December 6, 2015

MASTER'S THESIS APPLIED MATHEMATICS

OPTIMAL FORGETTING: DYNAMIC PRICING IN CHANGING MARKETS

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Preface

This thesis marks the end of my Masters study Applied Mathematics. The research was conducted at the University of Twente, at the chair of Stochastic Operations Research.

I would not have finished my studies and this thesis without the support of several people. First of all, I would like to thank my supervisor Arnoud den Boer for guiding me through this final project. You always had useful comments and helped me to keep the right focus. Secondly, I would like to thank Richard Boucherie and Pranab Mandal for taking place in the assessment committee. Thirdly, I would like to thank my friends, who provided the necessary distractions, but most of all a lot of support during all the times my body had given up on me. Finally, I would like to thank my family and in particularly my mother. She was always there for me when I needed it.

Lianne ter Heegde

"May not music be described as the mathematics of the sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music: music the dream, mathematics the working life."

- James Joseph Sylvester

Abstract

In this thesis we consider dynamic pricing and learning in changing markets. When estimating the demand function, a certain amount of previous sales data is taken into account. In case of changing markets it might not be optimal to take all available sales data into account in the estimation. The goal of this research is to find the optimal amount of sales data (N_t) that we have to include in the demand estimation in every time period.

Analytical results are given for a deterministic price set. A constant market is analysed as well as a market with a change point. We show that already in a constant market it is not as simple as we might think. The estimation of the intercept namely also depends on the points added in the estimation. For the change point model we show that it depends on the size of the change in the model parameters whether or not we improve the estimation of the slope parameter by adding pre-change data.

Simulations are performed and the Controlled Variance Pricing policy is used in them. The performance of five possible subsequences for N_t is compared in five different scenarios for the model parameters. In a market with no or only one small change point it is optimal to take all available data into account. In markets with one large change point, or with continuous change, it is optimal to choose N_t small. With the described method we can improve the performance of existing pricing policies.

Contents

Chapter 1

Introduction

According to Lin (2006), dynamic pricing is "a business strategy that adjusts the product price in a timely fashion in order to allocate the right service to the right customer at the right time". With dynamic pricing a company can adjust a price for a product or service periodically (e.g. per hour, per day, per week). The company can determine the new selling price based on previous sales by estimating how the demand depends on the price. The goal is to optimize some performance indicator, such as revenue or occupancy rate.

Dynamic pricing has been a widely researched topic for many years in several fields, such as operations research, economics, and computer science. In the last few years, dynamic pricing has gained more and more interest due to the fact that technology has made it easier to store and access sales data. Online sales grew explosively and with it the possibility to gain more insight into customer behaviour and in the relationship between price and demand. Furthermore, brick-to-mortar stores can use digital price tags nowadays, which makes it easier to change the prices fast and at practically no cost. In recent years also more attention has been given to pricing within changing environments.

In case of a changing environment, it might not be optimal to take all available sales data into account when estimating the demand function parameters. There is a trade-off: if more data is used, the estimation error is smaller, but the market could have behaved differently than it does at the current moment. However, if less data is used, the estimation errors are bigger, but the estimation itself is more precise. The question that we raise is: what data do we have to take into account? If we only look at the last N selling prices and associated demands, what choice of N is then optimal? The goal is to deduce this N from the available sales data. We will consider this question for a monopolist firm that can change the price of a product every time period at no extra cost. The demand function is assumed to be linear and the model parameters are unknown to the seller.

This report is structured as follows. A review of relevant literature is given in Chapter 2. In Chapter 3 we state the mathematical model for the problem. We continue in Chapter 4 with the solution approach, in which analytical theorems and proofs are given and also the set-up for the simulations. In Chapter 5 we show the results of the simulations. We conclude with Chapter 6, where the conclusions, a discussion on the results and recommendations for further research are given. Appendix A gives an overview of the variables used in this thesis.

Chapter 2

Literature review

In this chapter we will discuss relevant literature for the problem as described in the previous chapter. We will discuss literature about dynamic pricing in combination with price experimentation and in changing environments. For a more complete overview of literature about dynamic pricing see the survey of Den Boer (2015a). Section 2.3 concludes this chapter with literature about change point detection algorithms.

2.1 Dynamic pricing and learning

One of the first academics to add learning into pricing policies are Chong and Cheng (1975). They analyse pricing policies that consider multiple time periods for a monopolistic firm. The demand function is assumed to be unknown, but time invariant and linear. They show that the myopic pricing policy is the optimal policy, if no learning takes place. With this they mean that new obtained information about the demand is not used to update the demand function. When learning does occur, the myopic policy is not optimal. They also perform simulations, and these show that with experimenting with prices, the performance of the policy improves. The goal of Balvers and Cosimano (1990) is to show how learning affects the pricing policy. Learning influences future expected profits, but not the expected current profit. Therefore learning has no impact on the expectation of the forecast, though it affects its accuracy and variance. They introduce a speed of learning, dependent on the price. Due to this dependence, the speed of learning can be controlled. A low speed means that old information is regarded as more reliable. Lobo and Boyd (2003) also discuss pricing policies where price experimentation is incorporated. Their policy is the solution of a stochastic dynamic program. Due to intractability, the solution cannot be computed exactly, but it is approximated with a convex optimization program. Price experimentation is done by adding a random perturbation to the prices in the myopic policy. They show that this policy performs better than the myopic policy, unless the dithering of the prices becomes to high. The optimal dithering level depends on the problem parameters. Another policy that includes price variation is designed by Harrison et al. (2012). They propose a constrained variant of the myopic Bayesian policy and show that the expected performance gap relative to a clairvoyant is bounded by a constant under a binary prior distribution of the model parameters. The constraint on the myopic policy is such that the next price cannot lie too close to an uninformative, incumbent price. A similar policy is proposed by Den Boer and Zwart (2013), the so-called Controlled Variance Pricing (CVP). The CVP also puts a constraint on the myopic pricing policy. Every time period the certainty equivalent price is chosen, unless this would mean that a lower bound on the sample variance of the prices so far is not fulfilled. This bound is in that time period set at $ct^{\alpha-1}$, for some c and $\alpha \in (0,1)$. This creates a taboo interval for the next price. They prove that if $\alpha > 1/2$, the regret at time T is of order $\mathcal{O}(T^{1/2+\delta})$, with $\alpha = 1/2 + \delta$. In case of normally distributed demand and a linear demand function, the bound on the regret holds also for $\alpha = 1/2$, and then the regret is of order $\sqrt{T}$. Numerical results are given for several demand distributions and different values of c and T.

2.2 Dynamic pricing in changing environments

Garivier and Moulines (2011), Besbes et al. (2015) and Den Boer (2014) consider time-variant stochastic processes. Garivier and Moulines (2011) discuss the multi-armed bandit problem, where the distributions of rewards can have abrupt changes. Two algorithms are analysed, namely the discounted upper-confidence bound policy and the sliding window upper-confidence bound policy. The latter one has an upper bound of $\mathcal{O}(\sqrt(T\log(T))$. The performance of both policies depends on the discount factor or sliding window. The optimal values of these are not given by Garivier and Moulines (2011). Besbes et al. (2015) consider sequential stochastic optimization where the underlying cost functions may vary over time. They introduce the term 'variation budget', as a measure for the change. Based on this budget, bounds are derived for the order of regret that can be achieved. Den Boer (2014) studies one-step ahead forecasting for a time-variant stochastic process. Optimal weighed least-squares estimators are derived for four types of uncertainty sets, to minimize the expected squared prediction error.

In recent years dynamic pricing in changing environments has received more and more attention. Besbes and Sauré (2014) study the problem of dynamic pricing with shifts in the demand function. The time of the change is assumed to be unknown, as well as the demand function after the change. The initial demand function is presumed to be known. Optimal pricing policies have a monotone path until the change point. Besbes and Sauré (2014) also analyse the impact on the optimal policy of the model input. Change detection for bursty changes is used in the policy designed by Besbes and Zeevi (2011). The willingness-to-pay (WtP) distribution of the customers can change abruptly at one point in the time horizon. They assume that the WtP distributions are known, only the time of the change is unknown. A lower bound of order $N^{1/2}$, with N the total amount of customers, on the worst-case regret is derived and pricing policies are given that achieve this bound. The policies will only detect a change if the change is significant enough. For more gradual changes, the change might not be detected. However, the authors claim that "the lack of detection of a change in the response functions indicates that the new demand environment is still within an indifference zone relative to the current one, and hence using the latter model has only a minor impact on performance" (Besbes and Zeevi, 2011, p. 78). The difference between bursty changes and smooth changes is considered by Keskin and Zeevi (2013). The change in model parameters is measured using a quadratic variation metric. The authors design near-optimal pricing policies, which performance asymptotically match the derived lower bound on regret. It is shown that a better performance can be achieved in an environment with bursty changes than in an environment with smooth changes. A weighted least squared estimation is used in an environment with smooth changes. In the case of bursty changes, a combination of a pricing policy and a change detection policy is designed. Experimentation with prices is incorporated in both estimations. If the volatility of the market is very high and the regret is not sub-linear, no policy is long-run-average optimal. Keller and Rady (1999) and Den Boer (2015b) combine dynamic pricing with changing demand and experimentation with prices as well. Keller and Rady (1999) assume that the linear demand function is switching according to a Markov process between two possible states. Two very different policies are given. One policy contains extreme experimentation and tracking of the demand curve. This policy is optimal when the probability of switching is low or if there are low discount rates. The other policy has moderate experimentation and poor tracking and is optimal when there is a high probability of a demand function switch or a high discount rate. Den Boer (2015b) assumes that the demand function is the sum of a stochastic market process and a known function depending on the price. The stochastic process has unknown characteristics. The values of the process can change in time. Two estimators are given for the estimation of the market process. One estimation uses a forgetting factor (estimation with weights) and the other one uses a sliding window. An upper bound on the expected estimation error is derived for both estimators.

2.3 Change point detection algorithms

To detect changes in model parameters change point detection algorithms can be used. A lot of literature can be found about these algorithms. In this section we will give a brief summary. For a more complete overview see the surveys of Rosenberg (1973), Zacks (1983) or Chen and Gupta (2001).

The CUSUM (cumulative sum control chart) test is one of the most well-known tests for change detection. The CUSUM was developed by Page (1955). With CUSUM cumulative sums of the differences between measurements and a bench mark are calculated. If there is no change in the parameters then the values of these sums will lie around the bench mark value. Nyblom (1989) uses also the CUSUM test to detect changes in parameters that are modelled as martingales. This martingale assumption covers several types of change such as a change point or a more smooth change, like a random walk. Bagshaw and Johnson (1977) extend the CUSUM of Page (1955) to be able detect changes in parameters values of an autoregressive integrated moving average (ARIMA) model. The proposed test is primarily designed for detecting a single change. If more changes occur, the test still applies if there is a reasonable amount of time between the changes. A further extension of the CUSUM test is made by Lee et al. (2003). They consider a general parameter case and use the tests to detect changes in random coefficient autoregressive models. With the proposed test detection of changes in parameters is possible in a broad class of statistical models, such as threshold models and autoregressive conditional heteroskedasticity (ARCH) models. A comparison between the CUSUM test and an extended Wald test (Wald (1943)) is made by Andrews et al. (1996). They show that performance of the extended Wald test is quite well and that the performance of the CUSUM test is very poor. The extension on the Wald test is made to be able to detect an unknown number of change points. Another modification of the Wald test is made by Chen and Hong (2012). The authors propose a new consistent Wald-type test, which is able to detect both structural breaks and smooth changes.

Ploberger et al. (1989) propose the so-called fluctuation test. The position of the change points are assumed to be unknown. The difference with the CUSUM test is that the fluctuation test is based on successive parameter estimates, whereas the CUSUM test is based on the recursive residuals. Ploberger et al. (1989) shows that the performance of the fluctuation test is better than the performance of the CUSUM test. An even more powerful test is designed by Lin and Teräsvirta (1994). Lin and Teräsvirta (1994) present a Lagrange multiplier (LM) test. The performance of the LM test is compared to the CUSUM test and the fluctuation test is found to be more powerful.

Another change point detection test is the likelihood ratio test. Quandt (1958) proposes this test and also a test based on the F distribution. The parameters of a linear regression system are estimated with a maximum likelihood estimation. The parameters are assumed to switch from one regime to another and it is the goal to find the time point at which this switch occurs. If more switches occurs, then the number of switches must be known to be able to use the test. The power of both tests depend on the magnitude of the change. Kim (1994) analyses the properties of the likelihood ratio test. A generalized likelihood ratio test is defined and the power and the robustness of the test are discussed. Jandhyala and MacNeill (1991) propose a unified approach and derives Bayes-type statistics. These statistics can be used to test both one-sided and two-sided changes in single or multiple parameters. Compared to Quandts statistic (Quandt (1958)), the statistic of Jandhyala and MacNeill (1991) has better power for identifying small changes. Brown et al. (1975) present tests based both on the CUSUM test and on the likelihood ratio test.

The change point model can also be viewed as a model selection problem. Therefore, information criteria can be used to find the change point. Chen (1998) proposes the Schwarz Information Criterion (SIC, Schwarz et al. (1978), or Bayesian information criterion (BIC)) to find a change point in a linear regression model. Another information criterion is the Akaike information criterion (Akaike (1974)). As mentioned in Chen and Gupta (2001), when SIC's are very close, the difference among them might not be caused by an actual change, but by fluctuation of the data.

Two more test are designed by Chu et al. (1995) and Becker et al. (2004). Chu et al. (1995) investigate tests based on moving sums (MOSUM) of recursive and least-squares residuals. Chu et al. (1995) show that the tests are consistent and have non-trivial local power against a general class of alternatives. Furthermore, simulations demonstrate that the tests have a power advantage when there are double structural changes. Becker et al. (2004) propose the Trig-test, which is based on a trigonometric expansion to approximate the unknown form of the variation in the parameters. They show that it has excellent power to detect structural breaks and stochastic parameter variation. The power reduces when the change point moves towards the end of the time horizon.

