RAM ROBOTICS AND MECHATRONICS

Optimization of Bipedal Walker Gaits with Constant Knee Stiffness

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MSc Report

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Contents

1 Introduction

Energy efficient bipedal locomotion is one of the research areas of the Robotics and Mechatronics (RAM) group at the University of Twente. This MSc project continues that research with the optimization of bipedal walking gaits and the application of that optimization to a bipedal walker with constant knee stiffness.
Previous students investigated bipedal walking gaits with the spring loaded inverted pendulum (SLIP) model. A swing leg was added to the model [1] and the swing leg trajectories were chosen by an educated guess. The hip and swing leg trajectories of the segmented-SLIP (S-SLIP) model were used as references for the control of a bipedal robotic walker [2] [3]. Simulations showed that the gait cycle behaviour of the SLIP and S-SLIP model was not similar to that of the robotic walker. Therefore, using SLIP or S-SLIP trajectories as references for the control of a robotic walker did not seems like a viable idea. One of the recommendations in [3] indicated the need for higher quality reference trajectories.

The paper, presented in section 2, proposes a novel optimization method for the bipedal walker gaits with constant knee stiffness. Power of the used optimization method is that it optimizes over the state variables and inputs for the complete gait at once [4]. The bipedal walker has two legs, both with an upper and lower leg. There is a torsion spring on the knee. The parameters of the spring, the spring constant and the equilibrium position, are constant during walking. However, the spring parameters are part of the optimization.
The walker is optimized in three steps, the distinction between the three is how the knee spring is used in the optimization. First, the knee spring constant is set to zero. Second, the knee spring constant is part of the optimization. Third, the knee spring constant and equilibrium position are part of the optimization.
The optimizations show that the torque that is needed for a walking gait is reduced by adding the knee springs to the bipedal walker. Remarkable is that it also reduces the input torques to the hip joint that has no spring. The optimizations also show that the shape of the swing foot trajectories change with increasing gait velocities. So, imposing trajectories that have the same shape for different velocities, as was done before, will thus not result in optimal gaits with regard to energy efficiency.

Section 3 presents details of the optimization that are needed for the implementation of the optimization. The conclusions of the MSc project are in section 4. Final recommendations are reported in section 5.

2 Paper: Optimizing Bipedal Walking Gaits with Constant Knee Stiffness

Abstract – The paper focuses on bipedal locomotion and proposes an optimization method that simultaneously optimizes the state variables and the inputs and applies the optimization to a bipedal walker with constant knee springs. The optimization method is a direct local collocation method. The bipedal walking gait is optimized in three steps. First, the bipedal walker is optimized with the spring stiffness set to zero to get nominal gaits. Secondly, the walker gaits and the knee stiffness are optimized. Thirdly, the gaits are optimized together with the knee stiffness and its equilibrium position. The results are twofold. The shape of the swing foot trajectory changes when the velocity increases. So using the same shape of foot trajectories for all gait velocities does not create energy efficient walking gaits. The knee spring also reduces the amount of torque supplied to the walker by at most 91%. Remarkable is that adding knee springs to the bipedal walker also decreases the input torque to the hip joint.

Optimizing Bipedal Walking Gaits with Constant Knee Stiffness

Klaas J. Russcher*, Georg Still**, and Raffaella Carloni*

Abstract— **The paper focuses on bipedal locomotion and proposes an optimization method that simultaneously optimizes the state variables and the inputs and applies the optimization to a bipedal walker with constant knee springs. The optimization method is a direct local collocation method. The bipedal walking gait is optimized in three steps. First, the bipedal walker is optimized with the spring stiffness set to zero to get nominal gaits. Secondly, the walker gaits and the knee stiffness are optimized. Thirdly, the gaits are optimized together with the knee stiffness and its equilibrium position. The results are twofold. The shape of the swing foot trajectory changes when the velocity increases. So using the same shape of foot trajectories for all gait velocities does not create energy efficient walking gaits. The knee spring also reduces the amount of torque supplied to the walker by at most 91%. Remarkable is that adding knee springs to the bipedal walker also decreases the input torque to the hip joint.**

I. INTRODUCTION

The research of bipedal locomotion is inspired by human walking. Human walking is both energy efficient and robust. Most bipedal walking robots are either energy efficient or have stable walks. One approach of creating stable, energy efficient gaits is to control bipedal robotic walkers to energy efficient trajectories. However, it is a challenge to find energy efficient walker gaits.

The essentials of bipedal walking are captured by the spring-loaded inverted pendulum model (SLIP) [1]. The legs are modelled as springs. It walks on a limit cycle when the initial conditions are correctly chosen and the gait do not require an input torque. The hip trajectory and ground reaction forces resemble those of human walking. The limit cycles of the SLIP model are found with direct shooting optimization method. This method optimizes the initial conditions by simulation one gait step and checking if the gait step is on a cycle.

[2] extended the SLIP-model with a swing leg that has a knee. The way the foot is retracted was chosen by an educated guess about the optimal way to do it. The resulting trajectories of that model were later implemented on a real robotic walker [3] [4]. The dynamics of bipedal robotic walker are different from the SLIP model with swing leg. So forcing SLIP trajectories on the bipedal robotic walker is not energy efficient. Also the swing leg is not moved in a energy efficient way. This is the reason [4] indicated the need for higher quality reference trajectories to make the model more energy efficient.

This paper proposes a way to optimize gaits of bipedal walkers with the direct local collocation method [5]. This optimization does not require simulations, it updates the input and the state variables simultaneously. The bipedal walker gaits are optimized with respect to the input torques to the walker. The bipedal walker is modelled such that it makes continuous simulation possible. Optimizing that model makes the contact dynamics part of the optimization. The optimization is only constraint by the velocity of the walker. The optimization is applied to the walker to see if springs on the knees with fixed parameters can reduce the torque that is required for a certain gait velocity. Strength of the optimization method proposed in this paper is that it simultaneously optimizes the state variables as well as the inputs for the whole gait cycle.

The rest of this paper is as follows. Section II describes the model of the bipedal walker. The gait optimization is explained in section III. The results of the gait optimization are shown in section IV. The paper is concluded in section V.

II. BIPEDAL WALKER

This section describes the bipedal walker and its mathematical model. The bipedal walker is in the sagittal plane (Fig. 1). The walker consist of two legs, both with an upper and a lower leg. The configuration of the walker is described by six generalized coordinates q and six corresponding momenta p. There is a point mass m_{hip} at the hip, a point mass m_{ul} at the middle of the upper leg, and a point mass m_{ll} at the middle of the lower leg. This set-up is preferred over one with a point mass at the hip and at the feet, to avoid that the mass matrix, expressed in generalized coordinates, has dependent rows in some configurations. That makes the mass matrix not invertible and causes problems during simulation. The torsion springs on the knees are identical and have a fixed spring constant K and fixed equilibrium position q_{eq}. Inputs u_1, u_2 and u_3 are torques on respectively the hip joint and the knee joints. When the feet are in contact with the ground, a reaction force will prevent them from penetrating the ground.

