Master’s Thesis

AN ALLOCATION APPROACH OF SPONSORED SEARCH AUCTIONS

Ferry Kristanto

Examination Committee:

Dr. Walter Kern

Dr. Judith B. Timmer

Prof. dr. Marc J. Uetz

Acknowledgements

Firstly, I would like to give thanks to my supervisor, Walter Kern. Thanks to his guidance during my thesis work. Although he was very busy, he had time to have discussion. Moreover, his interest also encouraged me in my thesis work. His corrections in writing my thesis were also very useful. The other people who I would like to thank are Judith Timmer and Marc Uetz as part of my examination committee.

Last but not least, I would like to thank my family. Eventhough they are far away from me, their moral supports and prayers have increased my spirit. Finally, I am grateful to my girlfriend and my friends. Thanks for your times and supports which have pleased me. All of you have given motivation in doing my master thesis.

F. Kristanto

Enschede, July 23rd, 2012.

Contents

1. Introduction

Advertisement is a vital thing in nearly every business; without it there would not be any customers interested to the products. Recently, there are a lot of businesses promoting their product via internet. This comes with no surprise, since internet has massive number of users and hence it is very potential to be exploited. There are many ways in promoting products in the internet, for example: building a specific webpage, setting banner advertisements, or set a placement in the search engine. Among all of these forms of advertisements, setting a placement in the search engine becomes more and more popular. The aim of this advertisement is to increase the visibility of the product in the search engines. It is expected that as the product becomes more recognized, there would be more people getting interested to try it. Sponsored search auctions, in which advertisers pays the search engines (e.g. Google, Yahoo!) to get search results every time the search users type specific keywords, is one of the most popular ways to reach the aim.

Sponsored search auctions can be exemplified as follows. Suppose a pizza restaurant has a website and wants to promote it using search engines. One possible way would be like this: every time a search user types “salami”, the restaurant wants its website to be shown beside the generic search results. For that, the restaurant offers some money to the search engine. If the search engine accepts it, then the restaurant will get a placement in the search results. On the other hand, there might be a lot of other restaurants who also want to get placements. Each of them also offers some money, and the search engine has to determine which offer should be accepted. To accommodate multiple advertisers, the search engine can provide multiple advertisement slots. The search engine then chooses the most profitable advertisers and also charges each advertiser certain prices.

It can be seen that there is a competition within sponsored search auctions: each advertiser wants to win the advertisement slots, while the search engine wants to maximize its profit. Looking deeper at this situation, a smart strategy in offering the money and also fixing the prices is necessary. If an advertiser bids too low, then it has a small chance to win the placement. The search engine also cannot charge too high to all winning advertisers, otherwise there would not be any advertiser getting interested in the placements. The way the search engine gives the advertisement slots would also affect the total revenue. In principle, the search engine has to guarantee that the most valuable advertisement slot should be given to the advertiser who bids the slot most.

Offering, pricing, and allocating the slots strategically are some keys to success in sponsored search auctions. A game-theoretic approach would be the best choice to successfully deal with sponsored search auctions. This paper will discuss about game-theoretic analysis of sponsored search auctions. The structure of this paper is as follows. After having some introduction and problem description in chapter 1, some theoretical foundations in game theory and auction theory will be discussed in chapter 2. Chapter 3 discusses about known results in sponsored search auctions, which is then formulated into research questions. Chapter 4 deals with reinterpretation of sponsored search auction into allocation problem, and is continued further by analyzing a generalization of the problem. Finally, this paper is ended by drawing some conclusions and suggesting further research in chapter 5.

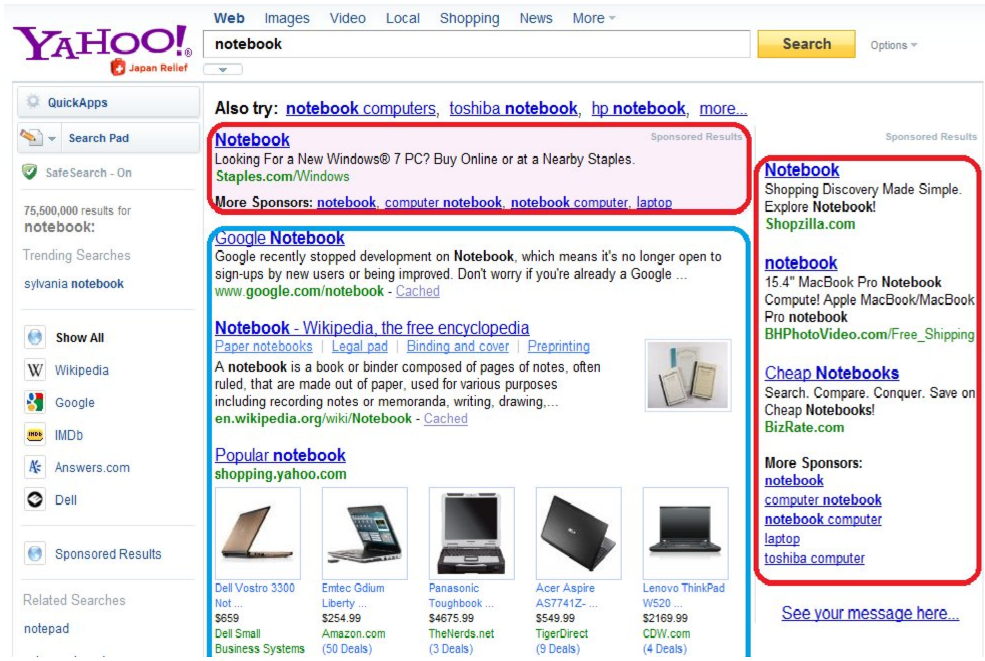


Figure 1. The red box shows sponsored search, while blue box is the standard algorithmic results.

Problem Description

In a nutshell, the problem can be described as follows. A search engine wants to auction off several advertisement slots. Bidders place their bids, then they are assigned to the slots. The search engine then determines a price for each bidder, following the mechanism being used by the search engine. We are interested to find a set of bids that forms an equilibrium.

Let us now describe the details of our assumptions and the mechanism. The search engine has m slots to be auctioned off to n bidders. Without loss of generality, assume $n = m$ (if one is smaller than the other, then add dummy bidders or slots). Each slot has a so-called *Click Through Rate* (*CTR*), which is the estimated number of clicks generated from that slot per period. The length of period is determined by the search engine, it might be in daily or even hourly basis. CTR is common knowledge, which means both the search engine and the bidders know the values. Generally, CTR may depends on both slot position and the bidder who obtained it. We denote CTR of slot i if occupied by bidder j as γ_{ij}. In the literature, CTRs are usually assumed to be bidder independent, which means that it is determined solely by position of the slots. This assumption enables us to order the slots in non-increasing order of CTR, i.e. the topmost slot has the highest CTR, the second highest slot has second highest CTR and so on. Each bidder has a *valuation* of slots v_j, which is the maximum amount of money he would be willing to spend for being assigned to a slot. Our objective is to allocate slots to bidders to *maximize social welfare*, that is, to maximize sum of product of valuation and CTR of all bidders. This objective is chosen since, by doing so, we guarantee that the slots with high CTR are allocated to bidders with high bids. Hence this objective is good for the bidders. The price of slots should be determined too, subject to the bids submitted by the bidders. The design of mechanism for maximizing search engine's revenue is another topic and will not be discussed here.

Sponsored search auction can be described as a maximum weighted matching problem, as can be seen in Figure 2. Intuition of maximum weighted matching. Left nodes are advertisement slots, right nodes are bidders. Weight of arcs are product of CTR and bidder's valuation. Our problem is equivalent to finding maximum matching. Each slot is assigned to exactly one bidder and vice versa, which is exactly a matching problem. This analogy is used to build a suitable mathematical model in chapter 3.

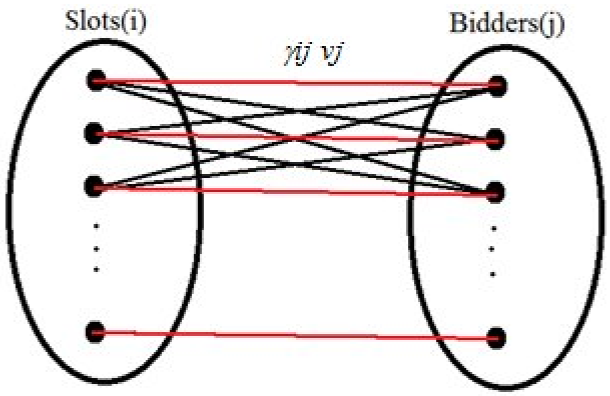


Figure 2. Intuition of maximum weighted matching. Left nodes are advertisement slots, right nodes are bidders. Weight of arcs are product of CTR and bidder's valuation. Our problem is equivalent to finding maximum matching.

2. Theory

2.1 Linear programming, duality, complementary slackness

Linear programming (LP) is an optimization problem in the following form.

$$\text{Maximize} \sum_j c_j x_j$$
$$\text{Subject to} \sum_j a_{ij} x_j \le b_i, \quad \forall i$$
$$x_j \ge 0, \quad \forall j$$

We can also define another problem related to this one, which is formulated as follows.

$$\text{Minimize} \sum_i b_i y_i$$
$$\text{Subject to} \sum_i a_{ij} y_i \ge c_j, \quad \forall j$$
$$y_i \ge 0, \quad \forall i$$

The first problem is called *Primal* LP, while the latter is called *Dual* LP. Their optimal values and solutions are nicely related. We shall see how they are related in these two theorems, which are taken from (Bertsimas & Tsitsiklis, 1997).

Theorem 2.1.1 (Strong duality). If Primal has an optimal solution, then so does Dual. Moreover, their optimal values are equal.

Theorem 2.1.2 (Complementary slackness). Let $\bar{x} = \{x_j\}$ be any Primal's optimal solution and $\bar{y} = \{y_i\}$ be an optimal solution of Dual. Then the following relation holds:

$$(1) \quad x_{j'} > 0 \Rightarrow \sum_i a_{ij'} y_i = c_{j'}$$
$$(2) \quad y_i > 0 \Rightarrow \sum_j a_{ij} x_i = b_i$$

We can interpret (1) as follows: if j-th variable of primal optimal solution is nonzero, then the j-th constraint of dual is tight. The other one, (2), can be interpreted by the same manner.

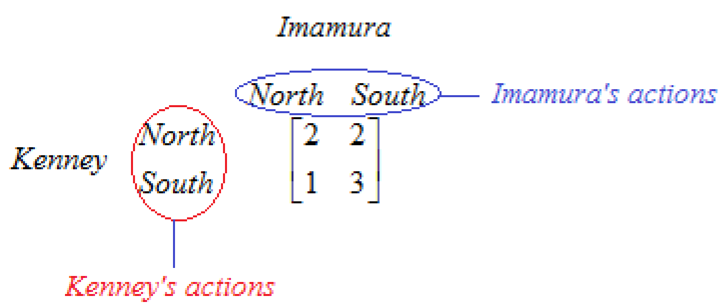
2.2 Game Theory

Game theory is a formal, mathematical discipline which studies situations of competition and cooperation between several involved parties (Peters, 2008). There are wide range of its application, such as economic or social problems of fair distribution to behavior of animals in competitive situations, from parlor games to political voting systems, and still many others. To get more insight about what kind of problems are tackled by Game theory, we present three examples which are taken from (Peters, 2008).

Example 2.2.1 (Battle of Bismarck Sea).

The game is set in the South-Pacific in 1943. The Japanese admiral Imamura has to transport troops across the Bismarck Sea to New Guinea, and the American admiral Kenney wants to bomb the transport. Imamura has two possible choices: a shorter Northern route (2 days) or a larger Southern route (3 days), and Kenney must choose one of these routes to send his planes to. If he chooses the wrong route he can call back the planes and send them to the other route, but the number of bombing days is reduced by 1. We assume that the number of bombing days represents the payoff to Kenney in a positive sense and to Imamura in a negative sense.

Solution. We note that this game has 2 players, namely Kenney and Imamura. Each player has two possible actions: go north or go south, in which every possible combination of actions yields different payoff for both players. The choice of one player is made simultaneously and independent to other player. This battle can be modeled into matrix below.



The entries of matrix represents payoff to Kenney. For example, if both Kenney and Imamura chooses to go to north, then Kenney gains 2 and Imamura gains -2. Furthermore, by choosing this action neither player can gain more payoff by changing his strategy unilaterally: if Kenney changes his direction to south while Imamura still chooses North, Kenney will gain 1, which is less than before. Analogously if Imamura goes to south while Imamura stays intact he still gains -2, which is not better than before. We call such a combination of action as *Nash equilibrium*, which is one of the main solution concepts of game theory. We shall see the formal definition of Nash equilibrium in the next section.

□

Our natural question will be: given a game, do the players can always find a set of strategy that forms a Nash equilibrium? The following example answers this question.

Example 2.2.2 (Matching pennies)

In the two-player game of matching pennies, both players have a coin and simultaneously show heads or tails. If the coins match, player 2 gives his coin to player 1; otherwise, player 1 gives his coin to player 2.

This example can be modeled in this matrix.

$$\begin{array}{cc} & \begin{array}{cc} Head & Tails \end{array} \\ \begin{array}{c} Head \\ Tails \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Solution. We notice that for any combination of actions taken by both players, one can always change his action and gain more: if both players choose Head, player 2 can change to Tails and gain 1, which is better than before. The same condition happens if both choose Tails. If player 1 chooses Tails and player 2 chooses Head, player 1 can change to Head and get better payoff of 1. We can apply similar analysis if player 1 chooses Head and player 2 chooses Tails. Hence, it seems that this example does not have any Nash equilibrium. But we can solve the problem by allowing the players to have probability over actions: let player 1 has probability ½ of choosing Head, and player 2 has probability q for choosing Head. Then the expected payoff of player 1 is

$$(1/2)\left[q\cdot 1+(1-q)(-1)\right]+(1/2)\left[\,q\cdot(-1)+(1-q)\cdot 1\right]=0$$

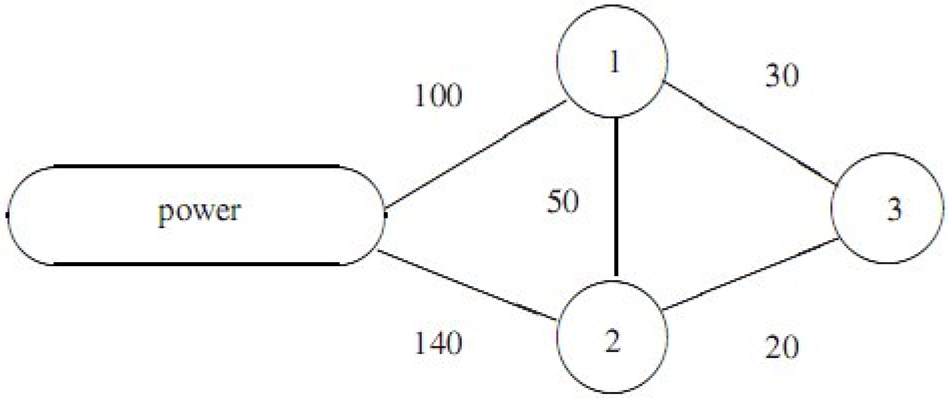
By letting player 2 to choose Head with probability ½ and player 1 chose Head with probability p, one can also confirm that his expected payoff is also 0. It can be seen that value 0 here plays role like Nash equilibrium. Hence choosing head with probability ½ is Nash equilibrium.

