Contributions to bin packing games

Master Thesis

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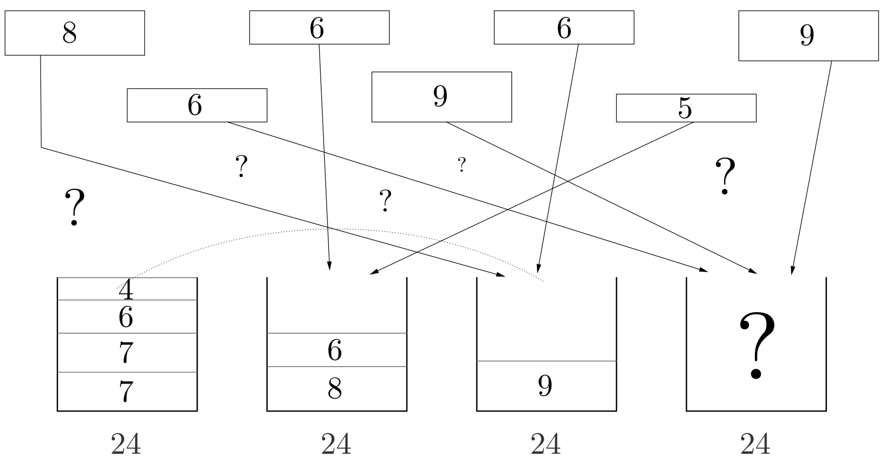

Preface

The report you are now reading is the report of my Final Project, which could also be named as my Master Thesis. Writing this report is my final assignment to obtain my Master degree, provided that this report will be graded with a mark from the set $S = \{6, 7, 8, 9, 10\}$.

The main person who will decide if the mark is indeed an element of S is Dr. Walter Kern, who is my supervisor as well. In September 2011 he told me about his research about bin packing and bin packing games, which sounded like a lot of complicated puzzles. That sounded perfect! I love solving puzzles! (Well, as long as they are solvable.) Then he asked me if I was interested in trying to make some contributions to this research as a final project. The answer was easy: sure!

The start was not easy. 'Easy' subclasses appeared to be quite hard and creating interesting instances also turned out to be harder than expected. However, when time went by, some new instances appeared and some results for small subclasses showed up. Although, I was not able to solve the problem for the main case, I am satisfied with the results and the report. Hopefully, you, as reader, will like it as well.

I would like to thank Dr. Kern for his help during my research. He offered me lots of suggestions about subclasses I should investigate, which was a great help for me. Furthermore, I would like to thank him for finding new angles, correcting my report and suggesting many improvements. Other people I would like to thank are the other members of the Graduation Committee, Prof. Dr. M. Uetz and Dr. ir. W.R.W. Scheinhardt, and Bas Joosten, because of his inspirational thesis and continuous interest.

I would also like to thank the people who were supporting me during my study: My family, house mates, the people of the DMMP Chair for giving help and relaxation during the breaks, W.S.G. Abacus and in particular my classmates and the members of the futsalteams 2 Vingers 4 and 2 Vingers 5, Navigators Studentenvereniging Enschede and all friends that are not included in the sets I have mentioned. Above all, I would like to thank the Lord Jesus Christ for sending love, wisdom and strength.

By the way, in my report I use '*we*' instead of '*I*'. This does not mean I am schizophrenic or something like that. I use it to get the reader more involved in understanding the results written in this report.

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1 Introduction

In this report *bin packing* is defined (Kern and Qiu [6]) by a set of k bins of capacity 1 each and n items of sizes $a_1, a_2, \ldots, a_n$. We assume w.l.o.g. $0 \leq a_i \leq 1$ for all a_i. Let A be the set of items and B be the set of bins. A *feasible packing* of an item set $A' \subseteq A$ into a set of bins $B' \subseteq B$ is an assignment of some (or all) elements in A' to the bins in B' such that the total size of items assigned to any bin does not exceed the bin capacity. Items that are assigned to a bin are called *packed* and items that are not assigned are called *not packed*. The *value* of a feasible packing is the total size of packed items.

A set F of items is called a *feasible set* if its total size does not exceed 1, the maximum binsize. Denote by $\mathcal{F}$ the set of all feasible sets. Let σ_F be the total size of all items in the feasible set F and $\sigma = (\sigma_F) \in \mathbb{R}^{|\mathcal{F}|}$. By y_F we denote how much feasible set F is used and $y = (y_F)$.

The integer linear packing program (IPP) corresponding to the integer bin packing is:

$$\begin{aligned} \text{(IPP)} \quad & \max \ \sigma^T y, \\ \text{s.t.} \quad & \sum_{F \in \mathcal{F}} y_F \leq k, \\ & \sum_{F \ni i} y_F \leq 1 \qquad (i = 1, \ldots, n), \\ & \quad y \in \{0,1\}^{\mathcal{F}}. \end{aligned}$$

If we relax the constraint that feasible sets should be packed as a whole, we get the fractional packing problem (PP):

$$\begin{aligned} \text{(PP)} \quad & \max \ \sigma^T y, \\ \text{s.t.} \quad & \sum_{F \in \mathcal{F}} y_F \leq k, \\ & \sum_{F \ni i} y_F \leq 1 \qquad (i = 1, \ldots, n), \\ & \quad y \in [0,1]^{\mathcal{F}}. \end{aligned}$$

We define the GAP as the difference between the optimal value v of (IPP) and the optimal value v' of (PP). Because $v' \geq v$, we have: GAP $= v' - v$.

2 Background

Of course the bin packing problem described in the introduction has some background. In this chapter we explain the connection with the closely related bin packing game and describe their corresponding linear problems.

2.1 Bin packing game

The reason why we study these two types of bin packing, is the close relation to something called *bin packing game*, which is defined as follows [5]: A *bin packing game* is a cooperative N-person game, where the set of players consists of k bins of size 1 and n items of sizes $a_1, \ldots, a_n$. The value of a coalition of bins and items is the maximum total size of items in the coalition that can be packed into the bins of the coalition.

A *cooperative (maximum value) N-person game* is defined by a set N of *players* and a *characteristic* (or *value*) *function* $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and N itself is the *grand coalition*. We define $v(S)$ as the total gain that coalition S can achieve if all its members cooperate.

In cooperative games one aims to 'fairly' allocate the total gain $v(N)$ of the grand coalition N among the individual players $i \in N$. An interesting concept essentially going back to von Neumann and Morgenstern [1] is the *core* of a cooperative game. The core consists the vectors $x \in \mathbb{R}^N$ that satisfy:

$$\text{(I) } x(N) \leq v(N),$$
$$\text{(II) } x(S) \geq v(S) \text{ for all } S \subseteq N,$$

where $x(S) = \sum_{i \in S} x_i$.

It is possible that the core is empty. Then there is not enough gain to get every player satisfied, which is undesirable of course. However, there are some ways to relax condition (II) such that the modified core is non-empty. Faigle and Kern [5] have introduced an ϵ-core, where $\epsilon \in (0, 1]$, as follows. Given $\epsilon > 0$, the ϵ-core consists all vectors $x \in \mathbb{R}^N$ satisfying condition (I) and (II'):

$$\text{(II') } x(S) \geq (1 - \epsilon)v(S) \text{ for all } S \subseteq N.$$

As stated in [6], we can regard ϵ as a tax rate, so that coalition S is allowed to keep only $(1-\epsilon)v(S)$ on its own. In order to approximate the core as close as possible, we would like to have the taxation rate ϵ as small as possible while keeping the ϵ-core non-empty.

2.2 (Integer) linear programs

If we only look at condition (II) and we want to minimize the total payments to the individual players, we get the linear program (LP):

$$\begin{array}{lll} \text{(LP)} & \min & x(N), \\ & \text{s.t.} & x(S) \geq v(S) \quad \text{for all } S \subseteq N. \end{array}$$

Note that we did not forget the constraint $x \geq 0$. If we consider S as being only one item or only one bin, its gain is 0. Therefore, the constraint in (LP) implies $x \geq 0$.

Because all bins have size 1, we may award all bins the same amount x_0. There are k bins, so the total payment to the bins is kx_0. The payment to item i is x_i. Hence we have to minimize: $kx_0 + \sum_{i=0}^{n} x_i$. The constraint of (LP) is replaced by a constraint that only considers the cases with only 1 bin and a set of items with a feasible total size ($\sum_{i \in F} a_i \leq 1$). This is sufficient to cover all cases. The resulting linear program is the allocation problem (AP):

$$\begin{array}{lllll} \text{(AP)} & \min & kx_0 & + \displaystyle\sum_{i=1}^{n} x_i, & \\ & \text{s.t.} & x_0 & + \displaystyle\sum_{i \in F} x_i \ \geq \ \sigma_F & \text{for all } F \in \mathcal{F}, \\ & & & x_0, x \ \geq \ 0. & \end{array}$$

Recall that we defined σ_F as the total size of all items in the feasible set F.

Remarkably, the dual of linear program (AP) is linear program (PP), which is defined in the introduction (Chapter 1). So, the allocation problem and the fractional packing problem are each other's dual! Thus, investigating the *bin packing game* is closely related to investigating the *bin packing* problem. That is why this research and thesis is about *bin packing* in the setting presented in the introduction.

As mentioned in the introduction, we denote the optimal value of (IPP) by v and the optimal value of (PP) by v'. Because of this duality, the optimal value of (LP) and (AP) is v' as well. There is a direct relationship between these optimal values and the non-emptiness of the ϵ-core. This is described in [5]:

Lemma 2.1 *ϵ-core$(v) \neq \emptyset$ if and only if $\epsilon \geq (v' - v)/v'$.*

Some interesting questions arise if we look at the ϵ-core. We are wondering if there is a minimal ϵ which results in a non-empty ϵ-core for all bin packing games or for all bin packing games in a specific subclass. From now on, we call this minimal ϵ: $\epsilon_{\min}$. There are more things we would like to investigate. This is explained in the next chapter.

3 Problem description

The previous sections might seem quite abstract and it could be hard to understand what is going on. In this section we give some examples and explain what we would like to investigate. Furthermore, we give an overview of earlier work that has been done.

3.1 Examples

So we have a set of n items of different sizes and k bins of size 1. We compare two methods of bin packing, namely the integer packing and the fractional packing. In the integer packing all feasible sets should be placed as a whole, which is straight-forward and intuitive. In contrast, in a fractional packing feasible sets may be split in a specific way. For example, if we want to place a part of the feasible set, say of width x, it uses width x of a bin and all items in this feasible set can be used in other feasible sets, up to width $1-x$ in total. We give two examples:

Example 1: We have two bins of size 1 and seven items with the following sizes:

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{2}{5}, \frac{1}{5}\right\}$$

Then the value of the optimal integer packing is equal to the value of the optimal fractional packing, that is $v = v' = \frac{29}{15}$. An optimal packing is shown in Figure 1.

