Master's thesis Applied Mathematics
(Chair: Stochastic System and Signal Theory, Track: Financial Engineering)
Faculty of Electrical Engineering, Mathematics and Computer Science (EEMCS)

Constructing an α-maximizing option trading strategy in a multi-dimensional setting

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October 5, 2013

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Preface

Dear reader,

Thank you for taking interest in this thesis, which I have written as final project for my Master's in Applied Mathematics at the University of Twente. The research for this thesis was conducted at the mathematics department of the Dublin City University, Ireland, under the primary supervision of Professor P. Guasoni. I would like to use this moment to express my gratitude to the people who have supported me throughout this project.

First of all, I would like to thank Professor P. Guasoni for giving me the opportunity of working with him and inviting me at the Dublin City University, giving me the chance to go abroad for my studies. It was a pleasure to work with him and I really appreciate the way in which he has supervised me. I am grateful for the informal, but professional atmosphere he provided and the way he always could find the time to meet with me to discuss the project.

Second, I want to thank Professor A. Bagchi for supervising me at the University of Twente, especially in the last phase of the project. I wish him all the best in his retirement.

Third, the people of the mathematics department of the Dublin City University, for welcoming me in Ireland and at the Dublin City University. It was a pleasure to work alongside this group. I would especially like to thank Dipl.-Ing. Dr. Eberhard Mayerhofer, who I came to know as a good friend, and who took the time to read the concept version of this thesis and provided me with a lot of useful feedback to establish the final version. Also, I would like to thank Christopher Belak, MSc. for being a good friend and introducing me to the city Dublin, as well as for setting up the database structure for the Optionmetrics database and teaching me the basics of MySQL language.

Finally, I would like to thank all other people who provided me with feedback on the concept version of the thesis, who supported me in academic or in personal ways throughout the project, and all my friends and family who visited me in Dublin during my time there, providing pleasant distractions during the weekends.

I have really enjoyed working on this project abroad and cherish the experience I have gained this way. I hope that you as reader can see this reflected in this thesis and enjoy whilst reading it.

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Abstract

In this thesis we derive a trading strategy which maximizes its excess returns, whilst controlling for the standard deviation of these excess returns, by dynamically investing in a portfolio of European call options on one or multiple benchmark assets and the benchmarks themselves. We show that this implies that the Sharpe ratio of these excess returns is maximized and that this is equivalent to the maximization of Jensen's alpha, the intercept in the ordinary least squares regression of these excess returns on the excess returns on the complete US equity market. The strategy is constructed such that the exposure of the obtained excess returns to the excess returns on the complete US equity market is statistically insignificant. The portfolio selection procedure for this strategy turns out to be a variant of Markowitz portfolio selection, adapted to admit derivatives in the selection.

The strategy's performance is studied in a theoretical framework, where the benchmarks follow geometric Brownian motions and options are priced at the benchmark's volatility with the Black-Scholes formula. We find that in this setting it is hard to generate superior performance as the statistical significance in the generated alphas is low. We also study the performance of our strategy with historical market data on four major stock indices on the US equity market over 1996-2013. In this setting we do find alphas that are significantly larger than zero and substantial Sharpe ratios, even in times of high volatility on the benchmarks, and that one obtains even better results when considering more benchmarks to invest on.

CHAPTER 1

INTRODUCTION

A growing literature suggests that a hedge fund manager can generate a positive return on top of the risk free rate (an excess return) by following strategies that repeatedly invest in dynamic portfolios that consist of one or more options on an underlying asset and have the possibility of holding the underlying asset itself. In previous research this has been shown through simulations and examples, but it remains unclear about the magnitude of the excess returns that can be achieved and how big the involved risk is.

Coval and Shumway [1] state that under the Black-Scholes model assumptions, options are redundant assets, but when one deviates from these assumptions, one can generate significant option returns. Broadie et al. [2] agree and report high returns on-out-of-the money (OTM) puts and straddles on the S&P 500 and explain this by mispricing effects of the market. Eraker [3], Jones [4], Kapadia and Szado [5], Liang et al. [6] and Santa-Clara and Saretto [7] all report similar findings on a variety of underlying assets, however, not all of them control for the involved risk. In these studies the term “risk” is used to denote the standard deviation of achieved excess returns.

This thesis is a follow up research on Guasoni et al. [8], in which a theoretical answer is provided for the questions that are left unanswered by the previous research. In this paper a trading strategy is derived which maximizes the alpha of the achieved excess returns, controlling for the risk. This paper explains the achieved excess returns by a single factor ordinary least squares (OLS) regression:

$$r^{pf} = \alpha + \beta r^{mkt} + \varepsilon, \tag{1.1}$$

where

- r^{pf} is the vector of excess returns generated by the investment strategy,
- α is the regression intercept, which captures the amount of excess return that is generated by the strategy itself (the excess returns linear projection orthogonal to the markets excess return), and is known as Jensen’s alpha,
- β is the sensitivity of the strategy excess returns with respect to the market excess returns,
- r^{mkt} is the vector of excess returns on the market itself, making βr^{mkt} the linear projection of the strategy’s excess returns on the market excess returns,

- ε is a vector of normally distributed shocks with zero mean, which only add variance to the excess returns.

The derived strategy is a variant of the buy-write strategy, involving long positions in benchmark assets and writing options with a continuous range of strikes on them. For the case where one considers a market that consists of one risky benchmark asset and a safe asset, the authors provide explicit formulae for the weights that one has to invest in each option to achieve the optimal portfolio on every moment on which is traded. The paper concludes from studies with simulated data that if common equity indices are used as benchmarks and if securities on these benchmarks are priced in the Black-Scholes framework, one could generate substantial alpha by trading frequently or holding options. However, such strategies carry a substantial risk as well, resulting in statistically insignificant alphas, and the probability of resulting in a negative alpha is close to one half. Hence, under the Black-Scholes model it is difficult for the hedge fund manager to generate superior performance from trading frequently in derivatives. Nevertheless, when the implied volatility of the derivatives is higher then the realized volatility of the benchmark asset, one is able to produce an alpha in the OLS that is statistically different from zero, even in absence of superior information.

This last conclusion rises the question of whether one is able to produce superior performance in practice by implementing this strategy on the actual derivatives market, which is the direct motive for the research conducted in this thesis.

In this thesis we formulate the alpha maximizing option trading strategy, which controls for the risk. The strategy repeatedly invests in a portfolio that consists of several options with discrete strikes on an benchmark asset and a position in this underlying asset itself. The selection of the optimal portfolio in this strategy turns out to be a variant of Markowitz portfolio selection theory, which thus far has not been studied very often with selecting derivatives, making it quite a novelty in this thesis. We also hedge out its sensitivity to market movements, to ensure that the generated alpha in regression (1.1) is generated only by our trading strategy. Thus, we want beta in this regression to be a factor of insignificant influence, meaning that its estimate in the OLS regression statistically insignificantly differs from zero at a reasonable significance level. We track the performance of our strategy with the Sharpe ratio, which is defined as the expected excess return generated by the strategy over the standard deviation of its excess returns, hence, a measure of the excess return one can generate per unit of standard deviation. We refer to this standard deviation with the term “risk”. We show that maximizing this Sharpe ratio is equivalent to maximizing the expected excess return given a constant level of variance. We test our strategy in a theoretical setting by simulating data and with historical market data. We use data from the Optionmetrics database, which contains market data from January 1996 to January 2013, to study whether it is possible in theory and in practice to generate a significant alpha under insignificant influence of market movements, and how high the Sharpe ratio that one can generate is.

We also investigate the effects of diversification, by developing an optimal option trading strategy which repeatedly invests in a range of options that can each depend on a different benchmark asset and in positions in these underlying themselves. We extend our Markowitz portfolio selection procedure for options with discrete strikes to a multidimensional setting, which enables us to test the performance of our strategy with simulated and historical data. In this discrete multidimensional setting we shall again hedge out the sensitivity to market movements, to ensure, so to say, "beta-neutrality". Another novelty of this thesis is that we also study the significance of the other Fama & French factors (the market capitalization factor and the book-to-market factor) in explaining our strategy's excess returns, by adding them to the regression. The main questions are whether one can produce alphas in the OLS regression with this multidimensional trading strategy that statistically significantly differ from zero, whether the strategy's excess returns are solely generated by the strategy itself, how high the Sharpe ratios obtained with this trading strategy can get, and whether the obtained Sharpe ratios are higher than those obtained with the one dimensional trading strategy, implying positive effects of diversification.

We find that with our investment strategy applied in the Black-Scholes framework it is hard to produce a significant alpha, in the case of a single benchmark asset, as well as in the case of multiple benchmark assets, agreeing with the conclusions in Guasoni et al. [8]. However, we do find that our trading strategy produces a significant, positive alpha and high Sharpe ratios under beta-neutrality when one uses historical market data of the S&P 500, the NASDAQ 100, the Russell 2000, and the Dow Jones 1/100th Industrial Average indices as benchmark assets, during the period of January 1996 - January 2013. The strategy even performs well in times of high volatility on the market (for example the Dot-com bubble period of 1996-2002 and the credit crisis/global recession period of 2007-2013), generating high Sharpe ratios. However, in these times the strategy generates more substantial betas than in periods of low volatility, but which are mostly statistically insignificantly different from zero. Considering more benchmarks to write derivatives on increases the Sharpe ratio of the strategy, thus the effects of diversification are positive. We also show that our trading strategy in multiple dimensions outperforms a naive multidimensional trading strategy which divides ones wealth equally over all considered underlying and then performs our one dimensional strategy on them. These effects increase when the correlation between the log returns on the underlying increases. We find the other Fama & French factors to be statistically insignificantly different from zero in the regression over the strategy's excess returns for most of our analyses, which strengthens our claim that the obtained strategy excess returns are generated by the strategy alone, and not by other factors.

The remainder of this thesis proceeds as follows: in chapter 2 we study and summarize the work of Guasoni et al. [8] as a basis for our research. In chapter 3 we derive the optimal trading strategy that invests in options with discrete strikes on a single benchmark asset and in the benchmark itself. In chapter 4 we derive the optimal trading strategy when one considers multiple benchmarks to invest on, by extending our Markowitz portfolio selection procedure to a multidimensional setting. We then test our derived strategies with data simulated from a Black-Scholes framework in chapter 5. In chapter 6 we first present and analyze the historical market data from the Optionmetrics database which we use in our strategy performance analyses, and then we test our strategies in a one and two dimensional setting, using different benchmark assets during different sub-periods of 1996-2013. In chapter 7 we present our conclusions and we make a few recommendations for further research.

CHAPTER 2

REVIEW OF PREVIOUS RESEARCH

This thesis is a follow up on Guasoni et al. [8]. In this chapter we introduce and summarize the research presented in the paper "Performance maximization of actively managed funds", in which the trading strategy is derived that trades in options with continuous strikes on a single benchmark asset and maximizes alpha in the regression (1.1), whilst controlling for the risk. The strategy turns out to be a variant of a buy-write strategy. Since this is a summary, the formulations in this chapter are sometimes very similar to the ones in the paper.

The motivation for the research conducted in this paper is that in previous literature it has been reported that positive regression alphas can be obtained by frequently trading in options, but leaves it unclear what the magnitude of these alphas can be and how big the risk involved is. A theoretical answer to these questions is derived in the paper, providing explicit formulae for the trading strategy that maximizes alpha by trading frequently in options with continuous strikes on a single benchmark, whilst controlling for the risk. In this thesis we bring the theory derived in the paper to practice by formulating the strategy for options with discrete strikes and testing the performance with historical market data. We also extend this strategy such that it can trade in options with discrete strikes that can each depend on a different benchmark.

The paper states that a lot of different trading strategies from the previous literature produce high alphas, but at a high level of risk, making the estimated alphas in the regression over the returns statistically insignificantly different from zero, and the probability of generating a negative alpha is close to one-half. Also, the question arises whether an investor could replicate the return generated by the strategy by investing in a set of benchmark assets. If not, how much more return can one generate with respect to the benchmark space? And what is the probability of actually generating more, in stead of less, return with respect to the benchmark space? The difference between the generated return and the return that can be generated with the benchmarks is labeled alpha, and is widely used in practice to measure the performance of a fund. The standard error of alpha, which measures the uncertainty of alpha, is used as the tracking error of the fund. The resulting ratio of alpha to its tracking error is referred to as the appraisal ratio. A high appraisal ratio indicates superior performance, because it indicates a high probability of a positive return. Maximizing this appraisal ratio is necessary for maximizing the Sharpe ratio, which is a widely used measure of fund performance. For a hedge fund, the appraisal ratio of the fund is itself the Sharpe ratio of the hedged position that neutralizes the benchmark risk. The appraisal ratio is the asymptotic T-statistic of the estimated alpha.

In practice, an investor who is evaluated by the performance of his fund relative to a benchmark (for example an index, like the S&P 500 and the NASDAQ 100), can hold this benchmark and write options on it. If the fund is fully invested in the benchmark, the fund return is a linear function of the market return with zero intercept. If the investor writes call options on the benchmark and invests the proceeds in a safe asset, the fund return is a nonlinear function of the market return. If the fund consists of a long position in the benchmark and short position in the options, the fund return has a non-zero alpha in the regression of the fund's excess return on the excess return of the index. In this framework, the authors pose the optimization problem and its solution.

First, define the excess return r_x on an actively managed fund, which is evaluated against a vector of excess returns $r_m = \{r_1, \ldots, r_k\}'$ on k benchmark assets. The fund is evaluated during the period from time 0 to T, divided in $\Delta t, 2\Delta t, \ldots, n\Delta t$, where $n\Delta t = T$, equally spaced time intervals on which returns are observed. Let r_{xi} and r_{mi} denote the observed excess returns on the fund and the benchmarks respectively, over the time interval from $(i-1)\Delta t$ to $i\Delta t$. The regression over the excess fund returns on the excess benchmark returns is

$$r_{xi} = \alpha + r'_{mi}\beta + \varepsilon_i$$

where the intercept α and the vector of slope coefficients β satisfy

$$\begin{aligned} \alpha &= \mathbb{E}\left[r_x - r'_m\beta\right], \\ \beta &= (\textit{var}\,(r_m))^{-1}\textit{cov}(r_m, r_x), \end{aligned}$$

and the term ε_i has zero mean and only adds variance to the excess returns. Then the risk associated with the benchmark is hedged out by adding a short position of β in the benchmark assets, and the return of the hedged position can be expressed as $r_x - r'_m\beta$, its expectation is α, and its tracking error is $\sqrt{\textit{var}\,(r_x - r'_m\beta)}$. The appraisal ratio of the hedged position is equal to the Sharpe ratio:

$$\text{APR} = \frac{\alpha}{\sqrt{\textit{var}\,(r_x - r'_m\beta)}}.$$

A high appraisal ratio implies high profitability for the hedged position, and as mentioned before, a high T-statistic of alpha in the ordinary least squares (OLS) regression.

The investor wants to maximize the appraisal ratio, so he should find a trading strategy such that the funds return r_x solves

$$\max_x \frac{\mathbb{E}[r_x - r'_m\beta]}{\sqrt{\textit{var}\,(r_x - r'_m\beta)}}.$$

One thus has to have a high alpha, but also a low tracking error, to maximize this appraisal ratio.

The space of payoffs available to the investor in a given period by trading in the securities available is denoted with X_a, the attainable space. The payoffs are assumed to have finite second moments, hence, $X_a \subset L^2(P, \Omega)$, where P is a probability measure and Ω is a sample space, and $L^2(P, \Omega)$ is the set of all measurable functions with finite second moments. The norm on this space is given by $||x|| = \mathbb{E}[x^2]^{1/2}$. The assumption of finite second moments is a minimal requirement to ensure that the sample estimates of linear regressions converge to their population counterparts. The space X_a allows the market to be incomplete, because it is allowed to be a strict subset of $L^2(P, \Omega)$. Its dimension could be infinite, which allows options with a continuous range of strikes and maturities on the benchmarks. The only constraint on the linearity of the payoff space.

The linear space of payoffs spanned by the benchmark assets, is denoted by X_b and referred to as the benchmark space. Its dimension is assumed to be $k+1$. Let $\{x_j\}_{j=0,\dots,k}$ be the payoffs of the $k+1$ independent assets that span X_b. Let the first payoff, x_0, be the constant payoff of a safe asset. A fund has abnormal return relative to its benchmarks if, and only if, its return or payoff falls outside the benchmark space. This can only be the case of X_b is a strict subset of X_a.

Let $\nu : X_a \mapsto \Re$ be the pricing function. Assume that the law of one price holds, thus ν is linear. Define a stochastic discount factor (SDF) for X_a as a random variable $m \in L^2(P, \Omega)$ such that $\nu(x) = \mathbb{E}[xm]$ for all $x \in X_a$. Let M_a denote the set of all SDFs for X_a with M_a. Then, by Riesz representation theorem, the exists some $m_a \in X_a$ such that $\nu(x) = \mathbb{E}[xm_a]$ for all $x \in X_a$, thus, $m_a \in X_a \cap M_a$. It follows that m_a is of smallest norm. The price function also applies to the set of benchmark assets, so the set of SDFs for X_b is $M_b = \{m \in L^2(P, \Omega) : \nu(x) = \mathbb{E}[mx] \text{ for all } x \in X_b\}$, and there exists a smallest norm SDF $m_b \in X_b \cap M_b$.

A trading strategy corresponds to a payoff $x \in X_a$. Assume that the payoff of the safe asset is $x_0 = 1$ and that its price is $\nu(1) > 0$. The return on the safe asset is thus $R_0 = 1/\nu(1)$. One can then write the excess return on the fund as $r_x = x - \nu(x)R_0$, and the excess return on the j-th benchmark as $r_j = x_j - \nu(x_j)R_0$ (for $i = 1, \dots, k$). Let $r_m = \{r_1, \dots, r_k\}'$. Based on observations (r_x, r_m') over n time intervals of equal length, one can obtain estimates of alpha (denoted $\hat{\alpha}_n$) and the appraisal ratio (denoted $\widehat{\text{APR}}_n$) by the OLS regression. As $n \to \infty$ the estimates converge to their population counterparts. The alpha and appraisal ratio depend on the fund's strategy x and are denoted with $\alpha(x)$ and APR(x).

The optimization problem translates to finding the strategy x that maximizes APR(x), denoted by

$$\text{APR}_{max} = \max\{\text{APR}(x) : x \in X_a\}.$$

The solution to this problem is found in a similar way as the construction of the mean-variance frontier and is given by

Theorem 2.0.1. *The alpha of any payoff $x \in X_a$ is*

$$\alpha(x) = R_0 \mathbb{E}[r_x(m_b - m_a)].$$

The maximal appraisal ratio over all payoffs in X_a is

$$APR_{max} = R_0 ||m_b - m_a||,$$

and the maximum is achieved for any payoff x of the form

$$x = z + \theta(m_b - m_a)$$

for some $z \in X_b$ and $\theta > 0$.

Proof. For the proof of this theorem we refer the reader to Guasoni et al. [8]. □

The term $m_b - m_a$ in the expression for the maximal appraisal ratio can be interpreted as the Hansen and Jagannathan (HJ) distance from m_b to the set of discount factors that price all payoffs in X_a. The HJ distance is

$$\delta = \min\{||m_b - m|| : m \in M_a\}.$$

One can write any SDF $m \in M_a$ as $m = m_a + (m - m_a)$ with $\mathbb{E}[(m - m_a)x] = 0$ for all $x \in X_a$. From $m_b - m_a \in X_a$ follows that

$$||m_b - m||^2 = ||m_b - m_a||^2 + ||m - m_a||^2,$$

and the minimum of $||m_b - m||$ over $m \in M_a$ is achieved when $||m - m_a|| = 0$, thus $\text{APR}_{max} = R_0\delta$. The maximal appraisal ratio can be related to the variance bounds, known as the Hansen Jagannathan bounds; one can reduce the expression for the maximal appraisal ratio to

$$\text{APR}_{max} = R_0\sqrt{\textit{var}\,(m_a) - \textit{var}\,(m_b)},$$

and according to Hansen and Jagannathan, *var* (m_a) is the greatest lower bound of the variance of the SDFs in M_a, and the same statement holds for *var* (m_b) and M_b. Furthermore, one can write the Sharpe ratios of both spaces as

$$\text{SHP}_i = R_0\sqrt{\textit{var}\,(m_i)}, \quad \text{for } i = a, b,$$

which implies

$$\text{APR}_{max} = \sqrt{\text{SHP}_a^2 - \text{SHP}_b^2}.$$

Theorem 2.0.1 solves the maximization problem of the appraisal ratio and provides the solution of the maximization of alpha itself, but in practice there could be some constraints on the maximization of alpha. The authors mention two constraints. One, the investor might not exceed a certain level of risk; typical risk management mandates that the tracking error of the strategy is to be lower than a certain upper bound. Second, investors can face collateral requirements, which depend on the riskiness of the total position, that limit their leverage. The authors provide the explicit expressions for the maximal alpha in these cases as well.

The paper then studies maximal performance in a complete market; the implications of Theorem 2.0.1 are studied under the assumption that the benchmark assets follow a geometric Brownian motion. An explicit solution for SHP_b is easily derived from the moments of the benchmark returns. An explicit solution for SHP_a is harder to obtain, because X_a could contain infinitely many security payoffs. If the market is complete, the space $X_a = L^2(P, \Omega)$ and one can obtain a minimum norm discount factor m_a in $L^2(P, \Omega)$. Under the assumption of geometric Brownian motion price processes and a complete market an explicit expression for SHP$_a$ is derived. With these explicit expressions for the Sharpe ratios, an explicit expression for the appraisal ratio is obtained.

The derived formulae remain valid in the presence of additional securities other than the benchmarks, which carry unpriced idiosyncratic risk. This means that using options on individual securities cannot improve the appraisal ratio if the returns on the benchmark assets span the risk factors and the options on the benchmarks are used optimally.

To outperform the strategy of trading in benchmarks, the authors suggest to write options on the benchmarks. They explore their intuition, by first applying Theorem 2.0.1 to describe the optimal option writing strategy and they argue that the relation between the pricing of the options and the process generating the benchmark returns is important for the assessment of the appraisal ratio that the optimal policy is likely to generate. The first case that is studied, is the case where benchmarks follow geometric Brownian motion and options are priced according to the Black-Scholes formula, hence, physical and implied volatilities coincide. The authors consider a benchmark space of one risky asset in addition to the safe asset, in this case the expression of the optimal payoff can be derived as a function of the benchmark return R_m.

Theorem 2.0.2. *Assume that the benchmark space consists of only one risky asset and its price follows a geometric Brownian motion with growth rate μ and volatility σ. Assume the continuously compounded safe rate is r. Then, for any numbers γ and ϕ and any positive number θ, the payoff satisfies*

$$x = \gamma + \phi R_m - \theta f(R_m),$$

where

$$f(R_m) = cR_m^{-b}, \quad \text{with } b = (\mu - r)/\sigma^2; \quad c = e^{[-r+0.5b(\mu+r-\sigma^2)]\Delta t},$$

solves the optimization problem of the appraisal ratio.

Proof. For this proof we refer the reader to Guasoni et al. [8] again. □

The payoff of the optimal strategy given in Theorem 2.0.2 is a nonlinear function of R_m, because f is nonlinear. The first derivative of f is negative ($f' < 0$) and the second derivative of f is positive ($f'' > 0$). In the analysis of this strategy, the authors choose θ to be one, and ϕ in such a way that the delta of the strategy with respect to the benchmark is one. The parameter γ is chosen so that the value of the strategy is one, to make x a return on a dollar investment. Δt is set to one, so the returns are annualized. The authors then explain that the optimal strategy can be implemented by writing options on the benchmarks, because the nonlinear part of $f(R_m)$ can be replicated by a portfolio of call and put options. Integration by parts shows that for any $K > 0$ and any twice differentiable function f, one has

$$f(R_m) = f(K) + f'(K)(R_m - K) + \int_0^K f''(k)(k - R_m)^+ \,\mathrm{d}k + \int_K^\infty f''(k)(R_m - k)^+ \,\mathrm{d}k.$$

The first integration represents long positions in put options and the second integral represents long positions in call options. The second derivatives in these integrations give one the weights one has to invest in options with strike k. Hence, the strategy works with a continuous range of strikes.

The authors test their strategy with simulated data in a Black-Scholes framework and conclude that in this setting it is very hard to produce a statistically significant alpha. Then they show through simulation, that if one prices options at an implied volatility that is higher than the realized volatility, one is able to generate a significant alpha.

This last conclusion is the immediate motive for the research in this thesis. The authors have performed their analysis with simulated options with higher implied volatilities than the realized volatilities on the benchmark. In our research, we derive an equivalent option trading strategy for options with discrete strikes and we then test this strategy with simulated and historical market data on traded options, for which the implied volatility generally differs from the realized volatility on the underlying. We also derive the equivalent strategy that considers multiple benchmarks to invest on, which trades in options with discrete strikes that can each depend on a different, individual benchmark, and we test the performance of this strategy with simulated and historical market data as well.

CHAPTER 3

PERFORMANCE MAXIMIZATION WITH A SINGLE BENCHMARK ASSET

As we have elaborated on in chapter 2, Guasoni et al. [8] derive a performance maximizing option trading strategy that repeatedly invests in a portfolio of options with continuous strikes on a single benchmark asset and a position in the underlying asset itself. However, such a theory could not be implemented in practice, as options are not available at every strike on the derivatives market. In this chapter we derive an equivalent trading strategy that repeatedly invests in a portfolio of European call options with discrete strikes on a single benchmark asset and in this underlying itself. We show that this strategy maximizes alpha, controlling for the risk, and that this is equivalent to the maximization of the Sharpe ratio of excess returns.

We find that constructing a Markowitz portfolio of European call options with discrete strikes for every trading period we consider maximizes the Sharpe ratio of the obtained excess returns. Dynamically investing in such a portfolio thus constitutes our trading strategy. The Markowitz portfolio selection procedure applied to derivatives in stead of equity securities is quite a novelty of this thesis. After formulating our model for the dynamics of the benchmark price, the option pricing framework and all involved assumptions, we therefore elaborate on why the Markowitz portfolio selection procedure meets our needs in maximizing the Sharpe ratio of the excess returns that it generates and how it is adapted to support options in the selection procedure. In the remainder of this chapter we use our model and assumptions to derive explicit expressions for the expected excess returns on each option available on the moment on which we decide to trade and the covariance matrix of these excess returns, which are vital components in the construction of portfolio weights. We show how to adjust the weights such that beta-neutrality is ensured, and we formulate the performance maximizing portfolio weights for each moment on which we trade in explicit expressions. Finally, we pose two extensions to our model that could make the model more realistic.

3.1 Model formulation

Throughout this whole thesis, we assume that the price of the underlying (or, benchmark) asset follows a geometric Brownian motion. The price process is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3.1}$$

and is solved by

$$S_t = S_0 \cdot e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{3.2}$$

in continuous time, where

- S_t is the price of the underlying at time t,
- S_0 is the price of the underlying at time 0,
- μ is the growth rate of the Brownian motion,
- σ is the volatility of the Brownian motion,
- W_t is a standard Brownian motion.

The parameter μ is assumed to be constant, but one can extend the model in such a way that this parameter does vary over time. We pose this model extension in section 3.3. In a Black-Scholes framework, the parameter σ is also assumed to be constant, but when one uses market data, one can use the historical/realized volatility process of the benchmark, which makes σ a time dependent variable. The assumption of a price process that follows a geometric Brownian motion implies that the log returns of the underlying are normally distributed. We justify the use of a geometric Brownian motion process to model the underlying asset price with a few, straightforward arguments: first of all, a geometric Brownian motion only assumes positives values, which ensures us that the underlying price will not drop below zero. Second, the geometric Brownian motion generates the same kind of 'random shocks' in the asset price, which we also observe on the market, and last, the expected returns of a geometric Brownian motion process are independent of the value of the process itself, which is also something that agrees with the reality.