Change point detection tests have unfortunately one large drawback.The CUSUM test is not very powerful when changes occur late in the time horizon. This drawback is already mentioned by Page (1955). The Trig-test proposed by Becker et al. (2004) has the same problem. Sastri (1986) also discusses the fact that when analysing non-homogeneous time series large blocks of data must be obtained, both before and after the change point, to improve the power of change detection tests. This shortcoming makes it hard to perform tests while data is still gathered and the change point has occurred late in the time horizon. Furthermore, in this thesis we are looking for the amount of data that needs to be used in the estimation, and we do not specifically need to know the time the change point occurs. Results in Chapter 4 also show that that it is not necessarily optimal to immediately forget pre-change data after a change point occurs.

Chapter 3

Mathematical model

This chapter introduces the mathematical formulation of the problem, together with the notation. An overview of all variables can be found in appendix A. Section 3.1 describes the mathematical model, Section 3.2 gives the market models and Section 3.3 explains the pricing policy.

3.1 Pricing model

We want to sell a certain product over a time horizon of T periods. At the beginning of each time period $t \in \{1, 2, \ldots, T\}$ we can set a price p_t. In the same period a demand d_t is generated. The subscript t denotes the time period. We assume that the demand is linearly dependent on the price:

$$d_t = a_t + b_t \cdot p_t + \epsilon_t. \tag{3.1}$$

a_t and b_t are the unknown model parameters and we assume $a_t > 0$ and $b_t < 0$, $\forall t$. ϵ_t is the disturbance term in the demand, and is assumed to be a random variable, independent and identically distributed (i.i.d.), with $\mathbf{E}(\epsilon) = 0$ and $\mathrm{Var}(\epsilon) = \sigma^2, \quad \sigma^2 < \infty$.

3.1.1 Estimation of a_t and b_t

Since we cannot observe the real values of a_t and b_t (we only observe our price p_t and demand d_t), we have to estimate these values. We will estimate the model parameters by the ordinary least squares method (OLS):

$$\left(\widehat{a}_{(t-1,N_t)}, \widehat{b}_{(t-1,N_t)}\right) = \arg\min_{(a,b)} \sum_{i=\max(1,t-N_t)}^{t-1} (d_i - a - b \cdot p_i)^2. \tag{3.2}$$

We assume that there are at least two different prices in the set of the prices used in the estimation. With $\widehat{a}_{(t-1,N_t)}$ we denote the estimation of a at the beginning of time period t, based on N_t data points. The same notation holds for the estimation of b. If $t - N_t < 1$, we take all available data points into account in the estimation. The value of N_t can change every period, or we can assume is remains constant throughout our entire horizon T. How to choose this N_t is the main question of this thesis and a solution approach is discussed in the Chapter 4.

3.1.2 Remarks

We make some additional assumptions, namely:

- We choose the distribution of ϵ_t such that the probability of negative demand is very small;
- We assume infinite inventory, so we cannot have a backlog;
- There are no costs charged for changing a selling price.

3.2 Modelling the market

There are several possibilities for the behaviour of the market. When no change occurs, then we have a constant market. When a change does occur, this can either be an abrupt shift or a more gradual change. Of course, combinations of these three types are possible. From now on we will make a distinction between three models for the market change:

- Constant model: a_t and b_t are stable and have one value for the entire time horizon, we will denote this with a and b.
- Change point model: there is one change point at some point t_{cp}. Before and after t_{cp}, a_t and b_t remain the same. In this case we have two values for a_t and b_t: a and a', and b and b' for the respective pre- and post change point values.
- Random walk model: at every time step the values of a_t and b_t can change slightly, so we will keep our notation for this model at a_t and b_t.

3.3 Pricing policy

If we have our estimates for a_t and b_t, we are able to choose our p_t. The goal is to optimize our revenue and our (expected) optimal price is therefore set at the price that leads to the highest expected revenue in the upcoming period. This pricing policy is called the myopic pricing policy or the certainty equivalent pricing policy. We assume that there is a minimum and maximum selling price, respectively p_l and p_h, and that $p_h > p_l > 0, \forall t$. For all t we define:

$$\hat{p}^*_{(t,N_t)} = \max_{p \in [p_l, p_h]} p \cdot (\hat{a}_{(t-1,N_t)} + \hat{b}_{(t-1,N_t)} \cdot p). \tag{3.3}$$

If $\hat{a}_{(t-1,N_t)}$ and $\hat{b}_{(t-1,N_t)}$ have the correct sign, then

$$\hat{p}^*_{(t,N_t)} = \frac{-\hat{a}_{(t-1,N_t)}}{2\hat{b}_{(t-1,N_t)}}, \tag{3.4}$$

provided that the price lies in the admissible range. If this is not the case, or $\hat{a}_{(t-1,N_t)}$ and/or $\hat{b}_{(t-1,N_t)}$ has the wrong sign, the price should be set at the minimum or maximum price, depending on which of those two prices give the most expected revenue. The real optimal price is:

$$p^*_t = \frac{-a_t}{2 \cdot b_t}. \tag{3.5}$$

3.3.1 Price experimentation

As already mentioned in Chapter 2, the myopic pricing policy is not a optimal policy. Therefore, we will include experimentation with prices into our pricing policy. We will use the Controlled Variance Pricing policy, as described by Den Boer and Zwart (2013). We will set a lower bound on the variance of the prices that are taken into account in the parameter estimation. The bound is $ct^{\alpha-1}$, with $\alpha = 0.5$ and c arbitrarily chosen, $\forall t$. This bound creates a taboo interval for the to be chosen price p_t:

$$TI(t) = \left(\overline{p}_{(\max(1,t-N_t):t-1)} - \sqrt{c \cdot [t^\alpha - (t-1)^\alpha] \cdot \frac{t}{t-1}}, \overline{p}_{(\max(1,t-N_t):t-1)} + \sqrt{c \cdot [t^\alpha - (t-1)^\alpha] \cdot \frac{t}{t-1}} \right), \tag{3.6}$$

with,

$$\overline{p}_{(\max(1,t-N_t):t-1)} = \frac{1}{(t-1) - (\max(1, t-N_t))} \cdot \sum_{i=\max(1,t-N_t)}^{t-1} p_i. \tag{3.7}$$

If the lower bound on the variance is exceeded we will set p_t as follows:

$$\hat{p}^*_{(t,N_t)} = \max_{p \in [p_l, p_h] \setminus TI(t)} p \cdot (\hat{a}_{(t-1,N_t)} + \hat{b}_{(t-1,N_t)} \cdot p). \tag{3.8}$$

Chapter 4

Solution approach

In this chapter we come to an approach to solve the problem as described in Chapter 1 and as modelled in Chapter 3. First, we simplify the problem and look at a deterministic price-set and the choice of N_t in the three described market models. Next, we look into the original problem and show that this problem is quite complex. We conclude with a solution method, based on simulations, in Section 4.4.

4.1 Constant market

Intuitively one would expect that in a constant market it is optimal to take all available data into account when estimating a_t and b_t, since more data means more information. We will show that when taking more deterministic prices (and corresponding, stochastic, demands) into account when estimating b_t, this is true. However, for the estimation of a_t in a deterministic price-set, we will see that this might not always be the case.

Throughout this entire section we assume that we have a set of $N+1, N \geq 2$, deterministic prices **p**, with at least two different prices within it and $\forall i, p_i \geq 0$, Furthermore, we assume that ϵ is a random variable (i.i.d.), with $\mathbf{E}[\epsilon] = 0$ and $\mathrm{Var}(\epsilon) = \sigma^2, \sigma^2 < \infty$. a and b are constant values. Then we obtain: $\mathbf{d} = a + b \cdot \mathbf{p} + \boldsymbol{\epsilon}$, with **d, p** and $\boldsymbol{\epsilon}$ vectors. We use ordinary least squares estimation for the estimation of a and b, which estimators we denote with $\hat{a}$ and $\hat{b}$.
Remark: we use deterministic prices in Theorems 4.1.3 and 4.1.4. For prices obtained by using the pricing policy given in Section 3.3, we cannot automatically assume the same results. Den Boer (2013) already showed that adding non-deterministic data to an ordinary least squares estimation in linear regression can worsen the quality of the estimation.

First of all, two lemmas are given, which we use in the proofs of Theorem 4.1.3 and Theorem 4.1.4.

Lemma 4.1.1. *If we have n data points and $d_i = a + b \cdot p_i + \epsilon_i$ and we estimate a and b with the ordinary least squares method, then*

$$\hat{a}_n = \overline{\boldsymbol{d}}_n - \hat{b}_n \cdot \overline{\boldsymbol{p}}_n, \tag{4.1}$$

$$\hat{b}_n = \frac{\mathrm{Cov}(\boldsymbol{p}_n, \boldsymbol{d}_n)}{\mathrm{Var}(\boldsymbol{p}_n)} \tag{4.2}$$

Proof. We want to minimize $S(a, b)$, with:

$$S(a, b) = \sum_{i=1}^{n} (d_i - a - b \cdot p_i)^2. \tag{4.3}$$

$S(a, b)$ is convex and differentiable in a and b, so the unique solution is given by the first-order conditions:

$$0 = \frac{\delta}{\delta \hat{a}_n} S(\hat{a}_n, \hat{b}_n) = -2 \cdot \sum_{i=1}^{n} (d_i - \hat{a}_n - \hat{b}_n \cdot p_i),$$
$$0 = \frac{\delta}{\delta \hat{b}_n} S(\hat{a}_n, \hat{b}_n) = -2 \cdot \sum_{i=1}^{n} (d_i - \hat{a}_n - \hat{b}_n \cdot p_i) \cdot p_i$$

Now we have for $\hat{a}_n$:

$$\begin{aligned}
0 &= \sum_{i=1}^{n}(d_i - \hat{a}_n - \hat{b}_n \cdot p_i), \\
&= \sum_{i=1}^{n}(d_i - n \cdot \hat{a}_n) - \hat{b}_n \cdot \sum_{i=1}^{n} p_i, \\
&= \frac{1}{n}\sum_{i=1}^{n}(d_i - \hat{a}_n) - \frac{\hat{b}_n}{n} \cdot \sum_{i=1}^{n} p_i, \\
&= \overline{\mathbf{d}}_n - \hat{a}_n - \hat{b}_n \cdot \overline{\mathbf{p}}_n, \\
\hat{a}_n &= \overline{\mathbf{d}}_n - \hat{b}_n \cdot \overline{\mathbf{p}}_n.
\end{aligned}$$

Then for $\hat{b}_n$:

$$\begin{aligned}
0 &= \sum_{i=1}^{n}(d_i - \hat{a}_n - \hat{b}_n \cdot p_i) \cdot p_i \\
&= \sum_{i=1}^{n}(p_i \cdot d_i) - \hat{a}_n \cdot \sum_{i=1}^{n} p_i - \hat{b}_n \cdot \sum_{i=1}^{n} p_i^2,
\end{aligned}$$

filling in $\hat{a}_n = \overline{\mathbf{d}}_n - \hat{b}_n \cdot \overline{\mathbf{p}}_n$:

$$\begin{aligned}
&= \sum_{i=1}^{n}(p_i \cdot d_i) - \overline{\mathbf{d}}_n \cdot \sum_{i=1}^{n} p_i + \hat{b}_n \cdot \overline{\mathbf{p}}_n \cdot \sum_{i=1}^{n} p_i - \hat{b}_n \cdot \sum_{i=1}^{n} p_i^2 \\
&= \sum_{i=1}^{n}\big(p_i \cdot (d_i - \overline{\mathbf{d}}_n)\big) - \hat{b}_n \cdot \sum_{i=1}^{n} p_i \cdot (p_i - \overline{\mathbf{p}}_n) \\
\hat{b}_n \cdot \sum_{i=1}^{n} p_i \cdot (p_i - \overline{\mathbf{p}}_n) &= \sum_{i=1}^{n} p_i \cdot (d_i - \overline{\mathbf{d}}_n) \\
\hat{b}_n &= \frac{\sum_{i=1}^{n} p_i \cdot (d_i - \overline{\mathbf{d}}_n)}{\sum_{i=1}^{n} p_i \cdot (p_i - \overline{\mathbf{p}}_n)} \\
&= \frac{\sum_{i=1}^{n}(p_i \cdot d_i) - \sum_{i=1}^{n} p_i \cdot \overline{\mathbf{d}}_n}{\sum_{i=1}^{n} p_i^2 - \overline{\mathbf{p}}_n \cdot \sum_{i=1}^{n} p_i} \\
&= \frac{\sum_{i=1}^{n}(p_i \cdot d_i) - \frac{1}{n} \cdot \sum_{i=1}^{n} p_i \cdot \sum_{j=1}^{n} d_j}{\sum_{i=1}^{n} p_i^2 - \frac{1}{n} \cdot \left(\sum_{i=1}^{n} p_i\right)^2}, \\
&= \frac{n \cdot \overline{\mathbf{p}_n \cdot \mathbf{d}_n} - \frac{1}{n} \cdot n \cdot \overline{\mathbf{p}_n} \cdot n \cdot \overline{\mathbf{d}_n}}{n \cdot \overline{\mathbf{p}_n^2} - \frac{1}{n} \cdot n^2 \cdot \overline{\mathbf{p}_n}^2} \\
&= \frac{\overline{\mathbf{p}_n \cdot \mathbf{d}_n} - \overline{\mathbf{p}_n} \cdot \overline{\mathbf{d}_n}}{\overline{\mathbf{p}_n^2} - \overline{\mathbf{p}_n}^2}, \\
&= \frac{\mathrm{Cov}(\mathbf{p}_n, \mathbf{d}_n)}{\mathrm{Var}(\mathbf{p}_n)}.
\end{aligned}$$

