Differential equations in Dutch secondary school: a conceptual approach

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Abstract
The goal of this article is to share our findings towards composing and conducting mathematics education in differential equations, using a conceptual approach. This conceptual angle is grounded on Realistic Mathematics Education (RME) – an educational approach relying heavily on a horizontal and vertical mathematization. Our research was done by conducting a lesson series aimed at a group of eight high school students, who over the course of four lessons were presented with an arrangement of topics related to differential equations. In this, we used an RME-based approach, with an emphasis on conceptual understanding, and less of an emphasis on procedural understanding. The topics we discussed during the course, and the manner in which they were presented, were put together before starting the course, and were modified, if necessary, as the lessons went along. Finally, the students' conceptual and procedural understanding was tested with a written exam. A question list was used as to provide insight into the students' opinions and personal views on the course.
A central theme in our results is the fact that secondary school students clearly are not used to a conceptual approach of mathematics. The students show themselves to be quite able to reproduce (which is procedural), but not to produce mathematics (which requires conceptual understanding). We also found that an RME-based lesson is conducted most efficiently by class discussion, especially when introducing new concepts. Between the results of the exams and the question lists, the students with the lower exam scores also opined most favourably towards a procedural approach. The students with the higher grades are rather less negative towards the conceptual approach, although they are not fully positive either.
The use of graphical software towards stimulating the students' understanding of underlying concepts of DEs was promising – although not fully realized in our research – and begs for future research. Generally, our findings could be used as a basis or guideline towards teaching differential equations in secondary school, using a conceptual approach of mathematics.

Keywords: mathematics education, conceptual approach, differential equations, realistic mathematics education.

1. Introduction

Mathematics, by all accounts, is known to be an abstract science. If not the broadest of the sciences, it can certainly be called the cognitive foundation towards the more physical sciences. In that respect, Gauss' description as mathematics being the 'queen of sciences' is well-deserved. However, despite its apparent cerebral homogeneity, two distinct general ways do exist towards the application of mathematics. These stem from the conceptual and procedural approaches.

This dichotomy basically amounts to the difference between 'knowing what you're doing' and 'knowing what to do'. Chiefly during the last few decades of the 20th century, finding a proper balance between conceptual and procedural knowledge in mathematics education has been the subject of much scrutiny. Mathematics education has known some global paradigmatic shifts in this time, most notably by the introduction of the 'new math' method of teaching, and the balance between the conceptual and procedural approach continues to be a subject of research today (e.g. Hattikudur & Alibali, 2011; Star, 2005; Miller & Hudson, 2007; also, Kilpatrick, Swafford &

Findell, 2000).

Of course, we can naught but acknowledge the importance of procedural understanding in mathematics. However, it has been our observation that mathematics education, particularly in Dutch secondary school, puts too much of an emphasis on procedure, thereby detracting from teaching conceptually. By this relatively procedurally oriented curriculum, the conceptual mathematical understanding of students is marred to such a degree as to cause a cognitive gap between secondary school mathematics and mathematics taught in tertiary education. This is illustrated by the fact that, starting a mathematics major at the University of Twente, students are required to complete a course in basic mathematical skills (should they fail at a test given at the start of the first year), in addition to their regular curriculum. This course, in essence, is a recap of the entire Dutch secondary school mathematics curriculum. One would say that graduating in the mathematics course required for this major in secondary school should suffice, while apparently it doesn't.

Our research capitalizes on the idea that a more conceptually oriented mathematics education in secondary school will help students cross this cognitive gap. More generally, we believe an emphasis on mathematical concepts to be beneficial towards students' grasp of the 'big picture' of mathematics. To have a grasp on underlying concepts, so we believe, helps students towards interrelating seemingly discrete processes (for example, the exponential function grants insight towards the relation between differential equations and probability theory). By this, mathematics will be regarded more as a functional whole than as a proverbial bag of tricks. To this end, we will ground our lessons on the Realistic approach of Mathematics Education (RME), as devised by Freudenthal (1973, 1984). Freudenthal's realistic approach draws upon a phenomenological philosophy – a mathematical basis is built upon real-world observations.

The topic by which we chose to execute our research is that of differential equations. As a topic, differential equations – a relative newcomer to the field of secondary school education research – presents itself as being readily available for conceptual teaching. After all, the ideas behind differential equations are firmly rooted in everyday dynamical systems, thus being relatively easy to conceptualize by visualization. Furthermore, handling differential equations can readily be tied to the students' prior knowledge of basic calculus. To this end we composed a lesson series, as conducted by a set of modules, each of them designed in such a way as to stimulate the development of a mathematical thinking model, in small elementary steps. This, by heavily drawing on conceptualizations. From this thinking model, a more general theoretical frame would 'naturally' follow – or at least, such was our aim.

That said, the Dutch secondary school system (particularly regarding mathematics) should be elaborated upon, after which we will formulate our research question.

The Dutch context

In the Netherlands, much like in the rest of the world, education is divided into three stages: primary, secondary and tertiary education. Depending on the courses followed in secondary school, tertiary education is subdivided into three categories (these being vocational education, attending a research university, or university proper).

For secondary school students are subdivided into several school-categories, based on their relative productive and intellectual merit, as determined by a standardized national test in the final year of elementary school. The upper category, called vwo ('voorbereidend wetenschappelijk onderwijs', literally translated as 'preparatory scientific education', but more succinctly translated as 'pre-university'), is a course lasting six years, normatively aimed at students aged 12 to 18. This course in itself is subdivided into two parts: 'onderbouw' (age 12-15) and 'bovenbouw' (age 15-18). These can be very roughly translated as 'foundation' and 'structure', respectively. The 'foundation' pertains introductions to the full spectrum of courses, ranging from laying the foundation for language studies, to history, geography, the physical sciences, and indeed, mathematics.

Come the third year, the students are presented with a choice from a foursome of assortments of courses, called 'profiles'. Each of these profiles is designed as to emphasize a roughly interrelated set of courses, whilst omitting others. As such, each of them pertains the students' prospective profession. For example, the profile of 'nature and technology' ('natuur en techniek') is aimed at beta-oriented students by putting little emphasis on the foreign languages, social studies, etc., and putting more emphasis on physics, chemistry, and mathematics. That said, optional courses can be chosen by the students. For example, should a beta-oriented student (having chosen the nature and technology profile) have a particular aptitude for say, German, it is possible for this student to opt for a more involved German course, in addition to the set assortment of courses as dictated by the chosen profile. In this, the student actually is obliged to choose several optional courses (the number of chosen options being dependent on these courses' relative 'weight'). The fourth year until graduation are then spent following this chosen profile. In these years, the student has the possibility to shift profiles by choice, or if he or she would be advised to do so. A student graduates (for most courses) by way of a nationally standardized exam at the end of the sixth year. In this, the profile (or more accurately, the chosen courses) in which a student graduates, influences the options of which university major he or she can follow. For example, a mathematics major cannot be chosen if the student hasn't graduated for the so-called 'mathematics B' course.

Which neatly brings us to the question of how secondary school mathematics works in the Netherlands. Mathematics is divided into four discrete courses, called mathematics A, B, C and D. In this, either A or B are mandatory, depending on the profile chosen: alpha-oriented profiles contain mathematics A by default, while beta-oriented profiles contain mathematics B. Mathematics C and D are optional courses, basically amounting to extra material for mathematics A and B, respectively. Specifically, mathematics A pertains the more, if you will, applied side of mathematics, putting an emphasis on probability theory and statistics, and less of an emphasis on algebra and analytical calculus, instead opting for numerical calculus in that respect (i.e. by relying more heavily on the use of graphic calculators). Mathematics C, as said, is an optional addendum to mathematics A, putting more of an emphasis on algebra and analysis (in the context of the given theory of mathematics A). In turn, mathematics B weighs much more heavily upon algebraic and analytical skill, in particular taking calculus to a higher level, as well as the individual topics being handled more rapidly, than mathematics A does. That said, less time is put in probability theory and statistics. Mathematics D (the optional addendum to mathematics B) remedies this in some respect, for it contains specialized theoretical bundles, a selection of which is made (normatively) by the teacher at hand. These specializations range from an introduction in complex numbers, to probability distribution functions, game theory, and differential equations.

Due to the course of mathematics D having this loosely defined choice of topics, no standardized national exam exists. Instead, the course is completed by exams as put up by the particular school. As such, completing mathematics D has no bearing on a student's prospects when it comes to applying for university majors. The course, effectively, 'merely' provides the students with an advantage in mathematical thinking in comparison to their peers at the university. As such, on the whole, mathematics D is not a popular optional course. Most students opting for this course do so either out of a particular interest in mathematics, or by a process of elimination (seeing as some optional course has to be chosen). In both cases, this constitutes few participants.

As stated before, our research capitalizes on the idea that mathematics D – or rather more exactly, introducing differential equations via a conceptual approach – provides the students with a significant advantage in their prospective (beta-oriented) university majors. Introducing secondary school students to differential equations conceptually is, heretofore, a relatively unexplored branch of education studies. Thus, our research revolved around the question: “What is the nature of the students' conceptual understanding of differential equations, after participating in a RME oriented lesson series?”

2. Theoretical framework

Freudenthal's realistic approach proved itself to be a crucial basis to our research – including, as said, an emphasis on the stimulation of the students' conceptual knowledge.

Teaching mathematics using this realistic approach amounts to immersing the students into some kind of context. Hence, the term 'realistic' – the students are presented with a reality. Note that this reality can well contain fictional elements – the point is to convey a relatable context. Within this reality, the students are confronted with contextual problems. In order to solve these, a mathematical perspective needs to be conceptualized. Due to the student's immersion in the context, mathematical parallels can be drawn towards certain aspects of the given situation. This process is called horizontal mathematization, and lays down a mathematical foundation. As to finding solutions, again, a parallel is to be drawn between the contextual solution, and how to represent it in mathematical terms. This process, in turn, is called vertical mathematization. This way, the realistic method's aim is for the students to not only develop some procedural skill in mathematics, but also to provide conceptual insight into what it is they're doing. Ideally, by this approach, students get a good 'feel' as to how mathematics works.

Bear in mind that this method stands in contrast to the more common method of teaching mathematics. Rather than immersing the students into some reality, the term 'realistic' is misinterpreted in such a way that abstract problems are addressed indirectly, in terms of everyday objects. Instead of being asked to calculate the length of a hypotenuse, the student is asked to calculate the length of a ladder leaning against a wall. One could say that this method is in fact thoroughly procedural, yet thinly veiled into appearing conceptual. In the Netherlands, this misinterpretation of realistic mathematics pervades as being the dominant method by which mathematics is taught in secondary schools.

As a theoretical and colloquial frame by which to evaluate our findings, we used Sfard's (1991) terms of 'structural' and 'operational' understanding. Additionally, Sfard introduces 'interiorization', 'condensation' and 'reification' as being more specific terms by which to characterize more specific phases of a student's understanding of mathematics.

The first two terms can be conceptualized as follows. Structural mathematical understanding means approaching mathematical concepts as being 'static' (and if you will, abstract). In contrast, operational understanding regards mathematical concepts as being dynamic (indeed, tangible) processes. For example, a circle, structurally, is the set of all points which are equidistant to a common given point. Operationally, a circle is the curve drawn by spinning a compass around a fixed point.

We interpreted the meaning of Sfard's terms of 'structural' and 'operational' understanding to mirror our initial choice of terms, 'conceptual' and 'procedural' teaching, respectively. The reasoning behind this alleged likeness is perhaps best explained by example. Understanding a circle conceptually is, by and large, understanding an ideal. According to Sfard, the structure of a circle is a formal description of this ideal. If one were to understand the mathematical nature (i.e. said structure) of this ideal, it wouldn't be a stretch to see why a tangent line passes exactly one point on the circle. In this light, structural knowledge could be regarded as an active (or applied) form of conceptual understanding. Likewise, Sfard's operational understanding is an active form of procedural knowledge. After all, having the wherewithal as to know when (and how) to use a compass or ruler to draw a circle or a tangent line (this being the operational knowledge), is essentially the equivalent of an active procedural understanding of mathematics.

We characterized Sfard's more specific terms towards categorizing mathematical understanding – interiorization, condensation and reification – in a nutshell, as follows. Interiorization is getting accustomed to basic mathematical operations, which in turn lead to an acknowledgement of more general mathematical concepts. For example, learning addition leads to a basic understanding of the concept of the set of natural numbers. Condensation means getting accustomed to certain mathematical operations to such a degree that intermediary steps can be omitted. For example, one could freely say that the equation $2x - 7 = 0$ is equivalent to $x = 7/2$. This is condensation made manifest. Sfard described reification as being a “qualitative quantum leap” (Sfard, 1991, p.20). For reification constitutes one's realization that a particular mathematical operation is part of a larger construct, so to say. For example, one would realize that $ax + b = 0$ is a general representation of $2x - 7 = 0$, and is equivalent to $x = -b/a$. In this, reification could well be described as putting a specific construction in the 'bigger picture' of mathematics.

In a likewise manner to Sfard, Gray & Tall (1994) divide mathematical constructions into three ideological categories: concept, process, and procept. This trichotomy is perhaps easiest explained as follows. Take for example the notion of a function f(x). Here, f(x) encapsulates both the concept of a function for general x, as well as the process of assigning a certain value to a given x. Thus, 'f(x)' symbolizes both a process as well as a concept: Grey and Tall call such an amalgamation of process and concept, a procept.

Tall (2008) also distinguishes three so-called 'worlds' of mathematics: "the conceptual-embodied world, based on perception of and reflection on properties of objects, initially seen and sensed in the real world but then imagined in the mind; the proceptual-symbolic world that grows out of the embodied world through action (such as counting) and is symbolised as thinkable concepts (such as number) that function both as processes to do and concepts to think about (procepts);" and "the axiomatic-formal world (based on formal definitions and proof), which reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set-theoretic definitions" (Tall, 2008, p. 3).

The works of Rasmussen (see Rasmussen & King, 2000; Rasmussen, 2001; Stephan & Rasmussen, 2002; Rasmussen, Zandieh, King & Teppo, 2005; Rasmussen & Marrongelle, 2006), in bearing thematic parallels towards our research in the context of mathematics education, provided us with some practical inspiration towards constructing our lesson plans. Stephan & Rasmussen (2002) describe a series of lessons concerning differential equations. Furthermore, Rasmussen et al. (2005) describe the efforts and effects of a practice-oriented series of lessons. By this, “the research team’s instructional design efforts were grounded in the instructional design theory of Realistic Mathematics Education” (Rasmussen et al., 2005, p.56). While the research goals of these articles are not fully in line with our own (not to mention the fact that they were aimed at college students as opposed to secondary school students), their results and methods helped us in the construction of our own lesson series.

Zwarteveen(-Roosenbrand), Verhoef, Hendrikse & Pieters (2009, 2010, 2011) as well as Verhoef, Zwarteveen-Roosenbrand, Van Joolingen & Pieters (2013) provided us with several concrete pointers towards teaching differential equations in secondary school. We integrated these in our lesson setup. Specifically these pointers were: (i) the advice of starting off the lesson series with a discussion as to the meaning of differential equations in terms of change of a variable, followed by a formal definition and a short test (this point is congruous with earlier studies by Machiels-Bongaerts & Schmidt (1990) and Peeck, Van den Bosch & Kreupeling (1982), which concern the importance of activating or 'mobilizing' the students' prior knowledge before introducing new material), (ii) teaching the composition of differential equations by way of four 'modeling phases' and (iii) the integration of a (slightly modified) assignment used in their research, which in turn was a modified version of an assignment adopted from Rasmussen's research.

3. Methodology

3.1 Participants

Our research was conducted at the Twickel College in Hengelo. It was participated by eight vwo ('pre-university') students (six male, two female), who were in the fifth year of their mathematics D course (age 16-17). Among them, one of the students repeated the fifth year (thus being about a year older than his peers). All of the students were heretofore unknown to the term 'differential equation', but they did already have a grasp of basic calculus. The individual students are indicated by an initial.

