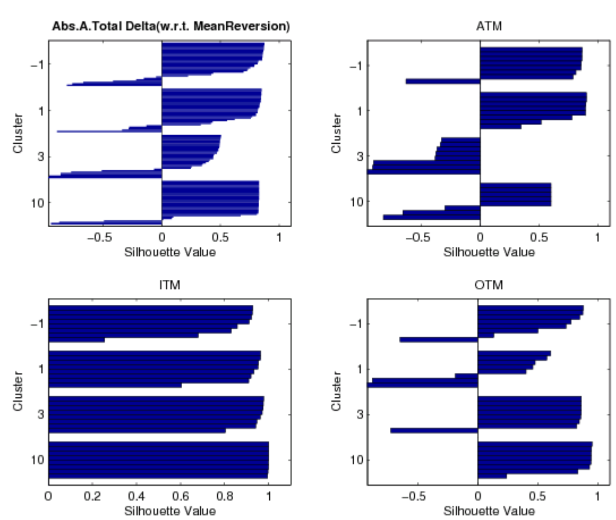

Figure (8.2.3.7). We evaluate then "Absolute.A.Total Delta" groups of the previous figure (8.2.3.1) w.r.t. α.The figure shows only the silhouettes for the Average Total Bucketed Delta w.r.t. mean reversion. The α parameter is important for the ITM deals since the hedging portfolio is almost composed from positions on vanilla swaps.

8.3 Hedge test results: 20 year Bermudan swaption

The last part of our experiments include the hedge test on 20 year deals as we have described at the beginning of the chapter. The number of simulation paths is set to 50K. We have chosen only 3 market scenarios starting at 0 and 510 trading day from our historical data. We will consider to test only ATM cases once they are more interesting from a model risk point of view. Therefore, in total we perform 8 hedge tests.

For computational reasons the yield curve will depend only on the 5, 10 and 30 years swap rates and the model will be calibrated to the 3 vanilla swaptions that expire at the first, tenth and twentieth exercise date of the Bermudan swaption respectively. Hence we will have 3 bucketed deltas and vegas.

This amount of experiments is statistically not important. Even though, we include the results to give a small example of how the model risk may appear on options of long maturity. The results show that the presence of model risk on these deals is much higher than the short maturity deals. The mean reversion risk proves to affect all the experimental values in comparison to the 5 years options. Below we provide information all the “experimental cases”. In appendix (H) the initial option price and the hedging error of each hedge test are listed in tables. As we can see from the tables the biggest contribution to the final hedging error comes from the initial option premium which is invested in cash at the beginning of the hedging. That means that the range of the final hedging error is mainly formed from the initial mispricing and not from the hedge test itself.

(A) Observation\Scenario	**0**	**510**
Initial Option Price	422.9895	458.5873
Range	54.047	59.235
St.Deviation	23.23469	25.47681

(B) Observation\Scenario	**0**	**510**
Final Hedging Error	8.729	-23.477
Range	71.596	49.966
St.Deviation	31.20136	21.43587

Table 8.6. The table (A) shows the initial derivative price and the table (B) shows the final hedging error.

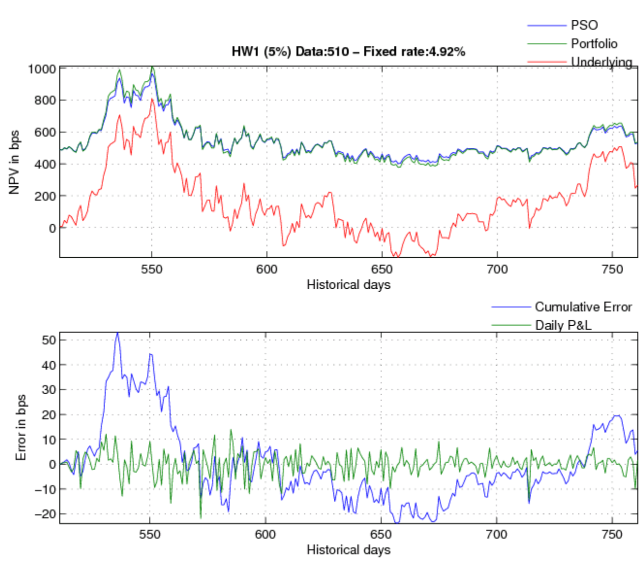

Figure (8.3.1.1). The figure shows the NPVs of an ATM 20 years Bermudan and its related hedging errors.

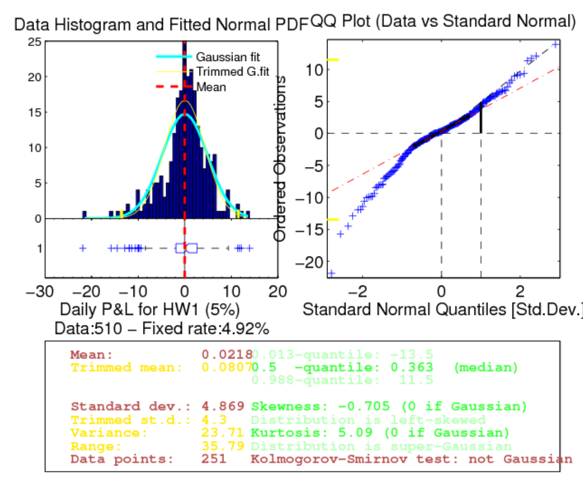

Figure (8.3.1.2). The daily errors of the replication form the daily P&L distribution of this experiment. It is interesting to compare the size of the standard deviation of this distribution to the deviation in figure (8.2.1.5)

8.3.1 High model risk deals

Here we present one of the 8 experiments on 20 years Bermudan swaptions. The experiment uses the same market scenario with that in figure (8.2.1.4) for the hedge test of a 5 years ATM Bermudan swaption. The similarity of the results will be enough to imply that both the 5 and 20 years deal have identical hedging error process since the are exposed on the risks of the same market scenario. The same applies for the rest of the 20 years deals. The hedging error shows identical behavior to the 5 years options. However is important to notice that the variance of the errors of the 20 years option is 4 times bigger than this of the 5 years in figure (8.2.1.4).

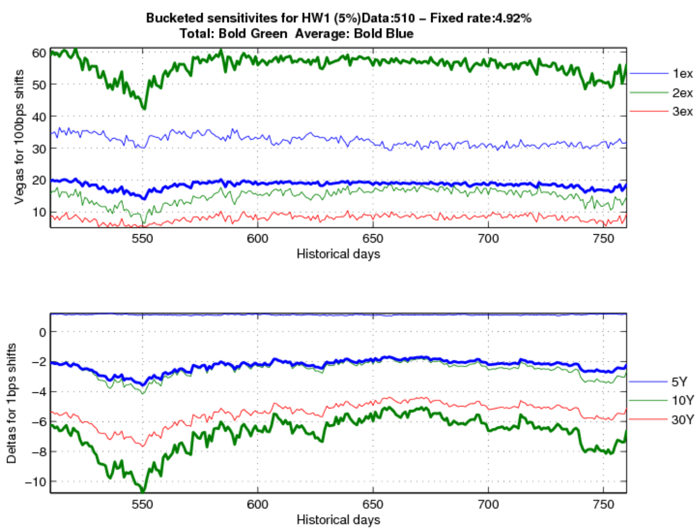

Figure (8.3.1.3). The sensitivities show the usual behavior of all the high risk deals, as we have described earlier. The size of the vegas is substantially higher compared to the vegas of the 5 year deal in figure (8.2.1.4).

8.3.2 Analysis of observed data

The analysis of the hedge tests for the 20 years Bermudans cannot be conclusive due to the number of the available tests. Nevertheless, the results give a promising indication for our future work. As it was expected the 20 years options are more sensitive to model risk than the 5 years options.

The scatter plot matrix[29] that follows next suggests several types of correlations along the whole set of the experimental characteristics. We examine relations between the initial option price (PSO), final hedging error (Error), the mean, the variance, the skewness and the kurtosis

[29] A group of scatter diagrams plotted in a matrix alignment. Scatter plot matrices are used to identify correlation structures between all the characteristics of the collected observations.

and the daily P&L (herein mean, std, skew and kurt respectively). The diagrams form clear types of correlations among all the combination of variables. The same scatter matrix for the 5 years Bermudan does not indicate significant correlation structures except from the initial price of the derivative. Furthermore the scatter plots of vega and delta are given as well. In general, all the characteristics apart from delta indicate a strong dependence on the mean reversion risk. The dependencies with respect to α are described on the following figures.

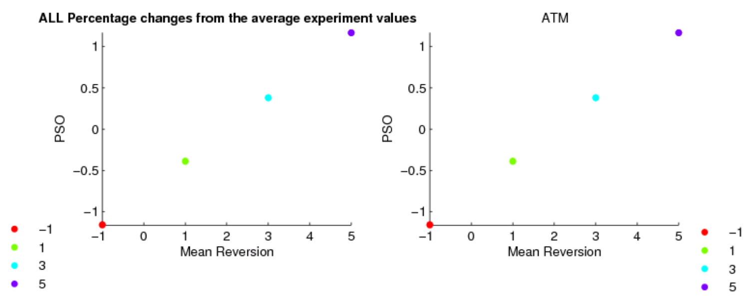

Figure (8.3.2.1). The results were expected according to the previous analysis of the 5 years deals. The initial option price is positive correlated to the mean reversion parameter. The higher the α the higher the price will be.