■

Lemma 4.1.2.

$$(N+1)\cdot \mathrm{Var}(\boldsymbol{p}_{N+1}) = N\cdot \mathrm{Var}(\boldsymbol{p}_N) + \frac{N}{N+1}\cdot (p_{N+1} - \overline{\boldsymbol{p}}_N)^2 \tag{4.4}$$

Proof.

$$\begin{aligned}
(N+1)\cdot \mathrm{Var}\left(\mathbf{p}_{N+1}\right) &= \sum_{i=1}^{N+1} \left(p_i - \overline{\mathbf{p}}_{N+1}\right)^2 \\
&= \sum_{i=1}^{N+1} \left(p_i - \frac{1}{N+1}\cdot \sum_{i=1}^{N+1} p_i\right)^2 \\
&= \sum_{i=1}^{N+1} \left(p_i - \frac{1}{N+1}\cdot \left(\sum_{i=1}^{N} p_i + p_{N+1}\right)\right)^2 \\
&= \sum_{i=1}^{N+1} \left(p_i - \frac{N}{N+1}\cdot \overline{\mathbf{p}}_N - \frac{p_{N+1}}{N+1}\right)^2 \\
&= \sum_{i=1}^{N} \left(p_i - \frac{N}{N+1}\cdot \overline{\mathbf{p}}_N - \frac{p_{N+1}}{N+1}\right)^2 + \left(p_{N+1} - \frac{N}{N+1}\cdot \overline{\mathbf{p}}_N - \frac{p_{N+1}}{N+1}\right)^2 \\
&= \sum_{i=1}^{N} \left(p_i - \frac{N}{N+1}\cdot \overline{\mathbf{p}}_N - \frac{p_{N+1}}{N+1}\right)^2 + \left(\frac{N}{N+1}\cdot (p_{N+1} - \overline{\mathbf{p}}_N)\right)^2 \\
&= \sum_{i=1}^{N} \left(p_i - \overline{\mathbf{p}}_N + \frac{1}{N+1}\cdot \overline{\mathbf{p}}_N - \frac{p_{N+1}}{N+1}\right)^2 + \left(\frac{N}{N+1}\cdot (p_{N+1} - \overline{\mathbf{p}}_N)\right)^2 \\
&= \sum_{i=1}^{N} (p_i - \overline{\mathbf{p}}_N)^2 + \sum_{i=1}^{N} \left(\frac{\overline{\mathbf{p}}_N - p_{N+1}}{N+1}\right)^2 + 2\cdot \frac{\overline{\mathbf{p}}_N - p_{N+1}}{N+1} \sum_{i=1}^{N} (p_i - \overline{\mathbf{p}}_N) \\
&+ \left(\frac{N}{N+1}\right)^2 \cdot (p_{N+1} - \overline{\mathbf{p}}_N)^2 \\
&= \sum_{i=1}^{N} (p_i - \overline{\mathbf{p}}_N)^2 + \frac{N\cdot (\overline{\mathbf{p}}_N - p_{N+1})^2}{(N+1)^2} + \frac{N^2\cdot (p_{N+1} - \overline{\mathbf{p}}_N)^2}{(N+1)^2} \\
&= N\cdot \mathrm{Var}\left(\mathbf{p}_N\right) + \frac{N}{N+1}\cdot (p_{N+1} - \overline{\mathbf{p}}_N)^2 .
\end{aligned}$$

■

Theorem 4.1.3. *The expected squared error of the estimated $b, \hat{b}_N$, based on N observations, is greater or equal to the expected squared error of the estimator $\hat{b}_{N+1}$:*

$$\mathbf{E}\left[\left(\hat{b}_N - b\right)^2\right] \geq \mathbf{E}\left[\left(\hat{b}_{N+1} - b\right)^2\right]. \tag{4.5}$$

Proof. Using lemma 4.1.1:

$$\begin{aligned}
\mathbf{E}\left[\left(\hat{b}_N - b\right)^2\right] &= \mathbf{E}\left[\left(\frac{\mathrm{Cov}(\mathbf{p}_N, \mathbf{d}_N)}{\mathrm{Var}(\mathbf{p}_N)} - b\right)^2\right], \\
&= \mathbf{E}\left[\left(\frac{\mathrm{Cov}(\mathbf{p}_N, a + b\cdot \mathbf{p}_N + \boldsymbol{\epsilon}_N)}{\mathrm{Var}(\mathbf{p}_N)} - b\right)^2\right], \\
&= \mathbf{E}\left[\left(\frac{\mathrm{Cov}(\mathbf{p}_N, a)}{\mathrm{Var}(\mathbf{p}_N)} + \frac{\mathrm{Cov}(\mathbf{p}_N, b\cdot \mathbf{p}_N)}{\mathrm{Var}(\mathbf{p}_N)} + \frac{\mathrm{Cov}(\mathbf{p}_N, \boldsymbol{\epsilon}_N)}{\mathrm{Var}(\mathbf{p})} - b\right)^2\right],
\end{aligned}$$

$$= \mathbf{E}\left[\left(0 + b + \frac{\mathrm{Cov}(\mathbf{p}_N, \boldsymbol{\epsilon}_N)}{\mathrm{Var}(\mathbf{p}_N)} - b\right)^2\right],$$

$$= \mathbf{E}\left[\left(\frac{\mathrm{Cov}(\mathbf{p}_N, \boldsymbol{\epsilon}_N)}{\mathrm{Var}(\mathbf{p}_N)}\right)^2\right], \tag{4.6}$$

using the definition of covariance:

$$= \mathbf{E}\left[\left(\frac{\overline{\mathbf{p}_N \cdot \boldsymbol{\epsilon}_N} - \overline{\mathbf{p}_N} \cdot \overline{\boldsymbol{\epsilon}_N}}{\mathrm{Var}(\mathbf{p}_N)}\right)^2\right],$$

$$= \frac{1}{(\mathrm{Var}(\mathbf{p}_N))^2} \cdot \mathbf{E}\left[(\overline{\mathbf{p}_N \cdot \boldsymbol{\epsilon}_N} - \overline{\mathbf{p}_N} \cdot \overline{\boldsymbol{\epsilon}_N})^2\right],$$

$$= \frac{1}{(\mathrm{Var}(\mathbf{p}_N))^2} \cdot \mathbf{E}\left[\left(\frac{1}{N} \cdot \sum_{i=1}^{N} (p_i \cdot \epsilon_i) - \overline{\mathbf{p}_N} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \epsilon_i\right)^2\right],$$

$$= \frac{1}{(\mathrm{Var}(\mathbf{p}_N))^2} \cdot \mathbf{E}\left[\left(\frac{1}{N} \cdot \sum_{i=1}^{N} \epsilon_i (p_i - \overline{\mathbf{p}_N})\right)^2\right],$$

$$= \frac{1}{(N \cdot \mathrm{Var}(\mathbf{p}_N))^2} \cdot \mathbf{E}\left[\sum_{i=1}^{N} \epsilon_i^2 \cdot (p_i - \overline{\mathbf{p}_N})^2 + \sum\sum_{i \neq j} \epsilon_i \cdot \epsilon_j \cdot (p_i - \overline{\mathbf{p}_N}) \cdot (p_j - \overline{\mathbf{p}_N})\right],$$

$$= \frac{1}{(N \cdot \mathrm{Var}(\mathbf{p}_N))^2} \cdot \left(\mathbf{E}\left[\sum_{i=1}^{N} \epsilon_i^2 \cdot (p_i - \overline{\mathbf{p}_N})^2\right] + \mathbf{E}\left[\sum\sum_{i \neq j} \epsilon_i \cdot \epsilon_j \cdot (p_i - \overline{\mathbf{p}_N}) \cdot (p_j - \overline{\mathbf{p}_N})\right]\right),$$

$$= \frac{1}{(N \cdot \mathrm{Var}(\mathbf{p}_N))^2} \cdot \left(\sum_{i=1}^{N} \mathbf{E}\left[\epsilon_i^2 \cdot (p_i - \overline{\mathbf{p}_N})^2\right] + \sum\sum_{i \neq j} \mathbf{E}\left[\epsilon_i \cdot \epsilon_j \cdot (p_i - \overline{\mathbf{p}_N}) \cdot (p_j - \overline{\mathbf{p}_N})\right]\right),$$

since ϵ_i and ϵ_j are independent:

$$= \frac{\left(\sum_{i=1}^{N} \mathbf{E}\left[\epsilon_i^2\right] \cdot \mathbf{E}\left[(p_i - \overline{\mathbf{p}_N})^2\right] + \sum\sum_{i \neq j} \mathbf{E}\left[\epsilon_i\right] \cdot \mathbf{E}\left[\epsilon_j\right] \cdot \mathbf{E}\left[(p_i - \overline{\mathbf{p}_N}) \cdot (p_j - \overline{\mathbf{p}_N})\right]\right)}{(N \cdot \mathrm{Var}(\mathbf{p}_N))^2},$$

filling in $\mathbf{E}\left[\epsilon_i^2\right] = \sigma^2$ and $\mathbf{E}[\epsilon_i] = 0$:

$$= \frac{1}{(N \cdot \mathrm{Var}(\mathbf{p}_N))^2} \cdot \sigma^2 \cdot \sum_{i=1}^{N} (p_i - \overline{\mathbf{p}_N})^2,$$

$$= \frac{\sigma^2}{N \cdot (\mathrm{Var}(\mathbf{p}_N))^2} \cdot \mathrm{Var}(\mathbf{p}_N),$$

$$= \frac{\sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}. \tag{4.7}$$

Now we obtain for the expected squared error of the estimate $\hat{b}$, based on $N+1$ observations:

$$\mathbf{E}\left[\left(\hat{b}_{N+1} - b\right)^2\right] = \frac{\sigma^2}{(N+1) \cdot \mathrm{Var}(\mathbf{p}_{N+1})}, \tag{4.8}$$

using lemma 4.1.2

$$= \frac{\sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot (p_{N+1} - \overline{\mathbf{p}_N})^2},$$

$$\leq \frac{\sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}$$

$$= \mathbf{E}\left[(\hat{b}_N - b)^2\right]. \tag{4.9}$$

■

This result shows us that if we take $N+1$ observations into account in estimating parameter b, we expect an equal or better estimate than the $\hat{b}$ based on N observations, in case of a constant market and a deterministic set of prices.

Theorem 4.1.4. *The expected squared error of the estimated $a, \hat{a}_N$, based on N observations is greater or equal to the expected squared error of the estimator $\hat{a}_{N+1}$:*

$$\mathbf{E}\left[(\hat{a}_N - a)^2\right] \geq \mathbf{E}\left[(\hat{a}_{N+1} - a)^2\right] \tag{4.10}$$

if and only if

$$p_{N+1} - \overline{\boldsymbol{p}_N} \notin \left(0, \frac{-2 \cdot \overline{\boldsymbol{p}_N} \cdot \mathrm{Var}(\boldsymbol{p}_N)}{\frac{\mathrm{Var}(\boldsymbol{p}_N)}{N+1} - \overline{\boldsymbol{p}_N}^2}\right). \tag{4.11}$$

Proof. First we will derive the expected squared error of $\hat{a}_N$. Then we will see that for Equation 4.10 to hold we need Condition 4.11.

Using Lemma 4.1.1:

$$\begin{aligned}
\mathbf{E}\left[(\hat{a}_N - a)^2\right] &= \mathbf{E}\left[\left(\overline{\mathbf{d}_N} - \hat{b}_N \cdot \overline{\mathbf{p}_N} - a\right)^2\right], \\
&= \mathbf{E}\left[\left(a + b \cdot \overline{\mathbf{p}_N} - \hat{b}_N \cdot \overline{\mathbf{p}_N} - a\right)^2\right], \\
&= \mathbf{E}\left[\left(b \cdot \overline{\mathbf{p}_N} - \hat{b}_N \cdot \overline{\mathbf{p}_N}\right)^2\right], \\
&= \mathbf{E}\left[\left(\overline{\mathbf{p}_N} \cdot (b - \hat{b}_N)\right)^2\right], \\
&= \overline{\mathbf{p}_N}^2 \cdot \mathbf{E}\left[\left(b - \hat{b}_N\right)^2\right], \\
&= \overline{\mathbf{p}_N}^2 \cdot \mathbf{E}\left[\left(\hat{b}_N - b\right)^2\right],
\end{aligned}$$

filling in Equation 4.7

$$= \frac{\overline{\mathbf{p}_N}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}. \tag{4.12}$$

For the estimator $\hat{a}_{N+1}$ then follows:

$$\mathbf{E}\left[(\hat{a}_{N+1} - a)^2\right] = \frac{\overline{\mathbf{p}_{N+1}}^2 \cdot \sigma^2}{(N+1) \cdot \mathrm{Var}(\mathbf{p}_{N+1})},$$

using Lemma 4.1.2

$$= \frac{\overline{\mathbf{p}_{N+1}}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot (p_{N+1} - \overline{\mathbf{p}_N})^2}. \tag{4.13}$$

We want the expected squared error of the estimation taking $N+1$ data points into account smaller or equal to the expected squared error of the estimation taking N data points into account:

$$\frac{\overline{\mathbf{p}_{N+1}}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot (p_{N+1} - \overline{\mathbf{p}_N})^2} \leq \frac{\overline{\mathbf{p}_N}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}. \tag{4.14}$$