The walker is modelled as a Hamiltonian system with nonlinear complementarity constraints. The Hamiltonian system describes the dynamics of the system, and the nonlinear complementarity constraints the contact with the ground. The model is a continuous simulation model, the events are caught by the complementarity constraints. Having a model suited for continuous simulation is important because

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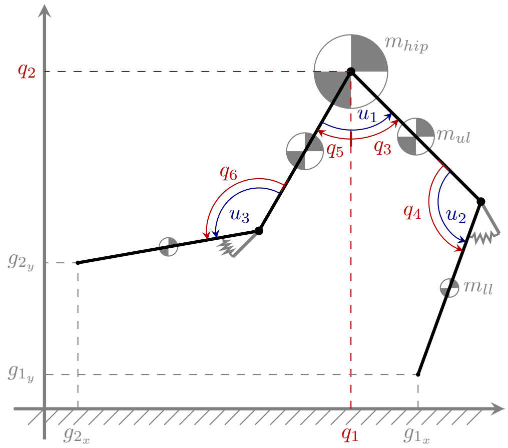

Fig. 1. The bipedal walker. Its pose is described by the coordinates q_1 to q_6. The inputs u_1, u_2, and u_3 are input torques. g_1 and g_2 are the end positions of the legs. The knee springs are zero-length rotational springs.

the optimization presented in this paper only works with continuous simulation models.

A. Mathematical Model

The Hamiltonian system is described by the energy function of the system, the Hamiltonian ($\mathcal{H}$), and the external forces on the system. The Hamiltonian is the sum of the kinetic energy T and the potential energy V:

$$\mathcal{H}(\boldsymbol{q},\boldsymbol{p}) = T(\boldsymbol{q},\boldsymbol{p}) + V(\boldsymbol{q}) \tag{1}$$

The kinetic energy is given by the equation:

$$T(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^\top \boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{p}$$

where $\boldsymbol{M}(\boldsymbol{q})$ is the mass matrix expressed in generalized coordinates. $\boldsymbol{S}(\boldsymbol{q})$ is the mapping of the generalized coordinates $\boldsymbol{q}$ to the x- and y-positions of the masses. $\dot{\boldsymbol{S}}(\boldsymbol{q})$ is the velocity of the masses in x- and y-direction. The kinetic co-energy is:

$$T_{co}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{S}}(\boldsymbol{q})^\top \boldsymbol{M}_0 \dot{\boldsymbol{S}}(\boldsymbol{q}) \tag{2}$$

Time derivative $\dot{\boldsymbol{S}}(\boldsymbol{q})$ is the Jacobian of $\boldsymbol{S}(\boldsymbol{q})$ times the velocities of the generalized coordinates:

$$\dot{\boldsymbol{S}}(\boldsymbol{q}) = \frac{\partial \boldsymbol{S}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_S(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3}$$

Equation (2) and (3) combined give the expression for $\boldsymbol{M}(\boldsymbol{q})$:

$$T_{co}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^\top \boldsymbol{J}_S(\boldsymbol{q})^\top \boldsymbol{M}_0 \boldsymbol{J}_S(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^\top \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

The potential energy of bipedal walker is the gravitational energy of the masses plus the energy stored in the knee springs:

$$\begin{aligned} V(\boldsymbol{q}) = q_2 m_{hip} g + y_{ul_1} m_{ul} g + y_{ll_1} m_{ll} g + y_{ul_2} m_{ul} g + \dots \\ y_{ll_2} m_{ll} g + \tfrac{1}{2}K(q_4 - q_{eq})^2 + \tfrac{1}{2}K(q_6 - q_{eq})^2 \end{aligned}$$

The derivatives of the Hamiltonian (1) to $\boldsymbol{q}$ and $\boldsymbol{p}$ are the dynamics of the system without external forces. The external forces are the torque inputs on the hip and knees ($\boldsymbol{u}$), and the ground reaction forces (GRF's) on the feet (λ). The GRF's act on the body via the transpose of the Jacobian of the feet positions g_1 and g_2.

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \\ \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{u} \dots \\ &\quad + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{J}_{g_1}(\boldsymbol{q})^\top \end{bmatrix} \begin{bmatrix} \lambda_{1_x}^+ - \lambda_{1_x}^- \\ \lambda_{1_y} \end{bmatrix} \dots \\ &\quad + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{J}_{g_2}(\boldsymbol{q})^\top \end{bmatrix} \begin{bmatrix} \lambda_{2_x}^+ - \lambda_{2_x}^- \\ \lambda_{2_y} \end{bmatrix}, \\ \boldsymbol{u} &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \; \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top \end{aligned} \tag{4}$$

The constraints to the Hamiltonian system are the non-linear complementarity constraints (5), which represent the contact of the feet with the ground [6]. A complementarity constraint connects two related non-negative variables whose product is equal to zero, which means that at least one of the two variables is zero. There is a set of complementarity constraints for each foot ($j = 1, 2$). The variable μ in (5b) is the friction constant.

$$0 \leq g_{j_y} \perp \lambda_{j_y} \geq 0 \tag{5a}$$
$$0 \leq \mu\lambda_{j_y} - \lambda_{j_x}^+ - \lambda_{j_x}^- \perp \gamma_j \geq 0 \tag{5b}$$
$$0 \leq \gamma_j + \boldsymbol{J}_{g_{j_x}}(\boldsymbol{q})\dot{\boldsymbol{q}} \perp \lambda_{j_x}^+ \geq 0 \tag{5c}$$
$$0 \leq \gamma_j - \boldsymbol{J}_{g_{j_x}}(\boldsymbol{q})\dot{\boldsymbol{q}} \perp \lambda_{j_x}^- \geq 0 \tag{5d}$$

These four equations together describe the contact of the foot with the ground. The complementarity constraints describe three kinds of behaviour. The foot can be off the ground, fixed to the ground, and slide over the ground. How that is represented by the complementarity constraints is explained next.

