MASTER THESIS

THE INVARIANT MEASURE OF RANDOM WALKS IN THE QUARTER PLANE

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22-08-2014

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Summary

In this report, we consider the problem that given the interior transition probabilities of a two-dimensional random walk, and a specified measure m, how to construct the transition probabilities on the horizontal axis, the vertical axis and the origin such that m is the invariant measure of the random walk. More precisely, the specified measure m is of the following form,

$$m(n_1, n_2) = \sum_{k=1}^{N} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}),$$

where $\rho_k, \sigma_k \in \mathbb{C}$. Define $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N)\}$. Moreover, let $\bar{\Gamma} = \{(\bar{\rho_1}, \bar{\sigma_1}), \ldots, (\bar{\rho_N}, \bar{\sigma_N})\}$ and $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$. We say that m is induced by $\hat{\Gamma}$.

First, we consider homogeneous transition probabilities on the boundaries. It is shown that if m is the invariant measure of the random walk, every geometric term should satisfy the interior balance equations individually. Moreover, Γ should be a pairwise-coupled set. On the other hand, the results of numerical experiments suggest that homogeneous transition probabilities can not be found.

Next, we consider inhomogeneous boundary transition probabilities. We find that under certain conditions, it is possible to construct inhomogeneous transition probabilities and a pattern can be found for the inhomogeneous boundary transition probabilities.

The contribution of the report is that, firstly, we consider a specified measure induced by complex numbers and their conjugates. Secondly, we consider constructing inhomogeneous transition probabilities on the boundaries.

Chapter 1

Introduction

In this section, we will give a brief introduction to the model and the problem considered in the report. Literature review will be given in Chapter 2. The detailed model description and problem statement will be given in Chapter 3.

1.1 General introduction

Random walks are a class of stochastic processes that have a wide range of applications in, for instance, telecommunication and logistics. In particular, random walks in the positive orthant have been studied widely in applied probability since they can be used to model a lot of queueing systems. Through analysis of the random walk, one can study the performance of the real-life queueing system. We are interested in the stationary performance of a random walk, which will lead to the study of the invariant measure of the random walk.

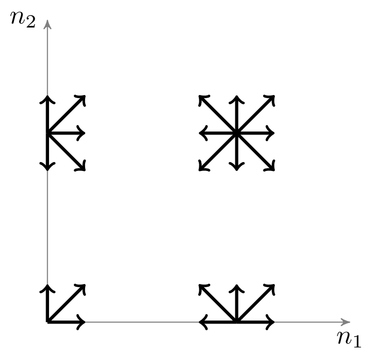


Figure 1.1: A two-dimensional random walk

In Figure 1.1, an illustration of a two-dimensional random walk is given. We consider a random walk in the quarter-plane, *i.e.*, a discrete-time Markov process in state space $S = \{0, 1, \dots\}^2$. A state is represented by its coordinates, *i.e.*, $n = (n_1, n_2)$ for $n \in S$. The random walk can jump from state n to its neighbors in the state space with certain probabilities and the random walk is defined by these transition probabilities. For instance, if the current state of the random walk is in the interior of the state space, then in the next time slot the random walk may jump to one of the eight neighboring states or stays at the same state. However, if the current state is on the horizontal or the vertical axis, there are only five neighboring states, as is shown in Figure 1.1. With these random transitions, the process will walk stochastically through the whole state space.

Let $m : S \to [0, \infty)$ be the invariant measure of the random walk and $f : S \to [0, \infty)$ be a performance measure, we are interested in the stationary performance described as follows,

$$\mathcal{F} = \sum_{n \in S} f(n)m(n). \tag{1.1}$$

If the invariant measure is in closed form, $\mathcal{F}$ can be derived immediately. However, it is well acknowledged that finding the closed-form invariant measure of a general random walk is difficult. Therefore, knowing the exact performance $\mathcal{F}$ of a general random walk is an open problem. Instead of looking at the exact performance, we consider bounding the exact performance of a general random walk, *i.e.*, finding $\mathcal{F}_l$ and $\mathcal{F}_u$ such that

$$\mathcal{F}_l \leq \mathcal{F} \leq \mathcal{F}_u.$$

In practice, a random walk is used to model a queueing system. If bounds on the stationary performance of the random walk is known, performance of the practical system is bounded. Those upper and lower bounds are often sufficient for characterizing the performance of practical systems.

To find upper and lower bounds on $\mathcal{F}$, we will analyze another random walk for which we know the invariant measure in closed form. The random walk is obtained by perturbing transition probabilities of the original random walk on the horizontal and vertical boundaries. More precisely, let R be a general random walk in the quarter plane, for which the closed-form invariant measure is unknown. Moreover, let $\tilde{R}$ be a random walk for which we know the invariant measure $\tilde{m}$ in closed form. Transition probabilities of R and $\tilde{R}$ are the same in the interior of the state space. On the other hand, transition probabilities of R and $\tilde{R}$ on the horizontal axis, vertical axis and the origin are distinct. Since we know $\tilde{m}$ in closed form, we can get $\tilde{\mathcal{F}}$ using formula described in Equation (1.1). If the difference between the transition probabilities of R and $\tilde{R}$ is not too large, it is expected that

$$|\tilde{\mathcal{F}} - \mathcal{F}| < \varepsilon,$$

where ε is sufficiently small. Upper and lower bounds on $\mathcal{F}$ can be found then.

1.2 Problem description

In this report, we consider the problem of finding the perturbed random walk for which the invariant measure is specified. Suppose that for the random walk $\tilde{R}$, transition probabilities in the interior of the state space are given, and the measure $\tilde{m} : S \to [0, \infty)$ is specified in closed form, then transition probabilities on the horizontal axis, vertical axis and the origin need to be constructed such that m is the invariant measure of $\tilde{R}$. We also consider a rescaled random walk, which means that if boundary transition probabilities for the given interior transition probabilities can not be found, we will rescale the interior transition probabilities and try to find boundary transition probabilities for the rescaled probabilities. Rescaling is allowed since it doesn't have influence on the invariant measure of the random walk. If we know the construction method, for a general random walk we can change its transition probabilities on the boundaries to those found by the construction method and get the perturbed random walk $\tilde{R}$, for which the closed-form invariant measure $\tilde{m}$ is known. If for state (n_1, n_2),

$$\tilde{m}(n_1, n_2) = \rho^{n_1} \sigma^{n_2},$$

we say that $\tilde{m}$ is a geometric measure induced by (ρ, σ). In this report, we consider that $\tilde{m}$ is a linear combination of a finite number of geometric measures induced by $\rho_k, \sigma_k \in \mathbb{C}$, for $k = 1, \dots, N$. To make sure that the measure is real for every state, we will also add the geometric terms induced by the complex conjugates of (ρ_k, σ_k). For any $\rho_k, \sigma_k \in \mathbb{C}$, denote by $\bar{\rho}_k, \bar{\sigma}_k$ the complex conjugates of ρ_k and σ_k. Define

$$\Gamma = \big\{(\rho_1, \sigma_1), \dots, (\rho_N, \sigma_N)\big\}.$$

Moreover, assume that for any $(\rho_k, \sigma_k) \in \Gamma$, $(\bar{\rho_k}, \bar{\sigma_k}) \notin \Gamma$. Corresponding to Γ, define

$$\bar{\Gamma} = \big\{(\bar{\rho_1}, \bar{\sigma_1}), \dots, (\bar{\rho_N}, \bar{\sigma_N})\big\}.$$

The measure we consider is of the form below

$$\tilde{m}(n_1, n_2) = \sum_{k=1}^{N} c_k (\rho_k^{n_1} \sigma_k^{n_2} + \bar{\rho_k}^{n_1} \bar{\sigma_k}^{n_2}). \tag{1.2}$$

Let $\tilde{\Gamma} = \Gamma \cup \bar{\Gamma}$, then we say that the measure $\tilde{m}$ is induced by $\tilde{\Gamma}$.

Cases where $\rho_k \in \mathbb{R}$ and $\sigma_k \in \mathbb{R}$ were studied in previous works. In this report, we will consider the case that $\rho_k, \sigma_k \in \mathbb{C}$ for $k = 1, \dots, N$. If we find the method to construct the random walk $\tilde{R}$ for a specified measure $\tilde{m}$, then through the extension to complex numbers and their conjugates, there are more random walks for which we know the invariant measure in closed form. Thus we have more options for the perturbed random walk $\tilde{R}$. The exact performance of the perturbed random walk $\tilde{F}$ can provide upper and lower bounds on the performance of the general random walk. Therefore, if we have more options for perturbed random walk, the upper and lower bounds will be improved.

The problem statement will be formulated formally in Chapter 3. For a measure $\tilde{m}$ of the form in Equation (1.2), first we consider random walks with homogeneous transition probabilities on the horizontal and the vertical axis. However, numerical results indicate that homogeneous transition probabilities can not be found hence it is not promising to consider homogeneous random walks. Therefore, we consider inhomogeneous transition probabilities on the horizontal and the vertical axis in Chapter 5. It turns out that under some conditions, inhomogeneous transition probabilities can be achieved.

The remainder of the report is structured as follows. Related research and previous works are described in Chapter 2. In Chapter 3, the two-dimensional model we consider in this report will be given. The problem will also be stated in details in that chapter. In Chapter 4, we consider how to construct homogeneous transition probabilities on the boundaries. We first give necessary conditions on the structure of Γ for $\tilde{m}$ to be the invariant measure. Next, we try to find transition probabilities on the horizontal and vertical axis for the specified measure $\tilde{m}$. Extensive numerical experiments and initial analytical results indicate that no homogeneous transition probabilities can be found. Thus in Chapter 5, inhomogeneous boundary transition probabilities are discussed and we find that, under certain conditions, inhomogeneous transition probabilities will be found. Examples of random walks with constructed boundary transition probabilities will be given in Chapter 6. In Chapter 7, we will discuss the conclusions in this report and look into future works.

Chapter 2

Related work

Most research on random walks is focused on the invariant measure of a random walk. However, it is well acknowledged that obtaining the closed-form invariant measure of a general random walk is an open problem. Theories about random walks in the quarter plane was described in [8], where general concepts were introduced. Boundary value problems were formulated and analytic approaches were used to get the generating function of the invariant measure [7, 8]. However, the problem can be solved explicitly only in a few special cases. In general, it is difficult to find the closed-form invariant measure of a random walk.

Adan, Wessels and Zijm developed a compensation approach for random walks in two dimensional space in [1], where they discussed the conditions under which the invariant measure is an infinite series of products of powers. Boxma and van Houtum applied this approach into an asymmetric 2×2 switch system with independent Bernoulli arrivals and found that the invariant measure is a sum of countably many geometric terms [3].

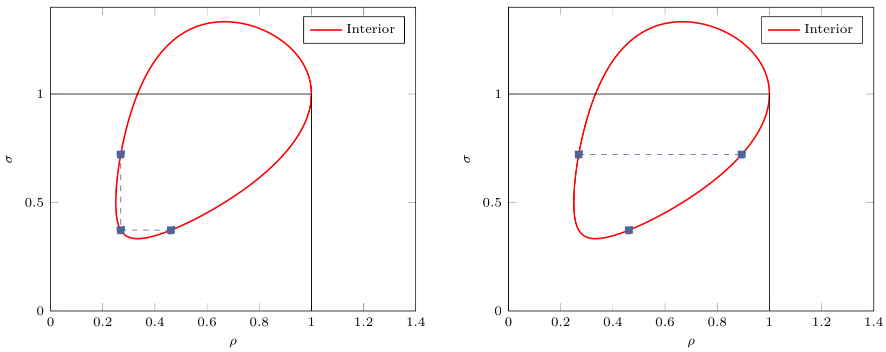
The structure of the space of product-form models was explored in [2] for continuous-time random walks. It was stated that for any product-form measure, transition rates of a random walk in the interior and on the boundaries can be selected independently such that the specified measure is the invariant measure of the random walk. Therefore, if the interior transition rates of a general random walk are given and a product-form measure m is specified, transition rates on the horizontal and vertical boundaries can be constructed such that m is the invariant of the random walk.

In this report, we specify that the invariant measure is a linear combination of a finite number of geometric terms, which is described in Equation (**??**). Moreover, the measure is induced by the set Γ, as defined in Chapter 1. Cases where $\rho, \sigma \in \mathbb{R}$ were studied in several papers. In [5] necessary conditions on the structure of Γ were given if the resulting measure is the invariant measure of a random walk. It was stated that each geometric term has to individually satisfy the balance equations in the interior of the state space. Moreover, the geometric terms must have a pairwise-coupled structure. The definitions of uncoupled partition and pairwise-coupled set are given below, following [5].

Definition 2.1 (Uncoupled partition)**.** *A partition* $\{\Gamma_1, \Gamma_2, \dots\}$ *of* Γ *is* horizontally uncoupled *if* $(\rho, \sigma) \in \Gamma_p$ *and* $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$ *for* $p \neq q$*, implies that* $\tilde{\rho} \neq \rho$*,* vertically uncoupled *if* $(\rho, \sigma) \in \Gamma_p$ *and* $(\tilde{\rho}, \tilde{\sigma}) \in \Gamma_q$ *for* $p \neq q$*, implies that* $\tilde{\sigma} \neq \sigma$*,* *and* uncoupled *if it is both horizontally and vertically uncoupled.*

Definition 2.2 (Pairwise-coupled set)**.** *A set* $\Gamma \subset C$ *is pairwise-coupled if and only if the maximal uncoupled partition of* Γ *contains only one set.*

In other words, a set $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N)\}$ is pairwise-coupled if and only if for any pair (ρ_j, σ_j), we can find another pair (ρ_k, σ_k) such that either $\rho_j = \rho_k$ or $\sigma_j = \sigma_k$. An example of pairwise-coupled set is illustrated in Figure 2.1(a) while an example is shown in Figure 2.1(b) for a set which is not pairwise-coupled.



(a) (b)

Figure 2.1: (a) A pairwise-coupled set. (b) A set that is not pairwise-coupled.

In Figure 2.1, the horizontal and the vertical axis represent $\rho \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ respectively. For (ρ, σ) on the red curve, balance equations in the interior of the state space are satisfied. If m induced by Γ is the invariant measure of the random walk, each geometric term has to satisfy the balance equations in the interior individually [5]. Thus we take three (ρ, σ) from the red curve and consider the measure induced by those pairs. In Figure 2.1(a), every (ρ_k, σ_k) has the same ρ_k or σ_k as one of the other pairs. Thus the set is pairwise-coupled. However, in Figure 2.1(b), there is an isolated term. Therefore, the set is not pairwise-coupled. Example of homogeneous random walks of which the invariant measures are sums of a finite number of geometric terms and method to construct such random walks are given in [5]. An example taken from [5] is given in Chapter 6.