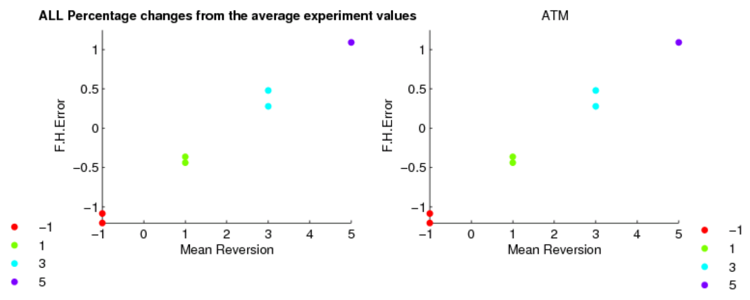

Figure (8.3.2.2). For 20 years deals we see that the mean reversion parameter does show a positive correlation relationship to the final hedging error.

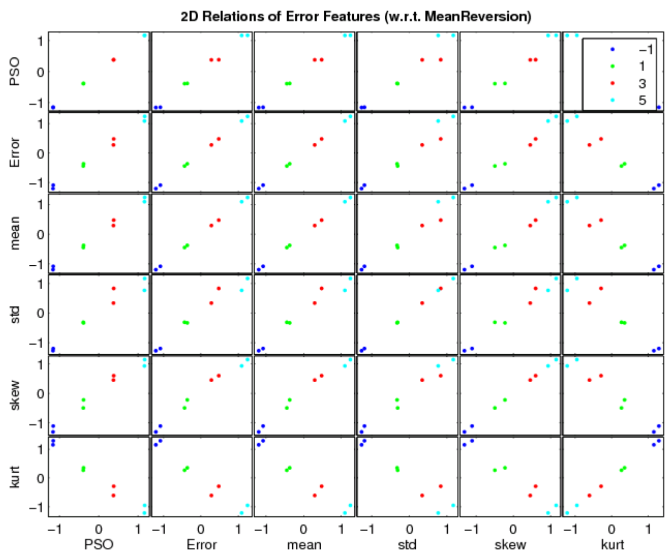

Figure (8.3.2.3). The scatter plot matrix suggests positive correlation for most of the combinations while kurtosis is negatively correlated w.r.t. the rest of the characteristics.

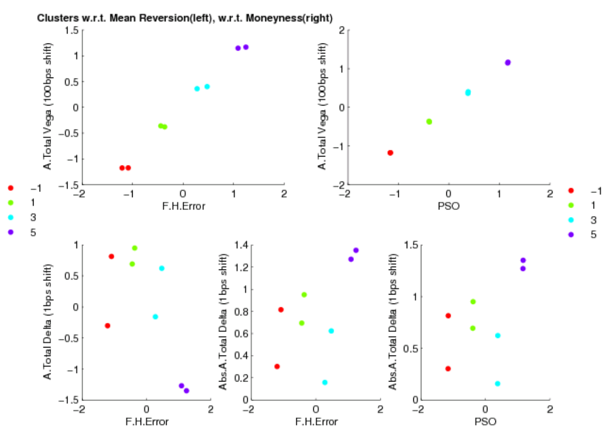

Figure (8.3.2.4). Once more vega suggest significant dependence with the parameter α. Delta even on these high risk experiments seems almost unaffected from the presence of difference α. As we have explained on the previous section the ATM deals are based on the vega positions rather than delta positions.

The results from the 20 years deal are quite interesting, even though, we are not allowed to continue on any comparison to the 5 years deals. The data that have been used on the hedge tests are not exactly the same with the 5 years deals since with have used less points on our yield and volatility term structures. As well as, the amount of the current experiments is not big enough for further conclusions. The results have been presented to give an indication of how big the model risk could be in longer maturity deals. These results suggest that the next work should focus more on higher maturity deals, where the effect of the model risk will be higher.

8.4 An estimate for model risk

As we saw on the previous two sections the hedging error is distributed according to different market factors. The unhedged error terms Θ and Γ seem to play an important role on the price of our replicating portfolios. However the discretization results and the model error contribute analogously to the final hedging error. Then, the equivalent expression of equation (3.15) for a ΔV-hedging portfolio will be,

$$\mathcal{E} = [{}_{\Delta V}\mathcal{E}]_{Unhedged} + \{{}_{\Delta V}\mathcal{E}_D\}_{Market} + \left({}_{\Delta V}\mathcal{E}_D^M +_{\Delta V} \mathcal{E}_{D,calib}^M\right)_{Model}$$

The previous examples have shown that none of the terms should be ignored or considered as more important than another. The size of each term depends on the market data and the nature of each deal, while model risk remains a hidden part of the total error. This makes the quantification of model risk difficult. To achieve this it is required good understanding of the hedging procedure in order to extract

$$\mathcal{E}^M =_{\Delta V} \mathcal{E}_D^M +_{\Delta V} \mathcal{E}_{D,calib}^M$$

out of $\mathcal{E}$. The extraction of $\mathcal{E}^M$ was illustrated with an example in figure(7.5) where a vanilla swaption is hedged with analytical and model pricing. This is not anymore the case for exotic options.

The easiest way of looking at model risk is to consider one "experimental case" of 4 mean reversions (look p.76 for terminology) . The error terms $[{}_{\triangle V}\mathcal{E}]_{Unhedged}$ and $\{{}_{\triangle V}\mathcal{E}_D\}_{Market}$ remain the same for the elements of one "experimental case". On the other hand, the distribution of $\mathcal{E}^M$ will contribute to the deviation of the 4 final hedging outcomes that belong to the same "experimental case".

In that sense, a possible estimate for the mean reversion uncertainty is the deviation of the $\mathcal{E}^M$ for one experimental case with 4 different errors due to the 4 mean reversions. Meanwhile the deviation per each experimental case differs according to moneyness and the market scenario, hence its value differs according to the total riskiness of the deal. Thus, we need to incorporate also this feature on our estimate.

For ATM and OTM deals vega is as measure of riskiness. Then a possible estimate of model risk of these deals can be the ratio

$$X_i = \frac{\textbf{Group's}\text{ St.Deviation of Final Hedging Error}}{\textbf{Group's}\text{ Mean of Average Total Vega}}$$

Where $i = 1, ..., 24$ as the number of the experimental cases (8 data partitions $\times$ 3 levels of moneyness). For the calculation of the Group st.deviation and mean refer to table (8.1.3).

The histogram of all X_i may suggest an upper bound for model risk. This can be done by setting a confidence interval on the histogram of the $X_i$distribution. The right quantile of the empirical distribution can be considered as the upper bound.

Figure (8.4.2) shows the histogram of X_i and the boxplot of the empirical distribution for a two-sided confidence interval of 95% applied to the results of 5 years deals. The right quantile indicates that we are 97.5% sure that the deviation of the final hedging error will not exceed the 1.2 times the vega of the deal.

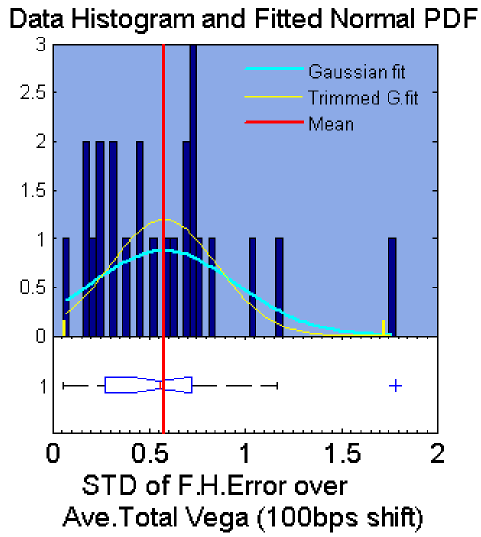

Figure (8.4.2). Here is presented the histogram of all X_i (above) together with the boxplot of the observed distribution (below) for a two-sided confidence interval of 95%. The right confidence level that the boxplot indicates is 1.2 times the average total vega of a deal.

Chapter 9
Conclusion

The initial plan of designing this project was to assess (or quantify) model risk for the Hull-White one factor short rate model. Our research which has been based on hedge testing have shown that the application of the existing model risk measures is not straightforward for unprocessed hedging outcomes.

As a result we focused on an extensive analysis of short and long maturity deals (5 and 20 years respectively). With this analysis we managed to create a clear picture for the behavior of the mean reversion risk of the Hull-White model on Bermudan swaptions. The conclusions upon these analysis enable us to test a practical estimate for the mean reversion risk, which has been applied on the set of our experiments.

9.1 Project evaluation

The first part of this project summarizes a long period of research related to model risk. On the theoretical part of the thesis we provide a formal description of the related model errors. This description aimed to present our intuition about model risk.

For the completion of our research we have designed and implemented a dynamic replicating strategy to perform the hedging simulations. The strategy can be realized under different hedging assumptions, hedging frequency, type of numerical sensitivities, different models and products.

A significant part of the thesis was spent to understand the hedging behavior of vanilla payoffs under different specifications and market scenarios. This offered significant knowledge to our team before we proceed to the final experiments for the model risk analysis.

The final results have shown that the hedging errors deviate within a range of $[-45, 45]$ basis points even after five years of hedging. Based on our hedging results we examine low and high model risk cases. The analysis shows that the model risk on ATM and OTM deals is mainly due to the vega of the deal while for ITM deals is due to delta. This is because when a deal is ATM or OTM the vega is big and the hedging is based mainly on vanilla swaptions. When the deal is ITM the hedging is based mostly on swaps.

For short maturity deals, like the 5 years Bermudan swaptions, we find that the final hedging error is not significantly correlated to the mean reversion risk. However, this is not the case for the 20y Bermudan swaptions where all the experimental parameters indicate clear forms of correlation with respect to mean reversion parameter. The impact of the mean reversion parameter on our results is translated to a range of 15bps for the 5 years Bermudans and 70bps for the 20 years Bermudans, expressed in EURO.