The participating instructors were (U) and (V) – both of us, students towards attaining a grade in teaching – and (F), an expert teacher at the Twickel College. An itinerary at the Twickel College, (U) was the practicing instructor during the lesson series, having (F) as a supervisor. As such, (F) provided us with feedback during the course of the lesson series. (V) observed the first and third lesson.

3.2 Research instruments

The main instruments toward our research were (i) an exam, (ii) a question list, and (iii) so-called lesson preparation forms.

The exam took place the week following the last lesson of the series, and was made by all eight students. The exam contains seven assignments, variously aimed at conceptual understanding or procedural skill. One of these assignments is a slightly modified version of an assignment used by Rasmussen & Marrongelle (2006), and later also by Zwarteveen et al. (2009, 2010, 2011) and Verhoef et al. (2013). We chose to include this assignment to provide us with a basis of comparison with these previous researches. The exam can be found in appendix A.

The question list was presented immediately after the exam. It constitutes nine open questions and short elaborations towards some of our choices in composing the lesson series. In total, four distinct topics are addressed in the question list: the students' motivation, their ideas about conceptual mathematics, their views on the context we used, and their experience in using computer software for some of the homework assignments. These topics, as well as the fact that the questions were open, served as a way to capture the students' personal opinions towards the way the lessons were conducted. The students' personal typification of conceptual mathematics is an indication towards how they view their own (conceptual) grasp on the theory. The other topics would give an indication towards certain factors, which in turn influence whether or not a measure of conceptual understanding is attained. The question list can be found in appendix B.

Our third instrument, the lesson preparation forms, were made prior to each individual lessen. These forms are essentially a schematic summary of the corresponding lesson, giving clear indications as to what topic (or lesson point in general) is addressed at what time. The forms also describe the learning goals and mode of execution of each topic. Thus, after each lesson, the lesson preparation form can be used as a reference when evaluating the lesson itself. For example, when the discussion of a particular topic takes more time than initially planned, possible implications on the way of conducting the lesson series as a whole can be discussed afterwards, and be incorporated in the next lesson.

Here, we will use the lesson preparation forms as a reference with which to pinpoint where certain problems arose and the implications these problems had on the subsequent lessons, as well as general recommendations for future use. The lesson preparation forms can be found in appendix C.

3.3 Material

The students' weekly lesson plan, as set up by the school, bound us to the following set of rules, in composing the lesson series. We were granted the last four lessons of the students' curriculum (the Fridays of the 24th and 31th of May, as well as those of the 7th and 14th of June in 2013), and finishing these off with an exam in the final exam period (specifically on Monday the 17th of June).

In this exam, we were asked not to put too much of an emphasis on conceptual understanding, thus incorporating procedurally oriented questions as well. Furthermore, each lesson would take 90 minutes (with a five minute halfway break).

That said, we set about putting together the theoretical contents of the course, and how these contents would be built up during the lesson series. With this rough outline, we composed modules for each lesson. During the course itself, after every lesson, we would evaluate this lesson and, if necessary, adapt the modules and didactic approach for the following lessons accordingly.

3.3.1 Context

Firstly, to the end of assuming the RME methodology, the actual context – the 'realistic setting' – of the modules was considered. In this, we opted for a historical perspective by making up some (mostly fictional) adventures of the pirate captain Blackbeard. In part, containing a humorous element. We are aware that we could have opted for a perspective pertaining ('real') current affairs, and in so doing, adopting a more serious approach, yet we deliberately chose not to. The reasoning behind this is that we thought this would come across as too 'dry', and that by a bit of a nonsensical (yet relatable) approach, we would reach to more students. The context would leave the students with a strong and clear image of the situation, in turn promoting their mathematical conceptual understanding, or at least so we thought. Additionally, the diversity of Blackbeard's adventures (i.e. the multitude of solid exemplary situations we presented) could easily be coupled to current practical affairs.

3.3.2 Approach

The first three lessons are spent towards laying down the theory, and the fourth lesson as a recap and summary of the previous three. Each of the first three modules contained homework assignments concerning the theory of the respective module, as well as building a theoretical bridge towards the theory of the subsequent week. The homework assignments of the fourth module constituted a diagnostic test, basically amounting to exam training.

The lesson plan incorporated RME by making the theoretical build-up of the first three modules context-based. In these, the students are challenged towards using their skills in order to find a solution to a contextual problem. The general theoretical frame would then be discussed as a group. The module for the final lesson would be a full summary considering all topics, as well as giving specific (numerical or analytical) examples. This educational approach (solving problems first, general theory second) happens to be the exact opposite of what the students are used to.

3.3.3 Content

Towards determining the exact contents and build-up of our lessons, we bore in mind the findings of the researches of Rasmussen & King (2000), Rasmussen (2001) and Zwarteveen et al. (2009, 2010, 2011) and Verhoef et al. (2013). Figures 1 and 2 show a summary of their (to us) most noteworthy findings and recommendations, respectively.

We then set about formulating concrete learning goals for the lesson series. In this, creating conceptual understanding had a priority over creating procedural understanding. More generally, the goal of this lesson series was to give a good and thorough first impression of DEs and their applications. To this end, specifically, we wanted to handle the definition of a (first order) DE, composing a DE by using a practical context, and giving some basic understanding towards solving DEs, most notably by using Euler's solution method. The specific goals to each lesson can be found in appendix C.

a) Students, at first, seem to associate DEs to the concept of exponential growth.
b) Students have no set strategy towards composing DEs and do not manage to find a specific goal toward doing so. Also, they have difficulty in acknowledging the concept of 'change', in the context of the DE.
c) The students do not apprehend how change can be described, using a derivative.
d) The students are used to working procedurally and lack structural understanding.
e) Modeling a dynamic process using a derivative is quite different from what the students are used to. They prefer to deal with 'direct' formulas or functions.
f) Students regard the concept of a function to be represented by its notation ('f(x)') – with that, a graph of a solution curve in a line element field is not regarded to be a function.
g) The students, in working with DEs, tend towards choosing the wrong variables, or lose sight of them.
h) Students tend to think that an equilibrium solution to a DE exists, whenever the differential quotient equals zero. The students seem not to regard a constant function as a function.

Figure 1: Rasmussen's and Zwarteveen's findings

a) Start off with simple dynamic processes, using various notations of a derivative.
b) At first, use the words 'change equation' instead of 'differential equation'.
c) Don't limit the theory to DEs concerning exponential growth.
d) Use common physical units in order to promote the students' structural understanding.
e) Attend to the students' understanding of a fraction as either a quantitatively procedural measure (this being a process), or as a measure of change (this being a concept). Give subtle indications as to the difference between the two.
f) Differentiate between discrete and continuous processes. It's best to handle continuous processes first.
g) The optimal situation would be for the students to ascertain independently that finding a direct function (rather than a DE) towards describing a dynamic process is, at best, impractical, if not impossible.

Figure 2: Rasmussen's and Zwarteveen's recommendations

3.3.4 Distribution of theory

After putting down our lesson goals we set upon determining which topics to handle as a group or individually, and which to assign as homework. As said, we composed modules containing contextual problems to be worked out during the lessons, as well as containing (contextual) homework assignments. We also put together explications towards the homework assignments, to be handed out the subsequent lesson.

A summary of the proposed topics per lesson, and the order and way in which they were handled, is given in figure 3.

Note that in lesson 2 we used the scheme proposed by Zwarteveen et al. (2010), which constitutes a design outline towards composing a DE. We will get into this later. Also, the homework following the third lesson contained several computer assignments. These assignments used the program Winplot – a handy piece of software which can be easily used towards plotting graphs of line element fields of DEs, as well as solution curves (for one, using Euler's method).

The modules, such as we presented them, can be found in appendix D.

3.4 Procedure

3.4.1 The lessons

Lesson 1

The first lesson presented us with some of the more immediate problems of the realistic method of teaching. The students were quite unready, or in some cases unwilling, towards adopting the more constructivist paradigm on which the lesson was based.

The first lesson point – setting in motion a class discussion as to determine the nature of a derivative – became a quick indication to this unreadiness. The students were quick to offer formular descriptions of derivatives, but they simply were at a loss in what to think or say, when asked to describe the nature of these formulas. (This nature being change – or more specifically, a proportional representation of change.) After some effort on both sides (with us dropping and the students considering hints), a description of this effect had to be offered up by us, rather than it being a result of a class discussion (which never truly took form).

Lesson 1 – An introduction to, and composing simple DEs

- Zeroing in on the concept of a derivative (class discussion).
- Compose a DE using Newton's law of cooling (independently, using the module).
- The definition of a DE (class instruction).

Homework assignments (independently, using the module).

- Composing more DEs using Newton's law of cooling.
- Composing a DE representing exponential growth.

Lesson 2 – Composing more involved DEs and an introduction to solving DEs.

- Discussion of the homework assignments (class discussion).
- Zwarteveen's scheme towards composing a DE (class instruction).
- Composing a DE using Newton's second law of motion (independently, using the module).
- Discussing the fact that a DE has multiple solutions, with the solutions being functions, which can be determined if starting values are given (class instruction).
- An introduction to line element fields and solution curves (class instruction).

Homework assignments (independently, using the module).

- Composing more involved DEs.
- Composing line element fields and solution curves.

Lesson 3 – Solving DEs using Euler's method, and verifying exact solutions by substitution.

- Discussion of the homework assignments, and in so doing, emphasizing the meaning of line element fields (class discussion)
- Introduction to Euler's solution method (independently, using the module).
- Explicating Euler's solution method (class instruction).
- Verifying exact solutions to DEs, using substitution (class instruction).

Homework assignments (independently, using the module).

- Using Euler's solution method.
- Using substitution towards verifying exact solutions.
- Using Euler's method and the general composition of line element fields using the Winplot computer program.

Lesson 4 – Recap and summary.

- A thorough run-through of each topic considered in the past weeks (class instruction and discussion).
- A presentation of several real-world applications of DEs (presentation).

Homework assignments (independently, using the module).

- Diagnostic test.

Figure 3: lesson summary

The second point of order was to set about constructing a relatively simple DE, using the first part of the module. The students were put to work, but their efforts were quickly mired by the way the questions were posed. In particular, this started on question (b), in which they were asked to determine certain values of a derivative, given a graph. Whilst very much able to see that said derivatives can be determined by calculating the slope of the corresponding tangent lines (drawn by hand), the students did not see that this was, in fact, the only way of doing so (with the given data). Consequently, this solution was beheld as something of an anticlimactic method, when we were forced to give it away. The subsequent questions held a similar pattern: the students struggling to apprehend the direction the question required them to take, then disappointment when faced with a (to them, anticlimactic) exposition. This pattern also caused most students to lose interest, with under half of them making it to question (d), quasi-independently, most of them giving up before that. At this point we intervened by explicating the questions and answers classically.

Following this, a general definition of DE's was given, followed by a short test. The students who slacked off previously managed to perk up, due to the straightforward instructional nature of this part. Remarkably however, regardless of their newfound enthusiasm, some of them seemed not to understand the (relatively simple) given definition. The test (amounting to identifying a couple of equations as being a DE or not) was met with several wrong answers. Their errors, however, were quickly corrected by the other students.

Several tentative hints towards the homework assignments marked the end of the first lesson.

On the whole, the lesson was wrought with a general feeling of aimlessness on part of the students. “Give us some theory already!” a student exclaimed, at one point. Due to this, as well as the words of the supervising teacher (F) gave us (“Bear in mind we're not dealing with university students here.”), we set about revising the lesson plans for the coming weeks.

Our new approach would favour direct class instruction, at least more so than it did before. More accurately, rather than putting the students to work on the modules individually, we would take them by the hand (so to speak), and work through the module as a group.

Another issue was the matter of time. Due to the relative lack of success of the first lesson (we felt that the students weren't sufficiently up to speed on this lesson's topics for them to adequately make the homework assignments), we felt obliged to include a point-by-point recap in the second lesson before commencing the planned topics. Effectively, this brought about a cascade shift in our program. The consensus was to omit the computer oriented part of the third lesson, and include it as homework assignments to the corresponding module.

Lesson 2

This new approach would prove to be fruitful. For the sake of clarity, we started off the second lesson by giving a rather more detailed explanation as to the nature and goal of this lesson series. This was then followed by the aforementioned recap of the first lesson.

It appeared at this point that the students were getting more accustomed (or at least open) to the, to them, unusual method of teaching mathematics this lesson series is subject to. We would venture to say that our efforts towards accommodating for their erstwhile confusion did not go unnoticed, and may have caused them to adopt a more sympathetic stance.

The lesson went on much as planned, though three particular exchanges are worthy of note. Firstly, in constructing a DE (using the second module), the students expressed some bewilderment when it became clear that the basic equation was a manifestation of Newton's second law of motion (here, net force = gravitational force minus friction). The law in itself caused no confusion (it was readily offered up by the students), but the simple fact that a law so grounded in the physical sciences was applied in a mathematical perspective was, to the students, unexpected.

Secondly, when asked to reduce the equation (which we put down in terms of forces) to a DE, the students expressed difficulty in identifying which of the variables brought about change. In other words, they were unable to identify the differential quotient (without a great deal of help). Only after a great deal of hints from our part did they notice acceleration to equal the change of velocity over time. Clearly, the notion of a DE as describing a process of change of a variable was not yet fully assimilated. Having said that, when presented with this fact (acceleration being change in velocity), they were quite quick to see the implications this would have on the resulting DE (for example, terminal velocity meaning that the acceleration equals zero).

Thirdly, come the end of the lesson, we were asked for answers to the homework assignments of the coming week. Our initial stance towards this was to give explicative hand-outs to the homework, the lesson after they were assigned. This, in order to promote a problem-solving atmosphere, independent of given answers to work 'towards'. The students on their part argued that they were quite willing to solve problems, such as they were posed, but were frustrated not to have anything with which to verify their findings. As indeed most of the students had tried to work on the assignments of the previous week to the best of their ability, we conceded to their request.

Lesson 3

The third lesson, though productive and largely proceeding as planned, presented us with another (perhaps quite typical) exchange. Bear in mind that the modules were written in such a way as to develop a frame of mind by building a mathematical construct in small, elementary steps. From this (relatively simple) construct, the general theoretical frame of the lesson would 'naturally' follow. As the module for this day constituted a particularly simple construction (graphical from the go, with direct and clear instructions as to what steps to take), we deemed it reasonable to let the students work on it independently, and see how it would go.

As in the first lesson, the students seemed unprepared for the simplicity of some of these steps. The general feeling is perhaps best described by a particular exchange between the instructor and one of the students. After some debate as to the steps required to take (as instructors we didn't want to give away too much in these kinds of debates), the student remarked “Well it's obvious that the line will cross the boundaries of the circle.” to which the instructor replied “That's what I wanted to hear. It's all there is to it.”

After this exchange, the rest of the module, and the encapsulating theory, was handled as a group. With our (more direct) guidance, the direction we were going for with the module was picked up rather more quickly. When discussing the general theory (Euler's method of solving DEs), the students even offered up that a more accurate approximation towards a solution of a DE could be attained by choosing a smaller step size for your calculations.

Lesson 4

The fourth lesson, being a (rather straightforward) summary on the topics of the previous weeks, followed by a short presentation on the applications of DEs, was met with some enthusiasm on part of the students. To them, the recap provided some much-needed oversight, some of the students even reaching (in Sfard's terms) a state of reification, as would become clear with the coming test. The presentation concerning the applications of DEs in particular managed to pique their interest, as it left them with a good impression as to their usefulness.

3.4.2 Data

We took in eight exams and question lists.

The exams have been analyzed in two ways. The first pertained the students' actions – by which we mean the proposed steps towards completing a given assignment in a mathematically correct manner. For example, producing a particular insight, or handling a mathematical operation correctly. Every one of these actions pertains to a certain topic we have discussed during the lessons. Thus, by analyzing them we can characterize the students' understanding towards these topics. The second analysis is a comparison of the students' procedural knowledge, relative to their conceptual knowledge.