Necessary conditions for the invariant measure of a random walk to be an infinite sum of geometric terms were explored in [6]. It was found in accordance with the result for a finite number of geometric terms that each geometric term should satisfy the interior balance equation and a pairwise-coupled structure must be satisfied. Furthermore, it was shown that the random walk cannot have transitions to the north, northeast, or east.

Suppose that we have a specified measure m that is induced by a finite number of pairwise-coupled geometric terms. The next question is how can we find a random walk m such that m is the invariant measure of R. Analysis was done for one geometric term $\rho^{n_1}\sigma^{n_2}$ with $\rho, \sigma \in \mathbb{R}$ in [4], where a naive scheme was given to find the boundary transition probabilities such that the geometric measure is the invariant measure of the random walk. It was stated that a rescaled random walk can always be found for the geometric measure. For measures that are linear combinations of geometric terms, the method was described in [5]. The construction depends on the locations of the intersections of the boundary and interior balance equations. It was concluded that choosing proper boundary transition probabilities is essential for the existence of the invariant measure.

After perturbing a general random walk, error bounds between the perturbed and the original random walk are considered. A Markov reward approach developed by van Dijk in [11] is used to bound the error. This Markov reward approach was applied to finite single-server tandem queues in [10] and reasonable bounds were given there. Perturbation effects of a discrete-time Markov reward process was studied in [12], where bounds were obtained for the absolute errors of the reward functions. Linear programming was formulated in [9] to provide the error bounds. In [9], the perturbed random walk has a product-form invariant measure. Approximation of performance and error bounds were also considered in [4], where the invariant measure of the perturbed random is a sum of three geometric terms. It was shown that, if the invariant measure is a sum of geometric terms, the resulting error bounds is better than the one obtained from product-form measure.

Chapter 3

Model and problem statement

In this chapter, the model considered in the report is described and the problem is stated. In Section 3.1, we give a general model description, where basic concepts and notations are introduced. Moreover, we consider the restrictive case and the relaxed case. The model and balance equations for the two cases are also given. In 3.2, a detailed problem statement is given. We also list the main goals of the report in the section.

3.1 Model description

In this report, we consider discrete-time random walks in the positive orthant of two-dimensional space, *i.e.*, the state space is

$$S = \{0, 1, \dots\}^2.$$

A state is represented by a two-dimensional vector, *i.e.*, $n = (n_1, n_2)$.

In addition, we consider a partition of S into disjoint components. Let $S_0 = \{(0,0)\}, S_1 = \{1,2,\dots\} \times \{0\}, S_2 = \{0\} \times \{1,2,\dots\}$ and $S_3 = \{1,2,\dots\} \times \{1,2,\dots\}$. We refer to these components as the origin, the horizontal axis, the vertical axis and the interior of the state space, respectively. The state space and all the components are shown in Figure 3.1.

For the random walks, we only allow short transitions between states. More precisely, a transition from state $n = (n_1, n_2)$ to $n' = (n'_1, n'_2)$ is possible only if

$$||n - n'||_\infty \leq 1.$$

Let $c(n)$ be the component of state $n \in S$, *i.e.*, $n \in S_{c(n)}$. Denote by D_k the neighbors of a state in S_k. More precisely, $D_3 = \{-1,0,1\} \times \{-1,0,1\}, D_1 = \{-1,0,1\} \times \{0,1\}, D_2 = \{0,1\} \times \{-1,0,1\}$ and $D_0 = \{0,1\} \times \{0,1\}$.

Let $p_d^k(n)$ be the probability of the random walk jumping from state $n \in S_k$ to $n+d$, where $d \in D_k$. In this report, we consider random walks with homogeneous transition probabilities in the interior of the state space, which means that

$$p_d^3(n) = p_d^3(n'), \quad \forall n, n' \in S_3, d \in D_3.$$

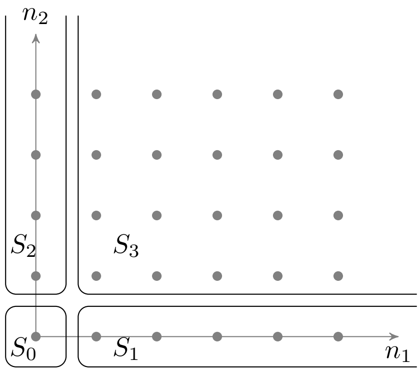


Figure 3.1: State space S and the components.

In the remainder, for simplicity of notation, we denote by p_d the homogeneous transition probabilities in the interior of the state space, *i.e.*, for $n \in S_3$, $d \in D_3$, $p_d = p_d^3(n)$.

We show all the neighbors in the components of the state space and the transition probabilities of a homogeneous random walk R in Figure 3.2, except the one that directs to the state itself.

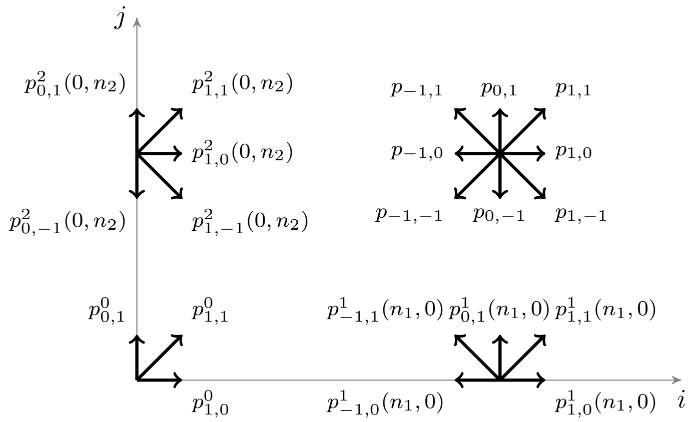


Figure 3.2: Transition diagram of random walk R

If for any $n_1, n_1' > 0$,

$$p_d^1(n_1, 0) = p_d^1(n_1', 0),$$

for $d \in D_1$, we say that the transition probabilities on the horizontal axis are homogeneous and denote by p_d^1 the transition probabilities on horizontal axis. Similarly, if for any $n_2, n_2' > 0$,

$$p_d^2(0, n_2) = p_d^2(0, n_2),$$

for $d \in D_2$, we say that the transition probabilities on the vertical axis are homogeneous and denote by p_d^2 those transition probabilities. It was assumed that $p^1_{s,1}(n_1, 0) = p_{s,1}$, $p^2_{1,t}(0, n_2) = p_{1,t}$ for any $n_1, n_2 > 0$ in [5]. In this report, we consider both the case where $p^1_{s,1}(n_1, 0) = p_{s,1}$, $p^2_{1,t}(0, n_2) = p_{1,t}$ for any $n_1, n_2 > 0$ and the other case where these equations do not hold. The models for the two cases will be given in detail in Subsection 3.1.1 and Subsection 3.1.2.

3.1.1 The restrictive case

In this case, we assume that for any $n_1, n_2 > 0$,

$$p^1_{s,1}(n_1, 0) = p_{s,1}, \forall s = -1, 0, 1$$

and

$$p^2_{1,t}(0, n_2) = p_{1,t}, \forall t = -1, 0, 1.$$

Consider a measure $m : S \to [0, \infty)$ of the random walk, then m is the invariant measure if and only if it satisfies all the balance equations, *i.e.*, for all $n_1 > 0$ and $n_2 > 0$,

$$\begin{aligned}
m(n_1, n_2) &= \sum_{s=-1}^{1} \sum_{t=-1}^{1} m(n_1 - s, n_2 - t) p_{s,t}, && (3.1)\\
m(n_1, 0) &= \sum_{s=-1}^{1} m(n_1 - s, 1) p_{s,-1} + \sum_{s=-1}^{1} m(n_1 - s, 0) p^1_{s,0}(n_1 - s, 0), && (3.2)\\
m(0, n_2) &= \sum_{t=-1}^{1} m(1, n_2 - t) p_{-1,t} + \sum_{t=-1}^{1} m(0, n_2 - t) p^2_{0,t}(0, n_2 - t), && (3.3)\\
m(0, 0) &= m(1, 1) p_{-1,-1} + m(1, 0) p^1_{-1,0}(1, 0) + \\
&\quad m(0, 1) p^2_{0,-1}(0, 1) + m(0, 0) p^0_{0,0}. && (3.4)
\end{aligned}$$

We will refer to Equations (3.1)-(3.4) as the balance equations in the interior, the horizontal axis, the vertical axis and the balance equation in the origin, respectively. The balance equation in the origin is implied by the balance equations for the other states. If the balance equations hold for the other states, then for sure it is satisfied in the origin. Thus in the following part of the report, if Equation (3.1) to (3.3) are all satisfied, we say the m is the invariant measure of the random walk.

3.1.2 The relaxed case

In this case, the balance equations are different from those of the restrictive case. m is the invariant measure of the random walk if and only if for all $n_1 > 1$, $n_2 > 1$,

$$
\begin{aligned}
m(n_1,n_2) &= \sum_{s=-1}^{1}\sum_{t=-1}^{1} m(n_1-s,n_2-t)p_{s,t},\\
m(n_1,0) &= \sum_{s=-1}^{1} m(n_1-s,1)p_{s,-1} + \sum_{s=-1}^{1} m(n_1-s,0)p^1_{s,0}(n_1-s,0),\\
m(n_1,1) &= \sum_{s=-1}^{1}\sum_{t=-1}^{0} m(n_1-s,1-t)p_{s,t} + \sum_{s=-1}^{1} m(n_1-s,0)p^1_{s,1}(n_1-s,0),\\
m(0,n_2) &= \sum_{t=-1}^{1} m(1,n_2-t)p_{-1,t} + \sum_{t=-1}^{1} m(0,n_2-t)p^2_{0,t}(0,n_2-t),\\
m(1,n_2) &= \sum_{s=-1}^{0}\sum_{t=-1}^{1} m(1-s,n_2-t)p_{s,t} + \sum_{t=-1}^{1} m(0,n_2-t)p^2_{1,t}(0,n_2-t).
\end{aligned}
$$

If the transition probabilities on the horizontal and vertical axis are found, then for state $(1,0)$, $(1,1)$ and $(0,1)$, transition probabilities can be found using similar approaches. The balance equation at state $(0,0)$ is implied by balance equations of the other states. Therefore, the equations for state $(1,1),(1,0),(0,1)$ and $(0,0)$ are omitted here since they are of minor importance to the analysis.

In this section, we have introduced basic concepts and notations used in the report. Furthermore, we have given models for two cases that will be considered in the report. In the next section, we will state the problem in detail.

3.2 Problem statement

In this report, we will consider how to find the random walk R, of which the invariant measure m is specified. As introduced in Chapter 1, we are interested in a measure that is a linear combination of a finite number of geometric terms induced by $\rho_k,\sigma_k \in \mathbb{C}$. Here we give the definition of the measure m.

Definition 3.1. *A measure m is called induced by $\hat{\Gamma}$ if*

$$
m(n_1,n_2) = \sum_{k=1}^{N} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}), \tag{3.5}
$$

where $\Gamma = \big\{(\rho_1,\sigma_1),\ldots,(\rho_N,\sigma_N)\big\}$ *with* $\rho_k,\sigma_k\in\mathbb{C}$*, and* $\bar{\Gamma} = \big\{(\bar{\rho_1},\bar{\sigma_1}),\ldots,(\bar{\rho_N},\bar{\sigma_N})\big\}$.

Moreover, we consider non-negative, finite measures in this report, thus we have the following assumption.

Assumption 3.2. *Assume that the invariant measure of the random walk is finite, i.e.,*

$$
\sum_{n\in S} m(n_1,n_2) < \infty. \tag{3.6}
$$

Moreover, assume that for any state (n_1,n_2),

$$
m(n_1,n_2) \geq 0.
$$

Define

$$\mathcal{U} = \left\{ z \in \mathbb{C} \middle| |z| < 1 \right\}.$$

Therefore, the measure is finite if and only if

$$(\rho_k, \sigma_k) \in \mathcal{U}^2,$$

for all $k = 1, \ldots, N$.

In the remainder of the report, when we consider $\rho_k, \sigma_k \in \mathbb{C}$, we always assume that $(\rho_k, \sigma_k) \in \mathcal{U}^2$. Suppose that the measure m is specified and it is induced by $\hat{\Gamma}$, we need to find the random walk R of which the invariant measure is m. We assume that transition probabilities in the interior of the state space are already given. Therefore, we only need to find the transition probabilities on the horizontal axis, the vertical axis and the origin. The goals of the report are described below.

1. For the given interior transition probabilities and a specified measure m, find the conditions on the structure of m under which the random walk can be constructed.

2. If the random walk can be constructed, explore the construction method, *i.e.*, find the following probabilities

$$\begin{aligned} p_d^1(n), n \in S_1, d \in D_1, \\ p_d^2(n), n \in S_2, d \in D_2, \\ p_d^0, n \in S_0, d \in D_0, \end{aligned}$$

such that m is the invariant measure of the constructed random walk.

The goals will be discussed in Chapter 4 for homogeneous random walks, where the structure of Γ will be given. On the other hand, it is indicated by numerical results that homogeneous transition probabilities on the boundaries can not be found for measures induced by complex numbers and their complex conjugates. Thus we turn to construct inhomogeneous transition probabilities, which will be considered in Chapter 5. The first and second goals are both discussed there. Furthermore, we show the way to find the boundary transition probabilities in Chapter 5. In Chapter 6, several examples are given to illustrate the construction method described in Chapter 5.

We also allow rescaling of the random walk, since rescaling does not have influence on the invariant measure of the random walk. When we construct the boundaries transition probabilities, we try to find probabilities which are between 0 and 1. However, if this is not possible but we can find values that are larger than 1, we can always rescale the interior transition probabilities and get boundary transition probabilities of the rescaled random walk. Thus, in the remainder of the report, when we try to find the boundary transition probabilities, we do not discuss whether these transition probabilities are between 0 and 1 or not.

In the next chapter, we consider homogeneous transition probabilities on the horizontal and the vertical axis. First we will find necessary conditions for the specified measure m to be the invariant measure of a random walk. Therefore, the structure of Γ will be discussed.