From now on we will set $u = p_{N+1} - \overline{\mathbf{p}_N}$:

$$\frac{\left(\overline{\mathbf{p}_N} + \frac{p_{N+1} - \overline{\mathbf{p}_N}}{N+1}\right)^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot (p_{N+1} - \overline{\mathbf{p}_N})^2} \leq \frac{\overline{\mathbf{p}_N}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}.$$

$$\frac{\left(\overline{\mathbf{p}_N} + \frac{u}{N+1}\right)^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot u^2} \leq \frac{\overline{\mathbf{p}_N}^2 \cdot \sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_N)}.$$

If $u = 0$, then we have an equality, and the estimation based on $N+1$ data points is as good as the estimation based on N data points. For u non-zero, we get:

$$\frac{\left(\overline{\mathbf{p}_N} + \frac{u}{N+1}\right)^2 \cdot \sigma^2 \cdot N \cdot \mathrm{Var}(\mathbf{p}_N) - \overline{\mathbf{p}_N}^2 \cdot \sigma^2 \cdot \left(N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot u^2\right)}{\left(N \cdot \mathrm{Var}(\mathbf{p}_N) + \frac{N}{N+1} \cdot u^2\right) \cdot N \cdot \mathrm{Var}(\mathbf{p}_N)} \leq 0$$

Since the denominator is strictly positive, the numerator has to be negative or 0. Rewriting gives us:

$$u^2 \cdot \left(\frac{\sigma^2 \cdot N}{(N+1)^2} \cdot \mathrm{Var}(\mathbf{p}_N) - \frac{\sigma^2 \cdot N}{N+1} \cdot \overline{\mathbf{p}_N}^2\right) + u \cdot \left(\frac{\sigma^2 \cdot N}{N+1} \cdot 2\overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N)\right)$$
$$+\sigma^2 \cdot N \cdot \overline{\mathbf{p}_N}^2 \cdot \mathrm{Var}(\mathbf{p}_N) - \sigma^2 \cdot N \cdot \overline{\mathbf{p}_N}^2 \cdot \mathrm{Var}(\mathbf{p}_N) \leq 0. \tag{4.15}$$

Equation 4.15 is of the form $x \cdot u^2 + y \cdot u \leq 0$, with $x = \frac{\sigma^2 \cdot N}{(N+1)^2} \cdot \mathrm{Var}(\mathbf{p}_N) - \frac{\sigma^2 \cdot N}{N+1} \cdot \overline{\mathbf{p}_N}^2$ and $y = \frac{\sigma^2 \cdot N}{N+1} \cdot 2\overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N)$. Since y is strictly positive, we are left with three possible solutions for the equation above.

- If $x = 0$, then $u \leq 0$;
- If $x > 0$, then $\frac{-y}{x} \leq u \leq 0$;
- If $x < 0$, then $u \leq 0$ or $u \geq \frac{-y}{x}$.

We will check if the third solution holds:

$$\frac{\sigma^2 \cdot N}{(N+1)^2} \cdot \mathrm{Var}(\mathbf{p}_N) - \frac{\sigma^2 \cdot N}{N+1} \cdot \overline{\mathbf{p}_N} \overset{?}{<} 0,$$

$$\frac{\mathrm{Var}(\mathbf{p}_N)}{N+1} - \overline{\mathbf{p}_N}^2 \overset{?}{<} 0,$$

$$\frac{\overline{\mathbf{p}_N^2} - \overline{\mathbf{p}_N}^2}{N+1} \overset{?}{<} \overline{\mathbf{p}_N}^2,$$

$$\frac{\overline{\mathbf{p}_N^2}}{N+1} \overset{?}{<} \frac{N+2}{N+1} \cdot \overline{\mathbf{p}_N}^2,$$

$$\overline{\mathbf{p}_N^2} \overset{?}{<} (N+2) \cdot \overline{\mathbf{p}_N}^2,$$

$$\frac{\sum_{i=1}^N p_i^2}{N} \overset{?}{<} (N+2) \cdot \left(\frac{\sum_{i=1}^N p_i}{N}\right)^2,$$

$$\frac{\sum_{i=1}^N p_i^2}{N} \overset{?}{<} \frac{N+2}{N^2} \cdot \left(\sum_{i=1}^N p_i^2 + \sum_{i \neq j}\sum p_i \cdot p_j\right),$$

$$\frac{\sum_{i=1}^{N} p_i^2}{N} < \frac{\sum_{i=1}^{N} p_i^2}{N} + \frac{2}{N^2} \cdot \sum_{i=1}^{N} p_i^2 + \frac{N+2}{N^2} \cdot \sum\sum_{i \neq j} p_i \cdot p_j. \tag{4.16}$$

x is indeed smaller than 0, since we assumed that we have at least two different prices and $\forall i, p_i \geq 0$. So, now we know that if $u \leq 0$ or $u \geq \frac{-y}{x}$, then our estimation based on $N+1$ data points is as good as or better than the estimation based on N points. The taboo interval for u is $(0, \frac{-y}{x})$ and $\frac{-y}{x}$ is as following:

$$\begin{aligned}
\frac{-y}{x} &= \frac{-2\sigma^2 \cdot N \cdot \overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N)}{N+1} \Bigg/ \frac{\sigma^2 \cdot N \cdot \mathrm{Var}(\mathbf{p}_N) - \sigma^2 \cdot N \cdot (N+1) \cdot \overline{\mathbf{p}_N}^2}{(N+1)^2} \\
&= \frac{-2\sigma^2 \cdot N \cdot \overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N) \cdot (N+1)^2}{(N+1) \cdot \left(\sigma^2 \cdot N \cdot \mathrm{Var}(\mathbf{p}_N) - \sigma^2 \cdot N \cdot (N+1) \cdot \overline{\mathbf{p}_N}^2\right)} \\
&= \frac{-2 \cdot \overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N) \cdot (N+1)}{\mathrm{Var}(\mathbf{p}_N) - (N+1) \cdot \overline{\mathbf{p}_N}^2} \\
&= \frac{-2 \cdot \overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N)}{\frac{\mathrm{Var}(\mathbf{p}_N)}{N+1} - \overline{\mathbf{p}_N}^2}
\end{aligned} \tag{4.17}$$

For large N, we obtain:

$$\frac{-y}{x} = \frac{-2 \cdot \overline{\mathbf{p}_N} \cdot \mathrm{Var}(\mathbf{p}_N)}{-\overline{\mathbf{p}_N}^2} = \frac{2 \cdot \mathrm{Var}(\mathbf{p}_N)}{\overline{\mathbf{p}_N}}. \tag{4.18}$$

Equation 4.18 shows us that, unless $\mathrm{Var}(\mathbf{p}_N)$ goes to zero for N large, the taboo interval for u does not become smaller and smaller as N grows and does not become empty. ■

Theorem 4.1.4 shows us that taking more data into account when estimating a, we do not always get a better estimate. It depends on what value the added price has, and whether it falls in the taboo interval. When we have a large variance in the prices and a relatively small mean price, this interval can become quite large.

4.2 Change point model

In case a change point might occur in our time horizon T, we would expect that as soon we are entering a new regime, we want to discard our old data and only take the data into account that is generated after t_{cp}. We could find t_{cp} with a change point detection algorithm, as described in Section 2.3. However, we are not looking for the exact moment at which the change point occurs, but for our optimal choice of N_t. Furthermore, with a change point detection algorithm we can only detect the change point after it takes place. We would rather know beforehand what N_t to choose.

In the following theorem we will see that when estimating parameter b, it depends on how large the change is, whether it is useful to take more, 'old', data into account or not. We again assume that we have a set of deterministic prices $\mathbf{p}$. Furthermore we assume that ϵ is a random variable (i.i.d.), with $\mathbf{E}[\epsilon] = 0$ and $\mathrm{Var}(\epsilon) = \sigma^2, \sigma^2 < \infty$. We assume $T = 2N, N \geq 2$. From $t = 1$ to N, we assume a and b constant. From $t = N+1$ to $2N$ we also assume that a and b are constant, but not equal to a and b in the first N time periods. We denote this a and b with respectively a' and b'. Let $\hat{b}_{(N+1:2N)}$ denote the estimator of b', based on the N last data points and let $\hat{b}_{(N:2N)}$ denote the estimator of b', based on the last $N+1$ data points. This means that in the last case, we take one extra, 'old', data point into account when estimating b'.

Theorem 4.2.1. *The expected squared error of the estimated $b, \hat{b}_{N+1:2N}$, based on N observations is greater or equal to the expected squared error of the estimator $\hat{b}_{N:2N}$:*

$$\hat{b}_{(N+1:2N)} \geq \hat{b}_{(N:2N)}. \tag{4.19}$$

if and only if

$$\begin{aligned}
&\mathbf{E}\left[((b-b') \cdot p_N + (a-a'))^2\right] \\
&\leq \left(\frac{\sigma^2}{N \cdot \mathrm{Var}(\boldsymbol{p}_{(N+1:2N)})} - \frac{\sigma^2}{(N+1) \cdot \mathrm{Var}(\boldsymbol{p}_{(N:2N)})}\right) \cdot \frac{((N+1) \cdot \mathrm{Var}(\boldsymbol{p}_{N:2N}))^2}{\theta^2}.
\end{aligned} \tag{4.20}$$

Proof. For the expected squared error of $\hat{b}_{(N+1:2N)}$ we have:

$$\mathbf{E}\left[\left(\hat{b}_{(N+1:2N)} - b'\right)^2\right] = \frac{\sigma^2}{N \cdot \mathrm{Var}(\mathbf{p}_{(N+1:2N)})}, \tag{4.21}$$

since we can use the result of the proof of Theorem 4.1.3, because b' is constant. For the expected squared error of the estimator $\hat{b}_{(N:2N)}$ we have:

$$\mathbf{E}\left[\left(\hat{b}_{(N:2N)} - b'\right)^2\right] = \mathbf{E}\left[\left(\frac{\sum_{i=N}^{2N} p_i \cdot d_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} d_i}{\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2} - b'\right)^2\right]. \tag{4.22}$$

We will use a new function for d_i, since it depends on the index i whether to use a or a' and b or b':

$$d_i = a' + b'p_i + \epsilon_i + (b - b')q_i + (a - a')r_i \tag{4.23}$$

with

$$q_i = \begin{cases} p_i & i < t_{cp} \\ 0 & i \geq t_{cp} \end{cases}$$

and

$$r_i = \begin{cases} 1 & i < t_{cp} \\ 0 & t \geq t_{cp}. \end{cases}$$

Then we have the following for $\hat{b}_{(N:2N)}$, filling in Equation 4.23:

$$\begin{aligned}
\hat{b}_{(N:2N)} \cdot \left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right) &= \sum_{i=N}^{2N} p_i \cdot (a' + b' \cdot p_i + \epsilon_i + (b - b') \cdot q_i + (a - a') \cdot r_i) \\
&\quad - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} (a' + b' \cdot p_i + \epsilon_i + (b - b') \cdot q_i + (a - a') \cdot r_i), \\
&= a' \cdot \sum_{i=N}^{2N} p_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} a' \\
&\quad + b' \cdot \left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right) \\
&\quad + \sum_{i=N}^{2N} p_i \cdot \epsilon_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} \epsilon_i \\
&\quad + (b - b') \cdot \left(\sum_{i=N}^{2N} p_i \cdot q_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot q_i\right) \\
&\quad + (a - a') \cdot \left(\sum_{i=N}^{2N} p_i \cdot r_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot r_i\right).
\end{aligned} \tag{4.24}$$

We can easily see that the first part of Equation 4.24 is zero:

$$a' \cdot \sum_{i=N}^{2N} p_i - \frac{\sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} a'}{N+1} = a' \cdot \sum_{i=N}^{2N} p_i - \frac{N+1}{N+1} \cdot a' \cdot \sum_{i=N}^{2N} p_i = 0.$$

For the last two parts of Equation 4.24 we have:

$$(b-b')\cdot\left(\sum_{i=N}^{2N}p_i\cdot q_i-\frac{1}{N+1}\cdot\sum_{i=N}^{2N}p_i\cdot q_i\right)$$

$$=(b-b')\cdot\left(p_N\cdot q_N+\sum_{i=N+1}^{2N}p_i\cdot q_i-\frac{1}{N+1}\cdot\left(\left(p_N+\sum_{i=N+1}^{2N}p_i\right)\left(q_N+\sum_{i=N+1}^{2N}q_i\right)\right)\right)$$

$$=(b-b')\cdot\left(p_N^2+0-\frac{1}{N+1}\left(p_N\cdot q_N+p_N\cdot\sum_{i=N+1}^{2N}q_i+q_N\cdot\sum_{i=N+1}^{2N}p_i+\sum_{i=N+1}^{2N}p_i\cdot\sum_{i=N+1}^{2N}q_i\right)\right)$$

$$=(b-b')\cdot\left(p_N^2-\frac{1}{N+1}\cdot\left(p_N^2+0+p_N\cdot\sum_{i=N+1}^{2N}p_i+0\right)\right)$$

$$=(b-b')\cdot\left(p_N^2-\frac{p_N^2}{N+1}-\frac{p_N}{N+1}\cdot\sum_{i=N+1}^{2N}p_i\right)$$

$$=(b-b')\cdot\left(\frac{N}{N+1}\cdot p_N^2-\frac{N}{N+1}\cdot p_N\cdot\overline{\mathbf{p}}_{(N+1:2N)}\right)$$

$$=(b-b')\frac{N}{N+1}\cdot p_N\cdot(p_N-\overline{\mathbf{p}}_{N+1:2N})$$

and

$$(a-a')\cdot\left(\sum_{i=N}^{2N}p_i\cdot r_i-\frac{1}{N+1}\cdot\sum_{i=N}^{2N}p_i\cdot r_i\right)$$

$$=(a-a')\cdot\left(p_N\cdot r_N+\sum_{i=N+1}^{2N}p_i\cdot r_i\frac{1}{N+1}\cdot\left(p_N\cdot r_N+p_N\cdot\sum_{i=N+1}^{2N}r_i+r_N\cdot\sum_{i=N+1}^{2N}p_i+\sum_{i=N+1}^{2N}p_i\cdot\sum_{i=N+1}^{2N}r_i\right)\right)$$

$$=(a-a')\cdot\left(p_N+0-\frac{1}{N+1}\cdot\left(p_N+0+\sum_{i=N+1}^{2N}p_i+0\right)\right),$$

analogous to the $(b-b')$ part:

$$=(a-a')\cdot\frac{N}{N+1}\cdot\left(p_N-\overline{\mathbf{p}}_{(N+1:2N)}\right).$$