The assignments in the exam can be differentiated into three main categories: (1) composing a DE, (2) understanding the concept of a DE, and (3) solving a DE. We classified every action required to complete these assignments into one of these categories. We checked the assignments of each individual exam, and noted whether or not the actions made therein were correct. We also checked the manner with which each action was made. Specifically, whether they were correct and thorough, correct but sloppy or incomplete, incorrect, or not made at all. Of course, it is possible that early actions made in a particular assignment, if made incorrectly, influence the course of subsequent actions, but we don't pay this special heed in our analysis. We also noted remarkable answers and (apparent) trains of thought.

In the second analysis of the exam, we classified the assignments (and their sub-assignments) as being either conceptually based or procedurally based.

The question list, in turn, is composed in such a way that it asks for the students' opinions regarding topics divided into four major categories. Every question can be answered in three general ways – positive, negative, or neutral (indifferently). We analyzed the question lists by classifying their answers as such. Here, also, we noted remarkable answers.

3.4.3 Data analysis

Our data analysis is subdivided into the three analyses described above: first, the two analyses on the exam, respectively, followed by the analysis on the question lists. The analyses pertaining the exam use the graphical classifications as described in figure 4.

[●]	the action is done thoroughly and correctly
[○]	the answer is correct, but the action is done sloppily or is incomplete
[x]	the action is done incorrectly
[-]	the action is not done
[ ]	(blank space): the action is irrelevant (due to an earlier incorrect action)

Figure 4: the classification of actions

<u>*Exam analysis 1: per action*</u>

Here, we present an exposition of the individual actions in the exam's assignments, divided into the categories of (1a) composing a DE, (1b) understanding the concept of a DE, and (1c) solving a DE.

1a – Composing a DE

Assignment 1 and 3 concern the composition of DEs. Specifically, the students are asked to compose a DE in assignment 1, as well as 3b and 3d. In this, assignment 1 is largely reproductive: the students have had numerous homework assignments concerning DEs of the same form, as well as their composition. Assignment 3 – being the aforementioned modified version of the assignment used in Rasmussen and Zwarteveen's respective researches – is rather different, inasmuch as the students are presented with an unfamiliar manner of questioning.

Towards completing these assignments, we introduced a scheme (figure 5) towards composing DEs as devised by Zwarteveen et al. (2010).

a) Understanding the situation (As) and identifying the relevant quantities (Ai).
b) Choosing the correct variables (Bv) and accompanying units (Bu).
c) Identifying the correct independent variable (Ci) and expressing into words how this variable changes (Cw), in the form of a difference equation (Cd).
d) Expressing the measure of change in a DE (Dd) perhaps preceded by a difference equation (Dp), along with the starting values (Ds).

Figure 5: Zwarteveen et al's classification scheme towards composing a DE

In our analysis of said assignments we used this classification scheme as a representation of the correct actions, although we omitted the subclassifications (As, Ai, and so forth). Thus, we only regard the classifications A, B, C and D.

For example, we classified (M)'s actions in making assignment 1 as "x ○ ● x". This can be interpreted as follows.

- The situation has not been fully understood or the relevant quantities have not been correctly identified ("x").
- The chosen variables and units were correct, but incomplete ("○").
- The independent variable has been correctly identified ("●"). (Note that, in this assignment,

putting the DE into words was not necessary.)
- The composition of the DE was incorrect (“x”). (The starting value was of no consequence.)

Besides this, assignment 3 bore several more actions towards correctly composing a DE. Sub-assignment 3a requires the students to correctly identify a net change in volume (by using in- and outflow). In our analysis, this constitutes one action. Assignment 3c, in turn, requires the students to correctly identify the value of the equilibrium state of a salt solution decreasing in density, and sketch a representative graph (two actions). In assignment 3d, we also noted the actions of correctly identifying the standard form of the DE representing the situation (this being Newton's law of cooling), and using linearization towards identifying the heat transfer coefficient, on top of the actions dictated by Zwarteveen's classification scheme.

1b – Understanding the concept of a DE

Assignments 2 and 4 tested the students' conceptual understanding of DEs. In assignment 2, which required the students to define a DE in words, we identified three actions for our analysis. These being, (1) mentioning the change of a quantity or variable, (2) mentioning that this change is expressed in terms of said quantity or variable, as well as (3) the quantity or variable it depends upon. Assignment 4 on itself could be made correctly, incompletely, or incorrectly, thus constituting one action.

1c – Solving a DE

Assignments 5, 6 and 7 tested the students' knowledge in in solving DEs. Assignment 5a required the students to calculate the terminal speed of a parachutist in free fall (one action), and 5b was an application of Euler's solution method. This was subdivided into two actions: composing the recursive formula, and choosing the correct value of n. For assignment 6a, verifying the solution to the DE by substitution constitutes one action. In turn, 6b constitutes three actions, the first two for showing the insight as to acknowledge that $dy/dx = 0$ and $y = 0$, and the third for solving the resulting set of equations. Assignment 7 requires one action.

Exam analysis 2: conceptual or procedural actions

Our second analysis pertained the conceptual or procedural basis of the assignments. In this, assignments 1 (reproducing the composition of a DE), the action of linearization in 3d, 5, 6 (except for the insight that $dy/dx = 0$), and 7, are all procedurally based. The remaining actions are conceptually based. Having made this distinction, actions of the sub-assignments, if more than one, were merged into one, pertaining to whether or not it was made correctly. For example, where in our first analysis (M)'s classification of assignment 1 was “x ○ ● x”, it becomes just “x” in this second analysis, seeing as the composition of the DE was incorrect.

In order to get a good image of the students' conceptual and procedural acumen respectively, we expressed their individual relative amount of correct answers (for the conceptually based as well as the procedurally based actions) in a percentage.

Analysis of the question lists

The question list is composed of questions pertaining, respectively, the students' motivation during the lesson series (question 1), their own views and opinions on the conceptual method (q.2-4), their views and opinions on the context (q.5-7), and their views on using the Winplot software (q.8-9).

The essence of their answers could be qualified as being 'positive', 'negative' or 'neutral' (alternatively, 'indifferent'), although question 8 cannot be answered neutrally. Figure 6 gives an overview of the graphical representations we used towards qualifying the students' answers in our analysis.

4. Results

Results of exam analysis 1

Table 1 shows the results of our first exam analysis. The three main topics of the course are put in bold, below which the assignments pertaining to each respective topic are shown. The upper row shows the initials of the students – the actions they made towards answering each assignment are represented in their respective column.

Motivation (q.1) + Motivation is higher than usual - Motivation is lower than usual +/- Motivation hasn't changed		
Conceptual thinking: personal understanding of DEs (q.2) + Fair understanding - Poor understanding +/- Limited understanding	Conceptual thinking: preference towards procedural or conceptual (q.3) P Preference to procedural C Preference to conceptual P/C No particular preference	Conceptual thinking: personal image of base concept and applications of DEs (q.4) + Attained a good image - Did not attain a good image +/- The image is limited
Context: helpful or not (q.5) + Yes - No +/- Neutral/indifferent	Context: Blackbeard's adventures (q.6) + Amusing - Not amusing +/- Neutral/indifferent	Context: serious/historical/...? (q.7) S Prefers serious context H Prefers historical context S/H Indifferent N Prefers no context
Computer assignments: made or not (q.8) + Yes - No	Computer assignments: useful or not (q.9) + Yes - No +/- Not sure	

Figure 6: Classification and meaning of the answers to the question lists

Notes

1a – Composing a DE

Six of the eight students manage to compose a correct DE in assignment 1, albeit in some cases, sloppily so (divisions were given decimal approximations, or left unsimplified). Besides that, three students put down the starting value, and only one mentions the independent variable. One student in particular (P) explicitly applies Zwarteveen's scheme towards composing the DE, and does so very tidily. Another student also attempts at using the scheme, but falters herein, and does not produce the correct answer.

In assignment 3b, five students manage to compose the correct DE. This, despite assignment 3a, being constructed with the purpose of assisting the students towards answering 3b, was done correctly by every student. In a likewise manner, 3c (drawing a sketch representing the situation) was supposed to evoke the right procedure towards answering 3d. The former was done correctly by five students, whilst only three students answered 3d correctly. That said, by being a relatable graphic representation of a well-known process, the sketch proves itself to be a vital crutch towards answering 3d.

It should be noted that the students do not apply Zwarteveen's scheme consistently. It is used rather more frequently towards answering assignment 3 than it is towards assignment 1. For instance, (G) does not use the scheme for assignment 1, does use it for 3b (flawlessly so), upon which he makes 3d again without using it. It should be noted that he did manage to get a correct DE in all three instances.

	A	G	J	K	M	P	T	W
Composing a DE								
Zwarteveen's scheme (1)	- - - ○	- - - ○	- - ● ○	- - ● ●	x ○ ● x	x ○ x ○	- - ● x	- - ● ○
Net change (3a)	●	●	●	●	●	●	●	○
Zwarteveen's scheme (3b)	- - - ●	● ● ● ●	● - ● ●	● ● ● x	- - - x	● ● x x	- - - ●	- - - ●
Sketch (3c)	● -	● -	● -	x -	● -	x -	● -	x x
Zwarteveen's scheme (3d)	- - - x	- - - ●	- - - ●	● ● x -	- - - -	● ● x x	- - - ●	- - - -
Using standard form of DE (3d)	-	●	●	-	●	x	●	○
Linearization (3d)	-	●	●	-	-	x	●	-
Understanding the concept of a DE								
Definition (2)	● ○ ○	- ● ●	● - ○	○ - -	○ ○ -	● ● ●	● - -	● ● ○
Equilibrium state (4)	○	-	-	-	-	-	-	●
Solving a DE								
Terminal velocity (5a)	●	●	●	●	●	●	●	●
Euler's method (5b)	● ●	● ●	x x	- -	● x	● x	● x	● ○
Verifying an exact solution (6a)	●	○	○	○	○	○	○	○
Solution curve (6b)	● ● x	- ● x	● ● ●	- ● ○	- - x	- ● x	- - x	● ● ●
Line element field and DE (7)	●	●	●	○	-	●	●	●

Table 1: Results of exam analysis 1

1b – Understanding the concept of a DE

One student produces a verbatim quote of the definition of a DE, such as we presented it. As for the other students, most mention the change of a variable, and half of them mention the fact that this change is represented in terms of said variable as well as another variable. The students appear to find difficulty in explicating the exact meaning of a DE.

Assignment 4 bears us witness to two general misinterpretations. Firstly, some students confuse the definition of equilibrium solutions, such as we presented it, with the concept of solution curves. To take (J)'s answer for example:

“Leaving from (4,4) as a starting point gives us a wholly different solution curve when compared to leaving from (1,3). These lines do not cross each other, so there exist multiple equilibrium solutions.”

Secondly, some students focus on the (supposed) asymptotes that exist in the given line element field. They propose that two lines can be drawn, both for which dy/dx = 0 holds (while actually, in the given line element field, this equation holds for a parabola passing the origin).

This misinterpretation can be explained. Some of the homework assignments explicitly concerned determining 'asymptotes' in a line element field. The students' answers leave us with the impression that they answered this exam assignment with these particular homework assignments in mind, as if to say “I guess this assignment concerns that topic”.

Only one student managed to give the correct answer, backed by the right reasoning. Another student did give the correct answer, but her reasoning was flawed.

1c – Solving a DE

Every student completed assignment 5a without fault. Note that this assignment pertained a very straightforward procedure (which was reiterated several times throughout the homework assignments). Euler's method (assignment 5b) is applied correctly by three students. Four students read the wrong value for n, opting for a value of (n+1) = 4 instead of the required (n+1) = 40. In other words, they do not see that the stated step size of one tenth of a second, implies the desired time of 4 seconds to correspond with 40 steps. One student makes no attempt toward an answer at all.

In assignment 6a (verifying a given exact solution to a DE), only one student explicitly states the resulting equation (yielded by substitution) holds 'for all x'. All other students only (though correctly) apply the procedure of substitution, and simplifying the resulting equation. In assignment 6b, three students correctly state that dy/dx = 0 and y = 0, from which two of these students yield the correct answer – the third makes a calculation error along the way. Three students, in turn, do see that y = 0, but attempt to further complete the assignment either by trying to read the given graph, or just by expressing c in terms of x. The two remaining students opt for a thoroughly incorrect approach altogether, with one reading off the value of an unrelated (for the purpose of the assignment) coordinate, and the other attempting to use linearization between two quasi-arbitrary coordinates.

Assignment 7 was done correctly by seven students. After writing down several thoughts, (M) states that she does not know what to do.

<u>Results of exam analysis 2</u>

Table 2 shows the results of our second exam analysis, which differentiated between procedural and conceptual actions. The table also shows the results pertaining our question list. Likewise to the previous table, the main topics (in the exam as well as the question list) are put in bold lettering, below which are shown the exam assignments or question list questions pertaining to each respective topic. We also show the students' exam grades (out of a possible 10 – bear in mind that in Dutch secondary schools, a 5.5 is the lowest possible passing grade). Again, the students' answers (and grades) are represented in their individual column. The rightmost column shows the average percentage of procedurally and conceptually based assignments answered correctly.

As we can see, every student has a higher percentage of correct answers to procedurally based actions than they do to conceptually based actions. (K) shows the largest relative difference between the two – 83% procedural to 11% conceptual; followed by (P) – 67% to 22%; in turn followed by (G) and (W) – both 100% to 56%. (J), (M) and (T) bear a smaller relative difference, with 86% to 67%; 40% to 33% and 67% to 56% respectively. On a procedural level, this means (G) and (W) bear the highest scores, both at 100%. On a conceptual level these are (A) and (J), both at 67%. Procedurally, (M) is the only student who scores comparatively low (40%), whilst (K), (P) and (M) score lowest conceptually (at 11%, 22% and 33%, respectively). Overall, (J) scored highest, (M) lowest.

<u>Results of the question lists</u>

None of the students state they have acquired a good conceptual understanding of DEs. Three explicitly qualify their conceptual understanding to be poor, whilst the others qualify it to be limited. Five students express their preference towards the procedural method, whilst one prefers the conceptual method. The remaining two express no preference. That said, the image as to the concept of a DE, as well as its applications, is generally stated to be quite positive.

	A	G	J	K	M	P	T	W	
Exam: Procedures									
Composing a DE (1)	○	○	○	●	x	○	x	○	
Linearization (3d)		●	●			x	●		
Terminal velocity (5a)	●	●	●	●	●	●	●	●	
Euler's method (5b)	●	●	x	-	x	x	x	○	
Verifying an exact solution (6a)	●	●	●	●	●	●	●	●	
Solution curve (6b)	x		●	○				●	
Line element field and DE (7)	●	●	●	○	-	●	●	●	
Percentage of correct actions	83%	100%	86%	83%	40%	67%	67%	100%	**78%**
Exam: Concepts									
Definition (2)	-	-	-	-	-	●	-	-	
Net change (3a)	●	●	●	●	●	●	●	○	
Composing a DE (3b)	●	●	●	x	x	x	●	●	
Sketch (3c)	●	●	●	x	●	x	●	x	
Composing a DE (3d)	x	●	●	-	-	x	●	-	
Using standard form of DE (3d)	-	●	●	-	●	x	●	○	
Equilibrium solution (4)	○	-	-	-	-	-	-	●	
For all x (6a)	●	-	-	-	-	-	-	-	
dy/dx=0 (6b)	●	-	●	-	-	-	-	●	
Percentage of correct actions	67%	56%	67%	11%	33%	22%	56%	56%	**46%**
Exam: Final grades	8.4	8.8	9	5.9	5.5	6.2	7.9	8.7	
Question list: Conceptual thinking									
Understanding (2)	-	+/-	-	-	+/-	+/-	+/-	+/-	
Preference (3)	P	P/C	C	P	P	P	P	P/C	
Image of DE concept/application (4)	+	+/-	+	+	+	+	+/-	+	
Question list: Motivation									
Motivation (1)	-	+/-	+	+/-	-	+/-	-	+	
Question list: Context									
Useful (5)	-	+/-	+/-	+	-	-	+/-	+	
Adventures (6)	-	+	+	+	+	+	+	+	
Serious (7)	S	S	S/H	S/H	S/H	N	S	H	
Question list: computer assignments									
Made (8)	+	+	-	-	-	-	-	+	
Useful (9)	-	+						+	

Table 2: Results of exam analysis 2, as well as the question lists

As far as motivation is concerned, the reactions to the question list are mixed. Two students react positively, three do so negatively, and three are neutral.