Chapter 4

Homogeneous transition probabilities on the boundaries

In this chapter, we consider homogeneous random walks and try to find transition probabilities on the horizontal axis, vertical axis and the origin. More precisely, assume that for any $n_1 > 0$,

$$p_d^1(n_1, 0) = p_d^1, d \in D_1,$$

and for any $n_2 > 0$,

$$p_d^2(0, n_2) = p_d^2, d \in D_2.$$

Suppose that the measure m is induced by the set $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$, where $\Gamma = \{(\rho_1, \sigma_1), \dots, (\rho_N, \sigma_N)\}$ with $\rho_k, \sigma_k \in \mathbb{C}$ and $\bar{\Gamma} = \{(\bar{\rho_1}, \bar{\sigma_1}), \dots, (\bar{\rho_N}, \bar{\sigma_N})\}$. In the next section, we will give a necessary condition on the structure of Γ for m to be the invariant measure of a homogeneous random walk.

4.1 Structure of the set Γ

Let $\Gamma = \{(\rho_1, \sigma_1), \dots, (\rho_N, \sigma_N)\}$, we consider a measure induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$, as defined by Definition 3.1.

In [5] it was shown that if $\rho_k, \sigma_k \in \mathbb{R}$ and $m(n_1, n_2)$ is the invariant measure of the random walk, then $(\rho_1, \sigma_1), \dots, (\rho_N, \sigma_N)$ should satisfy the interior balance equation individually and follow a pairwise-coupled structure. Here we show that the same conclusions apply when we consider $\rho_k, \sigma_k \in \mathbb{C}$.

First, we claim that if the measure m is induced by a single pair (ρ, σ) and its complex conjugate, then $\rho \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

Theorem 4.1. *Suppose that the measure m is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$. If $|\Gamma| = 1$, then $\rho \in \mathbb{R}$, $\sigma \in \mathbb{R}$.*

Proof. Let $\Gamma = \{(\rho, \sigma)\}$. Assume that at least one of ρ and σ is not real. Without loss of generality, assume that $\sigma \notin \mathbb{R}$. Consider states $(0, n_2)$ for all

$n_2 \geq 0$, then

$$\begin{aligned} m(0, n_2) &= c(\rho^0 \sigma^{n_2} + \bar{\rho}^0 \bar{\sigma}^{n_2}) \\ &= 2c \cdot \mathrm{Re}(\sigma^{n_2}). \end{aligned}$$

Let $\sigma = r_2 e^{i\theta_2}$ and assume that $\sigma \notin \mathbb{R}$, then $0 < r_2 < 1$ and $\theta_2 \in (0, 2\pi)$. Since m is a measure, then

$$m(0, n_2) = 2r_2^{n_2} \cos(n_2\theta_2) \geq 0.$$

Therefore, we get

$$\cos(n_2\theta_2) \geq 0, \tag{4.1}$$

for any $n_2 = 1, 2, \ldots$. On the other hand, $\cos(n_2\theta_2)$ is a periodic function in n_2 and the value fluctuates between -1 and 1. Thus, as n_2 goes to infinity, there exists an integer n_0 such that $\cos(n_0\theta_2) < 0$. m can not be non-negative for all states hence it is not a measure. Thus $\rho \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

□

From the theorem above, we see that the measure m can not be induced by a single pair (ρ, σ) and its complex conjugate if $\rho \notin \mathbb{R}$ or $\sigma \notin \mathbb{R}$. In the next theorem, we show that if m is a linear combination of a finite number of geometric terms, then each geometric term has to satisfy the balance equations in the interior of the state space individually.

To identify the geometric terms that satisfy the balance equations in the interior of the state space, we introduce the polynomial

$$P(x, y) = xy(1 - \sum_{s=-1}^{1} \sum_{t=-1}^{1} x^{-s} y^{-t} p_{s,t}). \tag{4.2}$$

It is easy to verify that if $P(\rho, \sigma) = 0$, then the balance equations in the interior of the state space hold for the geometric measure $\rho^{n_1}\sigma^{n_2}$. We define

$$I = \Big\{(\rho, \sigma) \in \mathcal{U}^2 \Big| P(\rho, \sigma) = 0\Big\}. \tag{4.3}$$

For any geometric measure induced by $(\rho, \sigma) \in I$, balance equations in the interior of the state space hold. In [5], it was proved that if $\rho_k, \sigma_k \in \mathbb{R}$ for $k = 1, \ldots, N$, then there exists integers w_1, w_2 such that $\rho^{w_1}\sigma^{w_2} \neq \rho_1^{w_1}\sigma_1^{w_2}$ for any $(\rho, \sigma) \in \Gamma \backslash (\rho_1, \sigma_1)$. When we consider $\rho_k, \sigma_k \in \mathbb{C}$, since we can rotate ρ_k, σ_k by an angle and get a different complex number, the statement in [5] does not always hold. Therefore, we first have the following assumption.

Assumption 4.2. *For any $(\rho_k, \sigma_k) \in \Gamma$ with $k = 1, \ldots, N$, there exist integers $w_1^k, w_2^k \in \mathbb{N}_+$ such that*

$$\rho_k^{w_1^k}\sigma_k^{w_2^k} \neq \rho^{w_1^k}\sigma^{w_2^k},$$

for any $(\rho, \sigma) \in \Gamma \cup \bar{\Gamma} \backslash \{(\rho_k, \sigma_k)\}$.

Next, we are going to give a theorem which demonstrates that every geometric measure in Equation (3.5) should satisfy the balance equations in the interior of the state space.

Theorem 4.3. *The measure m is the invariant measure of a random walk if and only if every geometric term satisfies the balance equations in the interior of the state space individually, i.e., for any $k = 1, \dots N$,*

$$(\rho_k, \sigma_k) \in I, (\bar{\rho_k}, \bar{\sigma_k}) \in I. \tag{4.4}$$

Proof. ($\Leftarrow$) It is trivial that if every geometric term satisfies balance equations in the interior of the state space, then m also satisfies the balance equations, since m is a sum over a finite number of geometric terms.

($\Rightarrow$) Let

$$m(n_1, n_2) = \sum_{k=1}^{N} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2})$$

be the invariant measure of a random walk, then from Equation (3.1), we get that

$$\begin{aligned}
&\sum_{k=1}^{N} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}) = \sum_{s=-1}^{1}\sum_{t=-1}^{1}\sum_{k=1}^{N} c_k(\rho_k^{n_1-s}\sigma_k^{n_2-t} + \bar{\rho_k}^{n_1-s}\bar{\sigma_k}^{n_2-t})p_{s,t} \\
\Leftrightarrow\ &\sum_{k=1}^{N} c_k\rho_k^{n_1}\sigma_k^{n_2}(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{-s}\sigma_k^{-t}p_{s,t}) + \\
&\sum_{k=1}^{N} c_k\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \bar{\rho_k}^{-s}\bar{\sigma_k}^{-t}p_{s,t}) = 0.
\end{aligned} \tag{4.5}$$

Without loss of generality, we only prove that $(\rho_1, \sigma_1) \in I$. From Assumption 4.2, there exists positive integer w_1^1, w_2^1 such that $\rho_1^{w_1^1}\sigma_1^{w_2^1} \neq \rho^{w_1^1}\sigma^{w_2^1}$ for any $(\rho, \sigma) \in \Gamma \cup \bar{\Gamma}\backslash\{(\rho_1, \sigma_1)\}$. We also assume that $\rho_j^{w_1^1}\sigma_j^{w_2^1} \neq \rho_k^{w_1^1}\sigma_k^{w_2^1}$ for any $(\rho_j, \sigma_j), (\rho_k, \sigma_k) \in \Gamma \cup \bar{\Gamma}$ with $j \neq k$. Otherwise, we partition the set Γ in the following way. If $\rho_j^{w_1^1}\sigma_j^{w_2^1} = \rho_k^{w_1^1}\sigma_k^{w_2^1}$ for some $j \neq 1, k \neq 1$, then we put (ρ_j, σ_j) and (ρ_k, σ_k) in the same element of the partition. Thus (ρ_1, σ_1) is the only pair in its element.

Consider the balance equations for states (dw_1^1, dw_2^1), with $d = 1, 2, \dots, 2N$, then we have

$$\begin{aligned}
&\sum_{k=1}^{N}[\rho_k^{w_1^1}\sigma_k^{w_2^1}]^d[c_k(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{-s}\sigma_k^{-t}p_{s,t})] + \\
&\sum_{k=1}^{N}[\bar{\rho_k}^{w_1^1}\bar{\sigma_k}^{w_2^1}]^d[c_k(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \bar{\rho_k}^{-s}\bar{\sigma_k}^{-t}p_{s,t})] = 0.
\end{aligned}$$

It is a system of linear equations in variables $c_k(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{-s}\sigma_k^{-t}p_{s,t})$, and $c_k(1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \bar{\rho_k}^{-s}\bar{\sigma_k}^{-t}p_{s,t})$. The coefficients matrix is

$$A = \begin{bmatrix}
\rho_1^{w_1^1}\sigma_1^{w_2^1} & \dots & \rho_N^{w_1^1}\sigma_N^{w_2^1} & \bar{\rho_1}^{w_1^1}\bar{\sigma_1}^{w_2^1} & \dots & \bar{\rho_N}^{w_1^1}\bar{\sigma_N}^{w_2^1} \\
(\rho_1^{w_1^1}\sigma_1^{w_2^1})^2 & \dots & \rho_N^{w_1^1}\sigma_N^{w_2^1\,2} & (\bar{\rho_1}^{w_1^1}\bar{\sigma_1}^{w_2^1})^2 & \dots & (\bar{\rho_N}^{w_1^1}\bar{\sigma_N}^{w_2^1})^2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
(\rho_1^{w_1^1}\sigma_1^{w_2^1})^{2N} & \dots & (\rho_N^{w_1^1}\sigma_N^{w_2^1})^{2N} & (\bar{\rho_1}^{w_1^1}\bar{\sigma_1}^{w_2^1})^{2N} & \dots & (\bar{\rho_N}^{w_1^1}\bar{\sigma_N}^{w_2^1})^{2N}
\end{bmatrix},$$

which has a Vandermonde structure. Since $\rho_j^{w_1^1}\sigma_j^{w_2^1} \neq \rho_k^{w_1^1}\sigma_k^{w_2^1}$ for any $j \neq k$, we have that $\det(A) \neq 0$. Therefore, for any $(\rho_k, \sigma_k) \in \Gamma$, $(\bar{\rho_k}, \bar{\sigma_k}) \in \bar{\Gamma}$,

$$\begin{aligned} 1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \rho_k^{-s}\sigma_k^{-t}p_{s,t} &= 0, \\ 1 - \sum_{s=-1}^{1}\sum_{t=-1}^{1} \bar{\rho_k}^{-s}\bar{\sigma_k}^{-t}p_{s,t} &= 0. \end{aligned}$$

We conclude that $(\rho_k, \sigma_k) \in I$, $(\bar{\rho_k}, \bar{\sigma_k}) \in I$, for $k = 1, \ldots, N$. □

Elements in $\bar{\Gamma}$ are the complex conjugates of Γ. If $\Gamma \subset I$, then $\bar{\Gamma} \subset I$. Thus $\hat{\Gamma} \subset I$ if and only if $\Gamma \subset I$. The next theorem claims that the representation of $\hat{\Gamma}$ is unique.

Theorem 4.4. *Let m be the invariant measure of a random walk, which is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \subset I$. If m is also induced by $\hat{\Gamma}'$, then $\hat{\Gamma} = \hat{\Gamma}'$.*

Proof. Since m is induced by both $\hat{\Gamma}$ and $\hat{\Gamma}'$, then for all $n_1 > 0$, $n_2 > 0$,

$$\begin{aligned} & \sum_{(\rho_k,\sigma_k)\in\Gamma} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}) - \sum_{(\rho_k,\sigma_k)\in\Gamma'} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}) \\ = & \sum_{(\rho_k,\sigma_k)\in\Gamma\cap\Gamma'} (c_k - c_k')\rho_k^{n_1}\sigma_k^{n_2} + \sum_{(\bar{\rho_k},\bar{\sigma_k})\in\bar{\Gamma}\cap\bar{\Gamma}'} (c_k - c_k')\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2} + \\ & \sum_{(\rho_k,\sigma_k)\in\Gamma\backslash\Gamma'} c_k\rho_k^{n_1}\sigma_k^{n_2} + \sum_{(\rho_k,\sigma_k)\in\Gamma\backslash\Gamma'} c_k\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2} - \\ & \sum_{(\rho_k,\sigma_k)\in\Gamma'\backslash\Gamma} c_k'\rho_k^{n_1}\sigma_k^{n_2} - \sum_{(\rho_k,\sigma_k)\in\Gamma'\backslash\Gamma} c_k'\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2} \\ = & \ 0. \end{aligned}$$

Similar to the proof of Theorem 4.3, we can find (w_1, w_2) such that $\rho_k^{w_1}\sigma_k^{w_2}$ are distinct. Consider states $(w_1, w_2), \ldots, (|\hat{\Gamma} \cup \hat{\Gamma}'|w_1, |\hat{\Gamma} \cup \hat{\Gamma}'|w_2)$, then we get a system of linear equations. The coefficients matrix has a Vandermonde structure. Therefore, we have $c_k - c_k' = 0$, if $(\rho_k, \sigma_k) \in \Gamma \cap \Gamma'$, $c_k = 0$ if $(\rho_k, \sigma_k) \in \Gamma\backslash\Gamma'$, and $c_k' = 0$ if $(\rho_k, \sigma_k) \in \Gamma'\backslash\Gamma$. As a consequence, $\Gamma = \Gamma'$. □

Consider a measure m induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \subset I$. Next we will prove that Γ should follow a pairwise-coupled structure. The definition of pairwise-coupled structure was already introduced in Chapter 2, where examples were given to illustrate the definition of the structure. If Γ is a pairwise-coupled set, so is $\bar{\Gamma}$. However, the set $\Gamma \cup \bar{\Gamma}$ may not be pairwise-coupled.