- The foot is off the ground. The normal force λ_{j_y} is then zero (5a) and the friction forces $\lambda_{j_x}^+$ and $\lambda_{j_x}^-$ are also zero (5b). The foot has a velocity and so the left-hand side (LHS) either (5c) or (5d) is non-negative, the slack-variable γ_j makes the other LHS also non-negative.
- The foot is fixed to the ground. g_{j_y} is zero and the normal force λ_{j_y} prevents the foot from penetrating the ground (5a). The LHS of (5b) is non-zero because the friction forces $\lambda_{j_x}^+$ and $\lambda_{j_x}^-$ are inside the friction cone and so the slack variable γ_j is zero. The velocity of the foot is zero because with $\gamma_j = 0$ the LHS's of (5c) and (5d) can only be non-negative when the velocity of the foot is zero. The friction forces $\lambda_{j_x}^+$ and $\lambda_{j_x}^-$ can have any non-negative value, as long as they keep the velocity in x-direction zero and are inside the friction cone.
- The foot is sliding over the ground. The foot is in contact with the ground and there is a normal force on the foot (5a). The friction force $\lambda_{j_x}^+$ or $\lambda_{j_x}^-$ is on the boundary of the friction cone, so the LHS of (5b) is zero. The slack-variable γ_j makes the LHS of either (5c)

or (5d) zero, this ensures that there is only a friction force in the direction opposite to the sliding direction.

The bipedal walker, represented by the Hamiltonian system with complementarity constraints, is completely described by a total of 23 variables.

III. GAIT OPTIMIZATION

The bipedal walking gaits are optimized with respect to the input torques applied to the walker. A walking gait is a continuous repetition of gait cycles, the time between two consecutive foot impacts of the same leg. Therefore, optimizing one gait cycle optimizes the complete walking gait. Optimal solution is a zero-input gait cycle, a gait which does not require input torques when it is on the gait cycle.

The gait cycle is, for the optimization, divided into N equal time steps. The optimization minimizes the objective function, which is a function of the inputs to the system over all N steps. The minimization is subject to the constraint that the walking gait obeys the Hamilton system dynamics. The Hamilton system is in turn subject to the contact complementarity constraints. So a mathematical program with complementarity constraints (MPCC) is obtained [7]. However, there are not many solvers that can handle MPCC's. Therefore, this paper proposes to approximate the complementarity constraints by standard equality constraints. When the standard equality constraints are met they have the properties of complementarity constraints.
All the constraints are now equality constraints and the bilevel program is formulated as a mathematical program with equilibrium constraints (MPEC). The state and input variables are simultaneously optimized by solving the MPEC. This optimization is categorized as a direct local collocation method [8].

As said, the gait cycle the divided into N time steps. Therefore, the total number of variables to optimize is N times the number of variables in the subjected system. This problem can be solved because the constraints at time step i only depend on the variables of the step $i-1$ and i. The Jacobian of the constraints is a sparse matrix and this makes it a sparse problem, which makes the problem solvable.

A. Objective Function

The goal of the gait optimization is to minimize the input torques u_1, u_2, and u_3 that are applied to the joints of the bipedal walker. This is represented by an objective function that squares the input torques and sums them over the N steps of the gait cycle.

$$F(\boldsymbol{u}) = \sum_{i=1}^{N} \tfrac{1}{2} \left(u_{1_i}^2 + u_{2_i}^2 + u_{3_i}^2 \right)$$

B. Dynamic Constraints

The dynamic constraints are represented by the implicit backward Euler integration method because that is more stable than explicit integration methods. The Euler integration is rewritten as equal to zero constraints. The time-derivatives of the state variables $\boldsymbol{q}$ and $\boldsymbol{q}$ are given by the Hamiltonian model in (4). The dynamic constraints connect each step to the previous step. This creates the cycle behaviour of the gait. The first step is connected to the last step by setting $i-1 = N$ for $i = 1$.

$$\begin{bmatrix} \boldsymbol{q}_i \\ \boldsymbol{p}_i \end{bmatrix} - \begin{bmatrix} \boldsymbol{q}_{i-1} \\ \boldsymbol{p}_{i-1} \end{bmatrix} - h_i \begin{bmatrix} \dot{\boldsymbol{q}}_i \\ \dot{\boldsymbol{p}}_i \end{bmatrix} = \boldsymbol{0}$$

C. Complementarity Constraints

The complementarity constraints are rewritten with the help of the Fischer-Burmeister complementarity function.

$$\phi_{FB}(a,b) = \sqrt{(c_a a)^2 + (c_b b)^2} - (c_a a + c_b b) = 0 \qquad (6)$$

The complementarity properties are satisfied when this function is zero, so when at least one of a or b is zero. The constants $c_a > 0$ and $c_b > 0$ scale the variables a and b to make them the same order of magnitude. Each of the complementarity constraints in (5) has its own scaling constant.

Problem for the optimization is that the derivative of this function is not defined at $(a,b) = (0,0)$. This causes problems when the variables want to cross this point during optimization. Suggestion for creating a smooth complementarity function is to square the Fischer-Burmeister complementarity function [9].

$$\varphi_{FB}(a,b) = \tfrac{1}{2}\phi_{FB}(a,b)^2 = 0$$

This makes the function smooth around the origin, while keeping the complementarity properties.

The choice of c_a and c_b in (6) is important for the numerical stability of the minimization method. After many simulations it turned out that the best results were obtained by using the next scaling constants.

$$\begin{aligned} c_a &= \frac{c_1}{\max(\, a_1, \ldots, a_N)} \\ c_b &= \frac{c_2}{\max(\, b_1, \ldots, b_N)} \end{aligned} \qquad (7)$$

c_1 and c_2 are parameters that are determined by tests. They only depend on the walker's parameters and not on the velocity of the walker or the forces applied to the walker. The scaling constants c_a and c_b are updated in each iteration of the optimization.

IV. RESULTS

The gait optimization (section III) is used for optimizing the gaits of the bipedal walker (section II). The optimization is implemented in MATLAB R2014a. MATLAB solves the optimization with the large-scale non-linear optimization package IPOPT [10]. IPOPT uses the linear solver MA57-solver, which is a solver for sparse symmetric systems of linear equations [11].
The first and the second half of the gait cycle are identical when the variables of leg 1 q_3 and q_4 are relabelled to the variables of leg 2 q_5 and q_6, and vice versa. This means that when the relabelling is done, it is sufficient to optimize half a gait cycle. This effectively cuts the number of optimizing

points in half. $N = 200$ is used for the optimization. The results are shown for one gait cycle.