Note, at this point, the apparent correlation between the students' relative motivation and their preference towards either the procedural or conceptual method. The two students who expressed their motivation, also stated to prefer the conceptual method (J) or a balance between the procedural and conceptual (G). All three students who expressed their relative lack of motivation also stated their preference towards the procedural method. Of the remaining three, two prefer the procedural method, and one is neutral.

Two students stated that our chosen context helped them in developing their understanding of DEs. In turn, three students stated the opposite. Two of them particularly lament the fact that our approach was too text-heavy, while instead they prefer a short and simple list of data towards completing the given assignment (thus, their complaints concern the amount of text we presented, rather than the actual context). Three of the students take an indifferent stance to our chosen context, neither feeling particularly distracted nor engaged by it.

Seven students found Blackbeard's adventures to be rather amusing, whilst one student, quite explicitly, did not (instead finding them rather “childish”). That said, three students would rather opt for a more serious context, while one student finds himself engaged by its historic setting. Three students express no particular preference, and one states to prefer no context at all (this is one of the students who already expressed difficulty towards the amount of reading involved).

The homework assignments concerning the Winplot computer program were made by only three students. Two of them state that the program helped towards developing their conceptual understanding, while one was not sure if it had.

A short combination of results

The question lists indicate that the procedural methodology is preferred over the conceptual. The exam results correspond with this idea, inasmuch as the relative procedural score is higher for each student. Also, in being among the two conceptual top-scorers, (J) was alone amongst all students in his preference of the conceptual method over the procedural.

5. Conclusions

Although the findings are varied and cover a broad spectrum, a major part revolve around the fact that the students simply are not used to a conceptual approach of mathematics. More specifically, our approach of mathematics was quite at odds with their expectations. The first lesson already gave a clear indication to this, but it carried on through the lesson series, and is also reflected in the exams and question lists. We will get into this presently. After that, we will regard the findings pertaining to the nature of students' actual knowledge of differential equations.

5.1 The conceptual approach versus students' expectations

The students' expectations – quite thoroughly set in the procedural approach – proved themselves, indeed, to mar our RME-based efforts. As indicated, two of the students in particular expressed displeasure at having to read so much, and more of them found difficulty in ascertaining the required mathematical data from the given context. With this, they show a certain unaptness towards horizontal mathematization. This is further indicated by the occurrences of particularly the first and third lessons (and also some of the exams), in which the students were quite dubious towards putting the given (contextual) data in familiar mathematical terms. Conversely, when guided towards (or presented with) a direct approach, their grasp of the situation and its inherent procedural knowledge proved to be exemplary, at least so during the lessons. What we can say at this point is that the students truly appear to view mathematics as a 'bag of tricks' (these 'tricks' sharing a common language – again, a thoroughly procedural view), rather than as an arrangement of interrelated ideals (bearing an interpretative application toward the real world – which is the conceptual view). One would say that, independently, the students are quite able to reproduce, but not to produce mathematics.

Nevertheless, as stated, the students were quite able to get a mathematical grasp on a situation, with proper guidance. Notice the difference in productiveness during the lessons, between when the students were put to work independently and when the topics were discussed as a group. Independently, the students were unready, or unwilling, to translate elementary contextual aspects into mathematical steps. On top of that, bearing in mind their remarks in the question lists concerning the accumulation of data from a given context, the students seemed put off (or even bored) by the sheer amount of (text-based) contextual information. Conversely, the students were more attentive during guided class discussion of the very same context, and the topics on the whole were handled much more efficiently whilst doing so.

To use Sfard's terms, the students are yet to reach a state of reification in what (to them) should be relatively elementary mathematical concepts. For example, question 6b of the exam states the x-axis to be a tangent line to a solution curve, which of course amounts to saying that $y = 0$ on $dy/dx = 0$ for this particular solution – relatively few students made this observation. Generally, would they have reached a state of reification, the solution to any elementary step (such as they were presented during the lessons) should be obvious and singular, and certainly be attainable independently. That is to say, the solutions should not be in doubt – for most of the students actually did find them obvious, but were put off by this obviousness. Again, this indicates the students' expectations concerning mathematics education. Rather than presenting them with new theory directly, they were to use theory already known to them to 'build up' new theory themselves.

That said, a distinctly positive development during the lessons was that the students became accustomed to the RME approach. This became apparent mostly during the final lesson. The recap and summary – both presented and discussed primarily on a conceptual level – were met with an apparent understanding on the students' part. Also, the instructor used call-backs to the more humorous elements of the modules' context in several examples, which were met appreciatively. This would indicate that, indeed, our chosen context had stuck, at least to some degree.

Furthermore, the students asked poignant questions pertaining certain mathematical details. For instance, (J) asked why the function $y(x) = ae^{\wedge}(cx)$ could be regarded as a general solution to $dy/dx = cy$, "just like that". His reasoning was that this choice of function seemed arbitrary, and thus, irresponsible on a mathematical level. The instructor explained that, simply put, said differential equation 'was invented first', and that the number e was 'invented' so as to conform to this differential equation. Likewise exchanges seemed to 'seal the gaps', as it were, to some students.

5.2 The students' understanding of differential equations

Bearing in mind Gray and Tall's (1994) notions of process, concept and procept, the start of the first lesson was a clear indication that the students started off with very little in the way of understanding differential equations as a concept. Case in point, when asked to describe the nature of a differential quotient, the students initially gave a formulaic expression, and noted that it represented the slope of a function (at given points). This, while correct, is a view that describes a differential quotient as a process, more so than as a broader concept (this concept being change), which we were aiming for. Some of the students, throughout the lesson series, retained a measure of difficulty in understanding a differential quotient (and by extension, DEs) in terms of being a procept (thus, a concept as well as a process), rather than solely a representation of a process.

That said, a glance at the exam grades would suggest that all students have an adequate understanding of differential equations. After all, each of them bore passing grades (but more specifically, one of them barely passed, two had an average grade, and the rest were above-average to exceptional). The overall course of the lesson series – the final lesson in particular – granted good reason to suspect as much. In this light, the way some of the students filled in their question lists seems contradictory, negative as most of them pose themselves to their own mastery of (the basic theory of) differential equations. On the other hand, however, they do judge themselves to have a good understanding of the concept of a DE (that of it being a representation of a dynamic process), and by extension, a good image of their applications.

Differentiating between the students' conceptual and procedural understanding, and scrutinizing them accordingly, grants some insight as to this apparent anomaly.

As stated, during the lessons the students' procedural knowledge was exemplary, and in great part this carried through to the exam. This is reflected in the fact that every student had a higher percentage of correct answers to procedurally based assignments relative to conceptually based assignments. Most of the procedural errors were either based on calculation errors and sloppiness, or lack of understanding on an underlying concept.

Considering this influence (that understanding an underlying concept has on one's procedure), most telling in this were the results of the exam assignments 3c and d (bear in mind that this assignment was also used in the previous researches of Rasmussen & King, 2001, and Zwarteveen et al., 2009, 2010, 2011), 5b, and 6b. These concern, respectively: composing an iteration of Newton's law of cooling by a situation described in words; applying Euler's solution method; and determining an exact solution to a DE by using conceptually worded data. Succinctly put, conceptual understanding weighs heavily in choosing the correct procedure towards answering these particular assignments. With each of these, there exists some correlation between the way the students answered the assignments, and how the students state to view their own understanding of DEs in the question lists. The students with (mainly) correct answers to these assignments are, on the whole, more positive as to their own understanding. By the same token, students with incorrect answers to these assignments had a rather more negative stance towards their own understanding of differential equations.

Notice also that answering assignment 3 (especially 3d) correctly, appears to correlate to the relative magnitude of the students' final grade. With the exception of (A), who did not answer 3d correctly but did well otherwise, all of the students who did well on assignment 3 also had a relatively high grade, whilst the passing grades of those who answered assignment 3 incorrectly were relatively low. This stands to reason – of all of the exam's assignments, assignment 3 was the one in which a balance between conceptual and procedural understanding was most important. To this end, 3a and 3c are supposed to evoke the concepts towards answering the somewhat more procedural 3b and 3d respectively. We see that students who had difficulty in coupling a concept to a correct procedure, scored comparatively low on the whole exam. Rather more specifically, the students who had difficulty with this third assignment also scored lowest on the conceptually based assignments. In the question lists, these particular students also advocated most strongly for a procedural approach in mathematics education.

Using Sfard's terms of interiorization, condensation and reification, we can shed a light on more specific aspects of the students' understanding of DEs. Here, we differentiate between their understanding of composing DEs and solving DEs (specifically, using Euler's method). The reason being that Euler's method was the only 'new' *calculative* procedure we introduced (i.e. we introduced a new formula) – whilst composing DEs is a *contextual* procedure (i.e. correctly using mathematical terms already known to the students, depending on the situation).

It appears the students have reached a state of condensation when composing DEs. Bear in mind that, in the composition of DEs, we focused on several standard forms (Newton's law of cooling and second law of motion). The students, bar a few exceptions, were quick to compose DEs without using intermediary steps (such as proposed by Zwarteveen et al's working scheme). In this, their errors (if any) were mostly based on sloppiness or faulty calculations. For the students with the higher grades, it could even be said they reached a state of reification, taking into account the deliberate manner with which they composed their DEs (and the assumptions they posed towards this end).

In solving DEs, there exists a more distinct separation between the individual students' level of understanding. Here, some students barely reached a state of interiorization – the calculative means towards attaining a solution (i.e. applying Euler's method) seemed not to have been fully integrated, if at all, while (during the lessons at least) the students did appear to have a good understanding of the underlying concept. In the exam, under half of the students managed to apply Euler's method correctly. In the question lists, Euler's method was also stated by (T) as being one of the hardest topics to thoroughly understand.

It appears that our emphasis on a conceptual methodology may have backfired here. During the lessons, not too much attention was paid to the calculative procedures corresponding to Euler's method, the reason being that we judged the calculations to be a relatively simple logical conclusion. Whilst we do maintain that in essence, when understanding the underlying concept of Euler's method, the calculations are little more than a procedural footnote, still this particular procedure should be emphasized in class rather more so than we did.

6. Discussion & recommendations

Rasmussen and Marrongelle (2006) vouch for class discussions as being the best way to make students understand DEs. Zwarteveen et al (2011), in turn, vouches for much the same, but notes that during such lessons, the first lesson will largely be spent on the students getting accustomed or getting around basic problems surrounding the understanding of DEs. Rasmussen and King (2000) had a likewise experience when they were forced to 'scale down' the relative difficulty of the DE they presented, when the students expressed their lack of understanding.

Our own experiences (especially during the first lesson) were quite comparable, despite the fact that we tried to make the first DE we presented quite simple. As said, the students were quite unready towards exercising horizontal mathematization, especially on an individual basis. The explication and recap at the beginning of the second lesson, as well as the recap and summary of the final lesson (both executed as class discussions) were met with rather more 'aha! moments'.

This calls to mind the research done by Sierpinska (2007), who lists a multitude of reasons as to why mathematics students are quite dependent on proper guidance by the teacher, for them to properly develop a measure of (conceptual as well as procedural) understanding. In this, she also postulates that attaining an adequate measure of conceptual understanding may not be dependent upon whether or not you're actually using a conceptual approach.

Regarding our own experiences in a didactic light, we can also say that a text-based approach is not optimal to begin with. Yet again, this comes down to the approach of mathematics the students are accustomed to. As said, the students were rather more receptive towards a class discussion of the context (and putting it in mathematical terms) with the teacher as a guide, than they were towards trying to immerse themselves into the context independently. This suggests that a rhetorical (rather than text-based) approach should be opted for. After all, an apt speaker would (generally) be more effective towards immersing students into some context, than pages of text would. That said, there are indications that our chosen context bears good prospects. Although the reactions of the students to the specific context were mixed, we do feel that, on account of its unusual nature, it stood out to such a degree so as to be memorable. We postulate that this will help the students evoke the concepts underlying DEs in the future.

The software-based homework assignments were also designed as to stimulate the conceptual understanding towards DEs. However, due to us being forced to reduce these to homework assignments rather than a lesson topic, the students were (in principle) not obliged to make them. As such, these assignments were made by only three students: (A), (G) and (W) – though one student, (P), stated his intentions towards doing so in his question list, but could not get the program to work. Of these three students, (G) and (W) regarded the assignments to be rather insightful, and (A) wasn't sure whether or not it helped (but was inclined toward the negative). On the whole, however, we would say that these results bear promise (a similar research into using graphical software, albeit concerning geometry, by Kilic (2013), was equally promising), and beg for further research into using graphical software towards stimulating the understanding of underlying concepts of DEs (also taking into consideration that the exam grades of these students were among the highest).

Assignment 3 of the exam – again, being a modified version of an assignment used in previous researches – also presented us with similar findings to said researches. Both Rasmussen & Marrongelle (2006) as Zwarteveen et al. (2010) note the difficulty that students have in composing a DE – specifically, whether the DE should be of the form $dS/dt = 6$ or $dS/dt = 6t$ (in this, the former DE is correct). Zwarteveen describes this lack of understanding in composing DEs to cause the students to fall back on the procedural habit of wanting to find a solution (i.e. a direct function).

In our research, the assignment (here, specifically, 3b) was modified in such a way that the correct answer was $dS/dt = 1$. In this, five of the students attained the correct answer, whilst (K) and (M) did indeed answer by giving $dS/dt = 1t$, and $dS/dt = t + 100$, respectively. In turn, (P) mistakenly tried to express the DE in the form of Newton's law of cooling. As stated before, it were these students who had the lowest scores on the conceptually based questions of the exam as a whole, which is also indicated in the question lists by their strong preference towards procedural mathematics.

As said, the results of our research have been varied, and cover a broad spectrum. And whilst bringing up some interesting answers, many questions concerning a RME-based secondary school mathematics education, especially concerning differential equations, still remain. Follow-up research is all but called for, and we heartily recommend our findings to be a basis for this. For seeing as our research spanned a 'mere' four weeks, our major constraint has been the matter of time – which is perhaps best reflected by (J)'s words in his question list, upon expressing interest in the conceptual method: "If only mathematics was the only course in secondary school."

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Appendix A: Exam

Exam for Atheneum 5, mathematics D **Differential equations** **June 2013**
Good luck!

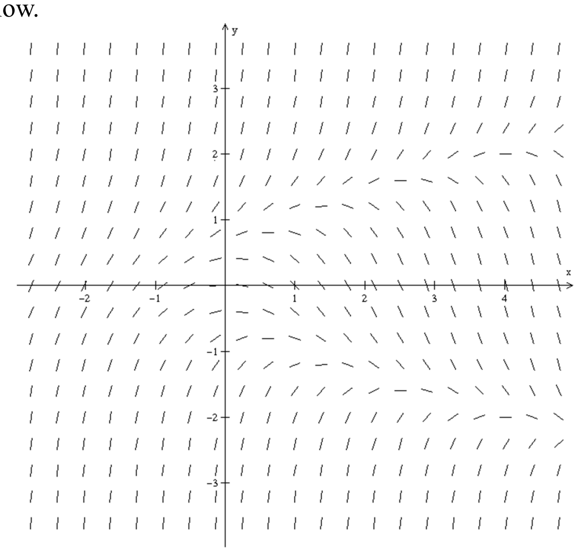
Assignment 1 (5 points)
A moped moves forward with a constant force of 50 N. The moped is subjected to frictional force. By air this is $F_A = 0.25v^2$ and by ground this is $F_g = 4v$. Here, F_A and F_g are in N, and v is in m/s.
Assume that there is no wind, and $v(0) = 0$. The moped and its driver have a combined mass of 150 kg.
Compose a DE describing the moped's acceleration.

