RISK REDUCTION IN SERVICE CONTRACT FULLFILMENT

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Abstract

In the service industry, performance-based contracts are a growing trend. Companies provide a service, and their yearly revenues depend on the quality or performance of that service during this year. This means that long-term averages are no longer sufficient for predicting cost-flow, the system also has to be analyzed during specific time intervals. This is usually done by considering an appropriate Markov chain, however closed form expressions for higher moments of the interval availability are only known for 2-state Markov chains. We have extended this model to a 3-state Markov chain, for which we have derived a closed form expression for the second moment of the interval availability under a few general constraints. We consider possible expansions of our model by examining specific cases, and for these cases we give results on the second moment and the constraints under which these results hold. We also provide a general framework for analyzing larger systems by considering their components individually. Using a few numerical examples, we outline the possible uses of our results.

Management Summary

Motivation
In the service industry, performance based contracts are a growing trend. For companies such as Thales Netherlands it's common to not only sell a product or system, but to also sell a support package often covering up to 20 years. During that time they provide service for the system, and their yearly revenues depend on the quality or performance of that service during this year. Since Thales would like to predict its costs-flow, the performance of the system should be analysed closely. However, critical parts are sometimes very unreliable, and there is great variability involved in the performance of the system. The main performance indicator is usually the system availability during a finite interval of time, the so-called interval availability.

Goals
The Goal of this research is to inquire insight into the variability of the system performance. With that goal in mind, we formulated the main research question as follows:

"How can Thales improve service contract performance by specifically focusing on reducing the variability in the system availability?"

We look at this from a theoretical angle. We aim to start by examining existing literature to try to find models that fit the situation at Thales Netherlands. We will then try to extend these models, in order to provide better predictions for the system behaviour

Approach
To answer our main research question, we study a currently existing model. We observe a simple case of the general system, to see if we can find a numerically efficient way of determining variability parameters. After analysing this simple case, we extend it in several possible ways and we check if the results still hold. We construct a general way of analysing large systems by focusing on specific (smaller) parts.

Results
We adapted the original model by providing an explicit way to determine certain variability parameters for a basic system. We then show how this system can be extended in several ways, and explain for which extensions the results still hold and how they are altered. After that we consider an approach that allows us to combine several smaller systems into larger systems, thus allowing a large system to be analysed by analysing the (much simpler) subsystems. We show how this can be used to compare different options when trying to determine an optimal service strategy.

Preface

This report is the result of my master project carried out at Thales Netherlands in Hengelo at the department of Logistic Engineering between March 2012 and January 2013.

Firstly I would very much like to thank Thales Netherlands for giving me the opportunity to do my graduation project in such a nice company. Specifically I am very thankful to Rindert Ypma for being my company supervisor, for keeping me motivated for the project, and for providing help when I needed it, even when I didn't always ask for it. I would also like to thank everyone else at the Logistic Engineering department, for providing a very pleasant working environment.

Secondly I would very much like to thank my supervisors at Universiteit Twente, Jan-Kees van Ommeren and Ahmad al Hanbali for their time, valuable input, and their support and encouragement. When I was low on motivation our meeting always kept me focussed and helped me move on, and your suggestions and critique were essential to my research. I would also like to thank Richard Boucherie for bringing me in touch with the project in the first place, and for presiding over my assessment committee. I would also like to thanks Johann Hurink for taking place in that committee and reading my report.

Finally I would like to thank everyone else who has supported me during my graduation period. My parents, my brother, flatmates, fellow students and friends. I appreciate that they listened to my stories about my graduation project, I even appreciate that they made fun of me when I had to stay home to work on the report or when I had to get up early for another day of 9 to 5. To work effectively you need to be able to relax and enjoy yourself in your time off, and there were always plenty of opportunities for that.

Sjoerd van Willigen
Enschede, January 2013

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1. Business Description of Thales Group & Thales Netherlands

In this section we will provide an overall picture of Thales, and more specifically of Thales Netherlands. We will briefly describe its history in section 1.1, and the organization of Thales Group and Thales Netherlands in section 1.2. To acquire some insight into the logistic area in which we conduct this research, we elaborate on the activities of the Business Unit Naval Systems and the department of Logistic Engineering – where I've conducted this research - in section 1.3.

1.1 History

Thales Group is a French based multinational company that was originally established in 1893 under the name *Compagnie Française Thomson-Houston* (CFTH). Over the years they created a large variety of electronic products, and in 1966 they merged their electronics arm with that of *Compagnie Générale de Télégraphie Sans Fil* (CSF) to form *Thomson-CSF*. In 1982 the company was nationalized by French prime minister François Mitterrand. During the eighties and nineties they remained a major electronic and defence contractor. To be able to apply more focus to their respected areas, in 1999 the defence and the consumer electronics part of the business were split by the French government before being privatized again. The consumer electronics business formed Thomson Multimedia (currently Technicolor SA), and the defence business changed their name to Thales Group in 2000.

The Dutch part of Thales originated in 1922, when "N.V. Hazemeyers fabriek van signaalapparaten" was established in Hengelo. Hazemeyer became one of the world's leading suppliers of naval surface systems, and a large contractor of the Dutch Royal Navy and later also of other European navies. During World War 2 the factory was hit hard (being close to the German border), however after the war the Dutch government decided to buy the company to be able to maintain (rebuild) a strong defence industry. The company was renamed to "Hollandse Signaalapparaten B.V.", and they developed systems for areas such as air traffic equipment, fire control systems, and most notably radar.

In 1956 Philips became the main shareholder of the company, and business was going so well that new plants were opened in Huizen, Delft and Eindhoven. However when the cold war ended the company faced a significant cut in their order intake due to defence budget cuts, and Philips decided that 'Defence and Control Systems' were not part of their core business. Therefore Hollandse Signaalapparaten B.V. was taken over by the French based multinational Thomson-CSF, which changed its name to Thales in 2000. In accordance to this, Hollandse Signaalapparaten B.V. was renamed to Thales Netherlands B.V.

1.2 Organisation

The Thales Group is a world leader in electrical systems and provides services for the aerospace, defence and security markets . The company is split up into roughly six core businesses: Aerospace, Air Systems, Land & Joint Systems, Naval, Security Solutions & Services, and Space. Together they generated an annual revenue of 13.03 billion euros in 2011, and the Thales Group employs 68.000 employees spread over 50 different countries. The company is ranked as the 475th largest company in the Fortune 500 list, and is the 11th largest defence contractor in the world. Globally 60% of the Thales Groups sales are military products.

Thales Netherlands houses the Naval section of the Thales Group, focusing mostly on radar and combat management systems. It is the largest defence company in the Netherlands, with roughly 2000 employees. Its business is divided into the categories Naval, Land & Joint Systems, Air Systems, Transport Security, and Services. This research concerns the business unit Naval Services, specifically the department of Logistic Engineering.

1.3 Naval Services and Logistic Engineering

Most of the products that Thales offers have a lifetime of twenty years or more. During this time the products require maintenance and sometimes even repair. The business unit Naval Services delivers this after sales support for the radar systems that Thales sells. They serve more than 85 customers spread over 42 different countries. The core services consist of delivering spare parts and carrying out repairs. Furthermore, Naval Services also offers upgrade programs, modifications, documentation and training. The exact form and amount of after-sales support a customer receives depends completely on the specific wishes of that customer. Depending on the customer and on the system properties, Naval Services can provide anywhere from only initial logistic support to full life-time support.

The Logistic Engineering departments plays a key role in the processes of Naval Services. For any contract that is made, they conduct a logistic support analysis to determine what kind of logistic support is needed for that specific system. They also provide input to the designers, to ensure that the products Thales creates are actually serviceable. Other tasks performed by the Logistic Engineering department are performing a life cycle cost analysis, supporting the technical authors with system knowledge, designing a specific service plan for specific customers, and determining the optimal allocation of spare parts by optimizing system availability while minimizing costs.

2. Research Design

In this chapter we will provide an outline of the research. In section 2.1 we will discuss the context of our research by giving an introduction to the long term service agreements that Thales offers and the products we consider. In section 2.2 we define the exact problem statement and in section 2.3 we define the corresponding research objective. Section 2.4 gives an outline of this thesis.

2.1 Context Description

2.1.1 Long-Term Service Agreements

Thales constantly has to deal with a large amount of technical- and customer specific developments. One of those developments is the closer relation between Thales and its customers. Performance-based contracts are a growing trend that aims to achieve this goal. Instead of offering separate services, Naval Services can take over all services at a fixed fee. This means Thales gets to determine the optimal support strategy, which should lead to more predictable and possibly lower costs. (One can assume Thales is more knowledgeable about its own systems than its customers are.)

In the case of a long-term service agreement, a certain performance is settled which covers a period of 5 to 25 years. The key performance indicator is the *system (or operational) availability*. Operational availability is the time a radar system is working, divided by the total system's operating time (uptime + downtime). According to Sherbrooke (2004) the operational availability is commonly expressed as

$$Operational\ Availability = \frac{Uptime}{Uptime + Downtime} = \frac{MTBM}{MTBM + MDT} \cdot 100\%\,.$$

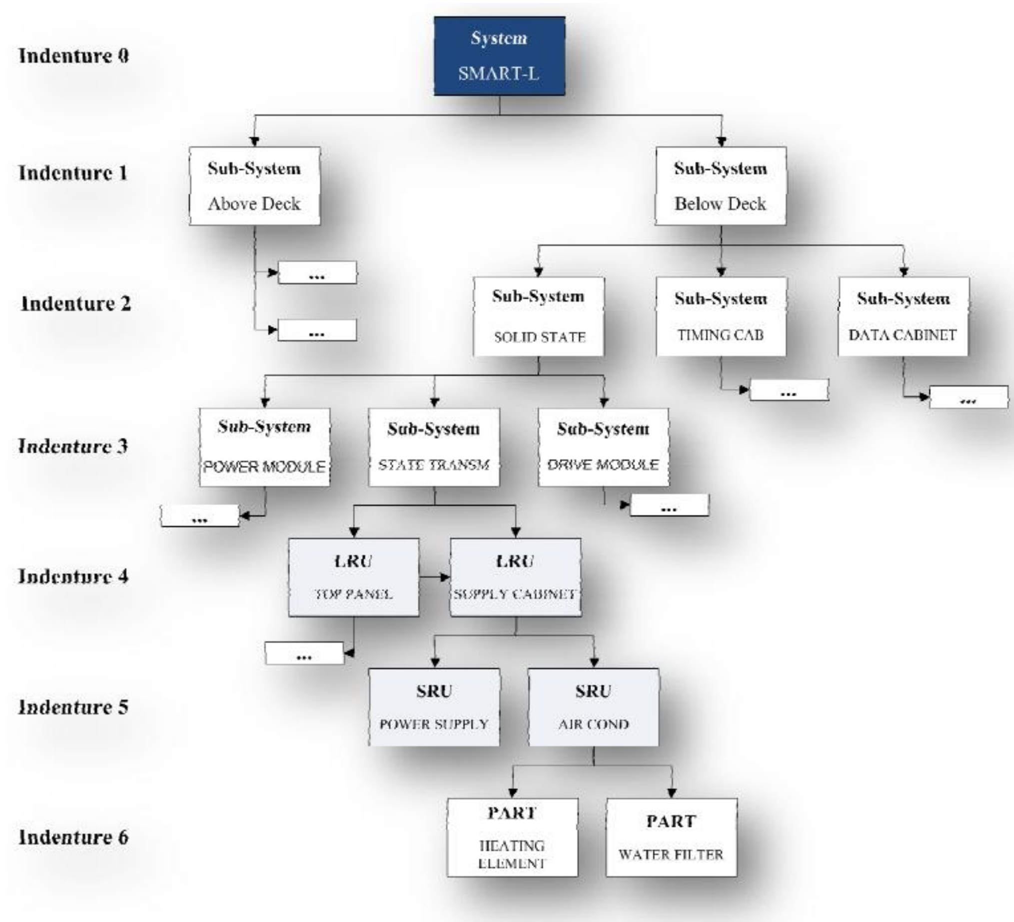
Increasing the mean time between maintenance (MTBM) or decreasing the mean down time (MDT) increases the operational availability. The MDT consists of mean preventive maintenance time (MPMT), mean corrective maintenance time (MCMT), and mean supply delay (MSD). However Thales defines downtime as a system waiting for a spare part, which is only the mean supply delay. Sherbrooke defines this kind of availability as *supply availability*. Supply availability is heavily influenced by the stocking policy, and is defined as

$$Supply\ Availability = \frac{MTBM}{MTBM + MSD} \cdot 100\%\,.$$

In this thesis we are interested in the Supply Availability as defined above, if at any time we mention 'availability' then we will imply the supply availability as defined above.

2.1.2 Product Information

Radar systems have a modular design. They consist of several subsystems, which all consist of several different modular units. The picture below displays an approximation of the product tree of the SMART-L radar system, one of Thales' main products.



We see that subsystems generally consist of LRU's (Line Replaceable Units). These are a complex collection of items that are designed to allow the entire LRU to be efficiently replaced as a whole. LRU's usually consist of several Shop Replaceable Units (SRU's), which in turn consist of multiple parts. A failure of one of these SRU's (or parts) will lead to a failure of the LRU, which in turn may lead to downtime for the entire radar system (depending on the criticality of the LRU). SRU's and LRU's are usually expensive and can possibly fail during missions (when the system is operating).