The total mass of the walker is $75\,kg$. The weight is distributed over the hip, upper leg , and lower leg and they account for respectively 60%, 15%, and 5% of the walkers weight, as in a average human [12]. The length of the legs is $1\,m$, $0.50\,m$ for both leg segments.
Boundaries are imposed on the configuration of the bipedal walker to prevent unnatural walking gaits. The upper legs cannot go above hip height, so the angles q_3 and q_5 are bounded by $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The lower legs cannot rotate through the upper leg and are prohibited to overstretch, so q_4 and q_6 are bounded by 0 and π. These boundaries are constraints in the optimization and are not physically enforced by end-stops on the walker. The momentum p_1 is bounded from below by 0 because backward walking gaits are not of interest. Additional constraint is that the step size of the foot is bound from below by the size that gives the required velocity. If the step size is not constraint from below, the input torques would be minimized by letting the walker stand still.

First, the walking gaits are optimized with the torsion spring stiffness K set to zero. The hip and knee torques needed for walking come solely from the input torques u_1, u_2, and u_3. The resulting gaits are nominal gaits. They are for analysis of the walking gaits and to verify if a torsion spring on the knee can improve the energy-efficiency of the bipedal walker gaits (section IV-A).
Second, the walking gaits are optimized together with the torsion spring K on the knee. This is done for two situations, one where equilibrium position of the spring is the stretched leg position $q_{eq} = \pi$, and one where the equilibrium position of the spring q_{eq} is also part of the optimization (section IV-B).

The bipedal walker gaits are optimized for different gait velocities. Small experiments show that the step-length of walking increases with increasing velocity. The slowest gait has a step-length of $0.20\,m$ with a step-time of $0.85\,s$. The fastest gait has a step-length of $0.50\,m$ with a step-time of $0.46\,s$. Seven gaits were created by distributing the step-length and the step-time evenly. These seven gaits have a velocity from 0.84 to $3.92\,km/h$.

A. Nominal Gaits

Fig. 2 shows the angles of the hip and knee for nominal gait cycles of the seven gait velocities. The trajectories of the swing foot with respect to the stance foot [0 0] are in figure 3. The stance foot is completely stretched, so the hip makes a circle movement with a radius of 1 (length of leg) around [0 0] that travels half the distance of the swing foot.

The results show that the angular rotation of the hip and knee increase with increasing velocity. The knee angle at lower gait velocities has two dents in the swing phase. The second dent increases with increasing velocity and the two dents 'melt' together at a certain undetermined gait velocity and form one large dent. This changes the trajectory of the swing foot from a pendulum swing motion at lower gait velocities to a parabolic motion at higher velocities.

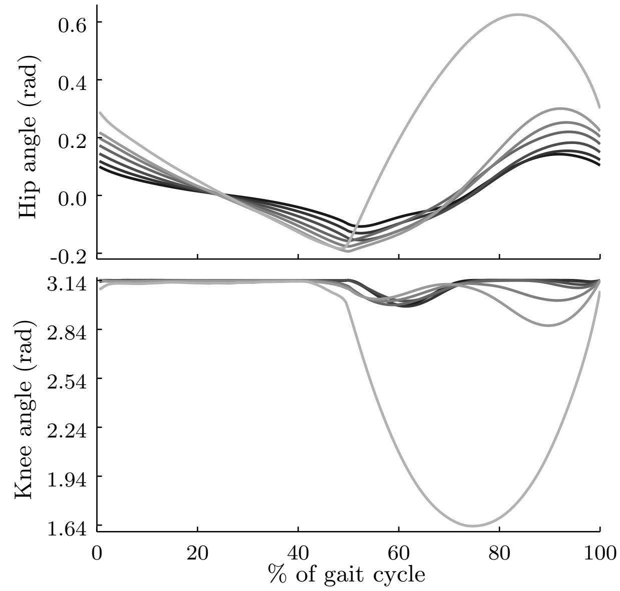

Fig. 2. Angles of one leg for a complete gait cycle for 7 velocities from 0.84 to $3.92\,km/h$. The lighter the colour, the higher the gait velocity.

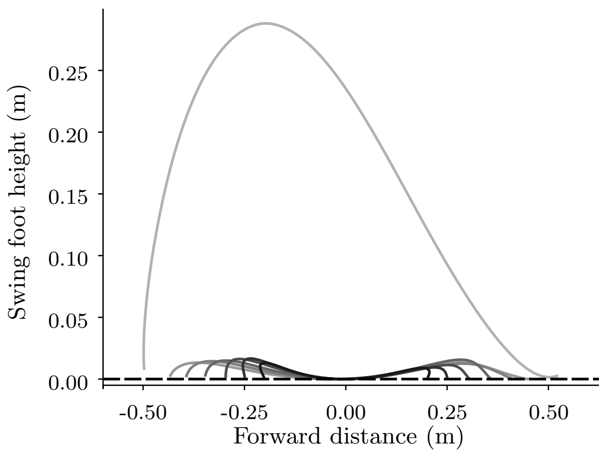

Fig. 3. Trajectories of the swing foot with respect to the stance foot for 7 velocities from 0.84 to $3.92\,km/h$. The lighter the colour, the higher the gait velocity.

B. Walking Gaits with Knee Springs

The results of optimization the walker gait with the torsion spring stiffness K and equilibrium position q_{eq} are shown in table I. Important value is $F_{\%}$ which is the percentage ratio of the objective function of the optimized gait with spring to objective function of the nominal gait. Adding a spring with $q_{eq} = \pi$ only decreases the input torques for lower gait velocities. Adding an optimized spring with an optimized equilibrium position decreases the objective function for all gait velocities. That optimizing the equilibrium position results in reduced input torques for all velocities is caused by the fact that the function of the knee springs changes with increasing velocity. At lower gait velocities the spring helps

TABLE I

SIMULATION RESULTS

Velocity	K optimized		K, q_{eq} optimized		
$(\frac{km}{h})$	$K(\frac{Nm}{rad})$	$F^1_{\%}$	$K(\frac{Nm}{rad})$	$q_{eq}(rad)$	$F^1_{\%}$
0.84	7.98	93	8.00	3.09	64
1.15	16.21	74	12.40	3.11	52
1.50	23.46	66	19.14	3.12	40
1.92	23.82	38	40.50	3.12	17
2.44	0.21	93	7.65	3.03	53
3.09	0.01	111	4.19	3.04	79
3.92	0.00	100	0.73	0.00	9

[1] $F_{\%} = 100 \cdot F/F_{nom}$. F is the objective function value of the optimized gait cycle, F_{nom} is the objective function value of the nominal trajectory for that gait velocity.

to stretch the leg, but at the higher gait velocities it helps to retract the leg.

Note that for the gait velocity of $3.09\,km/h$ the nominal trajectory is more efficient than the one with the optimized torsion spring K. This is due to the fact that IPOPT had problem to converge to the optimal solution. If adding a spring does not help, the ratio should obviously be reduced to 100 with a torsion spring constant of $K = 0$.