Assignment 2 (1 point)
Describe a DE in words.

Assignment 3 (12 points)
A barrel with a total volume of 500 litres contains 100 litres of water. 400 grams of salt are dissolved into this water. A water solution containing 2 grammes of salt a litre gets poured into the barrel at 3 litres a minute. The two solutions mix evenly and pour out at the bottom of the barrel at 2 litres a minute.

a) Determine how long it will take for the barrel to fill up, in minutes. *(1 point)*
b) Compose a differential equation representing the net change in volume of water in the barrel, taking t in minutes. *(4 points)*
c) We define $S(t)$ to be the function describing the concentration of salt in grammes per litre over time t in minutes. Draw a sketch of $S(t)$. Be sure to clearly indicate the value that $S(t)$ assumes over time. *(2 points)*
d) Compose a DE dZ/dt, describing $S(t)$ by approximation. *(5 points)*

Assignment 4 (1 point)
In differential calculus, we call a given function $y_E(x)$ an equilibrium solution to a DE $\frac{dy}{dx} = f(x, y(x))$, when $f(x, y_E(x)) = 0$ holds for any x.
Regard the line element field below.

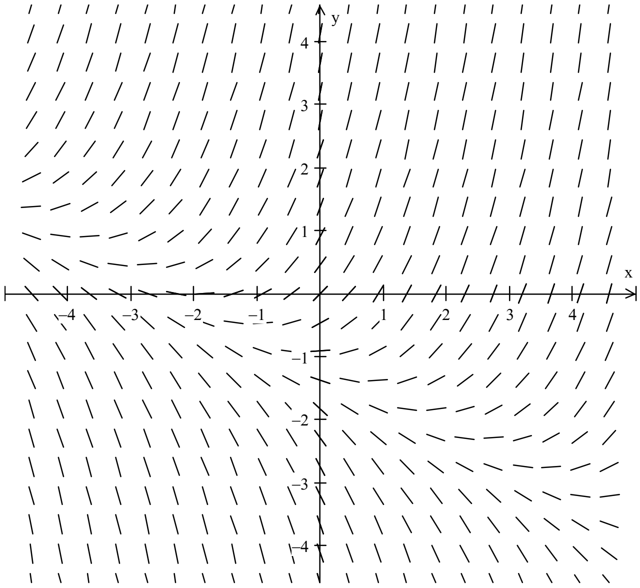


Argue whether or not the corresponding DE has one (or several) equilibrium solution(s).

Assignment 5 (8 points)
A lady parachutist jumps from an airplane. In free fall, her velocity is represented by the DE $\frac{dv}{dt} = 10 - 0.003v^2$, having $v = 0$ on $t = 0$. Here, is v is the velocity in km/h and t is the time in seconds.

a) Calculate the parachutist's terminal velocity (that is, her maximum velocity during free fall). *(3 points)*
b) Using Euler's method with step size 0.1, approximate her velocity at $t = 4$. *(5 points)*

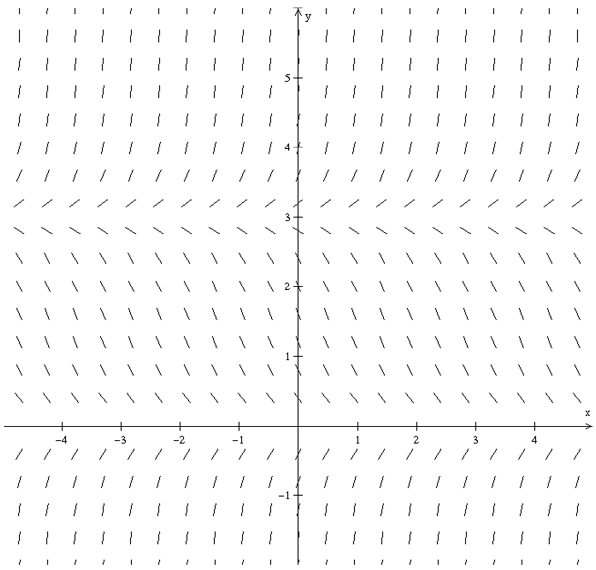
Assignment 6 (8 points)
Regard the DE $\frac{dy}{dx} = y + 0.5x + 1$. The graph below shows its line element field.



Any function $y(x) = ce^{x} - 0.5x - 1.5$ is a solution to this DE.

a) Prove this. *(4 points)*
b) The x-axis is a tangent to one of the solution curves. Calculate c for this solution. *(4 points)*

Assignment 7 (2 points)
Regard the line element field below.



Explicate which of the following DEs belongs to the line element field.

A. $\frac{dy}{dx} = 3y^2 - y$

B. $\frac{dy}{dx} = y^2 - 3y$

C. $\frac{dy}{dx} = y^2 + 3y$

Appendix B: Inquiry

Survey concerning the lesson series and exam concerning differential equations

You'll have noticed that the previous lessons were quite a bit different from the lessons you're used to, in that we did not use a book as a 'crutch' to your understanding of mathematics. Our intention was for you to brainstorm and throw around some ideas, and in so doing, to get in the right frame of mind for you to ascertain the right answers.

The idea behind this is for you to apprehend mathematics conceptually. Once in this conceptual frame of mind, mathematically exact answers should be evident. At least, this was our intention. By the book, this usually works the other way around.

Your personal experiences are important to our research. As such, we kindly ask you to be honest in filling in this enquiry. Try to avoid yes/no-answers, and be straightforward in your statements. If the lessons were boring or unclear to you, or exactly the opposite, we would like to know!

1. Did our manner of teaching motivate you more, less, or did your motivation go unchanged?

Let us distinguish 'conceptual' from 'procedural' mathematics. The difference between the two, most simply put, by example, is this. Procedural mathematics constitutes learning Euler's solution method by heart, and applying it appropriately. Conceptual mathematics means not learning anything by heart, but instead, getting a good idea of what you're working with: from this frame of thought, formulaic expressions become more or less self-evident. As said, the lesson series put an emphasis on thinking conceptually.

2. How did your exam go? Do you yourself think you have a good conceptual understanding of differential equations?

3. In mathematics, which one do you prefer: the conceptual or procedural approach? Was this lesson series an eye-opener in that respect?

4. Do you think you have a good image as to the applications of differential equations?

We opted for a contextual theoretical approach. To this end we put the theory in a historical perspective and made up some (partially nonsensical) adventures for captain Blackbeard.

5. Was this general context of any help to you understanding the underlying theory, or did it do the exact opposite by being a distraction?

6. How did you like Blackbeard's adventures?

We could have opted for a somewhat more serious contextual approach, concerning more actual problems (and leaving out history). The reason we didn't do this was because we thought that you would find this somewhat boring, and would be more appreciative of a more nonsensical approach.

7. Would you be more interested in a 'serious' context? Or doesn't it matter?

The homework assignments of module 3 presented you with some assignments for which a computer program was needed, called Winplot. The idea of these assignments was for you to 'fool around' with the line element fields and solution curves, thus stimulating your imagination in a mathematical sense (that is, your conceptual understanding).

8. These being homework assignments, you were (in principle) free to choose whether or not to download the program and make them. Did you do this, and why (or why not)?

9. If yes, do you think (perhaps retrospectively) the program helped you develop your conceptual understanding?

Any other remarks (for example, saying how handsome we are) or questions (for example, concerning how we could be so handsome) can be put below. Of course, you're free to address us personally or by email!

Appendix C: Lesson preparation forms

Note: due to the straightforward nature of the fourth lesson (it being a recap and summary, as well as bearing a short presentation on the multitude of applications of DEs, but bearing no new theory that the students will be tested on in the exam), no preparation form was made for this lesson.

Lesson preparation form 1
Class: 5Ad Instructor: Arnaud Uwland Date: 5-24-13

The students' learning process
Relevant knowledge
The students have a grasp of basic calculus, such as calculating the derivative of a function.

Lesson goals:
Come the end of this lesson, the students will understand that a derivative of a function represents the measure of change at a given point on the function. They will know the definition of a DE, and how a (relatively simple) DE can be composed. Also, they will know the standard form of Newton's law of cooling, and can determine the cooling constant by way of linearization.

Theoretical frame:
Module 1.

Relevant examples, assignments or definitions:
Proposed definition of a derivative:
'A derivative to a function describes the measure of change of said function on every point (for which said function is defined)' – or words to that effect.
Definition of a DE:
A differential equation is a mathematical representation, describing a measure of change by using the derivative of a function.
Formally: $dy/dx = f(x,y(x))$
A short list of equations, constituting the test:
a) $dy/dx = 5y – x$
b) $dN/dt = -0,6t$
c) $y' = 2$
d) $f(x) = 3x+2$
e) $N(t) = 5N'(t) + 3$

Time schedule

Time	Goal	Theoretical material	Mode of activity	The instructor's activity	The students' activity
20 min.	Introduction, and getting the students in the right frame of mind.	–	Class discussion.	Introducing the subject of DEs, first by giving a short overview on the topics of the coming lesson series. Then, initiating a class discussion by asking a worded definition of a derivative. Dropping constructive hints if necessary.	Discussion.
35 min.	Composing a simple DE.	Module 1	Working individually and in couples.	Putting the students to work on the module. Dropping constructive hints if necessary.	Working, independently for the first 10 min., then in couples.
10 min.	Discussing the students' previous work.	Module 1	Class discussion.	Discussing the answers to the assignments in module 1. In so doing, getting all students in the right frame of mind (if they weren't before).	Paying attention, discussion.
20 min.	Learning the definition of a DE. Testing the students' understanding.	–	Class instruction.	Offering the definition of a DE (as above). Testing the students' understanding of this definition by presenting a list of equations (as above), and asking whether or not the individual equations are Des.	Paying attention, discussion.
5 min.	Lesson end, assign homework.	The homework assignments of module 1	–	Assigning the students their homework. Should they come up, answering any questions the students may have.	Asking questions, if necessary.

Lesson preparation form 2
Class: 5Ad Instructor: Arnaud Uwland Date: 5-31-13

The students' learning process
Relevant knowledge
The students have had an introduction to DEs, and know the definition. However, we strongly suspect the students to have difficulty in understanding the concept of a DE, and by extension, understanding the composition of simple DEs.

Lesson goals:
Come the end of this lesson, the students can compose a somewhat more involved DE, specifically those involving Newton's second law of motion, as well as exponential growth. They are also introduced to the notion of a line element field, and how to draw one. To some extent, the students also know how to interpret a line element field (specifically: they know a solution curve to be represented by the line elements, combined with a given coordinate). The students know that a DE has multiple solutions, and that these solutions are direct functions. Also, given a starting value, they can determine the exact solution to a simple DE.

Theoretical frame:
Module 1 (recap) and module 2.

Relevant examples, assignments or definitions:
A step-by-step construction of the line element field to the DE dy/dx = (8/7)y. (Which is a DE used in the homework assignments.)

Notes
The students are not at all familiar with our approach of mathematics education – which they made quite clear during the previous lesson. We set some time apart to give a rather more detailed explanation as to the nature and goal of this lesson series.
On an unrelated note, (F) requested for us to set some time apart, so that the students could fill in a short inquiry. This would take about 15 minutes, which happened to be an opportune moment for the instructor to prepare for the subsequent lesson topic.

Time schedule

Time	**Goal**	**Theoretical material**	**Mode of activity**	**The instructor's activity**	**The students' activity**
5 min.	Explaining goals.	–	–	Giving a more detailed explanation as to the nature and goal of this lesson series.	Paying attention.
20 min.	Explicating upon last week's topics.	The homework assignments of module 1.	Interactive class instruction.	While we suspected the homework assignments to have been made poorly, said homework assignments all happen to revolve around a single form of DE. As such, a single homework assignment will be explicated upon quite thoroughly. Upon doing this, we expect the procedure to be clear, and some attention will be given to the conceptually based homework assignments.	Paying attention, cooperation.
5 min.	Taking note of the solutions to DEs.	Homework assignment 3b.	Class instruction.	Directing the students' attention to the fact that a DE has an unending amount of solutions, which in turn are determined singularly by using starting values.	Paying attention.
20 min.	Inquiry.	–	–	Drawing the element field (as above) in preparation to the following topic.	Filling in the inquiry.
15 min.	Introducing line element fields.	–	Class instruction.	Using homework assignment 3 towards composing a line element field. With assignment 3b as a conceptual reference, showing that a solution curve can be drawn upon choosing a starting coordinate.	Paying attention.
25 min.	Composing a more involved DE.	Module 2	Interactive class instruction.	Guiding the students through the assignments of module 2.	Paying attention, cooperation.

Lesson preparation form 3
Class: 5Ad Instructor: Arnaud Uwland Date: 6-7-13

The students' learning process
Relevant knowledge
The students are familiar with the concept of a line element field. Also, this lesson draws on their grasp of basic calculus, as well as their algebraic skill.

Lesson goals:
The students can use Euler's method towards solving DEs. In so doing, they can compose a recursive formula representing the DE, using a given step size, and input this in their graphic calculator. By entering a starting value, they can interpret the output of variables. Also, the students can verify exact solutions to DEs by substitution.

Theoretical frame:
Module 3.

Relevant examples, assignments or definitions:
To the end of instructing the students in using substitution towards verifying a proposed exact solution to a DE, we just show them the procedure once. Specifically, using the DE:
dy/dx = x^2 + y – 2
and the proposed solution
y(x) = -5 e^x – x^2 – 2x
which will result in the solution being verified.

The students are tested by presenting them with these proposed solutions:
a) y(x) = 3 e^x – x^2 -2x
b) y(x) = -5 e^x – x^2 + 2
This should result in the students stating that the former is a solution, while the latter is not.

Time schedule

Time	Goal	Theoretical material	Mode of activity	The instructor's activity	The students' activity
15 min.	Explicating upon last week's homework assignments, and coupling them to line element fields.	The homework assignments of module 2.	Class instruction.	Answering any questions to the homework assignments the students may have. Following this, using plots of line element fields (some of them pertaining to the homework assignments) to show how solution curves behave.	Asking questions if necessary, paying attention.
30 min.	Introducing Euler's solution method.	Module 3	Interactive class instruction.	Guiding the students through the assignments of module 3.	Paying attention, cooperation.
20 min.	Explicating Euler's solution method.	Slides.	Interactive class instruction.	Using the previously made assignments to try to make the students find a general expression for the resulting calculations (this general expression being Euler's method). After attempting this (successfully or not), showing this general expression, using slides with illustrations.	Paying attention, cooperation.
10 min.	Learning how to verify an exact solution to a DE.	–	Class instruction.	Showing how a proposed exact solution can be verified, using substitution. Testing the students by assigning them to substitute comparable equations into a DE, and seeing whether or not they reach the correct conclusions.	Paying attention, working.
15 min.	Lesson end, question time, assign homework.	-	-	Answering any questions the students may have, and assigning them their homework. In this, making special note of the computer-based assignments.	Asking questions if necessary, paying attention.

Appendix D: Modules

(cover)

The adventures of Blackbeard the pirate: an abstract perspective

Edward Teach, better known tot most as Blackbeard, was a pirate hailing from Bristol, England, who in the early 18th century terrorized the west coast of the American continent. Among the list of his (mis)deeds are the plundering of many a ship, the formation of an alliance of pirates, and ransoming a port town. Whilst doing battle, Teach was wont of weaving lit tapers in his beard, giving him a frightful appearance. All of this, as well as his sinister cognomen, got him into history as the prototype of the more romanticized pirate captains we use in popular stories today.

Figure 1 – One of the few remaining pictures of Blackbeard

In this module, and in those of the weeks to come, we'll follow the seabourne adventures of Blackbeard on possibly his most well-known ship – the Queen Anne's Revenge.