As a part of the model risk assessment, we have also presented a possible upper bound for model risk. This bound has shown that the model risk of the standard deviation of final hedging error will not exceed the limit of 1.2 average total vega (see section (8.1.1) for terminology) of a deal with probability 97.5%.

Additionally, the research provides evidence that the discretization and model errors of a hedge test can be decomposed under simple assumptions in case of vanilla products. Then, if indeed the hedging errors can be sufficiently described by the equation

$$\mathcal{E} = [{}_{\Delta V}\mathcal{E}]_{Unhedged} + \{{}_{\Delta V}\mathcal{E}_D\}_{Market} + \left({}_{\Delta V}\mathcal{E}_D^M +_{\Delta V} \mathcal{E}_{D,calib}^M\right)_{Model} \tag{9.38}$$

there might be a possibility of decomposing the error terms of more complex products. Our main concern is that without extracting the model error out of the observed final errors the quantification of mode risk will be very difficult to be achieved.

Methods that can implement a decomposition for equations of the form (9.38) are being used from the neural and signal processing community from mid-90s. The common objective of these techniques is to reproduce the original sources given a set of observations when the sources are unknown. This is usually called from researchers as “Blind signal separation” (see Hyvärinen [2000], Das, Routray & Dash [2007]). Several application of these algorithms can be found on magnetoencephalography, sound and image processing, on mobile and wireless telecommunications

9.2 Future work

This research needs to be extended to a wider range of payoffs and models, where the effect of model and parameter uncertainty can be different. In addition, we emphasize the absence of volatility smile in our experiments which is something that can give us even more interesting information.

Although the most important point that needs to be addressed, with respect to model risk, is the application of hedge tests with different model for pricing and different for hedging. Regarding the model for hedging, the choice is easy, while regarding the model for pricing the choice will be more difficult. The problem for illiquid products is that no market price is available. The choice of using a multifactor or a more complex model will only allow the comparison between two different models and not the assessment of model risk. However, by using different model for pricing as in figure (7.5) we might experience bigger hedging errors than what we have seen so far. If this still is not possible the only reasonable option, of studying model risk, is to compare the model-based price of the derivative (or the price of its hedging portfolio) minus the derivative's payoff for deals that have been active for an equal time period. These are certainly the most important tasks that need to be considered in a future work.

On the other hand, another interesting topic of model risk assessment is the definition of risk measures and consequently the definition of model risk reserves. For that reason we would recommend to concentrate on the decomposition of the hedging and pricing errors. The use of source decomposition algorithms might prove to be useful for that purpose. Without a precise methodology to study the model errors the problem of model risk quantification will continue bothering our books.

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Appendix A
Affine modeling

For the short rate modeling we take as an assumption that $r(t)$ is a martingale under a risk neutral measure $\mathbb{Q}$, which is defined on $(\Omega, \mathcal{F})$, and its dynamics are given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \tag{A.1}$$

where μ and σ are given a priori and $W(t)$ is a $\mathbb{Q}$-Wiener process.

Definition 7 *The instantaneous forward rate at t maturing at T is defined as*

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T)$$

where $P(t,T)$ is the price of a zero coupon bond at t maturing at T.

Definition 8 *The instantaneous short rate at t is defined by $r(t) = f(t,t)$.*

Lemma 5 *For $t \le s \le T$ the zero coupon bond can be seen as*

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$$

or as an expectation under a risk neutral measure $\mathbb{Q}$

$$P(t,T) = E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\} |\mathcal{F}_t\right]$$

Definition 9 *If the term structure $\{P(t,T) : 0 \le t \le T,\ T > 0\}$ has a form*

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{A.2}$$

where A and B are deterministic then the short rate model provides an affine term structure (ATS).

Proposition 6 *Assuming that μ and σ, the parameters of equation (A.1), have the form*

$$\mu(t, r(t)) = \alpha(t)r + \beta(t) \tag{A.3}$$

$$\sigma^2(t, r(t)) = \gamma(t)r + \delta(t) \tag{A.4}$$

then A and B, the affine term structure parameters, solve the following differential equations (see Bjork [2004])

$$\begin{aligned} \frac{\partial}{\partial t}B(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1 \\ B(T,T) &= 1 \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}A(t,T) &= \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) &= 0 \end{aligned} \tag{A.6}$$

Appendix B
Piecewise constant volatility

On our setting we assume that $r(t)$ is given as the sum of Gaussian processes $x_i(t)$ plus an additional deterministic term which facilitates the fitting to the initial zero-coupon bond prices. Then the $r(t)$ dynamics for the spot measure $\mathbb{Q}^0$ are

$$r(t) = \sum_{i=1}^{n} x_i(t) + \phi(t)$$

where $x_i(0) = 0$ and $\phi(0) = r(0)$. Then we can express the dynamics of all $x_i(t)$ as system of differential equations of the form

$$d\vec{X}(t) = -A\vec{X}(t)dt + Cd\vec{W}(t)$$

where $\vec{X}(t) = \{x_1(t), x_2(t), ..., x_n(t)\}'$, A a strictly positive diagonal $n \times n$ matrix, C a lower triangular $n \times n$ matrix and $\vec{W}(t) = \{W_1(t), W_2(t), ..., W_n(t)\}'$ a n-dimensional standard Wiener process while $W_i(t)$ are assumed to be $i.i.d.$ processes. This system can be seen, according to the representation of Dai&Singleton [2000], as

$$d\vec{X}(t) = -\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \vec{X}(t)dt + \begin{pmatrix} c_{1,1} & & 0 \\ \vdots & \ddots & \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix} d\vec{W}(t)$$

After that we assume that the price at t of a zero-coupon bond maturing at T is

$$P(t,T) = G\left(t, x_1(t), ..., x_n(t); T\right) = G(t, \vec{X}(t); T) = G^T(t, \vec{X}(t))$$

where G^T a smooth function. Formally G^T is denoted as

$$G^T(t, \vec{X}(t)) = \mathbb{E}^0 \left[\exp\left\{ -\int_t^T r(s)ds \right\} |\mathcal{F}_t \right] = e^{A(t,T)-B(t,T)'\vec{X}(t)} \tag{B.1}$$

with $B(t,T)$ a vector of the same dimension with $\vec{X}(t)$. Applying the multidimensional Itô rule we get

$$\begin{aligned}\frac{\partial G^T}{\partial t} - \sum_{i=1}^{n} a_i x_i \frac{\partial G^T}{\partial x_i} + \frac{1}{2}\mathbf{tr}\left[C^{'}HC\right] &= rG^T \\ G^T(T,\vec{X}(T)) &= 1\end{aligned}$$

where H a Hessian matrix with elements $H_{ij} = \frac{\partial^2 G^T}{\partial x_i \partial x_j}$.

Piecewise representation

We define as piecewise constant volatility $\sigma(t)$ the volatility of the equation (4.5) for the time partition $\tau_n : \{0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T\}$ where $\sigma(t)$ remains constant for the time interval $t \in (t_{i-1}, t_i]$ for $i = 1, ..., n$. The equation (4.5) can be expressed as

$$\begin{aligned}dx(t) &= -\alpha x(t)dt + \sigma(t)dW(t) \\ x(0) &= 0\end{aligned}$$

while $r(t) = x(t) + g(t)$ and $g(0) = r(0)$. Then equation (B.1) can be seen as $G^T(t_i, \vec{X}(t_i)) = e^{A(t_i,T) - B(t_i,T)^{'}\vec{X}(t_i)}$ with

$$\begin{aligned}B(t_i,T) &= \frac{1 - \exp\{-(T-t_i)a\}}{a} \\ A(t_i,T) &= \frac{1}{2}V(t_i,t_j) + \int_{t_i}^{T} g(s)ds\end{aligned} \tag{B.2}$$

where

$$V(t_i,T) = V(t_i,t_j) + \sum_{s=j}^{n-1} V(t_s,t_{s+1}) \quad , \quad \forall t \in (t_{i-1}, t_i]$$

while

$$\begin{aligned} V(t_s, t_z) &= \int_{t_s}^{t_z} \left(B(t_u, T)^2 \sigma^2(t_k)\right) dt_u \quad , \quad \forall (t_s, t_z] \in (t_{k-1}, t_k] \\ &= \frac{\sigma^2(t_k)}{2a^3} \left(e^{-2aT} \left(e^{at_z} - e^{at_s}\right) \left(e^{at_k} + e^{at_s} - 4e^{aT}\right) + 2a\left(t_z - t_s\right)\right) \end{aligned}$$

Moreover, $\phi(t)$ is the term that will give the fit to the initial yield curve. This can be done by obtaining the discount factors P(0,T) from the observed zero-coupon bond prices

$$P^*(t, T) = \exp(A(0, T)) = \exp\left(\frac{1}{2}V(0, T) - \int_0^T g(s)ds\right) \tag{B.3}$$

Using equation (B.2) we can find an expression for the integral

$$\begin{aligned} -\int_t^T g(s)ds &= -\int_0^T g(s)ds + \int_0^t g(s)ds \\ &= \ln P^*(0, T) - \frac{1}{2}V(0, T) - \left(\ln P^*(0, t) - \frac{1}{2}V(0, t)\right) \end{aligned}$$

and finally we find

$$A(t, T) = \ln \frac{P^*(0, T)}{P^*(0, t)} + \frac{1}{2}\left[V(t, T) - V(0, T) + V(0, t)\right]$$

Appendix C
Plotting functions

Scatter and silhouette plot

Here we provide a visual example of how silhouette values work in practice. A silhouette value for each point is a measure of how similar that point is to points in its own cluster compared to points in other clusters, and ranges from -1 to +1. The mathematical definition of this measure is available in Rousseeuw [1987]. Values close to -1 implies that the point cannot belong to its cluster, 1 implies prefect fit with its cluster and 0 that the point cannot be an element of any cluster. Silhouette plots are a common practice (see Duda, Hart & Stork [2001]) on the field of cluster analysis. These plots are used as an evaluation of the clustering process and the compactness of the derived clusters.