2.1.3 Repair Information

If an item in the system fails and a spare part is available, the broken item is immediately replaced. This is called repair-by-replacement. The failed item is brought to a repair facility, which depending on the complexity of the item can be either on board, locally at the shore, or at Thales. If the item cannot be repaired at a certain station, the part is sent to a higher echelon location. This kind of a repair network is called a multi-echelon network. It consists of different locations that all have different capacities with regards to stock, supply and repair. When an item fails, choices have to be made regarding whether to repair or discard the item (sometime buying a new one is cheaper) and where to repair the item. The main decision in multi-echelon network is usually the allocation of stock over the different echelons.

2.2 Problem Statement

A problem for the availability driven contracts is that Thales has to estimate the contract costs before the start of the contract. The estimated costs depend on the expected amount of failures and the interest- and inflation rate. This expected amount of failures (the demand) is not accurately known. Also the annual operating hours of the ships vary in practice, which obviously directly influences the demand for spare parts. These factors lead to difficulties in estimating the contract costs, and also affect the average availability during a contract. This in turn impacts the service perception of the customer.

A similar problem occurs when there are multiple ships involved in a single contract. If these ships go on a mission together, there will be high demand for spare parts during that time. Once the ships return (together), this demand will obviously become very low. If the ships operate independently however, the demand is more stable during the year. Also the repair throughput times are not accurately known, and Thales uses (roughly) estimated values of these. All these factors lead to a large variability in the system availability. Since Thales has to pay a penalty when the attained availability is too low and receives a bonus when the attained availability is high, they would like to predict the system availability as well as they can. However the availability is obviously a stochastic parameter, which depends heavily on the system parameters.

Currently Thales uses commercial software (INVENTRI) for determining optimal spare part stock levels to maximize the average availability. However this software uses Vari-Metric which does not take variability of the availability into account, and neither does other comparable software. This means that the resulting stock allocation may (should) yield a very high average availability, but could also result in a very large variation (that might be unacceptable for the customer). Having a radar system working for 9 years and then being broken for 1 year would result in a 90% availability over 10 years which could be quite good. However if the 1 year of downtime is exactly the only year out of the ten that the ship is actually on a mission, this 90% is not good at all (as opposed to roughly 1 month of downtime each year in between missions which would be fine).

The problem described above lies in the variability of the system availability. It is very hard to predict, and most currently used methods simply maximize the expected availability. However if Thales would have more insight into the factors that have a large effect on the variability, perhaps specific actions could be taken to reduce it.

2.3 Research Objective

The goal of this research is to acquire insight into the variability of the system performance. With this goal in mind, we formulate the main research question as follows:

“How can Thales improve service contract performance by specifically focusing on reducing the variability in the system availability?”

We will answer this main research question by answering the following sub questions:

1. *What currently existing literature is applicable to this research?*
2. *Which theoretical model and/or analysis currently provides the best approximation of the practical situation at Thales?*
3. *Can we extend or improve this model?*
4. *How well does our extended model represent the practical situation?*
5. *Which further advantages does our extended or improved model provide?*

2.4 Outline of Thesis

This thesis is structured as follows. In section three we provide a literature overview. We start with a general introduction into after sales business models and spare parts strategies. After that we give a short write-down of the METRIC algorithm as it is essential to the research field, and we follow up with a literature overview of interval availability.

In section four we introduce the theoretical models in which we conduct this research. Section 4.1 introduces a general model and some basic results. We extend those in section 4.2 by following the approach of [De Souza, 1986], and in section 4.3 where we give several results from [Al Hanbali, 2012] on which we build extensively. In chapter 5 we present our own work as additions/extensions to these models. Section 5.1 contains our adaptation of the model and one of our two main results. Sections 5.2 through 5.5 describe specific cases for which we can further evaluate our results. In section 5.5 we consider an approach which allows us to combine simple systems into larger systems.
Section six provides some numerical data backing up our results, and section seven concludes. The appendices contain some of the lengthier parameter values, as well as some of the more advanced matrix operations that are used in the research.

3. Literature Review

In this chapter we will provide a review of the currently existing literature that is applicable to this research. We will start by briefly examining the after sales business-models that correspond to the LTSA contracts that Thales offers. Furthermore we will provide an introduction to spare parts management, focusing mainly on the METRIC and VARI-METRIC models which are used by Thales to determine spare part allocations. We will also review existing literature about interval availability, and more specifically about variability of the attained (interval) availability.

3.1 After Sales Business Models used by Thales

Cohen et al. (2006) define multiple after-sales business models that companies can deploy in order to support their service products. These models are distinguishable by the service priority that the customers require for the specific product. They vary from products with no service priority at all (disposable products such as razor blades), to products with a very high service priority that generally play a critical role in keeping a system running (engine of an aircraft).

These models also differ by product ownership. Products with a low service priority are usually owned by the customer, whereas products with a very high service priority are often owned by the service provider so that they can guarantee a certain service level. In that case customers pay for the service but never own the actual product. Lease agreements (cars) are a common example of this.

The LTSA contracts that Thales offers can be classified as having a high service priority. When comparing the different contracts to the business models of Cohen et al. (2006) , we find the following classifications for the multiple LTSA contracts that Thales offers:

LTSA Contract	Service Priority	Guarantee Upon	Corresponding Business Model
Traditional	High	Support and Design Services	Cost-Plus
Spares Inclusive	High	Repair and Supply Services	Cost-Plus
Contract for Availability	Very High	System Availability	Performance Based
Contract for Capability	Very High	System Capability	Power by the Hour

3.2 Spare Part Strategies

Rustenburg (2000) defines different types of spare part strategies. These are classified by the price of the item (high or low), and by the maintenance concept (predictive or corrective). Predictive maintenance is performed to reduce the probability that an item breaks down, and thus is predictable since it is done regardless of the condition of the item in the system. Corrective maintenance is performed to restore an item once a breakdown or failure has occurred, and therefore has unpredictable behaviour.

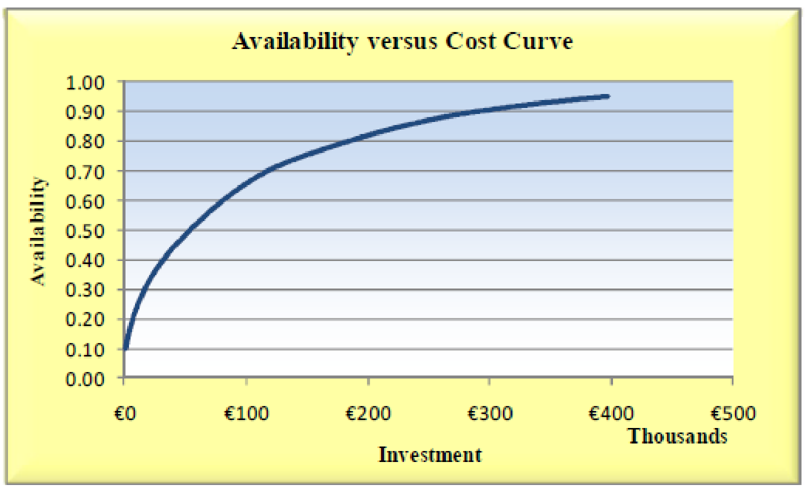
In the case of LTSA's at Thales, the probability of failures is generally low, spare parts are relatively expensive, and the focus lies on corrective maintenance. Therefore Rustenburg's classification puts us in the Spare Part Management Strategy. This is the most difficult (and arguably the most interesting) category, where spare parts are both expensive and critical. To manage this category effectively, advanced methods that are able to deal with unpredictability are required. A method that deals with this problem is the Multi-Echelon Technique for Recoverable Inventory Control (METRIC), developed by Sherbrooke (1968).

3.3 METRIC

3.3.1 Introduction

The METRIC model is based on maximizing the steady state system availability under a budget constraint. The goal is to determine optimal stock levels for each item at every location, such that the availability of the entire system is maximized. Note that the METRIC model uses the *system approach* where the availability of the entire system is optimized, and not the *item approach* where the optimal stock levels are determined for each item individually based on inventory-, holding- and stock out costs.

In the system approach the performance indicator is the system availability, the percentage of time that the system is available. Usually either the required availability or the budget constraint is considered as input. An availability-cost curve (see below) can often be determined, and is very useful for gaining insight in the relation between money spent and the attained availability. All points below the curve are considered inefficient, since either the same availability could be attained cheaper or a higher availability could be attained with the same investment.



Considering the importance of the METRIC model in current literature and the fact that Thales uses (an extension of) the METRIC model, we will provide a short overview of the model. For a complete overview, see [Sherbrooke, 2004].

3.3.2 Assumptions

Here we will discuss the assumptions that are made in the METRIC model. One of the most important assumptions is that we use an (s-1,s) inventory policy for every item at every echelon. This is the most common ordering strategy for items with a sufficiently low demand rate and a sufficiently high price. It means that for each item an inventory level s is determined, and if the stock falls below this level then an order for an additional unit will be placed immediately. If the order cannot be delivered immediately, it is backordered. The orders are placed for an individual item (one-for-one replenishment), the items are not batched for repair.

Furthermore the METRIC model assumes that breakdowns of items occur according to independent Poisson processes. This means that all failures are individually independent, and that items continue to fail upon system failure. The repair times are assumed to be generally distributed with a given mean, which can also obviously differ per item. If an item fails it is directly taken into repair and the repair shop has infinite capacity. This results in a repair shop that can be modelled as an (M/G/∞) queuing model.

Further on it is assumed that each backorder is equally important (no priorities). Obtaining a new part by taking it out of the stock of a parallel system is not allowed; only the depot resupplies the bases (no lateral supply).

3.3.3. Maximizing Availability

In the METRIC model the steady state system availability is maximized. One of the main arguments used is that maximizing the average system availability is achieved by minimizing the total expected number of backorders. This expected number of backorders can be calculated for each item individually, using that item's yearly demand and its repair throughput time.

The optimization procedure is based on using a marginal approach. First all stock levels are set to zero. Then for each possible choice of adding a unit of stock to any stock point, the marginal expected backorder reduction per invested euro is calculated. We then add the item for which this backorder reduction per invested euro is the largest. After that we recalculate the marginal reductions and pick the next item, and so forth until we reach our budget constraint. This constructs the most cost-efficient way of maximizing availability. For proofs or further elaboration see [Sherbrooke, 2004].

3.3.4 VARI-METRIC

The VARI-METRIC model is an extension of the METRIC model, and is the model that Thales currently uses for their stock allocation. The extension is that VARI-METRIC does not only use mean values for the amount of backorders, but also determines their variance. For each LRU, this is done by considering the demand that occurs during a products repair time, as well as the sum of the backorders of all its individual components (SRU's).

VARI-METRIC is definitely a more accurate version of the METRIC algorithm, however its goal is still the same: maximizing the expected system availability. This is exactly where our research intends to expand, we aim to get a grip on not only the expected interval availability but also its variability.

3.4 Interval Availability

In this section we review the existing literature on interval availability. In highly critical systems, steady state results can be very poor and do not always provide sufficient information for practical use. Therefore there is an increasing interest in calculating results for a limited time interval, instead of calculating long-term steady state averages. The interval availability, usually denoted by A(T), is defined as the fraction of time that the system is operational during the interval [0,T]. It is a random variable between 0 and 1.

Early research about interval availability was mostly about on-off (two state) Renewal Processes, see for example (Takács 1957). The results of this research are difficult to compute numerically. Therefore approximations were made by fitting phase-type distributions on the on and off periods, which yields an accurate result with a relatively small computation time, see (van der Heijden 1988). Smith (1997) made another approximation, which is based on fitting the approximated first two moments, the zero percent and the hundred percent probability in a beta distribution. The first two moments of the interval availability of an on-off two-state Markov chain are derived exact and in closed-form in (Kirmani and

Hood 2008). The main assumptions in all these studies are that the on periods are independent of the off periods, and furthermore that they are all independent of each other.

When extending the two-state Markov chain to a larger state space, an essential result was achieved in [De Souza, 1986]. They derived the cumulative distribution of the time spent in a subset of states of a Markov chain during a finite amount of time, in closed form. This subset could obviously be chosen as the set of states in which the system is operational. However computing the entire distribution of the interval availability using this result is numerically rather inefficient, the same can be said for the improved version derived in (Rubino and Sericola 1995). An efficient algorithm to determine the interval availability distribution was determined in (Carrasco 2004), however this result is limited to Markov chains that contain an absorbing set of states.

A different numerically efficient approach to determine the distribution of the interval availability was obtained in (Al Hanbali and van der Heijden, 2012), where the absorbing set of states is no longer a requirement. The authors determine the expectation, the variance, and the probability of a hundred percent interval availability using Markov analysis. They are then able to fit this using a combination of a beta distribution and a probability mass at one. Using simulation, the authors show that their approximation is highly accurate, especially for points of the distribution below the mean value which are practically most relevant. This thesis builds heavily on the results of (Al Hanbali and van der Heijden, 2012).

4. Theoretical Model

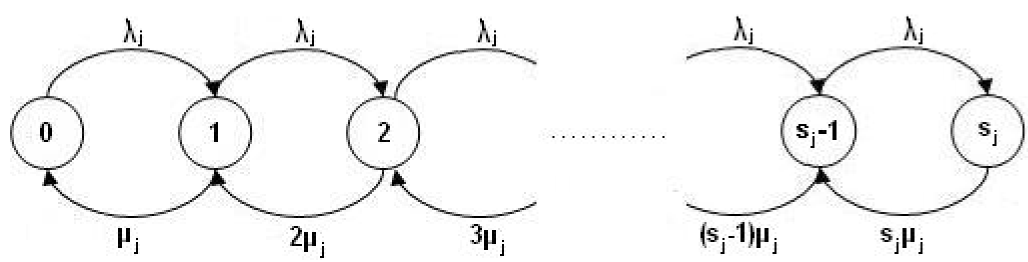
Explicit closed form expressions for the variance and survival probability of the interval availability were determined in [Al Hanbali 2012]. However these expressions are rather extensive and thus provide little intuitive (analytical) insight into which system parameters have the largest impact on variability.