Now, define "trading days" as the dates on which we trade, i.e., as the days on which we assemble our portfolio of options. Let these trading days take place on $t = 0, \Delta t, 2\Delta t, \ldots,$ $(k-1)\Delta t$ (for certain integer k), between which are equally spaced time intervals Δt. No further action is undertaken on other days. The payoff of options depends on the price of the underlying at expiration of the options, which in this setting is observed in discrete time. Therefore, we rewrite the underlying price process in equation (3.2) in a discrete setting. We assume that the options traded on trading day t have the same expiration date $T = t + \Delta t$. We can express the price of the underlying at expiration of the options traded on trading day t as:

$$S_T = S_t \cdot e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}\varepsilon}, \tag{3.3}$$

where

- S_T is the price of the underlying at expiration of the options in our portfolio,
- S_t is the price of the underlying on trading day t,

- $\Delta t = T - t$ is the time between the trading day and the time on which the options traded on this trading day expire, denoted in annualized terms,
- ε is a standard normally distributed variable.

The portfolio assembled on trading day t will thus be evaluated at day $T = t + \Delta t$. The excess return on the benchmark during this period is given by

$$r_t^{underlying} = e^{-r_f(t)\Delta t}\frac{S_T}{S_t} - 1, \tag{3.4}$$

where $r_f(t)$ is the annualized continuously compounded risk free interest rate on our trading day t, which we henceforth abbreviate to r_f, whilst noting it is still a function of time. In the Black-Scholes model, r_f is assumed to be constant over time.

Suppose that on a certain trading day t there are $n-1$ European call options available on the underlying, with strikes $K_1 > K_2 > \ldots > K_{n-1}$ and corresponding prices $C_1, C_2, \ldots, C_{n-1}$, and assume that all options have the same expiry. The price of these options under the risk neutral measure is given by $\mathbb{E}^{\mathbb{Q}}\left[e^{-r_f\Delta t}(S_T - K_i)^+\right]$, for which an explicit expression is given by the Black-Scholes formula:

$$C_i(S_t, \Delta t) = N(d_1)S_t - N(d_2)K_i e^{-r_f\Delta t}, \quad i = 1, 2, \ldots, n-1, \tag{3.5}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{\Delta t}}\left[\ln\left(\frac{S_t}{K_i}\right) + \left(r_f + \frac{\sigma^2}{2}\right)\Delta t\right], \quad d_2 = d_1 - \sigma\sqrt{\Delta t}, \tag{3.6}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\,\mathrm{d}z,$$

the cumulative distribution function (CDF) of the standard normal distribution. The probability density function (PDF) of the standard normal distribution is given by

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

For hedging purposes we introduce a final, the n-th, option, which we shall refer to as the "zero strike option". With this option we facilitate a position in the underlying asset itself, hence the strike of the option is zero and the price of the option is equal to the price of the underlying at our trading day, S_t. The realized excess return on each option is given by

$$r_i^{opt}(t) = \frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i} - 1, \quad i = 1, 2, \ldots, n, \tag{3.7}$$

and for the zero strike option, equation (3.7) reduces to equation (3.4). Let $\{r_1^{opt}(t), r_2^{opt}(t), \ldots, r_n^{opt}(t)\}' = r_t^{realized} \in \Re^{n\times 1}$ denote the vector containing the realized excess returns on all options available on this trading day t. We assume that the payoffs of all options have finite second moments, which is a minimal requirement to ensure that the sample estimates of linear regressions converge to their population counterparts.

When one uses market data, then one can observe the option prices C_i on the market and one can relax the Black-Scholes model assumptions for pricing the options. The fact that there is a difference between the prices dictated by the market and the ones given by the Black-Scholes model is referred to as “market mispricing”.

We omit put options in our portfolio selection, for most of the put options on the market are of American exercising style, making the models more complicated. With the put-call parity

$$P_i(S_t, \Delta t) = K_i e^{-r_f \Delta t} - S_t + C_i(S_t, \Delta t) \tag{3.8}$$

one can synthesize European style puts, and we can interpret positions in certain calls as positions in puts, cash and in the underlying.

Now that we have our model and options framework, we proceed with the issue of choosing the optimal portfolio on each trading day, which in this case is the portfolio with the highest expected excess return for a given level of standard deviation in these excess returns. We shall refer to the standard deviation of the obtained excess returns with the term “risk”, which deviates from more traditional formulations of risk, like the possibility of loss when a company defaults and the Value at Risk. We define a unit of standard deviation as a unit of risk. As typical risk management mandates that the tracking error of a strategy is to be lower than a certain upper bound, we construct the portfolio weights each trading day such that a predetermined upper bound on the risk will not be crossed. This limits the magnitude of a negative excess return generated by the portfolio, and in such a way limits the potential of a ‘big’ loss, but it also limits our upward potential.

In the next section we show that the Markowitz portfolio selection procedure gives us the optimal portfolio on each trading day and we derive explicit expressions for the figures needed in the assembly of this portfolio.

3.2 The Markowitz portfolio selection procedure

Our goal is to develop a trading strategy which maximizes our portfolios excess returns, controlling for the risk. Since the future is uncertain, we aim to maximize the expected portfolio excess returns. We find that the Markowitz portfolio selection procedure does exactly this, though it is not common to use this theory with derivatives. Using this selection procedure with derivatives is quite a novelty, but has been performed before, for example by Liang et al. [6]. We briefly introduce this selection procedure and its relevance for our research. We then derive the technicalities needed for our research.

In 1952, Harry Markowitz [9] publishes the article “Portfolio selection”, which is still used as a basis in modern portfolio selection. He derives an asset selection procedure that maximizes the discounted expected return of the portfolio, given a constant level of variance in these excess returns, based on relevant beliefs of future performances of the assets which one can choose from. He considers the discounted expected return, because the future is uncertain. The portfolio with maximum expected return is not necessarily the portfolio with minimum variance, so one is able to pick a portfolio with a very high expected return, but bearing a very high variance as well, which makes it an undesirable portfolio, that is why he seeks to maximize the expected return, given the risk preference of the investor.

So, this selection procedure is highly suitable to fit our goals, but there are two issues. First, the Markowitz selection procedure as presented in the paper of Markowitz considers a range of different assets to assemble ones portfolio with, not derivatives on these assets. We want to assemble a portfolio of European call options on one underlying asset, so we have to adapt the portfolio selection procedure to this setting. Second, the selection procedure in the paper is presented in a static setting, meaning that the investor assembles his portfolio on one point in time and then holds it. We want to develop a strategy that dynamically invests in a portfolio of options during a certain period, so selecting a portfolio of options once and holding this portfolio throughout this period does not yield the result we are aiming at.

We tackle this first issue by treating each available option on a trading day as if it were an asset available in the Markowitz portfolio selection procedure. In this setting, one is able to follow the same derivations as in Markowitz [9], but with somewhat different expressions, accounting for the fact that options have a different payoff than the underlying. The second issue is fairly easy to tackle; on each trading day we assemble our portfolio in a static setting, knowing exactly when the options expire and thus, when an excess return is generated on the portfolio. After all options have expired, the portfolio is useless and is thus dissolved. Assembling a Markowitz portfolio on each trading day with the available European call options on that trading day constitutes our dynamic trading strategy that maximizes alpha, and controls for the risk.

We first formulate the framework in which we are going to work for our portfolio selection on each trading day, and then we show in Proposition 3.2.1 that the Markowitz portfolio selection procedure also gives us the maximal Sharpe ratio. Maximizing the Sharpe ratio of every portfolio we assemble, maximizes the Sharpe ratio of all excess returns obtained with our strategy, and thus alpha in our regression (1.1). We then derive explicit expressions for the terms used in the Markowitz portfolio selection procedure on each trading day and extend the selection procedure with hedging arguments that will lead to beta-neutrality of the excess returns.

Consider a certain trading day t. On this day we invest in a range of options, including the zero strike option, hence in the underlying itself. Let $\pi_i(t)$ denote the weight (the proportion of our wealth) of the option with strike K_i we buy (when $\pi_i > 0$) in our portfolio or short (when $\pi_i < 0$) on this trading day. We directly drop the time index to abbreviate to π_i for practical reasons. Let $\{\pi_1, \pi_2, \dots, \pi_n\}' = \pi \in \Re^{n\times 1}$ denote the vector that contains all these weights. In equation (3.7) the realized excess return for each option is given, hence, we can write the realized excess return of our portfolio under choice of π as $R^\pi = \sum_{i=1}^n \pi_i \cdot r_i^{opt}(t) = \pi' r_t^{realized}$ (for which we also omit the time index). Its variance is denoted with *var* (R^π), and its standard deviation with $\sigma(R^\pi)$. The Sharpe ratio of the excess return generated by our portfolio choice on this trading day is now given by $\mathbb{E}[R^\pi]/\sigma(R^\pi)$. We formalize our claim that the Markowitz selection procedure maximizes the Sharpe ratio of excess returns in Proposition 3.2.1. The first two formulations in Proposition 3.2.1 are the formulations from which Markowitz derived his selection procedure, using R^π as the vector of excess returns generated by the different assets in the portfolio under the choice of weights π.

Proposition 3.2.1. *The following statements are equivalent:*

(i) $\max_\pi\{\mathbb{E}[R^\pi] : \textit{var}(R^\pi) = \sigma^2\}$,

(ii) $\min_\pi\{\textit{var}(R^\pi) : \mathbb{E}[R^\pi] = \mu\}$,

(iii) $\max_\pi\{\mathbb{E}[R^\pi]/\sigma(R^\pi)\}$.

Proof. Equivalence of the first two statements follows easily from introducing Lagrange multipliers and the fact that maximizing an expression f is the same as minimizing $-f$. Equivalence of the first two statements and the third follows from the fact that the first two statements find a point on the mean-variance frontier, which' slope is exactly the Sharpe ratio, as is argued in Cochrane [10]. □

So, using the Markowitz procedure to select our derivatives every trading day, we maximize the Sharpe ratio of our excess return generated by the portfolio. To choose our weights optimally on such a trading day, we follow the same derivations as Markowitz [9], but adapt them to options. In the selection procedure the vector of expected excess returns on the assets and the covariance matrix between them are used in the construction of the portfolio weights. We introduce $m_i(t) = \mathbb{E}[r_i^{opt}(t)]$, the expected excess return on an option with strike K_i traded on trading day t, $m(t) = \{m_1(t), m_2(t), \dots, m_n(t)\}' \in \Re^{n\times 1}$, the vector containing all these expected excess returns, which we shall abbreviate to m, and we introduce $S \in \Re^{n\times n}$, the covariance matrix of option excess returns, omitting its time index as well. Now we can write

$$\mathbb{E}[R^\pi] = \pi' m, \quad \textit{var}\,(R^\pi) = \pi' S \pi,$$

and we arrive at the following Lemma that gives us the optimal portfolio weights on a certain trading day when we do not incur any hedging.

Lemma 3.2.2. *The weights* π *that maximize* $\{\mathbb{E}[R^\pi] : \textit{var}(R^\pi) = \sigma^2\}$ *and the Sharpe ratio of excess returns on a certain trading day are given by*

$$\begin{aligned} \pi &= \lambda S^{-1} m, \\ \textit{with} \quad \lambda &= \sigma/\sqrt{m' S^{-1} m}, \end{aligned} \tag{3.9}$$

where m is the vector of expected excess returns on all options available on that trading day, and S is the covariance matrix of these excess returns.

Proof. We take the first statement from Proposition 3.2.1 and we introduce the Lagrange multiplier $\gamma/2$:

$$\max_{\pi}\{\mathbb{E}[R^{\pi}] : \textit{var}\,(R^{\pi}) = \sigma^2\} \Leftrightarrow \max_{\pi,\gamma}\{\mathbb{E}[R^{\pi}] - \frac{\gamma}{2}\textit{var}\,(R^{\pi})\} = \max_{\pi,\gamma}\{\pi' m - \frac{\gamma}{2}\pi' S\pi\}.$$

When we take the derivative of the target function w.r.t. π and set the equation equal to zero, we get

$$m - \gamma S\pi = 0 \Rightarrow \pi = \frac{1}{\gamma}S^{-1}m.$$

Now, setting $\lambda = 1/\gamma$, we get the first equality in equation (3.9). Finally, we need to meet the variance restriction *var* $(R^{\pi}) = \sigma^2$ by choosing λ correctly. We plug π into the variance expression:

$$\begin{aligned}&\textit{var}\,(R^{\pi}) = \pi' S\pi = (\lambda S^{-1}m)' S(\lambda S^{-1}m) = \lambda^2\tilde{\pi}' S\tilde{\pi} = \sigma^2, \quad \textit{with} \quad \tilde{\pi} = S^{-1}m \\ &\Rightarrow \lambda = \sigma/\sqrt{\tilde{\pi}' S\tilde{\pi}} = \sigma/\sqrt{(S^{-1}m)' S(S^{-1}m)} = \sigma/\sqrt{m' S^{-1} m},\end{aligned}$$

and we arrive at the result. □

With these weights we optimize our expected excess returns, adjusted for the risk σ. We can adjust λ in the weights to match our risk preference of σ per trading period. If one were to use these weights in practice, one needs expressions for m and S, which we derive in the next subsection. Furthermore, we want to hedge out our β-position w.r.t. the market, so we have to adjust the weights derived in Lemma 3.2.2 some more. This is elaborated on in section 3.2.2.

3.2.1 Explicit expressions for the expected excess returns and covariance of excess returns of options

As our portfolio selection each trading day depends on the expected excess returns vector m and the covariance matrix of excess returns S of that trading day, we need to find explicit expressions for these quantities. We first focus on the vector m.

The expected option excess return vector m

We have that $m = \{m_1, m_2, \ldots, m_n\}'$, and $m_i,\ i = 1, \ldots, n$ is the expected excess return that is generated by an option with strike K_i:

$$m_i = \mathbb{E}[r_i^{opt}] = \mathbb{E}\left[\frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i} - 1\right] = \frac{e^{-r_f\Delta t}\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+\right]}{C_i} - 1, \tag{3.10}$$

where r_f, K_i and C_i can be observed on the market, or where C_i is given under the risk neutral measure $\mathbb{Q}$ by the Black-Scholes model. Since we assume that the underlying follows a geometric Brownian motion, we can derive an explicit expression for the expected payoff of the option under the physical measure $\mathbb{P}$, hence, one can express m_i explicitly:

Theorem 3.2.3. *We can express the expected excess return of an European call option with strike K_i and price C_i, $i \in \{1, 2, \ldots, n\}$, longed on trading day t and with expiration on $T = t + \Delta t$, explicitly as*

$$m_i = \frac{e^{-r_f \Delta t} \left[S_t \cdot e^{\mu \Delta t} N(d_2) - K_i N(d_1) \right]}{C_i} - 1, \tag{3.11}$$

where

$$d_1 = \frac{\ln\left[S_t/K_i\right] + \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}, \quad d_2 = d_1 + \sigma\sqrt{\Delta t} = \frac{\ln\left[S_t/K_i\right] + \left(\mu + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}.$$

Proof. For the proof of this theorem we refer the reader to Appendix A.1. □

Now, one can assemble the vector m by calculating all m_i individually. For the zero strike option one has that $N(d_2) = N(\infty) = 1$, which reduces equation (3.11) to

$$m_n = \mathbb{E}[r_n^{opt}] = e^{(\mu - r_f)\Delta t} - 1, \tag{3.12}$$

which is exactly the result we would expect, as it is the discounted expectation of a geometric Brownian motion return.

The covariance matrix of option excess returns S

The covariance matrix of option excess returns S has the following structure:

$$S = \begin{bmatrix} \text{cov}\left(r_1^{opt}, r_1^{opt}\right) & \text{cov}\left(r_1^{opt}, r_2^{opt}\right) & \cdots & \text{cov}\left(r_1^{opt}, r_n^{opt}\right) \\ \text{cov}\left(r_2^{opt}, r_1^{opt}\right) & \text{cov}\left(r_2^{opt}, r_2^{opt}\right) & \cdots & \text{cov}\left(r_2^{opt}, r_n^{opt}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}\left(r_n^{opt}, r_1^{opt}\right) & \text{cov}\left(r_n^{opt}, r_2^{opt}\right) & \cdots & \text{cov}\left(r_n^{opt}, r_n^{opt}\right) \end{bmatrix}. \tag{3.13}$$

We derive an explicit expression for the general term, $\text{cov}\left(r_i^{opt}, r_j^{opt}\right)$, $i, j \in \{1, 2, \ldots, n\}$, in this matrix:

Theorem 3.2.4. *We can express the covariance between the excess returns on European call options with strikes K_i and K_j, with corresponding prices C_i and C_j, $i, j \in \{1, 2, \ldots, n\}$, longed on trading day t and with expiration on $T = t + \Delta t$, explicitly as*

$$\begin{aligned} \text{cov}\left(r_i^{opt}, r_j^{opt}\right) =& \frac{e^{-2r_f \Delta t}}{C_i C_j} \left[S_t^2 e^{(2\mu + \sigma^2)\Delta t} N(d_3) - S_t(K_i + K_j) e^{\mu \Delta t} N(d_4) + K_i K_j N(d_5) \right] \\ & - (m_i + 1)(m_j + 1), \end{aligned} \tag{3.14}$$

where

$$d_3 = \frac{\ln[S_t/\max(K_i,K_j)] + \left(\mu + \frac{3\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}},$$
$$d_4 = \frac{\ln[S_t/\max(K_i,K_j)] + \left(\mu + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}},$$
$$d_5 = \frac{\ln[S_t/\max(K_i,K_j)] + \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}.$$

Proof. For the proof of this theorem we refer the reader to Appendix A.2. □

One can assemble the covariance matrix S by calculating each individual component. For the covariance of excess returns of zero strike options, we get that $N(d_3) = N(\infty) = 1$, which reduces equation (3.14) to

$$\textit{cov}(r_n^{opt}, r_n^{opt}) = \textit{var}\left(r_n^{opt}\right) = e^{2(\mu - r_f)\Delta t}\left(e^{\sigma^2\Delta t} - 1\right), \tag{3.15}$$

which is exactly the result we would expect, as it is the discounted variance of a geometric Brownian motion return.

Having explicit expressions for m and S, we can compute the optimal portfolio weights π at every trading day. However, we still want our portfolio excess returns to be explained solely by our strategy and not by market movements, so, considering our regression (1.1), we want beta to be zero, or at least statistically insignificantly different from zero. With the current choice of weights, beta is very unlikely to be insignificantly different from zero, hence, we want to remove the sensitivity of our strategy excess returns w.r.t. market movements by altering our weights. We show how this is done in the next section.

3.2.2 Hedging

As we have mentioned in the previous section, we want our strategy excess returns to be explained by our strategy alone and not by the excess returns on the market. Therefore, we need to adjust our portfolio weights each trading day in such a way that beta in the regression (1.1) is statistically insignificantly different from zero at an acceptable level of significance. We perform a hedge in our portfolio that will realize this, in a similar way as Guasoni et al. [8], as described in chapter 2. In this case, with “hedging” we mean removing the portfolio’s sensitivity with the respect to the excess returns on the whole market, deviating from standard definitions of hedging that refer to removing the portfolios sensitivity with respect to its own underlying asset. Also, in traditional ways, hedges are often established by adding products to the portfolio. We shall establish the hedge by altering the weight of the zero strike option in our portfolio. Removing the portfolio’s sensitivity with respect to the market excess returns on each trading day will make beta in regression (1.1) an insignificant factor.

A natural, first thought would be to perform a delta-hedge, which is fairly easy to implement and does not need the zero strike option as a hedging position. We perform the hedge on the weights before the risk adjustment takes place. Define initial, unadjusted portfolio weights as $\tilde{\pi} = S^{-1}m$. If one has computed these weights, one can compute the total delta position of the portfolio. The delta of each option, Δ_i, is given by the $N(d_1)$ of the Black-Scholes equation (3.5). Let $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_n\}' \in \Re^{n\times 1}$ be the vector that contains the delta of all options. Then the total delta position of the portfolio is given by $\tilde{\pi}'\Delta = \Delta_{pf}$. To make this position zero, we add a constant c to each of the weights, the value of which is determined by

$$(\tilde{\pi} + c)'\Delta = \Delta_{pf} + c \cdot \sum_i \Delta_i = 0,$$

$$\Rightarrow c = \frac{-\Delta_{pf}}{\sum_i \Delta_i}.$$

Rebalancing the weights with this constant and then adjusting for the risk in each trading period makes the strategy delta-neutral, meaning that its sensitivity to movements in the price of its own underlying is hedged out. Unfortunately, this does not imply a zero or insignificant beta when one uses large time intervals. For example, when one uses a monthly interval, then this method will not work. Therefore, we need another approach.

To make our strategy beta-neutral, we perform a direct beta-hedge, by rebalancing the weight of the zero strike option. The beta of an option is given by $\mathit{cov}(r_i^{opt}, r_t^{mkt})/\mathit{var}\left(r_t^{mkt}\right)$ (see chapters 5 and 6 of Cochrane [10] on beta-representations), where r_t^{mkt} is the excess return on the market over the period of assembling our portfolio and expiration of the options in it. Unfortunately, measuring the covariance between an option excess return and the market excess return is not that easy, as they do not depend on the same underlying. Therefore, we shall hedge with option betas w.r.t. their own underlying, which are defined as

$$\beta_i = \frac{\mathit{cov}(r_i^{opt}, r_n^{opt})}{\mathit{var}\left(r_n^{opt}\right)}. \tag{3.16}$$

The beta of the zero strike option, β_n, is by definition equal to one. The total beta position of the portfolio prior to hedging is given by $\beta^{pf} = \sum_i \tilde{\pi}_i \beta_i$ and we aim to make it zero by adjusting the weight on the zero strike option. Now, the question is how to relate the zero strike option beta to the market excess returns. If we assume that the zero strike option is just as sensitive to its own underlying as it is to the market, then the total beta position of the portfolio is its sensitivity to the market and we can hedge it out by subtracting it from the weight of the zero strike option. However, when the zero strike option is more sensitive to the market then to its own underlying, we would get overcompensation effects when we simply extract the total beta position from the weight on the zero strike option, or when it is less sensitive to the market than to its own underlying, we would not hedge out the total position by simply subtracting it. In this case, we should scale the amount we subtract from the zero strike option to account for these effects. Define the "market beta" of the zero strike option, β^{mkt}, as the sensitivity of the zero strike option to the market excess returns, given by the covariance of excess returns on the underlying and the market divided by the variance of market excess returns. We have three options for choosing this market beta:

1. Assume that the market beta of the zero strike option is equal to one (implying that the strategy's sensitivity to the excess returns on its own underlying is equal to the sensitivity to the excess returns on the complete market).
2. Calculate the market beta of the zero strike option using a certain period prior to our first trade and keep constant over time.
3. Calculate the market beta of the zero strike option dynamically, using the same period as in option 2 to calculate the market beta used for our first portfolio, but then add every new observation of the strategy excess returns and the market excess returns to the vectors we use to calculate this market beta, and calculate it again on each trading day.

The first option is by far the easiest one, as no extra work has to be done. The second and third option demand extra calculations. The market beta of the zero strike option, β^{mkt}, can be computed by taking the excess returns of the market and the underlying over a certain period prior to our first trade and then calculate the covariance between these excess returns divided by the variance of the market excess returns, which can be done once (option 2) or every trading day (option 3).

To hedge, we subtract the portfolio beta position from the initial weight of the zero strike option. When using option 1, this can be done directly. Define $b = \{0, 0, \ldots, 0, \beta^{pf}\}' \in \Re^{n\times 1}$, and then subtract this vector from the initial weights and then adjust for the risk. When one does not assume that the market beta of the zero strike option is one, one needs to divide b by the calculated market beta first, to compensate for the fact that the beta position we hedge with is not equal to one; when it is larger than one, we need to subtract a smaller proportion of the weight on the zero strike option, when it is smaller than one, we need to subtract more to obtain beta-neutrality with respect to the market. This procedure makes the portfolio beta equal to zero each trading day, so we expect the beta of our realized excess returns to be close to zero as well.

Theorem 3.2.5. *On each trading day, the weights π that maximize $\{\mathbb{E}[R^\pi] : \mathit{var}(R^\pi) = \sigma^2\}$ and the Sharpe ratio of excess returns for that trading day, whilst hedging out the portfolio's market exposure, are given by*

$$\begin{aligned} \pi &= \lambda\left(S^{-1}m - b\right), \\ \textit{with} \quad \lambda &= \sigma/\sqrt{m'S^{-1}m}, \\ b &= \{0, 0, \ldots, 0, \beta^{pf}/\beta^{mkt}\}', \\ \beta^{pf} &= \textstyle\sum_i \tilde{\pi}_i \frac{\mathit{cov}(r_i^{opt}, r_n^{opt})}{\mathit{var}(r_n^{opt})}, \\ \tilde{\pi} &= S^{-1}m, \end{aligned} \tag{3.17}$$

and the elements of m are given by Theorem 3.2.3 and the elements of S by Theorem 3.2.4, and where one can choose β^{mkt} either equal to one or compute it from historical excess returns, as the covariance of the underlying excess returns and the market excess returns divided by the variance of the market excess returns beforehand and use it as a constant, or update it dynamically when new information is obtained each trading day.

Proof. The proof of this theorem is analog to the proof of Lemma 3.2.2, but it adds the hedging arguments provided in this section. □

If one where to choose his portfolio weights every trading day according to Theorem 3.2.5, one would have an alpha-maximizing option trading strategy in one dimension that controls for the risk and hedges out the sensitivity to market movements. This strategy also maximizes the Sharpe ratio of the strategy's excess returns, which under beta-neutrality is given by $\alpha/\sigma(r^{pf})$. In chapter 5 we test the performance of this strategy with simulated data from a Black-Scholes framework, and then we test the performance of the strategy using historical market data in chapter 6. We investigate whether alpha is statistically significantly different from zero, whether beta is an insignificant factor in the regression (1.1) and how high the Sharpe ratio can get. We will also test our three options for choosing β^{mkt} and see what the effects are on the strategy performance.

Finally, we can extend our model in a few ways to meet more market properties. We shortly discuss two of them in the next section.

3.3 Model extensions

The model assumptions made in section 3.1 are fairly basic. In this section we propose two model extensions, which could make the model more realistic. First, we have assumed that the growth rate of the underlying geometric Brownian motion is constant throughout time. This is not entirely realistic, so we propose an extension that makes μ a time-dependent variable. Second, we have assumed that one can buy and sell options at the same price. In practice, this is not the case, because there are trading costs involved; there is a substantial bid-ask spread on the options. We introduce this bid-ask spread in a simple manner to the model. We omit a bid-ask spread on the underlying itself, as it is far more liquid than the derivatives written on it, hence, if there were a spread, then it would be marginal. We analyze the effects of the implementation of these extensions with our market data analysis in chapter 6.

3.3.1 A dynamic growth rate μ

As mentioned before, μ is the growth rate of the geometric Brownian motion of the underlying asset price. In the geometric Brownian motion, this term is assumed to be constant, but we can argue that this term should be time-dependent. For example, if we would assemble our portfolio at every first trading day of the month and we would do this for several years, then it would be a strong assumption that the growth rate stays the same over all these years. We make it time-dependent in the following way:

$$\mu(t) = r_f(t) + \frac{1}{2}\sigma(t), \tag{3.18}$$

where $\mu(t)$ is the growth rate on trading day t, $r_f(t)$ is the annualized, continuously compounded risk free interest rate on trading day t and $\sigma(t)$ is the historical/realized volatility of the underlying on this day. The factor 1/2 is the average Sharpe ratio on the US equity market in postwar data; 8% average excess return divided by 16% average volatility (all are annualized figures). Hence, in this way $\mu(t)$ is the risk free rate plus the expected excess return on day t, making it the expected return on day t, which agrees with the definition of the parameter. The dynamic growth rate is easy to implement this way, because it only uses terms that were already introduced to the model. A simple substitution will do.