Fig. 4 and 5 show the hip and knee angles and torques for a gait velocity of $1.15\,km/h$. The results of the other gait velocities show similar behaviour. Using a spring with a not-optimized equilibrium position alters the hip and knee angles significantly. The angles of the nominal walking gaits are approached by also optimizing the equilibrium position. Remarkable is that adding a knee spring also decreases the maximum torques supplied to the hip. Note that by optimizing the equilibrium position q_{eq} of the spring, the input torques during the stance phase are higher.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a method for optimizing bipedal walker gaits and used it on a bipedal walker with knee torsion springs. Nominal walking gaits showed that the angle trajectories changes with the gait velocity. Optimizing the only the knee spring constant decreased the input torques for lower gait velocities, but not for the higher gait velocities. Optimizing the knee spring constant as well as the equilibrium position of the torsion spring reduced the required input torques for all gait velocities. Remarkable result is that not only the torques on the knees decreased, but also the maximum torque supplied to the hip.

The resulting trajectories were noisy for approximately half of the simulations. This is because the IPOPT solver is not well suited for this problem. Suggestion is to use other (commercial) solvers like SNOPT [13].

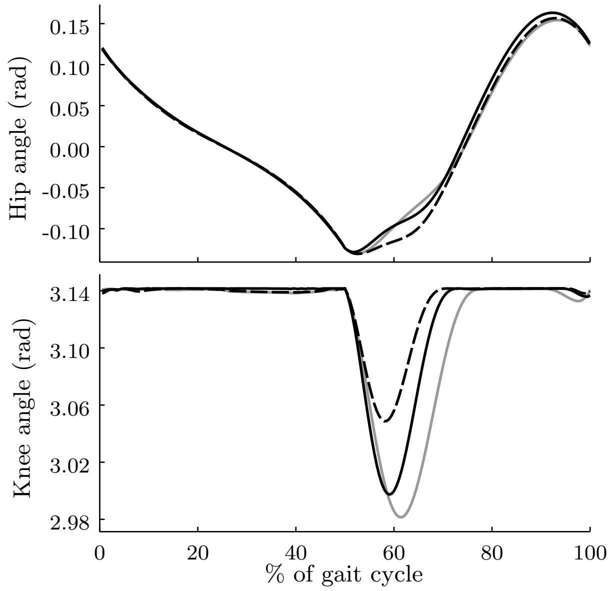

Fig. 4. Angles of the hip and knee for the nominal gait (—), optimization of only the knee torque (- -), and optimization of the knee spring and the equilibrium position (—). The gait velocity is $1.15\,km/h$.

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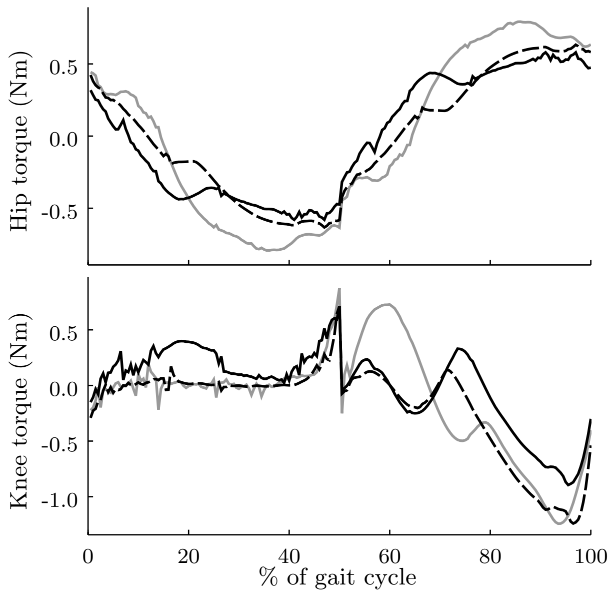

Fig. 5. Input torques on the hip and knee for the nominal gait (—), optimization of only the knee torque (- -), and optimization of the knee spring and the equilibrium position (—). The gait velocity is $1.15\,km/h$.

3 Implementation of Optimization

This section explains details of the paper ‘Optimizing Bipedal Walking Gaits with Constant Knee Springs’. For a proper understanding it is advised to first read the paper thoroughly. This section can then be used to implement the theory of the paper.

3.1 Implementation in MATLAB

Most solver require that the objective function, the constraints, and their first derivatives are supplied to the solver. The size of the equations is large, especially from the Jacobian of the constraints. Putting them by hand in MATLAB will certainly cause problems. Luckily these functions are constructed from a few matrices; the mass-matrix, the positions of the foot, and the potential energy. We also need the first and second order derivatives of these matrices.
Therefore, MuPAD is used to make it simpler. MuPAD is the symbolic engine build into MATLAB. The matrices are put into MuPAD. Advantage of MuPAD is that it can calculate the first and second derivatives of these matrices and it can export them to MATLAB. But before exporting the files to MATLAB, it first optimizes them so calculations are not done twice.

3.2 Iterior Point Optimizer (IPOPT)

The interior point optimizer (IPOPT) is a is a software package for large scale non-linear optimization. It is released as open source code under the Eclipse Public License (EPL). For more details about IPOPT and how to install it we refer to [5].

IPOPT uses the interior point method to solve the problem. The constraints in the interior point method are posed as barriers. There are a three options, that relate to how these barriers are treated, that are crucial for enabling IPOPT to solve the optimization. These three options force IPOPT not to be satisfied with the first local minimum that it finds. Further details are in the IPOPT documentation [6].

```
mu_strategy                 = 'adaptive';
adaptive_mu_globalization = 'kkt-error';
mu_oracle                   = 'probing';
```

3.3 Mathematical Details

The goal is to optimize a walking gait with respect to the input torques. This is represented by minimizing an objective function that is a function of those input torques. The constraints to the problem follow from the Hamiltonian system dynamics. First part of the constraints are the rewritten implicit backward Euler method for solving differential equations. Second part are the complementarity constraints related to the dynamics. $0 \leq a \perp b \geq 0$ means $a \geq 0$, $b \geq 0$ and

$a \cdot b = 0$. What these complementarity constraints exactly represent is explained in the paper.