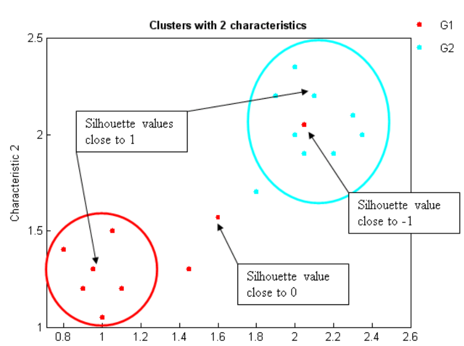


Figure (C.1). The figure shows 2 artificial groups and theis elements colored with respective colors.

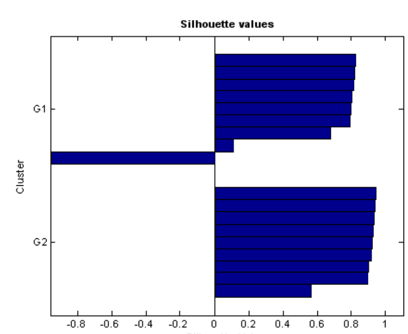


Figure (C.2). The figure shows the silhoutte values of G1 and G2 cluster from figure (C.1).

The example uses 2 artificial clusters where the elements are classified manually. On this classification includes an element of group “G1” spotted with red color that obviously should belong to group “G2”. This for instance has a silhouette close to -1. other cases of elements that might or might not belong to their group are the elements outside the groups' borders these values will have small silhouettes or close to 0.

Signal statistics plot

This function computes and plots statistical characteristics of a signal, including the data histogram, a fitted normal distribution, a normal distribution fitted on trimmed data, a boxplot, and the QQ-diagram. The estimates value are printed in a panel and can be read as output. (ref. Luca Finelli, CNL / Salk Institute - SCCN [2002]).

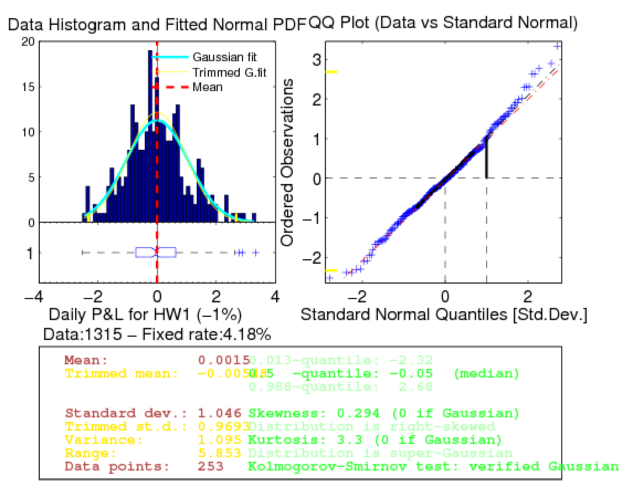
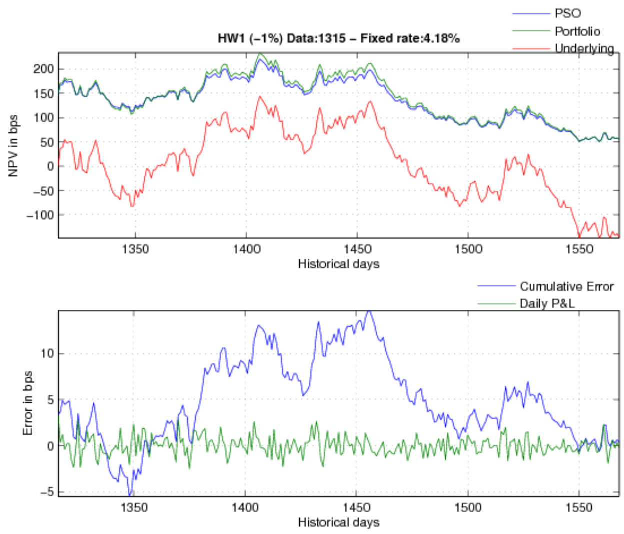
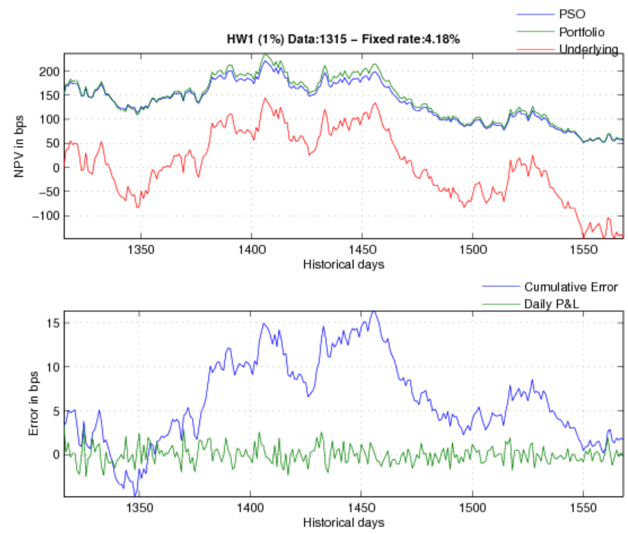


Figure (C.3). The signal statistics plot function shows the empirical distribution of the daily hedging errors and QQ-diagram w.r.t. to the theoretical Gaussian distribution and other statistical measures of the empirical distribution.

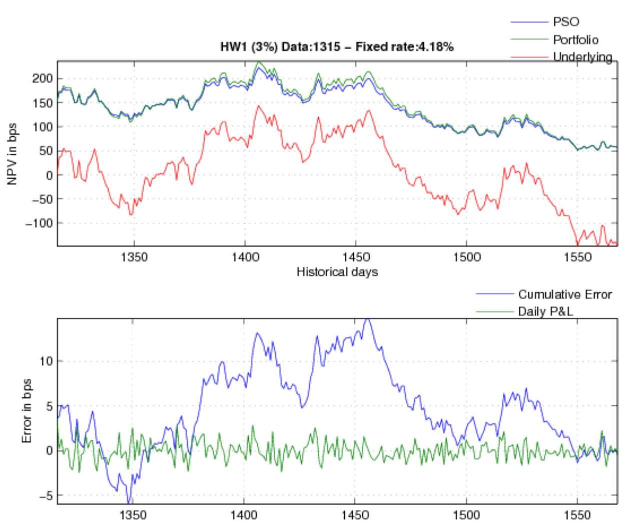
Appendix D
Results: The effect of mean reversion



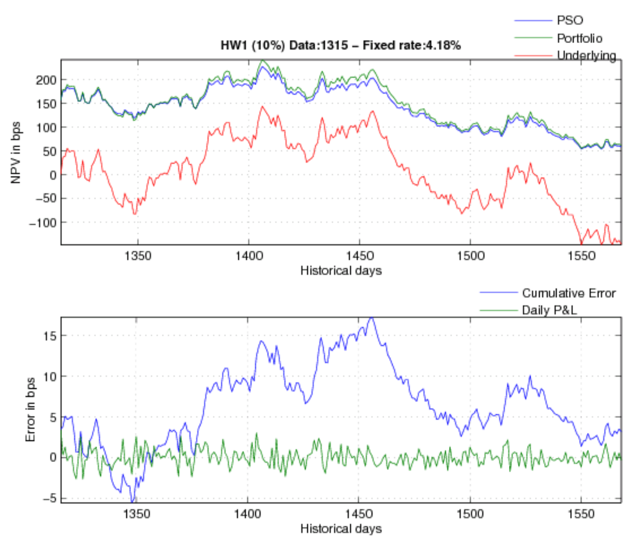
Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter -1%.



Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 1%.



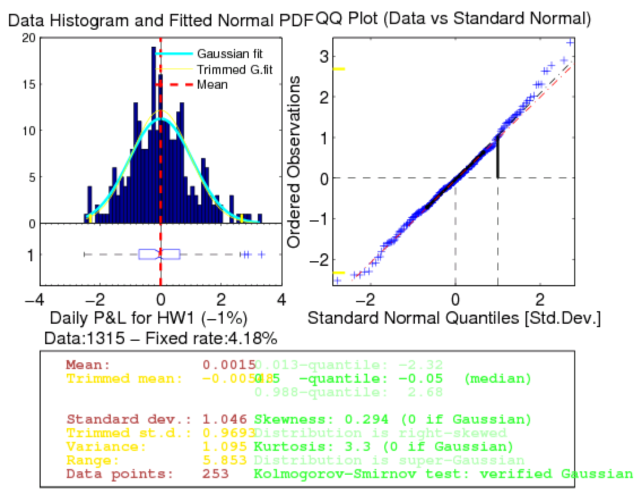
Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 3%.