□

Due to the possibility for randoming the strategies of both players, we call such strategy as *mixed strategies* (Peters, 2008). The equilibrium induced by mixed strategy is called *Mixed Nash Equilibrium*, where equilibrium without mixed strategy is called *Pure Nash Equilibrium*.

Example 2.2.3 (Three cooperating cities).

Cities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in the following picture. Each city can hire any of the transmission links. If the cities cooperate in hiring the links they save on the hiring costs (the links have unlimited capacity). The situation is represented in figure below.



We can model this situation as a game with three players 1,2, and 3. Denote set of all players as N = {1,2,3}. Players now can build a coalition S, so it can be any subset of N. For every coalition S, denote $v(S)$ as cost saving of that coalition. Hence if $c(S)$ is the cheapest routes connecting all the cities to power source by coalition S, the cost saving from coalition S (denoted by $v(S)$) is formulated by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S), \qquad \text{for each nonempty } S.$$

The following table lists all cost saving for all coalitions.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

Solution. The main question for this game is to find out which coalition should be chosen, and also how to distribute the cost savings to all players. It is usually assumed that the players choose "grand coalition", so $S = N$, and then we find a way to distribute $v(S)$ = 220 to all players. One may attempt to distribute it evenly so each player gets 220/3, but this is considered unfair since one player might contribute more than the others in the coalition. One solution concept for this problem is to seek a coalition such that no player has an incentive to break it. In our case, we try to find a distribution of cost saving (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 220$, $x_1 + x_2 \geq 90$, $x_1 + x_3 \geq 100$, $x_2 + x_3 \geq 120$, and of course $x_1, x_2, x_3 \geq 0$. Such a solution concept is called a *Core*. It is simple to compute, but it consists of many points and hence leaves too many choices of feasible solutions. There are other concepts i.e. Shapley value and Nucleolus, which produces only one solution. We shall not discuss Shapley value and Nucleolus here due to irrelevancy to our topic.

There is a clear difference between example 2.2.1 and example 2.2.3, i.e. the possibility to form a coalition. This difference categorizes game theory into two kinds: *non-cooperative game* and *cooperative game*. In the first case, the players do not form a coalition, the main problem is to find a set of actions or strategies such that under that strategy, no player has an incentive to deviate unilaterally. In the latter case, the players may form a coalition, and the main problem is to choose a suitable coalition and then distribute the cost saving. This clear difference factor does not categorize the game so sharply; there are still connections between both kind of games. The Core solution concept, for example, has a characteristic like Nash equilibrium: we have to find a set of points such that no player has incentive to break the coalition. There are also some examples that employ analysis from both non-coooperative and cooperative game, for example *bargaining problem*. In this problem, two players compete to obtain a portion of one *divisible* good (for example sharing a bottle of wine). Payoff for each players depends on how much portion they get. Here we have to find out how to divide the goods to maximize the payoff for both players. The details of this problem can be seen in (Peters, 2008). Nevertheless, in this thesis we shall discuss more about non-cooperative game.

Formal Definition of a Non-cooperative Game.

In a non-cooperative game, there are several players who choose several actions to get a payoff. Hence there are three ingredients of a game: players, actions, and also payoff. The value of payoff is determined by the actions choosed by the players. We have seen in example 2.2.2 that actions can be randomized, i.e. the players can choose the actions according to a probability distribution. It can be seen that this probability distribution is more general than the mere set of definite actions, hence it is important to use this idea to model a game. We shall call probability distribution over all actions of a certain player as *strategy* of the player.

Summarizing, a non-cooperative game consists of set of players N, set of strategies S_i, and also set of payoffs u_i. Hence we can make formal definition of non-cooperative game below.

Definition 2.2.1. A non-cooperative game is a $2n + 1$ tuple $G = \{N, S_1, \ldots, S_n, u_1, \ldots, u_n\}$, where:

- $N = \{1, \ldots, n\}$ is the set of players.
- For every $i \in N$, S_i is the strategy set of player i.
- For every $i \in N$, $u_i : S_1 \times \cdots \times S_n \to R$ is the payoff function to player i. Given that for every j, player j plays strategy $s_j \in S_j$, then player i will get payoff $u_i(s_1, \ldots, s_i, \ldots, s_n)$. In what follows it would often be convenient to use the notation (s_i, s_{-i}) for $(s_1, \ldots, s_n)$.

Now let us define formally what a strategy is. In a game, every player can choose among, say m choices of actions. Without loss of generality, we can assume that every player has the same number of possible actions. If one player has less actions than other, we can define dummy actions whose payoff is $-\infty$. Based on example 2.2.1 and example 2.2.2, this choice maybe definitive or random. Hence a strategy is a probability distribution on the set of actions of a player. We now can define a strategy formally.

Definition 2.2.2. A strategy s_i of player i is an m-tuple $s_i = (p_1^i, \ldots, p_m^i)$, where $\sum_{j=1}^{m} p_j^i = 1$ and $p_j^i \in [0,1]$ for every $j \in \{1, \ldots, m\}$.

There are two kinds of strategies based on the possibility of doing randomization over the actions. Strategy with definite choice of action is called *pure strategy*, while the one with randomized choice is called *mixed strategy*. We state each of them formally.

Definition 2.2.3. A strategy s_i is called:

- *Pure strategy*, if there exists j such that $p_j^i = 1$.
- *Mixed strategy*, if it is not pure strategy.

Formal Definition of Nash Equilibrium.

A strategy profile $(s_1, \ldots, s_n)$ where $s_i \in S_i$ for every $i \in N$ is called *Nash equilibrium* if and only if for every $i \in N$ and for every other strategy $s'_i \in S_i$, this condition holds:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

In words, a set of strategies is a Nash equilibrium if and only if no player can increase his payoff by changing his strategy unilaterally. An equilibrium $(s_1, \ldots, s_n)$ is called *Pure Nash Equilibrium* if for every player *i*, s_i is pure strategy. The term *Mixed Nash Equilibrium* is used if not all of $\{s_1, \ldots, s_n\}$ are pure strategy. In other words, a Nash equilibrium is pure if and only if every player has a definite choice of action.

We have seen that there is a Pure Nash Equilibrium in example 2.4.1, but this is not the case for example 2.4.2. By allowing mixed strategies, we can find a Mixed Nash Equilibrium for that example. The definition of mixed strategies turns out to be sufficient to guarantee the existence of Nash Equilibrium for so-called finite games, i.e., games with finitely many players and actions (Peters, 2008).

2.3 Auction Theory

An auction is one of many ways to sell items. In ordinary selling, the price of items is determined solely by the sellers, but in an auction both sellers and buyers determine the selling price. Being used since antiquity, auction turns out to generate higher prices, and hence becomes more and more popular. There are many kinds of goods which are sold by auctions. Some examples are art, real estate, publicly owned-assets, government's treasury bills, and lately internet advertisements

(Tian, 2010). It can be seen that there are a wide variety of items being sold using auctions, which gives us enough motivation to study it.

We categorize auctions into two kinds based on the number of items being sold: *single-item auction* and *multiple-item auction*. In this section we focus on the (simpler) case of single item auction, while the multiple one is treated in the next section, in the context with Sponsored Search Auctions. We first provide some examples, taken from (Tian, 2010), to get a rough idea about an auction. From now on we shall use the word *bidders* in referring to buyers, since the buyers actually place a bid for winning the item.

Example 2.3.1. One simple example of an auction is the so-called *English auction*. In this auction, the seller sells one item to many potential bidders. The seller mentions a starting price (low enough), and then watches how many bidders are interested to buy the item at the current price. The seller then raises it by a small increment, so as to reduce the number of interested bidders. This process continues until there is only one interested bidder. The winning bidder will get the item and pay the latest announced price.

Example 2.3.2. Another example is the so-called *Dutch auction*. In contrast with the English auction, the seller now sets the starting price as high as possible such that there are no bidders interested to buy the item at that price. The seller then decreases the price by a small decrement until there is only one bidder who is interested at the current price. The winning bidder will get the item and also have to pay the latest announced price.

In these two auctions, every bidder knows the bids of all other bidders since they announce their bid publicly. There are also other varieties of auction mechanisms in which the bid of each bidder is private information. We provide two examples which are very closely related.

Example 2.3.3 (Sealed-Bid First Price Auction). This auction asks each bidder to submit a sealed-bid, then award the item to bidder with highest bid. The price he has to pay is the highest bid.

Example 2.3.4 (Sealed-Bid Second Price Auction). This auction asks each bidder to submit a sealed-bid, then award the item to bidder with highest bid. The price he should pay is not the highest bid, but the second highest bid.

We shall call these two examples simply by *First Price Auction* and *Second Price Auction* for the sake of simplicity. These examples are the most common examples in auction theory, and hence it is important to know their properties. We shall discuss this later.

We can briefly summarize what happens in an auction. Generally the seller will fix a certain price for the item. The buyers then offer a better price to win the item. In other words, there is a “price competition” among buyers: each of them tries to give their best price, and then, according to certain rules, the seller will award the item to one of the buyers.

There are three ingredients for running an auction:

1. Price discovery. As seen in example 2.3.1, the seller may fixes a price. This is only an estimated price. The existence of bids determines the true price for the item, in other words we might be able to know how much the item worth for the bidders. Hence, we can view auction as a market test for valuing the item.
2. Item allocation. After all bidders placing the bids, the seller has to determine the winning bidder and give him the item.
3. Payment rule. The winning bidder has to pay a certain amount of money, which is determined by the auction mechanism.

It can be seen that item allocation and payment rule are two main things that must be determined within an auction, since price discovery is just an implication from these two ingredients.

We have seen that auction involves two parties: seller and bidder. Now how do we define our goal in conducting an auction? The obvious goal is to maximize the money earned by the seller. On the other hand, it is also important to allocate the item to the bidder who values the item most, since by doing so we guarantee that the item is given into the right hand. For example, the government wants to sell art works such as historical painting or sculpture. In this case, it would be better to award it to a bidder whom we can really trust rather than to a bidder with highest offered price, in order to preserve its quality. Hence, we can also allocate the item to the bidder who has the highest valuation for the item. Another possible goal is to combine both of them: maximize total revenue and valuation of both seller and bidders. Hence, there are at least three possible goals in conducting an auctions:

1. Revenue maximization: maximize seller's revenue regardless to whom the seller awards the item.
2. Social welfare maximization: maximize total valuation of winning bidders. This guarantees that the item goes to bidder who values the item most.
3. Total revenue maximization or social welfare maximization: maximize revenue of both seller and bidders.

Summarizing, an auction is a competition between *sellers* and *bidders*. Each bidder places a *bid* to win the item, while the seller has the authority to *allocate* the item and also *charge* the winning bidder. The way the seller allocate and charge the bidder is determined by the *mechanism*. Item allocation and price determination is chosen to fulfill one of three goals: maximize seller's revenue, maximize bidders' total valuation, or maximize seller's revenue and bidders' valuation combined.

We now try to discuss properties of two common auction mechanisms: First price and Second Price Auction. In analyzing them, we assume that the bidders are ordered in descending order of valuation. It means that if we denote the valuation of bidder j as v_j, then we assume $v_1 \geq v_2 \geq \ldots \geq v_m$.

First Price Auction

As mentioned, in this mechanism all bidders are asked to submit a sealed bid, and then the item will be awarded to the highest bidder at a price equal to his bid. Now let us try to analyze this auction in game theoretic-manner. First Price Auction is a non-cooperative game $G = \{N, B_1, \ldots, B_n, u_1, \ldots, u_n\}$ with:

- $N = \{1, \ldots, n\}$ is set of bidders.
- $B_i = [0, \infty)$ is set of possible bids of player i. A particular bid of player i is denoted by b_i.
- $u_i(b)$ is the payoff for player i under bid b. Thus $u_i(b) = v_i - b_i$ if bidder i submits the highest bid, and $u_i(b) = 0$ otherwise since there is only one winning bidder. In case of a tie, we choose the smallest index i such that $b_i = \max_j b_j$.

Done with formal definition, now let us continue to look for equilibrium properties of this mechanism. We start by proving its existence, and then derive one interesting property.

Proposition 2.3.1 (existence of equilibrium on First Price Auction). There always exists an equilibrium in First Price Auction.

Proof. Let n be the number of bidders. Arrange the bidders such that $v_1 \geq v_2 \geq \cdots \geq v_n$. We shall show that the set of bids $b^* = (v_2, v_2, 0, \ldots, 0)$ is an equilibrium. The proof is as follows. Under b^*, payoff of bidder 1 is $v_1 - v_2 \geq 0$, and other bidder's payoff is zero. If bidder 1 deviates by bidding lower than v_2, he will lose the item and hence the resulting payoff is zero. Raising his bid will make him pay higher and worsens his payoff. Move to bidder $j \neq 1$, raising his bid to more than v_2 makes him win the item, but he has to pay more than his valuation and hence he has negative payoff. Lowering the bid will not change anything, since he still loses the item. By this argument, we have just proved that $b^* = (v_2, v_2, 0, \ldots, 0)$ is indeed an equilibrium.

□

Proposition 2.3.2 (cf., e.g. (Tian, 2010)). Bidder 1 obtains the item in all Nash equilibria of First Price Auction.

Proof. Let $b^* = (b^*_1, \ldots, b^*_n)$ be any Nash equilibrium, assume $b^*_1 \geq \ldots \geq b^*_n$. Assume the contrary that the item is awarded to bidder $j \neq 1$. Of course bidder j does not bid above his valuation, otherwise he would get negative payoff. Hence we have $0 \leq b^*_j \leq v_j$. Now let bidder 1 deviate by bidding $\tilde{b}^*_1 = b^*_j + \varepsilon$, where ε is chosen such that $b^*_j + \varepsilon < v_1$. Then we have

$$u_1(\tilde{b}^*_1, \tilde{b}^*_{-1}) = v_1 - \tilde{b}^*_1 > 0 = u_1(b^*)$$

This contradicts the hypothesis that b^* is Nash equilibrium.

∎

These are two nice propositions for First Price Auction. The existence of equilibrium implies that we can predict how much the bidders would place their bids. Proposition 2.3.2 describes the allocation of item under any equilibrium: no matter which equilibrium the bidders are using, the item would always be given to the bidder with highest valuation. Hence, it can be said that any equilibrium of First Price Auction maximizes social welfare.

Second Price Auction

We can analyze this mechanism in a similar way with First Price Auction. In game theory, Second Price Auction can be formally defined as a non-cooperative game $G = \{N, B_1, \ldots, B_n, u_1, \ldots, u_n\}$ as follows.

- $N = \{1, \ldots, n\}$ is set of bidders.
- $B_i = [0, \infty)$ is set of possible bid of player i. A particular bid of player i is denoted by b_i.
- $u_i(b)$ is payoff for player i under bid b. Analogous with First Price Auction, $u_i(b) = v_i - \max_{j \neq i} b_j$ if i is the lowest index such that $b_i = \max_j b_j$, and $u_i(b) = 0$ otherwise.

We again cite one proposition about equilibrium, which says that telling bidders' true valuation is an equilibrium under Second Price Auction. We use the term *bidding truthfully* to refer to the condition that each bidder tells his true valuation.