To see if we can obtain these insights, we analyse the theoretical models that are applicable to our research. In section 4.1 we describe the general model that we work with, and show some of its basic properties. Section 4.2 contains an important result obtained in [De Souza, 1986], and its derivation. After that we discuss the work of [Al Hanbali 2012], which is the framework on which we build our own results in chapter 5.

4.1 The General Model

In this section we will give a general introduction to the model that we work with. We consider a system consisting of M items. Assume that the jth item fails according to a Poisson process with rate $\lambda_j,\ j = 1, \dots, M$. Moreover we assume that all items are individually independent, meaning the breakdown behaviour of one item has no effect on any of the other items. The repair time of the jth item is exponentially distributed with rate μ_j. Furthermore we assume infinite repair capacity, which supplies the fact that repairs are also individually independent.

Let s_j denote the number of items of type j that are in the system, including the one currently in use (stock level plus one). Given these values of s_j for all j, we can now easily set up a Markov chain which describes the state of the system. We only need to know how many parts are currently in repair for each item j. For that we define $R_j(t)$ as the number of type j items in repair at time t. Now for each item j we can define a Markov chain where the state denotes $R_j(t)$, this yields the following transition diagram:



This is the well-known M/M/s/s queue, which is generally used to describe a queue with exponential inter-arrival and repair times and a finite (s-sized) number of servers. This system is also known as the Erlang Loss Model. The steady state probabilities for this model are relatively easy to determine, and are given by

$$\pi_i(j) = \frac{\left(\frac{\lambda_j}{\mu_j}\right)^i / i!}{\sum_{n=1}^{s_j} \left(\frac{\lambda_j}{\mu_j}\right)^n / n!} \quad , \quad i \geq 0\,, \tag{1}$$

where in this case $\pi_i(j)$ denotes the steady state probability of having *i* items of type *j* in repair. The probability of item j being unavailable, $\pi_{s_j}(j)$ in this case, is also known as the blocking Probability.

We can also easily determine the generator matrix. Let G_j denote the transition generator of R_j, matrix G_j then looks as follows

$$G_j = \begin{pmatrix} -\lambda_j & \lambda_j & & 0 & \dots & 0 \\ \mu_j & -(\lambda_j + \mu_j) & & \lambda_j & \ddots & \vdots \\ 0 & 2\mu_j & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \lambda_j \\ 0 & \dots & 0 & s_j\mu_j & & -(s_j\mu_j) \end{pmatrix}. \tag{2}$$

Now having introduced the M/M/s/s queue, we see that our system can be seen as M individually independent Markov chains, all with their individual breakdown rates (λ_j), repair rates (μ_j) and item levels (s_j). Each individual item is up and running if one or more of the of the total item amount s_j are not in repair. In other words, the only non-operational state is the state on the very right where every single item of type j is in repair.

In more general terms, we define $X_j(t)$ as the state of item j at time t. This means that $X_j(t) = 1$ if item j is operational at time t, and $X_j(t) = 0$ if it is not. Linking this to our Markov chain, we see that $X_j(t) = 1$ if and only if $R_j(t) < s_j$. So we can state that our entire system is operational if $R_j(t) < s_j \;\; \forall j$.

The random variable we are interested in in this model is the interval availability. We define it as A[T], and it denotes the fraction of time during the interval [0,T] that the system is working. We can determine its expectation rather easily. If we assume that the system is in its steady state at the start of the interval, then the expected interval availability is equal to the steady state operational probability:

$$E[A(T)] = \prod_{j=1}^{M} P(X_j = 1) = \prod_{j=1}^{M} 1 - P(R_j = s_j) = \prod_{j=1}^{M} \left(1 - \frac{\left(\frac{\lambda_j}{\mu_j}\right)^{s_j} / s_j!}{\sum_{n=1}^{s_j} \left(\frac{\lambda_j}{\mu_j}\right)^{n} / n!} \right). \tag{3}$$

This is a very straightforward and intuitive result, giving the proof here would be rather superfluous (it is given in for example [Al Hanbali, 2012])

Before we go on to further analysis in the next section, we first introduce some notation. We define Ω_j as the state space of the Markov chain R_j, which consists of the set of operational states Ω_o and the set of non-operational states Ω_F. Next we define γ_j as a row vector of size equal to the cardinality of Ω_j. This vector is obtained by taking the steady state probabilities $\pi(j)$ and replacing these by zero for all malfunctioning states.

We also define the column vector f_j of size equal to the cardinality of Ω_j. This vector also has zero entries for all the non-operational states, and all entries corresponding to operational states are one. In our case this results in

$$\Omega_j = \{0,1,..,s_j\} \quad , \quad \gamma_j = [\,\pi_0(j) \quad \pi_1(j) \quad \dots \quad \pi_{s_j-1}(j) \quad 0] \quad , \quad f_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}. \tag{4}$$

Note that throughout this section and in fact throughout this report, we will assume that at time 0 the system is in steady state. This is a relatively small assumption to make, and it prevents a lot of complications in further derivations.

Now that we have presented a general model, we will present the model and a main result of de Souza in the next section.

4.2 The De Souza Model

In this section, we show the main results obtained in [De Souza, 1986]. In that paper they analyse the same kind of systems that we are currently looking at, though they consider an application based on repairable computer systems instead of radar components.

For their results they randomize (uniformize) the Markov chain. This uniformization method (also known as Jensen's Method) is commonly used in probability theory to compute transient solutions for finite state continuous time Markov chains. It involves the construction of an analogous discrete time Markov chain. This is done by picking a uniformization constant v which is at least as large as the largest transition rate out of any of the states in the original system. You then assume that all transitions occur at this rate, and you add self-transitions to compensate for the difference between v and the actual outgoing rate. This results in an equivalent uniformized Markov chain.

In [De Souza, 1986] they analyse O[T] which is defined as the sojourn time in a certain set of states up to time T. During this section we will assume that this set of states will be the set of operational states (Ω_o). This of course means that

$$O[T] = T \cdot A[T]. \tag{5}$$

The main result of de Souza is a formula for the cumulative distribution of O[T]. Since our result builds heavily on this result, we will give the theorem here and show its proof.

Theorem [De Souza]

The cumulative distribution of the sojourn time in a subset of states, is given by:

$$P(O(T) < x) = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{n!} \cdot \sum_{k=0}^{n+1} \alpha(n,k) \cdot \sum_{i=k}^{n} \binom{n}{i} \left(\frac{x}{T}\right)^i \left(1 - \frac{x}{T}\right)^{n-i} , \tag{6}$$

Where T denotes the length of the interval, v is the uniformization constant, and the $\alpha(n,k)$'s are probabilities that can be determined recursively.

The derivation of this formula is as follows.

To determine the probability of spending less time than a certain value x (or equivalently spending less time than the fraction $\frac{x}{T}$) in operational states, we condition on the number of transitions made during the interval of length T. In other words:

$$P(O(T) < x) = \sum_{n=0}^{\infty} P(O(T) < x \,|n\ transitions) \cdot P(n\ transitions)\ . \tag{7}$$

We know that in all states, the transitions occur at rate v. Hence

$$P(n\ transitions) = e^{-\nu T} \cdot \frac{(\nu T)^n}{n!}\ , \tag{8}$$

so

$$P(O(T) < x) = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{n!} \cdot P(O(T) < x \,|n\ transitions)\ . \tag{9}$$

Now given the fact that there are n transitions during the time interval $[0, T]$, this interval is divided into n+1 intervals of length $Y_1, \ \ldots \ , Y_{n+1}$. We then further condition on the number of times that the process visits one of the operational states (Ω_o) during interval [0,T], denoted by k ($0 \leq k \leq n+1$) :

$$P(O(T) < x) = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{n!} \cdot \sum_{k=0}^{n+1} P(O(T) < x \,|n\ transitions, k\ visits\ into\ \Omega_o) \cdot P(k\ visits\ into\ \Omega_o|n\ transitions) \tag{10}$$

We now denote $P(k\ visits\ into\ \Omega_o|n\ transitions)$ by $\alpha(n, k)$, so

$$P(O(T) < x) = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{n!} \cdot \sum_{k=0}^{n+1} \alpha(n, k) \cdot P(O(T) < x \,|n\ transitions, k\ visits\ \Omega_o)\ . \tag{11}$$

Now $P(O(T) < x \,|n\ transitions, k\ visits\ into\ \Omega_o)$ means that the sum of all the interval lengths Y corresponding to the operational intervals must be smaller than x.

To determine this probability, we consider the time-distribution of the n transitions. We know that for a given number of Poisson events during a certain interval, the distribution of these events over the interval will be uniform. When we apply this fact directly to our situation, we can state that the time-distribution of the n transitions made during the interval is in fact a uniform distribution over $[0, T]$. Now, we know that k of those visits are into operational states. In order for the total availability to be smaller than x, we need the sum of the interval lengths of these k operational intervals to be less than x.

We then use a well-known result on exchangeability (see for example [Ross 1996]):

If $\{Y_{i_1}, \ldots, Y_{i_k}\}$ and $\{Y_{j_1}, \ldots, Y_{j_k}\}$ are any two sequences of length k, then they have the same joint distribution if all Y's are equally distributed and individually independent. Note that by setting $k = 1$, this implies that $P(Y_i \le s) = P(Y_j \le s)$ for all $1 \le i, j \le n + 1$.

Using these results we can state that the probability of the sum of the k operational interval lengths being smaller than x, is equal to the probability that the sum of the *first* k interval lengths is less than x. For this to happen, we would need *at least* k out of the n Poisson events to occur earlier than x. Now we know that given the fact that there are n Poisson events in a certain interval, we can assume that they are all distributed uniformly on this interval [0,T]. So the probability of having at least k of these occurring before x, is simply a sum of binomial probabilities with success probability $\frac{x}{T}$. Hence:

$$P(O(T) < x \mid n \; transitions, k \; visits \; into \; \Omega_o) = \sum_{i=k}^{n} \binom{n}{i} \left(\frac{x}{T}\right)^i \left(1 - \frac{x}{T}\right)^{n-i}. \tag{12}$$

Filling this into equation (11) on the previous page yields our desired result:

$$P(O(T) < x) = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{n!} \cdot \sum_{k=0}^{n+1} \alpha(n,k) \cdot \sum_{i=k}^{n} \binom{n}{i} \left(\frac{x}{T}\right)^i \left(1 - \frac{x}{T}\right)^{n-i}. \tag{13}$$

Now that we have proven the de Souza result, we move on to the model of al Hanbali.

4.3 The Al Hanbali Model

The model of [Al Hanbali 2012] is an extension on the work [De Souza, 1986]. In the paper, the variance of the interval availability is computed in closed and exact form. This variance along with the expectation of the interval availability and the probability that the interval availability equals one, is then used to approximate the survival function (cumulative distribution) using a Beta distribution.

For our work, we are interested in the exact and closed-form result on the variance of the interval availability. Based on de Souza's theorem (or equation (13)), they present the following result on the m^th moment of the interval availability:

$$E[A(T)^m] = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+m)!} \cdot \sum_{k=1}^{n+1} \alpha(n,k) \cdot \prod_{l=k}^{k+m-1} l \,. \tag{14}$$

The exact derivation can be found in [Al Hanbali 2012]. Given de Souza's result, the derivation is rather straightforward and not of great importance to our work.

The good thing here is that filling m=2 into equation (14) yields the second moment of the interval availability, which is the most significant term when analysing its variability. Among other things, it allows the computation of the variance since we already have its expectation. Considering the importance of the second moment term, we will evaluate it further. An evaluation of the term was done in [Al Hanbali 2012], which we will discuss. We will then attempt to further evaluate this result in chapter 5.

Theorem [Al Hanbali]
The second moment of the fraction of time that the Markov Chain R(t) sojourns in the subset Ω_o during [0,T] is given by:

$$E[A(T)^2] = 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot\ p_0 \cdot \sum_{i=1}^{n} (n-i+1) P^i e_0 + 2\sum_{l \in \Omega_0} x_l \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2}, \qquad (15)$$

where P denotes the transition probability matrix of the uniformized process R(t) , x_i is the steady state probability of the Markov chain in state I, p_0 is the column vector with i-th entry equal to x_i if $i \in \Omega_o$ and zero otherwise, and e_o is the column vector with i-th entry equal to 1 if $i \in \Omega_o$ and zero otherwise.

The derivation of this formula is as follows. Filling m=2 into equation (14) yields

$$E[A(T)^2] = \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot \sum_{k=1}^{n+1} \alpha(n,k) \cdot k \cdot (k+1)\,. \qquad (16)$$

The thing left to do here is to determine the $\alpha(n,k)'s$, which denote the probability of visiting k operational states when making n transitions. To do this, we extend the $\alpha(n,k)'s$ to $\Gamma(n,k,j)'s$, where j denotes the state in which the Markov chain appears in time T. Now we let $\Gamma(n,k)$ denote a row vector with the j[th] entry equal to $\alpha(n,k,j)$. Obviously summing over the elements of $\Gamma(n,k)$ yields $\alpha(n,k)$. If we then examine the relation between the Γ's, we find that the vector $\Gamma(n,k)$ satisfies the following recursion:

$$\begin{aligned} &\Gamma(n,k) = \Gamma(n-1,k-1) \cdot P_0 + \Gamma(n-1,k) \cdot P_F \qquad \text{for } n \geq 1, k \geq 1\,, \\ &\Gamma(n,0) = \Gamma(n-1,0) \cdot P_F \qquad \text{for } n \geq 1, k = 0\,. \end{aligned} \qquad (17)$$

Where P_0 denotes the probability matrix with only the transitions into the operational states (and all other values zero), P_F denotes the probability matrix with only the transitions into the unoperational states (and all other values zero).