Maximal horizontally uncoupled sets are obtained by putting $(\rho, \sigma) \in \Gamma$ with the same ρ in the same element of the partition. Maximal vertically uncoupled sets are obtained by putting $(\rho, \sigma) \in \Gamma$ with the same σ in the same element. Moreover, maximal uncoupled sets are obtained by putting $(\rho, \sigma) \in \Gamma$ with the same ρ or the same σ in the same element of the partition. Let $\{\Gamma_1^h, \Gamma_2^h, \ldots, \Gamma_H^h\}$ be the maximal horizontally uncoupled partition of Γ and $\{\Gamma_1^v, \Gamma_2^v, \ldots, \Gamma_V^v\}$ be the maximal vertically uncoupled partition of Γ. Consequently, $\{\bar{\Gamma}_1^h, \bar{\Gamma}_2^h, \ldots, \bar{\Gamma}_H^h\}$ is the maximal horizontally uncoupled partition of $\bar{\Gamma}$

and $\{\bar{\Gamma}_1^v, \bar{\Gamma}_2^v, \ldots, \bar{\Gamma}_V^v\}$ is the maximal vertically uncoupled partition of $\bar{\Gamma}$. Elements of $\Gamma_p^h, p = 1, \ldots, H$ have the same horizontal coordinate $\rho(\Gamma_p^h)$ and elements of $\Gamma_q^v, q = 1, \ldots, V$ have the same $\sigma(\Gamma_q^v)$. Moreover, for $p = 1, \ldots, H$, $\rho(\Gamma_p^h) = \rho(\bar{\Gamma}_p^h)$ if and only if $\rho \in \mathbb{R}$. To illustrate the balance equations on the horizontal and vertical axis. We define

$$B^h(\Gamma_p^h) = \begin{cases} \sum_{(\rho_k,\sigma_k)\in\Gamma_p^h} c_k\big[1 - \sum_{s=-1}^{1}(\rho_k^{-s}\sigma_k p_{s,-1} + \rho_k^{-s} p_{s,0}^1)\big] & , \quad \rho(\Gamma_p^h) \notin \mathbb{R} \\ \sum_{(\rho_k,\sigma_k)\in\Gamma_p^h} c_k\big[1 - \sum_{s=-1}^{1}(2\rho_k^{-s}\operatorname{Re}(\sigma_k) p_{s,-1} + \rho_k^{-s} p_{s,0}^1)\big], & \rho(\Gamma_p^h) \in \mathbb{R} \end{cases}$$

$$B^v(\Gamma_q^v) = \begin{cases} \sum_{(\rho_k,\sigma_k)\in\Gamma_q^v} c_k\big[1 - \sum_{t=-1}^{1}(\rho_k\sigma_k^{-t} p_{-1,t} + \sigma_k^{-t} p_{0,t}^2)\big] & , \quad \sigma(\Gamma_q^v) \notin \mathbb{R} \\ \sum_{(\rho_k,\sigma_k)\in\Gamma_q^v} c_k\big[1 - \sum_{t=-1}^{1}(2\operatorname{Re}(\rho_k)\sigma_k^{-t} p_{-1,t} + \sigma_k^{-t} p_{0,t}^2)\big], & \sigma(\Gamma_q^v) \in \mathbb{R} \end{cases}$$

Moreover, define $B^h(\bar{\Gamma}_p^h)$ and $B^v(\bar{\Gamma}_q^v)$ in the similar way and we can see that $B^h(\bar{\Gamma}_p^h) = \overline{B^h(\Gamma_p^h)}$, $B^v(\bar{\Gamma}_q^v) = \overline{B^v(\Gamma_q^v)}$. The following lemma gives sufficient and necessary conditions for the measure m induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \subset I$ to be the invariant measure of the random walk.

Lemma 4.5. *Consider a measure m induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \subset I$, where $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N)\}$ and $\bar{\Gamma} = \{(\bar{\rho_1}, \bar{\sigma_1}), \ldots, (\bar{\rho_N}, \bar{\sigma_N})\}$ with $\rho_k, \sigma_k \in \mathbb{C}$. Then m is the invariant measure of a random walk if and only if for any $p = 1, \ldots, H$ and $q = 1, \ldots, V$,*

$$B^h(\Gamma_p^h) = 0 \quad , \quad B^h(\bar{\Gamma}_p^h) = 0,$$
$$B^v(\Gamma_q^v) = 0 \quad , \quad B^v(\bar{\Gamma}_q^v) = 0.$$

Proof. ($\Leftarrow$) It is easy to verify that if the conditions above are satisfied, then balance equations on the horizontal and vertical axis are satisfied. According to Theorem 4.3, m also satisfies the balance equations in the interior since m is induced by $\Gamma \subset I$ and $\bar{\Gamma} \subset I$. Therefore, all balance equations are satisfied and m is the invariant measure of the random walk.

($\Rightarrow$) Suppose that m is the invariant measure of the random walk, then m

satisfies the balance equations on the horizontal axis, *i.e.*, for any $n_1 = 1, 2, \dots,$

$$\begin{aligned}
& m(n_1, 0) = \sum_{s=-1}^{1} m(n_1 - s, 1) p_{s,-1} + m(n_1 - s, 0) p^1_{s,0}, \\
\Leftrightarrow\ & \sum_{k=1}^{N} c_k \rho_k^{n_1} \big[1 - \sum_{s=-1}^{1} (\rho_k^{-s} \sigma_k p_{s,-1} + \rho_k^{-s} p^1_{s,0})\big] + \\
& \sum_{k=1}^{N} c_k \bar{\rho_k}^{n_1} \big[1 - \sum_{s=-1}^{1} (\bar{\rho_k}^{-s} \bar{\sigma_k} p_{s,-1} + \bar{\rho_k}^{-s} p^1_{s,0})\big] = 0, \\
\Leftrightarrow\ & \sum_{p=1}^{H} [\rho(\Gamma_p^h)]^{n_1} \sum_{(\rho_k,\sigma_k)\in\Gamma_p^h} c_k \big[1 - \sum_{s=-1}^{1} (\rho_k^{-s} \sigma_k p_{s,-1} + \rho_k^{-s} p^1_{s,0})\big] + \\
& \sum_{p=1}^{H} [\overline{\rho(\Gamma_p^h)}]^{n_1} \sum_{(\rho_k,\sigma_k)\in\Gamma_p^h} c_k \big[1 - \sum_{s=-1}^{1} (\bar{\rho_k}^{-s} \bar{\sigma_k} p_{s,-1} + \bar{\rho_k}^{-s} p^1_{s,0})\big] \\
& = 0. \\
\Leftrightarrow\ & \sum_{p=1}^{H} \big[\rho(\Gamma_p^h)\big]^{n_1} B^h(\Gamma_p^h) + \sum_{p=1}^{H} \big[\overline{\rho(\Gamma_p^h)}\big]^{n_1} B^h(\bar{\Gamma}_p^h) = 0 \\
\Leftrightarrow\ & \sum_{\rho(\Gamma_p^h)\in\mathbb{R}} \big[\rho(\Gamma_p^h)\big]^{n_1} B^h(\Gamma_p^h) + \sum_{\rho(\Gamma_p^h)\notin\mathbb{R}} \big[\rho(\Gamma_p^h)\big]^{n_1} B^h(\Gamma_p^h) + \\
& \sum_{\rho(\Gamma_p^h)\notin\mathbb{R}} \big[\overline{\rho(\Gamma_p^h)}\big]^{n_1} B^h(\bar{\Gamma}_p^h) = 0. \qquad (4.6)
\end{aligned}$$

Notice that $\rho(\Gamma_p^h), p = 1, \dots, H$ are all distinct due to the definition of maximal horizontally partition. Moreover, if $\rho(\Gamma_p^h) \notin \mathbb{R}$, $\rho(\Gamma_p^h) \neq \overline{\rho(\Gamma_p^h)}$. Thus Equation (4.6) is a system of linear equations in variables $B^h(\Gamma_p^h)$ and $B^h(\bar{\Gamma}_p^h)$, for $p = 1, \dots, H$. The coefficients follow a Vandermonde structure. Therefore, for $p = 1, \dots, H$,

$$B^h(\Gamma_p^h) = 0, B^h(\bar{\Gamma}_p^h) = 0.$$

Using the same argument, we can also get that for $q = 1, \dots, V$,

$$B^v(\Gamma_q^v) = 0, B^v(\bar{\Gamma}_q^v) = 0.$$

This concludes the proof of this lemma.

□

With Lemma 4.5, we are ready to prove that if m induced by Γ and $\bar{\Gamma}$ is the invariant measure of the random walk, then Γ has to follow a pairwise-coupled structure.

Theorem 4.6. *Let m be a measure induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \subset I$. If m is the invariant measure of a random walk, then Γ has a pairwise-coupled structure.*

Proof. Let $\{\Gamma_p^h\}_{p=1}^H$ and $\{\Gamma_q^v\}_{q=1}^V$ be the maximal horizontally and vertically partition of Γ respectively. As a consequence, $\{\bar{\Gamma}_p^h\}_{p=1}^H$ and $\{\bar{\Gamma}_q^v\}_{q=1}^V$ are the maximal horizontally and vertically partition of $\bar{\Gamma}$. In addition, denote by $\{\Gamma_u\}_{u=1}^M$

and $\{\bar{\Gamma}_u\}_{u=1}^M$ the maximal uncoupled partition of Γ and $\bar{\Gamma}$ respectively. For any $u = 1, \ldots, M$, we can find $I_u \subset \{1, \ldots, H\}$ and $J_u \subset \{1, \ldots, V\}$, such that

$$\Gamma_u = \bigcup_{p \in I_u} \Gamma_p^h = \bigcup_{q \in J_u} \Gamma_q^v,$$
$$\bar{\Gamma}_u = \bigcup_{p \in I_u} \bar{\Gamma}_p^h = \bigcup_{q \in J_u} \bar{\Gamma}_q^v.$$

Consider the measure m, we have

$$\begin{aligned} m(n_1, n_2) &= \sum_{k=1}^N c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}) \\ &= \sum_{u=1}^M \big[\sum_{(\rho,\sigma)\in\Gamma_u} c(\rho,\sigma)\rho^{n_1}\sigma^{n_2} + \sum_{(\rho,\sigma)\in\Gamma_u} c(\rho,\sigma)\bar{\rho}^{n_1}\bar{\sigma}^{n_2}\big]. \end{aligned}$$

Define that

$$m_u(n_1, n_2) = \sum_{(\rho,\sigma)\in\Gamma_u} c(\rho,\sigma)(\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}),$$

then $m(n_1, n_2) = \sum_{u=1}^M m_u(n_1, n_2)$. We will show that for any $u = 1, \ldots, M$, m_u also satisfies all the balance equations. For state $(n_1, 0)$,

$$\begin{aligned} & m_u(n_1, 0) - (\sum_{s=-1}^{1} m_u(n_1 - s, 1)p_{s,-1} + m_u(n_1 - s, 0)p_{s,0}^1)] \\ = & \sum_{(\rho,\sigma)\in\Gamma_u} c(\rho,\sigma)\big\{\rho^{n_1}[1 - \sum_{s=-1}^{1}(\rho^{-s}\sigma p_{s,-1} + \rho^{-s}p_{s,0}^1)]\big\} + \\ & \sum_{(\rho,\sigma)\in\Gamma_u} c(\rho,\sigma)\big\{\bar{\rho}^{n_1}[1 - \sum_{s=-1}^{1}(\bar{\rho}^{-s}\bar{\sigma} p_{s,-1} + \bar{\rho}^{-s}p_{s,0}^1)]\big\} \\ = & \sum_{p\in I_u} \rho(\Gamma_p^h)^{n_1} \sum_{(\rho,\sigma)\in\Gamma_p^h} [\sum_{s=-1}^{1}(\rho^{-s}\sigma p_{s,-1} + \rho^{-s}p_{s,0}^1) - 1] + \\ & \sum_{p\in I_u} \overline{\rho(\Gamma_p^h)}^{n_1} \sum_{(\rho,\sigma)\in\Gamma_p^h} [\sum_{s=-1}^{1}(\bar{\rho}^{-s}\bar{\sigma} p_{s,-1} + \bar{\rho}^{-s}p_{s,0}^1) - 1] \\ = & \sum_{p\in I_u, \rho(\Gamma_p^h)\in\mathbb{R}} \rho(\Gamma_p^h)^{n_1} B^h(\Gamma_p^h) + \sum_{p\in I_u, \rho(\Gamma_p^h)\notin\in\mathbb{R}} \big[\rho(\Gamma_p^h)^{n_1} B^h(\Gamma_p^h) + \overline{\rho(\Gamma_p^h)}^{n_1} B^h(\bar{\Gamma}_p^h)\big] \\ = & \ 0. \end{aligned}$$

The last equality follows from Lemma 4.5. Using similar approaches, we can also find that $m_u(n_1, n_2)$ satisfies the balance equations on the vertical axis for $u = 1, \ldots, M$. Thus $m_u, u = 1, \ldots, M$ are the invariant measures of the random walk. If $M > 1$, then each m_u is the invariant measure of the random walk. It contradicts with Theorem 4.4, which states that the representation of the invariant measure is unique. Therefore, $M = 1$ and the set Γ is pairwise-coupled.

□

In this section, we have looked at the structure of the set Γ which can be used to induce the invariant measure. We draw the conclusion that each geometric term in m must satisfy the balance equations in the interior of the state space individually. In addition, the set Γ has to follow a pairwise-coupled structure. In the next section, given the interior transition probabilities of a random walk, we take a measure m which is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma} \in I$ where Γ is a pairwise-coupled set. We will explore how to find the transition probabilities on the horizontal, the vertical axis and the origin such that m is the invariant measure of the random walk.

4.2 The restrictive case

In this section, we consider the method to find transition probabilities on the horizontal and vertical axis. Moreover, we consider the restrictive case, *i.e.*,

$$p^1_{s,1} = p_{s,1}, \forall s = -1, 0, 1$$

and

$$p^2_{1,t} = p_{1,t}, \forall t = -1, 0, 1.$$

In Theorem 4.1, we have shown that $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$ can not be a measure if $\rho \notin \mathbb{R}$ or $\sigma \notin \mathbb{R}$, since for some states it is negative. However, we didn't consider boundary balance equations for this signed measure. In Subsection 4.2.1, we look at the signed measure $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$, and we get the conclusion that we can not find both horizontal and vertical transition probabilities. In Subsection 4.2.2, we consider the measure induced by a pairwise-coupled set and its complex conjugate set, and try to use the structure to find transition probabilities on the boundaries.

Suppose that the interior transition probabilities of the random walk, $p_{s,t}$ are given. Let $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \rho_N)\} \subset I$ be a pairwise-coupled set and m be the measure induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$.