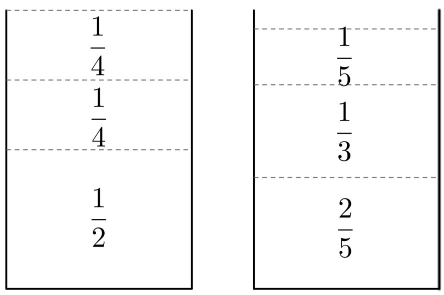

Figure 1: Both optimal integer and fractional packing

To make things clear: It is not allowed to slice a part of the unused item with size $\frac{1}{3}$, rotate it and put it on top of the items in the second bin to fill that bin as well. The only way to place items, is to create a set of items, which has a total size of at most 1, and place this set. If we visualize a feasible set as a pile of items with width 1 and height the sum of the sizes, it is allowed to reduce the width but is not allowed to reduce the height. And it is not allowed to rotate this set before placing it or to pack an item twice in one feasible set.

It occurs quite often that the optimal integer packing equals the optimal integer packing. Especially if all items have different sizes. Now we give an example with $v \neq v'$.

Example 2: There are two bins of size 1, five items with size $\frac{1}{3}$ and one item with size $\frac{1}{2}$. The optimal packings are given in the following figure. The items with size $\frac{1}{3}$ are called A, B, C, D and E for practical reasons.

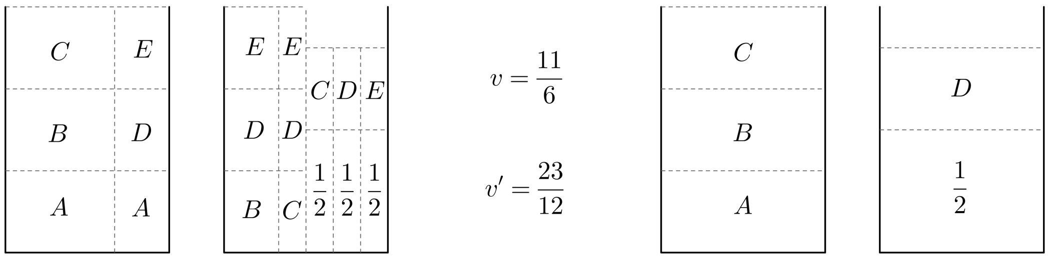

Figure 2: Different values for v and v'

The left packing in Figure 2 is an optimal fractional packing with value $v' = \frac{23}{12}$ and the right packing in Figure 2 is an optimal integer packing with value $v = \frac{11}{6}$.

3.2 Research goals

The original focus was on the bin packing game. One would like to know if it is possible to fairly distribute the total gain amongst the players in a bin packing game. A fair distribution could be defined as a distribution that is in the *core* of a cooperative game. This is already explained in the chapter 2.1.

If there does not exist such a distribution, one is interested in finding a distribution that violates the fairness constraints least possible, i.e. as fair as possible. This leads to the introduction of the ϵ-core. This ϵ-core is nonempty if and only if $\epsilon \geq (v' - v)/v$'. Therefore bin packing, as defined in the introduction, is connected to the bin packing game.

We would like to know what the minimal value of ϵ is such that all bin packing games have a non-empty ϵ-core. Or at least for specific subclasses like games with only two bins. Another interesting question concerns the range of the GAP. GAP is possibly uniformly bounded, which would have pleasant consequences. Generally speaking, a linear program needs less time to be solved than a corresponding mixed integer program. So calculating the optimal value of the non-practical linear program first, could give useful information. If, for example, a company has to deliver a large amount of boxes of different sizes and they need to order the amount of trucks they would like to hire soon. Then solving the fast linear program would give a good indication for the number of trucks they have to order. Later on, they can solve the slower integer program to determine the optimal packing.

Another application is cost distribution. As mentioned in Chapter 2 the dual of the linear program of the fractional packing problem is the linear program that decides how we should distribute value in a fair way amongst a set of players.

This research is part of the research of Dr. W. Kern (University of Twente). He started this research about twenty years ago and has made some progress in solving this problem. A few others also derived some results, which are explained in the next subsection.

3.3 Earlier work

There are already some results made by other researchers. In [2], Faigle and Kern observed that $\epsilon_{\min} \leq 1/2$: as long as there are items available, each bin could be filled to at least half its capacity. Hence, $\epsilon_{\min} \leq \frac{v'-v}{v'} \leq \frac{\frac{1}{2}k}{k} = \frac{1}{2}$. Furthermore, they gave the example showed in Figure 3. This example has an itemset consisting of three items of size $\frac{1}{2}$, called $\frac{1}{2}$, $\frac{1}{2}'$ and $\frac{1}{2}''$, and one item of size $\frac{1}{2} + \delta$ with $0 < \delta \leq \frac{1}{2}$. If we take the limit by letting $\delta \to 0$, we create an example with $v \to \frac{3}{2}$ and $v' \to \frac{7}{4}$, which results in GAP $\to \frac{1}{4}$ and $\epsilon \to \frac{1}{7}$. So, $\frac{1}{7} \leq \epsilon_{\min} \leq \frac{1}{2}$.

In [3], Woeginger proved that no example exist with $\epsilon > \frac{1}{3}$. He assumed there exist such examples and among those examples there should be one with a minimal number of players (bins and items). He then derived a contradiction, proving that $\epsilon_{\min} \leq \frac{1}{3}$. This is slightly improved by Kern & Qiu in [6] by using a different (greedy heuristic) approach which results in $\epsilon_{\min} \leq \frac{35}{108}$.

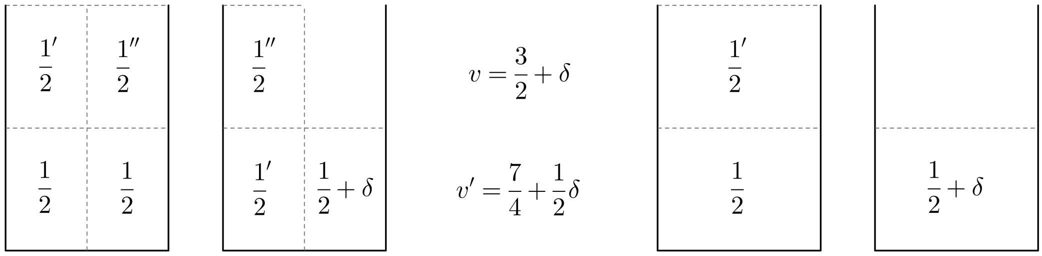

Figure 3: Example with $\epsilon \to \frac{1}{7}$

If all items have a size strictly larger than $\frac{1}{3}$, the $\frac{1}{7}$-core is always non-empty. This is proved by Kuipers in [4], using the fact that any feasible set contains at most two items. Note that the example in Figure 3 is part of this subclass. So for every $\epsilon < \frac{1}{7}$, there exist an example with an empty ϵ-core. In the spirit of the proof of Kuipers, Faigle and Kern [2] proved another theorem concerning items with $a_i > \frac{1}{3}$:

Theorem 3.1 [2] *If all itemsizes $a_i, \ldots, a_n$ in the bin packing game are strictly larger than 1/3, then $v' - v \leq \frac{1}{4}$.* □

Another useful result was made in [6]. Let $\delta \in (0, 1)$ and let N_δ be the subset of N that does not contain items with size $a_i \leq \delta$. We define ϵ_N and ϵ_{N_δ} as the minimal ϵ for N and N_δ respectively, that results in a non-empty core. Then,

Lemma 3.2 *If $\delta, \epsilon_{N_\delta} \leq \epsilon$, then $\epsilon_N \leq \epsilon$.*

The consequence of this lemma is that we can ignore 'small' items. For example, if we would like to prove that $\epsilon = \frac{1}{4}$ implies a non-empty core for all bin packing games, we can ignore all items that have a size $a_i \leq \frac{1}{4}$.

3.4 Conjectures

After all, we have some conjectures we would like to prove or to disprove. Some of them were made by other researchers and some of them were created during our study.

- The 1/7-core is non-empty for all bin packing games.
- There exists a constant C such that $v' - v \leq C$ for all bin packing games.
- For a bin packing game with 2 bins, GAP $= v' - v \leq \frac{1}{4}$.
- If $k \leq 5$, then $v' - v < \frac{1}{3}$.

4 Bin packing vs. 3-PART

A way to look at Bin Packing problems is seeking similarities with other (known) problems. In this section, instances of a problem called 3-partition are used to generate interesting instances for our Bin Packing problem. Interesting examples are for example small instances with a large GAP.

4.1 Background

If we have k bins of size 1 and $3k$ items whose sum of sizes equals k. Is it possible to place three items per bin such that all items are placed? This question is basically the well-known NP-complete 3-partition problem (3-PART). An interesting case, with applications to bin packing, has been investigated by Joosten [7]. He considered the ILP of 3-PART and removed the integrality, allowing all variables to have any value in [0,1], which turned the ILP into an LP. He found some instances where the LP has a solution, whereas the corresponding ILP does not have a solution. These instances were called *nearly-feasible* instances. In particular, he searched for instances where the solution vector of the LP only contains the values 0, $\frac{1}{2}$ and 1. Furthermore he tried to find instances which are minimal with respect to the number of bins. His main result was: Every *nearly-feasible* instance has $k \geq 6$. Moreover, there exist examples with 6 bins and 18 items.

He also gave some nearly-feasible instances in [7] and all these instances have integer itemsizes. For example, look at the instance:

$$\{0, 0, 0, 2, 2, 2, 3, 3, 4, 7, 7, 8, 8, 8, 9, 10, 11, 12\} \;\; (\text{Binsize} = 16)$$

These 18 elements are the 18 itemsizes and the total binsize available is the sum of all those itemsizes. In this case, the sum is 96, so each bin must have size 16. We are looking for $6t$ ($t \in \mathbb{N}$) sets of three different elements, such that every set has the same sum and all elements appear t times in the sets. An example with $t = 2$ is shown in Figure 4 [7].

A graph like this is called a *solution graph.* Every node is a feasible set and its elements are shown by their sizes. A ' symbol is added to distinguish between two elements of the same size. Furthermore, a line between two sets is drawn for every element they have in common. This results in a graph. In the case $t = 2$, all nodes have degree three, so it is a cubic graph.