We can fill this in in Equations 4.22 and 4.24:

$$\mathbf{E}\left[\left(\hat{b}_{N+1}-b'\right)^2\right]=\mathbf{E}\Bigg[\Bigg(\frac{0+b'\cdot\left(\sum_{i=N}^{2N}p_i^2-\frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N}p_i\right)^2\right)+\sum_{i=N}^{2N}p_i\cdot\epsilon_i-\frac{1}{N+1}\cdot\sum_{i=N}^{2N}p_i\cdot\epsilon_i}{\sum_{i=N}^{2N}p_i^2-\frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N}p_i\right)^2}$$

$$+\frac{(b-b')\cdot\frac{N}{N+1}\cdot p_N\cdot(p_N-\overline{\mathbf{p}}_{N+1:2N})+(a-a')\cdot\frac{N}{N+1}\cdot\left(p_N-\overline{\mathbf{p}}_{(N+1:2N)}\right)}{\sum_{i=N}^{2N}p_i^2-\frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N}p_i\right)^2}-b'\Bigg)^2\Bigg]$$

$$=\mathbf{E}\Bigg[\Bigg(\frac{\sum_{i=N}^{2N}p_i\cdot\epsilon_i-\frac{1}{N+1}\sum_{i=N}^{2N}p_i\cdot\epsilon_i+(b-b')\cdot\frac{N}{N+1}\cdot p_N\cdot(p_N-\overline{\mathbf{p}}_{N+1:2N})}{\sum_{i=N}^{2N}p_i^2-\frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N}p_i\right)^2}$$

$$+\frac{(a-a')\cdot\frac{N}{N+1}\cdot\left(p_N-\overline{\mathbf{p}}_{(N+1:2N)}\right)}{\sum_{i=N}^{2N}p_i^2-\frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N}p_i\right)^2}\Bigg)^2\Bigg]$$

We now set

$$x = \sum_{i=N}^{2N} p_i \cdot \epsilon_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} \epsilon_i, \tag{4.25}$$

$$y = (b - b') \cdot \frac{N}{N+1} \cdot p_N \cdot (p_N - \overline{\mathbf{p}}_{N+1:2N}), \tag{4.26}$$

$$z = (a - a') \cdot \frac{N}{N+1} \cdot \left(p_N - \overline{\mathbf{p}}_{(N+1:2N)}\right). \tag{4.27}$$

Then we have

$$\begin{aligned}
\mathbf{E}\left[\left(\frac{x+y+z}{\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2}\right)^2\right] &= \mathbf{E}\left[\frac{x^2 + (y+z)^2 + 2 \cdot x \cdot (y+z)}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right] \\
&= \mathbf{E}\left[\frac{x^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right] \\
&+ \mathbf{E}\left[\frac{(y+z)^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right] \\
&+ \mathbf{E}\left[\frac{2 \cdot x \cdot (y+z)}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1} \cdot \left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right].
\end{aligned} \tag{4.28}$$

The third part of Equation 4.28 is equal to zero, since:

$$\begin{aligned}
\mathbf{E}[x] &= \mathbf{E}\left[\sum_{i=N}^{2N} p_i \cdot \epsilon_i - \frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} \epsilon_i\right] \\
&= \mathbf{E}\left[\sum_{i=N}^{2N} p_i \cdot \epsilon_i\right] - \mathbf{E}\left[\frac{1}{N+1} \cdot \sum_{i=N}^{2N} p_i \cdot \sum_{i=N}^{2N} \epsilon_i\right] \\
&= \mathbf{E}\left[\sum_{i=N}^{2N} p_i \cdot \epsilon_i\right] - \overline{\mathbf{p}}_{(N:2N)} \cdot \mathbf{E}\left[\sum_{i=N}^{2N} \epsilon_i\right] \\
&= \sum_{i=N}^{2N} \mathbf{E}\left[p_i \epsilon_i\right] - \overline{\mathbf{p}}_{(N:2N)} \cdot \sum_{i=N}^{2N} \mathbf{E}\left[\epsilon_i\right] \\
&= \sum_{i=N}^{2N} \mathbf{E}[p_i] \cdot \mathbf{E}[\epsilon_i] - \overline{p}_{(N:2N)} \cdot \sum_{i=N}^{2N} 0 \\
&= \sum_{i=N}^{2N} \mathbf{E}[p_i] \cdot 0 \\
&= 0.
\end{aligned} \tag{4.29}$$

Now we have two parts left, and only in the second part of Equation 4.28 we have to take into account the change that has occurred. The first part actually gives us the same result as the estimation in a constant model:

$$\mathbf{E}\left[\frac{x^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right] = \mathbf{E}\left[\frac{\left(\sum_{i=N}^{2N} p_i\cdot\epsilon_i - \frac{1}{N+1}\cdot\sum_{i=N}^{2N} p_i\cdot\sum_{i=N}^{2N}\epsilon_i\right)^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right]$$

$$= \mathbf{E}\left[\left(\frac{\sum_{i=N}^{2N} p_i\cdot\epsilon_i - \frac{1}{N+1}\cdot\sum_{i=N}^{2N} p_i\cdot\sum_{i=N}^{2N}\epsilon_i}{\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2}\right)^2\right]$$

using the proof of Lemma 4.1.1:

$$= \mathbf{E}\left[\left(\frac{\mathrm{Cov}(\mathbf{p}_{(N:2N)}, \boldsymbol{\epsilon}_{(N:2N)})}{\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)^2\right]$$

using the proof of Theorem 4.1.3:

$$= \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})} \tag{4.30}$$

We have obtained the expected squared of the estimator $\hat{b}_{N:2N}$, using $N+1$ data points, with N points from the current regime and one point from the old regime:

$$\mathbf{E}\left[\left(\hat{b}_{(N:2N)} - b'\right)^2\right] = \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})} + \mathbf{E}\left[\frac{(y+z)^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right]. \tag{4.31}$$

Since the first part of Equation 4.31 is smaller or equal to the expected squared estimation error of $b_{(N+1:2N)}$, it depends on the second part if the estimation of b', based on $N+1$ data points, is better than the estimation of b', based on N data points. If y and z are zero, then we have no change, and hence again a constant model. If $y+z=0$, then we also have the same result as in a constant market. Both y and z are continuous in respectively $b-b'$ and $a-a'$, hence $y+z$ is continuous. So, if $y+z$ is sufficiently small and the last part of Equation 4.31 is small enough, then the expected squared error of $\hat{b}_{(N:2N)}$ is smaller (or equal) to the expected squared error of $\hat{b}_{(N+1:2N)}$. The condition that needs to hold for the change to be small enough is:

$$\mathbf{E}\left[\frac{(y+z)^2}{\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2\right)^2}\right] \le \frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})} - \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}$$

$$\mathbf{E}\left[(y+z)^2\right] \le \left(\frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})} - \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)\cdot\left(\sum_{i=N}^{2N} p_i^2 - \frac{1}{N+1}\cdot\left(\sum_{i=N}^{2N} p_i\right)^2\right)^2$$

$$= \left(\frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})} - \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)\cdot\left((N+1)\cdot\mathrm{Var}(\mathbf{p}_{N:2N})\right)^2$$

We had set $y = (b-b')\cdot\frac{N}{N+1}\cdot p_N\cdot\left(p_N - \overline{\mathbf{p}}_{N+1:2N}\right)$ and $z = (a-a')\cdot\frac{N}{N+1}\cdot\left(p_N - \overline{\mathbf{p}}_{(N+1:2N)}\right)$. We will go back to this notation and use $\theta = \frac{N}{N+1}\cdot\left(p_N - \overline{\mathbf{p}}_{(N+1:2N)}\right)$:

$$\mathbf{E}\left[\left((b-b')\cdot p_N\cdot\theta + (a-a')\cdot\theta\right)^2\right]$$
$$\le \left(\frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})} - \frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)\cdot\left((N+1)\cdot\mathrm{Var}(\mathbf{p}_{N:2N})\right)^2$$

$$\mathbf{E}\left[((b-b')\cdot p_N+(a-a'))^2\right]\cdot\theta^2$$
$$\leq\left(\frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})}-\frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)\cdot((N+1)\cdot\mathrm{Var}(\mathbf{p}_{N:2N}))^2$$
$$\mathbf{E}\left[((b-b')\cdot p_N+(a-a'))^2\right]$$
$$\leq\left(\frac{\sigma^2}{N\cdot\mathrm{Var}(\mathbf{p}_{(N+1:2N)})}-\frac{\sigma^2}{(N+1)\cdot\mathrm{Var}(\mathbf{p}_{(N:2N)})}\right)\cdot\frac{((N+1)\cdot\mathrm{Var}\left(\mathbf{p}_{N:2N}\right))^2}{\theta^2} \tag{4.32}$$

We have now a condition on the change to determine whether adding data improves the estimation. The right-hand side of Equation 4.32 is positive, and therefore it is possible to either have a change small enough to fulfil the condition or a change big enough to exceed it.

■

This results means that taking into account an extra data point, that contains 'old' information, gives a better approximation of b' than the approximation that leaves out this added data point, provided the change in a and b is small enough. If the change becomes too large, including 'old' data points leads to a worse approximation of b'.

4.3 Random walk model

In the random walk model it is possible to have a small change in the model parameters a_t and b_t every time step. As shown in Section 4.2, when a change occurs that is small enough, we do not immediately discard the old data. However, if we have drifted too far away from our initial a and b, the mutual change becomes too large and the old data is obsolete. This means that in a random walk model, it depends on the changes that have occurred how to choose N_t. So, it could happen that a_t and b_t have a different value for every t, but the overall change is still small enough that taking every data point into account in the estimation is still optimal.

4.4 Solution

In the previous sections we have seen that in the case of deterministic prices, it is not just as simple as our intuition tells us. Already in the constant model we showed that when estimating the intercept, adding data points does not necessary result in a better estimation. If a big change occurs, or we have slowly drifted too far away from our original parameters, then we need to forget the 'old' pre-change data. However, due to our pricing policy, we do not set our prices deterministically. Therefore we cannot simply take over these results. This means that we should test all possible sequences of N_t and see which specific sequence is optimal. But if we already have a large amount of data points and/or we want to compute over a large horizon T, this becomes an intractable problem, both analytically and computationally. That is why we now introduce five subsequences, for which we are going to evaluate their performance in different scenarios. The performance criterion is described in Subsection 4.4.2.

4.4.1 Subsequences of N_t

From now on we will look at five different subsequences of N_t.

1. Firstly, we will take all available data into account: $N_t = t-1, \forall t$.
2. The second subsequence we use is a sliding window. We choose a fixed $N_t = N_{\text{fix}}, \forall t$ and $N_{\text{fix}} \geq 2$. If there are no N_t data points available, we take $N_t = t-1$.
3. In the third subsequence we also use a sliding window, but the window grows larger, as we have more data available. We take a fixed percentage $\%_{\text{fix}}$ of the available data into account: $N_t = \max(2, \lfloor \%_{\text{fix}} \cdot t \rceil)$. We take the maximum of 2 and $\lfloor \%_{\text{fix}} \cdot t \rceil$, because we need at least two data points to be able to estimate a_t and b_t.

4. Fourth, we will choose N_t such that our expected revenue in the next period is maximized. We will calculate $\hat{a}_{t-1}$ and $\hat{b}_{t-1}$ for all possible N: $N = 2, 3, \dots, t-1$ and determine $p_{(t,N_t)}$ via the pricing policy described in Chapter 3. Then we choose N_t such that:

$$N_t = \arg\max_{2 \leq N \leq t-1} p_{(t,N_t)} \cdot (\hat{a}_{(t-1,N_t)} + \hat{b}_{(t-1,N_t)} \cdot p_{(t,N_t)}).$$

5. Lastly, we will choose N_t such that the expected squared prediction error of the last k periods is minimized. We can set k arbitrarily or evaluate all possible values of k and choose the optimal one.

$$N_t = \arg\min_{2 \leq N \leq t-2} \sum_{\tau=t-k}^{t-1} (\hat{a}_{(\tau-1,\min(N,\tau-1))} + \hat{b}_{(\tau-1,\min(N,\tau-1))} \cdot p_\tau - d_\tau)^2$$

This means that we are choosing the N_t that would have been the optimal choice in the last k periods.

So, concluding, the different subsequences we are going to evaluate are:

$$N_t = \begin{cases} t-1 & \forall t, \\ \min(N_{\text{fix}}, t-1) & \forall t \\ \max(2, \lfloor \%_{\text{fix}} \cdot t \rceil) & \forall t, \\ \arg\max_{2 \leq N \leq t-1} p_{(t,N)} \cdot (\hat{a}_{(t-1,N)} + \hat{b}_{(t-1,N)} \cdot p_{(t,N)}) & \forall t, \\ \arg\min_{2 \leq N \leq t-2} \sum_{\tau=t-k}^{t-1} (\hat{a}_{(\tau,\min(N,\tau-1))} + \hat{b}_{(\tau,\min(N,\tau-1))} \cdot p_\tau - d_\tau)^2 & \forall t. \end{cases} \tag{4.33}$$

4.4.2 Performance measure

To be able to compare the performance of the different subsequences, we need a measure to do so. We will use the cumulative regret percentage as a measure. Regret is the difference between the expected profit for a clairvoyant and the expected profit in the used policy. The cumulative regret percentage is calculated as follows:

$$R(T) = \frac{\sum_{i=1}^{T} p_i \cdot (a_i + b_i \cdot p_i) - p_i^* \cdot (a_i + b_i \cdot p_i^*)}{\sum_{i=1}^{T} p_i^* \cdot (a_i + b_i \cdot p_i^*)} \cdot 100, \tag{4.34}$$

with $p_i^* = -a_i/(2 \cdot b_i)$.