Figure 2 – The Queen Anne's Revenge

In so doing, we'll put these adventures in a mathematical perspective.

Part 1

Rather unlike our current view of pirate captains – that of them being ruthless tyrants over a band of equally ruthless maniacs – Blackbeard actually wasn't so bad. His crew accepted his leadership because he was an apt leader, not because he was particularly bloodthirsty.
For Blackbeard knew, as any captain worth his salt would, that in order to keep one's crew in check, he would have to give them a nice treat every now and again. Enter Blackbeard's patented apple pie.

Presently, we regard one of Blackbeard's apple pies, which he just took out of the oven. His crew can't wait to get their hands on it – but it'll have to cool down, so wait they shall.

What we are going to do is to put this cooling process in a mathematical context.
To this end, the apple pie's temperature at given points in time is represented in the table and graph below. In these, T is the temperature in degrees Celsius and t is the time in minutes.

t	0	5	10	15	20	25	30
T	180	140	110	88	72	60	51
$\frac{dT}{dt}$	-9.3	-6.9					

Table 1

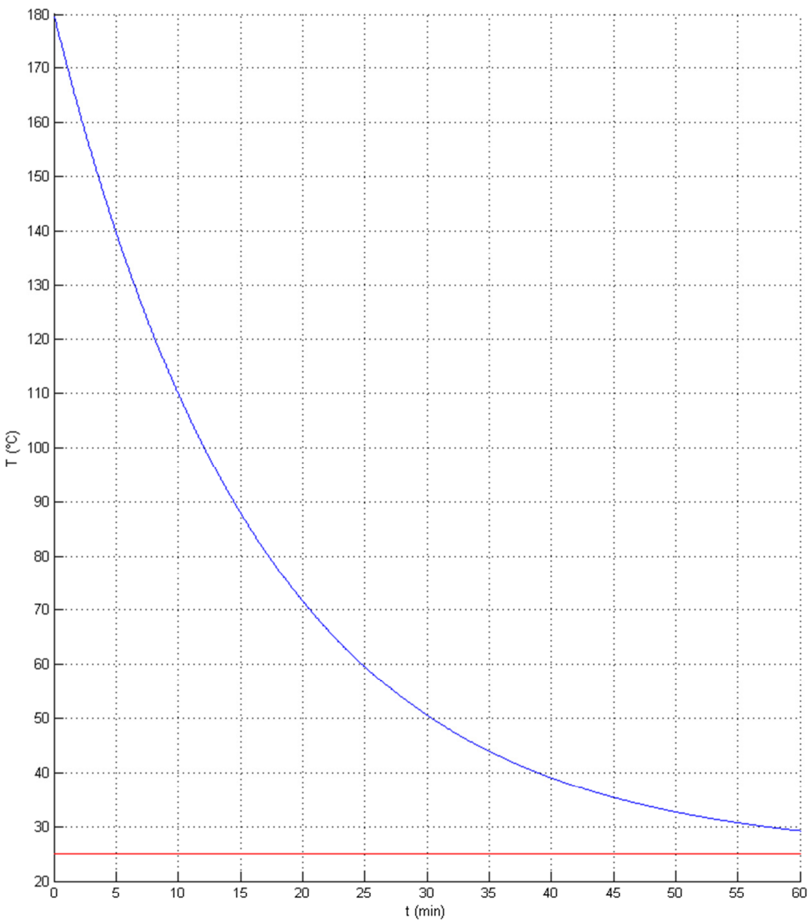


Figure 3 – Temperature over time (blue) and T_0 (red)

You'll have noticed a row in the table for $\frac{dT}{dt}$, as well as several given values.

- Explain what $\frac{dT}{dt} = -9.3$ and $\frac{dT}{dt} = -6.9$ mean in relation to each other, and how this corresponds with the graph.
- Complete the table by filling in the values of $\frac{dT}{dt}$.

Another thing you would have noticed is that the temperature steadily approaches a certain value. Not too strange, when you think about it – for it is the pie getting on the Caribbean equivalent of room temperature. Let's call this set temperature T_0. As you can see in the graph, $T_0 = 25$.

- Explicate what happens to $\frac{dT}{dt}$ in relation to T, over time.
- Use this explication to find an expression for $\frac{dT}{dt}$.
- Verify your expression by relating it to the table.

Homework assignments

Assignment 1
Nearing the end of November 1717, the Queen Anne's Revenge did battle with the Great Allen – a well-armed trading vessel – near the island of Saint Vincent. In sea battles, ships would fire their cannons at each other in order to damage the other ship, whereupon you could escape – or, conversely, so that the other ship could not escape, and be boarded. Beard aflame.

Be that as it may, firing a cannon was no small feat. Loading and reloading the cannon was quite a task, even more so considering that the barrel would get quite hot upon firing.

a) We consider a cannon which has just been fired at $t = 0$, having a felt temperature of 60 degrees Celsius at this point in time. 20 seconds later it has a felt temperature of 45 degrees, at which time the crew can commence reloading it. Normally, the cannon has a felt temperature of 15 degrees.
Compose a differential equation representing the cooling process of the cannon, using t in minutes. To this end, make use of the following. Firstly, use linear interpolation to approximate the temperature at $t = 10$. Secondly, use the difference quotient as an approximation to the differential quotient at time t.

b) Let's say that the barrel's temperature rises 45 degrees after each shot, and that the proportionality constant remains the same, regardless of the temperature. Sketch two graphs, representing temperature over time if:
1. The cannon would get enough time to cool down.
2. The cannon would be reloaded too quickly, effectively not getting enough time to cool down.

The wind rises, incidentally at a temperature of exactly 15 degrees Celsius. The cannon hatches are open, so the wind blows right through them – against the cannons.

c) How does this situation change the differential equation?

Assignment 2
Our friends, the pirates, have boarded the Great Allen. The poor crew of the trading vessel don't know how to handle a band of angry pirates in melee combat, so they are quick to surrender. On the upper deck they surrender quite immediately, one by one. As the pirates reach the lower decks, they encounter ever smaller pockets of resistance, who eventually surrender as well. At the end of the battle all 50 of the trade vessel's crew have surrendered.

We'll say that the number of crew members who surrender per minute is proportional to the difference between the total number of crew members and the amount of crew members who have already surrendered.

a) Sketch a graph, representing the amount of crew members who have surrendered over time.
b) The pirates board the ship at $t = 0$, at which time none of its crew have surrendered. In ten seconds, 5 have surrendered. Let's call the amount of crewmates who have surrendered, S. Compose two differential equations, both representing the increase of the amount of surrendered crewmates in terms of S. The first DE having t in minutes, the second with t in seconds.

The DE in minutes suggests that 23 crew members have surrendered after approximately one minute. On the other hand, the DE in seconds suggests that 22 crew members have surrendered at that time. (For the moment, assume this to be true. We'll get into determining these values later.)

c) Explain this difference. Take your base data into account. Which of the DEs is closest to the actual value of *S*?

Assignment 3

The battle fought, Blackbeard moored his ship by the island of Saint Vincent. Here, the pirates could resupply and repair battle damages. Saint Vincent is better known to some as Rabbit Island. At the time, the French colonists noted the abundance of rabbits on the island, thus naming it after Saint Vincent, who until this day is playfully called *le gourmand lapin*, on account of his appetite for rabbits.

You may have heard of how a population can grow exponentially. Using a DE, we will examine this concept.

a) Let's say that at $t = 0$, there are 1000 rabbits. The rate of growth of the rabbit population is proportional to the amount of rabbits itself. After one month, this amount is 1100. Taking t in years, compose a differential equation representing the growth of the population of rabbits.
b) Find the function $N(t)$ which conforms to this DE. To this end, take the value of $N(0)$ into account.

Part 2

Due to an unfortunate concurrence of events, Blackbeard's ship, the Queen Anne's Revenge, is in turmoil. Whilst anchored near Rabbit Island, the crew were getting ready to repair a malfunctioning cannon, when the pistol worn by a crewman (who happened to be passing by) accidentally fired. The bullet barely missed the cannon, but struck its taper, which lit. The cannon, still loaded, fired a cannonball straight up into the air. The cannonball reached a certain height before reversing course and falling straight down again. There's no wind to blow the cannonball off to either side, so if the Queen Anne's Revenge's crew won't move the ship, the deck will be struck by the falling cannonball, causing untold damage.

Figure 4 – The Queen Anne's Revenge, on anchor

As captain Blackbeard watches the accident unfold, he immediately senses the danger of the situation, as the cannon goes off with a loud bang. “All hands on deck!” he orders, “Weigh the anchor, sail ho!” The crew, experienced as they are, manage to raise the anchor and set sail in exactly one minute. It'll take another ten seconds for the ship to get out of harm's reach.

Now, what we are going to do is to find out whether or not Blackbeard and his crew manage to prevent the cannonball from hitting the ship.

The cannonball's mass is 6 kg and reaches a height of 1800 metres after exactly half a minute. When it reaches this height, it falls straight down. We define $t = 0$ to be the moment the cannonball reaches its apex.

a) What is the velocity v (in m/s) at $t = 0$?

Using the Newton's second law, $F = ma$, a differential equation representing the change in velocity of the cannonball during its fall can readily be made.

b) Which of the given quantities can be expressed by a differential quotient?

The cannonball is subject to a gravitational force F_g of 59 N, as well as a frictional force $F_f = 0.03v^2$. Here, F is in N and v is in m/s.

c) Give an expression for the net force imposed on the cannonball.
d) Use the above to compose a DE representing the cannonball's velocity from $t = 0$.
e) Calculate the terminal velocity of the cannonball (the greatest absolute velocity it reaches while falling.)
f) Explicate whether or not the Queen Anne's Revenge will reach safety in time.

Homework assignments

Assignment 4
The Queen Anne's Revenge is known to be quite a fast ship. All sails raised, and granting a good wind, its maximum velocity can readily be approximated. The ship has a mass of 40,000 kg, whilst the wind pushes it forth with a force of 10,000 N, starting at $t = 0$. The sea itself causes a friction F_f (in N), which is proportional to the square of the ship's velocity (in m/s). The proportionality constant is 500. The ship's velocity at $t = 0$ is 0 m/s.

a) Compose a DE representing the ship's velocity.

A ship's velocity is actually not represented in km/h or m/s, but in knots. A knot is defined as one sea mile per hour. In turn, one sea mile is exactly 1852 metres.

b) What is the terminal velocity of the ship in knots?

Assignment 5
Regard the DE $\frac{dy}{dx} = -\frac{x}{y}$.
a) Calculate $\frac{dy}{dx}$ in (0,5), (3,3), (-1,1) and (4,-1).
b) Draw all points for which $\frac{dy}{dx} = -1, \frac{dy}{dx} = 1$and $\frac{dy}{dx} = 0$ apply.
c) Draw a line element field for all points (x,y) within $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.
d) Give the equation for the solution curve passing (2,2).

Assignment 6

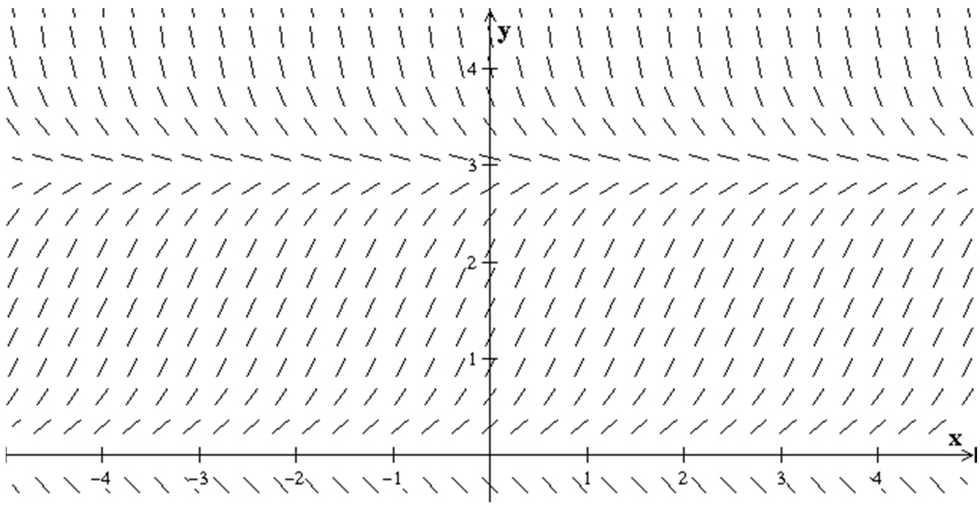


Figure 5

The line element field in figure 5 represents one of the following DEs. Explain which one it belongs to.
A. $\frac{dy}{dx} = 3 - y$
B. $\frac{dy}{dx} = y^2 - 3y$
C. $\frac{dy}{dx} = 3y - y^2$
D. $\frac{dy}{dx} = \frac{1}{3}y^2 - y$

Assignment 7
The Queen Anne's Revenge just partook in a battle and is left rudderless, somewhere in the ocean. Figure 6 is a line

element field representing the water current. The DE of this line element field is $\frac{dy}{dx} = -2y + x$.

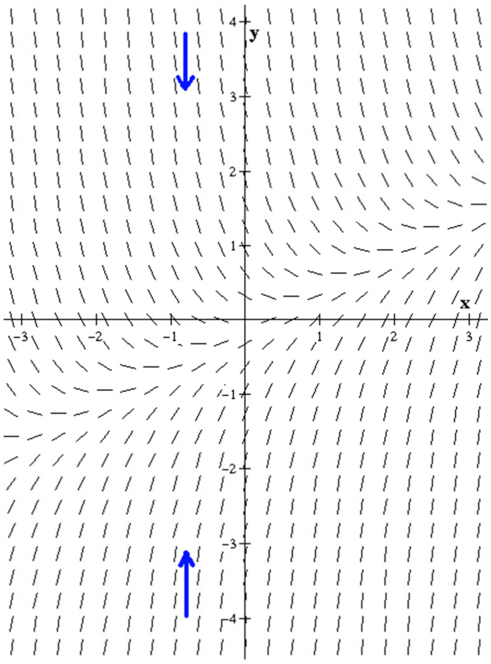


Figure 6

The arrows indicate the general direction of the current.

a) We don't know where the ship is, exactly, but we suspect it to be at one of these two coordinates: (-0.5,-3) or (-3,3). In figure 5, sketch the supposed courses of the drifting ship, starting at these coordinates.
b) The graph depicts a diagonal asymptote. Compose the equation to this line.

Part 3

One fateful day, during his travels, Blackbeard acquired a treasure map. The map would lead to the greatest treasure any pirate captain could wish for: the ultimate apple pie recipe.

However, as befits most treasure maps, the directions on the one Blackbeard found were quite cryptic. All it depicts are a map of Pie Island in a coordinate system, as well as the following clues:

- Start at the westernmost point.
- $\frac{dy}{dx} = x - y$
- 4 steps.
- Go with the flow.

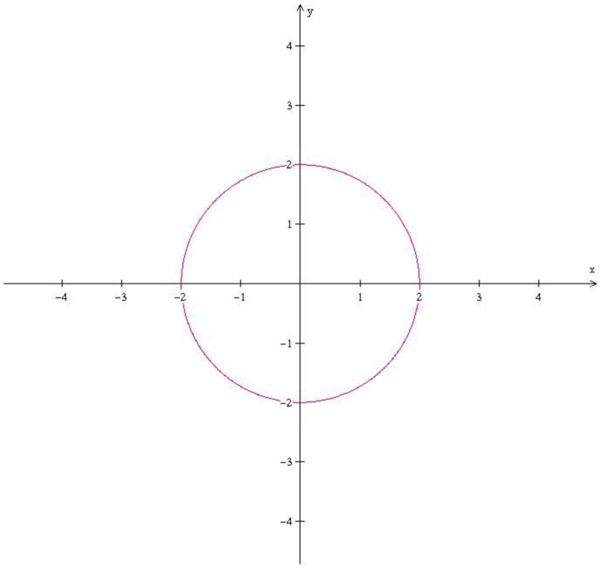


Figure 6 – Blackbeard's apple pie treasure map

Blackbeard and his crew just arrived at Pie Island, on the Queen Anne's Revenge. During the journey there, the captain has been busy deciphering the map. It took him quite some time, and a good think or two, but he is confident he knows how to find the treasure.