3.3.2 Trading costs

In practice, there are no frictionless markets. Hence, the assumption that one can buy and sell options at the same price is a strong one. We try to implement trading costs for options to the model in a very simple way. As mentioned before, we omit trading costs on the underlying, as it is far more liquid than the derivatives written on it.

Introducing trading costs beforehand by defining a bid price and an ask price complicates the strategy, because the strategy assumes one price for buying and selling. An overly simple way of introducing trading costs to our model, is to do it in hindsight, so we can at least study their impact on the strategy's performance. If our strategy consistently buys options of a certain level of moneyness and sells options of another level of moneyness, then we can adjust their prices by a certain factor beforehand; if one buys an option, one has to pay more than the amount of money one would get when selling the same option, so say, if there are trading costs of x percent of the option price, then we multiply the prices of the options we buy with a factor $1 + x$ and the options we sell with $1 - x$. Then we run the data analysis again, and if the strategy still consistently buys or sells the same options, then we can say something about the effects of the trading costs on the magnitude of the Sharpe ratio of the strategy's excess returns. However, if we observe that in this new setting one sells options which first we bought, and vice versa, then this approach produces erroneous results and is therefore useless.

CHAPTER 4

PERFORMANCE MAXIMIZATION WITH MULTIPLE BENCHMARK ASSETS

In the previous chapter we have derived a trading strategy that dynamically invests in a portfolio of European call options on a single benchmark asset. In this chapter we raise the question whether we can increase alpha (and thus the Sharpe ratio) by constructing a strategy that dynamically invests in a portfolio that consists of a range of European call options that can each depend on a different underlying benchmark asset. If the returns of all underlying are independent, one can add squared Sharpe ratios. When there is correlation, we still expect addition effects, but to a lesser extent. We expect an increase in the strategy's Sharpe ratio due to diversification in the portfolio. In this chapter we derive the alpha-maximizing strategy, controlled for the risk, for this setting with multiple benchmarks, by extending our Markowitz portfolio selection procedure of the previous chapter to one that admits European call options with discrete strikes that can each depend on a different, individual benchmark.

4.1 Markowitz portfolio selection for multiple benchmark assets

In this section we extend the Markowitz portfolio selection procedure derived in the previous chapter such that it supports multiple underlying assets and options with discrete strikes on them. By the same arguments as in section 3.2 and Proposition 3.2.1 we conclude that the Markowitz portfolio selection procedure still provides us with the maximal Sharpe ratio for each trading day. Investing dynamically in such a portfolio constitutes our strategy and maximizes the Sharpe ratio of its excess returns. We first pose our multidimensional model and then we derive new explicit expressions for the optimal portfolio weights on each trading day.

Suppose that we have N benchmark assets on which options can be traded, which each follow a geometric Brownian motion price process $S_t^{(i)}$, $i = 1, 2, \ldots, N$, with parameters $\mu_1, \ldots, \mu_N$ and $\sigma_1, \ldots, \sigma_N$, which can be correlated in the Brownian motion terms with parameter ρ_{ij}. Suppose that there is a annualized risk free rate r_f on every trading day.

Suppose that on a certain trading day t, $t \in \{0, \Delta t, 2\Delta t, \ldots, (k-1)\Delta t\}$, there are n_i European call options available on benchmark i, with strikes $K_1^{(i)} > K_2^{(i)} > \ldots > K_{n_i}^{(i)}$ and prices $C_1^{(i)}, C_2^{(i)}, \ldots, C_{n_i}^{(i)}$, where we again use the n_i^{th} option as a zero strike option for hedging purposes. Assume that all options on all underlying have the same expiration date $T = t + \Delta t$. We again assume that the payoffs of all options have finite second moments.

Denote the excess return on option with strike K_j on underlying i with $r_j^{(i)}$. Denote with $r^{(i)} = \{r_1^{(i)}, r_2^{(i)}, \ldots, r_{n_i}^{(i)}\}' \in \Re^{n_i \times 1}$ the vector of realized option excess returns on underlying i for the options traded on this trading day. We use $r^{opt} = \{r^{(1)}, r^{(2)}, \ldots, r^{(N)}\}' \in \Re^{\sum n_i \times 1}$ as the vector containing all realized excess return vectors of all underlying. As options only depend on the price of their own underlying at expiry, there does not change anything for the vectors of expected excess returns on the options. We denote with $m^{(i)}$ the vector of expected excess returns of the options on underlying i, with its elements defined as in theorem 3.2.3. We combine these vectors $m^{(i)}$ in vector $m = \{m^{(1)}, m^{(2)}, \ldots, m^{(N)}\}' \in \Re^{\sum n_i \times 1}$, the vector that contains all expected excess returns on all options on all underlying. We can express the covariance matrix of option excess returns as

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \in \Re^{\sum n_i \times \sum n_i}, \tag{4.1}$$

where S_{ii} for $i = 1, \ldots, N$ is given by Theorem 3.2.4 and where $S'_{ji} = S_{ij}$ for $i, j = 1, \ldots, N$. We are left with finding explicit expressions for the elements of S_{ij}, the covariance matrix of excess returns of options on underlying i and options on underlying j, where $i \neq j$. We express such a covariance matrix as

$$S_{ij} = \begin{bmatrix} cov(r_1^{(i)}, r_1^{(j)}) & cov(r_1^{(i)}, r_2^{(j)}) & \cdots & cov(r_1^{(i)}, r_{n_j}^{(j)}) \\ cov(r_2^{(i)}, r_1^{(j)}) & cov(r_2^{(i)}, r_2^{(j)}) & \cdots & cov(r_2^{(i)}, r_{n_j}^{(j)}) \\ \vdots & \vdots & \ddots & \vdots \\ cov(r_{n_i}^{(i)}, r_1^{(j)}) & cov(r_{n_i}^{(i)}, r_2^{(j)}) & \cdots & cov(r_{n_i}^{(i)}, r_{n_j}^{(j)}) \end{bmatrix}. \tag{4.2}$$

We want to have explicit expressions for its entries. To this end, we divide the matrix into three categories:

1. Covariances of the type $cov(r_k^{(i)}, r_l^{(j)})$, for $k = 1, \ldots, n_i - 1$ and $l = 1, \ldots, n_j - 1$,
2. Covariances of the type $cov(r_{n_i}^{(i)}, r_k^{(j)})$, for $k = 1, \ldots, n_j - 1$, and $cov(r_k^{(i)}, r_{n_j}^{(j)})$, for $k = 1, \ldots, n_i - 1$, which are equivalent,
3. Covariances of the type $cov(r_{n_i}^{(i)}, r_{n_j}^{(j)})$.

We will derive explicit expressions for each class when considering two benchmark assets as follows. The N-dimensional case follows the same derivations, but adds some terms. Extension is straightforward. The results for the two dimensional setting are presented in the following theorem.

Theorem 4.1.1. *For $N = 2$, consider the covariance matrix component S_{12} of option excess returns as in* (4.2) *and its three types of covariances. An explicit expression for covariances of* *type 1 is given by*

$$\text{cov}(r_i^{(1)}, r_j^{(2)}) = \frac{e^{-2r_f\Delta t}}{C_i^{(1)}C_j^{(2)}}\Big[S_t^{(1)}S_t^{(2)}e^{(\mu_1+\mu_2+\rho_{12}\sigma_1\sigma_2)\Delta t}MN(d_1,d_2) - K_i^{(1)}S_t^{(2)}e^{\mu_2\Delta t}MN(d_3,d_4) \\ -K_j^{(2)}S_t^{(1)}e^{\mu_1\Delta t}MN(d_5,d_6) + K_i^{(1)}K_j^{(2)}MN(d_7,d_8)\Big] \\ -(m_i^{(1)}+1)(m_j^{(2)}+1), \tag{4.3}$$

Where $MN(x,y)$ is the CDF of the standard bivariate normal distribution, and

$$\begin{aligned} d_1 &= \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1+\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}} + \rho\sigma_2\sqrt{\Delta t}, & d_2 &= \frac{\ln\left[S_t^{(2)}/K_j^{(2)}\right]+\left(\mu_2+\frac{\sigma_2^2}{2}\right)\Delta t}{\sigma_2\sqrt{\Delta t}} + \rho\sigma_1\sqrt{\Delta t}, \\ d_3 &= \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1-\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}} + \rho\sigma_2\sqrt{\Delta t}, & d_4 &= \frac{\ln\left[S_t^{(2)}/K_j^{(2)}\right]+\left(\mu_2+\frac{\sigma_2^2}{2}\right)\Delta t}{\sigma_2\sqrt{\Delta t}}, \\ d_5 &= \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1+\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}}, & d_6 &= \frac{\ln\left[S_t^{(2)}/K_j^{(2)}\right]+\left(\mu_2-\frac{\sigma_2^2}{2}\right)\Delta t}{\sigma_2\sqrt{\Delta t}} + \rho\sigma_1\sqrt{\Delta t}, \\ d_7 &= \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1-\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}}, & d_8 &= \frac{\ln\left[S_t^{(2)}/K_j^{(2)}\right]+\left(\mu_2-\frac{\sigma_2^2}{2}\right)\Delta t}{\sigma_2\sqrt{\Delta t}}. \end{aligned}$$

An explicit expression for covariances of type 2 is given by

$$\text{cov}(r_i^{(1)}, r_{n_2}^{(2)}) = \frac{e^{-2r_f\Delta t}}{C_i^{(1)}}\Big[S_t^{(1)}e^{(\mu_1+\mu_2+2\rho_{12}\sigma_1\sigma_2)\Delta t}N(d_1) - K_i^{(1)}e^{\mu_2\Delta t}N(d_2)\Big] - (m_i^{(1)}+1)(m_{n_2}^{(2)}+1), \tag{4.4}$$

where

$$d_1 = \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1+\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}} + \rho\sigma_2\sqrt{\Delta t}, \quad d_2 = \frac{\ln\left[S_t^{(1)}/K_i^{(1)}\right]+\left(\mu_1-\frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}} + \rho\sigma_2\sqrt{\Delta t},$$

for which the lower case indices of excess returns can be swapped to swap the roles of the assets.

An explicit expression for covariances of type 3 is given by

$$\text{cov}(r_{n_i}^{(i)}, r_{n_j}^{(j)}) = e^{-2r_f\Delta t}e^{(\mu_1+\mu_2+\rho_{12}\sigma_1\sigma_2)\Delta t} - (m_{n_1}^{(1)}+1)(m_{n_2}^{(2)}+1). \tag{4.5}$$

Proof. For the proof of this theorem we refer the reader to Appendix A.3. □

One now has the tools to assemble a Markowitz portfolio again, but now we can hedge on two positions; both zero strike options. Define $\tilde{\pi} = S^{-1}m$ as the initial weights again. One has the same three options as in the one dimensional case for the market betas of the zero strike options ($\beta_k^{mkt},\ k = 1,2$), and one can compute them in the same way as before. The total beta position of the portfolio w.r.t. the first underlying is calculated as:

$$\beta_1^{pf} = \sum_{i=1}^{n_1+n_2} \tilde{\pi}_i \frac{\text{cov}\left(r_i^{(1)}, r_{n_1}^{(1)}\right)}{\text{var}\left(r_{n_1}^{(1)}\right)},$$

and we again subtract this position from the weight of the zero strike option on the first underlying to hedge. In this way, new weights are formed, call them π^*, which are equal to the initial weights, but differ in the zero strike option position of the first underlying. With these new weights we calculate the total beta position of the portfolio w.r.t. the second underlying as

$$\beta_2^{pf} = \sum_{j=1}^{n_1+n_2} \pi_j^* \frac{\textit{cov}\left(r_j^{(2)}, r_{n_2}^{(2)}\right)}{\textit{var}\left(r_{n_2}^{(2)}\right)},$$

and we subtract this position from the zero strike option on the second underlying to hedge again.

The optimal portfolio weights weights are given by the hedged Markowitz portfolio selection procedure, presented in the next theorem.

Theorem 4.1.2. *Consider two benchmark assets to trade options on. Then, on each trading day, the weights π that maximize $\{\mathbb{E}[R^\pi] : \textit{var}(R^\pi) = \sigma^2\}$ and the Sharpe ratio of excess returns for that trading day, and hedge out the portfolio's market exposure, are given by*

$$\begin{aligned}
& \pi = \lambda(\pi^* - c), \\
\textit{with}\quad & \lambda = \sqrt{m' S^{-1} m}, \\
& \pi^* = \tilde{\pi} - b, \\
& \tilde{\pi} = S^{-1} m, \\
& b = \{0, 0, \ldots, 0, \beta_1^{pf}/\beta_1^{mkt}, 0, \ldots, 0\}', \\
& c = \{0, 0, \ldots, 0, \beta_2^{pf}/\beta_2^{mkt}\}', \\
& \beta_1^{pf} = \textstyle\sum_{i=1}^{n_1+n_2} \tilde{\pi}_i \frac{\textit{cov}\left(r_i^{(1)}, r_{n_1}^{(1)}\right)}{\textit{var}\left(r_{n_1}^{(1)}\right)}, \\
& \beta_2^{pf} = \textstyle\sum_{j=1}^{n_1+n_2} \pi_j^* \frac{\textit{cov}\left(r_j^{(2)}, r_{n_2}^{(2)}\right)}{\textit{var}\left(r_{n_2}^{(2)}\right)},
\end{aligned} \tag{4.6}$$

where $m = \{m^{(1)}, m^{(2)}\}$, with the components of $m^{(k)},\ k = 1, 2$, given by Theorem 3.2.3 and S is as in equation 4.1, with its components given by Theorem 3.2.4 and Theorem 4.1.1, and where one can choose $\beta_k^{mkt},\ k = 1, 2$, either equal to one, or compute it from historical excess returns, as the covariance of the underlying excess returns and the market excess returns over the variance of the market excess returns beforehand and use it as a constant, or update it dynamically when new information is obtained each trading day.

Proof. The proof of this theorem is equivalent to the proof of Theorem 3.2.5, but adds hedging arguments on each underlying, as discussed in this section. □

Theorem 4.1.2 is easily extended for more than two underlying assets, one just has to calculate m and S in higher dimensions and hedge subsequently with respect to each underlying asset, extending the way which we showed for two underlying. If one chooses his portfolio weights every trading day according to Theorem 4.1.2, one would have an alpha-maximizing option trading strategy in multiple dimensions that hedges out the sensitivity to market movements. This strategy also maximizes the Sharpe ratio of its excess returns.

In chapter 5 we test the performance of our two dimensional strategy with simulated data from a Black-Scholes framework. In chapter 6 we apply the strategy to historical market data and study whether statistically significant alphas can be generated under beta-neutrality and how high the Sharpe ratio we can generate is.

CHAPTER 5

SIMULATIONS

In this chapter we test the performance of our strategies for a single benchmark (as derived in chapter 3) and multiple benchmarks (as derived in chapter 4) in a theoretical setting. From a Black-Scholes framework we generate data and we study whether one can generate an alpha which is significantly different from zero. We investigate how high the Sharpe ratio one can achieve is and how well the beta-hedge performs. For both cases we explain how the simulations are performed in an algorithmic fashion. We start off with the one dimensional case.

5.1 Simulations with a single benchmark asset

In this section we test the performance of the investment strategy derived in chapter 3 with simulated data. We explain step by step how the simulation is performed and then we elaborate on the results. The steps that we take are as follows:

1. Generate an underlying price process from geometric Brownian motion with certain parameters, on which we can write options. We consider the market to consist of this benchmark, and there is a risk free asset available.

2. For every trading day, generate a range of strikes (including a strike of zero) of European call options that are available on the benchmark.

3. Use the Black-Scholes formula to calculate the call option prices corresponding to these strikes for each trading day. Use the benchmark price as the price of the zero strike option on each trading day.

4. Calculate realized excess returns on all available derivatives on each trading day. The excess return on the zero strike option is the excess return on the market.

5. Calculate the expected excess returns on all available derivatives and assemble the vector m on each trading day.

6. Calculate the covariance matrix of excess returns S for each trading day.

7. Calculate initial weights on each trading day.

8. Perform a beta-hedge for the weights on each trading day.

9. Adjust the weights for a certain level of risk on each trading day.

10. Calculate realized portfolio excess returns from each trading day.

11. Calculate the mean, standard deviation and Sharpe ratio of these realized portfolio excess returns.

12. Calculate the mean, standard deviation and Sharpe ratio of the market excess returns.

13. Regress the realized strategy excess returns on the market excess returns to estimate alpha and beta.

First, we simulate the underlying asset price process. We do this from the geometric Brownian motion (3.3). We simulate a process that is observed every month and we generate 2001 consecutive prices. We assemble a Markowitz portfolio of call options on each moment on which a price is observed, except for the moment of the last observation. The options we generate expire on the next trading day, so we shall have 2000 observations of strategy excess returns. We choose the price of the underlying at $t = 0$ to be $S_0 = \$100.00$. We choose the other parameters in the geometric Brownian motion to be

$$r_f = 0.034, \quad \mu = 0.08, \quad \sigma = 0.20, \quad \Delta t = 1/12, \quad T = t + \Delta t,$$

and we simulate the price process by iterating equation (3.3) for $t = 0, 1, \ldots, 2000$, drawing ε from the standard normal distribution for each iteration. We use a random seed of 42 for the random number generator, so one can replicate the same process. The choice for the risk free rate is arbitrary, but 3.4% is a value that up until the global recession of 2008 and onwards was a representative figure for this rate. The choice for μ equal to 8% is a choice that is often made by hedge fund managers. A σ equal to 20% is also arbitrary, but does reflect the uncertainty of the benchmark. The choice for Δt equal to 1/12 follows from that we work with monthly data. We plot the natural logarithm, from now on referred to as log, of the generated price process in figure 5.1. We see that the price process generally drifts up, but is fairly volatile as well.

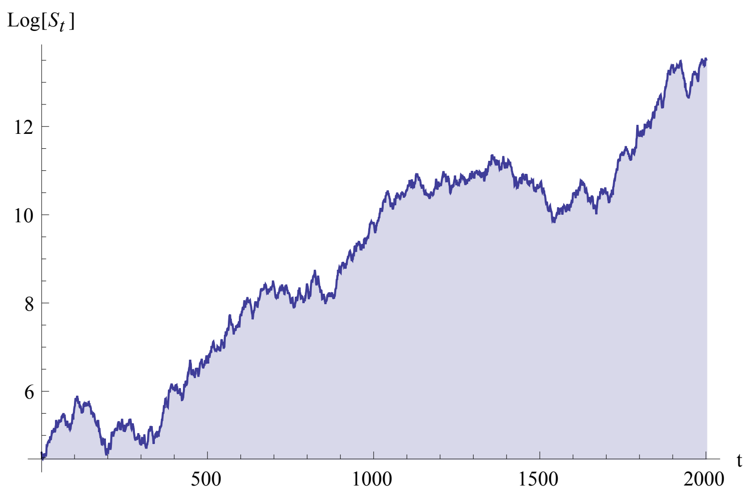


Figure 5.1: A price process simulated with geometric Brownian motion.

Throughout these simulations, we assume that the market only consists of this risky asset and a risk free rate. Therefore, we treat the underlying excess returns given by equation (3.4) as the market excess returns.

Next, we generate a range of options on the benchmark for each trading day. In the Optionmetrics database, there are 13 options available on each trading day and they are characterized by their delta, which is their sensitivity to movements in the underlying price process. In the database, the available options are options with deltas running from 0.20 through 0.80 with steps of 0.05. Therefore, we adapt our simulations to this framework. On each trading day we generate 13 options, and we start of with generating their strike prices. We solve the strikes from the theoretical delta of European call options, which is given by the term $N(d_1)$ in the Black-Scholes formula (3.5), which is given as

$$N\left(\frac{1}{\sigma\sqrt{\Delta t}}\left[\ln\left(\frac{S_t}{K_i}\right)+\left(r_f+\frac{\sigma^2}{2}\right)\Delta t\right]\right).$$

We see that the only unknown parameter in this expression is the strike K_i, hence, we can solve the equations

$$N(d_1)=\Delta_i, \quad \Delta_i \in \{0.20, 0.25, \ldots, 0.80\}$$

for K_i on each trading day, giving us 13 strikes per trading day. We repeat this for every trading day. Then, from this generated strikes, we generate the prices, C_i, for all these options with the Black-Scholes formula (3.5). Options with a delta of 0.5 are usually interpreted in finance as at-the-money (ATM) options, hence, options for which the strike price is equal to the current price of the underlying. This is formally not exactly the case, but the strike is very close to the current price of the underlying. A delta below 0.5 implies that the option is out-of-the-money (OTM), hence, the option has a higher strike price than the current underlying price. The lower the delta, the further the option is OTM. A delta above 0.5 implies that the option is in-the-money (ITM), hence, the option has a lower strike price than the current underlying price. The higher the delta, the further the option is ITM. We extend the range of strikes on each trading day with a strike price of zero and the option prices with the underlying price on that trading day.

Having a price process and options available on every trading day, we compute the realized excess return of each option on each trading day, using equation (3.7), creating the vector $r_t^{realized}$ for $t = 0, 1, \ldots, 1999$. The excess return of the zero strike option is equal to the excess return on the underlying itself. The mean, realized monthly excess return of the options in this simulation are

$$\{0.149916, 0.130867, 0.121305, 0.115353, 0.109943, 0.104326, 0.0991952,$$
$$0.0948217, 0.0892549, 0.0823697, 0.0736303, 0.0645233, 0.0557866, 0.00335623\},$$

where the first element is the mean, realized monthly excess return on the option with $\Delta = 0.20$, the second that of the option with $\Delta = 0.25$, continuing to the next to last element, which is the mean, realized monthly excess return on the option with $\Delta = 0.80$. The last element is the mean, realized monthly excess return on the zero strike option. Their standard deviations are respectively given by

$$\{3.27923, 2.81014, 2.46525, 2.19667, 1.97884, 1.79629, 1.6387,\\ 1.49928, 1.37468, 1.26148, 1.15736, 1.05862, 0.962105, 0.0591298\}.$$

We find that the mean, realized monthly excess return on the far OTM options is highest, but that their standard deviations are substantial as well, hence, the investments with a higher return, carry a greater risk, and might therefore be unattractive products.

We assemble the expected excess return vector m and covariance matrix S of these excess returns for each trading day, by plugging in the formulae derived in Theorem 3.2.3 and Theorem 3.2.4. Since the fractions used in these calculations remain constant over time by the way we constructed our options from the underlying at constant levels of delta, the vector m and matrix S are the same for each trading day. The expected monthly excess returns on the options are respectively given by

$$\{0.128254, 0.119651, 0.11223, 0.10562, 0.0995897, 0.0939856, 0.0886959,\\ 0.0836336, 0.0787258, 0.0739059, 0.069106, 0.0642481, 0.0592297, 0.00384069\},$$

which are close to the realized values we found. Considering the size of the covariance matrix, we omit printing it. From the covariance matrix we create the correlation matrix of excess returns, which we use to check whether the computations do not produce strange results. The correlation matrix in this case gives ones on the diagonal and numbers smaller than one, but bigger than zero on all other places. It is symmetric and correlation decreases when strikes are further apart. This is exactly what we would expect.

Having computed m and S for every trading day, we assemble our portfolios. We follow Theorem 3.2.5 to do this. For each trading day, we start of with computing the initial weights $\tilde{\pi} = S^{-1}m$. Then we perform the beta-hedge. Since our underlying is the market itself, we do not have to compute any β^{mkt}, so we set it equal to one. We compute β^{pf} and subtract it from the weight of the zero strike option. Last, we adjust our weights for the risk. We choose our risk preference to be 5% per month, which is about the standard deviation of the monthly excess returns on the S&P 500. We compute λ and scale our weights with this factor. Since m and S are the same for each trading day, the total portfolio beta position and λ are constant as well, and thus, so are the weights chosen at each trading day. The weights are respectively given by

$$\{-0.0154221, 0.0146114, -0.00789122, 0.0000894902, -0.00322045, -0.00264307, -0.00361614,\\ -0.00367368, -0.0066532, 0.00105103, -0.0360899, 0.102839, -0.168375, 2.31635\}.$$

We find that we invest substantially in the zero strike option, and that the rest of the weights are mainly put on the three most far ITM options. With the put-call parity in equation (3.8) we can interpret a far ITM call as a far OTM put and positions in the underlying and cash. Broadie et al. [2] report OTM puts to be highly favourable products, so our findings are similar.

Having constructed a vector π of portfolio weights for each trading day, we compute the realized excess return generated by our portfolio for each trading day by calculating $R^{\pi} = \pi' \cdot r_t^{realized}$. We store these excess returns in the vector $R^{pf} = \{R_0^{\pi}, R_1^{\pi}, \ldots, R_{1999}^{\pi}\}' \in \Re^{2000\times 1}$, where R_t^{π} is the realized excess return of our portfolio assembled at time t. We also assemble the excess returns on the market itself (the excess returns on the zero strike option) in the vector $R^{mkt} = \{r_0^{mkt}, r_1^{mkt}, \ldots, r_{1999}^{mkt}\}' \in \Re^{2000\times 1}$, where r_t^{mkt} is the excess return on the underlying given by equation (3.4). Mind here that all excess returns in this section are formulated in a monthly fashion, and are not yet annualized.

We now have all the data we need. We compute the mean excess return generated by our strategy by taking the mean of R^{pf} and we compute the standard deviation of our portfolio excess returns as $\sigma(R^{pf})$. The Sharpe ratio is then calculated as the mean excess return divided by the standard deviation of excess returns. We do this for the market excess returns as well. Then we perform the regression (1.1), where beta is estimated by *cov*$(R^{pf}, R^{mkt})/$*var* $\left(R^{mkt}\right)$, and alpha is estimated as the intercept in the regression. We denote the T-statistics of these alpha and beta, as well as their standard errors and p-values, from which we can conclude whether they are significantly different from zero, or not. If beta is insignificant, then alpha should be very close to the mean excess return generated by our strategy. The results are given in table 5.1a and table 5.1b, and the regression result is given in table 5.1c. The annualized figures in the tables are computed straight forward from the monthly figures by multiplying the mean excess return with $1/\Delta t$ and the standard deviation and the Sharpe ratio with $\sqrt{1/\Delta t}$. A scatter plot of the portfolio excess returns against the market excess returns can be found in figure 5.2.

As we can see in table 5.1a, the generated mean monthly excess return is very close to zero and negative. The standard deviation is around 5%, which is the level which we were aiming at. The Sharpe ratio is low. Furthermore, the strategy does not outperform the underlying asset itself, as the Sharpe ratio of the benchmark is greater than the one generated by the strategy. In table 5.1c we see that the estimated alpha and beta are both insignificantly different from zero at a big level of significance, as the p-values are considerably large. Thus, in this simulation, our strategy does not show superior performance. Eyeballing figure 5.2 makes it intuitive; we can see that the portfolio excess returns are only positive for a small range of market excess returns and are strongly negative for a far greater range.

We have performed the simulation with these parameters 2000 times (dropping the random seed, so multiple underlying processes are generated), and no significant alpha were generated. From this we conclude that also with a strategy that trades in options with discrete strikes

	Monthly figures	Annualized figures
Mean excess return	-0.000831578	-0.00997894
Std. Dev.	0.0512934	0.177686
Sharpe ratio	-0.0162122	-0.0561607

(a) Portfolio excess return statistics.