For the instance above there exists a solution with $t = 2$. We now show that in this example there does not exist a solution for $t = 1$. Note that

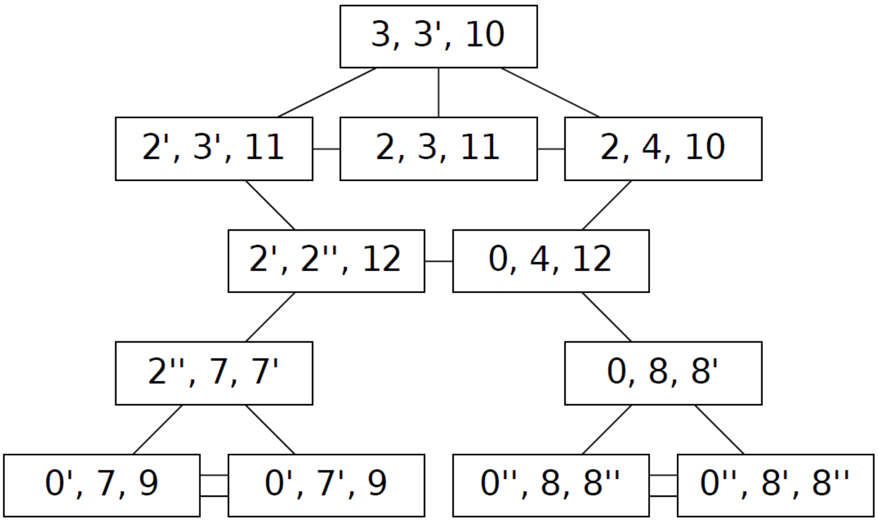

Figure 4: Example of a solution graph

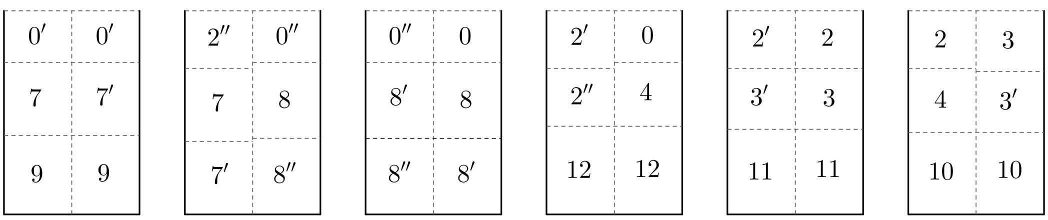

Figure 5: Corresponding Bin Packing

$t = 1$ implies that every element should be in exactly one set. Now look at the three items of size 8. To get such an item in a set, we have to find two other items whose sizes have sum 8. The only possibility is another item of size 8 and an item of size 0. So, an item of size 8 should be packed with another item of size 8. Thus, we can only pack an even number of items of size 8. However, we have three items of size 8. Hence, it is not possible to pack all of them. So, there does not exist a solution where all items are packed.

These instances can be applied to our bin packing problem. All sets in the 3-PART solution are used feasible sets in the bin packing with width $y_F = \frac{1}{t}$ for all used sets F. For example, the equivalent bin packing of the solution graph shown in Figure 4 is the packing in Figure 5. Because $t = 2$, the width of every set in the bin packing is $\frac{1}{2}$. However, bin packing does not have a constraint like 'every bin can contain at most three items'. So we have to rescale the itemsizes, such that at most three items fit in a bin.

One method to achieve a suitable instance is to add a constant C to all itemsizes and to add three times that constant ($3C$) to the binsizes. If the smallest item has a size bigger than a quarter bin, at most three items fit in a bin. Applying this technique, with $C = 17$, to the previous instance results into:

$$\{17, 17, 17, 19, 19, 19, 20, 20, 21, 24, 24, 25, 25, 25, 26, 27, 28, 29\} \quad (\text{Binsize} = 67)$$

Another way to get a feasible instance is to divide all sizes by a large number and add (almost) $\frac{1}{3}$ to all itemsizes to get bins with sizes 1. Let ϵ be a small number, but larger than 0. Dividing all itemsizes by $\frac{1}{3\epsilon}$ and adding $(\frac{1}{3} - 16\epsilon)$ results in the following itemsizes (with binsize 1):

$$\sigma_A = \left\{ \frac{1}{3} - 16\epsilon, \frac{1}{3} - 16\epsilon, \frac{1}{3} - 16\epsilon, \frac{1}{3} - 10\epsilon, \frac{1}{3} - 10\epsilon, \frac{1}{3} - 10\epsilon, \frac{1}{3} - 7\epsilon, \frac{1}{3} - 7\epsilon, \frac{1}{3} - 4\epsilon \right.$$
$$\left. \frac{1}{3} + 5\epsilon, \frac{1}{3} + 5\epsilon, \frac{1}{3} + 8\epsilon, \frac{1}{3} + 8\epsilon, \frac{1}{3} + 8\epsilon, \frac{1}{3} + 11\epsilon, \frac{1}{3} + 14\epsilon, \frac{1}{3} + 17\epsilon, \frac{1}{3} + 20\epsilon \right\}$$

Now take the limit: $\epsilon \to 0$. The gap between the optimal integer solution and the optimal fractional solution is at least the size of the smallest item, which is $\frac{1}{3}$ in the limit.

4.2 Modifying known instances

So every 3-PART example is a bin packing problem with three items per set and every nearly-feasible instance of 3-PART is an instance of a bin packing problem with a positive GAP, which approaches $\frac{1}{3}$ in the limiting case. In the previous subsection we had an example with a GAP of $\frac{1}{3}$ in the limit, which has 18 items and 6 bins. In this subsection we present a method of merging items to get less items and an example with a GAP of $\frac{1}{3}$ in the limit with only 15 items.

4.2.1 Merging items

If we look at Figure 4, we see two pairs of nodes with two lines connecting them. This implies that there are two pairs with two elements in common, namely {0”, 8, 8”} & {0”, 8’, 8”} and {0’, 7, 9} & {0’, 7’, 9}. Because every element occurs twice in the sets, those common items are always together so we can consider them as one item instead of two. This results in an instance with the same GAP and the same number of bins, but with less items. In this case we can merge 0” & 8” and 0’ & 9 resulting in a total of 16 items.

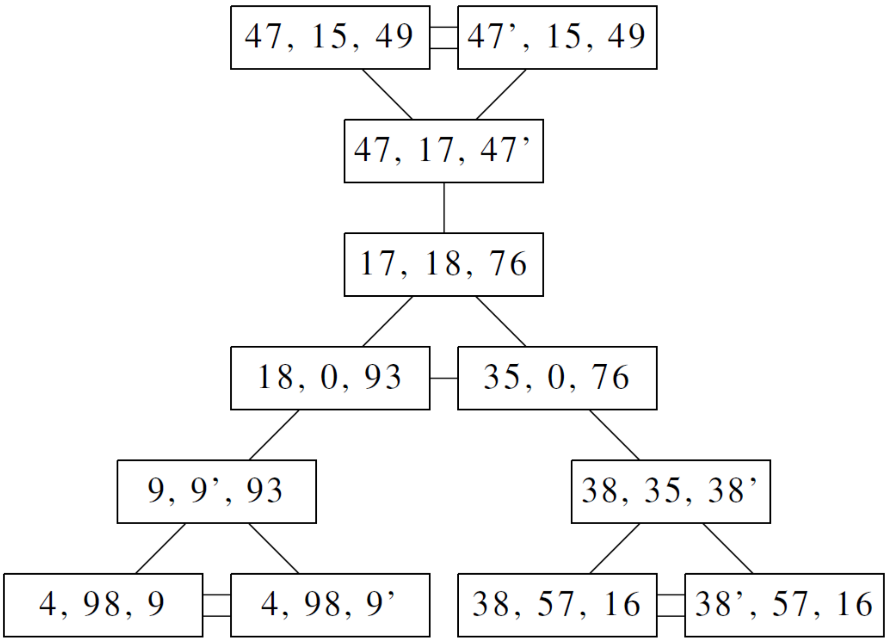

Figure 6: Three pairs of sets with two elements in common

There also exists an example with three pairs of ‘connected’ items. This is shown in Figure 6. In this case we can merge 4 & 98, 57 & 16 and 15 & 49 to form 102, 73 and 64 respectively. This results in an instance with 15 items, 6 bins and GAP$= \frac{1}{3}$, which is the smallest known example with GAP$\geq \frac{1}{3}$.

So we are not able to decrease the number of items of bins anymore, but we can decrease the itemsizes without creating non-integers. Such an example is mentioned below. Its itemsizes are:

$$\sigma_A = \{7, 8, 8, 9, 10, 10, 11, 11, 11, 13, 13, 17, 22, 23, 25\} \;\; (\text{Binsize} = 33)$$

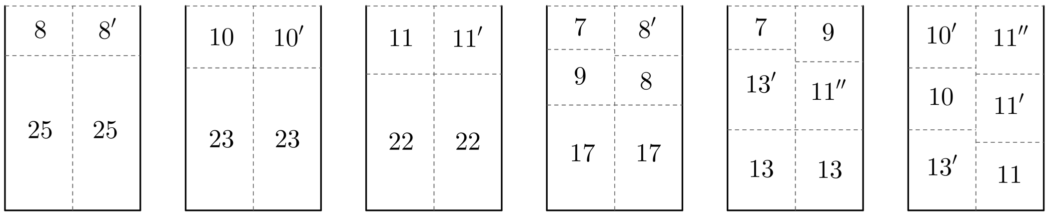

Figure 7: Fractional solution

The corresponding fractional bin packing is shown in Figure 7. If we modify this example to get bins of size 1 we get the following itemsizes:

$$\sigma_A = \left\{ \frac{1}{3} - 4\epsilon, \frac{1}{3} - 3\epsilon, \frac{1}{3} - 3\epsilon, \frac{1}{3} - 2\epsilon, \frac{1}{3} - \epsilon, \frac{1}{3} - \epsilon, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \right.$$
$$\left. \frac{1}{3} + 2\epsilon, \frac{1}{3} + 2\epsilon, \frac{1}{3} + 6\epsilon, \frac{2}{3}, \frac{2}{3} + \epsilon, \frac{2}{3} + 3\epsilon \right\} \;\; (\text{Binsize} = 1)$$

It is interesting to see that the optimal fractional packing, with these itemsizes, has the structure showed in Figure 8. All the (twelve) rectangles are used feasible sets of size 1. Two sets with a common item are connected by a line. This results in a graph with 12 vertices and 15 edges, which can be split into four complete graphs of three vertices and three edges connecting the three triangles to a connected graph. In the fractional packing we can use all twelve feasible sets half (that is $y_F = \frac{1}{2}$ for all used sets) to get a feasible solution, while it is not possible to pick more than four sets for the integer packing. Therefore, if there exist an integer packing with value v such that $v = v'$, there should be two other feasible sets with size 1 to get all items packed. If not, at least one item can not be packed, which results in a GAP of at most $\frac{1}{3}$ ($\frac{1}{3}$ in the limit).