Now we have already scaled our problem down by reducing the number of possible sequences of N_t. Nevertheless, the problem at hand is still too complex to be able to compute the solution analytically. There are still a lot of possible sequences that we would have to evaluate, because we still have to find the optimal values of N_{fix}, $\%_{\text{fix}}$ and k. Moreover, since every new price depends on the previous prices and corresponding demand, the calculations become more and more involved. We have to choose at least two starting prices. These can be deterministic, or drawn from a certain probability distribution. These starting prices influence all prices coming afterwards. Furthermore, we have to choose a value for c, to set the minimum variance in our prices. This c influences our prices and we can even find the optimal value for c, but finding this optimal value lies outside the scope of this research. We will just set c to a certain value.

Since it is hardly possible to solve the problem analytically, it will be simulated. In Chapter 5 various examples are shown for different market-models and scenarios. Considering that we cannot know how a market is going to behave, we have to make assumptions about this behaviour. Literature shows that the more we know about how the market is going to react, the better we can set our prices. Keskin and Zeevi (2013) use a budget of variation to describe the change. The lower the budget, the lower the bound on the regret and the better the performance of a policy. Below we will describe what the input and the output is for our simulations.

4.4.3 Simulations

All simulations are performed in MATLAB. From the scenario given as input we draw 1000 instances. These are evaluated in all the subsequences. First we evaluate Subsequences 2, 3 and 5 for every possible value of N_{fix}, $\%_{\text{fix}}$ and k. Due to computation time we will only evaluate 100 instances to determine the optimal choice of k. For the optimal k, k^* we run it again for the 1000 instances of the scenario. We will calculate the mean cumulative regret percentage at $t = T$ for every value and from this we can deduce the optimal values of N_{fix}, $\%_{\text{fix}}$ and k. Then we compare these optimized subsequences with the remaining subsequences, where respectively all data is taken into account (Subsequence 1) and where we maximize the expected revenue for the next period (Subsequence 4). We will again calculate the mean cumulative regret percentage at $t = T$ and then we can conclude which subsequence is optimal regarding our input.

Input

As input for our simulations we need the following:

- Possible scenarios for a_t and b_t with a corresponding probability distribution. For every $t \in \{1, 2, \ldots, T\}$ we have to state the possible values for a_t and b_t and the probability that goes with them. An example: suppose we know that there will be one change point between $t = 10$ and $t = 30$. We do not know our begin values of a_t and b_t, and we do not know how large the change is. Then we can state for our first period a probability distribution. Until $t = 10$, we will use the same value of a_t and b_t, as we drawn from our distribution in $t = 1$. From $t = 10$ to $t = 30$, we again have to state a probability distribution for the possible values of a_t and b_t. When a change occurs, we then set all a_t and b_t after t_{cp} to the new values. If there is also a probability distribution on the time at which our change point occurs, we can incorporate that distribution into the distribution of a_t and b_t.

- Starting prices: we need at least two starting prices, since we need at least two data points for our estimation of a_t and b_t. We can set these prices deterministically or provide a probability distribution for possible values. For these starting price we can calculate the associated demand.

- Probability distribution for ϵ_t, which are i.i.d.. From this distribution we draw a value for every t. The mean of ϵ_t must be 0 and the variance $\sigma(\epsilon) < \infty$.

- Value for c and α: when $c = 0$ no experimentation will take place. If this value is set too high, the prices will not converge to the optimal price, but they will alternate around this optimum In all simulations $\alpha = 0.50$, because it is proven in Den Boer and Zwart (2013) that $\alpha = 0.5$ is the optimal choice in case of normally distributed demand and a linear demand function.

- Time horizon T

- Values for p_l and p_h

Output

The program gives as output the mean cumulative regret percentage for every subsequence. The subsequence with the lowest mean is regarded as the optimal subsequence. From this we can easily deduce our optimal sequence of N_t. For cases 4 and 5 we will take the mean for every period of the computed sequences in every iteration, this is calculated automatically in the MATLAB program. For all results plots can be made.

Chapter 5

Numerical illustrations

In this chapter we will simulate five scenarios and give the corresponding results. The first scenario simulates a constant model, so a and b are constant throughout the entire time horizon T. The second and third scenario simulate the change point model, with respectively a small and a large change. In this case we have $a, \quad b \quad \forall t \leq t_{cp}$ and $a', \quad b' \quad \forall t \geq t_{cp}$. The last two scenarios simulate a random walk model. In Scenario 4 only a small change is possible for every time step in the model parameters a_t and b_t. For Scenario 5 this change can become larger.

We will compare the performance of the five subsequences in each scenario, as described in Chapter 4:

$$N_t = \begin{cases} t-1 & \forall t \\ \min(N_{\text{fix}}, t-1) & \forall t \\ \max(2, \lfloor \%_{\text{fix}} \cdot t \rfloor) & \forall t \\ \arg\max_{2 \leq N \leq t-1} p_{(t,N)} \cdot (\hat{a}_{(t-1,N)} + \hat{b}_{(t-1,N)} \cdot p_{(t,N)}) & \forall t \\ \arg\min_{2 \leq N \leq t-2} \sum_{\tau=t-k}^{t-1} (\hat{a}_{(\tau,\min(N,\tau-1))} + \hat{b}_{(\tau,\min(N,\tau-1))} \cdot p_\tau - d_\tau)^2 & \forall t \end{cases} \tag{5.1}$$

The following input holds for all scenarios:

- We set two deterministic starting prices: $p_1 = 3$ and $p_2 = 7$.
- The probability distribution for $\epsilon_t, \forall t$ is a Normal distribution with mean 0 and standard deviation: $\sigma = \min_t((a_t + b_t \cdot p_h)/3)$.
- Value for α: $\alpha = 0.50$. This value is shown to be optimal in Den Boer and Zwart (2013), since we have a linear dependence between price and demand an the demand is normally distributed.
- $c = 1.5$, unless specified otherwise.
- Time horizon $T = 100$.

5.1 Scenario 1: Constant model

In the first scenario we simulate the pricing algorithm in a constant market. For every iteration we draw a value of a and b from the intervals given below. These values remain constant in every iteration, for the entire time horizon T. The input is as follows:

- $a \in \mathcal{U}[1, 20]$
- $b \in \mathcal{U}[-a/6, -a/16]$. This ensures that the optimal price p^* lies between 3 and 8.
- $p_l = 1$ and $p_h = 10$

In Figure 5.1 the values of a and b of one realisation are plotted and we can see that a and b are indeed constant. In the rightmost plot the demand is plotted against the price, the asterisk denotes the optimal price.

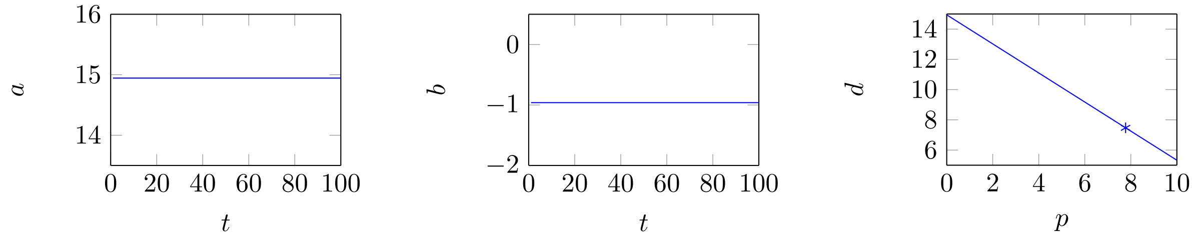


Figure 5.1: Values of one realisation of Scenario 1

Results

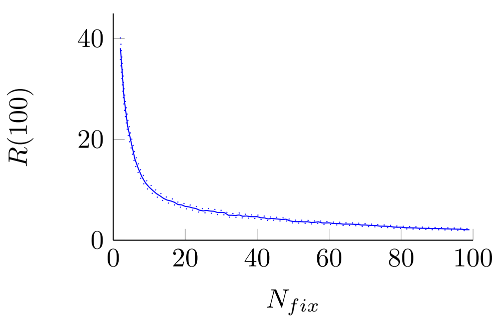
Table 5.1 gives the results for all the five subsequences, including the confidence intervals (CI). For Subsequences 2, 3 and 5, the optimal value for respectively N_{fix}, $\%_{\text{fix}}$ and k are given. Since the optimal values of N_{fix} and $\%_{\text{fix}}$ are such that all data is taken into account, the results for these two subsequences and the subsequence that evaluates all data (number 1) are the same. We can see that taking all data into account is optimal in this scenario. Table 5.2 shows the sample statistics of the parameters. In Figures 5.2a and 5.2b we can clearly see that the more data points we use in the estimation of a and b, the less regret we have. The confidence intervals are also given in these plots.

Subsequence	Optimal Value	$R(100)$	CI for $R(100)$
1	-	2.09	[1.87;2.30]
2	$N^*_{\text{fix}} = 99$	2.09	[1.87;2.30]
3	$\%^*_{\text{fix}} = 100$	2.09	[1.87;2.30]
4	-	38.45	[36.42;40.48]
5	$k^* = 52$	8.05	[7.50;8.61]

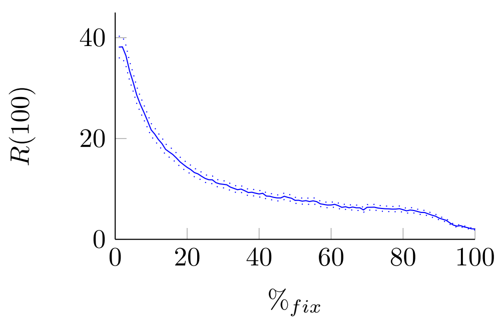
Table 5.1: Scenario 1: Average cumulative regret percentage at $t = 100$ and its confidence interval

	a	b	p^*
Mean	10.34	-1.16	4.76
Standard deviation	5.48	0.70	1.36
Maximum	19.91	-0.08	7.98
Minimum	1.10	-3.18	3.01

Table 5.2: Sample statistics of parameters for Scenario 1



(a) Subsequence 2: The mean cumulative regret percentage at $t = 100$ as a function of N_{fix}



(b) Subsequence 3: The mean cumulative regret percentage at $t = 100$ as a function of $\%_{\text{fix}}$

Figure 5.2: Scenario 1

Results for different values of c

The above simulations are also run for two different values of c, namely $c = 0$ and $c = 3$. If $c = 0$, no learning takes place and we have a myopic pricing policy. In Table 5.3 we can see the results of these simulations. It can be seen that almost all values of $R(100)$ are higher than the simulations run with $c = 1.5$, except for $c = 0$ in Subsequences 4 and 5. A paired sample t-test comparing the results in case of $c = 1.5$ and $c = 0$, on all values of $R(100)$ for Subsequence 1, shows that there is no significant difference (5% significance level). There is a significant difference for Subsequence 1 in case $c = 3$, compared to both $c = 1.5$ and $c = 0$. These results show that with incorporating learning into the pricing policy, we could improve our results. Furthermore, if the value of c is set too high, too much experimentation with prices takes place and the trade-off between earning and learning is unbalanced.

	$c = 0$			$c = 3$		
Subsequence	Optimal Value	$R(100)$	CI for $R(100)$	Optimal Value	$R(100)$	CI for $R(100)$
1	-	2.11	[1.88 ; 2.35]	-	2.43	[2.24 ; 2.62]
2	$N^*_{\text{fix}} = 99$	2.11	[1.88 ; 2.35]	$N^*_{\text{fix}} = 99$	2.43	[2.24 ; 2.62]
3	$\%^*_{\text{fix}} = 100$	2.11	[1.88;2.35]	$\%^*_{\text{fix}} = 100$	2.43	[2.24 ; 2.62]
4	-	37.68	[35.69 ; 39.67]	-	38.94	[36.90 ; 40.97]
5	$k^* = 61$	7.69	[7.17 ; 8.21]	$k^* = 15$	10.66	[10.02 ; 11.31]

Table 5.3: Scenario 1: Results for $c = 0$ and $c = 3$

5.2 Scenario 2: Change point model with a small change

In the second scenario we simulate the change point model, with a rather small change. For every iteration we draw the values for a, b, a' and b', from the intervals mentioned below. As input we have:

- $a \in \mathcal{U}[1, 20]$
- $a' \in \mathcal{U}[\max(a - 2.5, 1), a + 2.5]$
- $b \in \mathcal{U}[-a/6, -a/16]$. This ensures that the optimal price p^* lies between 3 and 8.
- $b' \in \mathcal{U}\left[\frac{-a}{2(p^*-0.5)}, \frac{-a}{2(p^*+0.5)}\right]$. This ensures that the optimal price p'^* lies in the interval $[p^*-0.5; p^*+0.5]$.
- $p_l = 1$ and $p_h = 10$

The change point can occur between $t = 40$ and $t = 60$, and is drawn uniformly from that interval for each iteration. Before the change point we have a and b as model parameters, after the change point we have a' and b'.

Figure 5.3 shows the values of a, a', b and b' of one realisation. In the right plot the price is plotted against the demand for both regimes. The asterisks denote the optimal prices. It can be seen that the optimal prices p^* and p'^* lie very close together.