	Monthly figures	Annualized figures
Mean excess return	0.00335623	0.0402748
Std. Dev.	0.0591298	0.204832
Sharpe ratio	0.0567604	0.196624

(b) Market excess return statistics.

	Estimate	Standard error	T-statistic	p-value
α	-0.000777181	0.00114889	-0.676463	0.498825
β	-0.016208	0.0194036	-0.835309	0.403644

(c) Regression results.

Table 5.1: Simulation results.

it is hard to obtain superior performance by trading frequently in derivatives. As Guasoni et al. [8] have already concluded this for a strategy with continuous strikes, this conclusion comes as no surprise, and agrees with Coval and Shumway [1] that in a Black-Scholes world options are redundant assets.

5.1.1 Varying parameters

We have elaborated on our simulation with parameters that are representative for the market, but to validate the model, we study what happens if one changes certain parameters. An interesting case, is the case where $\mu = r_f$. In this case, we expect to see no general drift in the price process and we expect all derivatives to have an expected excess return of zero. Therefore, we also expect that our strategy will not produce a significant alpha, nor a high Sharpe ratio.

Keeping the rest of the parameters constant and the random seed at 42, we get the price process from figure 5.3, which indeed shows no general drift. The expected excess returns on the options are all of order 10^{-14} or smaller, so practically equal to zero, as we expected. The mean realized excess return on options is very low as well. The regression table of the results is given in table 5.2, and as we can see, neither alpha, nor beta is significant. Also in this case the benchmark is not outperformed by the strategy.

It is also interesting to change the parameter σ. We use $\mu = 0.08$ again and keep the rest of the parameters and the random seed as they were. We first use a low σ equal to 0.05. Logically, we find a price process that shows almost no volatility, and which shows a steep upward drift. We find options to have a very high realized and expected excess return. We find that the strategy produces a higher alpha than with $\sigma = 0.2$, but the estimated alpha is still

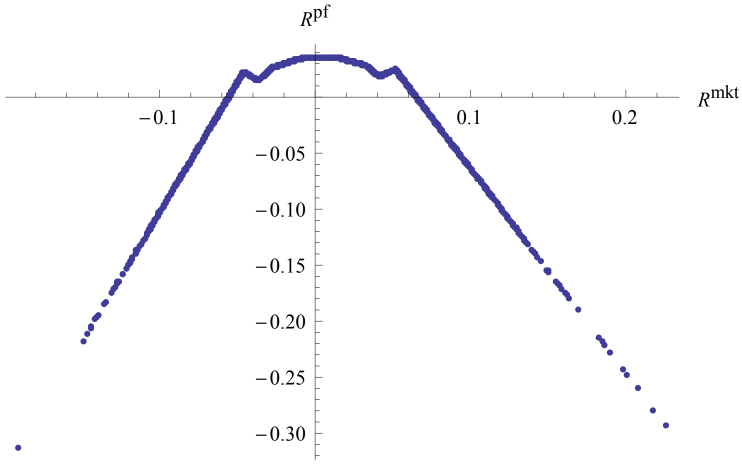


Figure 5.2: Scatter plot of portfolio excess returns against market excess returns.

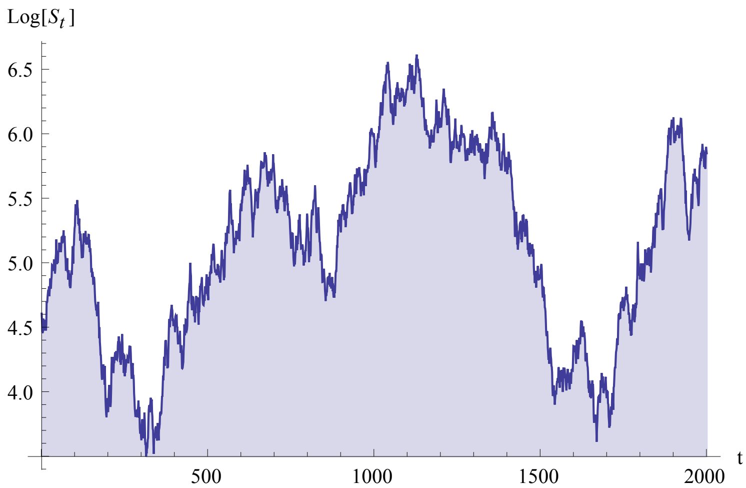


Figure 5.3: The underlying price process if $\mu = r_f$.

insignificant in the regression. Beta is insignificant as well. Almost all weight in the portfolio is placed on the zero strike option, which is not that surprising. As the volatility is very low, the underlying itself is a safe and profitable bet.

We study the strategy performance with σ to 0.4, which is quite high, keeping the rest of the parameters as they were. We get a very volatile underlying price process, where the volatility dominates the general drift, so no general drift can be spotted in the price process. We find realized and expected option excess returns to be very low and a strategy that produces insignificant alpha and beta.

Hence, varying parameters does not improve the strategy performance, as it still generates insignificant alphas and low Sharpe ratios. As suggested in Guasoni et al. [8] and Coval and Shumway [1], pricing the options at an implied volatility that differs from the realized volatility

	Estimate	Standard error	T-statistic	p-value
α	-0.000529743	0.0011316	-0.468135	0.639739
β	-0.00586368	0.0192153	-0.305157	0.760278

Table 5.2: Regression results for the case $\mu = r_f$.

on the underlying produces significant excess returns on options. In the chapter 6 we study the strategy performance in this setting, where we use actual market data, so the option prices deviate from the ones given by the Black-Scholes formula used with the volatility of the underlying.

5.2 Simulations with multiple benchmark assets

In this section we study the performance of the investment strategy derived in chapter 4 with simulated data. We study the strategy that dynamically invests in options on two benchmark assets. Again, we explain our simulation step by step and thereafter we elaborate on the results. The steps that we take are as follows:

1. Generate two, underlying price processes from geometric Brownian motions with certain parameters, which are correlated in the Brownian motion terms. Consider the excess returns on the first benchmark as the excess returns on the market.
2. For each trading day, generate a range of strikes (including the zero strike) of European call options on both of the benchmark assets.
3. Use the Black-Scholes formula to calculate the call option prices corresponding to these strikes for each trading day. Use the benchmark prices as the prices of the zero strike options on each trading day.
4. Calculate the realized excess returns on all available derivatives on both underlying on each trading day. The excess return on the zero strike option on the first benchmark is the excess return on the market.
5. Calculate the expected excess returns on all available derivatives on both underlying and assemble the vector m on each trading day.
6. Calculate the covariance matrix of excess returns S for each trading day.
7. Calculate initial weights for each trading day.
8. Perform a beta-hedge on the weights of both underlying on each day.
9. Adjust the weights for a certain level of risk on each trading day.
10. Calculate realized portfolio excess returns from each trading day.

11. Calculate the mean, standard deviation and Sharpe ratio of these realized portfolio excess returns.
12. Calculate the mean, standard deviation and Sharpe ratio of excess returns on both underlying.
13. Regress the realized strategy excess returns on the market excess returns to estimate alpha and beta.

We start with simulating the underlying price processes. We generate them both with a geometric Brownian motion, each with its own parameters, but we need to generate them in such a way that the geometric Brownian motions are correlated in the Brownian motion terms. We simulate processes that are observed monthly and we generate 2001 consecutive prices. We assemble a Markowitz portfolio of call options on each moment on which a price is observed, except for the moment of the last observation. The options we generate expire on the next trading day, so we shall have 2000 observations of strategy excess returns. We choose the price of the underlying at $t = 0$ to be $S_0^{(1)} = \$100$ and $S_0^{(2)} = \$420$. The parameters we use are as follows:

$$r_f = 0.034, \quad \mu_1 = 0.07, \quad \mu_2 = 0.08, \quad \sigma_1 = 0.20, \quad \sigma_2 = 0.22, \quad \rho_{12} = 0.9, \quad \Delta t = 1/12.$$

and we simulate the price processes by iterating equation (3.3) for $t = 0, 1, \ldots, 2000$ for both underlying, but to make them correlated, we do not simply draw ε_1 and ε_2 separately from the standard normal distribution. We first construct 2 strings of 2001 standard normal distributed drawings, x_1 and x_2. Then we define $\eta_1 = x_2$ and $\eta_2 = \rho_{12} x_1 + x_2 \sqrt{1 - \rho_{12}^2}$, which are now two strings with correlation ρ_{12}. Now, in stead of drawing ε_1 and ε_2 for each trading day, we use the next element in η_1 and η_2 respectively. The choice of parameters is again to match reality and are all but ρ_{12} explained by the same reasoning as in section 5.1 . We choose the correlation to be fairly high, because the indices which we work with in the market data analysis in chapter 6 exhibit a large correlation as well. We plot the log of the resulting price processes in figure 5.4, in which we see that the price processes show similar movements due to the correlation in their Brownian motion terms.

Next, we generate a range of options on both underlying on every trading day. We do this in the same way as in the one dimensional simulation for both underlying, but with less available strikes on both. As we have seen in the one dimensional case, most of the weight was put on far ITM options, and if we would use 14 options on both assets, the covariance matrix would become quite big, so we choose to use only the range of ATM to ITM options (deltas running from 0.50 to 0.80 in steps op 0.05) on both underlying. We again use the Black-Scholes formula (3.5) to compute all option prices and we extend both ranges of strikes with a zero strike on each trading day, and add their prices to our option prices (which are equal to $S_t^{(1)}$ and $S_t^{(2)}$ on trading day t respectively), resulting in 8 available European call options per underlying per trading day.

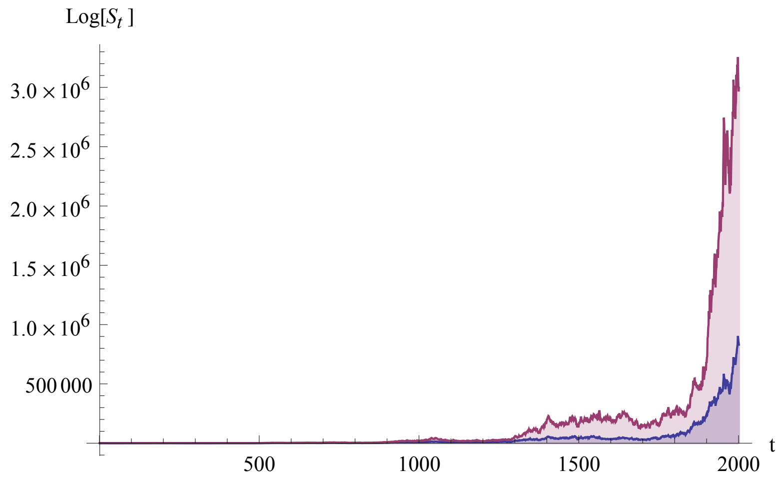


Figure 5.4: The underlying price processes, generated with correlated geometric Brownian motions. The blue process is that of the first asset (the market), the red process is that of the second asset.

We continue with the calculation of the realized excess returns on all options on every trading day. This is done in the same way as in the single asset case for both underlying. Then, on each trading day we combine all realized excess returns in one single vector $r_t^{realized}$, which' first 8 elements are the realized excess returns of the options on the first underlying, followed by those on the second. The mean, realized monthly excess returns of the options in this simulation are

$$\{0.100911, 0.0948905, 0.088579, 0.0823376, 0.0758537, 0.0679569, 0.0595883, 0.00342251,$$
$$0.0949841, 0.0877659, 0.0811328, 0.0746247, 0.0693676, 0.0637317, 0.0583194, 0.00370704\},$$

where the first line contains the mean, realized monthly excess returns of the options on the first underlying, running from the ATM option to the far ITM option, followed by the mean, realized monthly excess return on the zero strike options on the first underlying. The second line contains the equivalent for options on the second underlying. We note that the figures are all positive and the figures for the options with the same delta on different underlying are quite alike. Their standard deviations are respectively given by

$$\{1.62647, 1.49045, 1.36798, 1.25546, 1.15049, 1.05167, 0.95567, 0.0591939,$$
$$1.63734, 1.50107, 1.37795, 1.26499, 1.15847, 1.05731, 0.958362, 0.0650526\},$$

and we again note that the lower the delta of the option, the higher the realized excess return, but the higher the standard deviation again as well.

The next step is to assemble the expected excess return vector m. We compute the expected excess returns on all options as in the one dimensional case, as they only depend on their own underlying. We then combine these expected excess returns on each trading day in the same way as we have done with the realized excess returns. We again compute the mean expected excess returns:

$$\{0.0689959, 0.0650948, 0.0613083, 0.0575854, 0.0538736, 0.0501125, 0.0462224, 0.0030045, \\ 0.0807276, 0.0761497, 0.0717092, 0.0673457, \ 0.062998, 0.0585954, 0.0540445, 0.00384069\},$$

which are again close to the realized values.

We assemble the covariance matrix S on each trading day. Considering the representation for S given in equation (4.1), on each trading day we first calculate S_{11} and S_{22} as in the single asset case. Then, we calculate S_{12} in a few steps. We first calculate the covariances of type 1 with equation (4.3). Then we calculate the covariances of type 2 and 3 with the rest of the equations provided by Theorem 4.1.1. We combine the calculated terms to one matrix, and we transpose this matrix to get S_{21}. Then we combine all these matrices to one covariance matrix S, and from this covariance matrix, we can generate the correlation matrix. Again, all fractions used in the explicit formulae to calculate m and S are constant over time, so m and S are constant over time. Considering the sizes of the covariance and correlation matrices, we omit printing them. The correlation matrix does show the behaviour that we would expect; the correlations in the S_{12} and S_{21} parts of the correlation matrix decrease as strikes drift further apart.

Having computed m and S for every trading day, we again assemble our portfolios. For this, we use Theorem 4.1.2 on each trading day. The initial weights are computed, and then the beta-hedge is performed. We compute the beta with respect to the first underlying and subtract this weight from its zero strike option. Then we calculate the beta position with respect to the second underlying with these adjusted weights, and subtract this position from the weight of the zero strike option on the second underlying. Hence, we assume that the β_2^{mkt} is equal to one, which is not a bad assumption, as the correlation between the assets is very high. Then we adjust for 5% risk per month again, by calculating λ and adjusting the weights with this factor. Since m and S are constant over time, the weights will be constant as well. In the same order that we have used for the other vectors we presented, we find them to be

$$\{-0.00400262, 0.00656925, -0.00157441, 0.00120184, -0.000811391, 0.00481288, -0.00363784, -2.00319, \\ -0.0227464, 0.024274, -0.00917854, 0.00216984, -0.00789276, 0.0191966, -0.0324392, 2.03059\}.$$

For both assets we see that most weight is on the zero strike option again, and we cannot conclude that a certain kind of option is favourable along with the zero strike option, as all of the other weights are quite alike.

Having our vector π of portfolio weights on each trading day, we can again compute the realized excess returns generated by our portfolios by calculating $R^{\pi} = \pi' \cdot r_t^{realized}$. We store these excess returns in a vector R^{pf} again, like in the one dimensional case, and we assemble the excess returns on the market itself (the excess returns on the zero strike option of the first underlying) in the vector R^{mkt}, as in the one dimensional case. Note that these excess returns throughout this section are all formulated in a monthly fashion, and are not yet annualized.

We proceed as in the one dimensional case by calculating the mean of R^{pf} and its standard deviation. Then we compute the Sharpe ratio. We do this for both benchmark assets as well. All figures can be annualized in the same way as we described in the one dimensional case. Then, we regress the strategy excess returns on the market excess returns to estimate alpha and beta, and we denote the standard error of the estimates, their T-statistics and their p-values. We report the results in table 5.3.

	Monthly figures	Annualized figures
Mean excess return	-0.000929554	-0.0111546
Std. Dev.	0.050492	0.17491
Sharpe ratio	-0.0184099	-0.0637738

(a) Portfolio excess return statistics.

	Monthly figures	Annualized figures
Mean excess return	0.00342251	0.0410701
Std. Dev.	0.0591939	0.205054
Sharpe ratio	0.0578186	0.200289

(b) Market excess return statistics.

	Monthly figures	Annualized figures
Mean excess return	0.00370704	0.0444845
Std. Dev.	0.0650526	0.225349
Sharpe ratio	0.0569853	0.197403

(c) Second underlying excess return statistics.

	Estimate	Standard error	T-statistic	p-value
α	0.000321253	0.00102212	0.314301	0.753325
β	-0.365465	0.0172428	-21.1952	0

(d) Regression results.

Table 5.3: Simulation results.

As we can see in table 5.3a, the generated mean monthly excess return is very close to zero and negative. The standard deviation of strategy excess returns is around 5%, the level we were aiming at, and the Sharpe ratio is small. The strategy does not outperform either of the benchmarks, which generate higher Sharpe ratios, as can be found in tables 5.3b and 5.3c. In table 5.3d we see that the estimated alpha is insignificantly different from zero at a big level of significance. However, the beta is significant, which shows that the beta-hedge failed in this case. This is probably due to overcompensation in the hedging; we subtract the beta position with respect to the market from the weight on the zero strike option of the first underlying, making the market beta position of the portfolio equal to zero. Since the correlation between the underlying is high, this would almost imply that the beta position with respect to the second underlying is zero as well, but we still compensate for it. Therefore, there might be overcompensation. Therefore, we perform the strategy performance analysis again, but only hedge on the first underlying, to study whether there are overcompensation effects. The strategy performance results are given in table 5.4a. The regression results are given in table 5.4b. A scatter plot of the portfolio excess returns against the market excess returns is given in figure 5.5.

	Monthly figures	Annualized figures
Mean excess return	0.000410518	0.00492621
Std. Dev.	0.0504795	0.174866
Sharpe ratio	0.00813236	0.0281713

(a) Portfolio excess return statistics.

	Estimate	Standard error	T-statistic	p-value
α	0.000395894	0.00113091	0.350066	0.726326
β	0.00427271	0.0190781	0.223959	0.822812

(b) Regression results.

Table 5.4: Simulation results when one only hedges w.r.t. the first underlying.

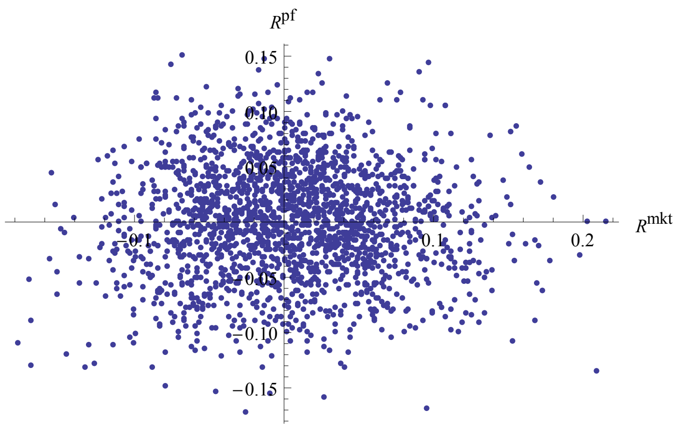


Figure 5.5: Scatter plot of portfolio excess returns against market excess returns.

As we can see in table 5.4a, the Sharpe ratio is still smaller than that of both underlying. In the scatter plot in figure 5.5 we can see that the strategy excess returns show no clear pattern and are neither mostly positive, nor mostly negative, so it is of no surprise that the mean excess return is close to zero. Table 5.4b shows us that now both estimated alpha and beta are insignificant at a high level of significance, so we conclude that there were indeed overcompensation effects in the hedging process, and that this phenomenon might also occur when working with historical market data.

We have performed this simulation with these parameters 2000 times (dropping the random seed), and again, no significant alphas were generated. So, we conclude that also in a two dimensional setting it is hard to obtain superior performance by trading frequently in derivatives on several underlying when options are priced by the Black-Scholes formula at the volatilities of the underlying.

5.2.1 Varying parameters

In the one dimensional case we have investigated the effects of varying μ and σ on the simulation results. We expect the results to be similar in the bivariate case. We study a single, very interesting case in this bivariate setting; suppose that we use the same parameters for the geometric Brownian motion of the first benchmark as before, but we set $\mu_2 = r_f$, $\sigma_2 = 0$ and $\rho_{12} = 0$, hence, the second asset is not correlated with the first and is not volatile at all, it functions as a risk free account without correlation with the market. Then, we expect that our strategy in a bivariate setting yields the same result as in the one dimensional setting, as options on the second asset should have a zero expected excess return and zero correlation with each other and the options on the first asset, which translates to S_{12}, S_{21} and S_{22} being matrices filled with zeros.

For numerical reasons, we set $\sigma_2 = 10^{-16}$, as setting it to zero would yield division by zero errors, and run the simulation. In the vector m, the string with expected excess returns of options on the second asset shows figures all of order 10^{-9} or lower, so they are practically zero. The covariance matrix S is filled with figures like this in the S_{12}, S_{21} and S_{22} blocks as well. This agrees with what we expect, but the fact that they are not exactly equal to zero causes enormous problems in the portfolio selection procedure. The covariance matrix is not invertible anymore, and we get an output that is dominated by numerical errors.

Therefore, we adapt our code to prevent errors like these. First, we rewrite it such that when an entry in m or S is smaller than order 10^{-8}, it is set equal to zero. Second, in the computation of the initial portfolio weights we do not use a standard matrix inversion command, but we solve the linear system $\tilde{\pi}'S = m$ for $\tilde{\pi}$, which is numerically more efficient and has the advantage that if m is filled with zeros on the second asset positions and S is filled with zeros in the S_{12}, S_{21} and S_{22} blocks, it still produces the weights for the options on the first asset, and gives weights equal to zero for the options on the second asset. Would we use matrix inversion, then we would again encounter the problem that S is not invertible if the S_{12}, S_{21} and S_{22} blocks are filled with zeros, as it is not of full rank.

With these adaptations, we find the results which we would expect. The vector m and covariance matrix S are filled with zeros on the places where we expect them, and the strategy only places weights on the options on the first asset. The portfolio performance results we find are similar to the ones found in the one dimensional case, especially when using the same parameters for the first asset. However, the computation time is enormous, since the covariance matrix construction has to process a lot of underflows in the computations.

We conclude that when very small numerical errors are encountered, the strategy can produce very erroneous results, hence, we use the alterations made to the code to solve these numerical errors in our market data analysis as well.

CHAPTER 6

MARKET DATA AND STRATEGY PERFORMANCE ANALYSIS

In this chapter we study the performance of the strategies derived in chapters 3 and 4 with historical market data. We test the strategies in a one dimensional setting and in a two dimensional setting. We use market data from the Optionmetrics database, running from January 4th 1996 to January 31st 2013. We consider the S&P 500, the NASDAQ 100, the Russell 2000 and the Dow Jones 1/100th Industrial Average in our analyses. Throughout our entire data analysis, we will consider the S&P 500 as "the market". The database contains all historical prices on these assets, the realized volatility of these assets and an annualized risk free rate on every date. Although the database contains data on all historical traded options on these assets, we use another table of the database for the option data; the volatility surface table. This table contains the interpolated volatility surface for each asset on each day, established using a methodology based on a kernel smoothing algorithm and the actual traded options. This table contains information on standardized options on every business day with a range of fixed expirations. The options are available at a range of deltas of 0.20 to 0.80 with steps of 0.05.

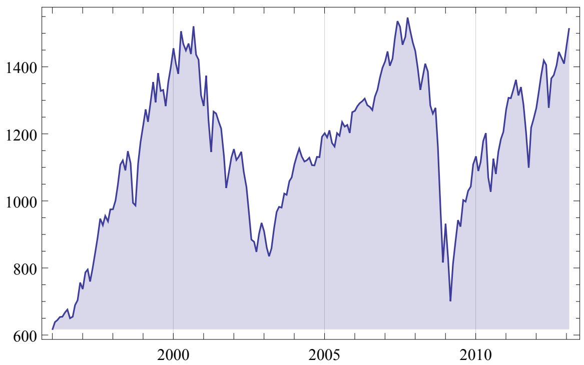
In the first section of this chapter we provide a visualization and analysis of the data on the underlying assets. In the second section we explain our analysis methodology used in the single asset case and we then perform our strategy performance analysis over different periods of time and with settings, and then present the results. The same is done for the two dimensional case in the third section.

6.1 Historical data on the benchmark assets

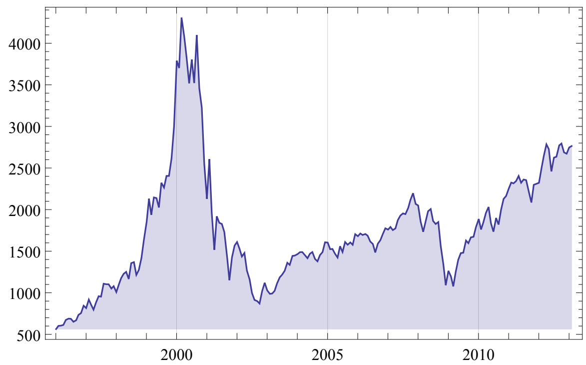
In this section we provide a visualization and an analysis of the underlying price processes we consider in our analyses, of their realized volatilities and of the risk free rate over time.

The S&P 500 index consists of 500 leading companies publicly traded in the US stock market. Therefore it is a suitable candidate to represent "the market", thus, throughout this whole chapter we use the excess returns on the S&P 500 as the market excess returns. The NASDAQ 100 consists of the 100 largest non-financial companies listed on the NASDAQ, which contains a lot of stocks in technological companies. The Russell 2000 is a small capitalization stock market index of the bottom 2000 stocks in the Russell 3000 index, and the Dow Jones 1/100th Average Industrial index is the Dow Jones Industrial Average, an index that shows the performance of 30 large publicly owned companies in the US, scaled by 1/100.

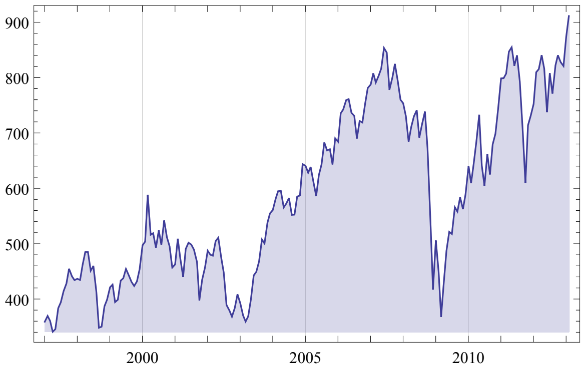
In figure 6.1 we present the underlying price processes of the four underlying assets. We observe the close of each underlying asset on every first trading day of the month, over the entire period of available data. We use these closing prices as the underlying prices in our analyses. We see that the S&P 500 and NASDAQ 100 have data available starting from January 1996, the Russell 2000 from January 1997 and the first available data of the Dow Jones 1/100th is from January 1998. For all indices we have data available until January 31st 2013. In figure 6.1 we clearly see the effects of the Dot-com bubble for all assets. This



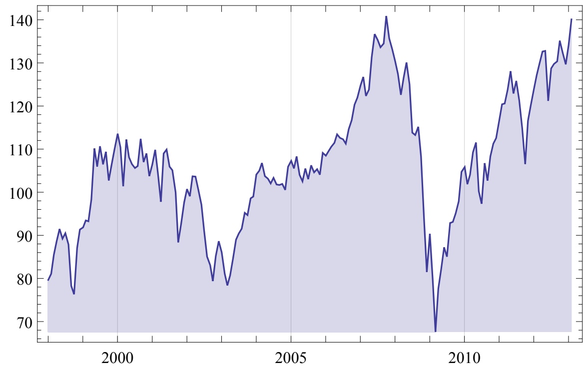
(a) S&P 500.



(b) NASDAQ 100.



(c) Russell 2000.