4.2.1 Boundary transition probabilities for a signed measure

In this subsection, we consider the signed measure $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$, and we try to find transition probabilities on the horizontal and vertical axis. Moreover, if $\rho, \sigma \in \mathbb{R}$, the conclusions were given in [5] and we assume that at least one of ρ, σ is not real. Hence we consider two cases. The first case is $\rho \in \mathbb{R}, \sigma \notin \mathbb{R}$ or $\rho \notin \mathbb{R}, \sigma \in \mathbb{R}$. The second case is $\rho \notin \mathbb{R}$ and $\sigma \notin \mathbb{R}$.

Case 1. $\rho \in \mathbb{R}, \sigma \notin \mathbb{R}$ **or** $\rho \notin \mathbb{R}, \sigma \in \mathbb{R}$

First consider that if $\rho \in \mathbb{R}, \sigma \notin \mathbb{R}$, then

$$m(n_1, n_2) = c(\rho^{n_1}\sigma^{n_2} + \rho^{n_1}\bar{\sigma}^{n_2}).$$

From Lemma 4.5, if m is the invariant measure of the random walk, then $B^h(\rho) = 0$, *i.e.*,

$$1 = \sum_{s=-1}^{1} 2\rho^{-s}\,\mathrm{Re}(\sigma)p_{s,-1} + \sum_{s=-1}^{1} \rho^{-s}p^1_{s,0}. \tag{4.7}$$

Inserting $\sum_{s=-1}^{1} p_{s,1} + \sum_{s=-1}^{1} p_{s,0}^{1} = 1$ into Equation (4.7), we get

$$(1-\rho)p_{-1,0}^{1} + (1-1/\rho)p_{1,0}^{1} = \sum_{s=-1}^{1} 2\rho^{-s}\operatorname{Re}(\sigma)p_{s,-1} - \sum_{s=-1}^{1} p_{s,1}. \tag{4.8}$$

There are two variables, $p_{-1,0}^{1}, p_{1,0}^{1}$, in Equation (4.8). If we use $p_{-1,0}^{1}$ as one coordinate on the two-dimensional plane and $p_{1,0}^{1}$ as the other coordinate, then all the solutions to Equation (4.8) lie on a straight line in the plane. Since $0<\rho<1$, the slope of the straight line is positive. Thus, non-negative solutions to Equation (4.8) can be found. Therefore, transition probabilities of a rescaled random walks can always be constructed.

On the vertical axis, since $\sigma \notin \mathbb{R}$, then $B^{v}(\sigma)=0$, *i.e.*,

$$1 - \sum_{t=-1}^{1} \rho\sigma^{n_2-t}p_{-1,t} - \sum_{t=-1}^{1} \sigma^{n_2-t}p_{0,t}^{2} = 0.$$

Plugging in $\sum_{t=-1}^{1} p_{1,t} + \sum_{t=-1}^{1} p_{0,t}^{2} = 1$, we have

$$(1-\sigma)p_{0,-1}^{2} + (1-1/\sigma)p_{0,1}^{2} = \sum_{t=-1}^{1} \rho\sigma^{-t}p_{-1,t} - \sum_{t=-1}^{1} p_{1,t}. \tag{4.9}$$

We need to find the variables $p_{0,t}^{2}, t=-1,0,1$ such that Equation (4.9) holds. Notice that in Equation (4.9), the coefficients can be complex and the variables $p_{0,-1}^{2}, p_{0,1}^{2}$ are real numbers. Let $x_v = [p_{0,-1}^{2}, p_{0,1}^{2}]^{T}$, then Equation (4.9) is equivalent to the linear system $A_v x_v = b_v$, where

$$\begin{aligned} A_v &= \begin{bmatrix} \operatorname{Re}(1-\sigma) & \operatorname{Re}(1-1/\sigma) \\ \operatorname{Im}(1-\sigma) & \operatorname{Im}(1-1/\sigma) \end{bmatrix}, \\ b_v &= \begin{bmatrix} \operatorname{Re}(\sum_{t=-1}^{1} \rho\sigma^{-t}p_{-1,t} - \sum_{t=-1}^{1} p_{1,t}) \\ \operatorname{Im}(\sum_{t=-1}^{1} \rho\sigma^{-t}p_{-1,t} - \sum_{t=-1}^{1} p_{1,t}) \end{bmatrix}. \end{aligned}$$

If $0 \le x_v \le 1$, then we can find the transition probabilities on the vertical axis for a rescaled random walk. Here the inequality works element-wise.

However, we have done a lot of numerical experiments on a wide range of different interior transition probabilities, and the numerical results suggest that the solutions to the system is negative.

If $\rho \notin \mathbb{R}, \sigma \in \mathbb{R}$, similar approaches can be used for the analysis on the vertical axis. Through numerical experiments, we also see that when we consider balance equations on the horizontal axis, the solutions are negative. Thus we can find transition probabilities on the vertical axis but not on the horizontal axis.

Case 2. $\rho \notin \mathbb{R}$ **and** $\sigma \notin \mathbb{R}$

If $\rho \notin \mathbb{R}$ and $\sigma \notin \mathbb{R}$, then

$$m(n_1,n_2) = c(\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}).$$

From Lemma 4.5, if m is the invariant measure of the random walk, then for any $p=1,\dots,H$ and $q=1,\dots,V$,

$$B^{h}(\Gamma_p^{h}) = 0, B^{v}(\Gamma_q^{v}) = 0.$$

Similar to the analysis in Case 1, we have the two linear systems for the signed measure,

$$\begin{aligned} A_h x_h &= b_h, \\ A_v x_v &= b_v. \end{aligned}$$

We have done many numerical experiments on various interior transition probabilities. The numerical results indicate that, for a signed measure $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$, only one of x_h and x_v is non-negative. If $x_h \geq 0$, then $x_v < 0$. If $x_v \geq 0$, then $x_h < 0$. Thus, we have the following conjecture.

Conjecture 4.7. *For a signed measure $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$ with $\rho, \sigma \in \mathbb{C}$, the balance equations on the horizontal axis and the vertical axis lead to two linear systems, $A_h x_h = b_h$ and $A_v x_v = b_v$. Exactly one of the following two statements is true.*

1. $x_h \geq 0$.
2. $x_v \geq 0$.

From Conjecture 4.7, we see that we can find transition probabilities either on the horizontal axis or on the vertical axis.

In this subsection, we see that if $\rho, \sigma \in \mathbb{C}$, for a signed measure $\rho^{n_1}\sigma^{n_2} + \bar{\rho}^{n_1}\bar{\sigma}^{n_2}$, either horizontal transition probabilities or vertical transition probabilities can be found. In the next subsection, we will use the property of pairwise-coupled structure and try to find boundary transition probabilities for a measure m induced by a pairwise-coupled set Γ and its complex conjugate set $\bar{\Gamma}$.

4.2.2 Boundary transition probabilities for m induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$

Let $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N) | \rho_k, \sigma_k \in \mathbb{C}\}$ be a pairwise-coupled set and m be the measure induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$. From Subsection 4.2.1, we know for a single signed measure $\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}$, either horizontal transition probabilities or vertical transition probabilities can be found. If we consider a pairwise-coupled set, we can use the structure of the set.

Suppose transition probabilities on the horizontal axis are found for the signed measure $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$ and transition probabilities on the vertical axis are found for the signed measure $\rho_N^{n_1}\sigma_N^{n_2} + \bar{\rho_N}^{n_1}\bar{\sigma_N}^{n_2}$, if we can find proper coefficients c_k such that conditions in Lemma 4.5 hold, then m is the invariant measure of the random walk.

First we give the following conjecture, which states that for two signed measures, $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$ and $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$, if (ρ_1, σ_1) and (ρ_2, σ_2) belong to the same element of the maximal horizontally or vertically uncoupled partition of Γ, then horizontal transition probabilities can be found for one of the signed measure and vertical transition probabilities can be found for the other.

Conjecture 4.8. *Suppose that the signed measures $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$ and $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$ both satisfy the balance equations in the interior of the state space. If $\rho_1 = \rho_2$ or $\sigma_1 = \sigma_2$, then exactly one of the following statements hold:*

1. *Transition probabilities* on the horizontal axis *can be found for* $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$*, and transition probabilities* on the vertical axis *can be found for* $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$.

2. *Transition probabilities* on the vertical axis *can be found for* $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$*, and transition probabilities* on the horizontal axis *can be found for* $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$.

Following from Conjecture 4.7 and 4.8, we have the lemma below.

Lemma 4.9. *Let* $\Gamma = \{(\rho_1,\sigma_1),\dots,(\rho_N,\sigma_N)|\rho_k,\sigma_k \in \mathbb{C}\}$ *be a pairwise-coupled subset of* I *and* m *be the measure induced by* $\hat{\Gamma}$*. Moreover, transition probabilities on the horizontal axis are found for the signed measure* $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$ *and transition probabilities on the vertical axis are found for* $\rho_N^{n_1}\sigma_N^{n_2} + \bar{\rho_N}^{n_1}\bar{\sigma_N}^{n_2}$*. If Conjecture 4.7 and 4.8 hold and* m *is the invariant measure of the random walk, then* $|\Gamma|$ *is even.*

Proof. Since transition probabilities are found for the signed measure $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$, then from Conjecture 4.7 and 4.8, only vertical transition probabilities are found for $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$. Keep using Conjecture 4.7, 4.8 and the pairwise-coupled structure, we get that only vertical transition probabilities are found for $\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}$ if k is even and only horizontal transition probabilities are found for $\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}$ if k is odd. Since transition probabilities on the vertical axis are found for $\rho_N^{n_1}\sigma_N^{n_2} + \bar{\rho_N}^{n_1}\bar{\sigma_N}^{n_2}$, then N is even. Thus $|\Gamma|$ is even. □

Next, we give the main result of this subsection. In the next theorem, we state that even if the pairwise-coupled structure is used, we can not find the boundary transition probabilities such that m is the invariant measure of the random walk.

Theorem 4.10. *Let* $\Gamma = \{(\rho_1,\sigma_1),\dots,(\rho_N,\sigma_N)\}$ *with* $\rho_k,\sigma_k \in \mathbb{C}$ *and* m *be the measure induced by* $\hat{\Gamma}$*. Moreover, transition probabilities on the horizontal axis are found for the signed measure* $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$ *and transition probabilities on the vertical axis are found for* $\rho_N^{n_1}\sigma_N^{n_2} + \bar{\rho_N}^{n_1}\bar{\sigma_N}^{n_2}$*. If Conjecture 4.7 and 4.8 hold, then no boundary transition probabilities can be found such that* m *is the invariant measure of the random walk.*

Proof. Since Γ is a pairwise-coupled set, then either $\rho_1 = \rho_2$ or $\sigma_1 = \sigma_2$. Thus we consider two cases in the following proof.

Case 1. $\rho_1 = \rho_2$

If $\rho_1 = \rho_2$, then (ρ_1,σ_1) and (ρ_2,σ_2) are in the same element of the maximal horizontally uncoupled set. Without loss of generality, assume that $\rho_1 = \rho_2 \notin \mathbb{R}$. Let $(\rho_1,\sigma_1),(\rho_2,\sigma_2) \in \Gamma_1^h$, then according to Lemma 4.5,

$$\begin{aligned} B^h(\Gamma_1^h) &= c_1\big[1 - \sum_{s=-1}^{1}(\rho_1^{-s}\sigma_1 p_{s,-1} + \rho_1^{-s}p_{s,0}^1)\big] + \\ &\quad c_2\big[1 - \sum_{s=-1}^{1}(\rho_2^{-s}\sigma_2 p_{s,-1} + \rho_2^{-s}p_{s,0}^1)\big] \\ &= 0. \end{aligned}$$

Since $p^1_{-1,0}, p^1_{1,0}$ are found for the signed measure $\rho_1^{n_1}\sigma_1^{n_2} + \bar{\rho_1}^{n_1}\bar{\sigma_1}^{n_2}$, then

$$1 - \sum_{s=-1}^{1} (\rho_1^{-s}\sigma_1 p_{s,-1} + \rho_1^{-s} p^1_{s,0}) = 0.$$

Thus we have

$$1 - \sum_{s=-1}^{1} (\rho_2^{-s}\sigma_2 p_{s,-1} + \rho_2^{-s} p^1_{s,0}) = 0,$$

which means that $\rho_2^{n_1}\sigma_2^{n_2} + \bar{\rho_2}^{n_1}\bar{\sigma_2}^{n_2}$ also satisfies the balance equations on the horizontal axis. This contradicts with Conjecture 4.7 and 4.8, which says that no horizontal transition probabilities can be found for $\rho_2^{n_1}\sigma_2^{n_2}$.

Case 2. $\sigma_1 = \sigma_2$

If $\sigma_1 = \sigma_2$ and transition probabilities on the horizontal axis are found for $\rho_1^{n_1}\sigma_1^{n_2}$, then by the pairwise-coupled structure of Γ, $\rho_2 = \rho_3, \sigma_3 = \sigma_4$, etc. According to Lemma 4.9, N is even. Thus $\sigma_{N-1} = \sigma_N$, *i.e.*, $(\rho_{N-1}, \sigma_{N-1})$ and (ρ_N, σ_N) are in the same element of the maximal vertically uncoupled partition. Suppose that $(\rho_{N-1}, \sigma_{N-1}), (\rho_N, \sigma_N) \in \Gamma^v_V$, then by Lemma 4.5, we have

$$\begin{aligned} B^v(\Gamma^v_V) &= c_{N-1}\big[1 - \sum_{t=-1}^{1} (\rho_{N-1}\sigma_{N-1}^{-t} p_{-1,t} + \sigma_{N-1}^{-t} p^2_{0,t})\big] + \\ &\quad c_N\big[1 - \sum_{t=-1}^{1} (\rho_N \sigma_N^{-t} p_{-1,t} + \sigma_N^{-t} p^2_{0,t})\big] \\ &= 0. \end{aligned}$$

Using the arguments in the proof of Case 1, we can get

$$1 - \sum_{t=-1}^{1} (\rho_{N-1}\sigma_{N-1}^{-t} p_{-1,t} + \sigma_{N-1}^{-t} p^2_{0,t}) = 0,$$

which contradicts with Conjecture 4.7 and 4.8. Therefore, no boundary transition probabilities can be found such that m is the invariant measure of the random walk.