Proposition 2.3.3 (cf., e.g. (Tian, 2010)). In a Second Price Auction, bidding truthfully is an equilibrium.

Proof. Let n be the number of bidders. Arrange the bidders such that $v_1 \geq v_2 \geq \cdots \geq v_n$. Take truthful bids $(v_1, v_2, \cdots, v_n)$. Under this bid, $u_1(b) = v_1 - v_2 \geq 0$ and $u_i(b) = 0$ for every $i > 1$. We split the proof by analyzing each bidders separately: the first is bidder 1, the second is other bidder.

Case 1: bidder 1. Since bidder 1 is the winning bidder, deviating his bid would not increase his payoff.

Case 2: bidder $i > 1$. Assume bidder i wants to deviate by changing his bid to $\tilde{b}_i$. To increase the payoff, he has to bid more than the current bid of bidder 1, or $\tilde{b}_i > v_1$. Under this bid, his payoff will be $v_i - v_1 \leq 0$.

With the consideration of case 1 and case 2, we conclude that truthful bidding is an equilibrium.

∎

In First Price Auction, bidder 1 always gets the item in all equilibrium. This is no longer true in Second Price Auction, as the following proposition shows.

Proposition 2.3.4 (cf., e.g. (Tian, 2010)). In a Second Price Auction, there is a Nash equilibrium in which the winner is not bidder 1.

Proof. Let the bidders be arranged in non-increasing valuation. We construct the bids $b^* = (b^*_1,...,b^*_n)$ satisfying certain properties. Fix one bidder j, then let $b^*_j > v_1$, $b^*_1 < v_j$, and $b^*_i = 0$ for every $i \notin \{1, j\}$. It implies that the item is awarded to bidder j. We now prove that this is an equilibrium.

Case 1: bidder 1. Let bidder 1 deviate by changing his bid to $\widetilde{b}_1$. To gain more payoff, he has to obtain the item, so $\widetilde{b}_1 > b^*_j$. Under bid $\widetilde{b}_1$, his payoff is $v_1 - b^*_j < v_1 - v_1 = 0$, which is not better than previous payoff.

Case 2: bidder ***j***. Bidder j wins the item, his payoff is $u_j = v_j - b^*_1 > v_j - v_j = 0$. By raising the bid he is still the winner and gains no better payoff. By decreasing the bid, if he is still winner then his gain remains unchanged, and if he loses the item then his payoff is zero, which is lower than before.

Case 3: bidder $i \notin \{1, j\}$. Let bidder i deviates by changing his bid to $\widetilde{b}_i$. To gain more payoff, he has to get the item, so $\widetilde{b}_i > b^*_j$. But $b^*_j > v_1 > v_i$, which means that eventhough bidder i wins the item by bidding $\widetilde{b}_i$, he has to pay more that his valuation, resulting in negative payoff.

From 3 cases we conclude that bid $b^* = (b^*_1,...,b^*_n)$ is indeed an equilibrium.

∎

Proposition 2.3.4 tells us one weakness of Second Price Auction: not all equilibrium guarantees that the item will be given to the bidder with the highest valuation, and hence this mechanism in general does not maximize social welfare. Trivially, the equilibrium does maximize social welfare if all bidders bid truthfully. It can be inferred too from the proof of this proposition that bidding more than the valuation might be beneficial for the bidder. At this point it is tempting to say that First Price Auction has more interesting properties than Second Price, but we shall see later than the addition of multiple item to be auctioned off changes the properties of First Price Auction drastically.

Mechanism

We recall that in an auction, there are two major ingredients, i.e. item allocation and price determination. Four examples have been given: English, Dutch, First Price, and Second Price Auction. It can be seen that each kind of auction has its own way in allocating the item and determining the price. In the first three examples, the item is awarded to the bidder with the highest bid and then charge him the highest current price. In the Second Price Auction, bidder with the highest bid still obtains the item but he pays the second highest price. It can be seen that there are many possibilities to define the auction mechanism. A mechanism is constructed to reach a certain goal. For example, if we allocate the item to highest bid, then it is expected that the seller's revenue is maximized; if the item is allocated to the bidder who values the item most, then we maximize social welfare.

Definition 2.3.1 (Tian, 2010). *A mechanism* is a collection of three sets (B, Π, μ), where:

- $\mathrm{B} = \{ B_1, \ldots, B_n \}$, B_i is set of possible bids for bidder i. We can view the bids as a function of valuation, i.e. $b_i = \mathrm{B}_i(v_i)$.
- $\Pi : \mathrm{B} \to [0,1]^n$ is an allocation rule, where for any $b \in \mathrm{B}$, $(\Pi_1(b), \ldots, \Pi_n(b)) \in [0,1]^n$ satisfies the property that $\sum_i \Pi_i(b) = 1$ and $\Pi_i(b)$ denotes the probability that bidder i obtains the item under bids b.
- $\mu : \mathrm{B} \to R^n$ is payment rule, which determines how much the winning bidder has to pay for the item. The payment rule is usually subject to the allocation rule.

Using this definition, we now can describe the allocation and payment rule in First Price Auction and Second Price Auction formally in table below. Recall the tie-breaking rule that we award the item to smallest i such that $b_i = \max_j b_j$.

	First Price Auction	Second Price Auction
Allocation rule	$\Pi_i^I(b) = \begin{cases} 1 & ,b_i = \max_j \{b_j\} \\ 0 & ,b_i < \max_j \{b_j\} \end{cases}$	$\Pi_i^{II}(b) = \begin{cases} 1 & ,b_i = \max_j \{b_j\} \\ 0 & ,b_i < \max_j \{b_j\} \end{cases}$
Payment rule	$\mu_i^I(b) = \begin{cases} b_i & ,b_i = \max_j \{b_j\} \\ 0 & ,b_i < \max_j \{b_j\} \end{cases}$	$\mu_i^{II}(b) = \begin{cases} b_{i+1} & ,b_{i+1} = \max_{j \neq i} \{b_j\} \\ 0 & ,b_i < \max_{j \neq 1} \{b_j\} \end{cases}$

Table 2.3.1. Formulation of First Price and Second Price Auction.

Furthermore, we see that in Second Price Auction, bidding truthfully is an equilibrium. This is interesting, since by bidding truthfully we may know the bidders' true valuation and then we can allocate the item efficiently; this is one of three goals within an auction. Of course, having a mechanism in which bidding truthfully is an equilibrium brings us advantage. We state two definitions related to this situation.

Definition 2.3.2 (Tian, 2010). A mechanism (V, Q, M) is called *direct* if it directly asks each bidder to report their true valuations, which means that $V = \{\{v_1\}, \ldots, \{v_n\}\}$.

Definition 2.3.3 (Tian, 2010). A direct mechanism is *incentive compatible* if for every $i \in N$ and for every other reported valuation v'_i of player i, we have $u_i(v_i, v_{-i}) \geq u_i(v'_i, v_{-i})$.

If bidding truthfully is an equilibrium, then the mechanism is said to have a *truthful equilibrium*. An incentive compatible mechanism means that for each bidder, the payoff by bidding truthfully is at least as good as payoff by bidding other bid.

At first it seems that the idea of direct mechanism seems a bit absurd, but actually it helps us to analyze general mechanism. There is a nice relation between "general" mechanism and direct mechanism as the following proposition shows.

Proposition 2.3.5. Given a mechanism and an equilibrium of that mechanism, there exists a direct mechanism in which (1) it is an equilibrium for each bidder to report his true valuation, (2) the truthful equilibrium outcomes (i.e. the allocation and payment) are the same as in the given equilibrium of the original mechanism.

Proof. Take an arbitrary mechanism (B, Π, μ), and let β be an equilibrium of that mechanism. Define a direct mechanism (V, Q, M) which $Q(v) = \Pi(\beta(v))$ and $M(v) = \mu(\beta(v))$.

(1) Assume bidder i can make a profitable deviation (within direct mechanism (V, Q, M)) by reporting a fake valuation $z_i \neq v_i$. This means in original mechanism, he can increase his payoff by bidding $\beta_i(z_i)$, which contradicts the hypothesis that $\beta_i(v_i)$ is an equilibrium.
(2) By definition of $Q(v) = \Pi(\beta(v))$ and $M(v) = \mu(\beta(v))$, it can be seen that the outcomes from direct mechanism corresponds to those from original mechanism.

□

This proposition shows that we only need to analyze direct mechanisms, since outcomes from a direct mechanism can be translated back into outcomes from a particular mechanism.

Having a direct, incentive compatible mechanism is very good, since by doing so the seller can expect to be reported the true valuation of all bidders, even if he does not ask them to provide it. We have seen two examples: first-price and second-price auction. Our natural question will be: do there exist other direct, incentive compatible mechanisms? The answer is yes, and that is a mechanism called *Vickrey-Clarke-Groves* (VCG).

Vickrey-Clarke-Groves mechanism (VCG)

This mechanism is constructed basically with one main purpose: encouraging all bidders to bid truthfully. We have seen that Second-Price Auction is a direct mechanism and also incentive compatible. VCG is actually a generalization of Second-Price Auction; it is constructed by analyzing the impact of the presence of winning bidder to all other bidders. Let us consider Second-Price Auction of one item with multiple bidders. Bidder 1 whose valuation is v_1 wins the item, while other bidders gets nothing. If bidder 1 were not there, then the item would go to bidder 2, who values the item at v_2. The other bidders still get nothing. So by the absence of bidder 1, bidder 2 to n cumulatively will have an increase of $v_2 + 0 + \ldots + 0 = v_2$. This is exactly the price the bidder 1 has to pay. In other words, each bidder pays the harm caused by his presence to the other bidders.

We now try to define VCG as a mechanism formally. By Proposition 2.3.5, in our discussion we always assume that any mechanism (B, Π, μ) is direct. First let us state one definition.

Definition 2.3.4 (Tian, 2010). An allocation rule $Q^*: \mathrm{B} \to [0,1]^n$ is called *efficient* if it maximizes social welfare, that is, for all $v \in \mathrm{B}$, $Q^*(v) \in \arg\max_{Q} \sum_{j \in N} Q_j v_j$. A mechanism with efficient allocation rule is called *efficient mechanism*.

We have seen an auction with efficient allocation, i.e. single item First-Price Auction. Furthermore if Q^* is an efficient allocation and v is the valuation of all bidders, we denote the maximum value of social welfare as $W(v) := \sum_{j \in N} Q^*_j(v) v_j$. Similarly, we define welfare of all agents other than i as $W_{-i}(v) := \sum_{j \neq i} Q^*_j(v) v_j$. We are now ready to describe VCG formally.

Definition 2.3.5 (Tian, 2010). *Vickrey-Clarke-Groves* (VCG) mechanism (B, Q^{VCG}, M^{VCG}) is an efficient mechanism with payment rule $M_i^{VCG}(v) = W(0, v_{-i}) - W_{-i}(v)$.

$M_i^{VCG}(v)$ is thus the difference between social welfare when bidder i bids zero and the welfare of other bidders under truthful bids v; assuming that in both cases we are employing efficient allocation Q^{VCG}.

It can be seen that in case of single item auction, VCG is equivalent with Second-Price Auction. The explanation is as follows. Consider bidder 1, whose valuation is the highest among all bidders. Since VCG maximizes social welfare, he would be the winning bidder. $W(0, v_{-1})$ is total social welfare under condition that bidder 1 bids 0, hence under this bid the item goes to bidder 2 and the total social welfare is $W(0, v_{-1}) = v_2$. Next, $W_{-1}(v)$ is total social welfare other than bidder 1 under truthful bids. Since all bidders bid truthfully, the item again goes to bidder 1 and hence total social welfare other than bidder 1 is $W_{-1}(v) = 0$. The price for bidder 1 is then

$M_1^{VCG}(v) = W(0, v_{-1}) - W_{-1}(v) = v_2$. Now consider bidder $i > 1$. Using similar analysis, we get $M_i^{VCG}(v) = W(0, v_{-i}) - W_{-i}(v) = v_1 - v_1 = 0$. This transforms the VCG into Second Price Auction.

We now prove that VCG is incentive compatible mechanism.

Proposition 2.3.6 (cf. e.g. (Tian, 2010)). VCG is incentive compatible.

Proof. Take any bidder i. Let him bid b_i, and let v_{-i} be the valuations of the other bidders. The payoff of bidder i is

$$
\begin{aligned}
&Q_i^{VCG}(b_i, v_{-i})v_i - M_i^{VCG}(b_i, v_{-i}) \\
&= Q_i^{VCG}(b_i, v_{-i})v_i - W(0, v_{-i}) + W_{-i}(b_i, v_{-i}) \\
&= Q_i^{VCG}(b_i, v_{-i})v_i - W(0, v_{-i}) + \sum_{j \neq i} Q_j^{VCG}(b_i, v_{-i})v_j \\
&= \sum_{j \in N} Q_j^{VCG}(b_i, v_{-i})v_j - W(0, v_{-i})
\end{aligned}
$$

The definition of Q_j^{VCG} implies that for all v_{-i}, the first term is maximized by choosing $b_i = v_i$, and since the second terms does not depend on v_i, it is optimal to bid $b_i = v_i$. Thus, equilibrium payoff of bidder i when the values are v is

$$Q_i^{VCG}(v)v_i - M_i^{VCG}(v) = W(v) - W(0, v_{-i})$$

which is just the difference in social welfare induced by i when he bids his true valuation v_i as opposed to his zero bid.

□

2.4 Sponsored Search Auctions

As mentioned in Introduction, sponsored search returns several search results to search users. (Aggarwal & Muthukrishnan, 2008) states that in general there are three involved players in sponsored search auctions:

- Advertisers, who want to place their website on search results.
- The auctioneer, the search engine such as Google, Yahoo, or MSN who conduct the auction.
- Search user, who use the search engine.

In our discussion, we shall focus on models using only the first two players: advertisers and auctioneer. By employing the knowledge in game theory and auction theory, sponsored search can be analyzed in a more formal way. Firstly, let us redefine the bidders. It is easy to see that the auctioneer is the one who builds the auction mechanism. The advertisers are those who place the bids. Hence, we can view the advertisers as the bidders. From now on, we shall use the word *bidders* to refer to the advertisers and *search engine* for referring the auctioneer. The items being sold are the search result's *slots*. Secondly, we redefine the two possible goals of auction. In context of sponsored search auctions, the goal would be either maximize search engine's revenue or maximize bidders' total valuation.

We adopt the notations from auction theory: denote b_j as bid of bidder j and v_j as valuation of bidder j. Throughout this discussion, it is assumed that valuations are non-increasing, i.e. $v_1 \geq \cdots \geq v_n$.

Nevertheless, there is a difference for the items being sold. A search engine may auctions off more than one advertisement slot, which means that this is an auction for multiple items. Hence finding out whether the properties of single-item auction also hold for multiple items would be an appropriate first step.

Three auction mechanisms for single item has been discussed: first-price, second-price, and VCG. These mechanisms can be used for analyzing multi-item auctions by making some generalization.