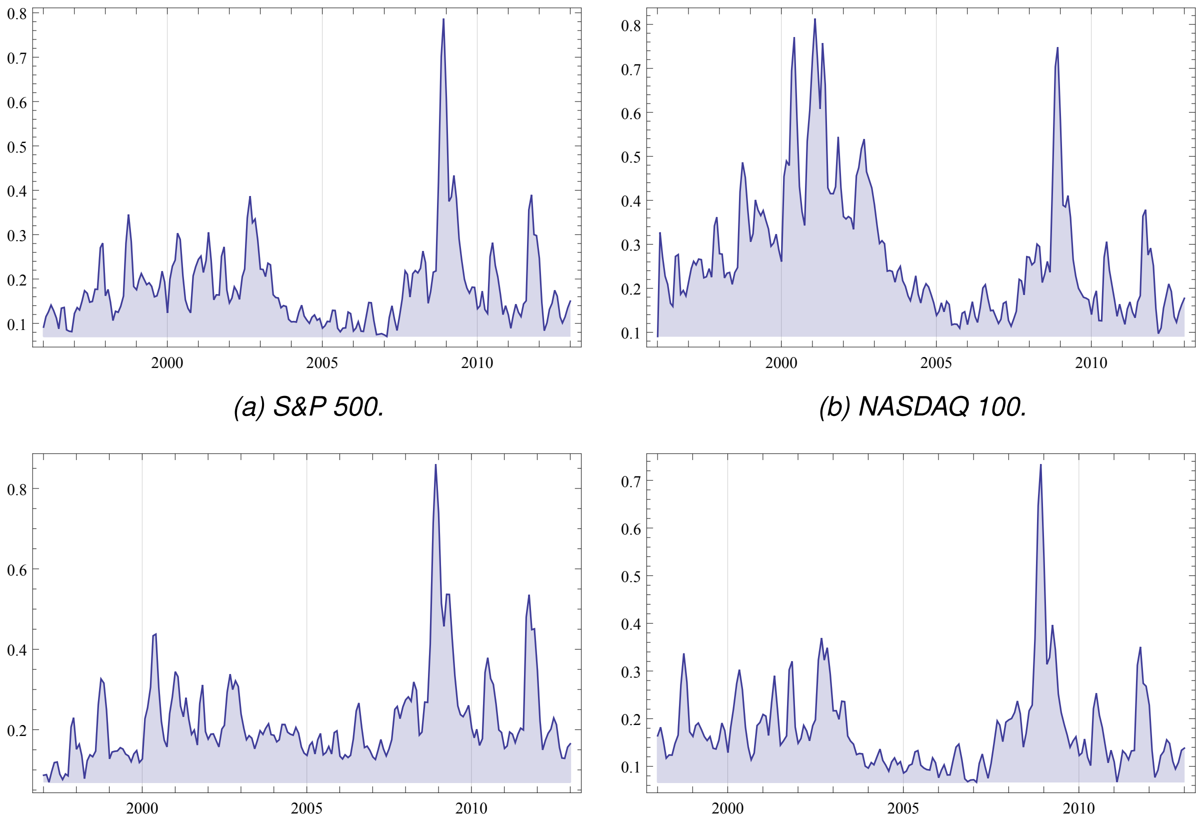


(d) Dow Jones 1/100th.

Figure 6.1: Historical underlying price processes, prices are given in US dollars.

Internet bubble was a historic speculative bubble during the period of 1997-2000, and burst in March 2000. The effects for the S&P 500 and NASDAQ 100 are most clear; the prices rise steeply in the period 1997-2000, even reaching 4000 for the NASDAQ 100, and then they make an enormous drop after March 2000. The magnitude of the effects on the NASDAQ 100 is explained by the fact that the NASDAQ 100 contains a lot of technological stocks, which were of course most affected by a bubble in the Internet section. We also see the effects of the recent credit crisis on all assets; after the burst of the Dot-com bubble we see prices rising again in early 2003. The prices continue to rise, until 2007, when the credit crisis kicked in, and we see them dropping again. The following global recession is clear as well; in 2008 the prices reach a historical low. In 2009 the prices start to rise again.

In figure 6.2 we graph the realized annualized volatility of the underlying observed on the first business day of each month, based on a period of 60 calendar days prior to that date. We use these volatility processes as input for our geometric Brownian motion and for our derived explicit formulae during our strategy performance analysis. We see that during the period of the Dot-com bubble the volatility of each underlying is substantial and quite volatile itself. After the bubble bursts, we especially see the NASDAQ 100 volatility shooting up, which is in line with the enormous price drop in the index at that time. In the post-bubble period we see that the volatility on each underlying is relatively low, whilst the growth in the underlying price is substantial. This is a quite strange phenomenon on the market. The volatility rises again during the credit crisis of 2007 and peaks around 2009, when the global recession has settled in.



(a) *S&P 500.* (b) *NASDAQ 100.*

(c) *Russell 2000.* (d) *Dow Jones 1/100th.*

Figure 6.2: Historical/realized volatility of the underlying price processes.

Considering this analysis, we choose to test the performance of our strategies in four different periods:

1. The period over the total range of available data, which is Jan. 1996/1998 - Jan. 2013.
2. The period Jan. 1996/1998 - Dec. 2002, the Dot-com bubble period.

3. The period Jan. 2003 - Dec. 2006, a period of strong increase in prices, whilst having a low realized volatility.

4. The period Jan. 2007 - Jan. 2013, the credit crisis and global recession period.

The Optionmetrics database also contains a zero curve table, which contains the zero-coupon continuously compounded interest rate curve on each day for different maturities, which is used to produce other figures in the database as well. The curve is derived from BBA LIBOR rates and settlement prices of CME Eurodollar futures. For the exact methodology of creating these curves, we refer to the Optionmetrics reference manual, provided by IvyDB. For our research, we extract a risk free rate from these curves. We want a risk free rate that is based on about a year of calendar days, therefore, we extract all interest rates based on a number between 275 and 400 calendar days from the database. From this set we extract the rates on each first trading day of the month, and then, if there are multiple rates for one trading day, we delete the latter ones, leaving the one based on the lesser number of calendar days. Following this procedure gives us an annualized risk free rate available on every trading day we consider. The observations of this rate are graphed in figure 6.3. In this picture we see that the interest rates between 1996 and 2000 are high, swinging between 5% and 6% per year. When the Dot-com bubble bursts, interest rates drop as well, to a low in mid 2003. After that, the rates increase again, until 2007, where the credit crisis is also the cause of them dropping again. In 2009 they drop below 1% per year and stay at an all time low up to the end of our dataset in 2013.

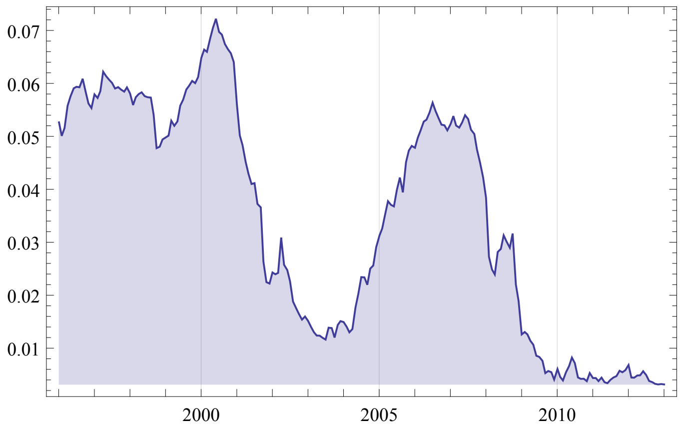


Figure 6.3: The annualized risk free rate process, observed monthly, denoted in decimals.

In the next sections we study the performance of our strategies that use the data described in this section for the underlying assets. In these sections we elaborate on the interpolated option data as well.

6.2 Strategy performance analysis with a single benchmark asset

In this section we study the performance of our derived strategy for the one dimensional case with historical market data. We first elaborate on how our analysis is performed and then test our one dimensional strategy on all four underlying assets, over all four periods mentioned in the previous section. We study the effect of the three options we have for the choice of β^{mkt} in Theorem 3.2.5. Then we implement the model extensions as proposed in section 3.3 and test them with the S&P 500. We shall also perform a regression on our strategy excess returns with the market returns and the other Fama & French factors as explanatory variables; the market capitalization factor (SMB) and the book-to-market factor (HML), which are available at Kenneth French's website. The tables on individual option analysis can be found in Appendix B.

6.2.1 Methodology

We trade on every first business day of the month, which thus will be our trading days. We assume that the underlying price process follows a geometric Brownian motion with $\mu = 0.08$ and σ equal to the realized volatility of the underlying at the current trading day. On every trading day, we have a range of 13 standardized European call options available from the Optionmetrics database, which all expire in 30 calendar days, thus, we will use a Δt of 1/12. The deltas of the options run from 0.20 to 0.80 in steps of 0.05 and the corresponding strike prices and option prices are provided in the Optionmetrics database. For each trading day we add the zero strike option with a price equal to the underlying price on the that trading day and an expiration of 30 calendar days to this set of options. Since expiration in 30 calendar days does not necessarily have to correspond to our next trading day, we construct a vector which contains the underlying price at expiration of the options, or, if they expire on a non-business day, the price on the closest preceding business day. This is thus the vector containing all values for S_T.

Given these values of S_T we calculate the realized excess return on all options and on the underlying itself in the same way as in section 5.1, but using the market dictated values for all variables.

We proceed with the calculation of the expected excess returns on all options on all trading days, the assembly of m on each trading day, and then with the calculation of the elements of the covariance matrix S and its assembly on each trading day. To avoid numerical errors due to plugging in zeros for these calculations with the zero strike option, we use equations (3.12) and (3.15) to calculate m_n and $cov(r_n^{opt}, r_n^{opt})$ on each trading day, where n indicates the zero strike option.

Then we calculate our portfolio weights on each trading day, in the same way as in section 5.1, but with the option of choosing β^{mkt} unequal to one, for which we explain the methodology at the end of this section. Then, we compute the excess return generated by our portfolio choice on each trading day and collect them in a vector R^{pf}. We compute the mean excess return, the standard deviation of the excess returns and the Sharpe ratio of our strategy, which is the fraction of these respective figures.

We then regress the strategy excess returns on the market excess returns. The market excess returns are in this case the realized excess returns on the S&P 500 during the trading periods, so we need to compute them once for every trading day and import them in the analyses on the other indices. We use the excess returns on the zero strike options on the S&P 500 for this cause. The regression gives us the estimates of alpha and beta, the standard errors of these estimates, their T-statistics and their p-values. From these statistics we can conclude whether the estimates of alpha and beta are statistically different from zero or not at a certain level of significance. We always use a 5% significance level, unless we mention otherwise. We also regress the realized portfolio excess returns on the market excess returns and on the other Fama & French factors, SMB and HML, and study whether the betas belonging to these factors are significant or not.

The remaining question is how to calculate β^{mkt} when one deviates from the assumptions that it is equal to one. We have two options; use a period prior to our first trading day to calculate it and then keep it constant throughout the entire period of analysis, or updating it dynamically by computing it each trading day, using a period prior to our first trading day to compute the β^{mkt} on our first trading day and then updating it on every new trading day, by adding the observed underlying and market excess returns from the previous trading day to the vectors of excess returns generated in the period before our first trade, used in the calculation of β^{mkt}.

The first option is straightforward. We choose the length of the period prior to our first trade and we compute the monthly excess returns on our underlying asset and on the S&P 500 during this period. We then take the covariance of these excess returns divided by the variance of the excess returns on the S&P 500, which gives us β^{mkt}. If we analyze our strategy running from 1996, then we can import the underlying data from Yahoo! Finance for the period. If we use periods that start later, we can use Optionmetrics data as well.

The second option is quite like the first, but we compute β^{mkt} on every trading day. We start by doing the same as with the fixed beta method; we use a period prior to our first trade to calculate excess returns on the underlying and the market and we then compute β^{mkt} from this data, which we use as β^{mkt} on our first trading day. Then, every next trading day we add the observed excess returns on the underlying and the market to our vectors used in the calculation of β^{mkt} and compute it again. In this way, the market beta of the underlying is adapted to events on the market.

6.2.2 Results

In this subsection we present the results generated with our one dimensional strategy. We analyze the strategy performance on our four indices during the four periods mentioned in section 6.1. We then analyze the effect of the different choices for β^{mkt} and we then test our model extensions.

We start with the analysis of our strategy performance during the entire available period, giving us 205 portfolio excess returns on the S&P 500 and NASDAQ 100, 193 for the Russell 2000 and 181 for the Dow Jones 1/100th. Initially, we work with β^{mkt} equal to one. We also study the individual available options and the performance of holding the underlying assets themselves. In table B.3 we list the mean realized option excess return for all options on all underlying and their standard deviations. A delta of one belongs to the zero strike option, as the sensitivity w.r.t. the underlying of the underlying itself is equal to one. In table B.7 we list the mean expected option excess return for all options on all underlying. Considering the size of the covariance matrix, we omit printing it.

As we can see in tables B.3 and B.7, options generally generate a negative excess return, except for the ones on the NASDAQ 100, which is very daft. The excess returns are greater in magnitude as the options are more OTM. One consistently has positive expected excess returns on the options. In figure 6.4 we see that the realized volatility of the S&P 500 is generally lower than the implied volatilities of the options written on it, and that how further an option is ITM, the higher its implied volatility is. The other indices produce similar figures.

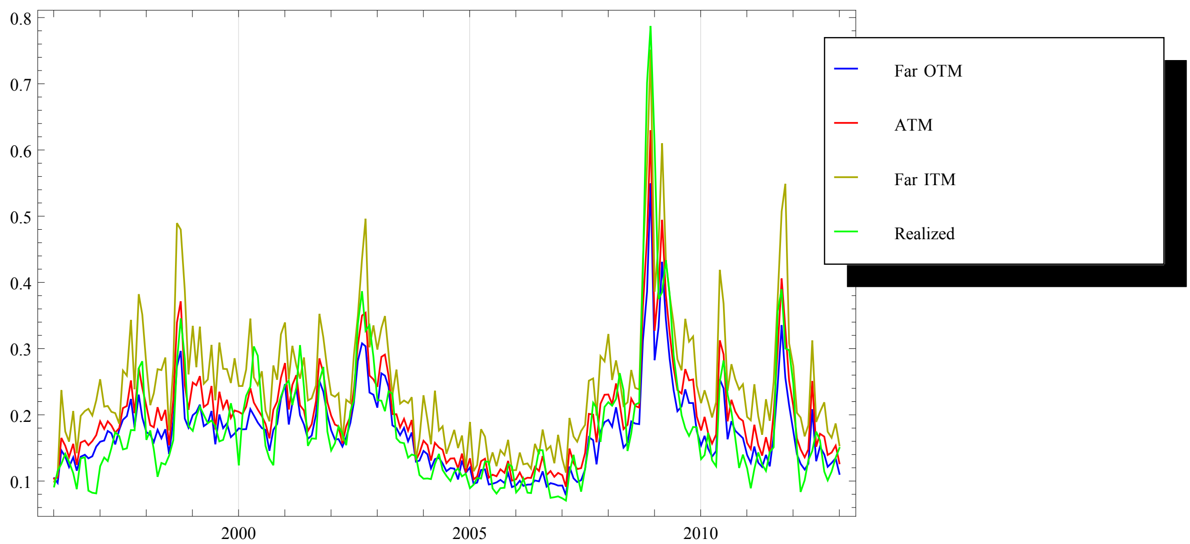
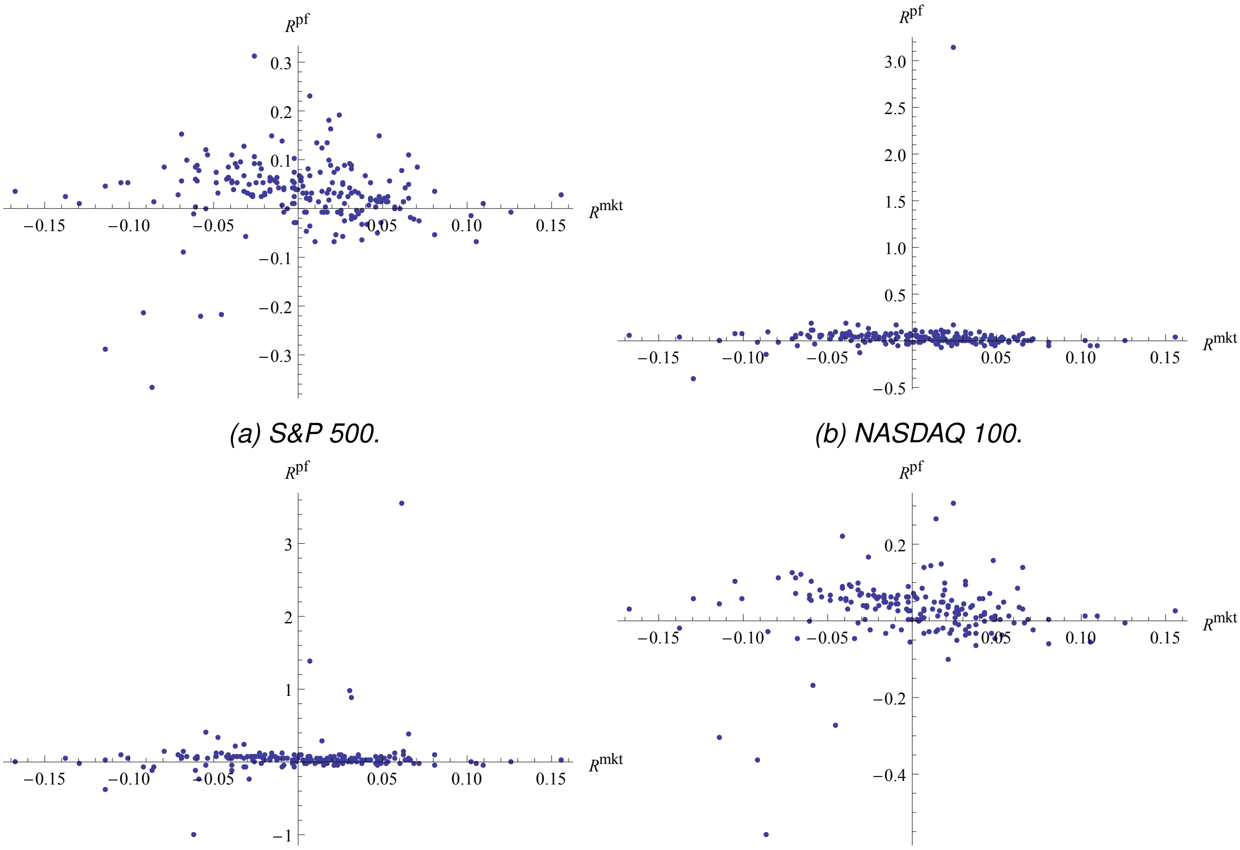


Figure 6.4: Monthly observed implied volatilities of the most far ITM and OTM options and the ATM option on the S&P 500 and the realized volatility of the S&P 500.

The strategy results can be found in table 6.1 and the regression results can be found in table 6.2. The regression results with the other Fama & French factors are listed in table 6.3 and we present scatter plots of the portfolio excess returns in figure 6.5. The mean of the absolute value of portfolio weights is given in table B.1. The results of holding the benchmarks are given in table 6.4.

In table 6.1 we find that our strategy generates quite substantial Sharpe ratios, which all



(a) S&P 500. *(b) NASDAQ 100.*

(c) Russell 2000. *(d) Dow Jones 1/100th.*

Figure 6.5: Scatter plots of portfolio excess returns against market excess returns, during the period of 1996/1998-2013.

	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
	monthly	annual	monthly	annual	monthly	annual	monthly	annual
Mean	0.032	0.384	0.042	0.504	0.065	0.780	0.028	0.336
Std. Dev.	0.072	0.249	0.225	0.779	0.306	1.060	0.085	0.294
Sharpe	0.448	1.552	0.185	0.641	0.212	0.734	0.324	1.122

Table 6.1: Monthly and annualized strategy performance statistics (mean realized excess returns, their standard deviations and Sharpe ratios) for all underlying during the period 1996/1998-2013.

outperform their own benchmark, as we can conclude from table 6.4, since the Sharpe ratios obtained with the strategy are substantially higher. The standard deviations of the excess returns on the NASDAQ 100 and the Russell 2000 are substantially higher than the 5% we were aiming at, but as we can see in the scatter plots in figure 6.5, this is mostly due to one single, enormous outlier. This outlier is generated at $t = 0$ and caused by a numerical error, which places a abnormally large weight on the option with $\Delta = 0.80$. If the outliers would be removed, then the standard deviations drop to a level much closer to the one we aim at and the Sharpe ratios will thus increase. The most substantial excess returns (positive and negative)

	Estimate	Standard error	T-statistic	p-value
α	0.032	0.005	6.387	$1.135 \cdot 10^{-19}$
β	$2.794 \cdot 10^{-4}$	0.105	0.003	0.998

(a) S&P 500

	Estimate	Standard error	T-statistic	p-value
α	0.042	0.016	2.638	0.009
β	0.032	0.329	0.097	0.923

(b) NASDAQ 100

	Estimate	Standard error	T-statistic	p-value
α	0.064	0.022	2.906	0.004
β	0.736	0.451	1.633	0.104

(c) Russell 2000

	Estimate	Standard error	T-statistic	p-value
α	0.027	0.006	4.350	$2.284 \cdot 10^{-5}$
β	0.081	0.130	0.627	0.532

(d) Dow Jones 1/100th

Table 6.2: Regression with the market excess returns as explanatory variable, during the period of 1996/1998-2013.

that we can see in the scatter plots are mostly generated by very sudden and substantial increases and decreases in the underlying price.

Table 6.2 shows us that all estimated alphas are significantly different from zero at a 5% significance level, so our strategy generates significant alphas. All estimated betas are insignificantly different from zero, even when they seem large, so our beta-hedge works and our strategy excess returns are explained only by the strategy itself and not by market movements. In table 6.3 we find that for all underlying also the other two Fama & French factors are of insignificant influence, which even strengthens our claim that the strategy excess returns are explained by only the strategy itself.

In table B.1 we find that the strategy places almost all weight in the portfolio at the zero strike option and the three most far ITM options, just as in the simulation. This is the case for all underlying, so we decide to work with only these four options in the rest of the analyses. This makes the strategy more simple for the hedge fund manager, since he has to perform less calculations. We first present the results for the complete available period again, and we then perform the rest of the analyses that we proposed in the introduction of this chapter.

We first discuss the strategy performance of the strategy that uses only the three most far ITM options and the zero strike option during the complete available period 1996/1998-2013. We keep all parameters equal to the ones used in the previous analysis. The separate option analysis and the underlying performance during this do not change, as we only change

	Estimate	Standard error	T-statistic	p-value
α	0.032	0.005	6.310	1.748$\cdot 10^{-9}$
β	0.012	0.107	0.114	0.909
SMB	-0.074	0.148	-0.497	0.620
HML	0.051	0.157	0.323	0.747

(a) S&P 500.

	Estimate	Standard error	T-statistic	p-value
α	0.043	0.016	2.680	0.008
β	0.078	0.335	0.234	0.815
SMB	-0.493	0.466	-1.059	0.291
HML	0.046	0.493	0.094	0.925

(b) NASDAQ 100.

	Estimate	Standard error	T-statistic	p-value
α	0.067	0.022	3.000	0.003
β	0.709	0.459	1.550	0.124
SMB	-0.410	0.641	-0.635	0.526
HML	-0.594	0.674	-0.882	0.380

(c) Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.027	0.006	4.235	3.66$\cdot 10^{-5}$
β	0.104	0.132	0.785	0.433
SMB	-0.031	0.186	-0.168	0.867
HML	0.195	0.191	1.023	0.308

(d) Dow Jones 1/100th.

Table 6.3: Regression results with respect to the other Fama & French factors during the period of 1996/1998-2013.

	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
Mean	0.002	0.008	0.004	9.248$\cdot 10^{-4}$
Std. Dev.	0.048	0.081	0.063	0.046
Sharpe	0.039	0.096	0.068	0.020

Table 6.4: Underlying mean realized monthly excess returns, their standard deviations and Sharpe ratios during the period 1996/1998-2013.

the strategy, not the products themselves. Therefore, we do not reprint those tables. We omit printing the scatter plots as well. The strategy results can be found in table 6.5 and the regression results are given in table 6.6. Table 6.7 contains the regression results with the other Fama & French factors.

In table 6.5 we see that the generated Sharpe ratios are very close to those generated with the full range of available options. The high standard deviations on the NASDAQ 100 and the Russell 2000 are again caused by a single outlier at the very start of the period. If we look at

	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
	monthly	annual	monthly	annual	monthly	annual	monthly	annual
Mean	0.033	0.396	0.042	0.504	0.065	0.780	0.028	0.336
Std. Dev.	0.072	0.249	0.222	0.769	0.309	1.070	0.087	0.301
Sharpe	0.468	1.621	0.187	0.648	0.209	0.724	0.318	1.102

Table 6.5: Monthly and annualized strategy performance statistics (mean realized excess returns, their standard deviations and Sharpe ratios) for all underlying during the period 1996/1998-2013, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.033	0.005	6.651	2.637$\cdot 10^{-10}$
β	0.109	0.104	1.045	0.297

(a) S&P 500

	Estimate	Standard error	T-statistic	p-value
α	0.041	0.016	2.648	0.009
β	0.146	0.325	0.450	0.653

(b) NASDAQ 100

	Estimate	Standard error	T-statistic	p-value
α	0.063	0.022	2.870	0.005
β	0.841	0.454	1.851	0.066

(c) Russell 2000

	Estimate	Standard error	T-statistic	p-value
α	0.027	0.006	4.278	3.060$\cdot 10^{-5}$
β	0.198	0.132	1.500	0.135

(d) Dow Jones 1/100th

Table 6.6: Regression with the market excess returns as explanatory variable, during the period of 1996/1998-2013, using less available options.

figure 6.6 on page 69, we see that for these two indices an abnormal large, negative weight is placed on the most far ITM option, causing the outlier. We are not sure about why this happens, but we have a strong suspicion that it is a numerical issue in the covariance matrix assembly on that trading day.

Considering table 6.6 we see that the estimated alphas are all still significant and that all betas are still insignificant. Some betas are higher than the ones generated with the full range of options. This is because a relative big weight is put on the zero strike option and there are less other options available to put more weight on. There are mispredictions in m and S, which make the beta-hedge imperfect. The relative bigger weight on the zero strike option thus makes the portfolio more sensitive to the market.

From the results in table 6.7 we conclude that also in this setting, none of the other Fama & French factors is of significant influence on our strategy excess returns. We thus conclude

	Estimate	Standard error	T-statistic	p-value
α	0.033	0.005	6.649	2.717$\cdot 10^{-10}$
β	0.125	0.106	1.176	0.241
SMB	-0.163	0.148	-1.104	0.271
HML	0.017	0.156	0.111	0.912

(a) S&P 500.

	Estimate	Standard error	T-statistic	p-value
α	0.042	0.016	2.693	0.008
β	0.189	0.330	0.573	0.567
SMB	-0.496	0.459	-1.080	0.281
HML	0.013	0.486	0.027	0.978

(b) NASDAQ 100.

	Estimate	Standard error	T-statistic	p-value
α	0.066	0.022	2.958	0.003
β	0.808	0.462	1.748	0.082
SMB	-0.355	0.646	-0.550	0.583
HML	-0.608	0.680	-0.894	0.372

(c) Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.027	0.006	4.160	4.965$\cdot 10^{-5}$
β	0.225	0.134	1.682	0.094
SMB	-0.040	0.188	-0.213	0.832
HML	0.239	0.194	1.231	0.220

(d) Dow Jones 1/100th.