We're going to 'map' his thought process in getting there.

“Firstly,” Blackbeard thought, “I'll have to represent this 'flow' somehow.”

a) Draw the line element field in figure 6, between $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

“Well this is easy,” Blackbeard thought, “all that remains for me to do is to go along with the flow for a step in the *x*-direction.” Using his ruler, he promptly drew a line as described. The result left him quite disgruntled.

b) Explain Blackbeard's displeasure.

At wit's end, the captain stared his one eye out trying to find a clue he missed, but to no avail. That is, until Cookie, Blackbeard's brilliant parrot, began squawking “It's on the rear! It's on the rear!”

Figure 7 – Blackbeard and Cookie

Blackbeard turned over the treasure map, and lo, there he beheld the last clue:

c) Step 0.5.

With this information, Blackbeard renewed his efforts.

He drew a line in the direction given at te starting point up to half a step in the *x*-direction. At the arrival point, so to speak, he determined the new direction and drew another line, again up to $0.5x$ to the left. He repeated this process up to four times. Looking at the fruits of his work, Blackbeard could not deny that he was quite pleased with himself. The ultimate apple pie recipe was as good as his!

d) Follow Blackbeard's steps as described. At what coordinates can he find the treasure?

Homework assignments

Assignment 8
Like any ship's captain, Blackbeard wears a trademark hat. Its primary function, of course, is protecting Blackbeard from sunburn, but it also happens to cover up his embarrassing bald spots. After years of use, the hat is worn and could use replacement. However, our captain can't find it in his heart to replace his 'crown'.
Thus, the weather gods decide to take the initiative: a sudden gust of wind yanks off Blackbeard's hat and blows it sky-high, leaving our captain disgruntled and hatless.

The hat reaches a height of 1000 metres, no less, when the wind finally releases it. Then it flutters back down in free fall. The hat has a mass of 200 grams, and is subjected to a gravitational force F_g of 2N and a friction $F_f = 0.016v^2$. F is in N and v is the velocity in m/s.

a) Compose the differential equation, representing the hat's change in velocity. Take $v = 0$ at $t = 0$.
b) What is the hat's velocity after two seconds? Use Euler's method, and choose an appropriate step size.

Assignment 9
The Queen Anne's Revenge has been laying on anchor for several weeks near an island in the Pacific Ocean. Quite a boring situation by all accounts. Blackbeard, a patient man, passes the time by watching his favourite plant grow. The plant's growth is subject to $\frac{dL}{dt} = L - 0.01L^2$. Here, L is its length in cm, and t is the time in weeks, with $L = 1$ at $t = 0$.

a) Approximate the plant's lenght at $t = 4$. Take a step size of one day.
b) Calculate the plant's maximum length.

Assignment 10
We present you with three DE's and three solutions to DE's.
$DE_1: \frac{dy}{dt} = -t^3 + y + 3t^2 - 1$; $DE_2: \frac{dy}{dt} = \frac{1}{4}t^2 - \frac{1}{2}ty + \frac{1}{4}t^2$; $DE_3: \frac{dy}{dt} = -\frac{2y}{t}$
$S_1 : y = \frac{2}{t^2}$; $S_2 : y = 2e^t + t^3 + 1$; $S_3 : y = t + 2$
To each one of the DE's belongs one of the solutions. Determine which belongs to which.

In order to make assignments 11, 12 and 13, you'll need a computer with internet access for downloading a program. This program, *Winplot*, is available for free download. You'll find instructions for downloading, starting, and using this program in the 'Winplot appendix'. Please read this appendix carefully before attempting the following assignments.

Assignment 11
Consider the DE $\frac{dy}{dx} = \frac{x-1}{1-y}$.

a) Enter the DE dy/dx = (x-1)/(1-y) into Winplot.
b) Plot the solution curves that go through (0,0), (-1,0), (-2,0) and (-2,2). Write down any suspicion you may have, regarding the solutions.
Note that the program plots part of the solution curve by using Euler's method.
c) Investigate the possibility of plotting a solution curve starting at (-1,1). What about (1,1)? Explain the results.
d) Compose a differential equation in which the solution curves are all circles with centre (-3,-2). Verify your DE by using Winplot.

Assignment 12
Regard the DE $\frac{dy}{dx} = \frac{x^2+2y}{xy-2}$.

a) Plot a line element field for this DE by entering dy/dx = (x^2+2y)/(x*y-2) into Winplot. Then, plot the curve (a function) going through all points, from which the solution curves of the DE have a horizontal tangent.
b) By the same token, plot the curve going through all points from which the solution curves of the DE have a vertical tangent.

Assignment 13

Use Winplot to plot the DE $\frac{dy}{dx} = \frac{x+y}{x-y}$.

The line element field of this DE, and by extension its solution curves, shape into something like a vortex. Observe the line element field below, which is also vortex-shaped. Give us the DE conforming to this field.

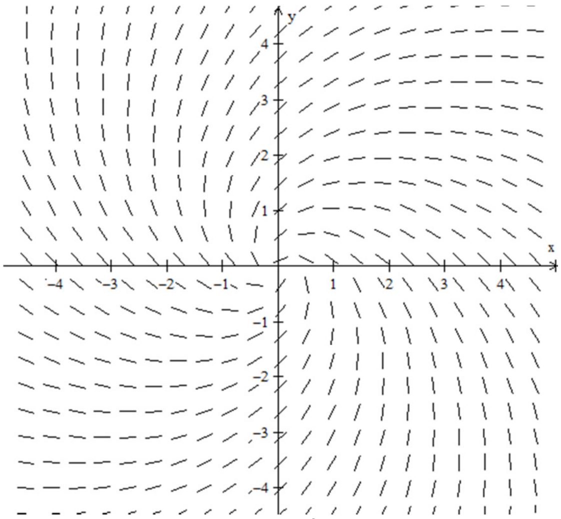


Figure 8

Winplot Appendix: Downloading and starting up Winplot

Open your internet browser and enter http://math.exeter.edu/rparris/peanut/wp32z.exe
You'll notice that the file "wp32z.exe" is being downloaded. When finished, open this file. Windows may give you a security warning at this point, in which case, click 'Run'. A window will open, asking you to 'unzip' the contained files. Choose or create a directory of your liking (the default setting is C:\peanut), then click 'Unzip'. The file 'winplot' is now available in your chosen directory.

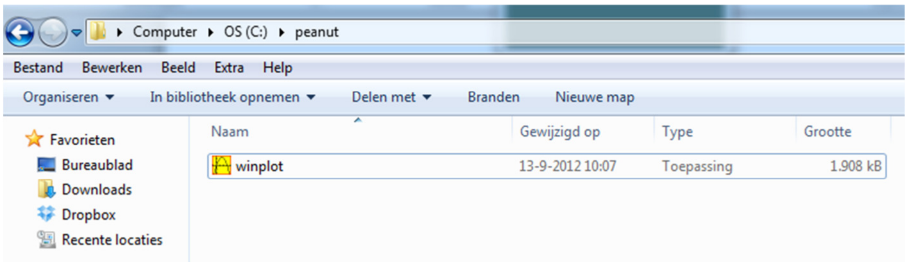


Figure 9

Double click this file to start Winplot. Two small windows will appear: Winplot itself, and a 'did you know that' window. The latter gives you useful hints into using Winplot. Close this window, leaving only Winplot itself.

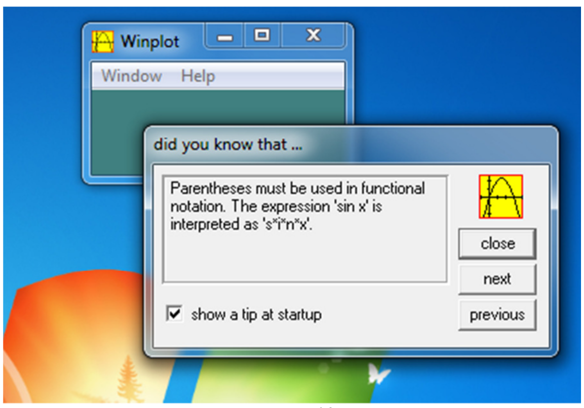


Figure 10

You have now started Winplot.

Plotting line element fields to DEs
After starting the program, click the 'Window'-menu, then click '2-dim'. A new window presents itself.

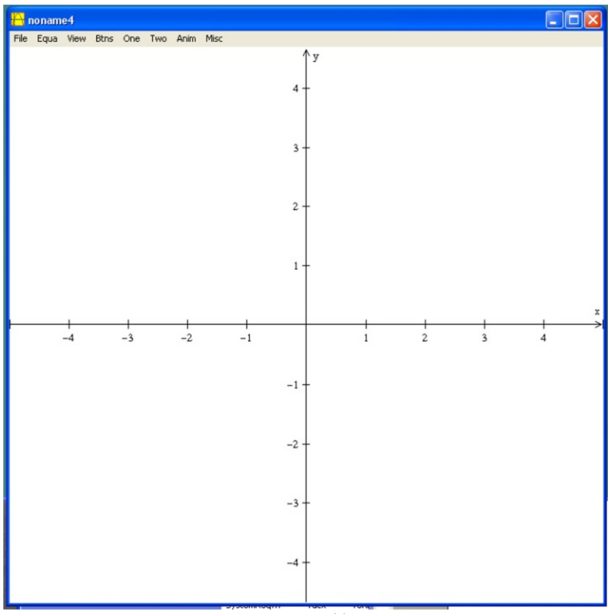


Figure 11

In this window, click the 'Equa'-menu, then click 'Differential'. This will present you with three options:
1. dy/dx
2. dy/dt
3. xdot(n)

Use the first (1. dy/dx) in order to enter a differential equation. A new window appears:

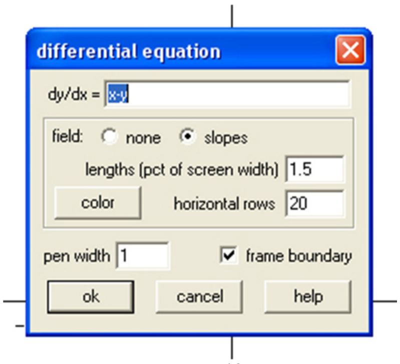


Figure 12

Enter your DE and click 'ok'. Note: when entering divisions, use parentheses and '/' to appropriately divide the numerator and denominator. For example (y-x)/(1+x). Multiplication of variables is done by '*', for example, x*y. If you like, you can change the settings of your line element field by clicking the 'edit' option on the 'inventory' window which has appeared.

Plotting a solution curve in a line element field
In the window depicting your line element field, choose the 'One'-menu, then click 'Initial-value problems', then '1. dy/dx trajectory'. The window below will appear.

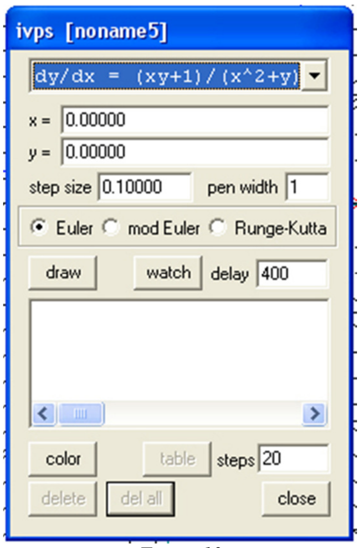


Figure 13

In this window you can enter values for x and y, which amount to the starting point for your solution curve. Having done this, click 'draw' (if you wish you can play around with the 'step size' and 'steps' beforehand, and see what happens). Part of the solution curve will now appear in the line element field.

A 'regular' function (that is, not a DE, but for example f(x) = x) can be plotted in your line element field as well. To this end, choose the 'Equa'-menu and click '1. Explicit'. A new window will appear, in which you can enter your function. Upon doing so, it will appear in your plot.

Part 4 – Summary

1) The definition of differential equations

A differential equation (henceforth DE) is an expression representing the measure of change of a certain variable, in terms of said variable and the variable it depends upon.

Formally: $\frac{dy}{dx} = f(y(x), x)$

The above is a hackneyed way of saying, for example, that the acceleration (change in velocity) of a falling object is not only time-dependent, but also dependent on its velocity at any given time.

Bear in mind that the variables (here, *y(x)* and *x*) needn't actually be present in the equation. In these cases the DE is still dependent on them, but they just happen to be zero within the context of the DE.

Remark: a function *y(x)* is defined to be a solution to a DE, when it conforms to the DE for all values of x on which *y(x)* exists. We'll handle solutions of DEs later on.

Example 1.1

- $\frac{dy}{dx} = x + y^2$
- $\frac{dN}{dt} = 2N$
- $f'(x) = -4$

are DEs (after all, they concern the measure of change of a variable), while

- $y(x) = 2x^2 + 3x - 1$

is not.

2) Composing DEs

2.1) A bullet list for composing DEs

a) Identify your measured quantities (velocity, time, volume, etc.)

b) Choose your variables and appropriate units (time in minutes or seconds, litres, etc.)
c) Determine which of your variables brings about change, and how, and express this measure of change in words.
d) Using the above, compose your DE.

Example 2.1
We are given an infinitely large barrel, in which water is poured at a rate of 10 litres a minute. There's a small hole at the bottom of the barrel, from which the water leaks out at a rate of 1 litre a minute. Compose a DE representing the change of the amount of litres of water in the barrel.

- The quantities are volume (V) and time (t).
- The units are litres and minutes, respectively.
- Every minute, 10 litres of water get in, while 1 litre gets out. Therefore, the net change in volume is that 9 litres are added every minute.
- $\frac{dV}{dt} = 10 - 1 = 9$ with t in minutes

(This example can readily be verified when you acknowledge the simple fact that $V(t) = 9t$.)

2.2) Composing DEs using a given worded expression

You may happen to stumble upon a worded expression to a DE, as opposed to an equation that is given directly. In these cases, a certain diligence towards analyzing and picking apart this expression is required, in order to compose the equation.

Example 2.2
From module 1, homework assignment 2:
"We'll say that the number of crew members who surrender per minute is proportional to the difference between the total number of crew members and the amount of crew members who have already surrendered."

We'll call the number of crew members who have surrendered S. The total number of crew members is 50 (as the assignment states). Now,

- "the number of crew members who surrender per minute" is $\frac{dS}{dt}$.
- "the difference between the total number of crew members and the amount of crew members who have already surrendered" is one number (namely, said difference) – this number being (50 – S)
- the difference is proportional to $\frac{dS}{dt}$, so there's a constant in front of it.

In short:

$$\frac{dS}{dt} = c(50 - S)$$

Provided that we lack additional data (for us to determine c), we're done here.

2.3) Newton's law of cooling

Our friend Newton was a smart man. One of his many accomplishments was putting the cooling process of a given object in mathematical terms.

If an object has a temperature T, deviating from the temperature T_0 of its environment, this object will assume the temperature of its environment over time. Or, as expressed by a DE:

$$\frac{dT}{dt} = c(T - T_0)$$

This DE really is quite elegant, more so than you might think at first glance. For it states, quite literally, that the change in temperature is proportional to the difference in temperature between the object itself and its environment. Here, this means that, as the difference in temperature gets smaller, the rate of temperature change gets smaller as well. When $T = T_0$, the change in temperature is equal to 0. Which makes perfect sense. After all, objects usually don't get hotter or colder by their own accord.

Additionally, c represents the 'speed' at which the object assumes the temperature of its environment. Recall assignment 1c from module 1 – the cannon cools off more rapidly when there's a wind blowing against it, so the magnitude of c

assumes a larger value.