□

In this section, we have shown how to construct the transition probabilities on the horizontal and vertical axis for a specified measure m to be the invariant measure of the random walk. However, we also see that although horizontal or vertical transition probabilities can be constructed for a single geometric measure, but there is a contradiction with the structure of Γ. Therefore, homogeneous transition probabilities can not be found for a specified measure induced by complex numbers and their complex conjugates. The restrictive is considered here. In the next section, we are going to generalize the conclusions we get from this section and state that even if we consider the relaxed case, homogeneous transition probabilities still can not be found.

4.3 The relaxed case

In this part, we consider the relaxed perturbation, which means that we do not require

$$p^1_{s,1} = p_{s,1}, \forall s \in \{-1, 0, 1\}.$$

From Subsection 3.1.2, we see that besides the balance equations of the restrictive case, there are more balance equations for states $(n_1, 1)$ and $(1, n_2)$, *i.e.*,

$$m(n_1, 1) = \sum_{s=-1}^{t=1} \sum_{t=-1}^{0} m(n_1 - s, 1 - t)p_{s,t} + \sum_{s=-1}^{1} m(n_1 - s, 0)p^1_{s,1}, \quad (4.10)$$

$$m(1, n_2) = \sum_{s=-1}^{0} \sum_{t=-1}^{1} m(1 - s, n_2 - t)p_{s,t} + \sum_{t=-1}^{1} m(0, n_2 - t)p^2_{1,t}. \quad (4.11)$$

We can analyze in the same way as that in Section 4.2, and consider $\rho \in \mathbb{R}$ or $\rho, \sigma \notin \mathbb{R}$. However, we find that all the lemmas, conjectures and theorems can be generalized here, since for balance equations at states $(n_1, 0)$, the sum $\sum_{s=-1}^{1} p_{s,1}$ is more important to the equation than the individual transition probability.

For the relaxed case, it still holds that for a single geometric term, we can not find both transition probabilities on the horizontal and the vertical axis. Moreover, either horizontal or vertical transition probabilities can be found. Therefore, all the conclusions follow from those in Section 4.2 even if we consider the relaxed case.

In this chapter, we try to construct a homogeneous random walk for a specified measure m induced by $\hat{\Gamma}$. First we give necessary conditions on the structure of the Γ for m to be the invariant measure of the random walk. Next, given a specified measure for which the conditions hold, we try to construct the transition probabilities on the horizontal and vertical axis. However, results of numerical experiments indicate that no boundary transition probabilities can be found due to the contradiction between the structure of the set and the transition probabilities constructed for one single signed measure. It seems not promising to work on homogeneous transition probabilities. Therefore, we turn to inhomogeneous transition probabilities on the boundaries.

In the next chapter, we are going to consider an inhomogeneous random walk, which means that the transition probabilities on the horizontal and vertical axis depend on the state. Assume that the random walk still has homogeneous transition probabilities in the interior of the state space. We will to use similar construction method to find horizontal and vertical transition probabilities of the random walk for a specified measure to be the invariant measure.

Chapter 5

Inhomogeneous transition probabilities on the boundaries

In this chapter, we consider random walks with inhomogeneous transition probabilities on the horizontal and vertical axis. Moreover, we assume that transition probabilities in the interior of the state space are still homogeneous. Therefore, from Theorem 4.3 in Chapter 4, each geometric term has to satisfy the balance equations in the interior individually, *i.e.*, the measure m is induced by a set $\hat{\Gamma} \subset I$.

Suppose that the measure m is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$, where the set $\Gamma = \big\{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N)\big\}$ and $(\rho_k, \sigma_k) \in \mathbb{C}$ for $k = 1, 2, \ldots, N$. In addition, assume that $\Gamma \subset I$, then the balance equations in the interior of the state space hold for m. From results in Chapter 4, it is difficult to find homogeneous transition probabilities on the horizontal and the vertical axis. In this chapter, we try to find inhomogeneous transition probabilities on the horizontal and the vertical axis such that m is the invariant measure of the constructed random walk. More precisely, the transition probabilities depend on the state. Let $p_{s,t}(n_1, 0), s \in \{-1, 0, 1\}, t \in \{0, 1\}$ be the transition probabilities at state $(n_1, 0)$ on the horizontal axis and $p_{s,t}(0, n_2), s \in \{0, 1\}, t \in \{-1, 0, 1\}$ be the transition probabilities at state $(0, n_2)$ on the vertical axis. In the notation, we omit the superscription since the state itself already gives the axis that it is on.

Through analysis in this chapter, it is found that inhomogeneous transition probabilities exist if certain conditions are satisfied and we have no constraints on the structure Γ. Hence, the measure m can be induced by any subset $\hat{\Gamma}$ of I. The measure m may contain both real and complex geometric terms. Besides, the inhomogeneous transition probabilities $p_{s,t}(n_1, 0)$ and $p_{s,t}(0, n_2)$ are bounded.

In this chapter, we only consider the restrictive case, since conclusions in the relaxed can be generalized from the restrictive case. First, we will show that under certain conditions, inhomogeneous transition probabilities can be found though a recurrence relation. Next, we will discuss the conclusions we find.

5.1 Inhomogeneous boundary transition probabilities

We consider the restrictive case, *i.e.*,

$$\begin{aligned} p_{s,1}(n_1,0) &= p_{s,1}, \\ p_{1,t}(0,n_2) &= p_{1,t}, \end{aligned}$$

for any $n_1, n_2 \geq 1$ and $s,t = -1,0,1$. Let m be a measure induced by $\hat{\Gamma} \subset I$, with $\rho_k, \sigma_k \in \mathbb{R}$. As a consequence, balance equations in the interior of the state space are satisfied. The balance equations on the horizontal and vertical axis are given below,

$$m(n_1,0) = \sum_{s=-1}^{1} m(n_1-s,1)p_{s,-1} + \sum_{s=-1}^{1} m(n_1-s,0)p_{s,0}(n_1-s,0) \quad (5.1)$$

$$m(0,n_2) = \sum_{t=-1}^{1} m(1,n_2-t)p_{-1,t} + \sum_{t=-1}^{1} m(0,n_2-t)p_{0,t}(0,n_2-t). \quad (5.2)$$

First, consider balance equations on the horizontal axis. By inserting

$$1 = \sum_{s=-1}^{1} p_{s,0}(n_1,0) + \sum_{s=-1}^{1} p_{s,1},$$

we obtain

$$\begin{aligned} & m(n_1,0)\big(\sum_{s=-1}^{1} p_{s,0}(n_1,0) + \sum_{s=-1}^{1} p_{s,1}\big) \\ = & \sum_{s=-1}^{1} m(n_1-s,1)p_{s,-1} + \sum_{s=-1}^{1} m(n_1-s,0)p_{s,0}(n_1-s,0) \\ \Leftrightarrow \quad & m(n_1,0)\big[p_{-1,0}(n_1,0) + p_{1,0}(n_1,0) + \sum_{s=-1}^{1} p_{s,1}\big] \\ = & \sum_{s=-1}^{1} m(n_1-s,1)p_{s,-1} + m(n_1+1)p_{-1,0}(n_1+1,0) + m(n_1-1)p_{1,0}(n_1-1,0). \end{aligned}$$

By rearranging the items in the equation above, we get

$$\begin{aligned} & [m(n_1+1,0)p_{-1,0}(n_1+1,0) - m(n_1,0)p_{1,0}(n_1,0)] \\ - & [m(n_1,0)p_{-1,0}(n_1,0) - m(n_1-1,0)p_{1,0}(n_1-1,0)] \\ = & m(n_1,0)\sum_{s=-1}^{1} p_{s,1} - \sum_{s=-1}^{1} m(n_1-s,1)p_{s,-1}. \end{aligned} \quad (5.3)$$

Notice that $m(n_1+1,0)p_{-1,0}(n_1+1,0) - m(n_1,0)p_{1,0}(n_1,0)$ represents the difference between the flow going from state $(n_1+1,0)$ to state $(n_1,0)$ and the flow coming back from state $(n_1,0)$ to state $(n_1+1,0)$. Define, for $n_1 \geq 1$,

$$d(n_1,0) = m(n_1,0)p_{-1,0}(n_1,0) - m(n_1-1,0)p_{1,0}(n_1-1,0), \quad (5.4)$$

then Equation (5.3) can be written as

$$d(n_1+1,0)-d(n_1,0)=m(n_1,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(n_1-s,1)p_{s,-1}. \qquad (5.5)$$

In the next lemma, we give a closed-form expression for $d(n_1,0)$ for any $n_1=1,2,\dots,$ if balance equations on the horizontal axis hold for measure m.

Lemma 5.1. *Let m be a measure induced by $\hat{\Gamma}\subset I$, i.e.,*

$$m=\sum_{k=1}^{N}c_k(\rho_k^{n_1}\sigma_k^{n_2}+\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2}),$$

where $\rho_k,\sigma_k\in\mathbb{C}$. If m satisfies balance equations on the horizontal axis, then for $n_1=1,2,\dots,$

$$d(n_1,0)=\sum_{k=1}^{N}\sum_{s=-1}^{1}2c_k\big[-\mathrm{Re}(\frac{\rho_k^{n_1}}{1-\rho_k})p_{s,1}+\mathrm{Re}(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k})p_{s,-1}\big]. \qquad (5.6)$$

Proof. Keep using the recurrence relation in Equation (5.5), we have

$$\begin{aligned}
d(n_1+1,0) &= d(n_1,0)+m(n_1,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(n_1-s,1)p_{s,-1}\\
&= d(n_1-1,0)+m(n_1-1,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(n_1-1-s,1)p_{s,-1}+\\
&\quad m(n_1,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(n_1-s,1)p_{s,-1}\\
&= d(n_1-1,0)+\sum_{j=n_1-1}^{n_1}\big[m(j,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(j-s,1)p_{s,-1}\big]\\
&= \dots\\
&= d(1,0)+\sum_{j=1}^{n_1}\big[m(j,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(j-s,1)p_{s,-1}\big].
\end{aligned}$$

Thus, for any $k=1,\dots,n_1$,

$$d(n_1+1,0)=d(k,0)+\sum_{j=k}^{n_1}\big[m(j,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(j-s,1)p_{s,-1}\big]. \qquad (5.7)$$

$d(n_1+1,0)$ goes to 0 as n_1 goes to infinity, since both $m(n_1+1,0),m(n_1,0)$ go to 0 and $0\le p_{-1,0}(n_1+1,0),p_{1,0}(n_1,0)\le 1$. Thus we have, for any $n_1=1,2,\dots,$

$$d(n_1,0)+\sum_{j=n_1}^{\infty}\big[m(j,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(j-s,1)p_{s,-1}\big]=0. \qquad (5.8)$$

If $m(n_1,n_2)=\sum_{k=1}^N c_k(\rho_k^{n_1}\sigma_k^{n_2}+\bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2})$, then

$$\begin{aligned}
&\sum_{j=n_1}^{\infty}\big[m(j,0)\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}m(j-s,1)p_{s,-1}\big]\\
=&\sum_{k=1}^{N}c_k(\frac{\rho_k^{n_1}}{1-\rho_k}+\frac{\bar{\rho_k}^{n_1}}{1-\bar{\rho_k}})\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}\Big[\sum_{k=1}^{N}c_k(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k}+\frac{\bar{\rho_k}^{n_1-s}\bar{\sigma_k}}{1-\bar{\rho_k}})\Big]p_{s,-1}\\
=&\sum_{k=1}^{N}2c_k\,\mathrm{Re}(\frac{\rho_k^{n_1}}{1-\rho_k})\sum_{s=-1}^{1}p_{s,1}-\sum_{s=-1}^{1}\sum_{k=1}^{N}2c_k\,\mathrm{Re}(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k})p_{s,-1}
\end{aligned}$$

Rearrange the items in the equation above and plug it into Equation (5.8), we get for any $n_1=1,2,\dots,$

$$d(n_1,0)=\sum_{k=1}^{N}\sum_{s=-1}^{1}2c_k\big[-\mathrm{Re}(\frac{\rho_k^{n_1}}{1-\rho_k})p_{s,1}+\mathrm{Re}(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k})p_{s,-1}\big].$$

□

From Lemma 5.1, expression for $d(n_1,0)$ is given in closed form. From Equation (5.4), we see that if for state $(n_1,0)$, $d(n_1,0)$ is known, then $p_{-1,0}(n_1,0)$ and $p_{1,0}(n_1-1,0)$ are two variables in Equation (5.4). Moreover, $p_{-1,0}(n_1,0)$ is uniquely determined by $p_{1,0}(n_1-1,0)$. Therefore, let $n_1=1$, we can fix $p_{1,0}(0,0)$ and get $p_{-1,0}(1,0)$. Next, consider state $(2,0)$ and fix $p_{1,0}(1,0)$, we get $p_{-1,0}(2,0)$. If we do this for every state, we can find inhomogeneous transition probabilities on the horizontal axis. Using similar approaches, inhomogeneous transition probabilities on the vertical axis can also be found.

To find whether these inhomogeneous transition probabilities are bounded or not, we fix $p_{1,0}(n_1,0)$ for every state $(n_1,0)$, *i.e.*, let $p_{1,0}(n_1,0)=p_h$, for any $n_1\geq 1$. Using Equation (5.4) for every state, we can get $p_{-1,0}(n_1+1,0)$. In the following theorem, we show that $p_{-1,0}(n_1,0)$ will be converging as n_1 goes to infinity.