Table 6.7: Regression results with respect to the other Fama & French factors during the period of 1996/1998-2013, using less available options.

that our strategy still performs well with this smaller range of available options.

We continue with our analysis of strategy performance during our second period of interest; the period running from January 1996/1998 to December 2002, the Dot-com bubble period, giving us 84 observations of excess returns for the S&P 500 and NASDAQ 100, 72 for the Russell 2000 and 60 for the Dow Jones 1/100th. For this period, we again present an analysis of the separate options and the performance of holding the underlying, as well as the strategy performance results and regressions.

In table B.4 we see that during this period, one generally loses money when one simply holds an option. Only the the NASDAQ 100 produces positive mean realized option excess returns. Opposed to the complete timespan, we now find negative expected option excess returns for every underlying in table B.8.

The strategy performance statistics are found in table 6.8, in which we can see that the

	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
	monthly	annual	monthly	annual	monthly	annual	monthly	annual
Mean	0.038	0.456	0.058	0.696	0.127	1.524	0.034	0.408
Std. Dev.	0.074	0.256	0.339	1.174	0.498	1.725	0.078	0.270
Sharpe	0.512	1.774	0.172	0.596	0.254	0.880	0.434	1.503

Table 6.8: Monthly and annualized strategy performance statistics (mean realized excess returns, their standard deviations and Sharpe ratios) for all underlying during the period 1996/1998-2002, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.038	0.008	4.669	$1.169 \cdot 10^{-5}$
β	0.063	0.168	0.374	0.710

(a) S&P 500

	Estimate	Standard error	T-statistic	p-value
α	0.058	0.037	1.565	0.121
β	0.038	0.763	0.050	0.961

(b) NASDAQ 100

	Estimate	Standard error	T-statistic	p-value
α	0.122	0.059	2.081	0.041
β	1.379	1.204	1.145	0.256

(c) Russell 2000

	Estimate	Standard error	T-statistic	p-value
α	0.033	0.010	3.261	0.002
β	0.090	0.222	0.406	0.686

(d) Dow Jones 1/100th

Table 6.9: Regression with the market excess returns as explanatory variable, during the period of 1996/1998-2002, using less available options.

Sharpe ratios produced are more or less equal to the ones that we get over our entire timespan. Hence, in this bubble period with high volatility, our strategy seems to perform just as well as over a period with high and low volatilities. The performance of our strategy is also only explained by the strategy itself, and not by market movements, as we can see from the fact that all betas in table 6.9 are insignificant. We see that the generated alphas are significant for all underlying, except for the alpha generated by our strategy on the NASDAQ 100. We must also note that the p-value of the alpha generated on the Russell 2000 is 0.041, which makes it significant at a 5% significance level, but just barely. These phenomena are caused by the high standard deviations on our strategy excess returns on the NASDAQ 100 and Russell 2000, which are each again caused by a single outlier at time zero. If these outliers would not be there, then the standard deviations would drop, the mean realized excess returns would rise, making the Sharpe ratios higher as well, and would make the estimated alpha significant in

	Estimate	Standard error	T-statistic	p-value
α	0.038	0.008	4.504	$2.244 \cdot 10^{-5}$
β	0.114	0.217	0.526	0.601
SMB	-0.003	0.217	-0.015	0.988
HML	0.108	0.270	0.399	0.691

(a) S&P 500.

	Estimate	Standard error	T-statistic	p-value
α	0.058	0.038	1.553	0.124
β	0.035	0.784	0.045	0.964
SMB	-0.250	1.774	-0.141	0.888
HML	-0.503	1.494	-0.337	0.737

(b) NASDAQ 100.

	Estimate	Standard error	T-statistic	p-value
α	0.113	0.059	1.901	0.062
β	1.298	1.227	1.058	0.294
SMB	2.649	2.808	0.943	0.349
HML	-2.510	2.304	-1.089	0.280

(c) Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.033	0.011	3.167	0.002
β	0.091	0.225	0.403	0.688
SMB	-0.033	0.465	-0.071	0.943
HML	-0.245	0.379	-0.647	0.520

(d) Dow Jones 1/100th.

Table 6.10: Regression results with respect to the other Fama & French factors during the period of 1996/1998-2002, using less available options.

	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
Mean	$-1.091 \cdot 10^{-4}$	0.008	$-8.762 \cdot 10^{-4}$	0.003
Std. Dev.	0.490	0.108	0.061	0.062
Sharpe	-0.002	0.077	-0.014	0.041

Table 6.11: Underlying mean realized monthly excess returns, their standard deviations and Sharpe ratios during the period 1996/1998-2002.

case of the NASDAQ 100. In table 6.10 we see that none of the other Fama & French factors is of significant influence in explaining our strategy excess returns on all of the underlying. If we compare the underlying performance during this period, given in table 6.11, with our strategy performance, we see that our strategy generates a higher monthly Sharpe ratio on all of the underlying than the Sharpe ratios obtained by holding the underlying themselves. Hence, our strategy outperforms each benchmark asset during this period, even with the high standard deviations on the NASDAQ 100 and Russell 2000.

Next, we perform the analysis of our strategy performance during our third period of interest; the period running from January 2003 to December 2006, the post-bubble period, in which we have 48 observations of excess returns for all underlying. For this period we again present an analysis of the separate options and the performance of the strategy, as well as the regressions over the strategy excess returns and the analysis of the performance of the underlying assets.

	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
	monthly	annual	monthly	annual	monthly	annual	monthly	annual
Mean	0.035	0.420	0.025	0.300	0.022	0.264	0.026	0.312
Std. Dev.	0.045	0.156	0.040	0.139	0.037	0.128	0.041	0.142
Sharpe	0.778	2.695	0.627	2.172	0.612	2.120	0.625	2.165

Table 6.12: Monthly and annualized strategy performance statistics (mean realized excess returns, their standard deviations and Sharpe ratios) for all underlying during the period 2003-2006, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.035	0.007	5.424	2.092$\cdot 10^{-6}$
β	-0.152	0.170	-0.895	0.375

(a) S&P 500

	Estimate	Standard error	T-statistic	p-value
α	0.024	0.006	3.962	2.567$\cdot 10^{-4}$
β	0.252	0.226	1.115	0.271

(b) NASDAQ 100

	Estimate	Standard error	T-statistic	p-value
α	0.022	0.005	3.925	2.882$\cdot 10^{-4}$
β	0.149	0.207	0.720	0.475

(c) Russell 2000

	Estimate	Standard error	T-statistic	p-value
α	0.024	0.006	3.996	2.306$\cdot 10^{-4}$
β	0.191	0.231	0.825	0.414

(d) Dow Jones 1/100th

Table 6.13: Regression with the market excess returns as explanatory variable, during the period of 2003-2006, using less available options.

As we can see in table B.5, for all underlying one again loses money when holding plain options, whilst one expects a positive excess return on them, as we can see in table B.9. The low realized volatility of the underlying this period makes the options relatively cheap, while there is an upward trend in the price of all underlying. This results in very high Sharpe ratios generated by the strategy on all underlying assets, which we can find in table 6.12. It is of no

	Estimate	Standard error	T-statistic	p-value
α	0.034	0.007	4.991	9.935$\cdot 10^{-6}$
β	-0.151	0.173	-0.877	0.385
SMB	-0.119	0.204	-0.586	0.561
HML	-0.202	0.225	-0.900	0.373

(a) S&P 500.

	Estimate	Standard error	T-statistic	p-value
α	0.025	0.006	4.195	1.301$\cdot 10^{-4}$
β	0.053	0.237	0.222	0.825
SMB	-0.164	0.254	-0.645	0.522
HML	0.550	0.243	2.263	0.029

(b) NASDAQ 100.

	Estimate	Standard error	T-statistic	p-value
α	0.022	0.006	4.038	2.125$\cdot 10^{-4}$
β	0.014	0.222	0.065	0.949
SMB	-0.144	0.239	-0.602	0.551
HML	0.377	0.228	1.654	0.105

(c) Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.024	0.006	4.002	2.374$\cdot 10^{-4}$
β	0.022	0.246	0.090	0.929
SMB	0.103	0.264	0.389	0.700
HML	0.421	0.252	1.667	0.103

(d) Dow Jones 1/100th.

Table 6.14: Regression results with respect to the other Fama & French factors during the period of 2003-2006, using less available options.

	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
Mean	0.002	0.004	0.003	0.006
Std. Dev.	0.039	0.065	0.043	0.026
Sharpe	0.056	0.064	0.065	0.243

Table 6.15: Underlying mean realized monthly excess returns, their standard deviations and Sharpe ratios during the period 2003-2006.

surprise that all generated alphas, listed in table 6.13, are significant. In the same table we find all beta to be insignificant again as well. Adding the other Fama & French factors to the regression leads to a single significant observation; in table 6.14 we find the book-to-market factor to be of significant influence on the strategy results generated on the NASDAQ 100, at a 5% significance level. The other factors are of insignificant influence on all underlying. We again find that the strategies outperform their benchmarks, as the Sharpe ratios listed in table 6.15 are lower for each underlying asset than the Sharpe ratios generated by our strategy.

Last, we analyze the performance of our strategy during our fourth period of interest; the period running from January 2007 to January 2013, the period of the credit crisis and the global recession, giving us 72 observations of excess returns for each underlying. We again present an analysis on the separate options and their expectations, as well as a strategy performance analysis, an analysis of the regressions over the strategy excess returns, and an analysis of the performance of the underlying assets.

	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
	monthly	annual	monthly	annual	monthly	annual	monthly	annual
Mean	0.028	0.336	0.032	0.384	0.019	0.228	0.030	0.360
Std. Dev.	0.053	0.184	0.050	0.173	0.041	0.142	0.065	0.225
Sharpe	0.523	1.812	0.638	2.210	0.456	1.580	0.458	1.587

Table 6.16: Monthly and annualized strategy performance statistics (mean realized excess returns, their standard deviations and Sharpe ratios) for all underlying during the period 2007-2013, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.028	0.006	4.571	2.032$\cdot 10^{-5}$
β	0.612	0.431	1.420	0.160

(a) S&P 500

	Estimate	Standard error	T-statistic	p-value
α	0.032	0.006	5.434	7.557$\cdot 10^{-7}$
β	0.151	0.102	1.478	0.144

(b) NASDAQ 100

	Estimate	Standard error	T-statistic	p-value
α	0.019	0.005	3.861	2.492$\cdot 10^{-3}$
β	0.108	0.084	1.279	0.205

(c) Russell 2000

	Estimate	Standard error	T-statistic	p-value
α	0.030	0.008	3.868	2.432$\cdot 10^{-4}$
β	0.142	0.134	1.572	0.294

(d) Dow Jones 1/100th

Table 6.17: Regression with the market excess returns as explanatory variable, during the period of 2007-2013, using less available options.

Again, we find that one generally incurs a negative excess return on holding options on every underlying asset, but at a lower standard deviation than the ones in other periods, as can be found in table B.6. In table B.10 we find that the expected excess return on options is positive again, and generally of the same magnitude as during other periods. In table 6.16

	Estimate	Standard error	T-statistic	p-value
α	0.029	0.006	4.624	1.734$\cdot 10^{-5}$
β	0.692	0.432	1.603	0.114
SMB	0.133	0.148	0.903	0.370
HML	-0.114	0.160	-0.714	0.478

(a) S&P 500.

	Estimate	Standard error	T-statistic	p-value
α	0.032	0.006	5.376	1.010$\cdot 10^{-6}$
β	0.130	0.116	1.120	0.267
SMB	-0.039	0.299	-0.133	0.895
HML	0.143	0.242	0.591	0.557

(b) NASDAQ 100.

	Estimate	Standard error	T-statistic	p-value
α	0.020	0.005	4.004	1.566$\cdot 10^{-4}$
β	0.111	0.095	1.168	0.247
SMB	-0.269	0.239	-1.128	0.263
HML	0.200	0.199	1.004	0.319

(c) Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.030	0.009	3.763	3.524$\cdot 10^{-4}$
β	0.132	0.154	0.856	0.395
SMB	0.067	0.384	0.173	0.863
HML	0.003	0.320	0.011	0.992

(d) Dow Jones 1/100th.

Table 6.18: Regression results with respect to the other Fama & French factors during the period of 2007-2013, using less available options.

	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
Mean	-0.001	0.004	-4.451$\cdot 10^{-4}$	9.610$\cdot 10^{-4}$
Std. Dev.	0.014	0.065	0.019	0.058
Sharpe	-0.084	0.064	-0.023	0.017

Table 6.19: Underlying mean realized monthly excess returns, their standard deviations and Sharpe ratios during the period 2007-2013.

we can see that our strategy generates substantial Sharpe ratios again, even in this period of higher volatility. Table 6.17 shows that all generated alpha are again undoubtedly significant and that all betas are insignificant. The other Fama & French factors are of no significant influence on the strategy excess returns, as we can find in table 6.18. Also over this period we find our strategy to outperform the benchmarks, comparing its Sharpe ratios to the smaller Sharpe ratios generated by holding the benchmarks, which are listed in table 6.19.

Considering the analysis of our strategy performance during the four periods, we conclude that our strategy performs well, as it generates substantial Sharpe ratios and outperforms holding the benchmarks in every period. As all generated betas in all periods are insignificantly different from zero at a 5% significance level, we conclude that the generated excess returns by the strategy are all explained by only the strategy itself, and their alphas to be significantly different from zero at a 5% significance level (except for the one on the NASDAQ 100 during 2003-2006 as result of a numerical error). In almost all cases we find the other Fama & French factors to be insignificant in the regression as well, which strengthens our claim of that the strategy excess returns are solely explained by the strategy itself. We note that the standard deviation of the excess returns generated by the strategy is sometimes far from 5% per month, mostly due to a single outlier, of which the cause cannot be explained by market crashes, etc, but is more likely to be generated by numerical issues in the analysis. When we compare our obtained Sharpe ratios with Sharpe ratios of previous research, they look good as well. Eraker [3] reports Sharpe ratios near one in yearly figures for portfolios with short positions in call options on the S&P 500 during the period 1996-2013 and Kapadia and Szado [5] report monthly Sharpe ratios between 0.130 and 0.160 on buy-write strategies with call options on the Russell 2000 during the period 1996-2006, but we do note that they use other models and periods of time for their analyses, so we cannot easily compare the results, but we do get an idea of the magnitude of Sharpe ratios found in previous research. Therefore, we conclude that our strategy generally performs outstanding in this one dimensional setting during these historical periods, even with the assumption of a fixed growth rate μ and the assumption that the underlying has a beta with respect to the market of one.

We now study whether the results improve when one uses a market beta unequal to one. As we use a more realistic beta, we expect that the estimated beta obtained in the regression (1.1) will be closer to zero, especially in the case where we update β^{mkt} dynamically with the extra information available on the market every next trading day. We perform this analysis over the period of January 2001 - January 2013 (resulting in 145 observations of strategy and market excess returns), as this enables us to use data from the Optionmetrics database in the periods prior to our first trade. For each underlying we choose the period to be the period from when we first have data on the underlying to December 2000, hence, for the NASDAQ 100 this is 1996-2000, for the Russell 2000 this is 1997-2000 and for the Dow Jones 1/100th this is 1998-2000. In this analysis we use the same parameters as in our previous analyses, and we work with the three most far ITM options and the zero strike option available in our portfolio selection.

In table 6.20 we list the results of the analysis; for every asset we present the mean monthly strategy excess return, its standard deviation, its Sharpe ratio, and the estimated alpha and beta in the regression (1.1). In the first column we present the results with a market beta equal to one, as a reference. In the second column we present the results generated with a fixed market beta, calculated over the period prior to our first trade. In the last column we list the results generated with a dynamically updated market beta. All the reported alphas are significant at a 5% significance level and all reported betas are insignificant at the same level of significance. As we can see, all results are quite alike. For none of the underlying assets does the Sharpe ratio increase significantly when using different market betas than one. Also, we do not see a real improvement in the estimated beta when using different market betas than one. For the NASDAQ 100 and the Russell 2000 they are actually worse and for the Dow Jones 1/100th we see only a slight improvement. However, they are all insignificant, also in the case of choosing a market beta equal to one. Thus, we conclude that the extra trouble of making the beta hedge more realistic is not worthwhile, as one achieves practically the same results when choosing the market beta of each underlying equal to one.

	$\beta^{mkt} = 1$	$\beta^{mkt} = 4.739$	Dynamic β^{mkt}
Mean.	0.029	0.030	0.029
Std. Dev.	0.063	0.063	0.063
Sharpe	0.463	0.469	0.463
α	0.029	0.030	0.029
β	0.102	0.189	0.145

(a) NASDAQ 100

	$\beta^{mkt} = 1$	$\beta^{mkt} = 0.744$	Dynamic β^{mkt}
Mean.	0.025	0.024	0.025
Std. Dev.	0.054	0.054	0.054
Sharpe	0.451	0.447	0.451
α	0.025	0.024	0.025
β	-0.004	-0.055	-0.018

(b) Russell 2000

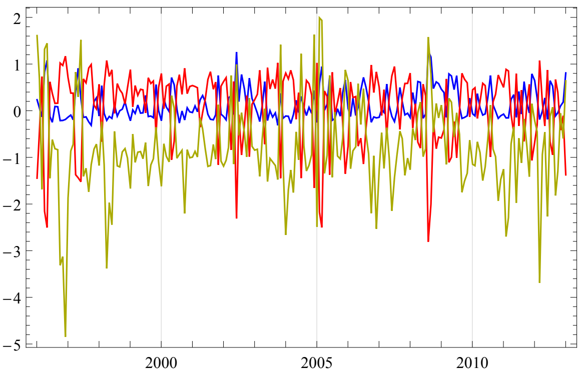
	$\beta^{mkt} = 1$	$\beta^{mkt} = 0.333$	Dynamic β^{mkt}
Mean.	0.025	0.023	0.025
Std. Dev.	0.087	0.078	0.087
Sharpe	0.283	0.300	0.283
α	0.025	0.023	0.025
β	0.177	-0.157	0.157

(c) Dow Jones 1/100th

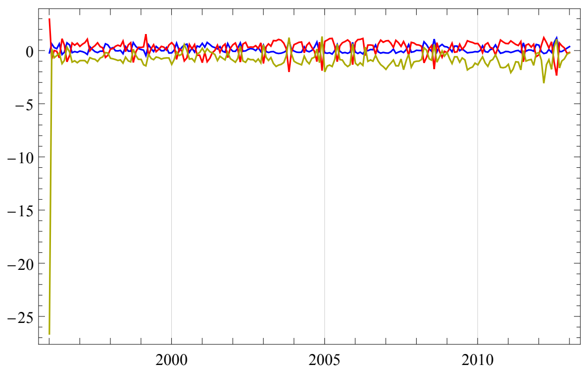
Table 6.20: Strategy performance results with different choices for β^{mkt} over the period 2001-2013, formulated in monthly terms.

Next, we study our model extensions as proposed in section 3.3. We test them on the S&P 500 during the period of 1996-2013 and we start with implementing the dynamic growth rate. This is fairly easy; the historical volatility of all underlying and the risk free rate are available in the Optionmetrics database, so the only thing that changes in our methodology is that we construct a vector $\mu(t)$ beforehand, as defined in equation (3.18), with the available data on each trading day. We then replace our constant μ in the calculations each trading day with the entry of this vector that corresponds to the trading day for which we do the calculations. We again use the three most far ITM options and the zero strike options, and we use a market beta of one. This way, a mean strategy excess return of 0.035 (in monthly terms) is reached at a standard deviation of 0.075, resulting in a Sharpe ratio of 0.469. The estimated alpha is 0.035 at a T-statistic of 6.676 and a p-value of $2.296 \cdot 10^{-10}$, which is thus significant at a 5% significance level. The estimated beta is 0.150 at a T-statistic of 1.366 and a p-value of 0.173, which is thus insignificant at the same level of significance. We compare the results to table 6.5, which contains the results on the S&P 500 over the same period and number of available options, but with a constant μ. The Sharpe ratio in this table for the S&P 500 is 0.468, so there is no significant increase when using a dynamic μ. In table 6.6 we note that the reported alpha of 0.033 also differs but marginally from the one obtained with a dynamic μ and that the beta of 0.109 is actually lower with a fixed μ. Both betas are insignificant.

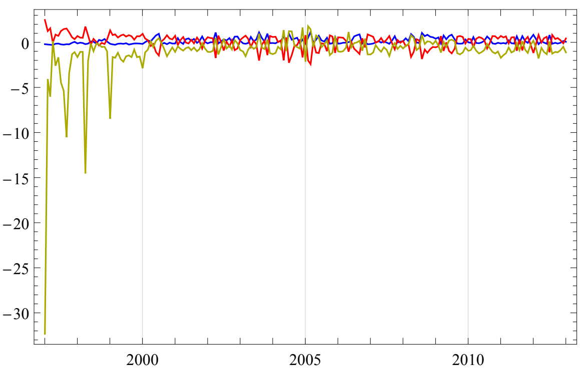
From this we conclude that the extension of a dynamic general drift in the geometric Brownian motion is an easy to implement model extension that does not bring a significant improvement to the results. As practically the same results are obtained as when using $\mu = 0.08$, we recommend to keep the growth rate of the geometric Brownian motion constant.



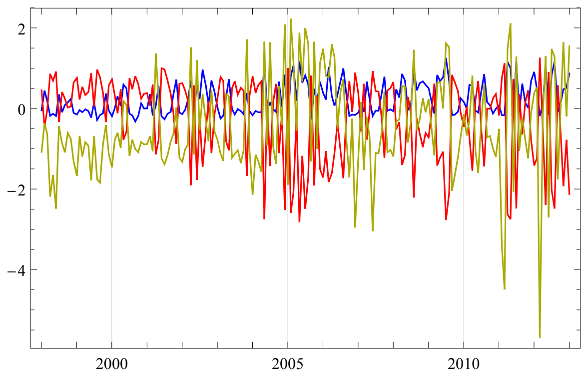
(a) S&P 500.



(b) NASDAQ 100.



(c) Russell 2000.



(d) Dow Jones 1/100th.

Figure 6.6: Weight dynamics in the portfolio over the period of 1996/1998-2013. The blue line represents the weight of the option with a delta of 0.7, the red that of one with delta 0.75 and the yellow that of one with delta 0.8. The weight placed on the zero strike option is substantially larger at every point in time, and for scaling effects is left out of the figures.

The second, and last, model extension we suggested, was to implement trading costs in an easy fashion. If the strategy would constantly buy one type of option and would sell another, then we could adjust the prices of these products in hindsight and perform the analyses again. However, when we look at figure 6.6, which describes the weight dynamics of the portfolio through time (with constant μ, $\beta^{mkt} = 1$ and only the most far ITM and the zero strike options available, over the period of 1996/1998-2013), we can see that for every underlying no type of option is solely bought or shorted in the period. The other investigated periods show similar dynamics, so unfortunately, we cannot simply implement the trading costs in the way we suggested. Implementing them otherwise is beyond the scope of this thesis, but we do recommend this addition for further research.

6.3 Strategy performance analysis with multiple benchmark assets

In this section we study the performance of our derived strategy for the multidimensional case with historical market data. We perform the analyses in a two dimensional setting with the S&P 500 and the Russell 2000 as underlying assets, in which we again use the S&P 500 as a representative of the market itself. We first elaborate on our methodology and we then perform our analyses during the same periods as in the previous section, but with a slight alteration to the first and second periods; since we now need data available on both underlying, we will let the first trading day be January 3rd 2000. As we have concluded in the previous session that the results do not improve with a dynamic μ or a market beta unequal to one, so we keep μ constant for both assets and the market beta equal to one. We elaborate on the strategy performance in a same kind of way as in the previous section and we again present the estimated alpha and beta in the regression (1.1), as well as the regression with the other Fama & French factors. We then compare the results obtained with our strategy to a naive strategy that divides ones wealth in two equal parts and uses each part to invest on a single underlying asset with our one dimensional strategy. If our multidimensional strategy works well, we expect to see an increase in the Sharpe ratio with respect to this naive strategy. We also test our strategy on two combinations of three separate assets in the Dow Jones index over the period of January 2001 to January 2013; we consider the Walt Disney Corporation, the McDonald's Corporation and Walmart Stores Incorporated, which are the three of the best performing assets in the Dow Jones index. The tables on individual option analysis can be found in Appendix B.

6.3.1 Methodology

Our methodology in this case is a lot like the one in the singe underlying case of the previous section and the one used in the simulations in section 5.2. We again trade on every first business day of the month. We assume both underlying follow a geometric Brownian motion with $\mu_1 = \mu_2 = 0.08$, where we use 1 to refer to the S&P 500 and 2 to refer to the Russell 2000, and for σ_1 and σ_2 we use the observed realized volatilities on the S&P 500 and the Russell 2000 respectively on each trading day. We determine the correlation between the two geometric Brownian motions by taking the daily log returns on both assets during a period prior to our first trade and compute the correlation between them. We keep this correlation constant throughout the whole analysis. One could calculate this correlation dynamically each trading day, based on a fixed period prior to the trading day, but whilst experimenting with this, we ran into numerical issues with the covariance matrix, so we chose to keep it constant.

We again use the monthly observed closing prices of the indices for $S_t^{(i)}$ and $S_T^{(i)}$, $i = 1, 2$, and we consider the same 13 standardized European call options available on each trading day for each of the underlying available in the Optionmetrics database, and add the zero strike option. Each option has an expiry of 30 days, so $\Delta t = 1/12$. We construct the vector $S_T^{(i)}$ for both assets in the same way as in the one dimensional case and we calculate the realized excess return on all options and the underlying themselves for each trading day, using the market data of prices, volatilities and interest rates. We combine the realized excess returns on options and the underlying themselves again in vectors, as in the simulations of section 5.2.

We then compute the entries of m and S as in section 4.1 and assemble them for every trading day. We then assemble our portfolio each trading day with weights given by Theorem 4.1.2. To prevent overcompensation in the hedging effect, we generally only hedge with respect to the S&P 500 (or in the case of the Dow Jones single components; we shall only hedge with respect to one of the assets). In this case, β_1^{mkt} is obviously equal to one and β_2^{mkt} is left out of the equation.

We then compute the realized excess return that follows from the portfolio choice each trading day by multiplying the weights and the realized excess returns vector, collect them in a vector R^{pf} and then compute the mean excess return, its standard deviation and its Sharpe ratio. Then we perform the regressions with the market excess returns as explanatory variable and the complete set of Fama & French factors as explanatory variables, to see whether the regression alphas are significantly different from zero and the regression betas insignificantly different from zero. We again always use a significance level of 5%, unless we mention otherwise.

When using the Dow Jones single components there is one minor adjustment to the methodology; thus far we have worked with actual closing prices on the underlying, as there were no reported splits and dividends in the Optionmetrics database for the underlying indices, however, these phenomena do occur with the single components of the Dow Jones. Therefore, we first adjust the closing prices of these stocks and the option prices and strikes for splits. Usually stock option prices are not adjusted for cash settled dividends of less than 10%, so we do not adjust for dividends.

The methodology of our naive strategy is simple; for the same period we calculate the excess returns generated by our one dimensional strategy on each underlying. On each trading day we then take half of each excess returns (or the mean of all generated excess returns, when considering more than two underlying) and add them to form a new excess return. We then collect these new excess returns in a vector again and calculate the mean, standard deviation and Sharpe ratio of this vector, and we perform the regressions with this vector and the market excess returns to estimate alpha and beta. If the underlying assets have zero correlation, this approach would give us the optimal Sharpe ratio, but when there is correlation, we expect to see improvements when using our multidimensional trading strategy.