Note as well the similarity of this DE to the DE representing the change in the number of crew members who have surrendered, above. In essence, the latter is a cooling-DE as well. Why, you ask? Well, let's consider the following. Taking a (say, apple pie flavoured) ice cream from the freezer will result in it assuming the temperature of its new environment – in other words, it will melt. In this light, 'law of cooling' could be regarded as a poor choice of words – after all, the law concerns 'assuming' the temperature of the environment, rather than just 'cooling down'.

We can justify this issue by regarding the mathematical side of things. To this end, note that the value of c, as well as the initial temperature of the object, are not inherent to the equation. So on the one hand, the value of c could be positive as well as negative (in the DE representing the number of surrenders, just imagine a minus sign in front of *c*, and it'll quickly assume the cooling-DE's standard form). On the other hand, the initial temperature can be higher or lower than the temperature of the environment (so the solution curve can go down as well as up).

2.4) Determining *c* in Newton's law of cooling

When given a cooling-DE, as well as some data, you may want to determine *c*.

In theory, of course, the cooling-DE is sound. However, in order to determine *c*, we need data. And sadly, due to our physical constrictions, us humans are quite unable to measure phenomena continually. Thus, we'll have to make do with an approximation.

This approximation is made by utilizing linearization.

Example 2.4

Consider the DE $\frac{dy}{dx} = c(y-5)$, which has a starting value of $y = 15$ on $x = 0$. For simplicity's sake, let us say that we're putting a leek (having a temperature of 15 degrees Celsius) in the fridge (programmed at 5 degrees Celsius). We also happen to know that $y = 11$ at $x = 0.1$ (in this example, the unit of time is of no consequence, so we disregard it).

What we are going to do is to approximate the 'real' change in temperature of the leek. We'll do this by drawing a line between our given coordinates, and then saying that this line segment approximately represents reality.

What we're really saying by this, is that the point exactly halfway the line segment approximately represents a point on the 'real' curve. With this, the difference quotient of the line segment (and by extension, the slope on the halfway point) approximates the 'real' slope. In other words: $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$.

Regard the graph below.

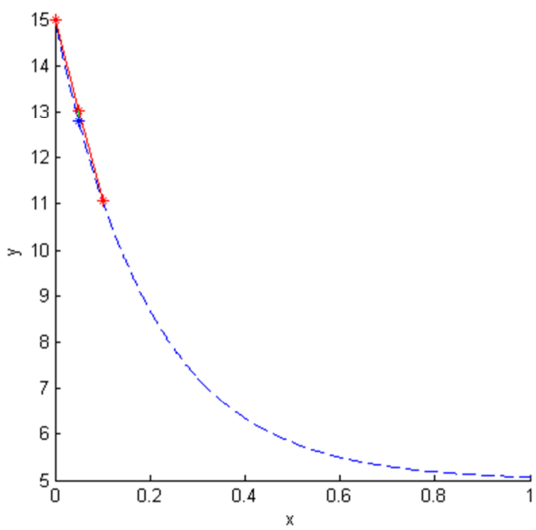


Figure 14 – Our approximation (the line segment) and reality (the dotted curve)
Bear in mind that we actually don't know how reality progresses (i.e. the function of the dotted curve)!

All that's left are some calculations.

Obviously, the x-value of our approximation is 0.05. The y-value is $\frac{15+11}{2} = 13$, the average of the given coordinates. The slope in (0.05,13) is $\frac{\Delta y}{\Delta x} = \frac{11-15}{0.1} = 40$.
So again, what we're saying is that these data are subject to the DE, by approximation. Substitution grants us:
$-40 = c(13 - 5)$
$c = \frac{-40}{8} = -5$

Of course, the workings of this method imply that the approximation gets more accurate as *Δx* (the time between our measurements) gets smaller. You can see that our linear approximation doesn't get anywhere near reality, should we wait a bit longer and measure the values of *y* at, say, *x* = 0 and *x* = 0.5 (figure 15).

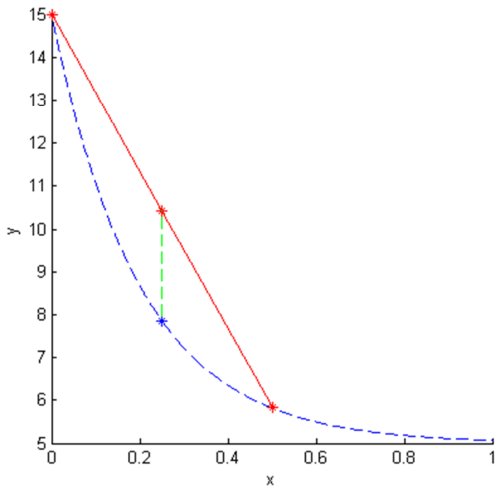


Figure 15: A poor approximation

3) Solving DEs

3.1) Multiple solutions

Given the lack of any initial values, a DE has countless solutions. Only when an initial value does rear its head (for example in the case of a starting temperature), we can pinpoint a singular solution. From an arithmetical perspective, this is best explained by using a simple example.

Example 3.1
Regard the DE $\frac{dy}{dx} = 2$.
The solutions to this DE are quite easily determined. Integration yields $y(x) = 2x + C$.
At this point, this means that the this function is valid in relation to the DE, for any possible value of *C* in *y(x)* – and these possibilities are endless. After all, for any arbitrarily chosen value of *C*, the slope (change over *x*) of the line *y(x)* is 2. The DE states this exactly.
Now let's say that we are given an initial value: $y(0) = 5$.
Quite immediately it becomes clear that $y(x) = 2x + 5$.

3.2) Solving a simple DE

DEs of the form $\frac{dy}{dx} = cy$ are, basically, Newton's law of cooling, having an environmental temperature of 0 (aren't they?)
It is relatively easy to solve this manifestation of Newton's law. To do this, just bear in mind how you would speak out the DE in words: “the derivative of the function *y(x)* is a constant times itself.”

A moment's thought will grant us the insight that this only holds for two general functions:

- $y(x) = ae^{cx}$
  After all, $y'(x) = cae^{cx} = cy(x)$
- Or: $y(x) = 0$

Seeing as we're dealing with the law of cooling, *c* can be determined in the way described above. Given the appropriate data, the solution is found as follows.

Example 3.2
We regard the DE $\frac{dy}{dx} = -2y$ with an initial value $y(0) = 10$.
Enter the general functions which comply to this DE.
Immediately you can say that $y(x) = 0$ is not valid, for $y(0) = 10$.
So that leaves $y(x) = ae^{-2x}$
$y(0) = ae^{0} = a = 10$
Therefore, $y(x) = 10e^{-2x}$.

3.3) Verifying exact solutions

Methods do exist towards solving more complex DEs, but they are quite involved, which is why we haven't given them pause in this series of lessons.
However, it is quite easy to verify an alleged solution to be an actual solution to a DE, using substitution.

Keep in mind that a function y(x) is a solution to a DE, when it conforms to the DE for all values of x on which y(x) is defined.

Example 3.3
Regard $\frac{dy}{dx} = x^2 + y - 2$
Also regard these functions:

- $y_1(x) = 3e^x - x^2 - 2x$
- $y_2(x) = -5e^x - x^2 + 2$

We'll just follow some easy steps towards verifying these functions to be solutions, as follows: $y_1'^{(x)} = 3e^x - 2x - 2$
Substitution yields:
$3e^x - 2x - 2 = x^2 + (3e^x - x^2 - 2x) - 2$
$3e^x - 2x - 2 = 3e^x - 2x - 2$
Which holds for any *x* on which $y_1(x)$ is defined, so $y_1(x)$ is a solution to the DE.
Next: $y_2'(x) = -5e^x - 2x$
Substitution yields:
$-5e^x - 2x = x^2 + (-5e^x - x^2 + 2) - 2$
$-5e^x - 2x = -5e^x$
Which only holds for x = 0, so $y_2(x)$ is not a solution to the DE.

3.4) Line element fields

A line element field to a DE, in essence, is a graphical representation of the DE's 'behaviour'.

The DE has a certain slope (or measure of change) for every given coordinate on the xy-plane. Just imagine being at a certain point, and the DE pushing you in a certain direction away from that point. Alternatively, imagine the pull of a water current, and you're not far off, conceptually speaking (it is no coincidence that DEs are applied widely in fluid dynamics).

A line element field to $\frac{dy}{dx} = f(y(x), x)$ is a drawing of a coordinate system, wherein the slope $\frac{dy}{dx} = f(y_0, x_0)$ is represented at every point (x_0, y_0). This representation is done by drawing a small line at every given coordinate.

Remark: when you're asked to draw a line element field between a given range of *x* and *y*, only draw line elements at the points (*a*,*b*), with *a* and *b* being integers. That is, unless you're assigned to do so differently.

Example 3.4
From module 2, homework assignment 5.

“Regard the DE $\frac{dy}{dx} = -\frac{x}{y}$
Draw a line element field for all points (x,y) within $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.”
Now, for instance, the slope at (1,2) is $\frac{dy}{dx} = -\frac{1}{2}$, so on (1,2) you draw a small line having the slope -0.5.
Some more of the same work will give you the following graph.

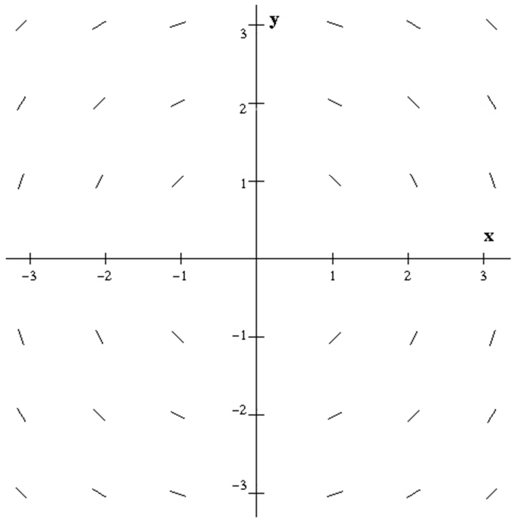


Figure 16 – The line element field

3.5) Solution curves in a line element field

Choosing any starting point, and letting yourself 'go with the flow' of the line element field, will result in your 'course' actually being a solution curve. Which is pretty neat. See the graph below.

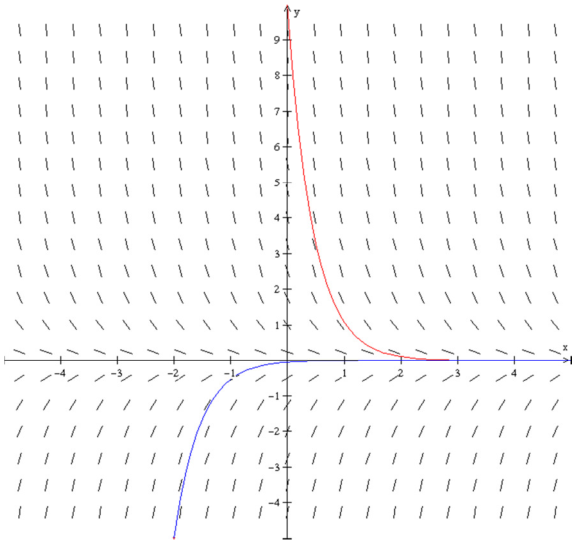


Figure 17 – A typical line element field, with two solution curves shown.

The reason here is that 'choosing any starting point', from which you draw your curve, essentially is the same as choosing your initial value. Figure 17 is a plot of the line element field of $\frac{dy}{dx} = -2y$, as well as the curves starting from (0,10), and (-2,-5).
This DE should look familiar to you – after all, we've seen it before in example 3.2.
By choosing (0,10) as a starting point, we're actually saying the initial value is $y(0) = 10$.
So what we're looking at in figure 17, is the curve emanating from (0,10) actually being the graph of $y(x) = 10e^{-2x}$, which is the solution we determined in example 3.2.
By the same token, the curve stemming from (-2,-5) is actually the graph of $y(x) = -5e^{-2x-4}$ (verify this).

3.6) Euler's method

As said, this lesson series does not concern determining the exact solutions to more complex DEs. This on account of the procedures involved taking too much time to thoroughly understand.
Having said that, we present you with an elegant method, as devised by Euler, toward approximating solutions to DEs, as opposed to determining them exactly.

The expression 'go with the flow' encapsulates the feel of this method. You're at a certain point in the 'flow' of the DE (i.e. some point in the line element field), and you're being pushed in a certain direction. Then you're at a new point, whence you're pushed into another direction. Repeating this, you're drawing your solution curve.

However, our human shortcomings dictate that we simply cannot determine which direction we are pushed towards on literally every point on the curve. We'll have to make do with an approximation, by choosing a certain step size.

In words, this is what we're going for: choose your starting point and use the DE to determine the direction. Draw a line from this point in the given direction, until you've reached the chosen step size in the x-direction. You've reached a new point, in which you again use the DE to determine the direction, and you repeat the process.

A graphical representation of this process is given by using the DE from module 3: $\frac{dy}{dx} = x - y$, with the starting point (-2,0).

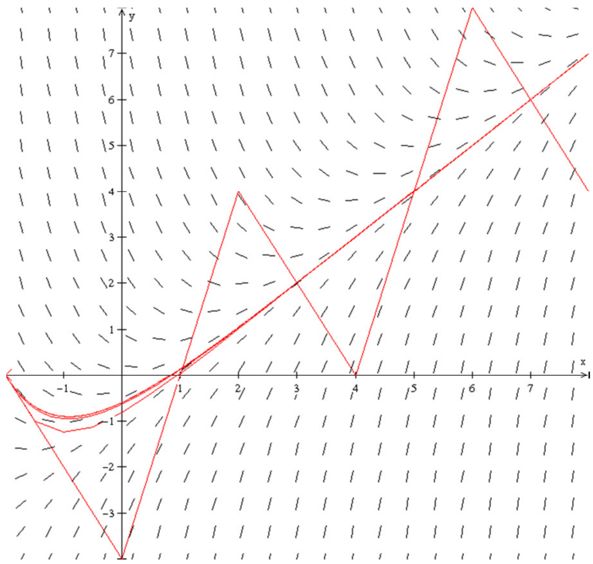


Figure 18 – The line element field, as well as some solutions, as given by Euler's method (having step size 2, 0.5, 0.1 and 0.01).
As the step size decreases, the approximation gets more accurate.

Now, formally:
We're given any DE, as well as an initial value $A(x_0, y_0)$. Euler's method states that the subsequent point on the solution curve, (x_1, y_1), can be approximated by:

$$x_1 = x_0 + h$$
$$y_1 = y_0 + h\left[\frac{dy}{dx}\right]_A$$

Here, the step size is h, and $\left[\frac{dy}{dx}\right]_A$ is the slope at A.

More generally, for the nth and (n+1)st step counts:

$$x_{n+1} = x_n + h = x_0 + h(n+1)$$
$$y_{n+1} = y_n + h\left[\frac{dy}{dx}\right]_N$$

Where $\left[\frac{dy}{dx}\right]_N$ is the slope at $N(x_n, y_n)$.

On your graphic calculator, you can use option (8): RECUR to input this recursive equation and read y_n.

Example 3.6
Regard $\frac{dy}{dx} = x - y$ having $y(-2) = 0$
We're asked to approximate $y(3)$ using Euler's method, with step size 0.5.

This gives us:
$y_{n+1} = y_n + 0.5[x_n - y_n]$
$x_n = -2 + 0.5n$

Or simply:
$y_{n+1} = y_n + 0.5[-2 + 0.5n - y_n]$

Now you grab hold of your graphic calculator and put this in:
$a_{n+1} = a_n + 0.5[-2 + 0.5n - a_n]$

Take care to adjust your table settings (F5: SET) to $a_0 = 0$ (because, of course, $y_0 = 0$), and take appropriate starting and ending values for n. We're looking for the value of $y(3)$ – which corresponds to the value of a_{10} in the table. After all, $x_{10} = -2 + 10 \cdot 0.5 = 3$.

Having done this, the table states $a_{10} = 2.0029 \approx y(3)$.