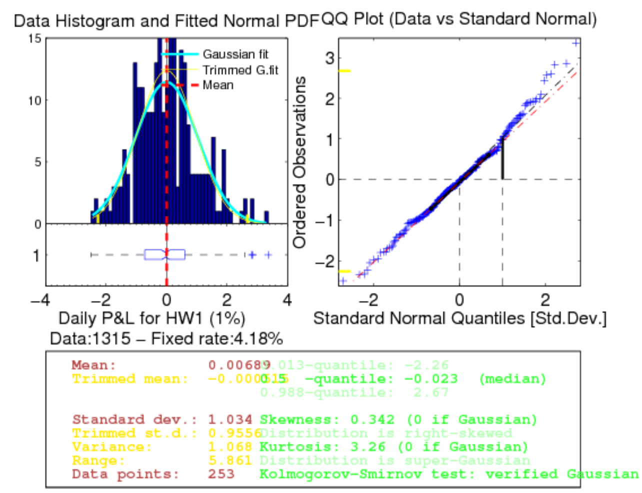
Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 10%.

Here we compare the effect of the mean reversion parameter to the actual prices and errors for one experimental case (the same market scenario, the same strike).

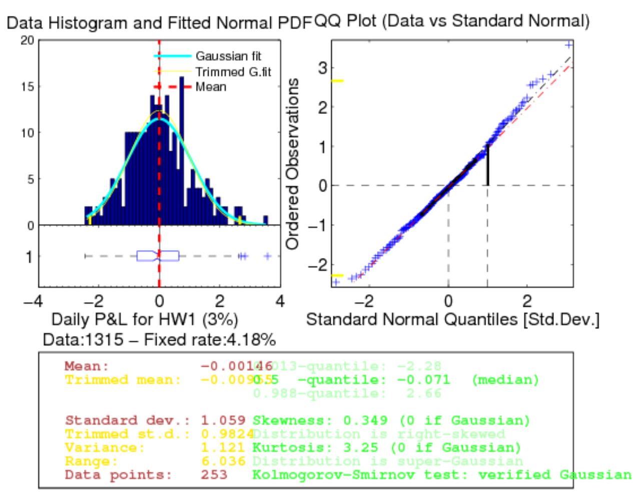
The corresponding statistics of the daily hedging errors are provided below. We assume that the reader is already familiar with the plotting functions.



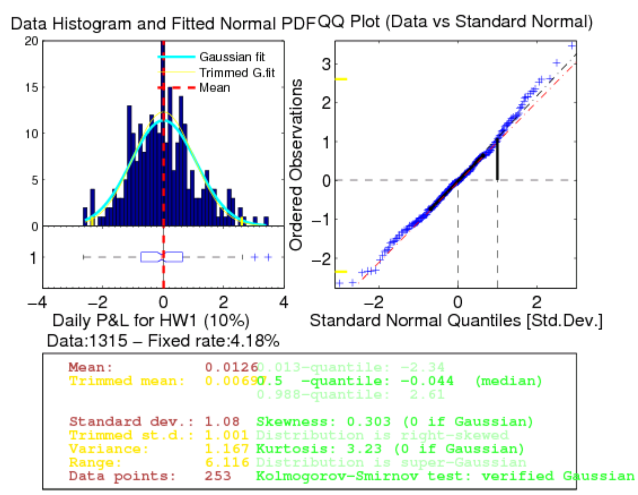
Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter -1%.



Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 1%.

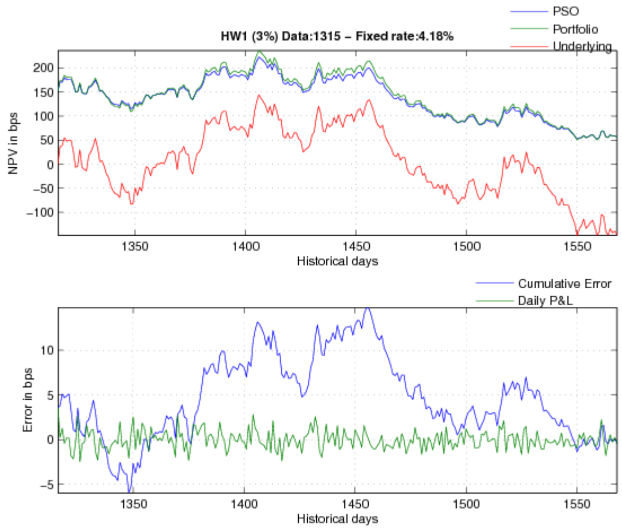


Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 3%.

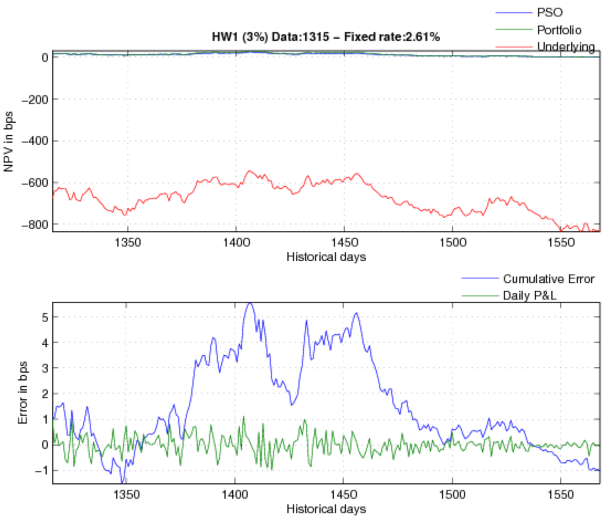


Experimental case (Scenario: 1315 start date - Strike: 4.18%) for mean reversion parameter 3%.

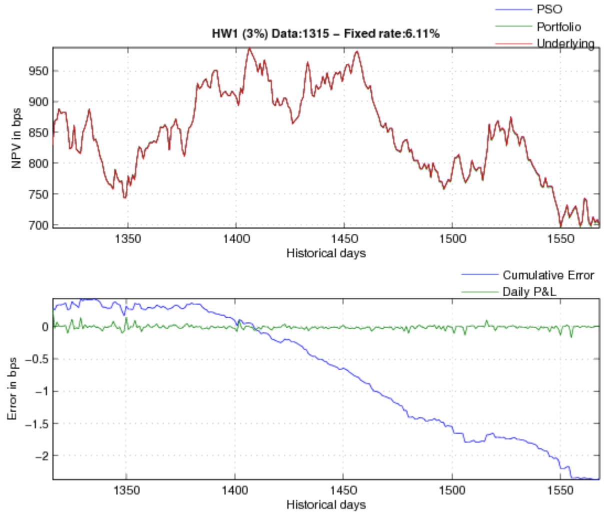
Appendix E
Results: The effect of moneyness



ATM



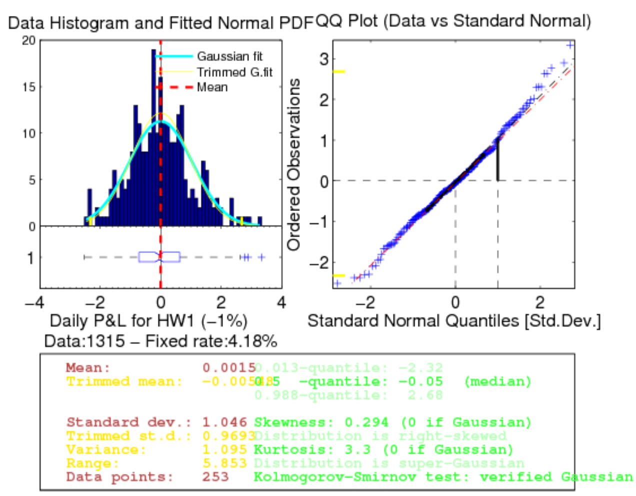
OTM



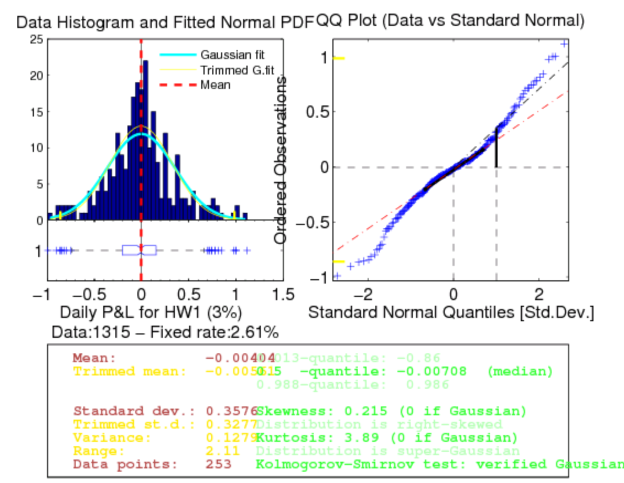
ITM

Here we show the effect of moneyness for the same market scenario and the mean reversion parameter. The effect on the daily hedging errors is severely big.

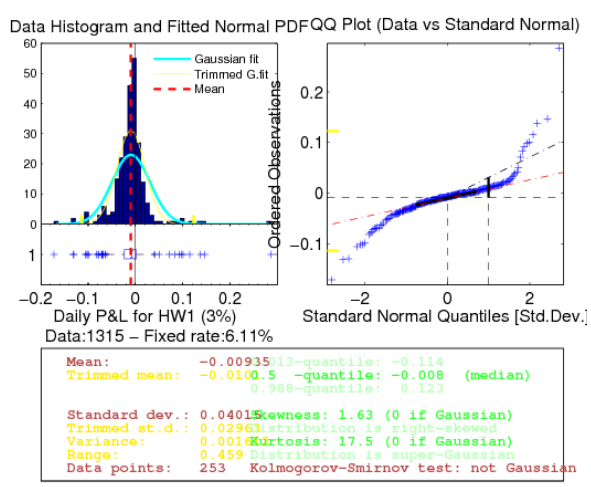
The corresponding statistics of the daily hedging errors are provided below. Observe the effect of moneyness on the P&L histograms. We assume that the reader is already familiar with the plotting functions.