Generalized First Price (GFP)

It is very straightforward to generalize the idea of First Price Auction for auction with multiple items. As written in Problem Description, without loss of generality it is assumed that number of slots is equal to the number of bidders. In sponsored search auctions, Generalized First Price (GFP) pools all bids, ranks them in non-increasing order of bids, then awards the i-th highest CTR slot to the bidder with i-th highest bid for every slot i. The winning bidder is charged a price equal to his bid. The total valuation of bidder j for being assigned to slot i is $\mu_i v_j$. This simple mechanism was used by Yahoo prior to 2004 (Lahaie, Pennock, Saberi, & Vohra, 2007). Unlike in single-item First Price auction, this mechanism does not have a Pure Nash equilibrium. We state this as a theorem.

Theorem 2.4.1, modified of (Lahaie, An Analysis of Alternative Slot Auction Designs for Sponsored Search, 2006). There is no Pure Nash Equilibrium under GFP unless $\mu_1 = \ldots = \mu_n$.

Proof. Let $b = (b_1, \ldots, b_n)$ be a Pure Nash Equilibrium. Arrange the bidders such that $v_1 \geq \cdots \geq v_n$, and we use tie-breaking rule as usual: allocate the item to the bidder having lowest index. Note that $b_i > b_{i+1}$ is impossible, otherwise bidder i can make a profitable deviation by instead bidding $b_i - \varepsilon > b_{i+1}$ for small enough $\varepsilon > 0$. This does not change the slots' allocation, but increase his profit by $\mu_i \varepsilon$. Hence for every i we must have $b_i = b_{i+1}$, or we can simply write the equilibrium as $b = (b_n, \ldots, b_n)$. Furthermore, $b_n = 0$, otherwise the last bidder can deviate by bidding $b_n - \varepsilon$ and hence increase his profit by $\mu_n \varepsilon$ without changing his allocation.

Now take any bidder j. Under b, he obtains slot j and gains $(v_j - 0)\mu_j$. Bidder j can deviate by bidding ε, hence obtaining slot 1 at price ε and gains $(v_j - \varepsilon)\mu_1$. This deviation is profitable if and only if $(v_j - \varepsilon)\mu_1 > v_j \mu_j$, or equivalently $\varepsilon < v_j(\mu_1 - \mu_j)/\mu_1$. ε must be positive, hence we can find such ε if and only if $\mu_1 - \mu_j \neq 0$ for every j. It means that there is a Pure Nash Equilibrium if and only if $\mu_1 = \ldots = \mu_n$.

□

Generalized Second Price (GSP)

As in GFP, we can also generalize the idea of second-price auction for multiple objects easily. Originally developed by Google (Easley & Kleinberg, 2010), this mechanism is used widely by search engines, after several trials of adopting other mechanisms.

The mechanism of Generalized Second Price (GSP) in sponsored search auctions works as follows.

1. Ask bidders to announce their bids.
2. Sort the bids in non-increasing order, then for every slot i, award slot with i-th highest CTR to bidder with i-th highest bid.
3. Charge bidder j with $(j+1)$-th highest bid.

In GSP, bids are interpreted as *price per click* the bidder is willing to pay. It is up to the bidders whether they bid their true valuation or not. The price charged to the winning bidders in step 3 also means price per click. In our case, bidder j will pay $\mu_j b_{j+1}$ in total (in expectation) for being assigned to slot j.

We now proceed to equilibrium analysis. It has been known that for case of single item, second-price auction has truthful equilibrium. Unfortunately, this is not the case for GSP, as the following example from (Easley & Kleinberg, 2010) shows.

Example 2.4.1. Given three slots with CTR 10, 4, and 0 respectively and 3 bidders with valuation 7, 6, and 1.

CTR	*Slots*	*Bidders*	*valuations*	*b_1*	*b_2*	*b_3*
10	●	*x* ●	*7*	*7*	*5*	*3*
4	●	*y* ●	*6*	*6*	*4*	*5*
0	●	*z* ●	*1*	*1*	*2*	*1*

Consider first the truthful bids b_1: (7, 6, 1). Under these bids, bidder x gets first slot, y gets second one, and z gets last slot. Under GSP, each player has to pay per click the bid just below him. Recall that payoff of a bidder is defined by the difference between total valuation and price of slot. Hence under bids b_1,

- Bidder x gains $10(7-6)=10$,
- Bidder y gains $4(6-1)=20$,
- Bidder z gains $0(1-0)=0$.

Now we show that this bid is not equilibrium. If bidder x changes his bid to 5, then he will obtain second slot with payoff $4(7-1)=24$. This payoff is better than before, so for this example bidding truthfully is not a Nash equilibrium.

This example has an equilibrium, though. Consider bids *b_2* (5,4,2). Under this bid x gains $(7-4)10 = 30$, y gains $(6-2)4 = 16$, and z gains $(1-0)0 = 0$. First note that nobody would prefer obtaining the third slot since it has zero CTR. If x were in the second slot, he would gain $(7-2)4 = 20$, which is worse. If y obtains slot 1 he will gain $(6-5)10 = 10$, which is worse. Bidder z would not want to obtain higher slot since to do so he has to pay more than his valuation. We conclude then that the set of bids *b_2* is equilibrium; moreover it is also socially optimal, in which the total valuation of all bidders is $70 + 24 + 0 = 98$.

Now consider the last set of bids *b_3* (3, 5, 1). First notice that y now gets the first slot, replacing x. Under this set of bids x gets $28 - 4 = 24$, y gains $60 - 30 = 30$, and z still gets zero gain. If y wants to deviate, he gets worse: 20. Raising the bid for x gives him gain $70 - 50 = 20$, while lowering his bid lead him to get zero gain. It is obvious result for z. This shows that bids (3,5,1) also forms an equilibrium, although it is not socially optimal.

Example 2.4.1 shows that there are multiple equilibria under GSP mechanism. We also remark that an equilibrium might not be socially optimal. Nevertheless, it can be shown that among all GSP's equilibria we can find at least one that is socially optimal (Easley & Kleinberg, 2010). The details will be discussed in chapter 3.

VCG

The Vickrey-Clarke-Groves (VCG) mechanism is a generalization of single item second-price auction (Easley & Kleinberg, 2010). It has been shown that GSP does not preserve the truthful bidding property. There is still one auction mechanism left, i.e. VCG. As shown, VCG is truthful mechanism. Our main question would be to find out how do the VCG allocation and payment rule look alike when being applied to multiple-item auction. The VCG allocation in the context of sponsored search auction is clear: the items are allocated to maximize sum of valuations of all bidders.

VCG payment is defined as follows. Recall the VCG payment formula $M_j^{VCG}(v) = W(0, v_{-j}) - W_{-j}(v)$. Here $W(0, v_{-j})$ means maximum allocation without the existence of bidder j (since bidder j bids zero), while $W_{-j}(v)$ means maximum allocation without the presence of bidder j and also the slot being awarded to him.

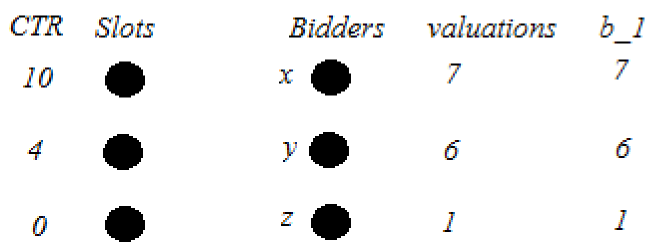
Now we try to describe the formula in terms of allocation. Let V_B^S denotes the value of a maximum allocation given that the set of slots is S and the set of existing bidders is B. Suppose VCG assigns slot i to bidder j, then VCG price of slot i is $V_{B-\{j\}}^{S} - V_{B-\{j\}}^{S-\{i\}}$. In words, the price of slot i is the decrease in maximum allocation if slot i is removed. Unlike GSP, the established price in VCG is *price per slot*.

We can summarize the VCG mechanism for sponsored search auction as follows (Easley & Kleinberg, 2010):

1. Ask all bidders to announce their valuation (not necessarily truthful).
2. Assign slots to bidders to get socially optimal allocation.
3. Compute the price of slot by formula $V_{B-\{j\}}^{S} - V_{B-\{j\}}^{S-\{i\}}$ for every slot i that is assigned to bidder j.

Now let us try to use VCG to compute allocation and price of slot for each bidder.

Example 2.4.1 revisited.



Let all bidders bid their true valuation (7, 6, 1). Step 2 of VCG mechanism allocates them to the slots to maximize social welfare, i.e. to maximize total valuation of all bidders. The table below lists all possible allocations.

Allocation	Total valuation
1 → x, 2 → y, 3 → z	10 (7) + 4 (6) + 0 (1) = 94
1 → x, 3 → y, 2 → z	10 (7) + 0 (6) + 4 (1) = 74
2 → x, 1 → y, 3 → z	4 (7) + 10 (6) + 0 (1) = 88
2 → x, 3 → y, 1 → z	4 (7) + 0 (6) + 10 (1) = 38
3 → x, 1 → y, 2 → z	0 (7) + 10 (6) + 4 (1) = 64
3 → x, 2 → y, 1 → z	0 (7) + 4 (6) + 10 (1) = 34

It can be seen that total valuation is maximized if we allocate first slot to bidder x, second slot to bidder y, and final slot to bidder z, with total valuation is 94.

Step 3 of VCG computes the price of each slot. By applying the formula, we get these results.

- Price of slot 1: $V_{B-\{x\}}^{S} - V_{B-\{x\}}^{S-\{1\}} = 10(6) + 4(1) - 4(6) - 0(1) = 40$.
- Price of slot 2: $V_{B-\{y\}}^{S} - V_{B-\{y\}}^{S-\{2\}} = 10(7) + 4(1) - 10(7) - 0(1) = 4$.

- Price of slot 3: $V_{B-\{z\}}^{S} - V_{B-\{z\}}^{S-\{3\}} = 10(7) + 4(1) - 10(7) - 4(1) = 0$.

Finally, we can determine the payoff for each bidder.

- Payoff of bidder x: $70 - 40 = 30$.
- Payoff of bidder y: $24 - 4 = 20$.
- Payoff of bidder z: $0 - 0 = 0$.

⍰

As stated in Auction Theory, VCG encourages each bidder to bid truthfully. Other than truthful bidding, an equilibrium can be constructed by fixing a reserved price, which means that the search engine tells all bidders to bid above a certain value. Complete characterization of equilibrium under VCG can be seen in (Blume, Heidhues, Lafky, Munster, & Zhang, 2008).

2.5 Assignment Problem

One of the main problems of auctioning multiple items is to determine the items' allocation to the bidders. In our discussion, we have to find a socially optimal allocation. The study about allocating the items to the bidders is actually special case of the so-called assignment problem. Here are several examples of assignment problems:

1. Suppose there are two copy machines and two available operators engaged at different rates to operate them. Which operator should operate which machine to maximize the profit?
2. There are some containers has to be transported to n different locations. There are n trucks available, in which each of them has different speed and fuel consumption rate. Which truck has to go to which location to minimize the total cost?

The general assignment problem is: we have to assign several jobs to workers for certain goal, such as to minimize the cost or to maximize the profit. We usually restrict that one worker can get at most one task and vice versa. If our goal is to maximize the total output, then the assignment problem can be formulated as LP below. Here c_{ij} denotes profit by workers j if being assigned to job i, where x_{ij} is binary variable determining which job is assigned to which worker.

LP FORMULATION OF ASSIGNMENT PROBLEM

Maximize

$$\sum_{i,j} c_{ij} x_{ij}$$

Subject to

$$\sum_{j} x_{ij} \leq 1 \qquad , \forall i$$

$$\sum_{i} x_{ij} \leq 1 \qquad , \forall j$$

$$x_{ij} \in \{0,1\}$$

There are several method to solve assignment problem, one of the most well-known methods is *Hungarian method* (Kuhn, 2005). In chapter 3, we shall see how sponsored search auction can be formulated exactly as an assignment problem.

3. Known Results

There are some known results in sponsored search auctions if we restrict the CTR to be bidder independent. In this discussion, we refer to two existing major results. The first one is from (Lahaie, Pennock, Saberi, & Vohra, 2007), in which they analyze the relation of VCG and GSP's equilibrium; furthermore they derive the result by viewing an auction as an allocation problem. The second result is from (Bu, Deng, & Qi, 2012), where every bidder may submit multiple bids (and hence may obtain multiple slots). They showed interesting equilibrium properties if multi-bidding is allowed. All results will be discussed in this section.

3.1. Single bidding, bidder-independent CTR

Let us denote the CTR of slot i for being assigned to bidder j as γ_{ij}. If it is bidder independent, γ_{ij} can be denoted simply as $\mu_i v_j$. We interpret μ_i as estimated number of clicks generated by slot i, regardless to which bidder who get that slot. The other parameter, v_j , is interpreted as the true amount of money (valuation) bidder j wants to gain for one click in any slot. Hence, the product $\mu_i v_j$ means true amount of money (valuation) bidder j wants to gain for being assigned to slot i. Since the CTR and bidders are now independent, we can order them in non-increasing order: $v_1 \geq \cdots \geq v_n$ and $\mu_1 \geq \cdots \geq \mu_n$. Within this subsection, we assume that these relation hold.

There are two mechanisms mainly used for analyzing sponsored search auctions, i.e. GSP and VCG. We give some results concerning analysis with these two methods. We shall start with VCG first, since in this section the result from GSP is built upon VCG's properties.

VCG mechanism can be described in form of primal-dual Linear Programming. The description below is due to (Lahaie, Pennock, Saberi, & Vohra, 2007), provided with assumption that the CTR is bidder independent. Using analogy of maximum allocation, this sponsored search auction can be modeled as a Linear Programming as follows. We define binary variable x_{ij}, in which $x_{ij} = 1$ if slot i is assigned to bidder j and $x_{ij} = 0$ otherwise.

PRIMAL

Maximize

$$\sum_{i,j} \mu_i v_j x_{ij}$$

Subject to

$$\sum_{j} x_{ij} \leq 1 \qquad ,\forall i$$

$$\sum_{i} x_{ij} \leq 1 \qquad ,\forall j$$

$$x_{ij} \in \{0,1\}$$

Table 1. LP formulation of sponsored search auction with bidder-independent CTR.

DUAL	
Minimize	
	$$\sum_i p_i + \sum_j q_j$$
Subject to	
	$$p_i + q_j \geq \mu_i v_j$$
	$$p_i, q_j \geq 0$$

Table 2. The Dual of LP.

It can be seen directly that PRIMAL's solution is socially optimal allocation. There is a nice interpretation of DUAL, but to derive it we need some properties from PRIMAL. Hence we shall discuss PRIMAL first.

We remark that PRIMAL can be solved efficiently (i.e. in polynomial time). The explanation is as follows. We have formulated sponsored search auction into LP as in table 3.1, which is equivalent with LP formulation of Maximum Matching Problem: given a graph, find a subset of edges with maximum cardinality such that no two edges have vertices in common. There is a well-known polynomial time algorithm for this problem: *blossom algorithm* (Cook, Cunningham, Pulleyblank, & Schrijver, 1998).

Now we proceed to major impact caused by bidder-independence. Unlike general matching problem, there is a very simple procedure for solving socially optimal allocation for bidder-independent CTR: simply assign slot to bidder in non-increasing order of CTR and bid. We state this as a proposition, which is taken from (Lahaie, Pennock, Saberi, & Vohra, 2007), but we prove it by ourselves.