To find an example with a GAP close to $\frac{1}{3}$ we have to find suitable values for a, b and c. Values are suitable if they do not create more feasible sets of size 1. A few examples of things that should not occur are: $a + b + c \neq 1$, $(1-2a)+b+c \neq 1$ and $(1-2a)+(1-2b)+c \neq 1$. There are more examples,

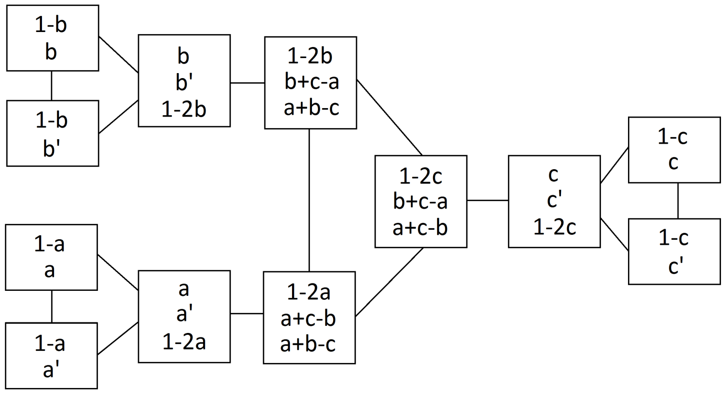

Figure 8: structure of the example

but this has not been investigated thoroughly.

To get the itemsizes we mentioned before, choose $a = \frac{1}{3} - 3\epsilon$, $b = \frac{1}{3} - \epsilon$ and $c = \frac{1}{3}$. If we want to have integer values, this case is equivalent to the case with binsize 33 (replace the 1 in Figure 8 by 33) and $a = 8$, $b = 10$ and $c = 11$.

If we look at Figure 8 an interesting question arises. Is it possible to replace the sets $\{1 - 2b, b + c - a, a + b - c\}$, $\{1 - 2c, b + c - a, a + c - b\}$ and $\{1 - 2a, a + c - b, a + b - c\}$ by a single set, which would lead to an example with only 5 bins? Unfortunately, that is not possible. Merging those sets implies that we do not have itemsizes $a+b-c$, $a+c-b$ and $b+c-a$ anymore and that the set $\{1 - 2a, 1 - 2b, 1 - 2c\}$ would be a set of size 1. However, this implies that $a + b + c = 1$ as well and we can create five disjoint sets of size 1, which is an feasible integer solution with value 5. So, the GAP would be zero.

4.2.2 Existence of cases

Joosten showed in [7] that it is not possible to find an example with $3k$ items, k bins, 3 items per bin and a GAP of $\frac{1}{3}$ if $k < 6$ and $t = 2$. In our previous examples we merged two items together to get less items, but still resulted in a feasible example. We are wondering if it is possible to have an infeasible example with $3k$ items ($k \leq 5$), where merging leads to feasibility. To investigate this, we start with a lemma:

Lemma 4.1 *If there are $3T$ items with a total value of T and 3 different items per feasible set, then $T-1$ disjoint feasible sets with value 1 imply that there exists another disjoint feasible set with value 1.*

Let the items have sizes a_i ($i \in \{1, 2, \dots, 3T\}$) and sort the items such that feasible set x contains the items a_{3x-2}, a_{3x-1} and a_{3x}. Then,

$$\sum_{i=1}^{3T} a_i = T \text{ and } a_1 + a_2 + a_3 = \dots = a_{3T-5} + a_{3T-4} + a_{3T-3} = 1 \text{ imply}$$

$$a_{3T-2} + a_{3T-1} + a_{3T} = \sum_{i=1}^{3T} a_i - \sum_{i=1}^{3T-3} a_i = T - (T-1) = 1.$$

Hence, there are T disjoint sets of size 1. □

Note that this Lemma also holds if we replace 3 by an arbitrary other positive integer.

In turns out that no such examples exist. We prove it in the next theorem and we also show the infeasibility of the cases with $3k$ items. These are already proved in [7], but we use a different approach.

Theorem 4.2 *If we have $k < 6$ bins, $n \leq 3k$ items with $\sum_i a_i = k$ and there exists a fractional packing with value $v' = k$, at most three items per bin and for every $F \in \mathcal{F}$ there holds $y_F = \frac{1}{2}$ or $y_F = 0$, then $v = k$ as well.*

To prove this we use a 'brute force' approach and prove in every case with k bins, that there exist $k-1$ disjoint feasible subsets of size 1. According to the previous Lemma there exists a k'th disjoint subset as well.

Case: $k = 1$
Trivial. There exists a feasible packing of size 1. Hence $v = 1 = k$.

Case: $k = 2$
Trivial. One feasible subset of size 1 implies that there exists another feasible set of size 1. Hence $v = 2 = k$.

Case: $k = 3$
If there are 3 bins, we have 6 used feasible sets. There are (at most) 9 items of which some of them could have size 0 to deal with cases that contain less than 9 items. We number these items from 1 to 9. Without loss of generality the first feasible set is $\{1, 2, 3\}$. Every item is in at most two used feasible

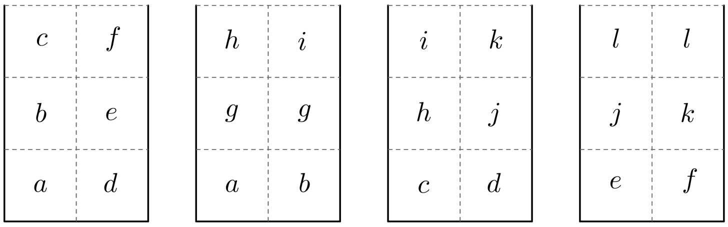

Figure 9: Placing of the 12 items

sets, so there are not more than three other used feasible sets with item 1,2 or 3. So there are two other used feasible sets disjoint with {1,2,3}. Hence, two disjoint feasible sets so there exist a third disjoint feasible set, which results in $v = 3 = k$.

Case: $k = 4$, $n \leq 11$

In this case we have 11 items, with possibly some items of size 0, 4 bins and 8 used feasible sets. If we have a feasible set with only one item, it occurs twice and the remaining filling problem is equal to the case with only three bins. So we can assume that all feasible sets have at least two items. Because every item occurs in exactly two used feasible sets, there are (at least) two sets that contain only two items. Call those sets F_1 and F_2. If $F_1 = F_2$, the resulting problem, filling the other three bins, is equivalent to the case with $k = 3$, so this results in $v = k$ as well.
If $F_1 \cap F_2 = \emptyset$, they intersect with maximal four other sets. While there are six sets left, there is a third disjoint set. Hence, $v = 4 = k$.
The remaining case is $F_1 \cap F_2 = \{a\}$ for some item a. If we now take a set of two items and a disjoint set of three items (which clearly exists), the union of these sets is non-disjoint with maximal 5 other sets. Because there are 6 other sets, there exist a third disjoint set as well. Hence, $v = 4 = k$.

Case: $k = 4$, $n = 12$

Without loss of generality there are two disjoint sets with the following items: $\{a, b, c\}$ and $\{d, e, f\}$. To prevent from getting a third disjoint set, all items a, b, c, d, e and f are also placed in the six other sets. Because the new formed sets containing a, b or c are disjoint with $\{d, e, f\}$, they have to share an item. Applying the same argument to the other three sets, it results in the distribution shown in Figure 9.

So there are no three disjoint used sets. However, this packing is just the union of two packings in two bins (one for items $\{a, b, c, g, h, i\}$ and one for items $\{d, e, f, j, k, l\}$), each for which we already proved that there exists an integer packing of value 2. Hence, $v = 2 \cdot 2 = 4 = k$.

Case: $k = 5$, $n \leq 11$

As mentioned before, we can assume that all feasible sets contain at least two items. This implies that we have at least 8 sets with only 2 items. These 8 sets contain 3 disjoint sets, with a total of 6 items. So these the union of these three is non-disjoint with maximal 3+6=9 sets. Hence, there is a fourth disjoint set, so $v = 5 = k$.

Case: $k = 5$, $n = 12$

Now we have 6 sets with 2 items. If there are 3 disjoint sets amongst them, there also exists a fourth disjoint set in the fractional packing and we have $v = 5$. Else, we are sure that there are always two disjoint sets amongst them. To prevent three disjoint sets, we have to place the items in a specific way. Namely, we have the sets:

$$\{a, b\}, \{c, d\}, \{a, e\}, \{b, e\}, \{c, f\}, \{d, f\}$$

However, this results in 6 items of size $\frac{1}{2}$, so we can still form three disjoint feasible sets of size 1. Because the other four sets contain other items, we can create four disjoint feasible sets. Hence, $v = 5 = k$.

Case: $k = 5$, $n = 13$

There are 4 sets consisting of two items, so there are (at least) two of them disjoint. These two sets are disjoint with at least one of the other six sets containing three items. Without loss of generality these three disjoint sets are the sets: $\{a, b\}$, $\{c, d\}$ and $\{e, f, g\}$. To prevent four disjoint sets, the other seven sets must contain each one of these items. Furthermore, if item e, f or g is in a two-item-set (together with item h), we would have three disjoint two-item-sets which implies a fourth disjoint itemset as well. So the items e, f and g are in different three-item-sets all disjoint with $\{a, b\}$ and $\{c, d\}$. If two of these three sets are disjoint, we would have four pairwise disjoint sets and we are done. So we can assume that these sets contain the items: $\{e, i, j\}$, $\{f, i, k\}$ and $\{g, j, k\}$. But now we have a bin packing for two bins (four feasible sets where every item occurs twice) which results in $v = 2$ on that part. Hence, $v = 5 = k$.

Case: $k = 5$, $n = 14$

Two sets have only two items. Call these sets F_1 and F_2. Then we consider three cases: $F_1 = F_2$, $F_1 \cap F_2 = \{a\}$ and $F_1 \cap F_2 = \emptyset$.

If $F_1 = F_2$, the items in these sets are fully used and our instance reduces to

the case: $k = 4$ & $n = 12$.
If $F_1 \cap F_2 = \emptyset$, we say that these sets are: $F_1 = \{a, b\}$ and $F_2 = \{c, d\}$. There exists a third feasible set, say $\{e, f, g\}$, and to prevent a fourth disjoint set, all these seven items should be in the other seven sets. Furthermore the three sets containing e, f or g are all disjoint with both F_1 and F_2, so they must intersect each other pairwise. Hence, the sets $\{e, f, g\}$, $\{e, h, i\}$, $\{f, h, j\}$ and $\{g, i, j\}$ exist and this is just a bin packing of two bins. Hence $v = 2$ on that part, reducing our instance to the case $k = 3$.

If $F_1 \cap F_2 = a$, we can assume that $F_1 = \{a, b\}$ and $F_2 = \{a, c\}$, which implies that item b and c have the same size. If b and c are never in the same set, we could swap to form $F_1 = \{a, c\} = F_2$ which results in $v = k$ by reduction to $k = 4$. Therefore we can assume that there exists a set $F_3 = \{b, c, d\}$ and $F_4 = \{d, e, f\}$ exists as well. If items e and f are together in another set, we can replace these two items by one item, which results in case $k = 5$ and $n = 13$. Therefore, $e \in F_5$ and $f \in F_6$. The remaining sets F_7, F_8, F_9 and F_{10} are disjoint with both F_1 and F_4, which are disjoint as well, so they must intersect each other pairwise. This results again in a two-bin packing which has value 2 as mentioned before. So this reduces our instance to the case $k = 3$ and we are done.