$$\min_{\{\boldsymbol{q}_1,..,\boldsymbol{q}_N,\boldsymbol{p}_1,..,\boldsymbol{p}_N,\boldsymbol{u}_1,..\boldsymbol{u}_N,\boldsymbol{\lambda}_1,..,\boldsymbol{\lambda}_N,K,q_{eq}\}} \sum_{i=1}^{N} \frac{1}{2}\left(u_{1_i}^2 + u_{2_i}^2 + u_{3_i}^2\right)$$

s.t. $\forall\, i \in \{1,..,N\}$,

$$\begin{bmatrix}\boldsymbol{q}_i\\ \boldsymbol{p}_i\end{bmatrix} - \begin{bmatrix}\boldsymbol{q}_{i-1}\\ \boldsymbol{p}_{i-1}\end{bmatrix} - \quad h_i \left(\begin{bmatrix} \frac{\partial H}{\partial \boldsymbol{p}} \\ -\frac{\partial H}{\partial \boldsymbol{q}} \end{bmatrix} + \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_u\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} + \begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \frac{\partial g_1}{\partial \boldsymbol{q}}^\top & \frac{\partial g_2}{\partial \boldsymbol{q}}^\top\end{bmatrix}\begin{bmatrix}\lambda_{1_x}^+ - \lambda_{1_x}^-\\ \lambda_{1_y}\\ \lambda_{2_x}^+ - \lambda_{2_x}^-\\ \lambda_{2_y}\end{bmatrix}\right)_i = \boldsymbol{0}, \tag{1}$$

$$\forall i: j \in \{1,2\} \quad \begin{array}{ccccccc} 0 & \leq & g_{j_\perp} & \perp & \lambda_{j_\perp} & \geq & 0, \\ 0 & \leq & \mu\lambda_{j_\perp} - \lambda_{j_\top}^+ - \lambda_{j_\top}^- & \perp & \gamma_j & \geq & 0, \\ 0 & \leq & \gamma_j + \frac{\partial g_{j_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}} & \perp & \lambda_{j_x}^+ & \geq & 0, \\ 0 & \leq & \gamma_j - \frac{\partial g_{j_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}} & \perp & \lambda_{j_x}^- & \geq & 0 \end{array}$$

The model has $23 \times N + 2$ optimization variables in total. 6 q, 6 p, 3 u, 6 λ, and 2 γ variables for each time step. The torsion spring constant K and its equilibrium position q_{eq} are constant for the whole gait.
All lower-case letters that are not bold represent a single variable. Bold lower-case letters represent a vector. There are two types of bold lower-case letters, $\boldsymbol{q}$ and $\boldsymbol{q}_1$.

$$\begin{aligned} \boldsymbol{q} &= \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^\top \\ \boldsymbol{q}_1 &= \begin{bmatrix} q_{1_1}, \ldots, q_{1_N} \end{bmatrix}^\top \end{aligned} \tag{2}$$

Upper-case letters are matrices, except of course N which is the number of time steps.
The vector derivatives of a scalar have the following form.

$$\frac{\partial H}{\partial \boldsymbol{q}} = \begin{bmatrix} \frac{\partial H}{\partial q_1} & \frac{\partial H}{\partial q_2} & \frac{\partial H}{\partial q_3} & \frac{\partial H}{\partial q_4} & \frac{\partial H}{\partial q_5} & \frac{\partial H}{\partial q_6} \end{bmatrix}^\top \tag{3}$$

Complementarity Constraints The complementarity constraints can be written as two inequality constraints and 1 equality constraint (4). These three constraints can be put into the optimization software, but this requires an optimization package that can handle complementarity constraints. Most of the packages do not offer that function and do therefore not know that the three constraints are related to each other. This makes the optimization process hard.

$$\begin{aligned} &0 \;\leq\; a \perp b \;\geq\; 0 \\ &a \leq 0, \quad b \leq 0, \quad a \cdot b = 0 \end{aligned} \tag{4}$$

Therefore, the complementarity constraints are rewritten with the help of complementarity functions. These functions pack the two inequality constraints and the equality constraint into one equality constraint. The complementarity function is the squared Fischer-Burmeister complementarity function [7]. Squaring the Fischer-Burmeister function makes is smooth around $[0\,0]$.

$$\varphi(a,b) = \tfrac{1}{2}\phi_{FB}(a,b)^2 = \tfrac{1}{2}\left(\sqrt{(c_a \cdot a)^2 + (c_b \cdot b)^2} - (c_a \cdot a + c_b \cdot b)\right)^2 = 0$$

3.3.1 Jacobian of the Constraints

In total there are $20 \cdot N + 1$ constraints; $12 \cdot N$ dynamic constraints, $8 \cdot N$ complementarity constraints, and 1 constraint on the distance of the foot. The total number of variables is $23 \cdot N + 2$. So the

Jacobian of the constraints has the dimension $(20 \cdot N + 1) \times (23 \cdot N + 2)$. This is a large, but sparse matrix. Sparse because the constraints at i are mainly a function of the variables that belong to step i. Only the dynamic constraints are also a function of the p and q variables of step $i-1$ and of the knee spring parameters K and q_{eq}. So one constraint depends at most on $23 + 12 + 2 = 38$ of the $23 \cdot N$ variables. The structure of the Jacobian is shown in fig. 1.