Proposition 3.1.1. If CTRs are bidder independent, then PRIMAL can be solved by assigning slot with highest CTR to bidder with highest bid, second highest CTR slot to second highest bid, and so on.

Proof. Assume S is optimal assignment not obeying the "law" that n-th highest slot is allocated to n-th highest valuation bidder. Let's say slot i is the highest slot that is not "allocated properly". Say, slot i is allocated to bidder k, and slot j to bidder i. Consider another assignment S', obtained from S by assigning slot i to bidder i, slot j to bidder k, and the rest are same. Then the difference of total valuation of that from S and S' is just total valuation from these two slots. In other words, the difference is $\mu_i v_k + \mu_j v_i - (\mu_i v_i + \mu_j v_k)$. Hence,

$$\mu_i v_k + \mu_j v_i - \mu_i v_i - \mu_j v_k = (\mu_i - \mu_j)(v_k - v_i) \leq 0$$

Equality occurs if and only if $\mu_i = \mu_j$ or $v_i = v_k$. Hence total valuation from S is at least as good as total valuation from other assignments.

∎

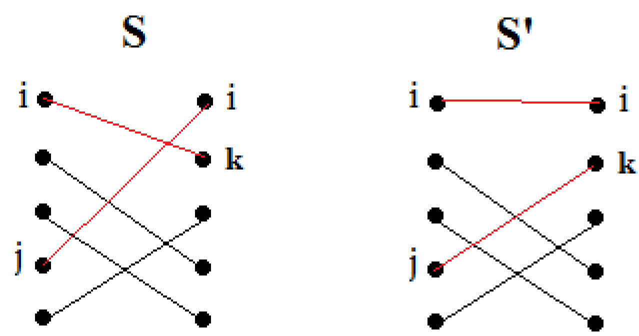


Figure 3. Heart of the proof of Proposition 3.1.1. We compare the total valuation with and without crossing edges.

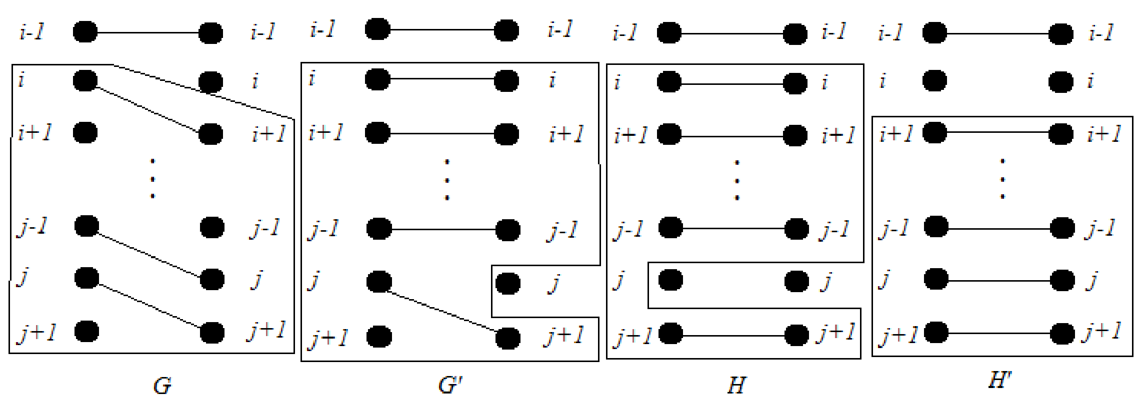
We now proceed to analysis of the DUAL problem. Firstly we notice that p_i corresponds to slot's constraint in PRIMAL, and q_j corresponds to bidder's constraint in PRIMAL. Our trivial guess is to interpret p_i as VCG price of slot i and q_j as payoff of bidder j. This interpretation is actually correct, and we shall state this as a proposition which is taken from (Lahaie, Pennock, Saberi, & Vohra, 2007), but the proof is our self-made.

Proposition 3.1.2. In LP formulation of Table 2. The Dual of LP., p_i is VCG price of slot i and q_j is payoff of bidder j.

Proof. By proposition 3.1.1., PRIMAL's optimal solution is obtained by assigning slot i to bidder i, i.e. assigning slot to bidder with the same index. We claim that $p_i := V^{S}_{B-\{i\}} - V^{S-\{i\}}_{B-\{i\}}$ and $q_i := \mu_i v_i - p_i$ solves the DUAL. We first show that the defined variables are feasible dual solution, then prove that the solution is optimal.

Step 1. Proof that the defined variables are feasible dual solution.

Without loss of generality, assume $i \le j$. See this figure for better illustration of matchings.



$$
\begin{aligned}
p_i + q_j &= V^S_{B-\{i\}} - V^{S-\{i\}}_{B-\{i\}} + \mu_j v_j - V^S_{B-\{j\}} + V^{S-\{j\}}_{B-\{j\}} \\
&= \left(\underbrace{V^S_{B-\{i\}}}_{G} - \underbrace{V^S_{B-\{j\}}}_{G'} \right) + \left(\underbrace{V^{S-\{j\}}_{B-\{j\}}}_{H} - \underbrace{V^{S-\{i\}}_{B-\{i\}}}_{H'} \right) + \mu_j v_j \\
&= \left(\mu_i v_{i+1} + \cdots + \mu_{j-1} v_j - \mu_i v_i - \cdots - \mu_{j-1} v_{j-1} \right) + \left(\mu_i v_i - \mu_j v_j \right) + \mu_j v_j
\end{aligned}
$$

Hence, proving $p_i + q_j \geq \mu_i v_j$ is equivalent to show that

$$\left(\mu_i v_{i+1} + \cdots + \mu_{j-1} v_j - \mu_i v_i - \cdots - \mu_{j-1} v_{j-1} \right) + \left(\mu_i v_i - \mu_i v_j \right) \geq 0 .$$

We evaluate the left hand side:

$$
\begin{aligned}
&\mu_i \left(v_{i+1} - v_j \right) + \mu_{i+1} \left(v_{i+2} - v_{i+1} \right) + \mu_{i+2} \left(v_{i+3} - v_{i+2} \right) + \cdots + \mu_{j-1} \left(v_j - v_{j-1} \right) \\
&\geq \mu_{j-1} \left(v_{i+1} - v_j \right) + \mu_{j-1} \left(v_{i+2} - v_{i+1} \right) + \mu_{j-1} \left(v_{i+3} - v_{i+2} \right) + \cdots + \mu_{j-1} \left(v_j - v_{j-1} \right) \\
&= \mu_{j-1} (0) \geq 0
\end{aligned}
$$

The defined variables satisfy all dual constraints, hence they are feasible. Moreover, since we define p_i as VCG price, then it is easy to see that $q_i := m_i v_i - p_i$ is difference between total valuation and VCG price, so q_j is gain of bidder j.

Step 2. Proof that the defined variables are optimal dual solution.

Using predefined p_i and q_j, the value of DUAL is

$$\sum_i p_i + \sum_j q_j = \sum_i p_i + q_i = \sum_i p_i + \mu_i v_i - p_i = \sum_i \mu_i v_i \ .$$

Now let $\overline{p}_i, \overline{q}_j$ be any feasible dual. Feasibility implies $\overline{p}_i + \overline{q}_i \geq \mu_i v_i$ for every i, hence the objective value is at least $\sum_i \mu_i v_i$. We can conclude that our predefined p_i and q_j are optimal dual solution.

□

Proposition 3.1.1 and 3.1.2 show the superiority of LP formulation of VCG mechanism: using single LP formulation, we can determine both the optimal allocation and also the price of slots.

Those are the major results concerning VCG mechanism in case of bidder-independent CTR. Now let us return to our discussion on GSP. As we have seen in section 2.4., GSP has multiple equilibria, which in general are not socially optimal. We shall see immediately that bidder-independent CTR enables us to define a socially optimal equilibrium under GSP. Firstly we notice that if CTR is bidder-independent, both GSP and VCG have the same procedure to allocate the slots, i.e. allocate them in non-increasing order of CTR and bids. Hence, if we somehow can construct the bids such that GSP charges each bidder the price equal to VCG price, then the bids will form an equilibrium. The resulting bid can be seen in the following proposition. Again, the proof is self-made.

Proposition 3.1.3 (Lahaie, Pennock, Saberi, & Vohra, 2007). Let the CTR be bidder independent, let p_{j-1} be as in DUAL and μ_{j-1} be CTR of slot $(j-1)$. Then the following bid forms equilibrium under GSP.

$$b_j = \begin{cases} v_1 & , j = 1 \\ p_{j-1}/\mu_{j-1} & , j \neq 1 \end{cases}$$

Moreover, the resulting allocation under GSP is socially optimal.

Proof. Let $b_j = p_{j-1}/\mu_{j-1}$ be the bid of bidder j. Recall the dual constraint $p_i + q_j \geq \mu_i v_j$, which is the key tool in this proof. We also use complementary slackness, which says that for all j, $p_j + q_j = \mu_j v_j$.

We firstly show that the bids are non-increasing. Consider $b_j = p_{j-1}/\mu_{j-1}$ and $b_{j+1} = p_j/\mu_j$. From dual constraint, we have $\mu_{j-1}v_j - p_{j-1} \leq q_j$ and $\mu_j v_j - p_j = q_j$. Since $\mu_j \neq 0$ for every j, we can modify them into these new relations:

$$v_j - \frac{p_{j-1}}{\mu_{j-1}} \leq \frac{q_j}{\mu_{j-1}} \text{ ...(1)}$$

$$v_j - \frac{p_j}{\mu_j} = \frac{q_j}{\mu_j} \text{ ...(2)}$$

Substituting $v_j = \frac{q_j}{\mu_j} + \frac{p_j}{\mu_j}$ into (1), finally we have

$$\frac{p_j}{\mu_j} - \frac{p_{j-1}}{\mu_{j-1}} \leq \frac{q_j}{\mu_{j-1}} - \frac{q_j}{\mu_j} \leq 0$$

This is what we want to show. Furthermore non-increasing bids implies that GSP would allocate n-th highest slot to n-th highest bidder, which is socially optimal.

Next we show that this bid forms equilibrium. We recall that payoff of bidder j is difference between total valuation and price of slot. Since prices in GSP are prices of slot, then payoff of bidder j for being assigned to slot i is $\mu_i v_j - \mu_i b_{j+1} = \mu_i v_j - p_i$.

Take any bidder j. Since the bids are non-increasing, GSP allocates j-th highest slot to j-th highest bidder. Under the abovementioned bid, he gets slot j and gains $\mu_j v_j - p_j$. If he deviates by changing his bid such that he gets other slot i, then his payoff would be $\mu_i v_j - p_i$. But by dual constraint and complementary slackness, we have $\mu_i v_j - p_i \leq q_j = \mu_j v_j - p_j$. Hence, changing the bid unilaterally will not increase the bidder's payoff.

⍰

This proposition has a striking consequence: bidder-independency of CTR allows us to derive socially optimal equilibrium for both VCG and GSP.

3.2. Multiple bidding, bidder-independent CTR

(Bu, Deng, & Qi, 2012) analyzed sponsored search auctions when every bidder is allowed to submit more than one bid. As in single-bidding environment, the bid is interpreted as a bid for single click. The CTR is bidder-independent. GSP is used as the sole mechanism. To fit the multi-bidding condition, they introduce extended version of second price auction called M-GSP, which means that each bidder may submit at most M bids.

The multi bidding situation with M-GSP is as follows. There are m slots and n bidders. For every j, bidder j has unique valuation v_j, which is the maximum price he is willing to pay for each click. The total valuation of slot i for being assigned to bidder j is $\mu_i v_j$. For this multi-bidding model, we assume strict inequality for CTR, i.e. $\mu_1 > \mu_2 > \ldots > \mu_m$. Each bidder submits M bids (without loss of generality, if a bidder submits less than M bids, then several zero bids are added to ensure that every bidder has exactly M bids). For every k, the search engine awards k-th highest slot to k-th highest bid, and charges the winning bidder the price next to the winning bid. In case of ties, the search engine will allocate the slot to the bidder with prior time stamp, i.e. to the bidder having lower index.

Formally, bidder j submits M bids $(b_{1j}, \ldots, b_{Mj})$. Let us collect all bids from all bidders, and denote b_k as the k-th highest bid from the list of ordered, non-increasing bids. Back to bidder j, if one of his bids is the k-th highest bid among all bids, then bidder j obtains k-th slot and pays b_{k+1} per click. His gain for obtaining that slot is $(v_j - b_{k+1})\mu_k$.

There are several interesting results about the equilibrium of multi-bidding M-GSP. We shall describe them in the following propositions, all are taken from (Bu, Deng, & Qi, 2012). Firstly, several necessary conditions about the existence of equilibrium are presented.

Proposition 3.2.1 (necessary conditions of equilibrium). If there exists a pure Nash equilibrium in M-GSP, then the following conditions must be true.

1. For any two bidders i and j with $v_i \neq v_j$, if bidder j gets at least one slot except slot m (the last slot) then bidder j gets exactly M slots.
2. For every bidder i, if bidder i gets slots $k, k+1, \ldots, k+l$ ($l < M$) then $b_{k+1} = b_{k+2} = \ldots = b_{k+l+1} + \varepsilon$ for small enough ε.
3. (Winner monotone) for every two bidders i,j, if $v_i < v_j$ and bidder i gets at least one slot then bidder j must also get at least one slot.
4. If the owner of slot m (the last slot) is bidder j, and v_{max} = max{v_i | $i \neq j$ and i gets less than M slots}, then $b_m \geq v_{max}$.

Proof. Let b^* be any equilibrium arranged in non-increasing order. Recall the assumption that the CTR satisfies $\mu_1 > \mu_2 > \ldots > \mu_m$, i.e. it is a strict inequality.

1. Take any bidder j who gets at least one slot except the last slot. Assume the contrary that under equilibrium he obtains less than M slots. Then at least one of his bids must be less than b_m. He can obtain m-th slot by bidding $b_m+\varepsilon$ for small enough ε, resulting in additional gain $\mu_m(v_j - b_m)$. If this difference is positive, then it is a profitable deviation which would contradict our equilibrium assumption. Let us prove it by contradiction: assume that $v_j = b_m$. Suppose bidder j obtains slot $l < m$. Clearly we have a relation $v_j = b_m = b_{m-1} = \ldots = b_l$; if there is a k such that $b_{m+k} < b_{m+k-1}$, then he can lower the bid b_{m+k-1} to $b_{m+k-1} - \varepsilon$ for small enough ε, which would improve his gain. Let bidder i be the bidder who obtains the last slot m. He gains $(v_i - b_{m+1})\mu_m$ for it.
   Case 1. If $b_{m+1} = v_j$, then we have relation $v_j = b_{m+1} \leq b_m = \ldots = b_l = v_j$, which means all terms in between v_js are equal to each other, or particularly $b_{m+1} = b_{l+1}$. By obtaining slot m, bidder i gains $(v_i - b_{m+1})\mu_m$. But by updating his bid such that he obtains slot l, his gain would be $(v_i - b_{l+1})\mu_l$. Since the CTR have strict inequality, then $(v_i - b_{m+1})\mu_m < (v_i - b_{l+1})\mu_l$, which means that it is a profitable deviation. This is a contradiction with equilibrium assumption.
   Case 2. If $b_{m+1} < v_j$, then bidder j can decrease his bid to obtain the last slot m. We have relation $(v_j - b_{l+1})\mu_l < (v_j - b_{m+1})\mu_m$, which means that decreasing the bid is a profitable deviation.
   Based on both cases, we conclude that $v_j - b_m > 0$.
2. Take bidder i, who gets slots $k, k+1, \ldots, k+l$. Assume the contrary that there is n such that $b_{k+n} < b_{k+n+1}$. In this case, lowering his bid into $b_{k+n+1} - \varepsilon$ for small enough ε will result in additional gain, which contradicts the equilibrium assumption.
3. Take two bidder i,j, with $v_i < v_j$ and bidder i gets at least one slot. Assume the contrary that bidder j does not get any slot. Let slot k be any slot allocated to bidder i in equilibrium, and let b_k be the winning bid. If bidder j bids $b_k + \varepsilon$, then he would get that slot and gains $(v_j - b_k)\mu_k$, which is positive. This is a contradiction with equilibrium assumption.
4. Let bidder j be the winner of slot m. Assume the contrary that $b_m < v_{max}$ and let $v_i = v_{max}$. In this case, bidder i can obtain slot m by bidding $b_m+\varepsilon$. He gets additional positive gain of $(v_i - b_m)\mu_m$, which is a contradiction with our equilibrium assumption.