The initial condition for the recursion is given by:

$$\begin{aligned} \Gamma(0,0) &= (x_0, x_1, x_2) \cdot P_F\,, \\ \Gamma(0,1) &= (x_0, x_1, x_2) \cdot P_0\,, \end{aligned} \qquad (18)$$

where (x_0, x_1, x_2) denotes the starting distribution (which we assume to be the steady state distribution).

For determining expressions for the second moment, we don't necessarily need expressions for the individual $\alpha(n,k)$'s though. What we need are the terms

$$\sum_{k=1}^{n+1} \alpha(n,k) \cdot k \cdot (k+1) \ . \tag{19}$$

To determine these, we use an approach based on generating functions. We multiply quation (17) with z^k and then sum over k. Using the recursion, this yields

$$\Delta_n(z) = \sum_{k=0}^{n+1} \Gamma(n,k) \cdot z^k = \Delta_0(z) \cdot \left(zP_0 + P_f\right)^n = (x_0, x_1, x_2) \cdot \left(zP_0 + P_f\right)^n. \tag{20}$$

If we then take the derivative of this equation to z, set z=1 and then multiply with the column vector e, we find

$$\sum_{k=0}^{n+1} k\alpha(n,k) = (n+1) \cdot \sum_{i \in \Omega_0} x_i \ , \tag{21}$$

where we used that $P \cdot e = e$ along with $(x_0, x_1, x_2) \cdot P = (x_0, x_1, x_2)$ and $(x_0, x_1, x_2) \cdot e = 1$ (remember that we start in steady state).

Now if we do the same with the second derivative of $\Delta_n(z)$ to z and setting in z=1, we find

$$\sum_{k=0}^{n+1} k(k-1)\alpha(n,k) = 2p_0 \cdot \sum_{i=1}^{n} (n-i+1)P^i e_0 \tag{22}$$

We clearly see that if we sum (22) with two times (21), this yields our required

$$\sum_{k=1}^{n+1} \alpha(n,k) \cdot k(k+1). \tag{23}$$

So, we can give the formula for the second moment:

$$\begin{aligned} E[A(T)^2] &= \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot \sum_{k=1}^{n+1} k(k+1)\alpha(n,k) \\ &= \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot \left(2(n+1) \cdot \sum_{i \in \Omega_0} x_i \ + \ 2p_0 \cdot \sum_{i=1}^{n} (n-i+1) P^i e_0 \right). \end{aligned} \tag{24}$$

We can evaluate this further by working out the brackets:

$$E[A(T)^2] = 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot\ p_0 \cdot \sum_{i=1}^{n} (n-i+1) P^i e_0 \\ +\ 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot (n+1) \cdot \sum_{i \in \Omega_0} x_i\ . \tag{25}$$

The second term can be evaluated as follows

$$2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot (n+1) \cdot \sum_{i \in \Omega_0} x_i = 2\sum_{i \in \Omega_0} x_i \cdot \frac{1}{(\nu T)^2} \sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^{n+2}}{(n+2)!} \cdot (n+1) \cdot \tag{26}$$

If we then substitute $m = n + 2$, we find that the last summation can be evaluated as

$$\sum_{n=0}^{\infty} e^{-\nu T} \frac{(\nu T)^{n+2}}{(n+2)!} \cdot (n+1) = \sum_{m=2}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^m}{(m)!} \cdot (m-1) \\ = e^{-\nu T} + 0 + \sum_{m=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^m}{(m)!} \cdot (m-1) \\ = e^{-\nu T} + \left(\sum_{m=0}^{\infty} m \cdot e^{-\nu T} \cdot \frac{(\nu T)^m}{(m)!} \right) - \left(\sum_{m=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^m}{(m)!} \right)\ . \tag{27}$$

Here we see that the first of the two summations is exactly the expectation of a Poisson process with rate νT, which is νT. And the second summation is a summation over all Poisson probabilities, which of course equals 1. Filling these values into equation (26), yields exactly our required result:

$$E[A(T)^2] = 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot\ p_0 \cdot \sum_{i=1}^{n} (n-i+1) P^i e_0 + 2\sum_{l \in \Omega_0} x_l \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2}\,, \tag{28}$$

In the next chapter we build on this result, by decomposing the P matrix and evaluating its eigenvalues and eigenvectors for specific cases.

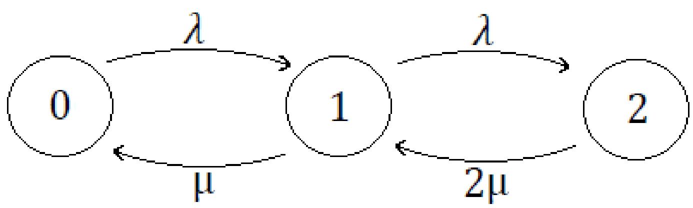
5. Specific Models

In this section we will analyse several specific versions for the models of [De Souza, 1986] and [Al Hanbali 2012] that we discussed in the previous chapter. Section 5.1 contains the most analysis, when we apply the model to a relatively simple case. We analyse this case, and aim for an explicit formula for the second moment of the interval availability. In section 5.2 we will discuss a generalization of our basic model. Section 5.3 considers a system with an additional unit of stock. After that, we consider a system with 2 items with both a single unit of stock in section 5.4. And finally section 5.5 considers an approach based on Kronecker operations which allows us to combine any of the systems mentioned above.

5.1 One Item, One Stock

In this section we look at the specific case of a system consisting of 1 item, which has 1 additional replacement unit in stock. The item breaks at an exponential rate λ. Once the item is broken the spare part will be placed into the system, and simultaneously the broken item will start being repaired with exponential rate μ. If this spare part breaks down before the original part is fixed, then the system is no longer working.

This system can be displayed by the following Markov chain, where the state denotes the number of broken items:



and the symbols corresponding to the arrows denote the transition rates.

We will apply the models of the previous chapter to this specific model, as we aim for an explicit expression for $E[A(T)^2]$. We continue where we left off in section 4.3, with the following expression:

$$E[A(T)^2] = 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot p_0 \cdot \sum_{i=1}^{n} (n-i+1) P^i e_0 + 2\sum_{l \in \Omega_0} x_l \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2}. \qquad (29)$$

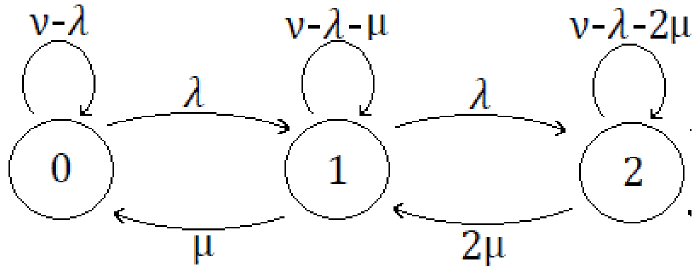
Now the only unknown here is the Matrix P^i. To be able to give an explicit expression for this, we try to determine its eigenvalues and corresponding eigenvectors in order to create a decomposition. If we were to find a decomposition in the form of

$$P = VDV^{-1}, \qquad (30)$$

then determining its i-th power would be easy since

$$P^i = VD^iV^{-1}. \qquad (31)$$

Keep in mind that the matrix P is the transition matrix of the uniformized process. To uniformize the process, we selected a uniformization constant ν and added self-transition to attain to following Markov Chain:



We now see that in all states the outgoing rate equals ν, so we can consider an analogous discrete time Markov Chain with transition probability matrix

$$P = \begin{bmatrix} 1-\frac{\lambda}{\nu} & \frac{\lambda}{\nu} & 0 \\ \frac{\mu}{\nu} & 1-\frac{\lambda+\mu}{\nu} & \frac{\lambda}{\nu} \\ 0 & \frac{2\mu}{\nu} & 1-\frac{2\mu}{\nu} \end{bmatrix}. \tag{32}$$

Where we should not forget the constraint that the uniformization constant has to be larger than or equal to all outgoing rates:

$$\nu \geq \lambda + \mu \ , \quad \nu \geq 2\mu \ . \tag{33}$$

So, we are looking for the eigenvalues of the P given in equation (32). Obviously since this is a stochastic matrix, θ=1 is one of the eigenvalues. It's corresponding eigenvector is either the steady state probability distribution (when using the left eigenvector) or the vector of ones (corresponding right eigenvector).

We influence the other eigenvalues by picking an appropriate value for ν. Solving det(P- θI)=0 in MAPLE yields:

$$\theta_1 = \frac{\nu - \lambda - \frac{3}{2}\mu - \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}}{\nu} \quad , \quad \theta_2 = \frac{\nu - \lambda - \frac{3}{2}\mu + \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}}{\nu} \quad , \quad \theta_3 = 1 \tag{34}$$

Now by picking an appropriate value for ν we can fix one more of the eigenvalues. Since we already have 1 as an eigenvalue, the next obvious choice would be to also aim for $\theta = 0$ as an eigenvalue. Two choices for ν assure this:

$$\nu_1 = \lambda + \frac{3}{2}\mu - \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}\,, \tag{35}$$

$$\nu_2 = \lambda + \frac{3}{2}\mu + \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}\,. \tag{36}$$

However we quickly see that ν_1 would not be an appropriate value for ν, since $\sqrt{4\lambda\mu + \mu^2} > \mu$ so $\nu_1 < \lambda + \mu$ which is not allowed.

So we pick ν equal to $\lambda + \frac{3}{2}\mu + \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}$, which yields our 3 eigenvalues

$$\theta_1 = 0 \quad , \quad \theta_2 = \frac{\sqrt{4\lambda\mu + \mu^2}}{\lambda + \frac{3}{2}\mu + \frac{1}{2}\sqrt{4\lambda\mu + \mu^2}} \quad , \quad \theta_3 = 1\,. \tag{37}$$

Now we can determine the corresponding eigenvectors that we need to complete the decomposition of P. The matrix V of (right) eigenvectors is given by

$$V = \begin{bmatrix} v_{1,1} & v_{1,2} & 1 \\ v_{2,1} & v_{2,2} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{38}$$

where the v's are given by

$$\begin{aligned}
v_{1,1} &= -\frac{1}{2}\cdot\frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)^2 \lambda^2(\lambda - 2\mu)}{\mu\left(\lambda\mu + 4\mu^2 + \lambda\sqrt{4\lambda\mu + \mu^2}\right)\left(2\lambda\sqrt{4\lambda\mu + \mu^2} + \mu\sqrt{4\lambda\mu + \mu^2} - 2\lambda^2 - 4\lambda\mu + \mu^2\right)}\,, \\
v_{1,2} &= -\frac{1}{4}\cdot\frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)^2 \lambda^2(\lambda - 2\mu)}{\mu\left(\lambda^2 - \mu^2 + \mu\sqrt{4\lambda\mu + \mu^2}\right)\left(\lambda\sqrt{4\lambda\mu + \mu^2} + 3\mu\sqrt{4\lambda\mu + \mu^2} - 5\lambda\mu - 5\mu^2\right)}\,, \\
v_{2,1} &= -\frac{1}{2}\cdot\frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)\lambda(\lambda - 2\mu)}{\mu\left(2\lambda\sqrt{4\lambda\mu + \mu^2} + \mu\sqrt{4\lambda\mu + \mu^2} - 2\lambda^2 - 4\lambda\mu + \mu^2\right)} \\
v_{2,2} &= \frac{1}{4}\cdot\frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)\lambda(\lambda - 2\mu)}{\mu\left(\mu\sqrt{4\lambda\mu + \mu^2} + \lambda^2 - \mu^2\right)}
\end{aligned} \tag{39}$$

As for V^{-1}, we can use the fact that it's rows are the left eigenvectors of P, or we can simply invert V. This results in

$$V^{-1} = \begin{bmatrix} \bar{v}_{1,1} & \bar{v}_{1,2} & 1 \\ \bar{v}_{2,1} & \bar{v}_{2,2} & 1 \\ \frac{2\mu^2}{\lambda^2} & \frac{\mu}{\lambda} & 1 \end{bmatrix}. \tag{40}$$

The values for the $\bar{v}$'s are of similar form as the v's.