6.3.2 Results

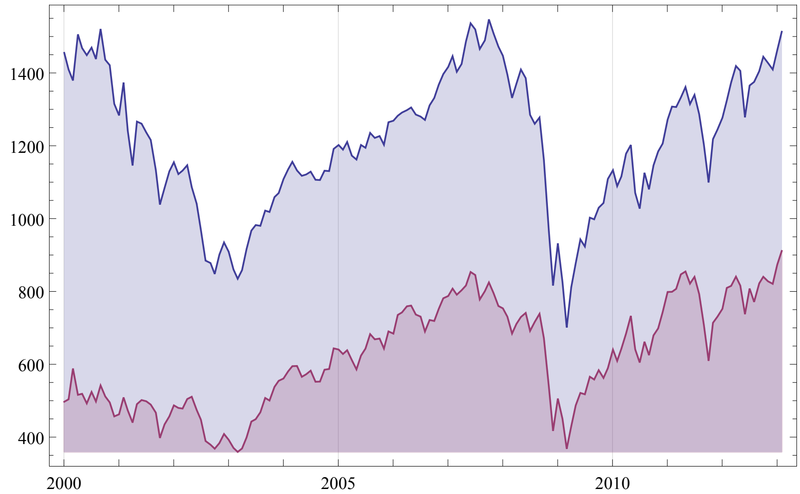
In this subsection we present the results generated with our multidimensional strategy, applied to two dimensional cases. We analyze the strategy performance using the S&P 500 and the Russell 2000 during the periods 2000-2013, 2000-2002, 2003-2006 and 2007-2013. We then compare these results to the results of applying the naive strategy on those two underlying. Then we analyze the strategy performance with two combinations of three components of the Dow Jones during the period 2001-2013 and compare them to the results of the naive strategy with them as well. The correlation between the log returns of both underlying is calculated from daily log returns during a period of January 1995 to the start of the period we study, using data imported from Yahoo! Finance. In figure 6.7 we visualize the monthly observed price processes of the S&P 500 and the Russell 2000 from 2000-2013 and their realized volatilities during that period. We see that the price processes are quite alike, but that the S&P 500 has a higher price throughout. Also, the volatility processes are alike. We thus expect a relatively large correlation between the underlying log returns.

We start with the analysis of our strategy performance during the period January 2000 - January 2013, giving us 157 excess returns generated by the strategy and a correlation of 0.763. We work with the parameters as stated in the methodology section and we first consider 13 available options on each trading day for both underlying.

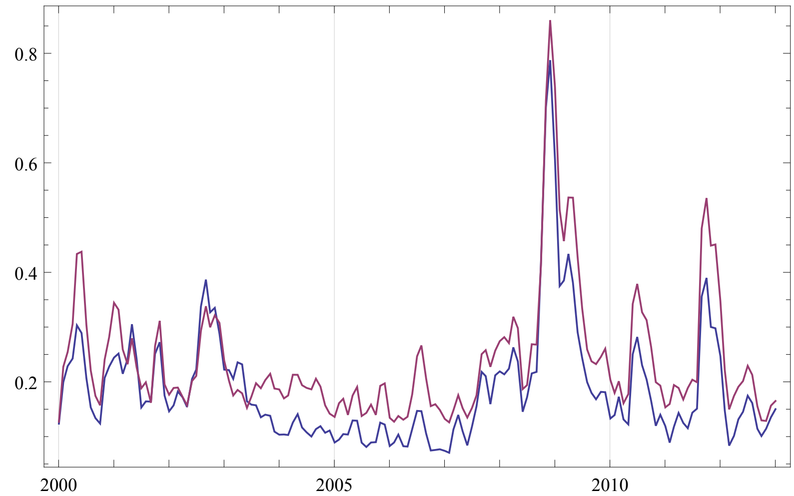
	Strategy	S&P 500	Russell 2000
Mean	0.039	-0.001	0.004
Std. Dev.	0.063	0.049	0.065
Sharpe	0.619	-0.025	0.064

Table 6.21: Strategy and underlying performance statistics (mean excess returns, their standard deviations and Sharpe ratios) in monthly figures, over the period 2000-2013.

As we can see in table 6.21, the multidimensional strategy outperforms both of its underlying, as its Sharpe ratio is considerably higher than that of its underlying. Now, the first trading day in this analysis is not equal to that one of the first trading dates (which differ for the underlying) in the one dimensional analysis, so it is hard to compare results. Though, the Sharpe ratio attained is higher than the ones reported for both underlying in table 6.1. The standard deviation as reported in table 6.21 is close to the 5% we were aiming at. When we look at



(a) Price processes.



(b) Realized volatilities.

Figure 6.7: The monthly price processes of the S&P 500 and the Russell 2000 during the period 2000-2013 and their realized volatilities. The blue lines are those of the S&P 500 and the red lines are those of the Russell 2000.

	Estimate	Standard error	T-statistic	p-value
α	0.039	0.005	7.747	$1.153 \cdot 10^{-12}$
β	-0.145	0.104	-1.394	0.165

Table 6.22: Regression with the market excess returns as explanatory variable, during the period of 2000-2013.

table 6.22, we can see that the estimated alpha is significant at a 5% significance level and that the estimated beta of the market excess returns is insignificant at this significance level. In table 6.23 we find that none of the other Fama & French factors are of significant influence on the strategy excess returns. When one studies the strategy performance when hedging on

	Estimate	Standard error	T-statistic	p-value
α	0.039	0.005	7.590	2.932$\cdot 10^{-12}$
β	-0.116	0.104	-1.114	0.267
SMB	-0.222	0.148	-1.501	0.135
HML	0.203	0.153	1.323	0.188

Table 6.23: Regression with all Fama & French factors as explanatory variables, during the period of 2000-2013.

both underlying assets with $\beta_2^{mkt} = 1$, one obtains similar results, but with an estimated beta of -0.149 at an T-statistic of -1.438 and p-value of 0.153, so the beta is somewhat worse, but statistically insignificant from zero. Hedging only on one underlying asset makes the strategy easier to implement, so we shall only hedge with respect to the first underlying. Thus, our strategy performs well during this period, but when we look at table B.2, we see that again most weight in the portfolio is placed at the three most far ITM options and the zero strike options on both underlying asset. Hence, to make the strategy simpler, we can neglect all other options in our portfolio selection every trading day. Therefore we perform the coming analyses in this reduced setting.

To check whether our strategy still performs well with less available options, we analyze the same period in this reduced setting. The number of observations and the correlation are of course the same. Table 6.24 gives us the portfolio performance analysis, from which we can see that the Sharpe ratio has even slightly increased, whilst the standard deviation is the same. In table 6.25 we find alpha to be significant and beta to be insignificant again, and table 6.26 shows us that still none of the other Fama & French factors is of significant influence in our regression. Hence, the assumption of that all other options to be irrelevant for the strategy seems to be valid. We continue with the analyses of the strategy during different periods.

	Strategy
Mean	0.042
Std. Dev.	0.063
Sharpe	0.669

Table 6.24: Strategy performance statistics (mean excess returns, their standard deviations and Sharpe ratios) in monthly figures, over the period 2000-2013, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.042	0.005	8.347	3.639$\cdot 10^{-14}$
β	-0.067	0.103	-0.652	0.515

Table 6.25: Regression with the market excess returns as explanatory variable, during the period of 2000-2013, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.042	0.005	8.306	$4.908 \cdot 10^{-14}$
β	-0.033	0.103	-0.326	0.745
SMB	-0.290	0.147	-1.977	0.050
HML	0.160	0.151	1.059	0.291

Table 6.26: Regression with all Fama & French factors as explanatory variables, during the period of 2000-2013, using less available options.

We first analyze the strategy performance during the period January 2000 - December 2002, in which we have 36 observations of excess returns and, naturally, still a correlation of 0.763. In table 6.27 we find that whilst both underlying are behaving very poorly, both having negative Sharpe ratios, the strategy achieves a high Sharpe ratio of 0.716 in monthly figures. This translates to a Sharpe ratio of 2.480 on a yearly basis. From the figures in table 6.28 we can again conclude that the estimated alpha in the regression is significant at a 5% significance level, and the beta to be insignificant again at this significance level, though, it would be significant when one uses a 10% significance level. Moreover, when one uses all Fama & French factors in the regression, we find the beta to be significant at a 5% significance level as well, whereas the other factors remain insignificant, as one can find in table 6.29. Thus, in this period of higher volatility, our strategy seems to be performing very well, achieving a high Sharpe ratio, but the beta-hedge seems to perform less well, though, still well enough. This does not seem to be the case in the one dimensional case over the period 1996/1997-2002, where table 6.8 reports a relatively high Sharpe ratio on the S&P 500 and a relatively low one on the Russell 2000 for the one dimensional case, and betas that are by far not significant, as we can see in table 6.9.

	Strategy	S&P 500	Russell 2000
Mean	0.046	-0.016	-0.006
Std. Dev.	0.064	0.051	0.068
Sharpe	0.716	-0.318	-0.093

Table 6.27: Strategy and underlying performance statistics (mean excess returns, their standard deviations and Sharpe ratios) in monthly figures, over the period 2000-2002.

	Estimate	Standard error	T-statistic	p-value
α	0.044	0.011	4.221	$1.709 \cdot 10^{-4}$
β	-0.356	0.209	-1.704	0.098

Table 6.28: Regression with the market excess returns as explanatory variable, during the period of 2000-2002, using less available options.

We continue with the period January 2003 - December 2006, giving us 48 observations of excess returns and a correlation of 0.800. In table 6.30 we find a staggering Sharpe ratio of

	Estimate	Standard error	T-statistic	p-value
α	0.037	0.011	3.407	0.002
β	-0.449	0.212	-2.121	0.042
SMB	0.728	0.531	1.371	0.180
HML	0.558	0.554	1.046	0.303

Table 6.29: Regression with all Fama & French factors as explanatory variables, during the period of 2000-2002, using less available options.

0.851 in monthly terms generated by our strategy, which is a Sharpe ratio of 2.950 in yearly terms. The realized standard deviation is even lower than the level we were aiming at. It is clear that the strategy outperforms both underlying. As we can see in table 6.12, also the Sharpe ratios in the single underlying case are high for both underlying assets. The achieved Sharpe ratio with the two dimensional strategy is still higher than both of them. Again we find alpha to be significant in the regression with the market excess returns as explanatory variable, as we can see in table 6.31, which also gives us that beta is by far not significant. Also the other Fama & French factors are not significant, as one can find in table 6.32. Hence, in this period of low volatility, but a steep underlying price increase, our two dimensional strategy performs outstanding.

	Strategy	S&P 500	Russell 2000
Mean	0.039	0.002	0.003
Std. Dev.	0.045	0.056	0.043
Sharpe	0.851	0.056	0.065

Table 6.30: Strategy and underlying performance statistics (mean excess returns, their standard deviations and Sharpe ratios) in monthly figures, over the period 2003-2006.

	Estimate	Standard error	T-statistic	p-value
α	0.039	0.007	5.831	$5.184 \cdot 10^{-7}$
β	-0.028	0.173	-0.164	0.870

Table 6.31: Regression with the market excess returns as explanatory variable, during the period of 2003-2006, using less available options.

Last, the period of January 2007 - January 2013, in which we have 73 observations of excess returns and a correlation of 0.805. In this period of high volatility, we see our mean excess return of the strategy dropping and its standard deviation rising (though, not to a level as high as during 2000-2002). The Sharpe ratio is therefore not that high this period, but still outperforms both underlying, as we can see in table 6.33. The alpha reported in table 6.34 is still significant, and beta is insignificant, but quite large (0.584 on monthly basis). Though it is insignificant in the regression with the other Fama & French factors as well, see table 6.35, the magnitude is big, so our beta-hedge seems to be working in this period, but not optimally. The

	Estimate	Standard error	T-statistic	p-value
α	0.038	0.007	5.659	1.068$\cdot 10^{-6}$
β	0.080	0.183	0.435	0.665
SMB	-0.039	0.297	-0.131	0.896
HML	0.460	0.281	1.641	0.108

Table 6.32: Regression with all Fama & French factors as explanatory variables, during the period of 2003-2006, using less available options.

	Strategy	S&P 500	Russell 2000
Mean	0.030	-0.001	-4.944$\cdot 10^{-4}$
Std. Dev.	0.056	0.014	0.019
Sharpe	0.531	-0.087	-0.025

Table 6.33: Strategy and underlying performance statistics (mean excess returns, their standard deviations and Sharpe ratios) in monthly figures, over the period 2007-2013.

	Estimate	Standard error	T-statistic	p-value
α	0.030	0.007	4.648	1.506$\cdot 10^{-5}$
β	0.584	0.456	1.282	0.204

Table 6.34: Regression with the market excess returns as explanatory variable, during the period of 2007-2013, using less available options.

	Estimate	Standard error	T-statistic	p-value
α	0.030	0.007	4.647	1.565$\cdot 10^{-5}$
β	0.558	0.467	1.194	0.236
SMB	-0.095	0.315	-0.302	0.764
HML	0.247	0.259	0.956	0.342

Table 6.35: Regression with all Fama & French factors as explanatory variables, during the period of 2007-2013, using less available options.

other Fama & French factors are again insignificant.

If we compare these results to the results from both underlying in the one dimensional case, we can see in table 6.16 that the Sharpe ratio we obtained with the two dimensional strategy barely outperforms the Sharpe ratio of the one dimensional strategy on the S&P 500 alone in this period. It does outperform the one on the Russell 2000 alone. In table 6.17 we also find an extraordinary high, but insignificant, beta in the regression on the excess returns on the one dimensional strategy on the S&P 500, while the beta we found with the Russell 2000 is small.

So, in this period our two dimensional strategy performs average. It is still a bit better than the one dimensional strategy applied on any of both underlying, but not overwhelmingly.

In all the analyses over the different sub-periods we consistently find larger betas when hedging on both underlying assets, but all insignificantly different from zero. We conclude that there are marginal overcompensation effects and that hedging only on the S&P 500 works just as well, which also makes the strategy easier to implement.

Now the question remains; does our two dimensional strategy outperform our naive strategy when using the S&P 500 and the Russell 2000? We study this over the period of January 2000 - January 2013, with the three most far ITM options and the zero strike option available at each trading day. We study how much the Sharpe ratio's differ and whether there is a difference in the regressions with the market excess returns as an explanatory variable. The results of our two dimensional strategy in this setting are given in table 6.24 and table 6.26. The results of the naive strategy can be found in table 6.36.

	Naive strategy
Mean	0.029
Std. Dev.	0.049
Sharpe	0.604

(a) Naive strategy performance

	Estimate	Standard error	T-statistic	p-value
α	0.010	0.002	4.273	3.349$\cdot 10^{-5}$
β	0.628	0.033	18.806	7.683$\cdot 10^{-42}$

(b) Regression results

Table 6.36: Naive strategy performance on the S&P 500 and Russell 2000 analysis during the period 2000-2013, using less available options, and formulated in monthly terms.

If we compare the results of the two dimensional strategy and the naive strategy, we see that the mean excess return and the standard deviation with the two dimensional strategy are both higher than with the naive strategy, but we also find a higher Sharpe ratio with our two dimensional strategy. Furthermore, the beta in our two dimensional strategy is insignificant, whereas the beta in our naive strategy is significant in the regression, causing the alpha estimated to be lower than with the two dimensional setting. Clearly, mixing two beta hedged positions in a naive way does not give us another beta hedged position. Therefore, we can conclude that with these indices our two dimensional strategy clearly outperforms the simple approach.

The correlation between log returns on the S&P 500 and the Russell 2000 is quite high. We test the performance of our two dimensional strategy in a setting of lower correlation as well and study whether it still outperforms the naive strategy. The assets we use for this analysis are three assets from the Dow Jones index; the Walt Disney Corporation, the McDonald's Corporation, and Walmart Stores Incorporated. We test two combinations with our two dimensional strategy; the combination of the McDonald's Corp. and Walmart Stores Inc. and the combination of the McDonald's Corp. and the Walt Disney Corp. We perform the analysis during the period of January 2001 - January 2013, giving us 145 observations of excess returns. We again work with the three most far ITM options and the zero strike option as available options on every trading day, which all expire in 30 days. We use the period from January 1995 - December 2000 to calculate the correlations between the underlying from the daily log returns. The correlation between the log returns on the McDonald's Corp. and the Walt Disney Corp. is 0.169 and the correlation between the log returns on the McDonald's Corp. and Walmart Stores Inc. is 0.284. We use $\mu_1 = \mu_2 = 0.08$ for each analysis, where we use 1 to indicate the first mentioned asset and 2 to indicate the other.

For comparison, we start of with using our one dimensional strategy on the three single components considered. In table 6.37 we find that the one dimensional strategy on each of the components outperforms its underlying benchmark, and in table 6.38 we find that all estimated alphas are significant, whilst all estimated betas are insignificant. Hence, the one dimensional strategy works very well on separate stocks during this period. We continue with our two dimensional strategy, for which the results are given in table 6.39, along with the results of the naive strategy. We see that the two dimensional strategy on both combinations outperforms their underlying assets and yields higher Sharpe ratios than when we use the one dimensional strategy on one of the underlying. The mean excess returns generated with the two dimensional strategy are higher than the ones generated with the naive strategy, also also the estimated alphas are higher, as we can find in tables 6.40 and 6.41. However, the standard deviations are also higher when using the two dimensional strategy, resulting in Sharpe ratios that are but marginally higher than the ones obtained with the naive strategy. The estimated betas are also higher than the ones we obtain with the naive strategy, however, they are all insignificant. The Sharpe ratio we obtain with our one dimensional strategy on the Dow Jones 1/100th over this period is 0.283 in monthly terms, so we see that performing the strategy on one of the three separate components already gives us a higher Sharpe ratio, which increases when we use our two dimensional strategy on two of them.

When one performs these analyses when hedging on both underlying assets, one again generates very similar results, but with marginally smaller betas, which are insignificantly different from zero as well. Hence, we can conclude that it is again easier to hedge on only one of the underlying, but that in this cases of low correlations between the underlying log returns the overcompensation effects do not show.

From these analyses we conclude that our strategies work well on stocks as well and that there is an increase in performance when one uses our multidimensional strategy. The correlation between the log returns on the stocks is low, and as we see that the Sharpe ratios from our two dimensional and our naive strategy are not far apart. This intuitively confirms our hypothesis that how closer the correlation between the log returns on the underlying is to zero,

	One dim. strategy	Walt Disney
Mean	0.033	0.007
Std. Dev.	0.069	0.073
Sharpe	0.482	0.107

(a) The Walt Disney Corp.

	One dim. strategy	McDonald's
Mean	0.039	0.008
Std. Dev.	0.069	0.064
Sharpe	0.572	0.120

(b) The McDonald's Corp.

	One dim. strategy	Walmart
Mean	0.042	0.003
Std. Dev.	0.060	0.052
Sharpe	0.697	0.052

(c) Walmart Stores Inc.

Table 6.37: One dimensional strategy and underlying performance of the Don Jones single components during the period of 2001-2013, formulated in monthly terms.

	Estimate	Standard error	T-statistic	p-value
α	0.033	0.006	5.786	$4.382 \cdot 10^{-8}$
β	0.074	0.117	0.629	0.530

(a) The Walt Disney Corporation

	Estimate	Standard error	T-statistic	p-value
α	0.039	0.006	6.860	$1.912 \cdot 10^{-10}$
β	0.038	0.118	0.320	0.749

(b) The McDonald's Corporation

	Estimate	Standard error	T-statistic	p-value
α	0.042	0.005	8.361	$5.019 \cdot 10^{-14}$
β	-0.022	0.102	-0.214	0.831

(c) Walmart Stores Inc.

Table 6.38: Regression results for the Dow Jones single components during the period 2001-2013.

	McDonald's Corp.& Walt Disney Corp.		McDonald's Corp & Walmart Stores Inc.	
	Bivariate strategy	Naive strategy	Bivariate strategy	Naive strategy
Mean	0.058	0.036	0.060	0.040
Std. Dev.	0.076	0.049	0.073	0.050
Sharpe	0.755	0.733	0.817	0.814

Table 6.39: Bivariate strategy and naive strategy performance for two combinations of two of the Dow Jones single components, during the period 2001-2013, formulated in monthly terms.

the less there is to gain with our multidimensional strategy as opposed to the naive strategy. The estimated alphas are still higher than the ones obtained with the naive strategy, so it is still favourable to use the multidimensional strategy, were it at a Sharpe ratio that is but marginally higher.

	Estimate	Standard error	T-statistic	p-value
α	0.058	0.006	9.076	8.136$\cdot 10^{-16}$
β	0.110	0.130	0.845	0.400

(a) Bivariate strategy

	Estimate	Standard error	T-statistic	p-value
α	0.036	0.004	8.772	4.742$\cdot 10^{-15}$
β	0.056	0.085	0.658	0.511

(b) Naive strategy

Table 6.40: Regression results of the bivariate strategy and naive strategy of the McDonald's Corp. and Walt Disney Corp. with the market excess returns as explanatory variable, during the period of 2001-2013.

	Estimate	Standard error	T-statistic	p-value
α	0.060	0.001	9.815	1.058$\cdot 10^{-17}$
β	0.064	0.126	0.511	0.610

(a) Bivariate strategy

	Estimate	Standard error	T-statistic	p-value
α	0.040	0.004	9.734	1.708$\cdot 10^{-17}$
β	0.008	0.085	0.092	0.927

(b) Naive strategy

Table 6.41: Regression results of the bivariate strategy and naive strategy of the McDonald's Corp. and Walmart Stores Inc. with the market excess returns as explanatory variable, during the period of 2001-2013.

Considering the analyses of our multidimensional strategy performance in all four periods, for indices and stocks, we conclude that our strategy performs very well. The generated Sharpe ratios are higher than those of the individual benchmarks and higher then the ones obtained with our one dimensional strategy on the individual benchmarks. The estimated alphas are significantly different from zero and all betas are insignificant, though, in times of high volatility they do tend to be high, but still insignificant. For underlying with high correlation one is better of hedging on only one of the assets, as there are slight overcompensation effects when one hedges on both, though the results do not differ much. When correlation is low, then the hedge works marginally better when hedging on both assets. Hedging on only one of the underlying makes the strategy easier to implement and gives one almost the same results. We also find that the multidimensional strategy outperforms our naive strategy and that the magnitude of outperformance increases when the correlation between the benchmarks increases.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

In this chapter we present the most important conclusions that we can draw from our research and a few recommendations we can make for further research. We start with our conclusions:

- ***In a Black-Scholes world, one cannot generate a significant alpha with our trading strategies on options with discrete strikes, neither in the one dimensional case, nor in the multidimensional case.***

Guasoni et al. [8] have already drawn this conclusion for the one dimensional case, using options with continuous strikes. As we have seen in our simulations in chapter 5, also when using European call options with discrete strikes, one cannot generate superior performance by trading dynamically in options according to our strategies when one works in a Black-Scholes world, where benchmark assets follow geometric Brownian motions with constant drift and diffusion, and options are priced according to the Black-Scholes formula, neither when considering one benchmark asset, nor when considering two.

- ***When using data obtained from the market, one can generate a statistically significant alpha with our trading strategies. The results improve when one considers more benchmark assets to invest on.***

From the results of chapter 6 we can conclude that our strategy for the one dimensional case works very well, as the reported Sharpe ratios obtained with the strategy on each considered benchmark asset are high and the generated alphas are significantly different from zero at a 5% significance level. Even in times of high volatility on the underlying assets, the strategy manages to outperform holding the benchmark asset. When one considers more than one benchmark asset to trade options on and uses our multidimensional strategy, even better results are obtained. The obtained Sharpe ratios are generally higher than the ones obtained in the one dimensional case and the alphas obtained are significant.

- ***Far in-the-money options and the zero strike option are favourable products in our strategies.***

From our simulations and market data analyses we can conclude that our portfolio selection each trading day places most weight on the most far in-the-money options and the zero strike option, thus, the benchmark asset itself. When considering only the three most far in-the-money options in our dataset and this zero strike option as available options every trading day, the results obtained even tend to be better than the ones obtained when considering all available options. This is the case for both the one dimensional strategy and the multidimensional strategy. Another positive side of this conclusion is that when a hedge fund manager wants to implement this strategy, he only has to consider a few products to invest on, instead of a wide range. This makes the strategy easier to implement.

- ***Our way of hedging out the strategy's sensitivity with respect to market excess returns works, but tends to produce high regression betas in times of high volatility on the market. When considering more than one benchmark asset in the strategy and when one uses all benchmarks to hedge out the sensitivity with respect to market excess returns, one can get overcompensated results that make the hedge worse.***

From our simulation results we can conclude that the beta-hedge works well when one considers a Black-Scholes framework. The obtained betas are very close to zero and are statistically insignificantly different from zero at a 5% significance level, for the one dimensional strategy, as well as for the multidimensional strategy. For the multidimensional strategy we encounter overcompensation effects when hedging on both underlying in the simulation, but when one hedges on only one of the underlying, the beta-hedge performs well. When one uses our one dimensional strategy with historical market data, the obtained betas are insignificant over every period we have analyzed, for every benchmark asset we have considered. The obtained betas over the period 2007-2013, where volatilities are high, are larger than those over the period of 2003-2006, where volatilities are low. However, the betas obtained over 1996/1998-2002, where volatility is high as well, do not differ much from those obtained over 2003-2006. The credit crisis and global recession period contains a very large peak in volatility for all considered benchmark assets, which we think causes the high betas during that period. Also, with our multidimensional strategy we find similar results and we conclude that the effect of the high volatilities is even greater, though the obtained betas are still insignificant, so the hedge is still performs adequate. In the analyses with historical data we only find the overcompensation effects of hedging on both underlying when the correlation between the log returns of the underlying is high, but on a marginal scale. The strategy is easier to implement when one hedges only with respect to one underlying and the results are very similar in the cases we have studied, so we recommend to omit the hedge on the second underlying.

- ***Our multidimensional strategy outperforms a naive strategy that spreads ones wealth and invests it on the considered benchmarks using our one dimensional strategy. The magnitude of this outperformance increases as the correlation between the considered benchmarks increases.***

When the correlation between the benchmarks would be zero, it would be optimal to split ones capital into equal parts and use each part to invest on a different benchmark using our one dimensional strategy. As we have seen for the S&P 500 and the Russell 2000, for which correlation is high, our two dimensional strategy outperforms this naive strategy, yielding a higher Sharpe ratio and an insignificant beta, whereas the naive strategy produces a significant beta. We have also investigated the differences between these strategies for two combinations of three stocks in the Dow Jones index, for which the correlations are lower. We see that the Sharpe ratios obtained with our multidimensional strategy are but marginally larger than those obtained with the naive strategy and for both strategies the generated betas are insignificant.

- ***When one hedges out the strategy's sensitivity with respect to the market excess returns using a position in the benchmark asset itself, but treating this benchmark asset as if it were the market works better than more complicated approaches. Also, improving the model with a dynamic growth rate in the geometric Brownian motion does not yield a significant improvement to the results.***

As we have seen in our one dimensional data analysis, one does not win much by not choosing β^{mkt} of the underlying equal to something else than one. The results do not improve, and sometimes actually get worse by constructing this sensitivity parameter from historical data and then using it as a constant, or updating it dynamically as one obtains more information on the benchmarks and the market during the period on which we use our strategy. Making the model more complicated, but more realistic, by making the growth rate in the geometric Brownian motion a function of time, does not bring any significant improvements to the results, so one can just treat this drift as a constant in the model.

We conclude this thesis with a few recommendations for further research, which in our opinion interesting to pursue:

- ***Improve the stability of the covariance matrix of option excess returns.***

The entries in the covariance matrix tend to get very, very small sometimes, leaving the covariance matrix prone to numerical errors. This can cause the covariance matrix to become non positive semi definite, which cannot be the case for covariance matrices. We have encountered this problems when using volatilities based on a historical period of 30 days, in stead of 60, which we use in our analyses, and when using dynamic correlation coefficients in the multidimensional case. We suspect that a dynamic correlation coefficient can improve the results. Thus far, we were not able to explain why the problem occurs exactly, so we recommend further research on this.