□

Complete sufficient conditions for pure Nash equilibrium are difficult to establish. They strongly depend on the number of bids submitted by all bidders. In some circumstances, the existence of pure Nash equilibrium is guaranteed (cf. Proposition 3.2.2 below). On the other hand, there are several instances in which there does not exist any pure Nash equilibrium. The following propositions and examples will demonstrate the existence of equilibrium.

Proposition 3.2.2. M-GSP always has a pure Nash equilibrium if $M \geq m$.

Proof. Rank the bidders in non-increasing valuations. Consider this bid construction: bidder 1 submits M bids which all are equal to the valuation of bidder 2, and all other bidders bid their true valuation. Formally, we have $b_{11} = ... = b_{M1} = v_2$, and $b_{1j} = ... = b_{Mj} = v_j$ for $j \neq 1$. Under this bid, the first bidder gets all slots at the price of v_2 per click. Hence the first bidder gets nonnegative gain and the other bidders gain nothing. Any change of bids of bidder 1 will not make any profit. All bidders other than the first bidder cannot make a profitable deviation, since to obtain a slot they have to bid more than their valuation and pays v_2. Hence, the bids are equilibrium.

□

What if $M \leq m$? The following example, which is taken from (Bu, Deng, & Qi, 2012), shows that under this condition the existence of pure Nash equilibrium is not guaranteed.

Example 3.2.1 (non-existence of pure Nash equilibrium). Consider 2-GSP (i.e. M-GSP with M = 2). There are three slots with $\mu_1 = 20$, $\mu_2 = 11$, $\mu_3 = 10$, and three bidders with valuations $v_1 = 5$, $v_2 = 4$, $v_3 = 1$.

Let us assume that there exists a pure Nash Equilibrium b^*. According to condition 1 and 3 of proposition 3.2.1, the winner must be bidder 1 and 2. There are two cases: first, bidder 1 gets slot 1, 2, and bidder 2 gets slot 3; second, bidder 1 gets slot 3 and bidder 2 gets slot 1, 2. By condition 2 and 4 of proposition 3.2.1, bidder 3 bids $b_4 \leq v_3$, and $b_2 = b_3 \geq v_3$. Now assume $b_2 = b_3 = x$.

Case 1. Bidder 1 gets slot 1, 2, and bidder 2 gets slot 3.

If b^* is a pure Nash equilibrium, then the following inequalities must be satisfied.

$$\begin{gathered}(v_1 - x)\mu_2 \geq (v_1 - b_4)\mu_3 \\ (v_1 - x)(\mu_1 + \mu_2) \geq (v_1 - b_4)(\mu_2 + \mu_3) \\ (v_2 - b_4)\mu_3 \geq (v_2 - x)(\mu_2 + \mu_3) \\ b_4 \leq v_3 \\ v_2 \geq x \geq v_3\end{gathered}$$

Case 2. Bidder 1 gets slot 3 and bidder 2 gets slot 1, 2.

Similarly, b^* is a pure Nash equilibrium if and only if the following inequalities are satisfied.

$$(v_2 - x)\mu_2 \ge (v_2 - b_4)\mu_3$$
$$(v_2 - x)(\mu_1 + \mu_2) \ge (v_2 - b_4)(\mu_2 + \mu_3)$$
$$(v_1 - b_4)\mu_3 \ge (v_1 - x)(\mu_2 + \mu_3)$$
$$(v_1 - b_4)\mu_3 \ge (v_1 - v_2)(\mu_1 + \mu_2)$$
$$b_4 \le v_3$$
$$v_2 \ge x \ge v_3$$

We can observe that if x is solution for Case 1, then it is also solution for Case 2. Hence, it suffices to analyze the existence of solution of Case 1. By solving inequalities in Case 1, we conclude that it has a solution if and only if at least one of these two conditions are satisfied.

1. $\dfrac{v_1 - v_2}{v_1 - v_3} \ge \dfrac{\mu_3^2}{\mu_2^2}$, $\mu_1\mu_3 \ge \mu_2^2$.
2. $\dfrac{\mu_1 - \mu_3}{\mu_1 + \mu_2} v_1 + \dfrac{\mu_2 + \mu_3}{\mu_1 + \mu_2} v_3 \ge \dfrac{\mu_2}{\mu_2 + \mu_3} v_2 + \dfrac{\mu_3}{\mu_2 + \mu_3} v_3$, $\mu_1\mu_3 \ge \mu_2^2$.

Substituting all values to condition 1, we have a contradictory inequality 1/4 ≥ 100/121. Hence, there cannot be any pure Nash Equilibrium for this instance.

□

So far, we discussed the results from GSP. In the single-bidding auction, we have seen that there is a relation between GSP's and VCG's equilibrium: there is an equilibrium under GSP whose revenue is equal to VCG's revenue. In multi-bidding model, this property also holds, as the following proposition shows.

Proposition 3.2.3 (Relation between M-GSP and VCG). In M-GSP, the auctioneer's revenue in any equilibrium when $M \ge m$ is equal to the revenue under VCG mechanism.

Proof. Use condition 1 and 3 of proposition 3.2.1. If $v_1 \ne v_2$, then for any equilibrium, bidder 1 is the only bidder who obtains all slots. By condition 2 and 4 of proposition 3.2.1., $b_2 = \ldots = b_{m+1} = v_2$. Since this is second-price auction, then the total payment from bidder 1 is $v_2 \sum_{j=1}^{m} \mu_j$. If $v_1 = v_2$, then similarly bidder 1 will get all slots with price v_2 for each click. In total, he has to pay $v_2 \sum_{j=1}^{m} \mu_j$. It can be shown that this sum is equal to the revenue under VCG as follows. If bidder 1 were not present, all slots would go to bidder 2, which results in total valuation of $v_2 \sum_{j=1}^{m} \mu_j$. All other bidders get nothing. Hence the total harm caused by bidder 1 to the rest of the others is $v_2 \sum_{j=1}^{m} \mu_j$.

□

As has been discussed in Theory, in general GSP is not truthful. This also holds for M-GSP in multi-bidding model as shown by Proposition 3.2.3. One may expect that VCG can be used if we want to employ a truthful mechanism. However, the pricing rule of a truthful mechanism can be totally different from GSP. Hence it is not sure whether under truthful mechanism it is worth for the bidders to submit multiple bids. In other words, it is not sure if obtaining many slots is more profitable than obtaining only one slot. As pointed out in (Bu, Deng, & Qi, 2012), there exists no truthful, socially optimal mechanism which also encourages all bidders to submit only one (nonzero) bid.

Despite the ample results from sponsored search auctions, there is still some room for improvements. Firstly, it has been seen that sponsored search is closely related to assignment problem. It would be interesting to investigate whether a more general model of sponsored search auction can be made using assignment approach. The general model should be such that we can translate it back to the case of bidder-independent CTR. Secondly, the known results are based on bidder independent CTR. Finding a way to construct a socially optimal allocation for more general CTR would be a good direction. Third, in line with the multi-bidding sponsored search auctions, we can try to analyze, under general CTR, if each bidder can submit multiple bids, where different bids correspond to different slots.

4. Discussion

4.1. Reformulation as assignment problem

In sponsored search auctions, the model being used and studied widely is the one based on CTR, bids, and valuations. Here we try to reformulate it as an assignment problem by simplifying the parameters. Instead of defining CTR μ_i and valuations v_j separately, we now state the product $\mu_i v_j$ simply as α_{ij}, and call it new valuation (of bidder j for slot i). The corresponding general model (with general α_{ij}) will be called new model and we refer to the bidder independent CTR-based model discussed in chapter 3 as the original model.

Interpretation of valuations and bids

The new valuation, α_{ij}, can be interpreted as total valuation of bidder j for being assigned to slot i. It is clear that the difference between the new valuation and the "old valuation" v_j is the number of independent variables being used to define it: the new valuation is now two-dimensional.

The bids can also be defined in a similar way: it depends on the bidder and the slot, hence it can be denoted it by b_{ij}. It is interpreted as bid of bidder j for slot i. The bids and valuations can be represented in matrix form as follows.

$$\begin{matrix} \textit{slots} & \textit{bids} & \textit{valuations} \\ \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} & \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} & \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \end{matrix}$$

The difference between original and new model

The generalization of valuation might alter the results in several aspects. One would expect two things that would differ significantly, i.e. the optimality structure in social welfare maximization and also the equilibrium result. These two things will be discussed in the next paragraphs.

The first is about the structure of optimality in social welfare maximization. In original model, socially optimal allocation can be done by awarding slots greedily to bidders in non-increasing order of CTRs and bids. This way of allocation is no longer true in the new model. Indeed the α_{ij} can be sorted in non-increasing order, and then allocate n-th highest slot to n-th highest bidder for every n, but this does not guarantee socially optimal allocation as the following example shows.

Example 4.1.1. Consider an instance with 3 slots and 3 bidders, with valuation matrix like below.

$$[\alpha_{ij}] = \begin{bmatrix} 20 & 15 & 8 \\ 18 & 12 & 6 \\ 16 & 9 & 3 \end{bmatrix}$$

Allocating them by n-th highest slot to n-th highest valuation bidder would result in total valuation of 20 + 12 + 3 = 35. If we instead assign slot 1 to bidder 2, slot 2 to bidder 1, and slot 3 to bidder 3, then the total valuation is 18 + 15 + 3 = 36, which is better.

□

It would be an interesting question how should we characterize α_{ij} such that allocating the slots and bidders by n-th highest slot to n-th highest valuation bidder guarantees socially optimal allocation. If it can be done, then we can generalize Proposition 3.1.3 for wider class of valuation.

Second, let us analyze the socially optimal equilibrium under GSP as has been stated in Proposition 3.1.3. According to this proposition, bid $\{b_j = p^*_{j-1}/\mu_{j-1}\}$ where p^*_j is VCG price per *click* forms socially optimal equilibrium under GSP. To analyze the validity of this theorem for the new model, this bid has to be redefined. It can be done simply by $\{b_j = p_{j-1}\}$, where p_j is VCG price of *slot* j. The interpretation of bid is as follows: bidder j bids p_{j-1} for (j-1)-th slot. Unfortunately, these bids in general do not form an equilibrium under new model, as this example shows.

Example 4.1.2. Consider an auction with 3 slots and 3 bidders. The valuation matrix is as follows.

$$[\alpha_{ij}] = \begin{bmatrix} 20 & 15 & 8 \\ 18 & 12 & 7 \\ 16 & 9 & 2 \end{bmatrix}$$

One can check easily that socially optimal allocation is obtained by matching slot 1 to bidder 2, slot 2 to bidder 3, slot 3 to bidder 1. Total valuation is 16 + 15 + 7 = 38.

Price of slots is computed with formula $V^S_{B-j} - V^{S-i}_{B-j}$. The result is as follows.

$$p_1 = V^S_{B-2} - V^{S-1}_{B-2} = 4,\ p_2 = V^S_{B-3} - V^{S-2}_{B-3} = 2,\ \text{and } p_3 = 0.$$

Moving to GSP, we construct set of bids $\{b_j = p_{j-1}\}$, which means $b_1 = 4 + M$, $b_2 = 4$, and $b_3 = 2$. Under this set of bids, bidder j will get slot j. The gain of each bidder is as follows.

$gain_1 = 20 - 4 = 16$
$gain_2 = 12 - 2 = 10$
$gain_3 = 2 - 0 = 2$

Now if bidder 3 deviates by bidding $\widetilde{b}_3 = 4 + \varepsilon$, he obtains second slot at price 4. His gain would be $7 - 4 = 3$, which is better than the gain from previous bid. Hence, bid $b_j = p_{j-1}$ is not equilibrium.

□

4.2. Click Auction

In practical situations, due to the large number of clicks, one would expect the clicks to be partitioned into several groups of pairwise equivalent clicks (same valuation and bid for all clicks within that group). Such a group may correspond to a slot in the original model or, say, the set of clicks from users with certain internet search characteristics. In addition, the clicks generated by the search users might have different valuations, depending on, say, the frequency of internet transactions carried out by the respective users. This results in non-equivalent clicks in which grouping into slots as in the original model is unlikely. Looking at this fact, while there have been a lot of results derived from slot auction approach, it would be more realistic to analyze the sponsored search auction problem by considering the clicks. Hence instead of auctioning the slots, the search engine now auctions off the clicks. In the original model, each bidder gets one slot only, but it consists of several clicks. It means that the "one slot constraint" has to be weakened: each bidder may possesses several clicks. Moreover there should be a limitation for the number of clicks being assigned to each bidder, otherwise all clicks would go to one bidder with highest bid. For that reason, the bidders are assumed to have a maximum number of clicks he wants to get. This setting will be referred as Click Auction. The details are given in the next paragraph.

The Click Auction problem is defined as follows. There are several clicks to be auctioned off, and there are several interested bidders. Valuation of each bidder is defined in a rather general way, i.e., valuation of bidder j for being assigned to click i is α_{ij}. The bids can be two-dimensional, b_{ij}. Each bidder has capacity c_j, i.e. the maximum number of clicks he is ready to buy. Without loss of generality, assume further that the total number of clicks is equal to the total capacity of all bidders, otherwise add dummy or remove the least valuable clicks . The mechanisms being used to analyze this problem are VCG and GSP.

The bidders and clicks have to be well-ordered: the first click is more valuable than the second click, the second click worths more than the third one, and so on. This condition is summarized into the following definition.

Definition 4.2.1. A set of valuations is called dominant iff for all i and j, $\alpha_{ij} \geq \alpha_{i+1,j}$ and $\alpha_{ij} \geq \alpha_{i,j+1}$.

$$\overset{valuations}{\begin{bmatrix} \alpha_{11} & \geq & \cdots & \geq & \alpha_{1n} \\ \geq & & & & \geq \\ \vdots & & \ddots & & \vdots \\ \geq & & & & \geq \\ \alpha_{n1} & \geq & \cdots & \geq & \alpha_{nn} \end{bmatrix}}$$

Figure 4. Illustration of bidder-dominant definition.