ATM

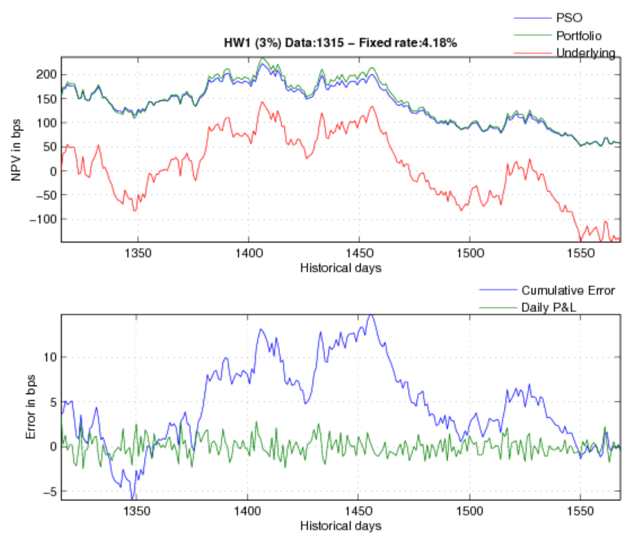


OTM

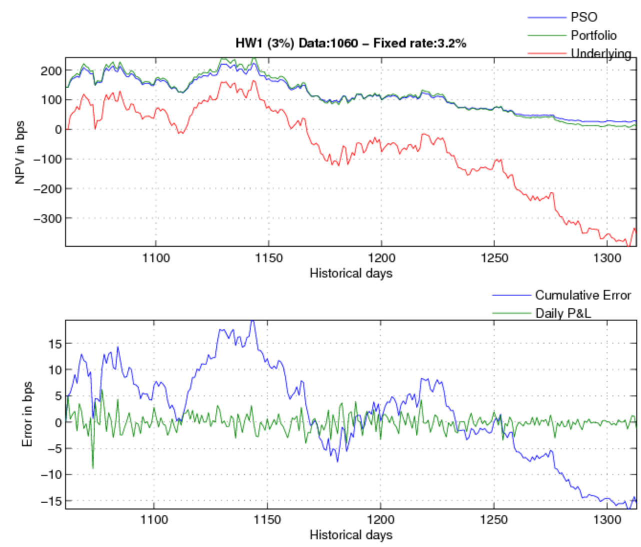


ITM

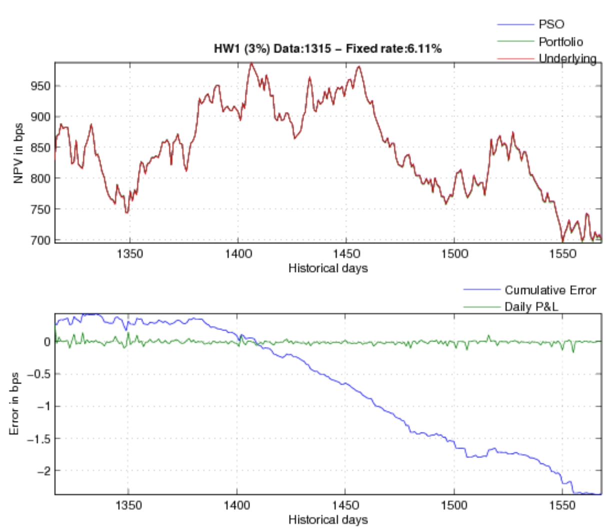
Appendix F
Results: The effect of market risk



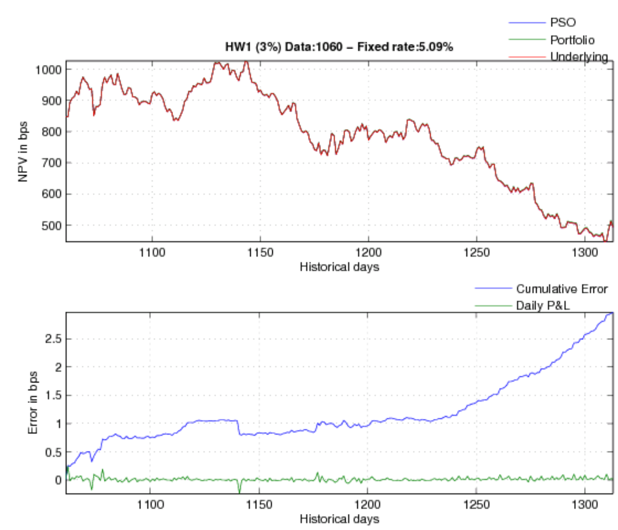
ATM - Scenario 1315 (the start date)



ATM - Scenario 1060 (the start date)



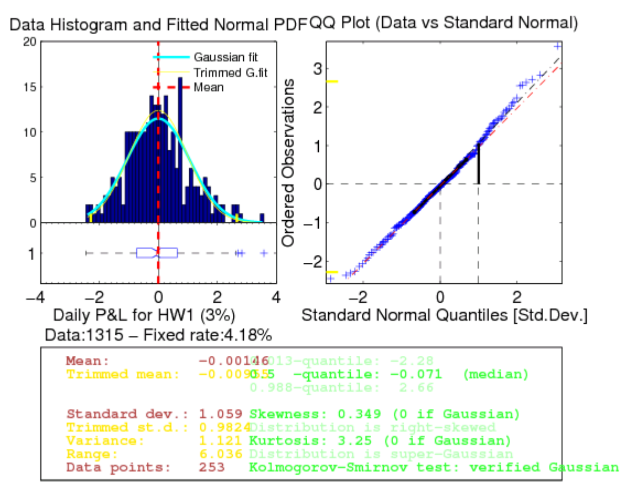
ITM - Scenario 1315 (the start date)



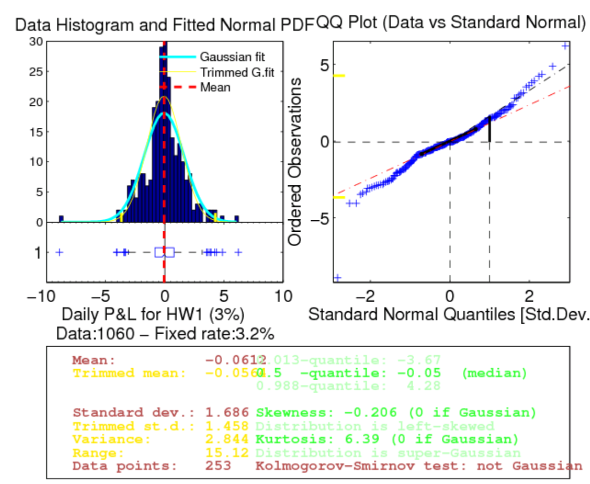
ITM - Scenario 1060 (the start date)

The effect of market risk is shown on the following figures. We show an ATM (above) and ITM (below) of the same mean reversion and moneyness on 2 different market risk scenarios.

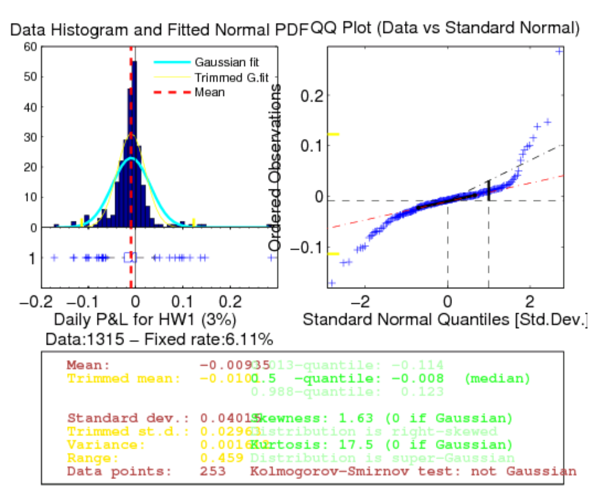
The corresponding statistics of the daily hedging errors are provided below. We show an ATM (above) and ITM (below) of the same mean reversion and moneyness on 2 different market risk scenarios. We assume that the reader is already familiar with the plotting functions.