Theorem 5.2. *Let m be the measure induced by $\hat{\Gamma}$, where $\Gamma=\{(\rho_1,\sigma_1),\dots,(\rho_N,\sigma_N)\}$, and $\rho_k,\sigma_k\in\mathbb{C}$. Moreover, let $\rho_{max}\in\{\rho_1,\dots,\rho_N\}$ be the one with maximum modulus, and assume that $|\rho_{max}|>|\rho_k|$ for any $\rho_k\neq\rho_{max}$. Let $p_{1,0}(n_1,0)=p_h$, for any $n_1=1,2,\dots$. If $\rho_{max}\in\mathbb{R}$, then the transition probabilities $p_{-1,0}(n_1,0)$ are converging.*

Proof. From Lemma 5.1, we can find closed-form expression for $d(n_1,0)$,

$$d(n_1,0)=\sum_{k=1}^{N}\sum_{s=-1}^{1}2c_k\big[-\mathrm{Re}(\frac{\rho_k^{n_1}}{1-\rho_k})p_{s,1}+\mathrm{Re}(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k})p_{s,-1}\big].$$

Consider Equation (5.4), plug in the expression of $d(n_1,0)$ and $m(n_1,0)$, we get

$$\begin{aligned} p_{-1,0}(n_1,0) &= \frac{m(n_1-1,0)}{m(n_1,0)}p_{1,0}(n_1-1,0)+\frac{d(n_1,0)}{m(n_1,0)} \\ &= \frac{\sum_{k=1}^N c_k(\rho_k^{n_1-1}+\bar{\rho_k}^{n_1-1})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})}p_h - \frac{\sum_{k=1}^N c_k(\frac{\rho_k^{n_1}}{1-\rho_k}+\frac{\bar{\rho_k}^{n_1}}{1-\bar{\rho_k}})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})}\sum_{s=-1}^{1}p_{s,1}+ \\ &\quad \sum_{s=-1}^{1}\Big[\frac{\sum_{k=1}^N c_k(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k}+\frac{\bar{\rho_k}^{n_1-s}\bar{\sigma_k}}{1-\bar{\rho_k}})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})}\Big]p_{s,-1}. \end{aligned}$$

Let c_{max} and σ_{max} be the coefficient and σ corresponding to ρ_{max}, then

$$\begin{aligned} & \frac{\sum_{k=1}^N c_k(\rho_k^{n_1-1}+\bar{\rho_k}^{n_1-1})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})} \\ = & \frac{\rho_{max}^{n_1-1}\big[\sum_{\rho_k\neq\rho_{max}} c_k((\frac{\rho_k}{\rho_{max}})^{n_1-1}+(\frac{\bar{\rho_k}}{\rho_{max}})^{n_1-1})+2\cdot c_{max}\big]}{\rho_{max}^{n_1}\big[\sum_{\rho_k\neq\rho_{max}} c_k((\frac{\rho_k}{\rho_{max}})^{n_1}+(\frac{\bar{\rho_k}}{\rho_{max}})^{n_1})+2\cdot c_{max}\big]} \end{aligned}$$

Since $|\rho_{max}|>|\rho_k|$ for any $\rho_k\neq\rho_{max}$, then

$$\begin{aligned} \lim_{n_1\to\infty}(\frac{\rho_k}{\rho_{max}})^{n_1-1} &= \lim_{n_1\to\infty}(\frac{\rho_k}{\rho_{max}})^{n_1}=0, \\ \lim_{n_1\to\infty}(\frac{\bar{\rho_k}}{\rho_{max}})^{n_1-1} &= \lim_{n_1\to\infty}(\frac{\bar{\rho_k}}{\rho_{max}})^{n_1}=0. \end{aligned}$$

Hence

$$\lim_{n_1\to\infty}\frac{\sum_{k=1}^N c_k(\rho_k^{n_1-1}+\bar{\rho_k}^{n_1-1})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})}=\frac{1}{\rho_{max}}.$$

Using similar approaches, we get

$$\lim_{n_1\to\infty}\frac{\sum_{k=1}^N c_k(\frac{\rho_k^{n_1-s}\sigma_k}{1-\rho_k}+\frac{\bar{\rho_k}^{n_1-s}\bar{\sigma_k}}{1-\bar{\rho_k}})}{\sum_{k=1}^N c_k(\rho_k^{n_1}+\bar{\rho_k}^{n_1})}=\frac{\rho_{max}^{-s}\operatorname{Re}(\sigma_{max})}{1-\rho_{max}},$$

for any $s=-1,0,1$. Therefore, $p_{-1,0}(n_1,0)$ is converging and

$$\lim_{n_1\to\infty}p_{-1,0}(n_1,0)=\frac{1}{\rho_{max}}p_h-\frac{1}{1-\rho_{max}}\sum_{s=-1}^{1}p_{s,1}+\sum_{s=-1}^{1}\frac{\rho_{max}^{-s}\operatorname{Re}(\sigma_{max})}{1-\rho_{max}}p_{s,-1}.$$

□

Similarly, we can fix $p_{0,1}(0,n_2)$ for each $n_2\geq 1$ on the vertical axis and use similar approaches to get $p_{0,-1}(0,n_2+1)$. We also have that, if, among all σ, the one with maximal modulus is real, then $p_{0,-1}(0,n_2)$ is converging.

In this section, we see that if ρ_{max} is real and we fix $p_{1,0}(n_1,0)$ for every state, then $p_{-1,0}(n_1,0)$ is converging as n_1 goes to infinity. In the next section, we will discuss the bounds on the limiting probability.

5.2 Bounds on inhomogeneous transition probabilities.

In Section 5.1, we see that $p_{-1,0}(n_1, 0)$ is converging and the limit depends on p_h and ρ_{max}. If the measure m is specified, then ρ_{max} is known. Moreover, the limiting probability is increasing in p_h and decreasing in $\sum_{s=-1}^{1} p_{s,1}$. First we consider the lower bounds on the limiting probability. If $p_h = 0$ and $\sum_{s=-1}^{1} p_{s,1} = 1$,

$$\begin{aligned} \lim_{n_1\to\infty} p_{-1,0}(n_1, 0) &= -\frac{1}{1-\rho_{max}} + \sum_{s=-1}^{1} \frac{\rho_{max}^{-s}\,\mathrm{Re}(\sigma_{max})}{1-\rho_{max}} p_{s,-1} \\ &= \frac{1}{1-\rho_{max}}(-1 + \sum_{s=-1}^{1} \rho_{max}^{-s}\,\mathrm{Re}(\sigma_{max}) p_{s,-1}). \end{aligned}$$

The results of numerical results indicate that $-1+\sum_{s=-1}^{1} \rho_{max}^{-s}\,\mathrm{Re}(\sigma_{max})p_{s,-1} \leq 0$, then we get that the lower bound of limiting probability is non-positive. Therefore, we can increase p_h and decrease $\sum_{s=-1}^{1} p_{s,1}$ such that the limiting probability of $p_{-1,0}(n_1, 0)$ is non-negative. The relaxed case will not be considered here thus we will not go into detailed discussion in the relaxed case. Here we use the relaxed case to indicate that the limiting probability can be bounded if we choose the parameters properly.

Suppose that $0 \leq \lim_{n_1\to\infty} p_{-1,0}(n_1, 0) \leq 1$ for some p_h. Now we are going to discuss the value of $p_{-1,0}(n_1, 0)$ for some finite state $(n_1, 0)$. If for some $n_1 \geq 0$, $p_{-1,0}(n_1, 0) \geq 1$, we can change the value of $p_{1,0}(n_1 - 1, 0)$. More precisely, let $p_{1,0}(n_1 - 1, 0) < p_h$, and then $p_{-1,0}(n_1, 0)$ will be decreased. Moreover, changing the value of $p_{1,0}(n_1 - 1, 0)$ does not have any influence on the transition probabilities at other states or the limiting probability. In fact, through numerical experiments, we can see that if ρ_{max} is real, the probabilities $p_{-1,0}(n_1, 0)$ are usually monotonic in n_1.

In this chapter, we see that inhomogeneous transition probabilities on the boundaries can be constructed such that a specified measure m is the invariant measure of the random walk. Moreover, if we choose proper parameters, these probabilities can be bounded. Some examples will be given in the next chapter to illustrate the homogeneous or inhomogeneous transition probabilities constructed on the horizontal and vertical axis.

Chapter 6

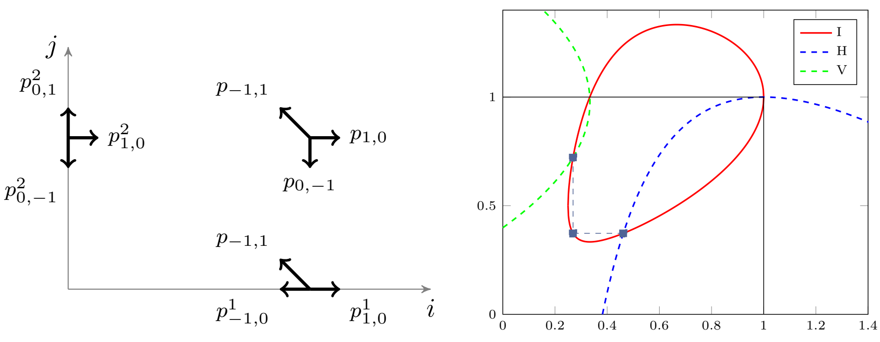
Examples

In this chapter, examples of random walks of which the invariant measure is specified are given.

In the first example, the measure is induced by real geometric terms. This example is taken from [5]. This example illustrates the pairwise-coupled structure and how geometric terms are found. Since the measure is induced by real geometric terms, we can draw the curve which represents the set I on the plane. Besides, we will plot the curves that are the equivalents of I for the balance equations on the horizontal and the vertical axis.

Example 6.1 (Figure 6.1)**.** *Consider the random walk with* $p_{1,0} = 0.05, p_{-1,1} = 0.15, p_{0,-1} = 0.15, p_{0,0} = 0.65, p^1_{1,0} = 0.15, p^1_{0,0} = 0.55, p^2_{0,1} = 0.0929, p^2_{0,0} = 0.7071, p^2_{0,-1} = 0.15$ *and all other transition probabilities* 0*.*

The measure is $m(n_1, n_2) = \sum_{k=1}^{3} c_k \rho_k^{n_1} \sigma_k^{n_2}$*, where* $(\rho_1, \sigma_1) = (0.4618, 0.3728)$, $(\rho_2, \sigma_2) = (0.2691, 0.3728)$, $(\rho_3, \sigma_3) = (0.2691, 0.7218)$, $c_1 = 0.1722, c_2 = -0.2830$ *and* $c_3 = 0.2251$*.* m *satisfies all the balance equations and* m *is the invariant measure of the random walk.*



(a) (b)

Figure 6.1: Example 1. (a) Transition diagram of the random walk. (b) Balance equations. The elements in Γ are denoted by blue squares.

In the next example, we take three real pairs that do not follow a pairwise-coupled structure. According to Section 4.1, the necessary condition for m to be the invariant measure of a homogeneous random walk is that Γ is pariwise-coupled. Thus homogeneous boundary transition probabilities do not exist for m to be the invariant measure. We show that inhomogeneous transition probabilities can be found for this measure.

Example 6.2 (Figure 6.2)**.** *Consider the measure* $m(n_1, n_2) = \sum_{k=1}^{3} c_k \rho_k^{n_1} \sigma_k^{n_2}$, *where* $(\rho_1, \sigma_1) = (0.5, 0.8838)$, $(\rho_2, \sigma_2) = (0.3, 0.2336)$, $(\rho_3, \sigma_3) = (0.6, 0.3248)$, $c_1 = c_2 = c_3 = 0.0220$.

The interior transition probabilities of the random walk are $p_{1,0} = 0.05$, $p_{0,1} = 0.05$, $p_{-1,1} = 0.2$, $p_{-1,0} = 0.2$, $p_{0,-1} = 0.2$, $p_{1,-1} = 0.2$, $p_{0,0} = 0.1$ *and all the other transition probabilities are* 0.

Let $p_{1,0}(n_1, 0) = 0.05$ *for all* $(n_1, 0)$ *on the horizontal axis and* $p_{0,1}(0, n_2) = 0.3$ *for all* $(0, n_2)$ *on the vertical axis. The inhomogeneous transition probabilities* $p_{-1,0}(n_1, 0)$ *and* $p_{0,-1}(0, n_2)$ *are shown in Figure 6.2. In the figures, the horizontal coordinate is* n_1 *or* n_2 *and the vertical coordinate is* $p_{-1,0}(n_1, 0)$ *or* $p_{0,-1}(0, n_2)$. *In the limiting case,* $p_{-1,0}^{\infty} = 0.1413$ *and* $p_{0,-1}^{\infty} = 0.0223$. *Moreover,* $\rho_{max} = 0.6$ *and* $\sigma'_{max} = 0.8838$, *we can verify that the limiting probabilities given in Theorem 5.2 are consistent with numerical results.*

m *satisfies all the balance equations and* m *is the invariant measure of the random walk.*

From Example 6.2, we see that if the measure m is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$ where Γ is not pairwise-coupled, we can still construct inhomogeneous transition probabilities on the horizontal and vertical axis such that m is the invariant measure of the random walk.

In the next example, we consider a measure that is a sum of one real geometric term $\rho_1^{n_1} \sigma_1^{n_2}$ and one complex term $\rho_2^{n_1} \sigma_2^{n_2} + \rho_2^{n_1} \bar{\sigma_2}^{n_2}$, where $\rho_2 \in \mathbb{R}, \sigma_2 \in \mathbb{C} \backslash \mathbb{R}$.

Example 6.3 (Figure 6.3)**.** *Consider the measure* $m(n_1, n_2) = c_1 \rho_1^{n_1} \sigma_1^{n_2} + c_2(\rho_2^{n_1} \sigma_2^{n_2} + \rho_2^{n_1} \bar{\sigma_2}^{n_2})$, *where* $(\rho_1, \sigma_1) = (0.3, 0.6603)$, $(\rho_2, \sigma_2) = (0.2, 0.2929 + 0.1193i)$, $c_1 = 0.0987$ *and* $c_2 = 0.0494$.

The interior transition probabilities of the random walk are $p_{-1,0} = 0.2$, $p_{1,-1} = 0.1$, $p_{-1,0} = 0.2$, $p_{0,0} = 0.3$, $p_{1,0} = 0.05$, $p_{-1,1} = 0.1$, $p_{0,1} = 0.05$ *and all the other transition probabilities are* 0.

We omit the transition diagram and the curves representing the balance equations, since if we consider complex numbers, the geometric terms can not be marked on the two-dimensional plane.

Let $p_{1,0}(n_1, 0) = 0.05$ *for all* $(n_1, 0)$ *on the horizontal axis and* $p_{0,1}(0, n_2) = 0.2$ *for all* $(0, n_2)$ *on the vertical axis. The inhomogeneous transition probabilities* $p_{-1,0}(n_1, 0)$ *and* $p_{0,-1}(0, n_2)$ *are shown in Figure 6.3. In the figures, the horizontal coordinate is* n_1 *or* n_2 *and the vertical coordinate is* $p_{-1,0}(n_1, 0)$ *or* $p_{0,-1}(0, n_2)$. *In the limiting case,* $p_{-1,0}^{\infty} = 0.4555$ *and* $p_{0,-1}^{\infty} = 0.1717$.

m *satisfies all the balance equations. Therefore, it is the invariant measure of the random walk.*

In the next example, we consider the measure m induced by a real geometric term and $\hat{\Gamma}$ where Γ is a pairwise-coupled set. The real geometric term is added to make sure that $m(n_1, n_2) \geq 0$ for all states (n_1, n_2).