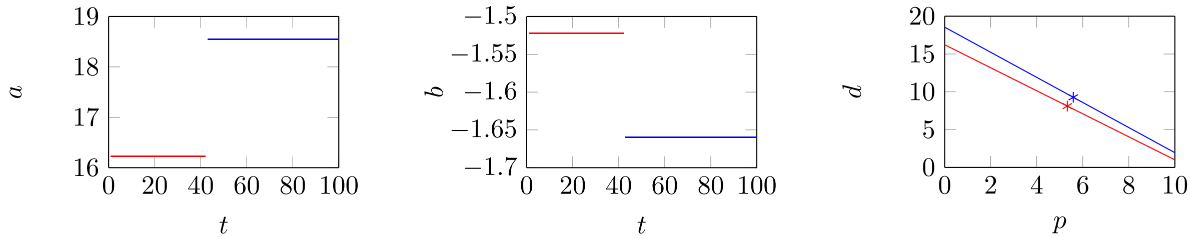


Figure 5.3: Values of one realisation of Scenario 2

Results

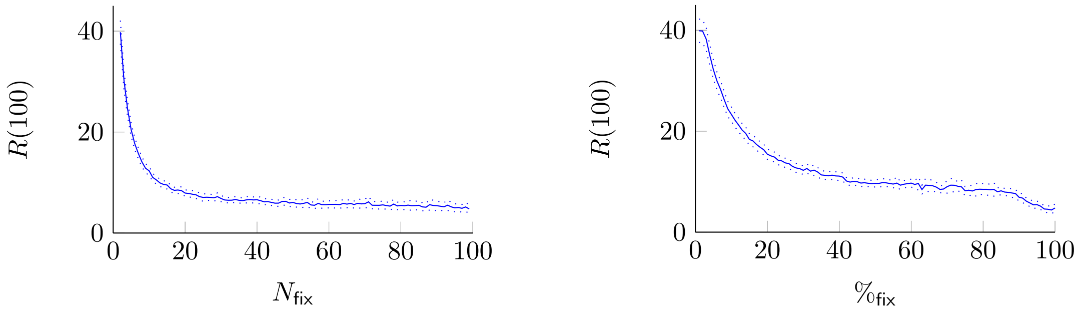
In Table 5.4 the results are given for the simulation of Scenario 2. It can be seen that taking 99% of the data into account is optimal. We have found an optimal value for $\%_{\text{fix}}$ of 99 and not 100. Although, the values of $R(100)$ for respectively $\%_{\text{fix}} = 99$ and $\%_{\text{fix}} = 100$ lie very close together (see Figure 5.4b), the paired sample t-test shows that there is a significant difference (at 5% significance level). We can also see that the cumulative regret percentage at $t = 100$ is about 2.5% higher than in the previous scenario.

Subsequence	Optimal Value	$R(100)$	CI for $R(100)$
1	-	4.81	[4.03 ; 5.58]
2	$N^*_{\text{fix}} = 99$	4.81	[4.03 ; 5.58]
3	$\%^*_{\text{fix}} = 99$	4.42	[3.73 ; 5.12]
4	-	40.87	[38.59 ; 43.15]
5	$k^* = 9$	11.15	[10.41 ; 11.88]

Table 5.4: Scenario 2: Average cumulative regret percentage at $t = 100$ and its confidence interval

	a	a'	b	b'	p^*	p'^*	$\|p^* - p'^*\|$	t_{cp}
Mean	10.49	10.62	-1.20	-1.22	4.73	4.73	0.24	50.02
Standard deviation	5.61	5.56	0.74	0.74	1.33	1.34	0.14	5.86
Maximum	20.00	22.35	-0.08	-0.08	8.00	8.32	0.50	60
Minimum	1.03	1.01	-3.11	-3.65	3.00	2.52	0.0017	40

Table 5.5: Sample statistics of parameters for scenario 2



(a) Subsequence 2: The mean cumulative regret percentage at $t = 100$ as a function of N_{fix}

(b) Subsequence 3: The mean cumulative regret percentage at $t = 100$ as a function of $\%_{\text{fix}}$

Figure 5.4: Scenario 2

5.3 Scenario 3: Change point model with a large change

In Scenario 3 again a change point occurs as in Scenario 2, but now the change is quite large. For every iteration we draw the values for a, b, a' and b', from the intervals mentioned below. As input we have:

- $a \in \mathcal{U}[1, 20]$,
- $a' \in \mathcal{U}[20, 40]$,
- $b \in \mathcal{U}[-a/6, -a/16]$. This ensures that the optimal price p^* lies between 3 and 8.
- $b' \in \mathcal{U}\left[\frac{-a}{2\cdot(p^*+2)}, \frac{-a}{20}\right]$. This ensures that the optimal price p'^* lies between $p^* + 2$ and 10.
- $p_l = 1$ and $p_h = 12$,

The change point is set again between $t = 40$ and $t = 60$, and is drawn uniformly from this interval for each iteration. Before the change point we have a and b as model parameters, after the change point we have a' and b'. The maximum price is set a bit higher, such that the optimal price after the change point does not lie too close to it.
Figure 5.5 gives the values of a, a', b and b' of one realisation. In the right plot the price is plotted against the demand for both regimes. The asterisks denote the optimal prices. It can be seen that the optimal prices p^* and p'^* are much further apart than in the previous scenario.

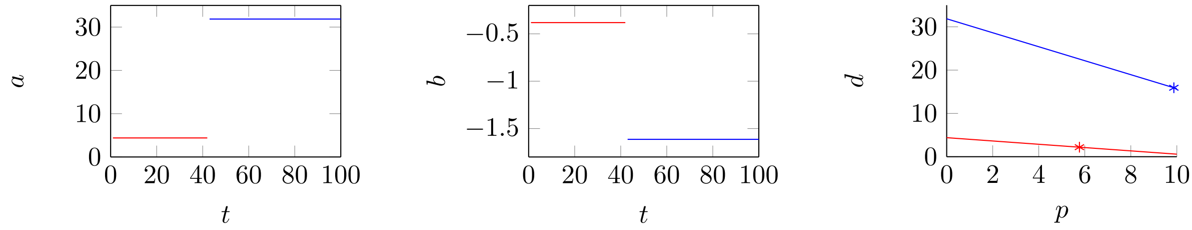


Figure 5.5: Values of one realisation of Scenario 3

Results

Now we can clearly see in Table 5.6 that taking into account all data is not optimal any more, but actually gives the worst result. The reason of this is that too much 'old' information is taken into account when estimating the model parameters. With $N_{\text{fix}} = 11$ we obtain the best result. Also, Subsequence 4 performs much better than in the previous two scenario's. Furthermore, we can see that in the best case the cumulative regret percentage is about 5% higher than in the previous scenario, where there was a small change between the two market regimes.

Subsequence	Optimal Value	$R(100)$	CI for $R(100)$
1	-	23.56	[22.54 ; 24.58]
2	$N^*_{\text{fix}} = 11$	9.65	[9.26 ; 10.04]
3	$\%^*_{\text{fix}} = 18$	12.88	[12.28 ; 13.47]
4	-	19.41	[18.46 ; 20.35]
5	$k^* = 30$	11.19	[10.73 ; 11.64]

Table 5.6: Scenario 3: Average cumulative regret percentage at $t = 100$ and its confidence interval

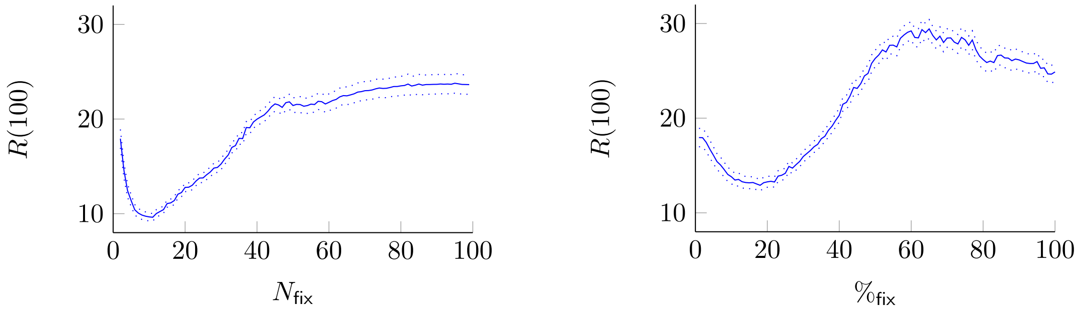
	a	a'	b	b'	p^*	p'^*	$\lvert p^* - p'^*\rvert$	t_{cp}
Mean	10.59	30.00	-1.25	-1.85	4.64	8.33	3.69	50.17
Standard deviation	5.35	5.75	0.74	0.47	1.37	1.26	1.26	5.80
Maximum	19.99	40.00	-0.07	-1.01	7.98	10.00	6.95	60
Minimum	1.05	20.00	-3.28	-3.38	3.00	5.12	2.00	40

Table 5.7: Sample statistics of parameters for scenario 3

5.4 Scenario 4: Random walk model

In Scenario 4 we simulate the random walk model. At every time step, a_t and b_t can change a small amount. We have as input:

- $a_1 \in \mathcal{U}[1, 20]$;
- $a_t = \begin{cases} a_{t-1} & \text{w.p.} \frac{1}{3} \forall t = 2, \dots 100 \\ \in \mathcal{U}[a_{t-1} - 1, a_{t-1} + 1] & \text{w.p. } \frac{2}{3} \ \forall t = 2, \dots 100 \end{cases}$



(a) Subsequence 2: The mean cumulative regret percentage at $t = 100$ as a function of N_{fix}

(b) Subsequence 3: The mean cumulative regret percentage at $t = 100$ as a function of $\%_{\text{fix}}$

Figure 5.6: Scenario 3

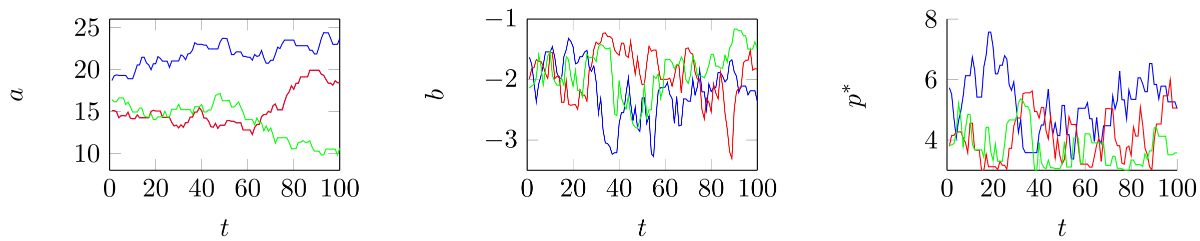


Figure 5.7: Three sample paths of Scenario 4

- $b_1 \in \mathcal{U}[-a/6, -a/16]$. This ensures that the optimal price p_1^* lies between 3 and 8.
- $b_t = \begin{cases} b_{t-1} & \text{w.p.} \frac{1}{3} \forall t = 2, \dots 100 \\ \in \mathcal{U}\left[\frac{-a}{2(p_{t-1}^*-1)}, \frac{-a}{2(p_{t-1}^*+1)}\right] & \text{w.p. } \frac{2}{3} \ \forall t = 2, \dots 100 \end{cases}$

  The optimal price at time t differs at most 1 from the optimal price at time $t-1$ with these settings.
- $p_l = 1$ and $p_h = 10$.

Figure 5.7 shows three sample paths of a_t and b_t and the corresponding optimal prices in every time period.

Results

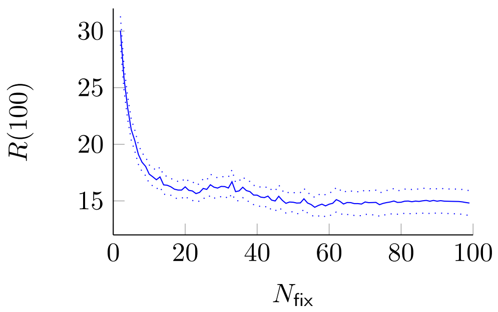
Table 5.8 shows the average cumulative regret percentage at $t = 100$ for the simulations of Scenario 4. Subsequence 2 performs best, with $N_{\text{fix}}^* = 56$. In Figure 5.8a however, we can see that for larger values of N_{fix} the average cumulative regret percentages lie very close together. Figure 5.8b shows almost the same pattern as in Scenario 1 and Scenario 2, except for the drop at the end of the time horizon. A possible explanation for this drop is given in the results of Scenario 5, where the same behaviour occurs. Subsequence 4 performs a lot worse again than the other subsequences.

Subsequence	Optimal Value	$R(100)$	CI for $R(100)$
1	-	15.49	[14.42 ; 16.57]
2	$N_{\text{fix}}^* = 56$	14.44	[13.59 ; 15.29]
3	$\%_{\text{fix}}^* = 100$	15.49	[14.42 ; 16.57]
4	-	31.02	[29.75 ; 32.39]
5	$k^* = 24$	16.20	[15.40 ; 17.00]

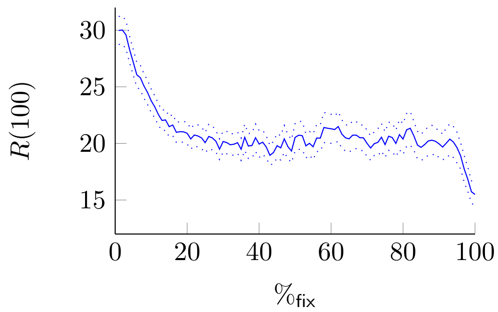
Table 5.8: Scenario 4: Average cumulative regret percentage at $t = 100$ and its confidence interval

Results for different values of c

For this scenario we provide results again for the simulations that are done with different values of c. Table 5.9 shows these results. We can see that quite different values of N_{fix} are now optimal. Besides that, we see



(a) Subsequence 2: The mean cumulative regret percentage at $t = 100$ as a function of N_{fix}



(b) Subsequence 3: The mean cumulative regret percentage at $t = 100$ as a function of $\%_{\text{fix}}$

Figure 5.8: Scenario 4

that all values of $R(100)$ are higher if $c = 3$. This means that we experiment too much with the prices and the balance between price experimentation and revenue maximization is off.