Case: $k = 5$, $n = 15$

We now have 10 sets, each containing 3 items, and 15 items, each placed in two different sets. For every two disjoint sets, there exist a third disjoint set. So w.l.o.g. the following sets exist: $F_1 = \{a, b, c\}$, $F_2 = \{d, e, f\}$ and $F_3 = \{g, h, i\}$. The remaining seven sets should all contain at least one of these items to prevent a fourth disjoint set. We can now distinguish three subcases:

- $F_4 = \{a, d, g\}$.
- $F_4 \supseteq \{a, d\}$ & $F_5 \supseteq \{b, e\}$.
- $F_4 \supseteq \{a, d\}$ & $F_5 \supseteq \{b, g\}$.

Note that if two items are both times in the same set, we can replace them by a single item which results in the bin packing problem with 14 items, a case that has been treated before.

If $F_4 = \{a, d, g\}$, then all other six sets have one item in common with the union of F_1, F_2 and F_3 and are disjoint with F_4. Furthermore, if $b \in F_5$ and $c \in F_6$, then both sets are disjoint with both F_2 and F_3 so F_5 and F_6

should intersect. We now apply the same argument to the other four sets, which results in: $F_5 = \{b, j, .\}$, $F_6 = \{c, j, .\}$, $F_7 = \{e, k, .\}$, $F_8 = \{f, k, .\}$, $F_9 = \{h, l, .\}$ and $F_{10} = \{i, l, .\}$ where the items m, n and o have not been distributed yet. However, it does not matter how we distribute the items m, n and o, there are always three disjoint sets amongst the last six sets. Together with F_4 they form four disjoint sets and there holds: $v = 5 = k$.

If $F_4 \supseteq \{a, d\}$ and $F_5 \supseteq \{b, e\}$, the remaining sets meet: $c \in F_6$, $f \in F_7$, $g \in F_8$, $h \in F_9$ and $i \in F_{10}$. So F_8, F_9 and F_{10} are disjoint with both F_1 and F_2. This implies that F_8, F_9 and F_{10} are not allowed to be disjoint. Hence: $F_8 = \{g, k, l\}$, $F_9 = \{h, k, m\}$ and $F_{10} = \{i, l, m\}$. Together with F_3, this forms a bin packing of two bins. Hence, $v = 5 = k$.

The final case is where $F_4 \supseteq \{a, d\}$ and $F_5 \supseteq \{b, g\}$. Then $c \in F_6$, $e \in F_7$, $f \in F_8$, $h \in F_9$ and $i \in F_{10}$. If $F_7 \cap F_8 = \emptyset$, then F_1, F_3, F_7 and F_8 form four disjoint sets. So we have to place item j in both F_7 and F_8. The same argument applied to F_9 and F_{10} results in $k \in F_9$ and $k \in F_{10}$. The current overview is shown in Figure 10.

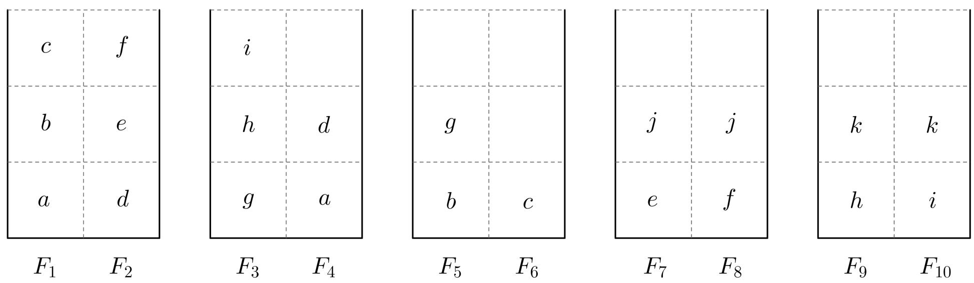

Figure 10: Overview

Without loss of generality, $F_6 = \{c, l, m\}$ and we have the following subsubcases:

- $l \in F_4$ and $m \in F_5$. Then the sets F_4 and F_5 have an empty intersection and they are both disjoint with F_7, F_8, F_9 and F_{10}. These four sets have 8 different items in its union, so $\{F_7, F_8, F_9, F_{10}\}$ contains at least two disjoint sets, which results in a total of 4. Hence, $v = 5 = k$.

- $l \in F_4$ and $n \in F_5$. Then $F_4 \cap F_5 = \emptyset$ and the items m, n and o (twice) have to be placed. If item m is placed in set F_7 or F_8, then set F_9 or F_{10} (or both) contains item o. This results in four disjoint sets. The case with $m \in F_9$ or $m \in F_{10}$ is similar.

- $n \in F_4$ and $o \in F_5$. If $l \in F_7$ or F_8 and $m \in F_9$ or F_{10}, the sets F_4, F_5 and the sets containing l and m are four disjoint sets and we are done. Else w.l.o.g.: $F_7 = \{e, j, n\}$, $F_8 = \{f, j, o\}$, $F_9 = \{h, k, l\}$ and $F_{10} = \{i, k, m\}$. Then $\{F_3, F_4, F_6, F_8\}$ are four disjoint sets.

- $n \in F_4$, $n \in F_5$, $l \in F_7$ and $m \in F_8$. Then $\{F_2, F_5, F_6, F_9\}$ form four disjoint sets.

- $n \in F_4$, $n \in F_5$, $l \in F_7$ and $m \in F_9$. Then $\{F_2, F_5, F_6, F_9\}$ form again four disjoint sets.

Now we have covered all cases. □

Remark that these proofs also hold for cases with less items, because items can be considered as items with size 0. However, we also gave proofs for these easier cases to show that these cases can be treated in an easier way.

Immediate consequence:

Lemma 4.3 *There is no nearly-feasible instance of 3-PART for $k \leq 5$.*

We now deal with the case with 6 bins in the following lemma:

Lemma 4.4 *Assume $k = 6$ bins, $n < 14$ items with $\sum_i a_i = 6$ and there exists a fractional packing with value $v' = 6$, at most three items per bin and $y_F = \frac{1}{2}$ or $y_F = 0$ for all $F \in \mathcal{F}$. Then $v = 6$ as well.*

The cases with less than 12 items contain items of size 1 and therefore reduce to cases with less than 6 bins, which are discussed in the previous theorem. So we have two cases left: $n = 12$ and $n = 13$. In both cases, there are at least 10 feasible sets of two items, so there always exist four disjoint sets. This can be shown quite easily: We call the first set with two items F_1 and this set contains the items a and b. Then, F_1 intersect with at most two other sets and there are at least seven other sets of two items disjoint with F_1. We call one of these sets F_2 and so on.

<u>Case: $n = 12$</u>

In this case all feasible sets contain two items. To prevent 5 disjoint sets, we have to place the items in a specific way which is shown in Figure 11. However, this implies that all items have size $\frac{1}{2}$, which clearly leads to an integer packing with value 6.

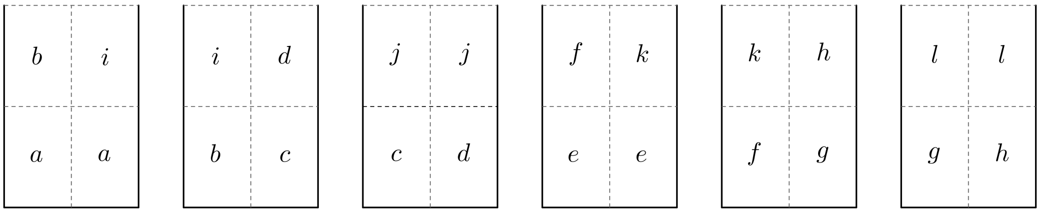

Figure 11: Fractional packing for 12 items

<u>Case: $n = 13$</u>
There are 10 sets containing two items and therefore there are four of them disjoint, say $F_1 = \{a, b\}$, $F_2 = \{c, d\}$, $F_3 = \{e, f\}$ and $F_4 = \{g, h\}$. Then the items of at least two of these sets are in other sets of size two as well. W.l.o.g. $a \in F_5$, $b \in F_6$, $c \in F_7$ and $d \in F_8$. Note that these sets contain no item from F_3 and F_4. Hence F_5 and F_6 are disjoint with F_2, F_3 and F_4, so F_5 and F_6 should have a common item. So, $F_5 = \{a, i\}$ and $F_6 = \{b, i\}$ and applying the same argument results in $F_7 = \{c, j\}$ and $F_8 = \{d, j\}$. However, F_1, F_5 and F_6 imply that $a = b = i = \frac{1}{2}$ and F_2, F_7 and F_8 imply that $c = d = j = \frac{1}{2}$. So, $i + j = 1$ and we have a feasible set disjoint with F_1, F_2, F_3 and F_4. Hence, $v = 6$.

This completes the proof. $\square$

The case with $n = 14$ ended with a remarkable result. We were expecting that there does not exist such an example with a positive GAP and we tried to prove this. However, it resulted in the example shown in Figure 12:

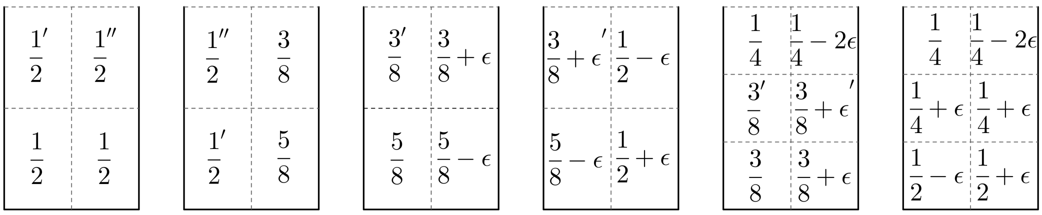

Figure 12: Fractional packing for 14 items

In Figure 12 we see 12 sets of which at most 4 are disjoint. The itemsize $\frac{1}{2}$ occurs three times, while the itemsizes $\frac{3}{8}$ and $\frac{3}{8} + \epsilon$ occur twice. Its solution graph is shown in Figure 13.

If there would exist an integer packing with value $v = 6$, all three items with size $\frac{1}{2}$ should be packed. However, the only possibility to pack these items in a set of size 1, is packing it along with another item of size $\frac{1}{2}$. Because we have three items of size $\frac{1}{2}$, this is impossible. Hence, $v < 6$. Because the smallest item has size $\frac{1}{4} - 2\epsilon$, there holds: $v = \frac{19}{4} + 2\epsilon$ and GAP $= \frac{1}{4} - 2\epsilon$.