Example 6.4 (Figure 6.4)**.** *Consider the measure is* $m(n_1, n_2) = \sum_{k=1}^{3} c_k(\rho_k^{n_1} \sigma_k^{n_2} + \bar{\rho_k}^{n_1} \bar{\sigma_k}^{n_2})$, *where* $(\rho_1, \sigma_1) = (0.5, 0.8838)$, $(\rho_2, \sigma_2) = (0.2{+}0.1i, 0.3656{+}0.3168i)$,

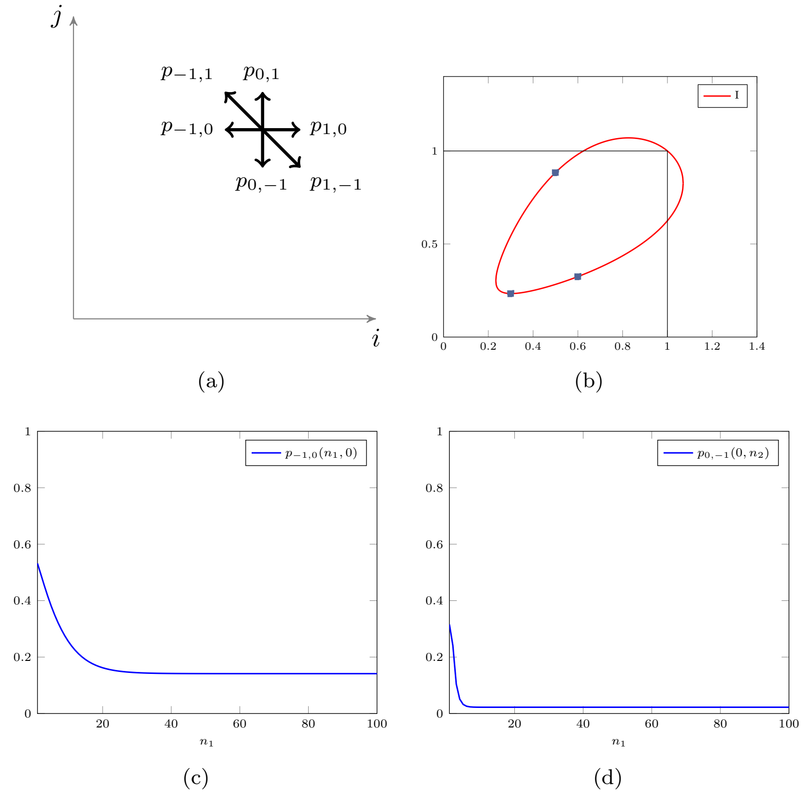


Figure 6.2: Example 2. (a) Transition diagram of the random walk. (b) Balance equations. The elements in Γ is denoted by blue squares. (c) Transition probabilities on the horizontal axis. (d) Transition probabilities on the vertical axis.

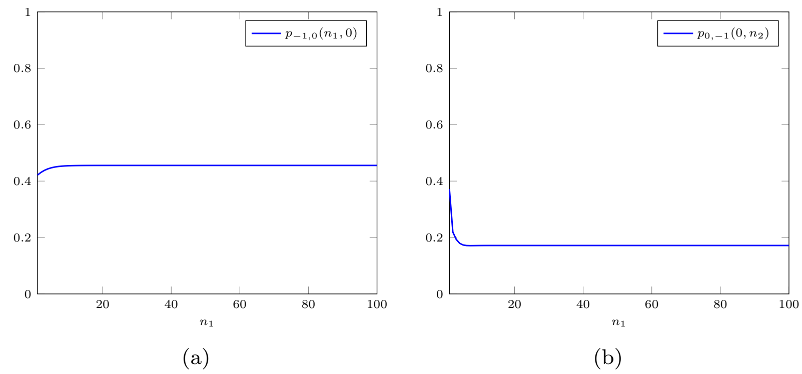


Figure 6.3: Boundary transition probabilities of Example 3. (a) Transition probabilities on the horizontal axis. (b) Transition probabilities on the vertical axis.

$(\rho_3, \sigma_3) = (0.2 + 0.1i, 0.1758 + 0.0203i)$, $c_1 = 0.0291$, $c_2 = 0.0145$ *and* $c_3 = -0.0145$.

The interior transition probabilities of the random walk are $p_{0,-1} = 0.2$, $p_{1,-1} = 0.2$, $p_{-1,0} = 0.2$, $p_{0,0} = 0.1$, $p_{1,0} = 0.05$, $p_{-1,1} = 0.2$, $p_{0,1} = 0.05$ *and all the other transition probabilities are* 0.

Let $p_{1,0}(n_1, 0) = 0.05$ *for all* $(n_1, 0)$ *on the horizontal axis and* $p_{0,1}(0, n_2) = 0.3$ *for all* $(0, n_2)$ *on the vertical axis. The inhomogeneous transition probabilities* $p_{-1,0}(n_1, 0)$ *and* $p_{0,-1}(0, n_2)$ *are shown in Figure 6.4. In the figures, the horizontal coordinate is* n_1 *or* n_2 *and the vertical coordinate is* $p_{-1,0}(n_1, 0)$ *or* $p_{0,-1}(0, n_2)$*. In the limiting case,* $p^{\infty}_{-1,0} = 0.6606$ *and* $p^{\infty}_{0,-1} = 0.0223$.

m *satisfies all the balance equations. Therefore, it is the invariant measure of the random walk.*

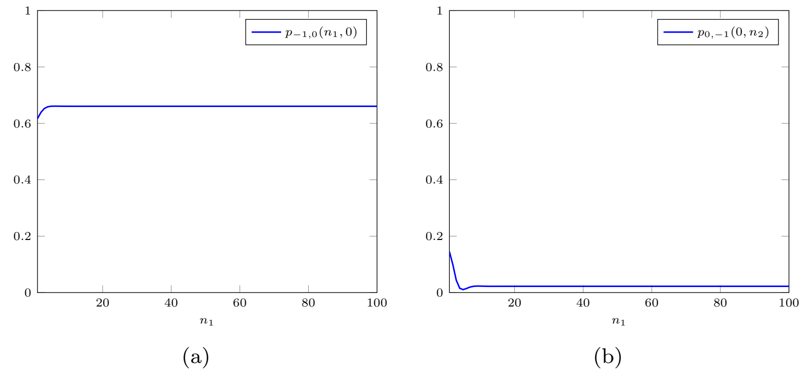


Figure 6.4: Boundary transition probabilities of Example 4. (a) Transition probabilities on the horizontal axis. (b) Transition probabilities on the vertical axis.

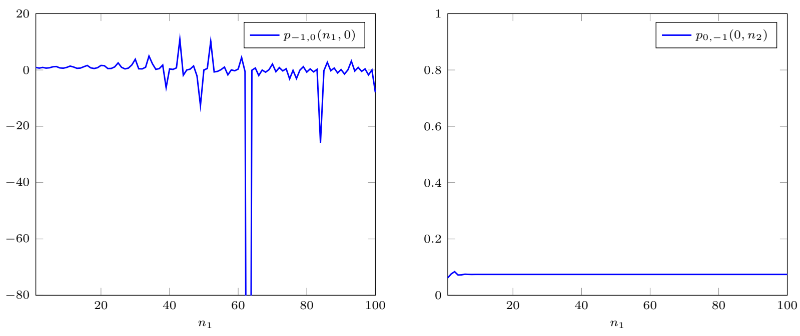
In the next example, we show that if ρ_{max} is not real, then probabilities $p_{-1,0}(n_1, 0)$ may not be converging.

Example 6.5 (Figure 6.5)**.** *Consider the measure* $m(n_1, n_2) = \sum_{k=1}^{3} c_k(\rho_k^{n_1}\sigma_k^{n_2} + \bar{\rho_k}^{n_1}\bar{\sigma_k}^{n_2})$*, where* $(\rho_1, \sigma_1) = (0.37, 0.9901)$, $(\rho_2, \sigma_2) = (0.0785 - 0.3791i, 0.1 - 0.1i)$ *and* $(\rho_3, \sigma_3) = (0.05 - 0.05i, -0.1086 + 0.4314i)$*. The coefficients are* $c_1 = 0.0031$, $c_2 = 0.0006$ *and* $c_3 = -0.0006$.

The interior transition probabilities of the random walk are $p_{0,-1} = 0.2$, $p_{1,-1} = 0.1$, $p_{-1,0} = 0.2$, $p_{0,0} = 0.2$, $p_{1,0} = 0.05$, $p_{-1,1} = 0.2$, $p_{0,1} = 0.05$ *and all the other transition probabilities are* 0.

Let $p_{1,0}(n_1, 0) = 0.05$ *for all* $(n_1, 0)$ *on the horizontal axis and* $p_{0,1}(0, n_2) = 0.2$ *for all* $(0, n_2)$ *on the vertical axis. The inhomogeneous transition probabilities* $p_{-1,0}(n_1, 0)$ *and* $p_{0,-1}(0, n_2)$ *are shown in Figure 6.5. In the figures, the horizontal coordinate is* n_1 *or* n_2 *and the vertical coordinate is* $p_{-1,0}(n_1, 0)$ *or* $p_{0,-1}(0, n_2)$*. In the limiting case,* $p^{\infty}_{0,-1} = 0.0741$ *on the vertical axis since* $\sigma'_{max} = 0.9901$*. However,* $\rho_{max} = 0.0785 - 0.3791i$ *and on the horizontal axis, the limiting probabilities do not exist.*

From all the examples above, under certain conditions, inhomogeneous transition probabilities can be constructed on the horizontal and vertical axis. Moreover, with this method, we can break the pairwise-coupled structure that is required in Chapter 4. Besides, we can find transition probabilities for the mea-



(a) (b)

Figure 6.5: Transition probabilities of Example 5. (a) Transition probabilities on the horizontal axis. (b) Transition probabilities on the vertical axis.

sure induced by complex numbers. Therefore, we extend the class of random walks for which we know the invariant measure in closed form.

Chapter 7

Conclusions and discussion

In the report, we consider random walks in the quarter plane and the invariant measure of the random walks. We focus on the problem that given the interior transition probabilities of the random walk and a specified measure m induced by the set $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$, how to construct the transition probabilities on the horizontal, vertical axis and the origin such that m is the invariant measure of the random walk. More precisely, we consider that m is induced by $\hat{\Gamma} = \Gamma \cup \bar{\Gamma}$, where $\Gamma = \{(\rho_1, \sigma_1), \ldots, (\rho_N, \sigma_N)\}$ with $\rho_k, \sigma_k \in \mathbb{C}$, *i.e.*,

$$\sum_{k=1}^{N} c_k (\rho_k^{n_1} \sigma_k^{n_2} + \bar{\rho}_k{}^{n_1} \bar{\sigma}_k{}^{n_2}).$$

We have considered homogeneous transition probabilities in Chapter 4 and inhomogeneous transition probabilities in Chapter 5. The conclusions of this report are listed below.

1. We find that if m is the invariant measure of the random walk, then each geometric term in m has to satisfy the balance equations in the interior of the state space individually.
2. Consider homogeneous transition probabilities on the boundaries, if m is the invariant measure of the measure, then Γ has to be pairwise-coupled.
3. We consider the restrictive case and the relaxed case for boundary transition probabilities. However, numerical results suggest that homogeneous transition probabilities can not be found such that m is the invariant measure of the random walk.
4. Consider inhomogeneous transition probabilities on the boundaries. If we choose $p_{1,0}(n_1, 0)$ to be a fixed probability for every state, then under certain conditions, $p_{-1,0}(n_1, 0)$ is converging as n_1 goes to infinity. Moreover, we can control the limiting probability by choosing a proper fixed value for $p_{1,0}(n_1, 0)$.

On the other hand, there are still some problems that remain open. Firstly, when we consider homogeneous transition probabilities, our conclusions are based on two conjectures. These conjectures are supported by extensive numerical experiments. A proof of these conjectures is needed. Secondly, we don't discuss the relaxed case in detail for the inhomogeneous boundary transition probabilities. Although we mention that for the relaxed case, transition probabilities $p_{s,1}(n_1, 0)$ for $s = -1, 0, 1$ can be changed a little to bound the probabilities $p_{-1,0}(n_1, 0)$, detailed analysis and numerical experiments should be worked on in future.

In future, after we have bounds on the inhomogeneous boundary transition probabilities, we will consider making perturbation on a general random walk and finding error bounds. Next, we will try to consider the high-dimensional random walks and apply the conclusions we have obtained in two-dimensional space.

Acknowledgements

I would like to express my gratitude to a lot of people who have helped me during my master studies and final project.

Firstly, I would like to thank the admission committee and scholarship office at University of Twente for offering me this opportunity to study here. Moreover, I would like to thank the inheritors of Professor de Winter and his wife, who have provided me the Professor de Winter Scholarship for the two years. Without their financial support, it would not be possible for me to have this wonderful experience here. Besides, I also want to give thanks to the Twente Gradute School(TGS) for offering a scholarship in the second year of my study. I have learned a lot from their workshops and activities, which will be helpful for my future study and research.

Secondly, I have had a great time in the Stochastic Operations Research(SOR) group during the period I'm doing my final project. I thank everyone who has contributed to it. In particular, I would like to thank Jasper, who has supervised me all the way from my internship to my master project. He also provides a lot of help when I work on my research proposal for TGS award and my application for a PhD scholarship. I benefit a lot from all the instructions and suggestions he has given me. I would also like to thank Jan-Kees, who never hesitates to offer help when I come to him. He also gives me a lot of important suggestions on my thesis draft. I am also grateful to Richard, who offers me guidance in my master study and helps me with my application for the PhD scholarship. Although he is very busy, he always spares his time and gives me suggestions when I need help. In addition, I want to give many thanks to Thyra, who helps me with practical issues; to Yanting and Mihaela for answering all the questions about my research and latex; to my officemates Niek and Corine for having a good time during these months; to Clara and Anne for the time spent together working on master projects and to everyone in SOR group. I like the time when we have coffee and lunch together.

Thirdly, I would thank my friends here, Yu Tu, Chao Ni, Zhenlei Li, Keli Chen, Lu Wang, Liuyi Shen, etc. I could not list all the names due to limited space. Having all of them around me makes my life here amazing and the memories we have together will be a big treasure to me. I would also like to thank my friends far away in China, Ling Hu and Ziran Ma, for their support and comfort when I'm feeling down.

At last, I want to leave the space to my family, for their unconditional love and trust.

Enschede, 19 August, 2014

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