In the model with separable valuation ($\alpha_{ij} = \mu_i v_j$), it has been shown that the resulting graph from social welfare maximization has no crossing edges. This condition is also assumed to be true for Click

Auction model such that an appropriate comparison between two models can be made. To be able to use it, a formal definition of the non-crossing edges condition is necessary.

Definition 4.2.2. A set of valuations $\{\alpha_{ij} \mid i \text{ is click}, j \text{ is bidder}\}$ satisfies Non-crossing optimality condition if and only if for every two clicks i,k and two bidders j,l with $i \le k$ and $j \le l$, the following inequality holds:

$$\alpha_{ij} + \alpha_{kl} \ge \alpha_{il} + \alpha_{kj}$$

This definition simply means that the graph of optimal allocation of a set of valuations satisfying non-crossing optimality condition does not contain crossing edges. Example 4.2.1 explains this definition further.

Example 4.2.1. Consider an auction with three clicks and two bidders, let $c_1 = 2$ and $c_2 = 1$. Two sets of valuations are given as follows.

$$[\alpha_{ij}] = \begin{bmatrix} 10 & 9 \\ 9 & 8 \\ 7 & 3 \end{bmatrix}, [\beta_{ij}] = \begin{bmatrix} 10 & 9 \\ 9 & 7 \\ 6 & 6 \end{bmatrix}$$

Figure 5. The first graph is the optimal allocation based on α_{ij}, the second graph is from β_{ij}. shows the graph of the optimal allocation. It can be seen that the first set of valuation does not satisfy non-crossing optimality condition, but the second set does.

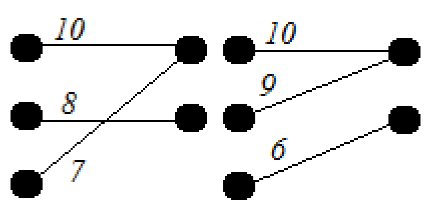


Figure 5. The first graph is the optimal allocation based on α_{ij}, the second graph is from β_{ij}.

Non-crossing optimality condition and dominant valuation condition are not equivalent. Consider an auction with three clicks and two bidders. Assume bidder 1 has capacity 2 and bidder 2 has capacity 1. Define the valuations as follows: $\alpha_{11} = 90, \alpha_{21} = 80, \alpha_{31} = 60, \alpha_{12} = 80, \alpha_{22} = 70, \alpha_{32} = 30$. These valuations are dominant, but the optimal allocation contains crossing edges as can be seen from Figure 6. The graph of optimal allocation. The crossing edges are drawn in red.: bidder 1 gets 1st and 3rd click, the rest is for bidder 2.

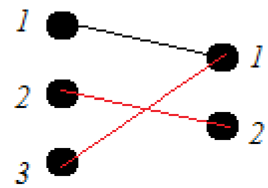


Figure 6. The graph of optimal allocation. The crossing edges are drawn in red.

Done with the definitions, we are ready to analyze the Click Auction with VCG and GSP (actually slightly modified version of GSP). Proposition 3.1.3 gives us an insight about the construction of an equilibrium: use the variables from the Dual linear program. Our first step is then to build a Linear Program for Click Auction whose objective is maximizing social welfare.

LP formulation of Click Auction with bidder's capacity.

Let i be index for clicks and j be index for bidders. The problem of maximizing social welfare can be formulated as LP below. Recall that α_{ij} is valuation of bidder j for click i, and c_j is capacity of bidder j.

PRIMAL 1

Maximize

$$\sum_{i,j} \alpha_{ij} x_{ij}$$

Subject to

$$\sum_j x_{ij} \le 1 \qquad , \forall i$$

$$\sum_i x_{ij} \le c_j \qquad , \forall j$$

$$x_{ij} \in \{0,1\}$$

DUAL 1

Minimize

$$\sum_i p_i + \sum_j c_j q_j$$

Subject to

$$p_i + q_j \ge \alpha_{ij}$$

$$p_i, q_j \ge 0$$

In original model (which was $\alpha_{ij} = \mu_i v_j$), the structure of the optimal allocation is simple: allocate n-th highest slot to n-th highest valuation bidder. It would be a good starting point to analyze the structure of optimal solution for our Click Auction model. This goal could be achieved by deriving several propositions below.

Proposition 4.2.1. Optimal allocation of PRIMAL 1 exhausts all bidders' capacities.

Proof. Let A be any optimal allocation. Assume that within A, there is a bidder j who gets less than c_j clicks. Let E_j be set of edges emanating from bidder j which are not selected in the optimal allocation A. Recall that by definition, valuations are nonnegative. If there exist one edge from E_j with positive valuation, then selecting that edge will increase the objective value, which is contradiction with the assumption that A is optimal allocation. If no edges of E_j having positive weight (which means all weights are zero), then we can select some of them until bidder j's capacity is exhausted without changing the optimal value. It means that without loss of generality, optimal allocation always exhausts all bidders' capacities.

□

Proposition 4.2.1 together with non-crossing optimality condition gives us an idea for constructing the optimal allocation. Since the graph of optimal allocation contains no crossing edges, we could simply allocate the clicks (start from the most expensive one) to the bidder with highest valuation whose remaining capacity is nonzero. This simple algorithm will compute such allocation, as well as give us the structure of the optimal allocation.

Algorithm *OptAllocation*

Set $i := 1$.
WHILE $i \leq n$ DO
  Select i-th highest click, allocate it to the highest bidder j whose residual capacity is nonzero.
  $c_j := c_j - 1$.
  $i := i + 1$.
ENDWHILE

Proposition 4.2.2. Algorithm OptAllocation computes the optimal allocation of PRIMAL 1.

Proof. Consider the situation when the algorithm tries to allocate click i, and let bidder j be the current highest bidder with nonzero remaining capacity. Assume that the algorithm is not optimal, and let OPT' be the optimal allocation. Since the algorithm is not optimal, OPT' will allocate click i to bidder $k > j$. On the other hand, proposition 4.2.1 says that bidder j has to get c_j clicks. Since the i-th click is not allocated to him, he will get lower click $h > i$ (see second picture of Figure 7. Heart of the proof of proposition 4.2.2. If bidder j has nonzero remaining capacity and click i is not allocated to bidder j, then "crossing edges" will occur, which violates the Non-crossing optimality condition.). But then the bipartite graph will have crossing edges, i.e. edge (i,k) and (h,j). This violates the non-crossing optimality condition, hence OptAllocation is optimal.

□

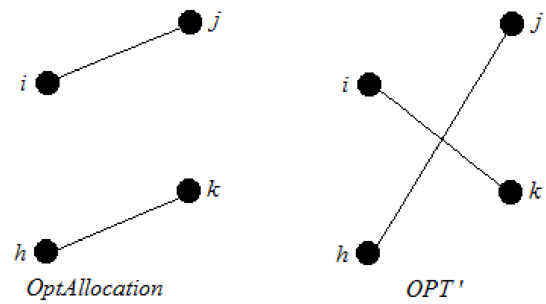


Figure 7. Heart of the proof of proposition 4.2.2. If bidder j has nonzero remaining capacity and click i is not allocated to bidder j, then "crossing edges" will occur, which violates the Non-crossing optimality condition.

This algorithm shows that the optimality structure of Click Auction is very similar with the structure from the original model.

Done with the primal, next we show a simple algorithm to compute the optimal dual solution using the complementary slackness theorem. The optimal allocation of PRIMAL 1 contains no crossing edges, and due to the fact that the valuations are dominant, then we can take this approach to construct dual solution: simply increase the value of the variables such that for each assigned pair of click and bidder (i,j), the dual constraint gets tight.

The complete algorithm is as follows. Assume that number of bidders is n and number of clicks is m, with $m = c_1 + \cdots + c_n$.

Algorithm *DualOpt*

Set $c_0 := 0$, $p_i := 0$ and $q_j := 0$ for all i and j.
Set $j := n$.
WHILE $j \geq 1$ DO
    Define q_j such that $p_{c_0+\cdots+c_n} + q_j = \alpha_{c_0+\cdots+c_n, j}$.
    FOR $i \in \{c_0 + \cdots + c_{j-1} + 1, \ldots, c_0 + \cdots + c_j - 1\}$ DO
        Define p_i such that $p_i + q_j = \alpha_{ij}$.
    ENDFOR
    Define $p_{c_1+\cdots+c_{j-1}}$ such that $p_{c_1+\cdots+c_{j-1}} + q_j = \alpha_{c_1+\cdots+c_{j-1}, j}$.
    $j := j - 1$.
ENDWHILE

Proposition 4.2.3. Algorithm DualOpt is optimal for DUAL 1, with running time $O(m + n)$.

Proof. The proof is done in two steps: Step 1 is to prove the feasibility, Step 2 is proof of optimality. Let p, q be the solution computed by DualOpt.

Step 1. Take any click k and bidder j. Assume that in optimal allocation, click k is allocated to bidder i. Without loss of generality, assume that $i < j$. We have to show that $p_k + q_j \geq \alpha_{kj}$, or equivalently

$p_k + q_j - \alpha_{kj} \geq 0$. For every bidder j, denote $\bar{j}$ as the last click being allocated to him (see the first graph of Figure 8. The graphical interpretation of proof of proposition 4.2.3.). This problem is again split into two cases.

Case 1. k is the last click being allocated to bidder i.

In this case k can be written as $\bar{i}$. We expand the expression $p_{\bar{i}} + q_j - \alpha_{\bar{i}j}$ as follows.

$$
\begin{aligned}
p_{\bar{i}} + q_j - \alpha_{\bar{i}j} &= \alpha_{\bar{i},i+1} - q_{i+1} + q_j - \alpha_{\bar{i}j} \\
&= \alpha_{\bar{i},i+1} - \alpha_{\overline{i+1},i+1} + p_{\overline{i+1}} + q_j - \alpha_{\bar{i}j} \\
&= \alpha_{\bar{i},i+1} - \alpha_{\overline{i+1},i+1} + \alpha_{\overline{i+1},i+2} - \alpha_{\overline{i+2},i+2} + p_{\overline{i+2}} + q_j - \alpha_{\bar{i}j} \\
&\vdots \\
&= \alpha_{\bar{i},i+1} - \alpha_{\overline{i+1},i+1} + \cdots + \alpha_{\overline{j-1},j} - \alpha_{\bar{j},j} + p_{\bar{j}} + q_j - \alpha_{\bar{i}j} \\
&= \alpha_{\bar{i},i+1} - \alpha_{\overline{i+1},i+1} + \cdots + \alpha_{\overline{j-1},j} - \alpha_{\bar{j},j} + \alpha_{\bar{j}j} - \alpha_{\bar{i}j} \\
&= \underbrace{\left(\alpha_{\bar{i},i+1} + \cdots + \alpha_{\overline{j-1},j}\right)}_{M1} - \underbrace{\left(\alpha_{\overline{i+1},i+1} + \cdots + \alpha_{\overline{j-1},j-1} + \alpha_{\bar{i}j}\right)}_{M2}
\end{aligned}
$$

Notice that both $M1$ and $M2$ represent total valuation of an allocation from a set of clicks $\{\bar{i}, \overline{i+1}, \ldots, \overline{j-1}\}$ to a set of bidders $\{i+1, \ldots, j\}$, in which each bidder has unit capacity. $M1$ is an allocation without crossing edges, while $M2$ has crossing edges. Hence by non-crossing optimality condition, $M1 - M2 \geq 0$ (see second and third graph of Figure 8. The graphical interpretation of proof of proposition 4.2.3.).

Case 2. k is not the last click being allocated to bidder i.

We expand the expression $p_k + q_j - \alpha_{kj}$ as follows.

$$
\begin{aligned}
p_k + q_j - \alpha_{kj} &= \alpha_{ki} - q_i + q_j - \alpha_{kj} \\
&= \alpha_{ki} - \alpha_{\bar{i},i} + p_{\bar{i}} + q_j - \alpha_{kj} \\
&\geq \alpha_{ki} - \alpha_{\bar{i},i} + \alpha_{\bar{i},j} - \alpha_{kj} \\
&= \underbrace{\left(\alpha_{ki} + \alpha_{\bar{i},j}\right)}_{M1} - \underbrace{\left(\alpha_{\bar{i},i} + \alpha_{kj}\right)}_{M2}
\end{aligned}
$$

Using similar analysis, both $M1$ and $M2$ represent total valuation of an allocation form a set of clicks $\{k, \bar{i}\}$ to a set of bidders $\{i, j\}$, in which each bidder has unit capacity. Since k is not the last click being allocated to bidder i, then $k < i$. As a consequence, $M1$ is an allocation without crossing edges. Hence by non-crossing optimality condition, $M1 - M2 \geq 0$ (see the fourth and fifth graph of Figure 8. The graphical interpretation of proof of proposition 4.2.3.).

Step 2. By construction of the algorithm, for every p_i we can find q_j such that $p_i + q_j = \alpha_{ij}$. Hence, $\sum_i p_i + \sum_j c_j q_j = \sum_{\substack{i,j \\ i \text{ connects } j}} \alpha_{ij}$, which is exactly the same as the solution of PRIMAL 1.

We conclude that this is optimal.

For the running time of the algorithm, notice that each dual variable is assigned only once. Since there are $m + n$ dual variables, then the complexity is $O(m + n)$.

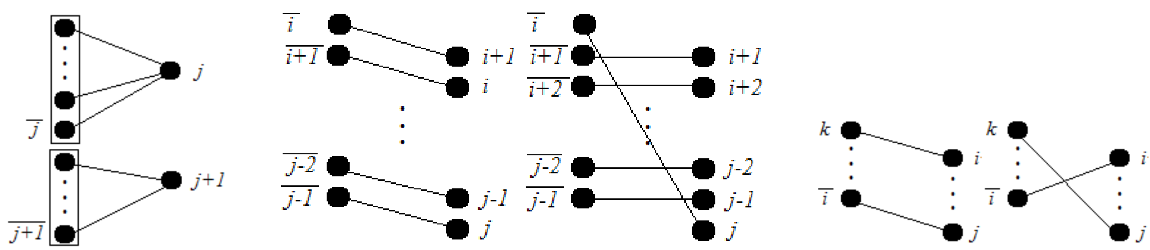


Figure 8. The graphical interpretation of proof of proposition 4.2.3.

Hence instead of solving Linear Programming by commercial solvers, these efficient algorithms can be used to solve them.

Reinterpretation of VCG

We are now ready to derive some results concerning VCG and GSP. Since Click Auction is a modification of the slot auction, one may expect that VCG mechanism might alter slightly to fit the problem description.

Recall that in auction theory, VCG charges each bidder the total harm he causes by his presence to the rest of the bidders. Using this definition, the VCG price of a click can be defined as the difference between maximum allocation of all bidders by reducing one bidder's capacity by one (see the second graph of Figure 9. Definition of VCG price of click i. The first is the graph of optimal allocation, the second and third describe the change of total valuation by reducing bidder *j*'s capacity by 1.) and the maximum allocation of all bidders excluding the valuation of click *i* to bidder *j* (see the third graph of Figure 9. Definition of VCG price of click i. The first is the graph of optimal allocation, the second and third describe the change of total valuation by reducing bidder *j*'s capacity by 1.).