Now that we have given the matrices V and D, we can use them to finally diagonalize P:

$$P = VDV^{-1}\,. \tag{41}$$

This gives us all that we need for determining $E[A(T)^2]$:

$$E[A(T)^2] = 2\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot \begin{bmatrix} x_0 & x_1 & 0 \end{bmatrix} \cdot \sum_{i=1}^{n} (n-i+1) V D^i V^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\sum_{l \in \Omega_0} x_l \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2} . \tag{42}$$

We simplify this further by working out both summations. First of all we note that all matrices and vectors except for D^i contain no i's or n's, so can be moved out of the summations, this results in

$$E[A(T)^2] = 2 \cdot \begin{bmatrix} x_0 & x_1 & 0 \end{bmatrix} \cdot V \left(\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \sum_{i=1}^{n} (n-i+1) D^i \right) V^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\sum_{l \in \Omega_0} x_l \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2} . \tag{43}$$

We can then evaluate the two summation terms inside the brackets when we simply fill in matrix D:

$$\sum_{i=1}^{n} (n-i+1) D^i = \sum_{i=1}^{n} (n-i+1) \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sum_{i=1}^{n} (\mathrm{n}-\mathrm{i}+1) \cdot \theta_2^i & 0 \\ 0 & 0 & \frac{n(n+1)}{2} \end{bmatrix} . \tag{44}$$

Here the central term is simply a power series, which can be further evaluated as follows:

$$\sum_{i=1}^{n} (\mathrm{n}-\mathrm{i}+1) \cdot \theta_2^i = \frac{{\theta_2}^{n+2} - (n+1){\theta_2}^2 + n\theta_2}{(\theta_2 - 1)^2} . \tag{45}$$

We can now work out the first summation (the summation over n) as well, which is given by

$$\sum_{n=0}^{\infty} e^{-\nu T} \cdot \frac{(\nu T)^n}{(n+2)!} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{{\theta_2}^{n+2} - (n+1){\theta_2}^2 + n\theta_2}{(\theta_2 - 1)^2} & 0 \\ 0 & 0 & \frac{n(n+1)}{2} \end{bmatrix} . \tag{46}$$

This is simply a combination of power series, which are all known.

Evaluating the summations results in the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 + a_2 + a_3 & 0 \\ 0 & 0 & a_4 \end{bmatrix}, \tag{47}$$

where the expressions for the a's are:

$$\begin{aligned} a_1 &= \frac{e^{-\nu T} \cdot \left(e^{\theta_2 \nu T} - 1 - \nu T \theta_2\right)}{(\theta_2 - 1)^2 \cdot (\nu T)^2}, \\ a_2 &= -\frac{e^{-\nu T} \cdot (1 + (\nu T - 1) \cdot e^{\nu T}) \cdot {\theta_2}^2}{(\theta_2 - 1)^2 \cdot (\nu T)^2}, \\ a_3 &= -\frac{\theta_2 \cdot e^{-\nu T} \cdot (2 + (\nu T - 2) \cdot e^{\nu T} + \nu T)}{(\theta_2 - 1)^2 \cdot (\nu T)^2}, \\ a_4 &= \frac{1}{2} \cdot \frac{-2 + ((\nu T)^2 - 2\nu T + 2)}{(\nu T)^2}. \end{aligned} \tag{48}$$

We are now (finally) ready to present the main result of this section:

Theorem 1

In a one-item one-stock system where items breakdown at an exponential rate λ and are repaired at an exponential rate μ (or 2μ if they are both broken), the second moment of the interval availability during an interval [0,T] is given by:

$$E[A(T)^2] = 2 \cdot [x_0 \quad x_1 \quad 0] \cdot VAV^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot (x_0 + x_1) \cdot \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2}$$

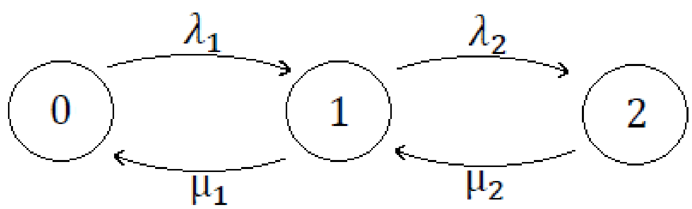
Where (x_0, x_1, x_2) denotes the steady state distribution, the matrices V and V^{-1} are given by equations (38), (39) and (40), and A is given by (47) and (48).

Note that this result contains no infinite sums, and no implicitly defined variables. Every parameter is defined explicitly. This is a very interesting result, as it not only gives a closed form and exact expression for the second moment of the interval availability, it is also an efficient method since due to our matrix decomposition we were able to evaluate the infinite sums to eventually attain a straightforward expression.

In the next few sections, we will analyse similar models to see if this result also translates to slightly more advanced cases.

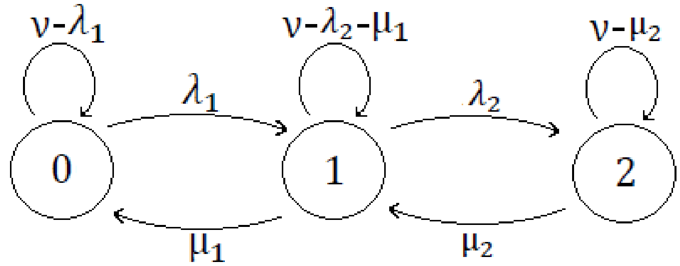
5.2 A Generalization

In this section we try to determine if the result from the previous section also holds for a generalized version of the model we used there. The system we now look at can be graphically displayed as follows:



This is obviously a generalized version of the original model, in the previous section λ_2 was equal to λ_1 and μ_2 equalled $2\mu_1$. To see if we can attain the same result, we go through the same steps performed in section 5.1.

First we uniformize the system with uniformization constant ν. Adding the self-transitions yields:



If we then consider the analogous discrete time Markov chain, it has the following transition matrix:

$$P = \begin{bmatrix} 1-\frac{\lambda_1}{\nu} & \frac{\lambda_1}{\nu} & 0 \\ \frac{\mu_1}{\nu} & 1-\frac{\lambda_2+\mu_1}{\nu} & \frac{\lambda_2}{\nu} \\ 0 & \frac{\mu_2}{\nu} & 1-\frac{\mu_2}{\nu} \end{bmatrix}. \tag{49}$$

Where we have the obvious constraint that the uniformization constant has to be larger than or equal to all outgoing rates:

$$\nu \geq \lambda_1 \quad , \quad \nu \geq \lambda_2 + \mu_1 \quad , \quad \nu \geq \mu_2 \ . \tag{50}$$

Now we take a look at the eigenvalues for this matrix, some algebra shows us that they are

$$\begin{aligned} \theta_1 &= \frac{1}{\nu}\left(\nu - \frac{1}{2}(\lambda_1+\lambda_2+\mu_1+\mu_2) - \frac{1}{2}\sqrt{f(\lambda_1,\lambda_2,\mu_1,\mu_2)}\right), \\ \theta_2 &= \frac{1}{\nu}\left(\nu - \frac{1}{2}(\lambda_1+\lambda_2+\mu_1+\mu_2) + \frac{1}{2}\sqrt{f(\lambda_1,\lambda_2,\mu_1,\mu_2)}\right), \\ \theta_3 &= 1\,. \end{aligned} \tag{51}$$

Where $f(\lambda_1, \lambda_2, \mu_1, \mu_2)$ is a long but relatively easy function containing combinations of the first and second powers of its parameters (see Appendix A)

Similar to the approach in the previous paragraph, we then pick a value for ν such that one of the eigenvalues becomes zero. In this case, we use

$$\nu = \frac{1}{2}(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \frac{1}{2}\sqrt{f(\lambda_1, \lambda_2, \mu_1, \mu_2)} \tag{52}$$

which sets θ_1 equal to zero, and simultaneously turns θ_2 into such a horrendously large function that we will not even put it in the appendix (it's pretty straightforward, just very big). Next we have a look at the eigenvectors of the P matrix. We see that they are of the same form as in the previous section:

$$V = \begin{bmatrix} \dot{v}_{1,1} & \dot{v}_{1,2} & 1 \\ \dot{v}_{2,1} & \dot{v}_{2,2} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{53}$$

where the $\dot{v}$'s are given by large terms of the λ's and μ's. In the same spirit V^{-1} can be found by simply inverting the V matrix.

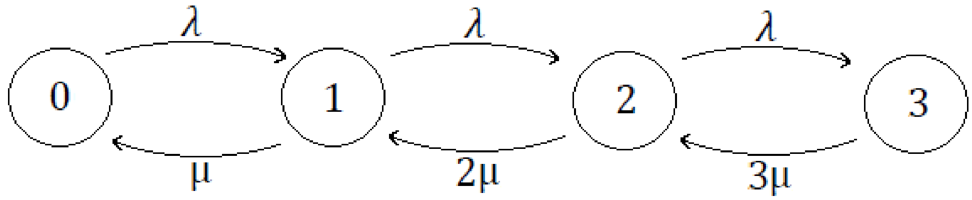
We now have our three eigenvalues (though one of them requires some evaluation), our V matrix and our V inverse matrix. This means that we can diagonalize P explicitly, and we can then use Theorem 1 to determine the second moment of the interval availability.

One of the main uses of this more general model, would be to use it to fit larger models to. Currently if one would want to analyse variability for a larger system, these systems are often fitted into (2-state) on-off processes. This allows you to determine the variability parameters for the aggregated system (for which they are known), however for the aggregation you would somehow have to aggregate all the on-states into one single state and all the off-states into one single state. Since we have now shown that we have an exact and closed form expression for the second moment of the interval availability in this 3-state model, we can use this extra state to make a much more significant distinction between the states in the large system.

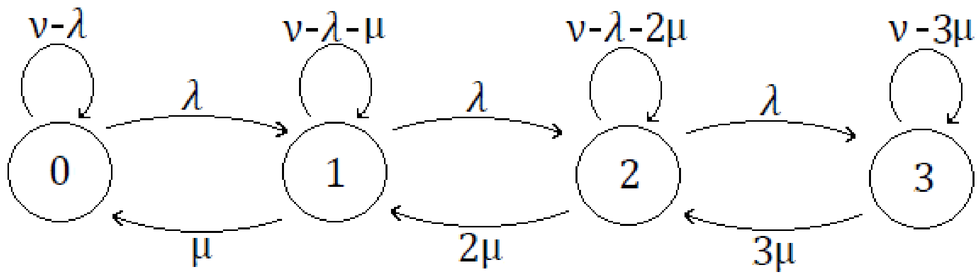
You could for example aggregate all down-states into state 2, all up-states for which a single failure would constitute downtime into state 1, and all other up-states could be aggregated to state 0. This is just one possible way of aggregating a large system into this 3-state system, there are many more possibilities that are all much better approximations than fitting it to a 2-state system. For the fitting purpose the model discussed in this section is also obviously more suitable than the model in section 5.1, due to the fact that this model has two extra parameters which can be selected.

5.3 An Additional Unit of Stock

In this section we look at another slightly extended version of our original model, we consider the case of one additional unit of stock. This means that we have 1 item for which we now have 2 extra units in stock. Breakdowns occur at rate λ, and the server has infinite repair capacity (in this case infinity equals three, interesting) and a repair rate of μ per broken item. This is shown by the following figure:



Similar to our approach in section 5.2, we try to perform the same steps as in our original model to see if the result holds. We start by uniformizing the system with a uniformization constant ν. Adding the self-transitions yields:



with the corresponding transition matrix for the analogous discrete time Markov chain:

$$P = \begin{bmatrix} 1-\frac{\lambda}{\nu} & \frac{\lambda}{\nu} & 0 & 0 \\ \frac{\mu}{\nu} & 1-\frac{\lambda+\mu}{\nu} & \frac{\lambda}{\nu} & 0 \\ 0 & \frac{2\mu}{\nu} & 1-\frac{\lambda+2\mu}{\nu} & \frac{\lambda}{\nu} \\ 0 & 0 & \frac{3\mu}{\nu} & 1-\frac{3\mu}{\nu} \end{bmatrix}. \tag{54}$$

Keeping in mind the constraints on the uniformization constant:

$$\nu \geq \lambda + 2\mu \quad , \quad \nu \geq 3\mu \ . \tag{55}$$

Now if we have a look at the eigenvalues of this system, we find them to be:

$$\theta_1 = \frac{1}{\nu}\left(\nu + g_1(\lambda,\mu)\right) \qquad \theta_2 = \frac{1}{\nu}\left(\nu + g_2(\lambda,\mu) - g_3(\lambda,\mu)\right)$$
$$\theta_4 = 1 \qquad \theta_3 = \frac{1}{\nu}\left(\nu + g_2(\lambda,\mu) + g_3(\lambda,\mu)\right), \tag{56}$$

where the g_i-functions are in the 'easy but very long'-category.

Here we see that we do have a slightly more complicated situation than in the previous two sections. We still have the possibility of selecting a suitable value for ν, however this only allow us to set one of the three complicated eigenvalues equal to zero. The other two remain large terms. This does not mean that it is no longer possible to use our method, however most of the numerical efficiency is no longer attained.

Let's say we pick ν to be equal to $-g_1(\lambda, \mu)$, this results in $\theta_1 = 0$. This leaves us with two hopelessly long expressions for θ_2 and θ_3. Also both V and V^{-1} lose the reasonably nice structural properties they had. For any given values of λ and μ they can still be computed of course, but it is no longer possible to explicitly give the expressions for them without filling entire pages.

Without going into too much detail, we can still work towards the result on the second moment of the interval availability by working through equations (42) through to (48). Eventually we will derive a matrix A that is of the form

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dot{a_1} & 0 & 0 \\ 0 & 0 & \dot{a_2} & 0 \\ 0 & 0 & 0 & \dot{a_3} \end{bmatrix}, \tag{57}$$

where the $\dot{a}$'s are summations of terms similar to the a's given in equation (48). Using this A matrix and the V and V^{-1}, we can give a modified version of theorem 1 for this extended case:

Theorem 1b

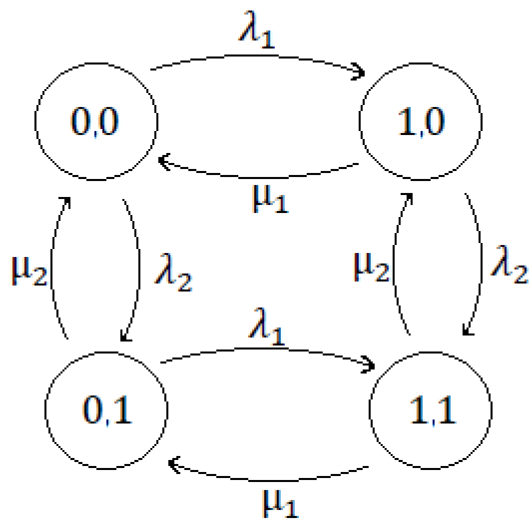
In a one-item two-stock system where items breakdown at an exponential rate λ and are repaired at an exponential rate μ per broken item(infinite repair capacity), the second moment of the interval availability during an interval [0,T] is given by:

$$E[A(T)^2] = 2 \cdot [x_0 \quad x_1 \quad x_2 \quad 0] \cdot VAV^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot (x_0 + x_1 + x_2) \cdot \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2},$$

where (x_0, x_1, x_2, x_3) denotes the steady state distribution, V is the matrix of eigenvectors and A is given by (57).