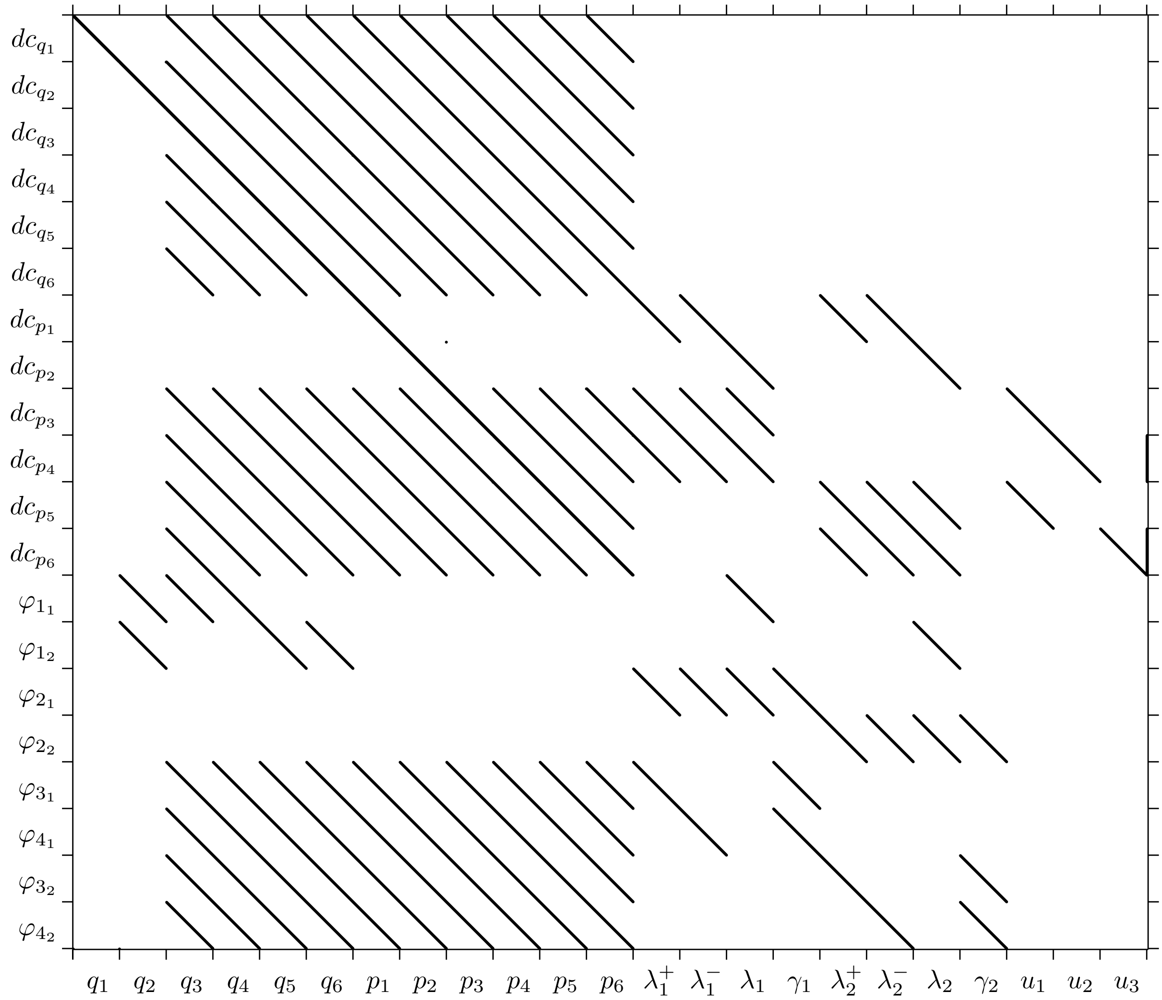

Figure 1: Structure of the Jacobian of the constraints. The spaces between the ticks are discretized from 1 to N. dc are the dynamics constraints and φ are the complementarity functions. The constraint on the distance travelled by the foot is below φ_{4_2}. The spring constant K and the equilibrium position of the spring are next to u_3.

3.3.2 Dynamic Constraints Jacobian

$$\begin{bmatrix} \boldsymbol{q}_i \\ \boldsymbol{p}_i \end{bmatrix} - \begin{bmatrix} \boldsymbol{q}_{i-1} \\ \boldsymbol{p}_{i-1} \end{bmatrix} - h_i \left(\begin{bmatrix} \frac{\partial H}{\partial \boldsymbol{p}} \\ -\frac{\partial H}{\partial \boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} 0 \\ B_u \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{\partial g_1}{\partial \boldsymbol{q}}^\top & \frac{\partial g_2}{\partial \boldsymbol{q}}^\top \end{bmatrix} \begin{bmatrix} \lambda_{1_x}^+ - \lambda_{1_x}^- \\ \lambda_{1_y} \\ \lambda_{2_x}^+ - \lambda_{2_x}^- \\ \lambda_{2_y} \end{bmatrix} \right)_i = 0, \qquad (5)$$

The dynamic constraints (5) are 12 equilibrium constraints for each time step. The derivatives of the first two state vectors are clear. The derivative of the first vector is a vector of ones and of the second a vector of minus ones.

The Jacobian of the part that is multiplied by h_i is more complex. The derivatives to u and λ are just one or minus one times the corresponding input matrix value. The derivatives to $\boldsymbol{q}$ and $\boldsymbol{p}$ are what make the derivatives complex. They consist of the second order derivatives of the Hamiltonian $\mathcal{H}$ and of the second order derivatives of the foot positions g_i.

The first order derivatives of the Hamiltonian $\mathcal{H}$ to $\boldsymbol{q}$ and $\boldsymbol{p}$ are shown next. i and j in (6) to (14) have the values $1-6$ and stand for the generalized coordinates and momenta.

$$\mathcal{H} = \frac{1}{2}\boldsymbol{p}^\top M(\boldsymbol{q})^{-1}\boldsymbol{p} + V(\boldsymbol{q}, K, q_{eq}) \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = M^{-1}\boldsymbol{p} \tag{7}$$

$$-\frac{\partial \mathcal{H}}{\partial q_i} = \frac{1}{2}\boldsymbol{p}^\top M^{-1}\frac{\partial M}{\partial q_i}M^{-1}\boldsymbol{p} - \frac{\partial V(\boldsymbol{q}, K, q_{eq})}{\partial q_i} \tag{8}$$

The second order derivatives of the Hamiltonian $\mathcal{H}$ are needed for the Jacobian of the constraints. Be aware that (10) seems to be identical to (11), this is not true. (10) is the transpose of (11). The next four equation still have to be multiplied by $-h$ for the Jacobian.

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_j} = \left(M^{-1}\right)_{[i,j]} \tag{9}$$

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_j} = \left(-M^{-1}\frac{\partial M}{\partial q_j}M^{-1}\boldsymbol{p}\right)_{[i]} \tag{10}$$

$$-\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_j} = \left(M^{-1}\frac{\partial M}{\partial q_i}M^{-1}\boldsymbol{p}\right)_{[j]} \tag{11}$$

$$-\frac{\partial^2 \mathcal{H}}{\partial q_i \partial q_j} = \frac{1}{2}\boldsymbol{p}^\top M^{-1}\left(-\frac{\partial M}{\partial q_j}M^{-1}\frac{\partial M}{\partial q_i} + \frac{\partial^2 M}{\partial q_i \partial q_j} - \frac{\partial M}{\partial q_i}M^{-1}\frac{\partial M}{\partial q_j}\right)M^{-1}\boldsymbol{p} - \frac{\partial^2 V(\boldsymbol{q}, K, q_{eq})}{\partial q_i \partial q_j} \tag{12}$$

Below are the derivatives of $-h_j \cdot \frac{\partial g_i}{\partial \boldsymbol{q}} \cdot \lambda$.

$$-h_j \cdot \frac{\partial^2 g_{1_y}}{\partial q_i \partial q_j} \cdot \lambda_{1_y} \qquad -h_j \cdot \frac{\partial^2 g_{1_x}}{\partial q_i \partial q_j} \cdot \left(\lambda_{1_x}^+ - \lambda_{1_x}^-\right) \tag{13}$$

$$-h_j \cdot \frac{\partial^2 g_{2_y}}{\partial q_i \partial q_j} \cdot \lambda_{2_y} \qquad -h_j \cdot \frac{\partial^2 g_{2_x}}{\partial q_i \partial q_j} \cdot \left(\lambda_{2_x}^+ - \lambda_{2_x}^-\right) \tag{14}$$