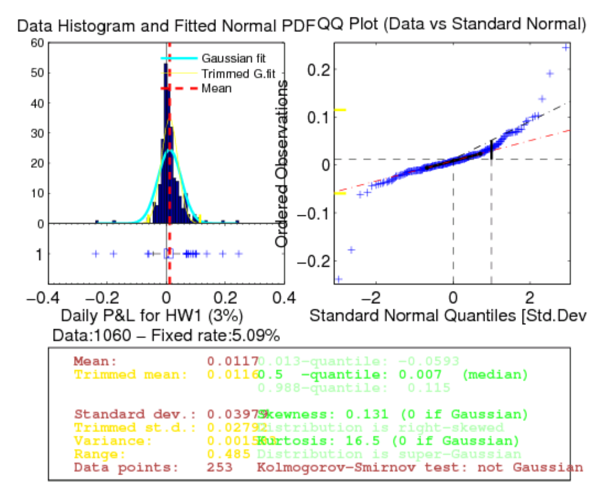
ATM - Scenario 1315 (the start date)



ATM - Scenario 1060 (the start date)

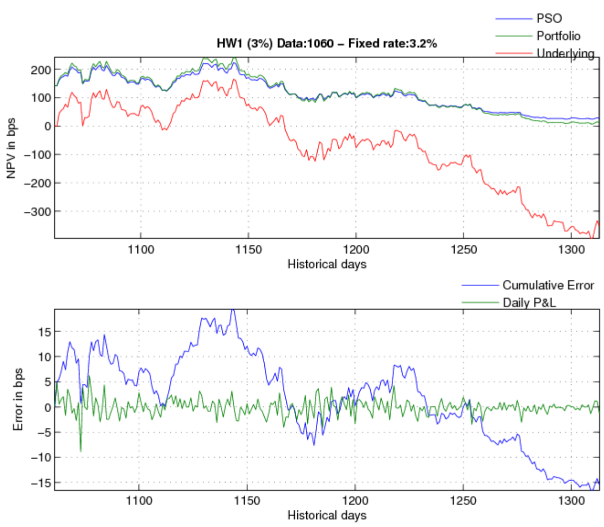


ITM - Scenario 1315 (the start date)

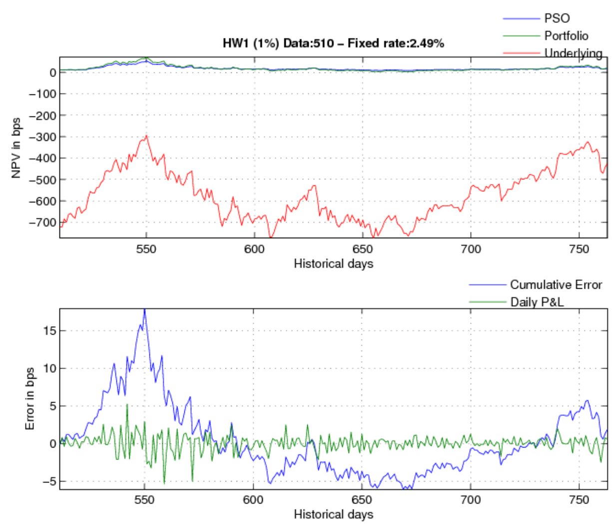


ITM - Scenario 1060 (the start date)

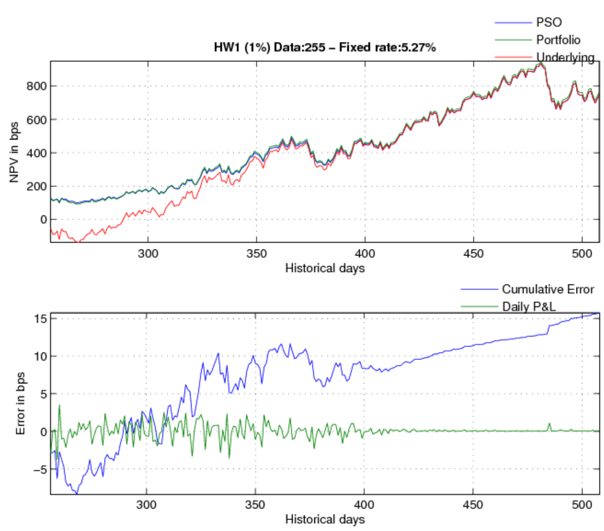
Appendix G
Results: High and low risk deals



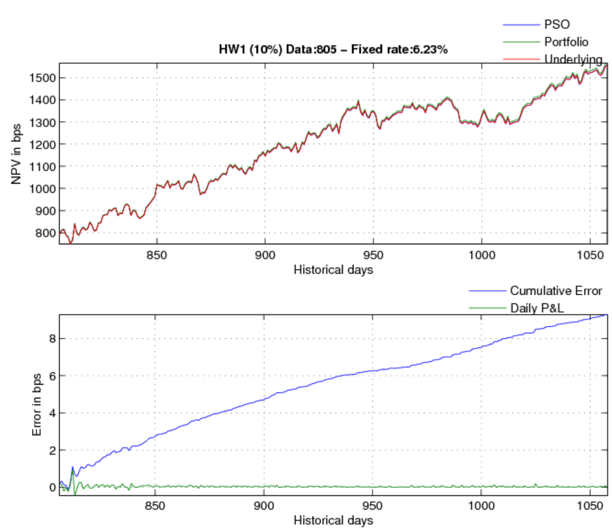
ATM that turns to OTM deal. (High vega risk case)



OTM deal. (High vega risk case)



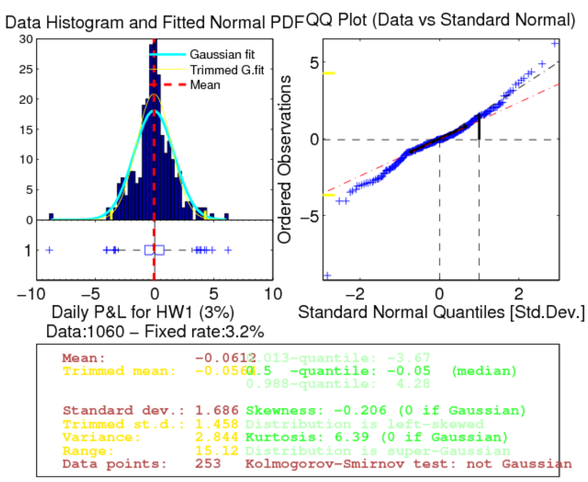
ATM that turns to ITM deal. (Low vega risk case)



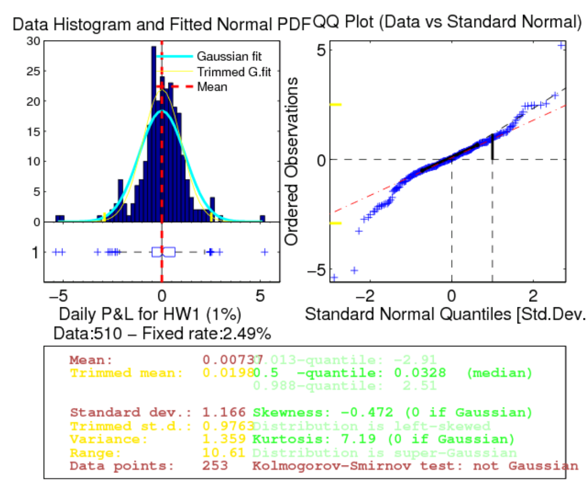
ITM deal. (Low vega risk case)

Here we present the 4 most typical combination of riskiness (or moneyness) during the life of a hedge test.

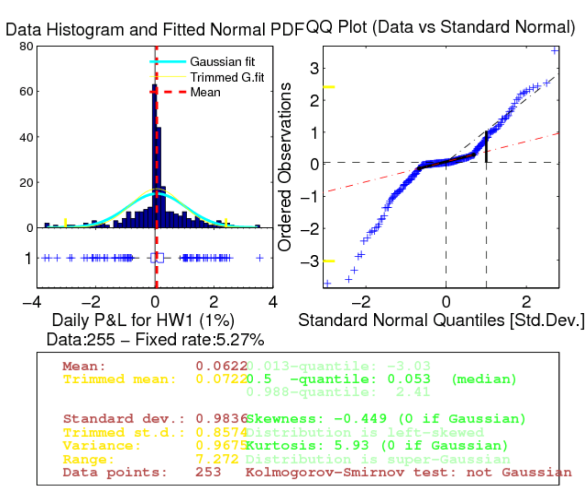
The corresponding statistics of the daily hedging errors are provided below. We assume that the reader is already familiar with the plotting functions.



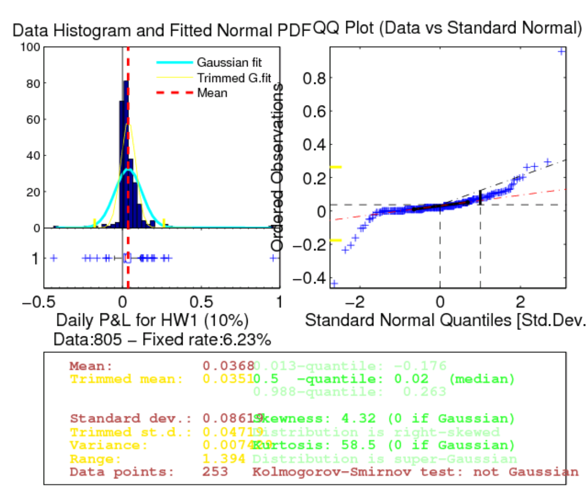
ATM that turns to OTM deal. (High vega risk case)



OTM deal. (High vega risk case)

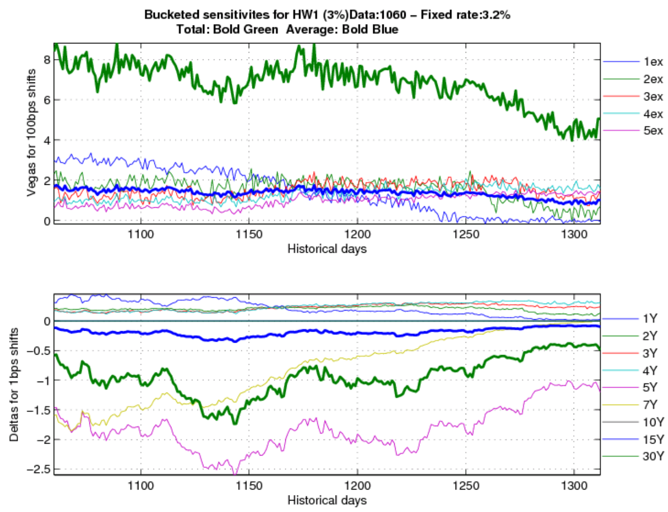


ATM that turns to ITM deal. (Low vega risk case)