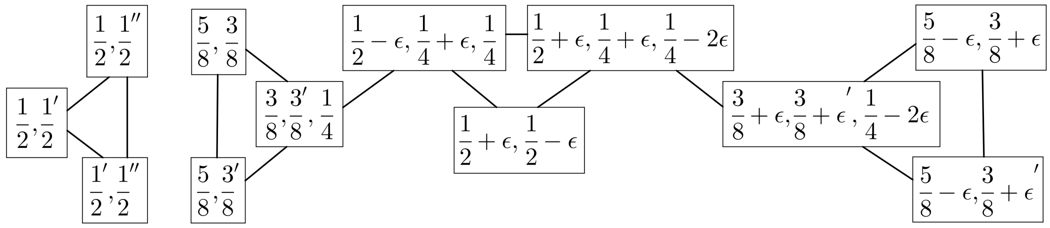

Figure 13: Solution graph

In Figure 13 we see the solution graph, which is remarkably disconnected. Its left part has three nodes consisting of two items each, which makes it impossible to get an equivalent 3-PART instance.

5 Two bins

The easiest case to investigate is the case where there is only 1 bin. However, this is trivial because only the largest feasible set is used in the optimal fractional packing, which is an integer packing as well. So there is no need to look at cases with $k = 1$. The case $k = 2$ is the first non-trivial case and that is why we are studying bin packing with two bins. We distinguish two cases: general and $v' = 2$.

5.1 General

In this subsection we treat the general case, i.e. there are two bins and no other constraints.

Lemma 5.1 *If there are two bins, the GAP, the difference between the optimal value of the fractional solution v' and the optimal value of the integer solution v, is at most $\frac{1}{2}$.*

Let F be the used set with the largest value. We call this value σ_F (≤ 1). Using F to fill the first bin, we can fill the second bin to at least half its capacity as stated in section 3.3 [2]. Then,

$$v \geq \sigma_F + \frac{1}{2} \geq 2\sigma_F - \frac{1}{2} \geq v' - \frac{1}{2}$$

Hence, $v' - v \leq \frac{1}{2}$. □

Creating an integer packing
To create an integer packing, it seems to be a good guess to take the largest feasible set F_1 and the largest feasible set disjoint from F_1. However, these two sets might not form the optimal packing. We can also take G_1, the second largest feasible set and the largest feasible set disjoint from G_1. This might lead to a better result, but could also be non-optimal.

We now give an example where we should ignore the $p - 1$ largest feasible sets ($p \geq 2$) to get an optimal packing:

$$\sigma_A = \left\{\frac{1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} - p\delta, \frac{1}{2} - p\delta, 2p\delta, (2p-1)\delta, (2p-2)\delta, \ldots, (p+2)\delta\right\}$$

Then, the total sum of all items is $2+\frac{1}{2}(3p^2-p+2)\delta$ and the largest feasible sets are:

$$
\begin{aligned}
&F_1 = \left\{\frac{1}{2}-p\delta, \frac{1}{2}-p\delta, 2p\delta\right\} \quad \sigma_{F_1} = 1 \\
&F_2 = \left\{\frac{1}{2}-p\delta, \frac{1}{2}-p\delta, (2p-1)\delta\right\} \quad \sigma_{F_2} = 1-\delta \\
&F_3 = \left\{\frac{1}{2}-p\delta, \frac{1}{2}-p\delta, (2p-2)\delta\right\} \quad \sigma_{F_3} = 1-2\delta \\
&\vdots \\
&F_{p-1} = \left\{\frac{1}{2}-p\delta, \frac{1}{2}-p\delta, (p+2)\delta\right\} \quad \sigma_{F_{p-1}} = 1-(p-2)\delta \\
&F_p = \left\{\frac{1}{2}+\delta, \frac{1}{2}-p\delta\right\} \quad \sigma_{F_p} = 1-(p-1)\delta
\end{aligned}
$$

Using one of the first $p-1$ feasible sets implies that we do not use both items of size $\frac{1}{2}+\delta$. We then get an integer packing of value at most $\frac{3}{2}+\frac{1}{2}p(3p-1)\delta$, while using F_p results in an integer packing with value $v = 2-2(p-1)\delta$, which is larger if δ is small enough. Moreover, if we take $\delta \to 0$, this difference is approaching $\frac{1}{2}$, which is the worst-case as stated in the previous lemma.

Lemma 5.2 *Using the largest feasible set may yield a value gap with respect to the optimal integer packing of size* $\approx \frac{1}{2}$.

So considering the use of the largest feasible set seems to be useless if we want to prove that the maximal GAP is less than $\frac{1}{2}$.

5.2 $v' = 2$

In this case we assume that there are two bins and the optimal fractional packing has value $v' = 2$.

Let $y = (y_F)$ be an optimal fractional packing and let s_y be the cardinality of the support of y (that is, the number of used sets in the fractional packing). Clearly, $s_y \geq 2$. If the optimal fractional packing contains two disjoint sets (of size 1), those sets form an optimal integer packing of size 2 as well. We are looking for examples with a (large) GAP so these cases are not interesting anymore. Therefore we assume in what follows that any two used sets in the fractional packing have at least one element (item) in common.

Before we distinguish cases with a different s_y we give some general results.

Lemma 5.3 *If we have two bins, $a_i > \frac{1}{4}$ for all items and there exists a fractional packing of value 2, then GAP $< \frac{1}{3}$.*

Proof Because $v' = 2$, the entire width of the two bins is used, so $\sum_i y_{F_i} = 2$. We now take an arbitrary used set and call this set F_1. If F_1 contains not more than two items it is disjoint with a certain other used set, because its items do not cover more than $y_{F_1} + 2(1 - y_{F_1}) < 2$ of the total width. So F_1 contains at least three items. Because all items have a size larger than $\frac{1}{4}$, F_1 should contain exactly three items.

Assume that $F_1 = \{a, b, c\}$. Then the following sets are non-empty: $A = \{F_i | F_i \cap F_1 = \{a\} \wedge \sigma_{F_i} = 1\}$, $B = \{F_i | F_i \cap F_1 = \{b\} \wedge \sigma_{F_i} = 1\}$ and $C = \{F_i | F_i \cap F_1 = \{c\} \wedge \sigma_{F_i} = 1\}$.

If item a, b or c has a size smaller than $\frac{1}{3}$, we can use F_1 and a item from set A, B or C respectively to form an integer packing with size $> \frac{5}{3}$ by using item a, b or c once.

Else, the items a, b and c have size $\frac{1}{3}$. Applying the same argument to other sets, leads to the observation that all items have size $\frac{1}{3}$. Then we can easily form an integer packing with value 2. So, GAP $< \frac{1}{3}$. $\square$

Lemma 5.4 *If we have two bins, $v' = 2$ and the most used set has width w, then GAP $\leq \frac{1-w}{2-w}$.*

Call the most used set F_1, so $y_{F_1} = w$. The remaining used sets (with a total width of $2 - w$) should all contain at least one item from F_1. Assume there does not exist an used set, where the total size of the items in the intersection with F_1 is at most $\frac{1-w}{2-w}$. Then, the total size of the items form F_1 exceeds $w \cdot 1 + (2 - w) \cdot \frac{1-w}{2-w} = 1$, which is impossible. Contradiction, so there exists such a set. Now take this set together with F_1 to form an integer packing by deleting the intersection. This intersection has a total size of at most $\frac{1-w}{2-w}$, so GAP $\leq \frac{1-w}{2-w}$. $\square$

If $w = 1$, the GAP is obviously 0 (there always exists a disjoint used set to form an integer packing of value 2). If we take $w \to 0$, then GAP $\to \frac{1}{2}$, a value we already knew. However, if $w \to 0$, there are a lot of different used sets and a lot of items as well. It feels intuitive that the GAP is small if there are a lot of items with different sizes. So, it should be possible to make some improvements on this area.

$s_y = 2$
If $s_y = 2$, both sets are fully used. That is, if we call the sets F_1 and F_2, there holds $y_{F_1} = y_{F_2} = 1$, which is an integer packing as well. Hence, no GAP.

$s_y = 3$
If $s_y = 3$, no optimal packing exists where every two sets have a common item. To show this, call the used sets F_1, F_2 and F_3. We use the fact that every item occurs in total at most once in the packing. (This is just the constraint: $\sum_{F \ni i} y_F \leq 1$.) So if two sets, F_i and F_j, have a common item, the inequality $F_i + F_j \leq 1$ must hold. Therefore, we get the inequalities:

$$y_{F_1} + y_{F_2} \leq 1, \;\; y_{F_1} + y_{F_3} \leq 1, \;\; y_{F_2} + y_{F_3} \leq 1,$$

which imply: $y_{F_1} + y_{F_2} + y_{F_3} \leq \frac{3}{2}$ and $v' \leq \sum y_{F_i} \leq \frac{3}{2}$, a contradiction. Hence, an example with $s_y = 3$ does not exist.

$s_y = 4$
An example with $s_y = 4$ does exist and we need at least seven items to achieve that: To have all pairs of sets non-disjoint, we need (at least) six items. If there were exactly six items, they need to be distributed in the following way:

$$\{a, b, c\}, \; \{a, d, e\}, \; \{b, d, f\}, \; \{c, e, f\}$$

However, this implies that $a + b + c + d + e + f = 2$ and together with $a + b + c = 1$ it is easy to see (Lemma) that $\{d, e, f\}$ is another feasible set of size 1. Hence, the optimal integer packing also has value 2.

This result also proves the following Lemma:

Lemma 5.5 *If there are two bins, $s_y = 4$, the optimal fractional packing has value $v' = 2$ and all items have a size strictly larger than $\frac{1}{4}$, that is $a_i > \frac{1}{4}$ for all i. Then, $v = 2$.*

If $a_i > \frac{1}{4}$ for all i, then at most three items fit in a bin. So the optimal fractional packing contains either two disjoint sets or just six items. In case there are two disjoint sets, these sets both have size 1, which leads to an integer packing with size $v = 2$. In the case there are just six items, the system we just described also implies $v = 2$. □

An example where it is impossible to get an integer packing with value 2 is one with the following itemsizes:

$$\sigma_A = \left\{ \frac{1}{4}, \frac{1'}{4}, \frac{1''}{4}, \frac{1'''}{4}, \frac{3}{8}, \frac{3'}{8}, \frac{3''}{8} \right\}$$

To distinguish between itemsizes of the same size, we added ' signs. The sum of all those items is $2 + \frac{1}{8}$ and all items have a size of at least $\frac{1}{4}$. Hence, it is impossible to get an integer packing with value 2. But it is possible to get a fractional packing with value 2. Consider the following sets (with $y_{F_i} = \frac{1}{2}$ for all i).

$$F_1 = \left\{ \frac{1}{4}, \frac{1'}{4}, \frac{1''}{4}, \frac{1'''}{4} \right\}$$

$$F_2 = \left\{ \frac{3}{8}, \frac{3'}{8}, \frac{1}{4} \right\}$$

$$F_3 = \left\{ \frac{3}{8}, \frac{3''}{8}, \frac{1'}{4} \right\}$$

$$F_4 = \left\{ \frac{3'}{8}, \frac{3''}{8}, \frac{1''}{4} \right\}$$

The constraint $F_i \cap F_j \neq \emptyset$ holds for all i, j, all sets have size 1 and no item is used more than twice. Hence, this is a feasible fractional packing of size 2. The optimal integer packing has value $\frac{15}{8}$, so the GAP equals $\frac{1}{8}$. This is shown in Figure 14.