3.3.3 Complementarity Constraints Jacobian

$$\begin{array}{ccccccc} 0 & \le & g_{i_y} & \perp & \lambda_{i_y} & \ge & 0, \\ 0 & \le & \mu\lambda_{i_y} - \lambda_{i_x}^+ - \lambda_{i_x}^- & \perp & \gamma_i & \ge & 0, \\ 0 & \le & \gamma_i + \frac{\partial g_{i_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}} & \perp & \lambda_{i_x}^+ & \ge & 0, \\ 0 & \le & \gamma_i - \frac{\partial g_{i_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}} & \perp & \lambda_{i_x}^- & \ge & 0 \end{array} \tag{15}$$

The complementarity constraints have all the same general form as shown in 16. Each step has 8 of these equality constraints. Note (17) needs still to be multiplied by the derivative of a or b.

$$\varphi(a,b) = \frac{1}{2}\left(\sqrt{(c_a \cdot a)^2 + (c_b \cdot b)^2} - (c_a \cdot a + c_b \cdot b)\right)^2 = \frac{1}{2}\phi_{FB}(a,b)^2 = 0 \tag{16}$$

$$\frac{\partial \varphi(a,b)}{\partial a} = \phi_{FB}(a,b) \cdot \left(\frac{c_a^2 \cdot a}{\sqrt{(c_a \cdot a)^2 + (c_b \cdot b)^2}} - c_a\right) \tag{17}$$

Most of the derivatives of the complementarity constraint variables are pretty straight forward. Only functions that are a little more complex are g_{i_y} and $\frac{\partial g_{i_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}}$. The derivatives of g_{i_y} are already calculated for $\frac{\partial g_i}{\partial \boldsymbol{q}}$, which is used in de dynamic constraints. The derivatives of $\frac{\partial g_{i_x}}{\partial \boldsymbol{q}}^\top \dot{\boldsymbol{q}}$ to $\boldsymbol{q}$ and $\boldsymbol{p}$ are shown below.

$$\frac{\partial}{\partial q_i}\left(\frac{\partial g_{1_x}}{\partial \boldsymbol{q}}^\top M^{-1}\boldsymbol{p}\right) = \frac{\partial^2 g_{1_x}}{\partial \boldsymbol{q}\partial q_i} M^{-1}\boldsymbol{p} - \frac{\partial g_{1_x}}{\partial \boldsymbol{q}}^\top M^{-1}\frac{\partial M}{\partial q_i} M^{-1}\boldsymbol{p} \tag{18}$$

$$= \left(\frac{\partial^2 g_{1_x}}{\partial \boldsymbol{q}\partial q_i} - \frac{\partial g_{1_x}}{\partial \boldsymbol{q}}^\top M^{-1}\frac{\partial M}{\partial q_i}\right) M^{-1}\boldsymbol{p} \tag{19}$$

$$\frac{\partial}{\partial p_i}\left(\frac{\partial g_{1_x}}{\partial \boldsymbol{q}}^\top M^{-1} p\right) = \frac{\partial g_{1_x}}{\partial \boldsymbol{q}}^\top \left(M^{-1}\right)_{[:,i]} \tag{20}$$

4 Conclusion

The results of this MSc project are twofold. An optimization method is presented that optimizes trajectories with contacts. In this project the optimization method was used to minimize the input torques of a bipedal walking gait. But optimization method is not restricted to that. It can optimize any trajectory with contacts. So it can also be used to optimize for instance the grasping of an object with a robot hand.

The gait optimization has two conclusions. The shape of trajectory of the swing foot changes with increasing gait velocity. So [3] was right that using the same shape of swing foot trajectories for all gait velocities is not optimal.

Adding a knee spring reduces the amount of torque needed for a walking gait. The higher the gait velocity, the more the knee spring reduces the required input torques. Remarkable result is a spring on the knees also reduces the input torque to the hip joint.

5 Recommendations

Improving the Optimization of Bipedal Walking Gaits

- *Use a different solver package.* The IPOPT solver poses the problem as a primal-dual problem in order to solve it. Simulations showed that IPOPT is not good at solving the problems posed in this paper. IPOPT starts with converging to an optimal solution. But when it is close to the solution, the dual error becomes dominant and the solution divererges from the optimal solution in an attempt to reduce the dual error. So the problem is the approach that IPOPT uses to solve the problem. [4] used the sparse non-linear solver (SNOPT) to solve this kind of problems and they reported good results. Reason IPOPT was used in this MSc project is that it is one of the few large scale solvers with MALTAB interface that are available under an academic license.

- *Make the time step h a optimization parameter.* The time step h was kept constant in this MSc project for simplicity. The largest constraint violations occur around the impact positions and these errors will at least slow down the convergence of the solver to the optimal solution. A variable time step increases the resolution at the impact points and that will help reducing the errors. It is recommended to optimize h over all N time steps independently.

- *Use higher quality initial gaits.* The initial gaits used in this MSc project were constructed by imposing trajectories of the foot and hip. These trajectories were divided into N steps and inverse kinematics was used to find the knee and hip angles. This resulted in a very low quality initial gait. Using higher quality gaits will decrease the number of iterations that are needed for finding an optimal solution.

Future Research

- *Replace the fixed knee springs with VSA's.* The torsion spring on the knees with fixed parameters reduces the torques required for walking. However, the spring stiffness and equilibrium are different for each gait velocity. So for implementation on a robotic walker, a variable stiffness actuator (VSA) is required to control stiffness and equilibrium position to the optimal values. The VSA can also be used to change the spring parameters during a gait. This can decrease the torques required for walking even more. Note that changing the stiffness and equilibrium position of the spring does also cost energy.

- *Investigate the stability of the gait trajectories.* Important for implementation on a robotic walker is how well disturbances to the optimal trajectories can be rejected. The optimization presented in this MSc project does not say anything about the stability of the optimized trajectories. It would be interesting to see if disturbances are rejected more energy efficient by torque control or by control of the knee spring parameters.

- *Investigate the use of the optimization for designing a model predictive controller (MPC's).* If using another solver makes the optimized gaits converge to a stable solution it is possible to use the resulting input torques for feed forward control. These controllers are called receding horizon controllers or model predictive controllers [8]. MPC's calculate an optimal control sequence for each point of the state space to the optimal trajectory. This control law can be calculated off line or online.

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