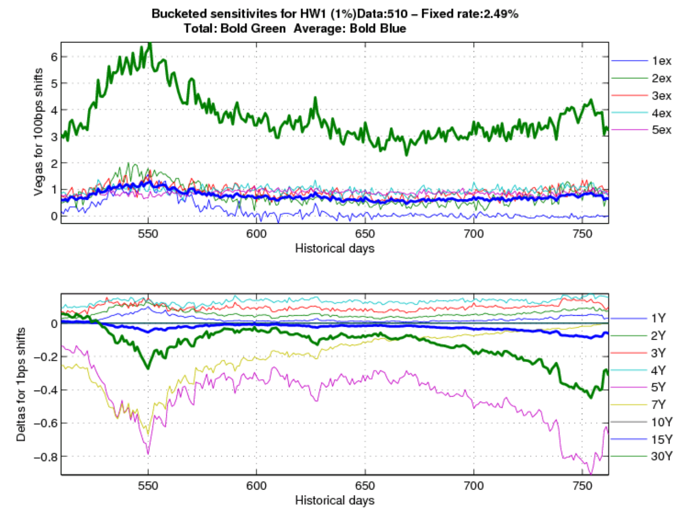


ITM deal. (Low vega risk case)

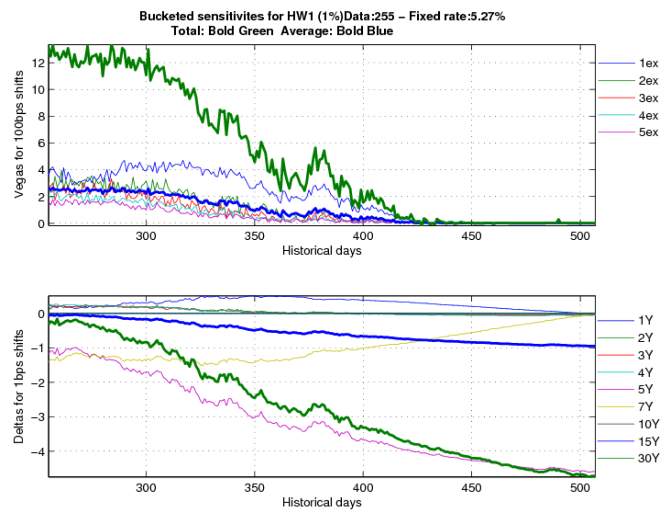
The corresponding bucketed sensitivities are provided below. We assume that the reader is already familiar with the plotting functions.



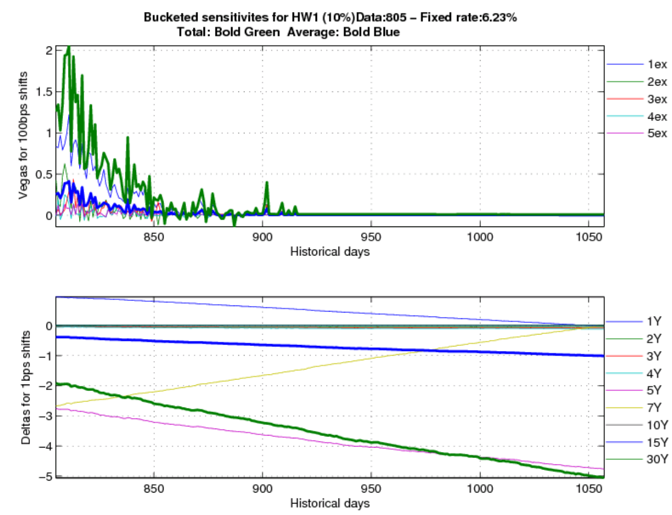
ATM that turns to OTM deal. (High vega risk case)



OTM deal. (High vega risk case)



ATM that turns to ITM deal. (Low vega risk case)



ITM deal. (Low vega risk case)

Appendix H
Tables: Initial price & Final hedging error

Scenario	Moneyness	Alpha	Initial Price	Final Hedging Error
0	'ITM'	1	799.41	8.919
0	'ITM'	-1	799.364	8.419
0	'ITM'	3	799.477	8.926
0	'ITM'	10	799.757	9.453
0	'OTM'	1	13.017	-41.5308
0	'OTM'	-1	12.8117	-42.7364
0	'OTM'	3	13.1704	-43.8558
0	'OTM'	10	13.8658	-38.979
0	'ATM'	1	142	-18.4343
0	'ATM'	-1	140.818	-26.7627
0	'ATM'	3	143.179	-16.1381
0	'ATM'	10	147.435	-12.4591
255	'ITM'	1	774.042	10.32
255	'ITM'	-1	773.944	9.95
255	'ITM'	3	774.174	9.32
255	'ITM'	10	774.73	9.17
255	'OTM'	1	14.1213	21.6025
255	'OTM'	-1	13.9361	20.146
255	'OTM'	3	14.3625	25.8564
255	'OTM'	10	15.1427	29.7048
255	'ATM'	1	152.617	19.034
255	'ATM'	-1	151.27	14.996
255	'ATM'	3	153.906	22.095
255	'ATM'	10	158.519	15.866

Table H.7. Results of 5 years Bermudans (part 1).

Scenario	Moneyness	Alpha	Initial Price	Final Hedging Error
510	'ITM'	1	807.485	15.02
510	'ITM'	-1	807.39	14.66
510	'ITM'	3	807.642	15.61
510	'ITM'	10	808.248	14.36
510	'OTM'	1	10.858	1.3208
510	'OTM'	-1	10.6536	1.96
510	'OTM'	3	11.0318	1.705
510	'OTM'	10	11.7317	6.2557
510	'ATM'	1	158.452	6.964
510	'ATM'	-1	157.265	3.949
510	'ATM'	3	159.678	7.333
510	'ATM'	10	163.856	12.91
805	'ITM'	1	794.38	9.27
805	'ITM'	-1	794.246	9.03
805	'ITM'	3	794.572	9.17
805	'ITM'	10	795.303	9.3
805	'OTM'	1	12.1413	28.8486
805	'OTM'	-1	11.9147	28.1864
805	'OTM'	3	12.3563	28.4913
805	'OTM'	10	13.1318	30.0046
805	'ATM'	1	161.346	29.493
805	'ATM'	-1	160.099	28.667
805	'ATM'	3	162.604	31.782
805	'ATM'	10	167.138	34.198

Table H.8. Results of 5 years Bermudans (part 2).

Scenario	**Moneyness**	**Alpha**	**Initial Price**	**Final Hedging Error**
1060	'ITM'	1	849.133	3.034
1060	'ITM'	-1	849.099	3.06
1060	'ITM'	3	849.184	2.949
1060	'ITM'	10	849.335	3.545
1060	'OTM'	1	7.94954	-3.91873
1060	'OTM'	-1	7.81004	-3.37131
1060	'OTM'	3	8.08325	-5.02701
1060	'OTM'	10	8.5644	-4.06918
1060	'ATM'	1	137.267	-3.2169
1060	'ATM'	-1	135.86	-1.7135
1060	'ATM'	3	138.557	-2.6821
1060	'ATM'	10	143.104	2.5487
1315	'ITM'	1	830.184	-2.172
1315	'ITM'	-1	830.111	-2.194
1315	'ITM'	3	830.272	-2.367
1315	'ITM'	10	830.619	-1.62
1315	'OTM'	1	13.1907	-1.26739
1315	'OTM'	-1	12.9547	-1.17433
1315	'OTM'	3	13.3622	-0.96121
1315	'OTM'	10	14.1057	0.1118
1315	'ATM'	1	150.658	1.8842
1315	'ATM'	-1	149.332	0.6204
1315	'ATM'	3	151.979	0.0818
1315	'ATM'	10	156.651	3.5053

Table H.9. Results of 5 years Bermudans (part 3).

Scenario	Moneyness	Alpha	Initial Price	Final Hedging Error
1570	'ITM'	1	837.264	-5.202
1570	'ITM'	-1	837.257	-6.962
1570	'ITM'	3	837.3	-5.34
1570	'ITM'	10	837.41	-3.05
1570	'OTM'	1	9.91578	-2.5333
1570	'OTM'	-1	9.74508	-4.40482
1570	'OTM'	3	10.0825	-2.8303
1570	'OTM'	10	10.609	-2.563
1570	'ATM'	1	133.342	1.376
1570	'ATM'	-1	132.085	1.751
1570	'ATM'	3	134.567	4.641
1570	'ATM'	10	138.96	16.93
1825	'ITM'	1	857.572	-25.25
1825	'ITM'	-1	857.535	-22.69
1825	'ITM'	3	857.609	-17.44
1825	'ITM'	10	857.882	-13.6
1825	'OTM'	1	12.3758	25.2548
1825	'OTM'	-1	12.1161	22.1701
1825	'OTM'	3	12.5676	33.4667
1825	'OTM'	10	13.334	33.0874
1825	'ATM'	1	143.219	28.952
1825	'ATM'	-1	141.897	25.601
1825	'ATM'	3	144.535	24.324
1825	'ATM'	10	149.111	26.506

Table H.10. Results of 5 years Bermudans (part 4).

Scenario	Moneyness	Alpha	Initial Price	Final Hedging Error
0	'ATM'	1	414.036	-2.613
0	'ATM'	-1	396.003	-28.857
0	'ATM'	3	431.869	23.647
0	'ATM'	5	450.05	42.739
510	'ATM'	1	448.584	-32.893
510	'ATM'	-1	429.158	-46.734
510	'ATM'	3	468.214	-17.513
510	'ATM'	5	488.393	3.232

Table H.11. Results of 20 years Bermudans.