	$c = 0$			$c = 3$		
Subsequence	Optimal Value	$R(100)$	CI for $R(100)$	Optimal Value	$R(100)$	CI for $R(100)$
1	-	13.82	[12.75 ; 14.90]	-	17.11	[16.01 ; 18.21]
2	$N^*_{\text{fix}} = 99$	13.82	[12.75 ; 14.90]	$N^*_{\text{fix}} = 18$	15.65	[14.99 ; 16.31]
3	$\%^*_{\text{fix}} = 100$	13.82	[12.75 ; 14.90]	$\%^*_{\text{fix}} = 100$	17.11	[16.01 ; 18.21]
4	-	31.63	[30.35 ; 32.90]	-	31.36	[30.06 ; 32.66]
5	$k^* = 18$	16.22	[15.51 ; 16.93]	$k^* = 38$	16.97	[16.03 ; 17.90]

Table 5.9: Scenario 4: Results for $c = 0$ and $c = 3$

5.5 Scenario 5: Random walk model

In Scenario 5 we simulate the random walk model again, but now the change in a_t and b_t can be larger every time step. We have as input:

- $a_1 \in \mathcal{U}[1, 20]$;
- $a_t = \begin{cases} a_{t-1} & \text{w.p. } \frac{1}{3} \quad \forall t = 2, \ldots 100 \\ \in \mathcal{U}[a_{t-1} - 5, a_{t-1} + 5] & \text{w.p. } \frac{2}{3} \quad \forall t = 2, \ldots 100 \end{cases}$
- $b_1 \in \mathcal{U}[-a/6, -a/16]$. This ensures that the optimal price p^*_1 lies between 3 and 8.
- $b_t = \begin{cases} b_{t-1} & \text{w.p.} \frac{1}{3} \forall t = 2, \ldots 100 \\ \in \mathcal{U}\left[\frac{-a}{2(p^*_{t-1}-3)}, \frac{-a}{2(p^*_{t-1}+3)}\right] & \text{w.p. } \frac{2}{3} \quad \forall t = 2, \ldots 100 \end{cases}$
  The optimal price at time t differs at most 3 from the optimal price at time $t - 1$ with these settings.
- $p_l = 1$ and $p_h = 10$.

Figure 5.9 displays the values of a_t, b_t and p^*_t of three sample paths. We see that the changes in the parameters can become quite large, and hence also the change in optimal price.

Results

The results of the simulation of Scenario 5 are given in Table 5.10. The best performance is achieved by Subsequence 5. The optimal value of k is 1. This means that we are looking at only the previous period and choosing the value of N_t that would have been optimal in that period. In Figure 5.10 the on average optimal choice of N_t is plotted, together with its confidence interval. We can see that N^*_t grows almost linear, except for the beginning of the time horizon. At $t = 100$ the optimal choice of N_{100} for Subsequence 5 is on average

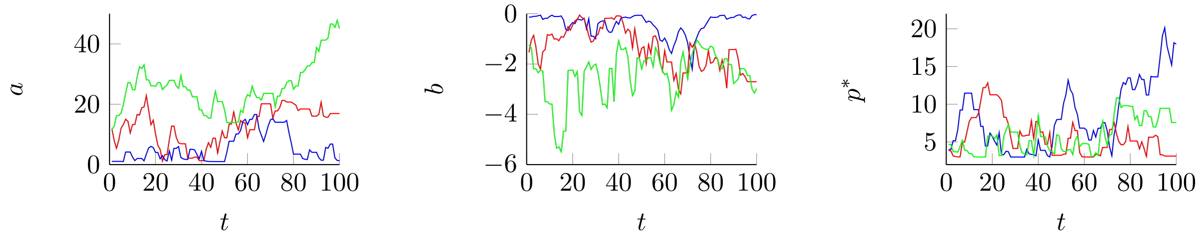


Figure 5.9: Three sample paths of Scenario 5

26. Figure 5.11 shows the performance of different values of N_{fix} and $\%_{\text{fix}}$. It can be seen in Figure 5.11a that a small value of N_t is optimal. After $t = 60$, the performance stabilizes. In Figure 5.11b we see a large drop in the line at $t = 98$. This can be explained by the sample variance of the prices. Figure 5.12 shows the sample variance as a function of $\%_{\text{fix}}$. We can see that at the end of the time horizon the sample variance grows quickly and this could be reason for the large drop in Figure 5.11b. The growth in sample variance can be explained by the fact that when using a large value of $\%_{\text{fix}}$ the starting prices are taken longer into account when estimating the demand.

Subsequence	Optimal Value	$R(100)$	CI for $R(100)$
1	-	30.41	[29.28 ; 31.55]
2	$N^*_{\text{fix}} = 6$	27.07	[26.35 ; 27.79]
3	$\%^*_{\text{fix}} = 7$	27.36	[26.66 ; 28.05]
4	-	31.63	[30.85 ; 32.42]
5	$k^* = 1$	26.29	[25.64 ; 26.93]

Table 5.10: Scenario 5: Average cumulative regret percentage at $t = 100$ and its confidence interval

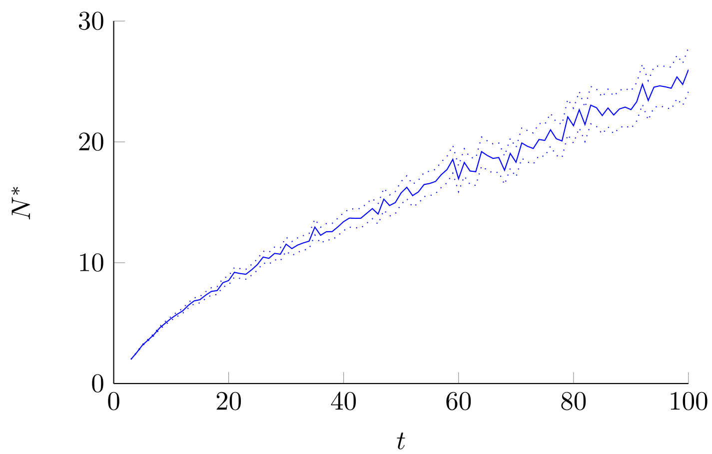
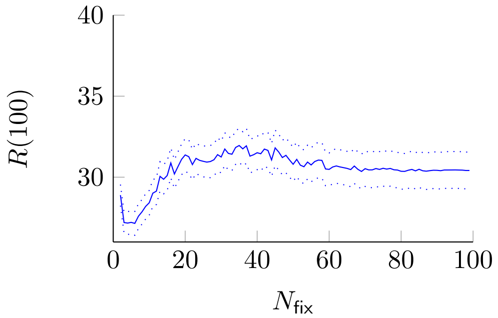
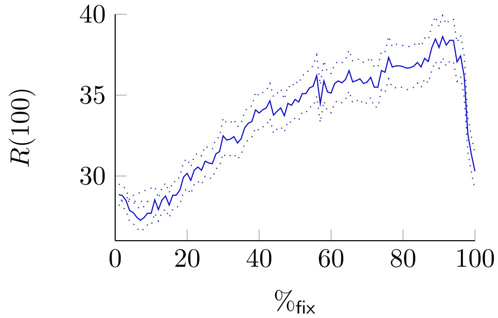


Figure 5.10: Scenario 5: The mean optimal choice of N for Subsequence 5



(a) Subsequence 2: The mean cumulative regret percentage at $t = 100$ as a function of N_{fix}



(b) Subsequence 3: The mean cumulative regret percentage at $t = 100$ as a function of $\%_{\text{fix}}$

Figure 5.11: Scenario 5

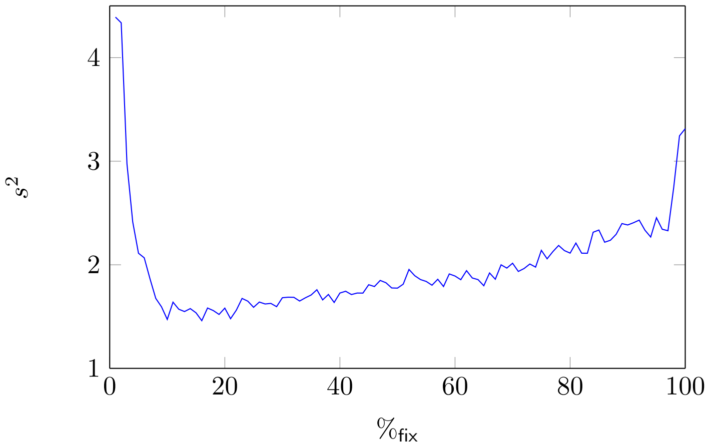


Figure 5.12: Scenario 5: The sample variance of the prices as a function of $\%_{\text{fix}}$

5.6 Summary of numerical results

The results of the five simulated scenarios show that when there is no change or only a small change in the model parameters, it is optimal to take all available data into account when estimating these parameters. However, when there is a larger change, whether it be a single change point or some consecutive smaller changes, it is not optimal any more to take all available data into account. When the change in a_t and b_t can become large every single time step, it is optimal to look at the previous period and find out what choice of N_{t-1} would have been optimal in that period and set N_t to this value.

Furthermore, we showed that the more change present in the market, the higher the cumulative regret percentage becomes. In Scenario 1, the constant model, we had a value of approximately 2 for $R(100)$ for the optimal subsequence. In Scenario 5, where a_t and b_t can change significantly every time step, this value had grown to 26 in the best-case scenario.

We have also shown that the value of c also influences our results. Finding the optimal value of c is not a part of thesis, and is subject of further research.

Chapter 6

Conclusions

The question that we raised in this thesis was: how do we choose N_t, the number of data taken into account when estimating the demand function? In a changing environment finding the answer to this question is quite hard.

Firstly, we simplified the problem and showed that in case of deterministic prices in a constant market the problem already becomes quite involved. In a constant market the estimator of the slope b improves when we add data. For the estimator of the intercept a this is not necessarily true. The possible improvement depends on the mean and the variance of the prices so far and on the choice of the added data point. In a market model with one change point, it depends on the size of the change in a and b if the estimator of b improves by adding pre-change data.

Due to the intractability of the problem, both analytically and computationally, we designed a set-up for simulations. In the simulations we used the Controlled Variance Pricing policy. The performance of five different possible subsequences for N_t is compared in five scenarios. The average cumulative regret percentage served as the performance measure.

The simulations show that when no change occurs, it is optimal to take all available data into account. If there is only one small change point, it is still optimal to use all available data. If a large change point occurs or when the market is constantly changing it is no longer optimal to take all data into account. Taking a small, fixed number of data points into accountis then the best choice. In a very volatile market it is even better to look at the previous period and find out what would have been the optimal number of data points for the estimation for this period and choose this number for the new estimation. These computations are however not as simple as the computations for a fixed number and also take a lot more time. Taking a fixed number into account is almost as good. The performance is still about ten percent better compared to the performance based on considering all data. Furthermore, taking a fixed (small) number is also advantageous for the amount of data storage. Not all data needs to be stored then, and in case of a large data set this can reduce the size of the data set considerably.

The simulations also show that when more change can be present in market, the higher the regret becomes. This means that in a very volatile market it is hard to set your prices optimally. However, with our described method the performance of existing pricing policies does improve. In a volatile market this improvement can realise a decrease in losses of about ten percent. In a market with a large bursty change this can even become more than fifty percent.

6.1 Recommendations for further research

In this section we will make some recommendations for further research. First of all, this is a theoretical work and therefore it would be nice to perform the simulations on real-life data. Secondly, we have not optimized the choice of c. For further research it is recommended to look into the amount of the variation in prices that is necessary to obtain the best results. A third suggestion is to research the use of weighted least squares estimation, and how to set the weights accordingly. This might enhance the performance of the pricing policy and comparisons can be made with the results in this thesis. Finally, some assumptions that we made in our mathematical model could be altered. For instance, we could investigate non-linearity of demand or the combination of a pricing policy with inventory restrictions.

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Appendix A

Overview of variables

$\%_{\text{fix}}$	Parameter for the size of the sliding window used in Subsequence 3
$\%^*_{\text{fix}}$	Optimal choice of $\%_{\text{fix}}$
α	Parameter for determining minimum variance in prices. The value if α is 0.5
a_t	Intercept parameter during period t
$\widehat{a}_{t-1,N_t}$	Estimated intercept parameter after period $t-1$, based on N_t data points
b_t	Slope parameter during period t
$\widehat{b}_{t-1,N_t}$	Estimated slope parameter after period $t-1$, based on N_t data points
c	Parameter for determining minimum variance in prices
d_t	Demand during period t
ϵ_t	Disturbance term for period t
k	Number of periods over which the expected squared prediction error is minimized
k^*	Optimal choice of k
N_{fix}	The size of the sliding window used in Subsequence 2
N^*_{fix}	Optimal choice of N_{fix}
N_t	Number of data points taken account into the estimation of a_t and b_t
p_t	Price for period t
p^*_t	Optimal price for period t
$\hat{p}^*_{(t,N_t)}$	Estimated optimal price for period t, based on N_t data points
p_l	Minimum selling price
p_h	Maximum selling price
$R(t)$	Cumulative regret percentage at period t
T	Time horizon
t	Time period, $\in \{1, 2, \ldots, T\}$
t_{cp}	Time period in which the change point occurs