5.4 Two Items, No Stock

In this section we look at a system consisting of two different items, but no stock. The items have their individual breakdown rates λ_i and repair rates μ_i. The system is represented by the following figure:



Here the first number denotes the number of broken items of type one, and the second number denotes the number of broken items of type two. Note that this system has only one operating state, namely the (0,0) state. If one of the two items is broken the system can be considered unoperational.

To see if our result holds for this system, we follow the same approach as in section 5.1. We start by uniformizing with a uniformization constant ν. This results in the following transition matrix:

$$P = \begin{bmatrix} 1-\frac{\lambda_1+\lambda_2}{\nu} & \frac{\lambda_1}{\nu} & \frac{\lambda_2}{\nu} & 0 \\ \frac{\mu_1}{\nu} & 1-\frac{\mu_1+\lambda_2}{\nu} & 0 & \frac{\lambda_2}{\nu} \\ \frac{\mu_2}{\nu} & 0 & 1-\frac{\lambda_1+\mu_2}{\nu} & \frac{\lambda_1}{\nu} \\ 0 & \frac{\mu_2}{\nu} & \frac{\mu_1}{\nu} & 1-\frac{\mu_1+\mu_2}{\nu} \end{bmatrix}. \tag{58}$$

With the uniformization constraint

$$\nu \geq \lambda_1+\lambda_2 \;\;,\;\; \nu \geq \lambda_1+\mu_2 \;\;,\;\; \nu \geq \mu_1+\lambda_2 \;\;,\;\; \nu \geq \mu_1+\mu_2 \;. \tag{59}$$

Now if we have a look at the eigenvalues, we see that they are given by

$$\begin{aligned} \theta_1 &= 1-\frac{\lambda_1+\lambda_2+\mu_1+\mu_2}{\nu}, & \theta_2 &= 1-\frac{\lambda_1+\mu_1}{\nu}, \\ \theta_4 &= 1, & \theta_3 &= 1-\frac{\lambda_2+\mu_2}{\nu}. \end{aligned} \tag{60}$$

An obvious choice here would be to pick ν equal to $\lambda_1 + \lambda_2 + \mu_1 + \mu_2$, this results in:

$$\theta_1 = 0 \qquad \theta_2 = 1 - \frac{\lambda_1 + \mu_1}{\lambda_1 + \lambda_2 + \mu_1 + \mu_2}$$
$$\theta_4 = 1 \qquad \theta_3 = 1 - \frac{\lambda_2 + \mu_2}{\lambda_1 + \lambda_2 + \mu_1 + \mu_2} \tag{61}$$

The corresponding eigenvectors are given by:

$$V = \begin{bmatrix} \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_1}{\mu_1} & 1 \\ -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2} & 1 & 1 \\ -\frac{\lambda_1}{\mu_1} & 1 & -\frac{\lambda_1}{\mu_1} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{62}$$

This system is computationally a lot easier than the original case even. If we work out equations (42) through (47), we will find an A matrix of the form

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 + a_2 + a_3 & 0 & 0 \\ 0 & 0 & \breve{a}_1 + \breve{a}_2 + \breve{a}_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}. \tag{63}$$

Where the values for a_1, a_2, a_3 and a_4 are those given in (48), and the $\breve{a}'s$ are also simply the a's of (48) but with θ_3 substituted for θ_2.

Since we now have all we need, we can provide another modified version of theorem 1:

Theorem 1c

In a <u>*two-item no-stock*</u> *system where items break down at rates λ_1 and λ_2 and are repaired at rates μ_1 and μ_2, the expectation of the squared interval availability during an interval T is given by:*

$$E[A(T)^2] = 2 \cdot [x_0 \quad 0 \quad 0 \quad 0] \cdot VAV^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot (x_0) \cdot \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2},$$

where (x_0, x_1, x_2, x_3) denotes the steady state distribution, V is the matrix of eigenvectors given in (62) and A is given by (63).

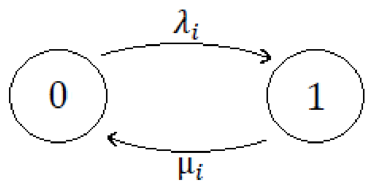
5.5 The Kronecker Approach

In this paragraph we consider an approach based on Kronecker sums and products. For a short introduction to this topic and several important operations and theorems, see appendix B. Kronecker sums and products are often used to combine matrices, in this section we see if we can use this approach to combine two or more smaller systems into one large system. In section 5.5.1 we show how our approach works by applying it on the two most basic systems. In section 5.5.2 we state our main result, and we show how it can be used for larger systems.

5.5.1 The Basic Case

We start by trying out the easiest case, combining two single item no stock systems with each other. If we combine them correctly, the result should be the system we analysed in the previous section (5.4).

So we have two systems of the form



To combine them we use Theorem B.1 in appendix B, which uses the Kronecker Sum. An important thing to note here is that we must combine the systems before we uniformize them. Combining two uniformized systems would be an unnecessary approximation, as we would then most likely have different uniformization constants in the two systems for which we will then have to compensate in our combined system (most likely the largest one will be selected).

So instead of the transition matrix of the uniformized system, we look at the generator matrices for the original systems:

$$Q_i = \begin{bmatrix} -\lambda_i & \lambda_i \\ \mu_i & -\mu_i \end{bmatrix} \qquad i = 1,2 \tag{64}$$

Now we attempt to combine two of those into the generator $\bar{Q}$, which we are hoping will turn out to be exactly the generator matrix of a two-item no-stock system (the system in section 5.4). We use the Kronecker Sum on the two generator matrices:

$$\begin{aligned}
\bar{Q} = Q_1 \oplus Q_2 &= \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{bmatrix} \oplus \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{bmatrix} + \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ \mu_1 & -\mu_1 & 0 & 0 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 \end{bmatrix} + \begin{bmatrix} -\lambda_2 & 0 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ \mu_2 & 0 & -\mu_2 & 0 \\ 0 & \mu_2 & 0 & -\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -\mu_1-\lambda_2 & 0 & \lambda_2 \\ \mu_2 & 0 & -\lambda_1-\mu_2 & \lambda_1 \\ 0 & \mu_2 & \mu_1 & -\mu_1-\mu_2 \end{bmatrix}
\end{aligned} \tag{65}$$

Here we first used the definition of the Kronecker Sum, and then the definition of the Kronecker Product (as given in Appendix B). We see that $\bar{Q}$ is exactly the generator matrix of a two-item no-stock system. Now to check if we can also determine the eigenvalues of the combined system, we use Theorem B.1 found in Appendix B which states that the eigenvalues of a Kronecker Sum are in fact all combinations of the eigenvalues of the two components. In this case, the eigenvalues (and eigenvectors) of the original systems are easily determined:

$$Q_i = \begin{bmatrix} -\lambda_i & \lambda_i \\ \mu_i & -\mu_i \end{bmatrix} \quad , \quad \theta_i = \begin{bmatrix} 0 \\ -(\lambda_i + \mu_i) \end{bmatrix} \quad , \quad V_i = \begin{bmatrix} 1 & -\frac{\lambda_i}{\mu_i} \\ 1 & 1 \end{bmatrix} \tag{66}$$

So if we look at the possible sums out of the two pairs of eigenvalues, we find:

$$\begin{aligned} &\theta_1 = 0 + 0 = 0 && \theta_2 = -(\lambda_1 + \mu_1) \\ &\theta_4 = -(\lambda_1 + \mu_1 + \lambda_2 + \mu_2) && \theta_3 = -(\lambda_2 + \mu_2) \end{aligned} \tag{67}$$

as eigenvalues for the generator of the combined system. However in our derivation we only look at eigenvalues for the uniformized system. Luckily, for the uniformized system which is given by

$$P = I + \frac{1}{\nu}\bar{Q} \tag{68}$$

we can easily determine the eigenvalues, because

$$eig(P) = eig\left(I + \frac{1}{\nu}\bar{Q}\right) = 1 + \frac{1}{\nu}eig(\bar{Q}), \tag{69}$$

where the $eig(\bar{Q})$'s are given in equation (67). So, when we fill these into equation (67) we find:

$$\begin{aligned} &\theta_1 = 1 && \theta_2 = 1 - \frac{\lambda_1 + \mu_1}{\nu} \\ &\theta_4 = 1 - \frac{\lambda_1 + \lambda_2 + \mu_1 + \mu_2}{\nu} && \theta_3 = 1 - \frac{\lambda_2 + \mu_2}{\nu} \end{aligned} \tag{70}$$

And these are exactly the same eigenvalues we found in the previous section (see equation (61)). Following theorem B.1 in appendix B, we can also determine the eigenvectors of the combined system as they are given by the Kronecker products of the eigenvectors of the two small systems.

$$\begin{aligned} \bar{V} = V_2 \otimes V_1 &= \begin{bmatrix} -\frac{\lambda_2}{\mu_2} & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} -\frac{\lambda_1}{\mu_1} & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_1\lambda_2}{\mu_1\mu_2} & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_1}{\mu_1} & 1 \\ -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2} & 1 & 1 \\ -\frac{\lambda_1}{\mu_1} & 1 & -\frac{\lambda_1}{\mu_1} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{aligned} \tag{71}$$

And this matrix is exactly the same as the one we found in equation (62) when we observed the 2 by 2 system. Note that in equation (71) we interchanged the two columns of the original V matrices, so that the ordering of the vectors matches the ordering in section 5.4.

So we have shown that for determining the eigenvectors and eigenvalues of a system consisting of two parts with no stock, we can use Kronecker operations on two one-part systems to combine them and determine the eigenvalues and eigenvectors of the combined system. In this simple case the approach doesn't save a whole lot of time or effort though, as the combined system is still relatively easy (see section 5.4). However for combining more or larger systems using this Kronecker approach is very effective, as we will illustrate in the next paragraph.

5.5.2 The General Model

Here we present our result on using the Kronecker approach for combining systems (possibly larger than 2 by 2). We then give the example of combining a 2-state system with our 3 state system, and show how it could be used to determine which of two items to add to stock.

Theorem 2

Consider a system consisting of M items, with stock levels s_j, breakdown rates λ_j, and repair rates μ_j ($j \in (1, .. M)$). This system can be constructed and analysed by combining the M individual systems. The generator matrix of this combined system will be given by

$$\bar{Q} = Q_1 \oplus Q_2 \oplus \dots \oplus Q_M,$$

its eigenvalues will be given by

$$\bar{\theta} = \theta_1 \oplus \theta_2 \oplus \dots \oplus \theta_M,$$

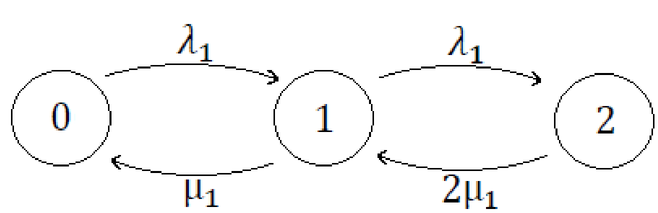
and its corresponding eigenvectors will be given by

$$\bar{V} = V_M \otimes \dots \otimes V_2 \otimes V_1,$$

where Q_i denotes the generator matrix of item i, θ_i is the vector containing its eigenvalues, and V_i is the matrix containing the corresponding eigenvectors.

The proof for this theorem directly follows the derivation in the previous paragraph. To show a slightly more advanced case, we will now proceed to apply this theorem by combining a 2 state system with a 3 state system.

We consider a system consisting of 2 items, item 1 which has one unit in stock, and item two which has zero units in stock. In order to be able to apply theorem 2, we first need to determine the individual generator matrix (Q), and its eigenvalues (θ) and eigenvectors (V). For the one stock model (as analysed in section 5.1):



We quickly find that

$$Q_1 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 \\ 0 & 2\mu_1 & -2\mu_1 \end{bmatrix}, \theta_1 = \begin{bmatrix} 0 \\ -\lambda_1 - \frac{3}{2}\mu_1 - \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} \\ -\lambda_1 - \frac{3}{2}\mu_1 + \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} \end{bmatrix}, V_1 = \begin{bmatrix} 1 & v_{1,2} & v_{1,1} \\ 1 & v_{2,2} & v_{2,1} \\ 1 & 1 & 1 \end{bmatrix} \quad (72)$$

Where the values for the v's are given in equation (39). Note that the first and third column have been exchanged in the V matrix. This is due to the fact that the eigenvalues of the generator matrix are not the same as those of the uniformized matrix. Uniformizing alters the eigenvalues (and thus scrambles their order), but has no effect on the eigenvectors.