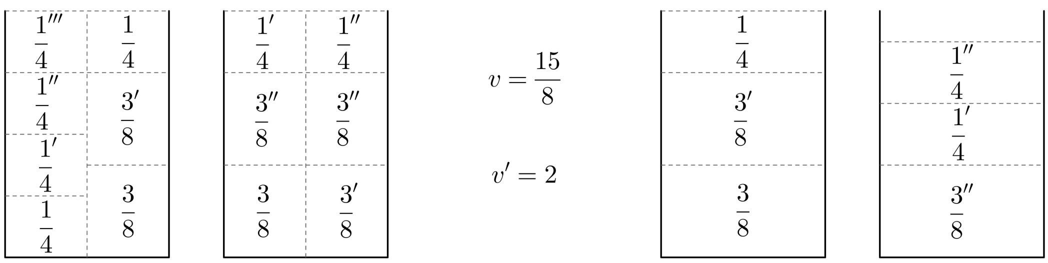

Figure 14: Optimal packings

The previous example is the limiting case of the structure shown in Figure 15. We assume that this structure holds and try to calculate a packing such that the GAP is maximized.

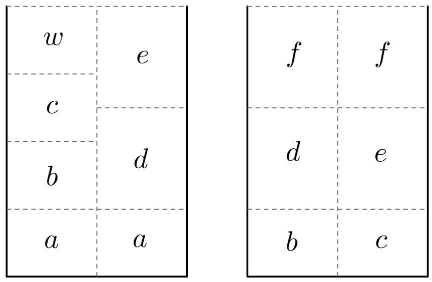

Figure 15: Structure

It contains four feasible sets:

$$\{a, b, c, w\},\ \{a, d, e\},\ \{b, d, f\}\ \{c, e, f\}$$

And all these sets have size 1. This leads to four constraints:

$$\begin{aligned} a + b + c + w &= 1 \\ a + d + e &= 1 \\ b + d + f &= 1 \\ c + e + f &= 1 \end{aligned}$$

Which results in the itemsizes shown in Table 1.

Item	**Size**
a	a
b	b
c	c
d	$\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c$
e	$\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c$
f	$\frac{1}{2} + \frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}c$
w	$1 - a - b - c$
Sum:	$\frac{5}{2} - \frac{1}{2}(a + b + c)$

Table 1: Itemsizes

The values of a, b and c can still be chosen, while the values of the other four variables depend on a, b and c. The sum of all variables is $\frac{5}{2} - \frac{1}{2}(a + b + c)$, which is equal to $2 + \frac{1}{2}w$. This is what we expect, because w is the only itemsize that occurs only once in the fractional packing as shown in Figure 15.

If all items have a size of at least w, the integer packing has a value of at most $2 + \frac{1}{2}w - \min\{a, b, c, d, e, f, w\}$. Assuming that the optimal integer packing equals this value, optimality is reached if all items have a size of at least $\frac{1}{4}$. (It is not possible to get a lower value of the optimal integer packing because of the constraint: $a+b+c+w = 1$.) This is the case in our previous example.

Lemma 5.6 *If there are two bins and $s_y = 4$, then the optimal y can be chosen to be half-integral: $y_F = \frac{1}{2}$ or $y_F = 0$ for all F.*

All sets do intersect each other pairwise. So $y_{F_i} + y_{F_j} \leq 1$ for all F_i and F_j. Because we use four sets, the result follows. $\square$

Lemma 5.7 *If there are two bins, $v' = 2$ and $s_y = 4$, then $GAP < \frac{1}{4}$.*

Proof Because $s_y = 4$, there are four sets of size 1 with $y_F = \frac{1}{2}$. We call these sets F_1, F_2, F_3 and F_4. If there are two disjoint sets the result follows immediately, so we can assume all sets do pairwise intersect. We can distinguish the items of a set into four groups: The first three groups are the items that are also in the other three sets, and the remaining group are items that are not placed in other sets. For now we assume that every group consists of only one item. (Otherwise combine the elements in a group to one new larger element.) In that case, w.l.o.g. the feasible sets are the following sets:

$$F_1 = \{a, b, c, w\}, F_2 = \{a, d, e, x\}, F_3 = \{b, d, f, y\}, F_4 = \{c, e, f, z\}$$

The sum of all these items is: $2 + \frac{1}{2}(w + x + y + z)$. We assume that $w \geq x \geq y \geq z$.

If item a has size $< \frac{1}{4}$, then the integer packing $F_1 + F_2 \setminus \{a\}$ results in $v > \frac{7}{4}$. Hence, by applying the same argument, the items a, b, c, d, e and f have a size of at least $\frac{1}{4}$. This implies that the items w, x, y and z have a size of at most $\frac{1}{4}$. So by deleting one or more items from the set $\{w, x, y, z\}$ out of the set of all items, it is possible to get a sum of all itemsizes $\in (\frac{7}{4}, 2]$.

More precisely, we delete the smallest set with a total size bigger than $\frac{1}{2}(w + x + y + z)$. There are four cases and in all four cases we can form an integer packing by using all remaining items so there always exists an integer packing with $v > \frac{7}{4}$:

If $\{x, y\}$ should be deleted, then use F_1 and $\{d, e, f, z\}$.
If $\{w, z\}$ should be deleted, then use F_2 and $\{b, c, f, y\}$.
If $\{x, y, z\}$ should be deleted, then use F_1 and $\{d, e, f\}$.
If $\{w\}$ should be deleted, then use F_2 and $\{b, c, f, y, z\}$.

So it is always possible to create an integer packing with a value larger than $\frac{7}{4}$. □

$\underline{s_y = 6}$
An example with $s_y = 6$:

$$\sigma_A = \left\{ \frac{3}{16}, \frac{3}{16}', \frac{3}{16}'', \frac{1}{4}, \frac{1}{4}', \frac{5}{16}, \frac{5}{16}', \frac{3}{8} \right\}$$

The sum equals $\frac{33}{16}$ and the smallest item has size $\frac{3}{16}$. Therefore, the value of the integer packing is not larger than $\frac{15}{8}$. This is shown in Figure 16.
To get a fractional packing with value 2, we can use the following sets (with all $y_{F_i} = \frac{1}{3}$):

$$F_1 = \left\{ \frac{3}{8}, \frac{5}{16}, \frac{5}{16}' \right\}$$

$$F_2 = \left\{ \frac{3}{8}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}' \right\}$$

$$F_3 = \left\{ \frac{3}{8}, \frac{1}{4}', \frac{3}{16}, \frac{3}{16}'' \right\}$$

$$F_4 = \left\{ \frac{5}{16}, \frac{5}{16}', \frac{3}{16}', \frac{3}{16}'' \right\}$$

$$F_5 = \left\{ \frac{5}{16}, \frac{1}{4}, \frac{1}{4}', \frac{3}{16} \right\}$$

$$F_6 = \left\{ \frac{5}{16}', \frac{1}{4}, \frac{1}{4}', \frac{3}{16}' \right\}$$

Note that, with these itemsizes, it is not possible to get two sets with all items in common, which would contradict the statement that $s_y = 6$. So, with these itemsizes we need six sets to get value 2. Furthermore, there is only one possibility, apart from permutation of items with the same size, to get an optimal fractional solution with $s_y = 6$, provided that there are 3 equal parts per bin. And if we try to get a solution with $s_y = 4$, the optimal value would be $\frac{63}{32}$.

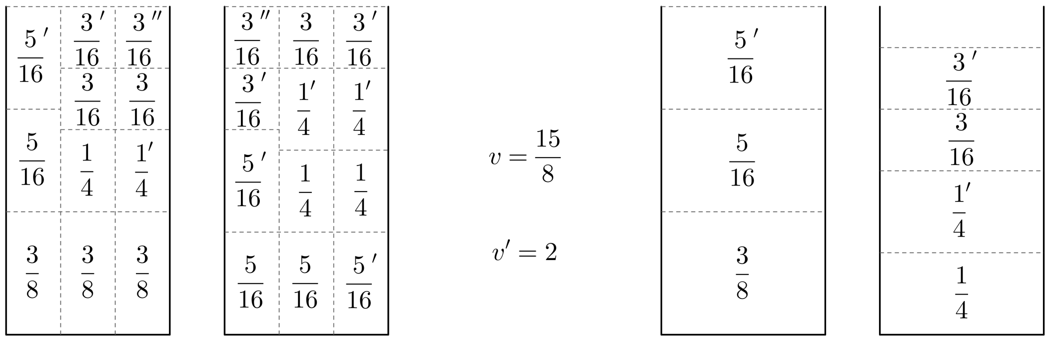

Figure 16: Optimal packings

To prove the uniqueness of this solution, consider that with these items there are four ways to get a set of size 1:

$$\alpha : \left\{\frac{3}{8}, \frac{5}{16}, \frac{5}{16}\right\}, \beta : \left\{\frac{3}{8}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}\right\}, \gamma : \left\{\frac{5}{16}, \frac{5}{16}, \frac{3}{16}, \frac{3}{16}\right\}, \delta : \left\{\frac{5}{16}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}\right\}$$

The letters α, β, γ and δ denote the number of times each set occurs.
If we look at the two items with size $\frac{1}{4}$ for example, we know that each item occurs in 3 sets. So, in total the itemsize $\frac{1}{4}$ occurs six times, which leads to the constraint: $\beta + 2\delta = 6$. Applying this technique to the other items results in four constraints:

$$\alpha + \beta = 3, \quad 2\alpha + 2\gamma + \delta = 6, \quad \beta + 2\delta = 6, \quad 2\beta + 2\gamma + \delta = 8$$

Where the last constraint equals 8 instead of 9, because one item of size $\frac{3}{16}$ is only part of two sets (instead of three). This due to the construction of these instance.

These four constraints lead to the feasible solution: $\alpha = 1$, $\beta = 2$, $\gamma = 1$ and $\delta = 2$. So it happens twice that the two items with size $\frac{1}{4}$ are together in the same set, which implies that the two other sets that contain an item with size $\frac{1}{4}$ have two different items with size $\frac{1}{4}$. Applying the same argument for items of size $\frac{5}{16}$ results in the observation that no two sets have the same items. Thus this packing is unique provided that 3 equal parts per bin are used. $\square$

Actually, there exists another solution that uses 6 different sets. This solution is shown in Figure 17 and needs 4 parts per bin.