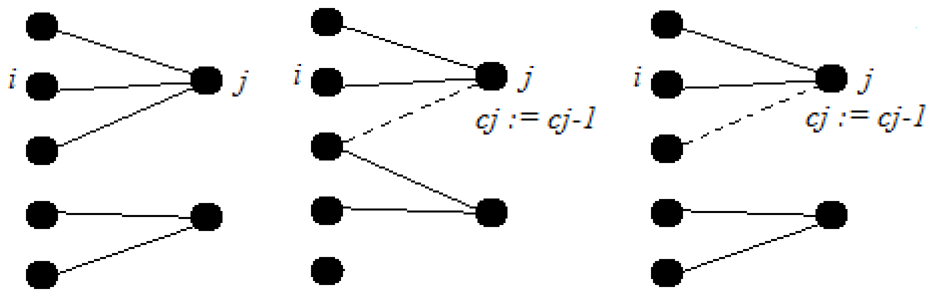


Figure 9. Definition of VCG price of click i. The first is the graph of optimal allocation, the second and third describe the change of total valuation by reducing bidder *j*'s capacity by 1.

This slightly modified notation is used to distinguish the "standard" VCG and the redefined one.

$$V^{S}_{C_j-1} - V^{S-\{i\}}_{C_j-1}$$

Furthermore, algorithm DualOpt enables us to define the VCG formula *explicitly*. The complete derivation is described within the following section.

Relation between Dual Variables and VCG prices

In the original model with separable valuation ($\alpha_{ij} = \mu_i v_j$), the dual variables correspond to the VCG prices. Hence we want to investigate whether the same result can be derived for Click Auction. The result turns out to be true, as the following proposition shows.

Proposition 4.2.4. In the DUAL 1, p_i is VCG price of click i and q_j is gain of bidder j.

Proof. Let p and q be optimal dual solution; take an arbitrary p_i. Assume that in the optimal allocation, click i is allocated to bidder j. Let $M1$ = $V^{S}_{c_j-1}$ and $M2$ = $V^{S-i}_{c_j-1}$. Using the definition of VCG that has been described before, if the difference of $M1$ and $M2$ is equal to p_i, then indeed p_i is VCG price of click i. This is the main idea of the proof.

By algorithm DualOpt, $M1$ is the sum of all values of dual variables except q_j (since bidder j's capacity is reduced by 1) and p_m (since bidder n obtains $(m\text{-}1)$-th click, then click m is allocated to nobody). Similarly, $M2$ consists of all values of dual variables except p_i and q_j. Hence,

$$M1 - M2 = \left(\sum_k p_k + \sum_l c_l q_l - p_m - q_j \right) - \left(\sum_k p_k + \sum_l c_l q_l - p_i - q_j \right) = p_i - p_m$$

By algorithm DualOpt, $p_m = 0$, so the expression above is equal to p_i. Hence p_i is VCG price of click i. Again by construction of algorithm DualOpt, it is clear that $q_j = \alpha_{ij} - p_i$, hence q_j can be interpreted as the gain of bidder j.

□

LP formulation is still equivalent to VCG mechanism. The next natural question would be to find out whether the dual variables can be used to construct a socially optimal equilibrium under GSP. We are able to derive it by slightly modifying the GSP as follows. Each bidder is asked to submit single bid. The bids are ordered in non-increasing order, and then the clicks are awarded to the highest bidder with nonzero remaining capacity. The winning bidder pays the second highest bid for the last click, for other clicks i the prices are determined by variable p_i as in DUAL 1.

Definition 4.2.5 (modified GSP). Assume there are n bidders, bidder j has capacity c_j. For our convenience, let $c_0 = 0$. Let $b = (b_1, \ldots, b_n)$ be the bids in non-increasing order submitted by the bidders. Modified GSP allocates clicks $\{c_0+\ldots+c_{j-1}+1, \ldots, c_0+\ldots+c_j\}$ to bidder j. Price of $(c_0+\ldots+c_{j-1}+k)$-th click is $p_{c_0+\cdots c_{j-1}+k}$, and price of $(c_0+\ldots+c_j)$-th click is simply b_{j+1}.

Proposition 4.2.5. Let α_{ij} be dominant valuations satisfying non-crossing optimality condition and p_{j-1} be as in DUAL 1. Fix $c_0 = 0$. Then the following bid forms a socially optimal Nash Equilibrium under modified GSP.

$$b_j = \begin{cases} M & , j = 1 \\ p_{c_0+\cdots+c_{j-1}} & , j \neq 1 \end{cases}$$

Here M is a big number. In words, each bidder other than the first one bids the price of click just before the first click he would obtain.

Proof. The proof is again split into two steps.

Step 1. Proof that the bids are non-increasing.

It means that we have to prove $b_{j-1} \geq b_j$, or equivalently $p_{c_0+\cdots+c_{j-1}} \geq p_{c_0+\cdots+c_j}$ for every j. Constraint of DUAL 1 says that $p_{c_0+\cdots+c_{j-1}} \geq \alpha_{c_0+\cdots+c_{j-1},j} - q_j$. Using dual constraint and complementary slackness,

$$\begin{aligned} p_{c_0+\cdots+c_{j-1}} &= \alpha_{c_0+\cdots+c_{j-1},j} - q_j \\ &= \alpha_{c_0+\cdots+c_{j-1},j} - \alpha_{c_0+\cdots+c_j,j} + p_{c_0+\cdots+c_j} \\ &\geq 0 + p_{c_0+\cdots+c_j} \\ &= p_{c_0+\cdots+c_j} \end{aligned}$$

For case $j = 1$, we have to show that $M \geq p_{c_1}$. This is clear by choosing M big enough.

Step 2. Proof that the defined bids are equilibrium.

To prove it, take any bidder j. Under the defined bids he obtains $(c_0 + \cdots + c_{j-1} + 1)$-th click until $(c_0 + \cdots + c_j)$-th click. His gain is

$$gain_j = \left(\alpha_{c_1+\cdots+c_{j-1}+1,j} - p_{c_1+\cdots+c_{j-1}+1}\right) + \cdots + \left(\alpha_{c_1+\cdots+c_j,j} - p_{c_1+\cdots+c_j}\right).$$

Now assume that he deviates his bid into $\tilde{b}_j$ and it causes him to obtain $(c_1 + \cdots + c_{k-1} + 1)$-th click until $(c_1 + \cdots + c_k)$-th click. His gain is

$$\widetilde{gain}_j = \left(\alpha_{c_1+\cdots+c_{k-1}+1,j} - p_{c_1+\cdots+c_{k-1}+1}\right) + \cdots + \left(\alpha_{c_1+\cdots+c_k,j} - p_{c_1+\cdots+c_k}\right)$$

Now compute the difference.

$$\begin{aligned} &gain_1 - \widetilde{gain}_1 \\ &= \underbrace{\left(\alpha_{c_1+\cdots+c_{j-1}+1,j} - p_{c_1+\cdots+c_{j-1}+1}\right) - \left(\alpha_{c_1+\cdots+c_{k-1}+1,j} - p_{c_1+\cdots+c_{k-1}+1}\right)}_{1} + \cdots + \underbrace{\left(\alpha_{c_1+\cdots+c_k,j} - p_{c_1+\cdots+c_j}\right) - \left(\alpha_{c_1+\cdots+c_k,j} - p_{c_1+\cdots+c_k}\right)}_{k-j+1} \end{aligned}$$

Each term can be analyzed in similar way, so let's pick an arbitrary term. Here is the result:

$$\alpha_{c_1+\cdots+c_{j-1}+l,j} - p_{c_1+\cdots+c_{j-1}+1} = q_j \geq \alpha_{c_1+\cdots+c_{k-1}+l,j} - p_{c_1+\cdots+c_{k-1}+1}.$$

The equality is due to complementary slackness, while the inequality holds by dual constraint. The above inequality shows that each term is nonnegative, hence sum of all terms will be nonnegative too. It implies that changing the bid does not increase the gain, so the bid is equilibrium. Moreover since the objective of the LP is maximizing social welfare, then the equilibrium is also socially optimal.

□

Thanks to the Non-crossing optimality condition, we succeeded in deriving VCG and LP's equivalency and also socially optimal Nash Equilibrium of modified GSP.

Comparison of original model and Click Auction

Click Auction is strongly related with the original model as follows. Firstly, the relation between slot's CTR and bidder's capacity. Both represent a bunch of clicks, however there is a difference in the way they are determined: CTR is usually an estimation, while capacity is determined precisely by the bidders. In original model it is the search engine who fixes the number of slots, while in Click Auction model the total number of clicks is determined by bidders' capacity. Secondly, both models have similar structure of optimality: they allocate highest slot (or click) to highest bidder. Third, the socially optimal equilibrium under GSP. The original model has this equilibrium, while Click Auction also has it under modified GSP. Both equilibria are constructed using dual variables. Another interesting point is to relate capacity with bidder's budget. Since capacity is determined precisely, it is like the budget: each bidder submits the number of clicks (money) he wants to buy (invest). This is interesting although loosely related to our discussion.

Another point of interest is comparing Click Auction with multi-bidding sponsored search as seen in (Bu, Deng, & Qi, 2012). There are several things that can be equalized and also contrasted. For the similarities, both models allow each bidder to obtain several slots (clicks). In addition, the bids in Click Auction can be interpreted as multi-bids: If bidder j having capacity c_j submits one bid, then it can be viewed as c_j identical bids, in which one bid can only be used to win one click. There is a difference though: in Click Auction we never allow a bidder to submit bids more than the number of clicks. We also have a similiarity for equilibrium result: in one equilibrium of multi-bidding model under GSP, its revenue is equal to the VCG's revenue.

On the other hand, we can contrast these two models in two ways. Firstly, the condition of valuation. (Bu, Deng, & Qi, 2012) considers separable valuations, while Click Auction employs a wider class of valuations: it requires only non-crossing optimality condition and dominancy. Such valuations do not necessarily be separable. In this sense, Click Auction (without non-crossing optimality condition) is indeed a generalization of the multi-bidding model. Secondly, the equilibrium analysis is where the main difference between these two auctions can be detected. In general, the multi-bidding model does not ensure the existence of equilibrium under GSP. In the Click Auction (with non-crossing optimality condition), we have seen that socially optimal equilibrium always exists regardless the capacity of the bidders, if we use modified GSP.

We shall demonstrate explicitly now that using modified GSP, we can find a Nash Equilibrium for an instance where multi-bidding with M-GSP failed to do so.

Example 3.2.1 revisited. Recall the setting: there are three slots with $\mu_1 = 20$, $\mu_2 = 11$, $\mu_3 = 10$, and three bidders with valuations $v_1 = 5$, $v_2 = 4$, $v_3 = 1$. This example has no equilibrium under M-GSP, as seen in example 3.2.1. In multi-bidding, 2-GSP is used. This can be translated into Click Auction by defining $c_1 = 2$, $c_2 = c_3 = 1$. We can also add 4^{th} click as a dummy click with zero valuation to ensure that sum of capacities is equal to the number of clicks, although it will not affect the computation. The valuations would be as follows.

$$\alpha_{ij} = \begin{bmatrix} 100 & 80 & 20 \\ 55 & 44 & 11 \\ 50 & 40 & 10 \end{bmatrix}$$

To construct an equilibrium, an LP formulation is needed but we skip it since it is trivial. We use proposition 4.2.5 to construct the bids. Using algorithm DualOpt, the optimal solution of Dual problem is $p_1 = 59$, $p_2 = 14$, $p_3 = 10$, $q_1 = 41$, $q_2 = 30$, $q_3 = 0$. By proposition 4.2.5, the bids (50, 14, 10) should be an equilibrium. Now let us prove it by analyzing each bidder separately.

Bidder 1. Under current bid, he gets slot 1 and 2, and gains (100-59) + (55-14) = 82. If he prefers slot 2 and 3, then he should lower his bid to b_1', where 14 > b_1' > 10. His gain would be (55-14) + (50-10) = 81, which is not better than before.

Bidder 2. Under current bid, he gets slot 3 and gains 40-10 = 30. Lowering his bid would make him lose the slot. Raising the bid to get slot 1 would alter his gain to 80-50=30, which is again not beneficial.

Bidder 3. Under current bid, he gets no slots and gains 0. Raising the bid to get slot 3 would alter his gain to 10-14 < 0, while obtaining slot 1 would make him gain 20-50 < 0.

Based on these argument, it is argued that bids (50,14,10) forms an equilibrium under modified GSP.

□

Click Auction could have some further applications: the clicks can be interpreted as slots. Hence we have a more general model of slot auction having general valuation and each bidder can obtain several slots.

5. Conclusion and Recommendation

A generalization of sponsored search auctions with allocation approach called Click Auction has been constructed. By modifying the valuations to satisfy non-crossing optimality condition, this generalized model turns out to have some similar properties with the original model and also have an equilibrium property that is not possessed by multi-bidding model. The first one is the structure of social welfare maximization. If the valuation in Click Auction satisfies non-crossing optimality condition, then assigning highest click to highest bidder is socially optimal. The second result, which is the most important result from this research, is equilibrium of Click Auction: a socially optimal equilibrium under modified GSP can always be constructed based on dual variables. This has to be seen in contrast with multi-bidding situation as seen in (Bu, Deng, & Qi, 2012) where GSP cannot guarantee the existence of equilibrium.

This research leaves further research directions as follows. It has been shown that under non-crossing optimality condition, a socially optimal equilibrium under GSP is guaranteed. It would be interesting to analyze the other direction: find all necessary and sufficient conditions to guarantee the existence of socially optimal equilibrium under GSP. An in-depth study about comparison of modified GSP and original GSP would also be a right research direction.

Bibliography

Aggarwal, G., & Muthukrishnan, S. (2008). Theory of Sponsored Search Auctions. *49th Annual IEEE Symposium on Foundation of Computer Science*, (p. 7). Philadelphia.

Bertsimas, D., & Tsitsiklis, J. N. (1997). *Introduction to Linear Optimization.* Athena Scientific.

Blume, A., Heidhues, P., Lafky, J., Munster, J., & Zhang, M. (2008). All equilibria of the multi-unit Vickrey auction. *Science Direct*, 729-741.

Bu, T.-M., Deng, X., & Qi, Q. (2012). Multi-bidding strategy in sponsored search auctions. *Journal of Combinatorial Optimization*, 356-372.

Cook, W., Cunningham, W., Pulleyblank, W., & Schrijver, A. (1998). *Combinatorial Optimization.* Wiley-Interscience.

Easley, D., & Kleinberg, J. (2010). *Networks, Crowds, and Markets:.* Cambridge: Cambridge University Press.

Kuhn, H. (2005). The Hungarian Method for the Assignment Problem. *Naval Research Logistics*, 7-21.

Lahaie, S. (2006). An Analysis of Alternative Slot Auction Designs for Sponsored Search. *Proceedings of the 7th ACM conference on Electronic Commerce*, (pp. 218-227). Michigan.

Lahaie, S., Pennock, D. M., Saberi, A., & Vohra, R. V. (2007). Sponsored Search Auctions. In N. Nisan, T. Roughgarden, E. Tardos, & V. V. Vazirani, *Algorithmic Game Theory* (pp. 699-716). Cambridge: Cambridge University Press.

Peters, H. (2008). *Game Theory.* Berlin Heidelberg: Springer.

Tian, G. (2010). *Lecture Notes Auction Theory.* Texas.