The parameters for the second item (that doesn't have any stock) were already determined in the previous section:

$$Q_2 = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{bmatrix}, \qquad \theta_2 = \begin{bmatrix} 0 \\ -(\lambda_2+\mu_2) \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 1 & -\frac{\lambda_2}{\mu_2} \\ 1 & 1 \end{bmatrix} \quad (73)$$

We can now determine the parameters for the combined system. First we determine its generator matrix:

$$\bar{Q} = Q_1 \oplus Q_2 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 \\ 0 & 2\mu_1 & -2\mu_1 \end{bmatrix} \oplus \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 \\ 0 & 2\mu_1 & -2\mu_1 \end{bmatrix} + \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1+\lambda_2) & \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & 2\mu_1 & -(2\mu_1-\lambda_2) & 0 & 0 & \lambda_2 \\ \mu_2 & 0 & 0 & -(\lambda_1+\mu_2) & \lambda_1 & 0 \\ 0 & \mu_2 & 0 & \mu_1 & -(\lambda_1+\mu_1+\mu_2) & \lambda_1 \\ 0 & 0 & \mu_2 & 0 & 2\mu_1 & -(2\mu_1+\mu_2) \end{bmatrix}, \quad (74)$$

We can then determine its eigenvalues, they are given by all possible combinations of an element of θ_1 plus an element of θ_2:

$$\bar{\theta} = \begin{bmatrix} 0 \\ -(\lambda_2 + \mu_2) \\ -\lambda_1 - \frac{3}{2}\mu_1 - \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} \\ -\lambda_1 - \frac{3}{2}\mu_1 - \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} - (\lambda_2 + \mu_2) \\ -\lambda_1 - \frac{3}{2}\mu_1 + \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} \\ -\lambda_1 - \frac{3}{2}\mu_1 + \frac{1}{2}\sqrt{4\lambda_1\mu_1 + \mu_1^2} - (\lambda_2 + \mu_2) \end{bmatrix}. \tag{75}$$

The corresponding eigenvectors can then also be determined:

$$\begin{aligned} \bar{V} = V_2 \otimes V_1 &= \begin{bmatrix} 1 & -\frac{\lambda_2}{\mu_2} \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & v_{1,2} & v_{1,1} \\ 1 & v_{2,2} & v_{2,1} \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & v_{1,2} & v_{1,1} & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2}v_{1,2} & -\frac{\lambda_2}{\mu_2}v_{1,1} \\ 1 & v_{2,2} & v_{2,1} & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2}v_{2,2} & -\frac{\lambda_2}{\mu_2}v_{2,1} \\ 1 & 1 & 1 & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2} & -\frac{\lambda_2}{\mu_2} \\ 1 & v_{1,2} & v_{1,1} & 1 & v_{1,2} & v_{1,1} \\ 1 & v_{2,2} & v_{2,1} & 1 & v_{2,2} & v_{2,1} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned} \tag{76}$$

We can now use these parameters to determine the second moment of the interval availability of the combined system. For that we still need to select a uniformization constant though. For the uniformization constraint we know it needs to be at least as large as the largest outgoing rate, so:

$$\nu \geq \max_i \left|\bar{Q}_{i,i}\right| \tag{77}$$

We can then determine the eigenvalues for the uniformized process, using that

$$eig(\bar{P}) = eig\left(I + \frac{1}{\nu}\bar{Q}\right) = 1 + \frac{1}{\nu}eig(\bar{Q}). \tag{78}$$

We know that the eigenvectors stay the same. So when we put the elements of $\bar{\theta}$ in the diagonal matrix $\bar{D}$, we have found the decomposition for our new matrix $\bar{P}$:

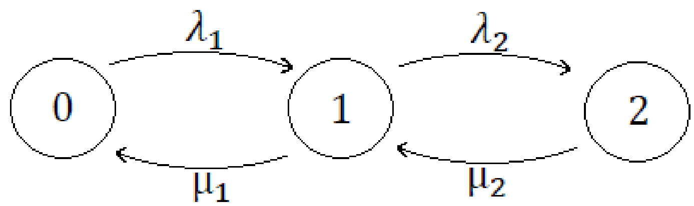
$$\bar{P} = \bar{V}\bar{D}\bar{V}^{-1} \tag{79}$$

This means that we can explicitly evaluate the second moment of the interval availability, as we have done in sections 5.1 through 5.4.

Note that we could use this method to compare several possible choices for allocating stock. Imagine having a two-item system with no units of stock for either item (the system described in section 5.4.1). When adding a unit of stock you would have to choose between the two, and you would like to pick the item that would reduce the variability the most. You can analyse this by applying theorem 2 twice, first combining the 2 systems where the first item has a unit of stock (as shown in the example above), then combining the 2 systems where the second item has a unit of stock. That last option would in this case simply mean substituting λ_1 for λ_2 and μ_1 for μ_2 in equations (74) through (79) of course, but for combining systems with a large amount of stock the method works in exactly the same way.

It is also worth noting that the use of theorem 2 is not limited to combining two systems, any number of systems can be combined. Simply add a system, determine the combined generator matrix, eigenvalues and eigenvectors, then add another system. Though this is a lot of work, it is still numerically efficient compared to determining all the parameters without using the individual systems.

The biggest possibility for applying theorem 2 might lie in combining it with the aggregation method discussed at the end of section 5.2. One could aggregate each individual system into the 3 state Markov chain given by:



Theorem 2 could then be used to combine any number of these relatively small systems (small compared to their original system at least). In that way the results for the second moment of the interval availability can be obtained for very large systems.

The accuracy of these results would heavily depend on the accuracy of the aggregation though. Examining the best way to aggregate a large system into the 3 state system shown above is a very interesting topic, but regrettably one that does not fit within the timeframe of this research.

6. Numerical Results

In this section we provide some numerical results, to serve as both a validation for the theory and as an interesting chapter in itself. Where possible, we will use our model given in section 5.5 to determine the system parameters. For some cases however we will use the approximation of [Al Hanbali, 2012] as presented in section 4.3.

6.1 Basic Numerical Model

The numerical case we observe is a basic 1-echelon model. We consider 1 system of 3 components, for all 3 components stock levels can be held. All three items have an expected breakdown time of 40.000 hours (roughly 4-5 years) and an expected repair time of 20.000 hours. We use these numbers as a basis for our numerical examples, for each specific example we will modify either some of the repair times or some the breakdown times by a certain factor. This allows us to observe the effect of adding (or not adding) one of these altered items to your stock. Note that though these repair and breakdown times are reasonably realistic, they are mostly selected to display interesting results in this numerical example.

We consider time intervals of one year, and we use the formulas derived in chapter five to determine the expected interval availability and the variance of the interval availability. For determining the survival rate of the interval availability we use the approximation of [Al Hanbali, 2012] as given in section 4.3.

6.2 Modifying Repair and Breakdown Times

We start by modifying both the repair and breakdown times, while keeping the ratio constant. To be able to clearly observe the effect of this, we make item 1 into a ‘slow mover’ by making the breakdown and repair times ten times as large as they originally were, and we make item 3 into a ‘fast mover’ by making the repair and breakdown times ten times smaller.

We then observe a system without stock, and consider all 3 possible ways of allocating a single unit of stock. Using the results of the previous chapter, this system can be constructed by combining one 3-state Markov Chain with two 2-state Markov chains. We find the following results for the expected interval availability, the variance of the interval availability, the coefficient of variation, and the 80% survival probability:

Constant (λ/µ) Ratio	E[A(T)]	Var[A(T)]	CV	P(A(T)>0.8)
Original Case (0,0,0)	29,5%	0.12	1.17	15,3%
+1 Slow Mover (1,0,0)	40,9%	0.12	0.85	20,8%
+1 Average mover (0,1,0)	40,9%	0.13	0.89	22,9%
+1 Fast Mover (0,0,1)	40,9%	0.18	1.07	33,0%

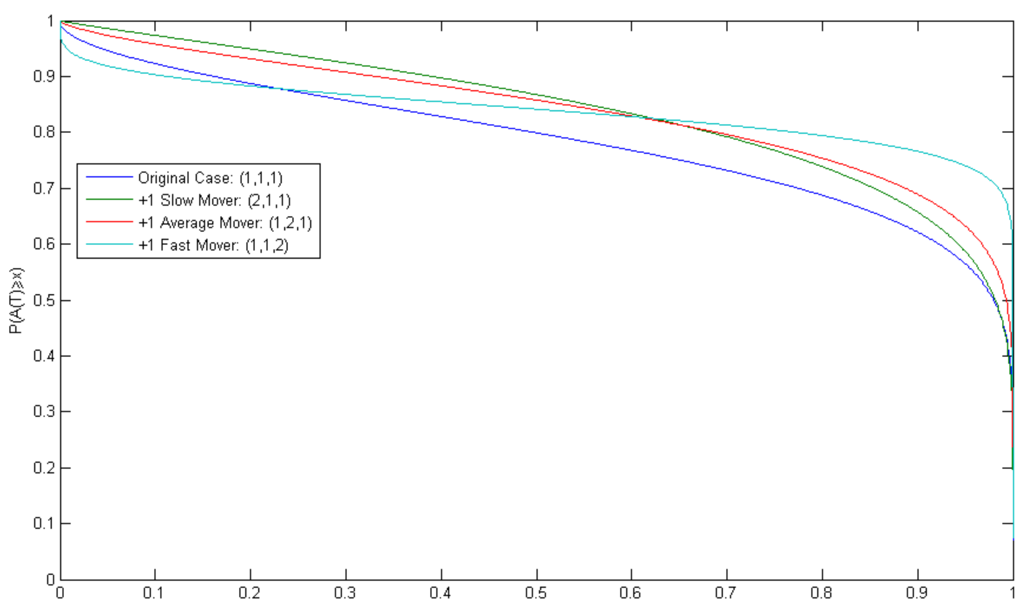
The first thing we see here is that all 3 of the options provide the same increase to the expected availability. This was to be expected, since for determining the expected availability we require only the expected number of backorders, which in turn is a function of (λ_j / μ_j) terms. The λ_j and μ_j never occur individually, thus since the ratio is the same for all three items, adding a unit of stock to any of the three yields the same increase in the expected interval availability.

However if we observe the results for the survival rate, we clearly see that the last option (adding an additional unit of stock for the fast mover) yields a significantly larger survival probability. Apparently focusing on the faster moving items in this case yields significantly better results in that aspect. Also note that the option that yields the highest survival rate in fact also yields the highest variance, whereas intuitively you would argue that a high variance is naturally a bad thing. However if your goal is simply to be over a certain percentage (in this case 80%), then you will need a high variance if your average is not high enough.

Keep in mind that these were results for a very low availability though. We now consider the same set of possibilities for adding a single unit of stock, in a situation where the base stock is already slightly higher (there's already one unit of stock for each item). Using the results of chapter five, this system could be constructed by combining two 3-state Markov chains (analysed in section 5.1) with one 4-state Markov chain (analysed in section 5.3). We find the following results:

Constant (λ/μ) Ratio	E[A(T)]	Var[A(T)]	CV	P(A(T)>0.8)
Original Case (1,1,1)	78,3%	0.10	0.41	68,7%
+1 Slow Mover (2,1,1)	83,8%	0.07	0.31	73,9%
+1 Average mover (1,2,1)	83,8%	0.08	0.33	75,4%
+1 Fast Mover (1,1,2)	83,8%	0.10	0.39	79,4%

Here we observe exactly as in the previous case. We see that for the average interval availability it doesn't matter which item is added, however for the survival probability adding the fast mover yields significantly better results. This is very interesting considering the fact that the conventional metric approach might not have picked this item since it only considers the average availability, where the fast mover is doing just as well. The following figure displays the survival curves of the four different stock levels:



Here we clearly see that adding an additional unit to the fast movers stock yields the largest increase for higher availability. Since each of the three options yields the same expected interval availability, the area below these three curves has to be equal. This also means that for low availability it's in fact preferable to add a slow mover. However since in all realistic scenarios we are interested in relatively high availabilities, we can say that adding a unit to the fast mover is clearly preferable for the survival rate.

To determine whether the effect observed here is perhaps driven by either one of the altered parameters (repair- or breakdown-time), we now consider the case where we keep one constant.

6.3 Only Modifying Repair Times

In this example we start with the basic system as described in section 6.1 (1 system, 3 components), but now we only change the repair times of the items while keeping the breakdown times constant. This changes the (λ/μ) ratio which has a significant effect on the system parameters, whereas in the last paragraph we kept this ratio constant.

So we make item 1 into a slow repairer (10 times slower) and item 3 into a fast repairer (10 times faster). Since (only) the repair times are now altered, the stock levels will have to be altered as well. To compensate for the slow repairing item there will have to be more of them in stock, whereas the fast repairing item will require less stock. To clearly demonstrate this effect we increase all breakdown rates by a factor of 10. A reasonably balanced stock level would now be (50,9,2), meaning 50 stock of the slow repairing item, 9 of the normally repairing item, and 2 of the fast repairing item. Note that these values were specifically selected to provide a reasonable balance between the 3 items, and to display numerically interesting results.

If we were to analyse this system use our methods derived in the previous chapter, we would have to aggregate two of the Markov chains (the ones for the item with 50 stock and the item with 9 stock). Since this is not something we managed to evaluate within this project, we will use the approximation by [Al Hanbali, 2012] to determine the results for this example.