- ***Introduce trading costs and margins to the model.***

We have worked with frictionless markets, as this is convenient for research. Our simple approach to introducing trading costs was too simple. To make the analysis more realistic, we suggest to implement trading costs to the model. Also, we take quite substantial short positions in some options in our portfolio. In practice, one has to make a down payment, a 'margin', to be allowed to do this. We omit these margins, as there is not much data available on them. The effects on the results might be interesting, though.

- ***Extend the strategies to allow put options and American exercise style options.***

A lot of traded options are of American exercise style and a lot of previous research suggests out-of-the-money puts to be favourable products. One can see our position in far in-the-money calls as positions in out-of-the-money puts, in the underlying, and in cash by the put-call parity, but introducing the actual product might provide interesting results. The problem with American put options is the fact one can exercise them before the expiry date, so the strategy has to be adapted to account for this.

- ***Test the strategies with actual traded options.***

For convenience, we have worked with interpolated options throughout this research, which are not actually traded on the market, but are based on options that are. We suggest to extend the strategies in such a way that one can work with actual traded options, on which the Optionmetrics database also contains a lot of data. The problem with the traded options is that the options do not all have the same time to expiration and that in practice it can happen that on a certain trading day there are options available on one benchmark, but not on the other. As our strategy assumes that all options always expire at the same date and are traded at the same date, it needs some extensions to account for actual traded options.

- ***Implement a stochastic volatility model with jump components.***

We have seen in our market data analysis that very sudden, large movements in the underlying can lead to substantial excess returns (positive and negative) on our portfolio choice, as we ‘bet’ on a certain scenario, but end up in a completely different one. These large movements can be caused by market crashes etc. A model that can account for sudden jumps in the price process and the volatility process might reduce the magnitude of these negative returns, but might also decrease the magnitude of our strong positive results. We think the effects of this implementation could be very interesting, but the explicit expressions for the expected option excess returns and covariances between them might be much harder to obtain. Broadie et al. [2] and Eraker [3], for example, study these kind of models for their trading strategies.

Bibliography

[1] J. D. Coval and T. Shumway, “Expected option returns,” *The Journal of Finance*, vol. 56, pp. 983–1009, June 2001.

[2] M. Broadie, M. Chernov, and M. Johannes, “Understanding index option returns,” *The Review of Financial Studies*, vol. 22, pp. 4493–4529, May 2009.

[3] B. Eraker, “The performance of model based option trading strategies,” *Review of Derivatives Research*, vol. 16, pp. 1–23, April 2013.

[4] C. S. Jones, “A nonlinear factor analysis of s&p 500 index option returns,” *The Journal of Finance*, vol. LXI, pp. 2325–2363, October 2006.

[5] N. Kapadia and E. Szado, “The risk and return characteristics of the buy write strategy on the russel 2000 index,” *Journal of Alternative Investments*, vol. 9, pp. 39–56, Spring 2007.

[6] J. Liang, S. Zhang, and D. Li, “Optioned portfolio selection: Models and analysis,” *Mathematical Finance*, vol. 18, pp. 569–593, October 2008.

[7] P. Santa-Clara and A. Saretto, “Option strategies: Good deals and margin calls,” *Journal of Financial Markets*, vol. 12, pp. 391–417, January 2009.

[8] P. Guasoni, G. Huberman, and Z. Wang, “Performance maximization of actively managed funds,” *Journal of Financial Economics*, vol. 101, pp. 574–595, September 2011.

[9] H. Markowitz, “Portfolio selection,” *The Journal of Finance*, vol. 7, pp. 77–91, March 1952.

[10] J. H. Cochrane, *Asset Pricing*. 3 Market Place, Woodstock, Oxfordshire: Priceton University Press, 2001.

Appendix A

Proofs

A.1 Proof of Theorem 3.2.3

Since the only unknown in equation (3.10) is $\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+\right]$, we will focus on this term and use that the underlying price process is a geometric Brownian motion. Since the ε in equation (3.3) is standard normal distributed under the physical measure, we have:

$$\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+\right] = \int (S_T - K_i)^+ N'(x)\,\mathrm{d}x = \int \left(S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} - K_i\right)^+ N'(x)\,\mathrm{d}x,$$

for which the integrand is only nonzero if

$$\begin{aligned}
S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} - K_i &> 0 \\
e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} &> K_i/S_t \\
\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x &> \ln\left[K_i/S_t\right] \\
x &> \frac{\ln\left[K_i/S_t\right] - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}} = \tilde{d}_1,
\end{aligned}$$

so, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+\right] &= \int \left(S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} - K_i\right)^+ N'(x)\,\mathrm{d}x \\
&= \int_{\tilde{d}_1}^{\infty} \left(S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} - K_i\right) N'(x)\,\mathrm{d}x \\
&= \int_{\tilde{d}_1}^{\infty} S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}x} N'(x)\,\mathrm{d}x - \int_{\tilde{d}_1}^{\infty} K_i N'(x)\,\mathrm{d}x \\
&= S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \int_{\tilde{d}_1}^{\infty} e^{\sigma\sqrt{\Delta t}x} N'(x)\,\mathrm{d}x - K_i N(-\tilde{d}_1),
\end{aligned}$$

where for the last integral we use that the normal distribution is symmetric. To find an explicit expression for the first integral, we will write the exponential times the PDF as a standardized normal PDF with mean μ^* and standard deviation σ^*, by completing the square in the exponential:

$$\begin{aligned}
&-\frac{(x-\mu^*)^2}{2{\sigma^*}^2} = \frac{-1}{2{\sigma^*}^2}\left[x^2 - 2\mu^* x + {\mu^*}^2\right] = \sigma\sqrt{\Delta t}x - \frac{1}{2}x^2 + C \\
&\Rightarrow \sigma^* = 1, \quad \mu^* = \sigma\sqrt{\Delta t}, \quad \textit{and} \quad C = -\frac{\sigma^2}{2}\Delta t.
\end{aligned}$$

Since we do not have C in our initial exponential, we have to compensate for this term by multiplication with e^{-C}, and since we are now using a standardized normal distribution, we must subtract μ^* from our lower integration bound, yielding:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+\right] &= S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \int_{\tilde{d}_1}^{\infty} e^{\sigma\sqrt{\Delta t}x} N'(x)\,\mathrm{d}x - K_i N(-\tilde{d}_1) \\
&= S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} e^{\frac{\sigma^2}{2}\Delta t} \int_{\tilde{d}_1 - \mu^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu^*)^2}{2}}\,\mathrm{d}x - K_i N(-\tilde{d}_1) \\
&= S_t \cdot e^{\mu\Delta t} \int_{\tilde{d}_2}^{\infty} \tilde{N}'(x)\,\mathrm{d}x - K_i N(-\tilde{d}_1) \\
&= S_t \cdot e^{\mu\Delta t} N(-\tilde{d}_2) - K_i N(-\tilde{d}_1) \\
&= S_t \cdot e^{\mu\Delta t} N(d_2) - K_i N(d_1),
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= -\tilde{d}_1 = -\frac{\ln\left[K_i/S_t\right] - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}} = \frac{\ln\left[S_t/K_i\right] + \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}, \\
d_2 &= -\tilde{d}_2 = -(\tilde{d}_1 - \sigma\sqrt{\Delta t}) = d_1 + \sigma\sqrt{\Delta t} = \frac{\ln\left[S_t/K_i\right] + \left(\mu + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}.
\end{aligned}$$

Plugging this expression in to equation (3.10) yields the result.

A.2 Proof of Theorem 3.2.4

We first rewrite *cov* $\left(r_i^{opt}, r_j^{opt}\right)$ in terms of expectations:

$$\begin{aligned}
\textit{cov}\left(r_i^{opt}, r_j^{opt}\right) &= \textit{cov}\left(\frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i} - 1, \frac{e^{-r_f\Delta t}(S_T - K_j)^+}{C_j} - 1\right) \\
&= \textit{cov}\left(\frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i}, \frac{e^{-r_f\Delta t}(S_T - K_j)^+}{C_j}\right) \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i} \cdot \frac{e^{-r_f\Delta t}(S_T - K_j)^+}{C_j}\right] \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[\frac{e^{-r_f\Delta t}(S_T - K_i)^+}{C_i}\right] \mathbb{E}^{\mathbb{P}}\left[\frac{e^{-r_f\Delta t}(S_T - K_j)^+}{C_j}\right] \\
&= \frac{e^{-2r_f\Delta t}}{C_i C_j} \mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+(S_T - K_j)^+\right] - (m_i + 1)(m_j + 1),
\end{aligned}$$

in which the only unknown term is $\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+(S_T - K_j)^+\right]$. We obtain this term through integration, in a similar fashion as with the proof of Theorem 3.2.3:

$$\mathbb{E}^{\mathbb{P}}\left[(S_T - K_i)^+(S_T - K_j)^+\right] = \int (S_T - K_i)^+(S_T - K_j)^+ N'(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int \left(S_t \cdot e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} x} - K_i \right)^+ \left(S_t \cdot e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} x} - K_j \right)^+ N'(x)\,\mathrm{d}x \\
&= \int_{\tilde{d}_5}^{\infty} \left(S_t \cdot e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} x} - K_i \right) \left(S_t \cdot e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} x} - K_j \right) N'(x)\,\mathrm{d}x \\
&= S_t^2 e^{\left(2\mu - \sigma^2\right)\Delta t} \int_{\tilde{d}_5}^{\infty} e^{2\sigma\sqrt{\Delta t} x} N'(x)\,\mathrm{d}x - S_t(K_i + K_j) e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \int_{\tilde{d}_5}^{\infty} e^{\sigma\sqrt{\Delta t} x} N'(x)\,\mathrm{d}x \\
&\quad + K_i K_j \int_{\tilde{d}_5}^{\infty} N'(x)\,\mathrm{d}x,
\end{aligned}
$$

in which we again complete the square in the first two integrals, subtract the corresponding μ^* from the lower integration bound and compensate for the corresponding e^C-terms that are not in our initial integrals. This yields:

$$
\begin{aligned}
&= S_t^2 e^{\left(2\mu + \sigma^2\right)\Delta t} N(-\tilde{d}_3) - S_t(K_i + K_j) e^{\mu \Delta t} N(-\tilde{d}_4) + K_i K_j N(-\tilde{d}_5) \\
&= S_t^2 e^{\left(2\mu + \sigma^2\right)\Delta t} N(d_3) - S_t(K_i + K_j) e^{\mu \Delta t} N(d_4) + K_i K_j N(d_5),
\end{aligned}
$$

where

$$
\tilde{d}_5 = \frac{\ln[\max(K_i, K_j)/S_t] - \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}, \qquad d_5 = -\tilde{d}_5 = \frac{\ln[S_t/\max(K_i, K_j)] + \left(\mu - \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}},
$$

$$
\tilde{d}_4 = \tilde{d}_5 - \sigma\sqrt{\Delta t}, \quad d_4 = -\tilde{d}_4 = \frac{\ln[S_t/\max(K_i, K_j)] + \left(\mu + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}},
$$

$$
\tilde{d}_3 = \tilde{d}_5 - 2\sigma\sqrt{\Delta t}, \quad d_3 = -\tilde{d}_3 = \frac{\ln[S_t/\max(K_i, K_j)] + \left(\mu + \frac{3\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}.
$$

Plugging this expectation into the equation for the covariance we derived at the start of this proof gives us the result.

A.3 Proof of Theorem 4.1.1

We write out the derivation for covariances of type 1 to full extent. The other two are special cases of this case.

For $i = 1, \ldots, n_1 - 1$ and $j = 1, \ldots, n_2 - 1$ we have

$$
\begin{aligned}
\textit{cov}(r_i^{(1)}, r_j^{(2)}) &= \textit{cov}\left(\frac{e^{-r_f \Delta t}(S_T^{(1)} - K_i^{(1)})^+}{C_i^{(1)}} - 1, \frac{e^{-r_f \Delta t}(S_T^{(2)} - K_j^{(2)})^+}{C_j^{(2)}} - 1 \right) \\
&= \textit{cov}\left(\frac{e^{-r_f \Delta t}(S_T^{(1)} - K_i^{(1)})^+}{C_i^{(1)}}, \frac{e^{-r_f \Delta t}(S_T^{(2)} - K_j^{(2)})^+}{C_j^{(2)}} \right) \\
&= \frac{e^{-2r_f \Delta t}}{C_i^{(1)} C_j^{(2)}} \mathbb{E}^{\mathbb{P}}\left[(S_T^{(1)} - K_i^{(1)})^+ (S_T^{(2)} - K_j^{(2)})^+ \right] \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[\frac{e^{-r_f \Delta t}(S_T^{(1)} - K_i^{(1)})^+}{C_i^{(1)}} \right] \mathbb{E}^{\mathbb{P}}\left[\frac{e^{-r_f \Delta t}(S_T^{(2)} - K_j^{(2)})^+}{C_j^{(2)}} \right] \\
&= \frac{e^{-2r_f \Delta t}}{C_i^{(1)} C_j^{(2)}} \mathbb{E}^{\mathbb{P}}\left[(S_T^{(1)} - K_i^{(1)})^+ (S_T^{(2)} - K_j^{(2)})^+ \right] - (m_i^{(1)} + 1)(m_j^{(2)} + 1),
\end{aligned}
$$

in which we see that again the only unknown term in the equation, is the expectation. We derive this term with the same methodology as in the single underlying case.

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[(S_T^{(1)} - K_i^{(1)})^+ (S_T^{(2)} - K_j^{(2)})^+\right] \\
&= \iint (S_t^{(1)} e^{(\mu_1 - \frac{\sigma_1^2}{2})\Delta t + \sigma_1 \sqrt{\Delta t} x_1} - K_i^{(1)})^+ (S_t^{(2)} e^{(\mu_2 - \frac{\sigma_2^2}{2})\Delta t + \sigma_2 \sqrt{\Delta t} x_2} - K_j^{(2)})^+ \cdot MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_{d_1^2}^{\infty} \int_{d_1^1}^{\infty} (S_t^{(1)} e^{(\mu_1 - \frac{\sigma_1^2}{2})\Delta t + \sigma_1 \sqrt{\Delta t} x_1} - K_i^{(1)}) (S_t^{(2)} e^{(\mu_2 - \frac{\sigma_2^2}{2})\Delta t + \sigma_2 \sqrt{\Delta t} x_2} - K_j^{(2)}) \cdot MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2,
\end{aligned}$$

where $MN'(x_1, x_2)$ is the PDF of the standard bivariate normal distribution, and

$$d_1^1 = \frac{\ln\left[K_i^{(1)}/S_t^{(1)}\right] - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)\Delta t}{\sigma_1\sqrt{\Delta t}}, \quad d_1^2 = \frac{\ln\left[K_j^{(2)}/S_t^{(2)}\right] - \left(\mu_2 - \frac{\sigma_2^2}{2}\right)\Delta t}{\sigma_2\sqrt{\Delta t}}.$$

We continue the derivation:

$$\begin{aligned}
&= S_t^{(1)} S_t^{(2)} e^{(\mu_1 - \frac{\sigma_1^2}{2} + \mu_2 - \frac{\sigma_2^2}{2})\Delta t} \int_{d_1^2}^{\infty} \int_{d_1^1}^{\infty} e^{(\sigma_1 x_1 + \sigma_2 x_2)\sqrt{\Delta t}} MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&- K_i^{(1)} S_t^{(2)} e^{(\mu_2 - \frac{\sigma_2^2}{2})\Delta t} \int_{d_1^2}^{\infty} \int_{d_1^1}^{\infty} e^{\sigma_2 x_2 \sqrt{\Delta t}} MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&- K_j^{(2)} S_t^{(1)} e^{(\mu_1 - \frac{\sigma_1^2}{2})\Delta t} \int_{d_1^2}^{\infty} \int_{d_1^1}^{\infty} e^{\sigma_1 x_1 \sqrt{\Delta t}} MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&+ K_i^{(1)} K_j^{(2)} \int_{d_1^2}^{\infty} \int_{d_1^1}^{\infty} MN'(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

For each integrand we again complete the square to make it a standardized bivariate normal PDF. Just as in the one dimensional case, we assume that $\sigma_1^* = \sigma_2^* = 1$. The covariance matrix of $MN'(x_1, x_2)$ is given by

$$\Sigma = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix},$$

thus, when we write out the exponentials of each integrand plus a constant for which we have to compensate and set it equal to the exponential of the standardized bivariate normal PDF, we get the general equation (which holds for higher dimensions of N as well):

$$\begin{aligned}
C + \alpha' x - \frac{1}{2} x' \Sigma^{-1} x &= -\frac{1}{2}(x - \mu^*)' \Sigma^{-1} (x - \mu^*) \\
C + \alpha' x - \frac{1}{2} x' \Sigma^{-1} x &= -\frac{1}{2} x' \Sigma^{-1} x + x' \Sigma^{-1} \mu^* - \frac{1}{2} \mu^{*\prime} \Sigma^{-1} \mu^*,
\end{aligned}$$

in which all vectors are two dimensional column vectors. The vector α is composed of the elements in the first exponential of each integrand that are multiplicated with the elements of the vector $x = \{x_1, x_2\}'$. For example, for our first integral we find $\alpha = \{\sigma_1\sqrt{\Delta t}, \sigma_2\sqrt{\Delta t}\}'$, making the term $e^{(\sigma_1 x_1 + \sigma_2 x_2)\Delta t}$, for the second we find $\alpha = \{0, \sigma_2\sqrt{\Delta t}\}'$ and for the third we find $\alpha = \{\sigma_1\sqrt{\Delta t}, 0\}'$. The fourth does not need an alteration. From these equations we deduce

$$\begin{aligned}
&\Rightarrow C = -\frac{1}{2}\mu^{*\prime}\Sigma^{-1}\mu^*, \quad \alpha' x = x' \Sigma^{-1} \mu^*, \\
&\Rightarrow \mu^* = \Sigma\alpha.
\end{aligned}$$

So now we have for each integral the constant exponential for which we have to compensate and the shift μ^* in the lower boundaries of the double integrals. The integrands can now all be expressed as standardized bivariate normal PDFs, so we can express the double integrals in terms of the CDF of the standard bivariate normal distribution, with entries derived from the shifted lower boundaries as we did in the single underlying case. Combining the terms results in the first claim of the theorem and its values for $d_1, d_2, \ldots, d_8$, which arise from the shifted d_1^1 and d_1^2 terms for each integral. In higher dimensions of underlying, one will get the same sort of expressions, but since α and Σ are of higher dimension, one will get more correlation terms in the constant for which we have to compensate and more standard deviation terms in our μ^*-expressions. The same formulas can be used for every S_{ij}.

Now, for covariances of the second kind, one follows the same derivation, but the last two integrals disappear, because one has a strike equal to zero. Also, because of one strike is zero, only one of the double integrals has a lower boundary, the other one (integrating over the underlying which has strike zero) will have a lower boundary of $-\infty$ and will thus integrate the bivariate normal density to a normal density, hence, one gets the result in terms of the normal CDF, as made in our second claim.

The covariances of the third kind are both on options with strike zero, hence, all integrals with terms of K before them will disappear and all lower boundaries will remain $-\infty$, hence, after standardizing these integrands, one remains with a double integral over a probability density over the entire real axis, which yields one. Therefore, only the terms before the integral remain, and one arrives at the final claim.

APPENDIX B

TABLES

In this appendix we present the tables with results from our data analysis in chapter 6 which we did not present in the chapter itself. The tables present analyses of the performance of holding individual options for all underlying and periods considered, expected excess returns on these options and the mean absolute weights of these options in the portfolios in the one and two dimensional case.

B.1 Mean absolute portfolio weights

Option Δ	S&P 500	NASDAQ 100	Russell 2000	DowJones 1/100th
0.20	0.020	0.017	0.021	0.027
0.25	0.051	0.039	0.040	0.069
0.30	0.054	0.038	0.030	0.052
0.35	0.070	0.041	0.032	0.040
0.40	0.077	0.050	0.043	0.044
0.45	0.078	0.058	0.047	0.048
0.50	0.82	0.062	0.054	0.050
0.55	0.94	0.070	0.065	0.059
0.60	0.087	0.076	0.064	0.053
0.65	0.097	0.074	0.063	0.066
0.70	0.156	0.173	0.129	0.142
0.75	0.500	0.557	0.490	0.534
0.80	0.950	0.971	1.204	0.934
1	8.758	5.999	10.251	8.510

Table B.1: Mean absolute portfolio weights for all underlying during the period 1996/1998-2013, using the one dimensional strategy

Option Δ	S&P 500	Russell 2000
0.20	0.017	0.012
0.25	0.044	0.026
0.30	0.047	0.021
0.35	0.064	0.023
0.40	0.061	0.032
0.45	0.066	0.033
0.50	0.068	0.039
0.55	0.077	0.047
0.60	0.072	0.045
0.65	0.081	0.042
0.70	0.122	0.074
0.75	0.388	0.264
0.80	0.705	0.418
1	6.418	2.594

Table B.2: Mean absolute portfolio weights for the S&P 500 and Russell 2000 during the period 2000-2013, using the two dimensional strategy.

B.2 Realized option excess returns analysis

Option	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
Δ	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
0.20	-0.49	1.38	-0.02	2.36	-0.36	1.97	-0.40	1.74
0.25	-0.39	1.36	0.02	2.12	-0.22	1.78	-0.36	1.60
0.30	-0.33	1.30	0.04	1.91	-0.17	1.63	-0.31	1.47
0.35	-0.25	1.22	0.05	1.73	-0.13	1.50	-0.25	1.34
0.40	-0.21	1.16	0.04	1.59	-0.11	1.38	-0.21	1.24
0.45	-0.17	1.10	0.04	1.46	-0.08	1.28	-0.17	1.15
0.50	-0.14	1.03	0.04	1.34	-0.07	1.18	-0.14	1.07
0.55	-0.12	0.96	0.04	1.22	-0.05	1.09	-0.12	0.99
0.60	-0.10	0.90	0.03	1.12	-0.04	1.00	-0.11	0.93
0.65	-0.08	0.84	0.03	1.03	-0.03	0.93	-0.10	0.86
0.70	-0.07	0.77	0.02	0.94	-0.03	0.85	-0.08	0.79
0.75	-0.06	0.70	0.02	0.85	-0.02	0.77	-0.07	0.72
0.80	-0.04	0.62	0.03	0.75	-0.01	0.69	-0.05	0.63
1	$1.85 \cdot 10^{-3}$	0.05	$7.82 \cdot 10^{-3}$	0.08	$4.28 \cdot 10^{-3}$	0.06	$9.24 \cdot 10^{-4}$	0.05

Table B.3: Mean realized monthly excess returns and option excess return standard deviation for all options on all underlying during the 1996/1998-2013 period.

Option	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
Δ	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
0.70	-0.11	0.84	0.05	1.06	-0.10	0.87	-0.19	0.80
0.75	-0.09	0.76	0.05	0.96	-0.09	0.79	-0.17	0.74
0.80	-0.08	0.68	0.04	0.86	-0.08	0.70	-0.14	0.66
1	-1.09$\cdot 10^{-4}$	0.05	8.29$\cdot 10^{-3}$	0.08	-8.76$\cdot 10^{-4}$	0.06	-3.15$\cdot 10^{-3}$	0.05

Table B.4: Mean realized monthly excess returns and option excess return standard deviation for all options on all underlying during the 1996/1998-2002 period, using less available options.

Option	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
Δ	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
0.70	-0.25	0.76	-0.27	0.82	-0.22	0.85	-0.24	0.60
0.75	-0.16	0.63	-0.18	0.71	-0.14	0.72	-0.16	0.50
0.80	-0.10	0.52	-0.11	0.61	-0.08	0.60	-0.10	0.42
1	2.19$\cdot 10^{-3}$	0.04	5.05$\cdot 10^{-3}$	0.06	2.81$\cdot 10^{-3}$	0.04	1.41$\cdot 10^{-3}$	0.03

Table B.5: Mean realized monthly excess returns and option excess return standard deviation for all options on all underlying during the 2003-2006 period, using less available options.

Option	S&P 500		NASDAQ 100		Russell 2000		Dow Jones 1/100th	
Δ	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
0.70	-0.30	0.23	-0.32	0.25	-0.30	0.26	-0.29	0.23
0.75	-0.21	0.20	-0.22	0.21	-0.21	0.22	-0.20	0.19
0.80	-0.14	0.16	-0.15	0.17	-0.14	0.18	-0.13	0.16
1	-1.22$\cdot 10^{-3}$	0.01	-1.22$\cdot^{-3}$	0.02	-4.45$\cdot 10^{-4}$	0.02	-1.33$\cdot 10^{-3}$	0.01

Table B.6: Mean realized monthly excess returns and option excess return standard deviation for all options on all underlying during the 2007-2013 period, using less available options.

B.3 Expected option excess returns analysis

Option Δ	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
0.20	0.37	0.32	0.22	0.41
0.25	0.26	0.23	0.14	0.30
0.30	0.19	0.16	0.09	0.22
0.35	0.13	0.11	0.05	0.16
0.40	0.09	0.08	0.02	0.12
0.45	0.06	0.05	-1.21$\cdot 10^{-4}$	0.09
0.50	0.04	0.03	-0.01	0.07
0.55	0.03	0.02	-0.02	0.05
0.60	0.02	0.01	-0.02	0.04
0.65	0.01	1.41$\cdot 10^{-3}$	-0.02	0.03
0.70	0.01	-1.52$\cdot 10^{-3}$	-0.02	0.03
0.75	0.01	-2.00$\cdot 10^{-3}$	-0.01	0.03
0.80	0.02	-4.10$\cdot 10^{-4}$	-4.65$\cdot 10^{-4}$	0.03
1	3.83$\cdot 10^{-3}$	3.84$\cdot 10^{-3}$	3.95$\cdot 10^{-3}$	4.10$\cdot 10^{-3}$

Table B.7: Mean expected monthly excess returns for all options on all underlying during the 1996/1998-2013 period.

Option Δ	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
0.70	-0.03	-0.02	-0.07	-0.04
0.75	-0.02	-0.02	-0.06	-0.03
0.80	-0.02	-0.02	-0.04	-0.02
1	2.44$\cdot 10^{-3}$	2.44$\cdot 10^{-3}$	2.52$\cdot 10^{-3}$	2.67$\cdot 10^{-3}$

Table B.8: Mean expected monthly excess returns for all options on all underlying during the 1996/1998-2002 period, using less available options.

Option Δ	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
0.70	0.04	5.16$\cdot 10^{-3}$	0.02	0.06
0.75	0.04	5.84$\cdot 10^{-3}$	0.02	0.06
0.80	0.05	6.39$\cdot 10^{-3}$	0.02	0.06
1	4.07$\cdot 10^{-3}$	4.07$\cdot 10^{-3}$	4.07$\cdot 10^{-3}$	4.07$\cdot 10^{-3}$

Table B.9: Mean expected monthly excess returns for all options on all underlying during the 2003-2006 period, using less available options.

Option Δ	S&P 500	NASDAQ 100	Russell 2000	Dow Jones 1/100th
0.70	0.04	0.02	0.02	0.06
0.75	0.04	0.01	0.02	0.06
0.80	0.04	0.02	0.02	0.06
1	$5.33 \cdot 10^{-3}$	$5.33 \cdot 10^{-3}$	$5.33 \cdot 10^{-3}$	$5.33 \cdot 10^{-3}$

Table B.10: Mean expected monthly excess returns for all options on all underlying during the 2007-2013 period, using less available options.