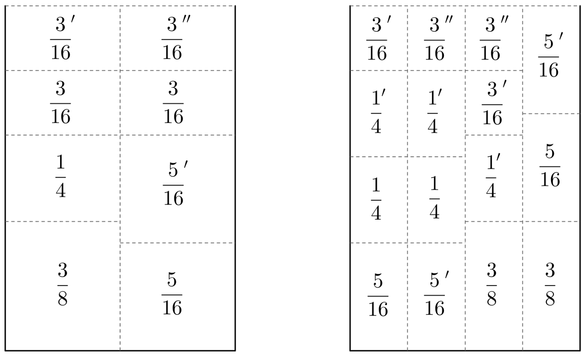

Figure 17: 6 sets

We are wondering if there exist more examples like this for $s_y = 2s$ with $s \geq 4$. To derive such an example, it may be useful to look at the strategy we used to derive the previous example with $s_y = 6$. We looked at bins of size 16 (=$8(s-1)$) and created a set of items with a total itemsize of 33 (=$16(s-1)+1$) and a minimal itemsize of 3 (=s). Then the optimal packing has a value of at most 30 and if we are lucky it may be possible to get a fractional packing with value 2. Furthermore, we would like to use $2s$ different feasible sets to get $s_y = 2s$, while it is not possible to get an optimal fractional packing if $s_y < 2s$.

We discovered that it is quite hard to do so. Although it is easy to find an example that needs s parts to reach optimality, it is hard to find an example that uses $2s$ different sets. We now give an example with $s = 5$, but $s_y \leq 7$:

$$\sigma_A = \left\{ \frac{5}{32}, \frac{5}{32}', \frac{5}{32}'', \frac{3}{16}, \frac{3}{16}', \frac{3}{16}'', \frac{7}{32}, \frac{7}{32}', \frac{1}{4}, \frac{5}{16} \right\}$$

In Figure 18 we show that it is possible to get a packing of value 2 by using 7 different sets.

If we would have used less than 5 parts per bin, it would be impossible to get a fractional packing of value 2. This can be shown by observing that the sum of all itemsizes is $\frac{65}{32}$ and the smallest item has size $\frac{5}{32}$. So at least $\frac{1}{5}$'th should be sliced of an item to get a total size of 2.

Another example is:

$$\left\{ \frac{1}{6}, \frac{1}{6}', \frac{5}{24}, \frac{1}{4}, \frac{1}{4}', \frac{7}{24}, \frac{1}{3}, \frac{3}{8} \right\}$$

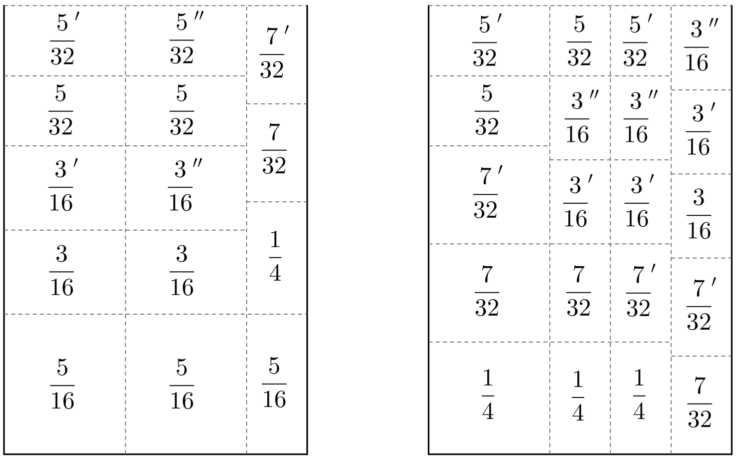

Figure 18: An optimal packing that needs 7 sets.

Its packing is shown in Figure 19.

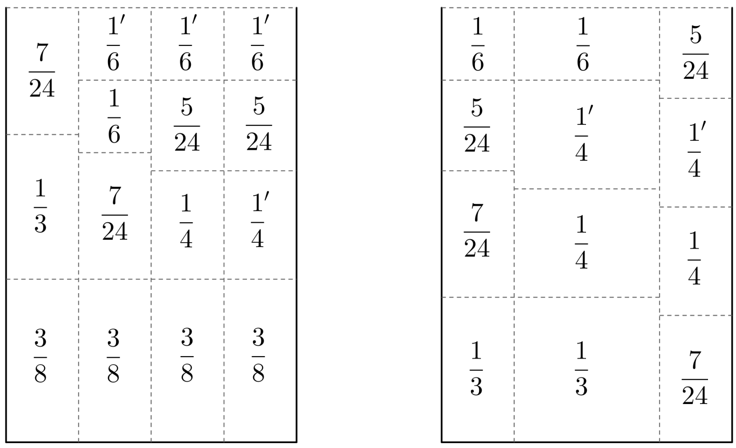

Figure 19: Another optimal packing that needs 7 sets

6 Conclusion & Discussion

After all, this research is not finished yet. The general problem has not been solved yet and there are still a lot of unsolved subclasses left. For example, we did not prove any of the conjectures mentioned in section 3.4. However, we have some conjectures about some subcases we have studied.

In section 5 the case with only 2 bins has been described. We gave some interesting examples and proved some lemmas for certain cases. However, we think that the bounds of some lemmas can be improved. These lemmas are lemma 5.1, 5.3 and 5.7.

If we look first at lemma 5.1 from page 26, it says that the upper bound of the GAP is $\frac{1}{2}$. However, the example with the largest known GAP is Figure 3 from page 11 which has GAP $\frac{1}{4}$. So there are still improvements to make.

Lemma 5.3 from page 28 states that if every item has a size larger than $a_i > \frac{1}{4}$, the GAP is less than $\frac{1}{3}$. Actually, we could not create a single instance with a positive GAP at all!

The third example is lemma 5.7 from page 32. This lemma states that GAP $< \frac{1}{4}$, while the largest known GAP is $\frac{1}{8}$ what is shown in Figure 14 from page 30.

In section 4 we did some research about the similarities between Bin Packing and the 3-partition problem. In lemma 4.3 we derived the same result as was derived in [7]: There does not exist a nearly-feasible instance of 3-PART for $k \leq 5$. However, when we tried to prove the case for $k = 6$ up to 14 items, we found a bin packing instance with a remarkable GAP of $\frac{1}{4}$ with an even remarkable disconnected solution graph.

The question now is: What things can we do to get more improvements? Starting with small subcases, like considering only two bins, should be a good start, but in our research we were not able to solve the problem for this case. It seems to be a better strategy to use software to create the instance that maximizes the GAP out of a specific subclass of instances. So far, we only tried to do so by hand and checked the validity of our instances by using AIMMS (c.f. section 7). So there may exist instances with a larger GAP.

7 AIMMS

It is hard to create examples without useful software. Although most examples we created where made by hand, we used the software AIMMS to check the validity of our instances. AIMMS is optimization software and optimizes by solving a set of (non)-linear equations, where some variables can be integer (or have other types of boundaries).

Although the bin packing problem can be written as the linear program described in the introduction (Chapter 1), we changed it a bit to make it suitable for our research. The described linear problem uses a set of all feasible sets to maximize. This set should be generated before to make it possible to solve this problem. However, this needs a long calculation time so we did not do this. (Remark that if there is a minimum bound on the itemsizes, generating can still be done in polynomial time.) Instead, we wrote two programs that avoided generating all feasible sets.

The first program splits every bin in a equal number of equal parts. This is sufficient to deal with our subproblems, but it might be non-optimal for the general case. This splitting in equal parts has, of course, a reason. In our constructed examples we saw this happening all the time and it is also logical for this reason: In the optimal situation some items are fully used, so the feasible sets that contain one of these items fit exactly in one bin. If feasible sets contain more than one of these items, there does not always exist an optimal solution with integer y_F. For example: an optimal solution with constraints $y_1 + y_2 \leq 1$, $y_1 + y_3 \leq 1$ and $y_2 + y_3 \leq 1$ could contain $y_1 = y_2 = y_3 = \frac{1}{2}$ (depending on the other constraints). However, sets of constraints like $\sum_{i \in C} y_i \leq 1$, with a certain set C, lead to rational solutions.

The second program uses the Column Generation method to derive the optimal fractional packing. We now describe both programs. The first program includes some redundant variables and parameters which are used to easily read some properties of the optimal solution.

Bin splitting Program

Sets:

Set name	**Letter**	**Range**
Bins	b	Integer from 1 to B
Items	i	Integer from 1 to I
Parts	p	Integer from 1 to P

Parameters:

Letter	**Description**
B	Number of bins
I	Number of items
P	Number of parts
S_i	Size of item i
T_b	Size of bin b

Variables:

Letter	**Description**
X_{ipb}	Binary which tells if item i is in part p of bin b
Y_{ib}	Binary which tells if item i is in bin b (*Redundant*)
Z_b	The total size of items in bin b (*Redundant*)

Constraints:

$\sum_i X_{ipb} \cdot S_i \leq T_b \quad \forall\ b, p$	Total itemsize in part does not exceed binsize
$\sum_{b,p} X_{ipb} \leq P \quad \forall\ i$	Every item is used in at most P parts
$Y_{ib} = \sum_p X_{ipb} \quad \forall\ i, b$	Number of times item i is in bin b (*Redundant*)
$Z_b = \frac{1}{P}(\sum_{i,p} X_{ipb} \cdot S_i) \quad \forall\ b$	Total size of items in bin b (*Redundant*)

Objective:

$$\max \ \frac{1}{P} \sum_{b,p,i} X_{ipb} \cdot S_i$$

Column Generation Model

Sets:

Set name	Letter	Range
Items	i	Integer from 1 to I
Feasible sets	f	Integer from 1 to F

Parameters:

Letter	Description
B	Number of bins
I	Number of items
F	Number of feasible sets
a_i	Size of item i
R_{if}	Binary which indicates if item i is in feasible set f
z_f	Size feasible set f

Variables:

Letter	Description
y_f	Usage of feasible set f

Constraints:

$\sum_f y_f \leq B$	Total usage fits in the bins (Bin constraint)
$\sum_f y_f \cdot R_{if} \leq 1 \;\forall\, i$	Every item is used at most once (Item constraint)

Objective:

$$\max \sum_f y_f z_f$$

Initialization

$F = 1$, $R_{11} = 1$ and $z_1 = a_1$.

Creating a new feasible set

Parameters:

Letter	Description
x_0	Shadowprice bin constraint
x_1 to x_I	Shadowprices item constraints
a_i	Size of item i

Variables:

Letter	Description
b_i	Binary which indicates if item i is in the new formed feasible set.

Constraints:

$$\sum_i b_i \cdot a_i \leq 1 \quad \text{The new formed set has to be feasible.}$$

Objective:

$$\min v = x_0 + \sum_i b_i \cdot x_i - \sum_i b_i \cdot a_i$$

Adding the new feasible set

Increase F by 1.
$R_{iF} = b_i$ for all i.
$z_F = \sum_i b_i \cdot a_i$.

Procedure

1. Run Initialization.
2. Find optimal packing given the generated feasible sets.
3. Create a new feasible set.
4. If $v < 0$, add the new set and go back to step 2.
5. Show packing.

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