We observe the choices we have for adding one additional unit of stock, and find the following results:

Constant Breakdown Rate	E[A(T)]	Var[A(T)]	CV	P(A(T)>0.8)
Original Case (50,9,2)	86,9%	0.05	0.27	78,7%
+1 Slow Repairer (51,9,2)	87,9%	0.05	0.25	80,2%
+1 Average Repairer (50,10,2)	87,8%	0.05	0.26	80,1%
+1 Fast Repairer (50,9,3)	87,9%	0.05	0.26	80,5%

Here we see that though adding each of the three items provide roughly the same increase to the expected interval availability, the survival rate increases the most when adding to the fast repairer (which currently has the smallest amount of stock). Whether this is because this part repairs faster, or because adding a single unit of stock is relatively the largest increase in stock level (50%), cannot be concluded at this point. Given the fact that the model of [Al Hanbali, 2012] is an approximation (which has been verified by simulation to give accurate results though), is it also the question whether the difference in the survival probability are significant.

6.4 Only Modifying Breakdown Times

We now do a similar numerical experiment for modifying the breakdown times. Starting from the basic system as described in section 6.1, we now modify item 1 to breakdown at a slower rate and item 3 to breakdown faster. A reasonably balanced stock level would now be (2,9,50), where obviously item 1 that has less breakdowns requires significantly less stock than item 3 that breaks down a lot (and takes the same time to repair).

We observe the three choices of adding one additional unit of stock:

Constant Repair Rate	E[A(T)]	Var[A(T)]	CV	P(A(T)>0.8)
Original Case (2,9,50)	85,7%	0.02	0.18	73,2%
+1 Slow Breakdown (3,9,50)	86,7%	0.02	0.16	75,6%
+1 Average Breakdown (2,10,50)	86,6%	0.02	0.17	75,2%
+1 Fast Breakdown (2,9,51)	86,6%	0.02	0.18	75,2%

Here we see that for the survival rate it seems preferable to add an additional unit of stock to the item that breaks down slower (and thus has the least amount of stock). However in this case this is also the option that METRIC would prefer, since it also provides the largest increase to the expected interval availability.

From these numerical data we've obtained reason to believe that in situations where adding a unit of stock to different items yields the same increase in the expected interval availability (balanced stock levels), for the survival rate it would be preferable to add the unit of stock to the item that current holds the smallest stock level. This yielded the largest increase in both the case where repair rates were different and breakdown rates were constant, and in the case where repair rates were constant and the breakdown rates varied.

If the ratio between the breakdown rate and the repair rate (λ/μ) is roughly equal for all possible options, it is clearly preferable to add a unit of stock to the fastest moving item. In this case this is the item where the breakdown rate (and thus the repair rate) is the highest, the item that 'moves' the most.

7. Conclusions

7.1 Conclusions

We conclude by answering our research questions:

1. What currently existing literature is applicable to this research?
We examined a large amount of literature in the general area of this research. First off all we examined the after sales business models given in [Cohen, 2006] and the spare part strategies given in [Rustenburg, 2000]. Combining this with the information that Thales provided regarding its processes, were able to classify said processes.

We also examined the METRIC model, as it is key to the entire spare parts industry. However our research specifically focussed on the area that METRIC disregards (variability), and in chapter 6 we have shown that disregarding variability can result in sub-optimal decision making.

Most importantly, we focussed on literature regarding interval availability. The two main articles we evaluated were [De Souza, 1986] and of [Al Hanbali, 2012]. The model described in [De Souza, 1986] was written with the specific application of repairable computer systems in mind, however its theoretical analysis and results can be translated directly to our research.

This is exactly what was done in [Al Hanbali, 2012], this article can certainly be seen as the basis and motivation for this research. The model of [De Souza, 1986] was applied to the spare parts management at Thales, and several nice results regarding the variance and survival function of the interval availability were derived.

2. Which theoretical model and/or analysis currently provides the best approximation of the practical situation at Thales?
This is without a doubt the model presented in [Al Hanbali, 2012]. It was specifically created for Thales, and provides a reasonable approximation of the practical situation there. Though some approximations were made, they are strictly necessary in order to be able to derive any kind of theoretical results.

A lot of similar models exist that use a two-state Markov Chain and then analyse interval availability. For example in [vdHeijden, 1988] the on and off periods were fitted using phase type distributions, which yielded a reasonable result. Another example, and the most notable model for two-state interval analysis, is the work of [Kirmani, 2008]. They derive the first two moments of the interval availability in exact and closed form. However these models are only suitable for on-off processes, and fitting larger systems into two states will be too much of an approximation to be able to draw any significant conclusions.

3. Can we extend or improve this model?
We looked for ways to extend or improve on the model of [Al Hanbali, 2012] and its results, and we found two ways to do that.

Our first main result involves a further evaluation of the expression for the second moment of the interval availability. By decomposing the transition probability matrix and picking an appropriate uniformization constant, we were able to create a much more efficient way of calculating this second moment for certain specific cases. In section 5.1 we derived our basic results for the one item one stock case in Theorem 1. Furthermore we considered several similar but slightly more advanced models in the sections 5.2 through 5.4, and in these models we give results on the second moment and the constraints under which these results hold.

Our second main result is derived in section 5.5, and involves using Kronecker operations to build up larger systems out of multiple smaller systems. Theorem 2 states exactly how to combine two systems and how use the properties of the original system to analyse that of the combined system. We give two concrete examples of how our result works, using the basic systems we analysed in the previous sections to build a larger system. We also mention the possibilities of combining this result with the aggregation method we mentioned in section 5.2. Though this might seem very promising, regrettably it did not fit within our project.

4. How well does our extended model represent the practical situation?
This is a difficult question to answer. Though our initial intentions were for this research to have a reasonably applied nature, it turned out to be mostly theoretical. However in section 6 we were able to use our results to determine parameters for the numerical evaluation, and we found that they matched the simulation results (as derived in [Al Hanbali, 2012]) very well. For the relatively small cases that we examined, we found our model to be a good representation of the practical situation.

5. Which further advantages does our extended or improved model provide?
Our first main results is a further evaluation of the expression for the second moment of the interval availability for specific cases. This result allows us to calculate this second moment much more effectively. The original result contained infinite sums and implicitly defined variables, which cause numerical difficulty especially for larger systems. Our result contains none of this, and is very straightforward and efficient. In sections 5.1 through 5.4 we have outlined the basic systems for which our results can be applied.

Our second main results allows us to combine small systems into larger systems using Kronecker operations. This is a very effective way to construct larger systems, as computational difficulty increases very quickly with size. We can use the results that we have derived for smaller systems, to evaluate a large system by combining its components. This could for example be used as an efficient way to compare several possible choices of allocating stock.

7.2 Further Research

Our research leaves several possibilities for further studies. First of all the result of theorem 1 might be applied to a larger set of systems. In our research we only evaluated it for systems that were reasonably small and manageable, however there is a good chance the result will hold in a much more general setting. Furthermore it might be interesting to try to determine if an approach similar to the one we did might also yield a more efficient way to determine higher moments of the interval availability.

The most interesting direction for further research in my opinion would be attempting to combine the Kronecker approach given in section 5.5 with the aggregation model mentioned in section 5.2. If there were to be an effective way to aggregate large systems into the 3 state system given in section 5.2, then this would be a very effective way to analyse them. However there are many possible ways of distributing the large amount of states over the three states, so there are plenty of possibilities for trying to determine the most efficient way to do so. If an efficient way were to be found, then combining this with Theorem 2 of section 5.5 would results in a very effective way to analyse larger systems.

Another topic for further research could be a more advanced application of my results. Though I have currently restricted myself to relatively simple examples, maybe a software implementation can be constructed to allow for analysis of more general cases. This would allow a broader scale of numerical results, as well as a way to evaluate exactly how good of an approach this is for larger systems.

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Appendices

Appendix A: Parameter Values

For the matrix V given in equation (38)

$$V = \begin{bmatrix} v_{1,1} & v_{1,2} & 1 \\ v_{2,1} & v_{2,2} & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

the v's are given by:

$$v_{1,1} = -\frac{1}{2} \cdot \frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)^2 \lambda^2(\lambda - 2\mu)}{\mu\left(\lambda\mu + 4\mu^2 + \lambda\sqrt{4\lambda\mu + \mu^2}\right)\left(2\lambda\sqrt{4\lambda\mu + \mu^2} + \mu\sqrt{4\lambda\mu + \mu^2} - 2\lambda^2 - 4\lambda\mu + \mu^2\right)},$$

$$v_{1,2} = -\frac{1}{4} \cdot \frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)^2 \lambda^2(\lambda - 2\mu)}{\mu\left(\lambda^2 - \mu^2 + \mu\sqrt{4\lambda\mu + \mu^2}\right)\left(\lambda\sqrt{4\lambda\mu + \mu^2} + 3\mu\sqrt{4\lambda\mu + \mu^2} - 5\lambda\mu - 5\mu^2\right)},$$

$$v_{2,1} = -\frac{1}{2} \cdot \frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)\lambda(\lambda - 2\mu)}{\mu\left(2\lambda\sqrt{4\lambda\mu + \mu^2} + \mu\sqrt{4\lambda\mu + \mu^2} - 2\lambda^2 - 4\lambda\mu + \mu^2\right)}$$

$$v_{2,2} = \frac{1}{4} \cdot \frac{\left(\sqrt{4\lambda\mu + \mu^2} - 2\lambda - 3\mu\right)\lambda(\lambda - 2\mu)}{\mu\left(\mu\sqrt{4\lambda\mu + \mu^2} + \lambda^2 - \mu^2\right)}$$

The function used in equation (51) is given by

$$f(\lambda_1, \lambda_2, \mu_1, \mu_2) = \lambda_1^2 + \lambda_2^2 + \mu_1^2 + \mu_2^2 - 2\lambda_1\lambda_2 - 2\mu_1\mu_2 + 2\lambda_1\mu_1 + 2\lambda_2\mu_2 - 2\lambda_1\mu_2 - 2\lambda_2\mu_1 .$$

Appendix B: Matrix properties & Operations

In this appendix we will provide some results on Kronecker operations, which can be found in [Horn, 1991] for example.

The Kronecker Product, also known as direct product or tensor product, is a useful tool for studying and working with matrix equations. It is defined for two matrices or arbitrary sizes.

Definition B.1 *The Kronecker product of matrix $A = [a_{ij}] \in M_{m,n}$ and matrix $B = [b_{ij}] \in M_{p,q}$ is denoted by $A \otimes B$ and is defined to be the block matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in M_{mp,nq}$$

Notice that $A \otimes B \neq B \otimes A$ in general.

We also define the Kronecker Sum:

Definition B.2 *The Kronecker sum of square matrices $A = [a_{ij}] \in M_n$ and $B = [b_{ij}] \in M_m$ is denoted by $A \oplus B$ and is defined to be the mn by mn matrix given by*

$$A \oplus B = (I_m \otimes A) + (B \otimes I_n)$$

The following theorem regarding eigenvalues and eigenvectors using Kronecker operations, is perhaps the most important theorem for this research

Theorem B.1 *Let $A \in M_n$ and $B \in M_n$ be given. Now if θ is an eigenvalue of A with x as its corresponding eigenvector, and if φ is an eigenvalue of B with corresponding eigenvector y, then $\theta + \varphi$ is an eigenvalue of $\oplus B$, and $y \otimes x$ is its corresponding eigenvector. Every eigenvalue of the Kronecker sum arises as such a sum of eigenvalue of A and B.*

Samenvatting

In de service industrie zijn contracten op basis van performance steeds gebruikelijker. Bedrijven bieden een dienst aan, en de jaarlijkse betaling die ze hiervoor ontvangen is afhankelijk van de kwaliteit van de aangeboden dienst gedurende dat jaar. Dit betekent dat voor het effectief voorspellen van de kosten en inkomsten het niet langer voldoende is om lange termijn gemiddeldes te gebruiken, er moet geanalyseerd worden hoe het systeem zich gedraagd gedurende specifieke tijdsintervallen.

Als toepassing hiervan kijken we naar Thales Nederland. Hier worden radar systemen ontworpen, gebouwd en verkocht, en naast het systeem zelf wordt meestal ook een bijbehorend onderhoudscontract afgesloten. Dit houdt in dat Thales verantwoordelijk is voor het werkend houden van de geleverde systemen, en dat de opbrengsten (of boetes) die Thales jaarlijks ontvangt hier direct aan gerelateerd zijn. Om het systeem zo vaak mogelijk beschikbaar te houden kan Thales reserve onderdelen aanschaffen en op locatie bewaren. Het budget hiervoor is echter beperkt, dus de vraag is hoe we dit budget zo effectief mogelijk kunnen besteden over reserve onderdelen.

Het theoretische model dat we hiervoor bestudeerd hebben is gebaseerd op Markov ketens. Bestaande modellen die geschikt zijn voor grotere systemen zijn meestal grove benadering, exacte (expliciet gedefinieerde) resultaten over de jaarlijkse beschikbaarheid bestaan voornamelijk voor Markov ketens met 2 toestanden. Aan de hand van het model opgesteld door [Al Hanbali, 2012], hebben wij met behulp van een matrix-decompositie vergelijkbare resultaten verkregen voor een Markov keten met 3 toestanden. We beschrijven meerdere basis modellen waarvoor onze resultaten gelden.

Daarnaast geven we een algemene methode om grote systemen te analyseren door de kleinere componenten van deze systemen individueel te analyseren. Aangezien de numerieke complexiteit snel stijgt bij grotere systemen, is dit een relatief efficiënte methode. We geven enkele voorbeelden om duidelijk te maken hoe deze methode toegepast kan worden, en we geven indicaties van de verdere mogelijkheden die deze methode biedt bijvoorbeeld voor het selecteren van een onderdeel om extra op voorraad te leggen.