# Onderzoek van Onderwijs

## Van Hiele levels applied to solving combinatorial reasoning problems

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## Abstract

Building on prior research, we redefine the Van Hiele levels to be applied to combinatorial counting problems. Combinatorial counting is a difficult subject for both teacher and student and the current curriculum does not seem to help students develop a relational network of knowledge. After defining the levels for combinatorics, we collect data to examine the accuracy of the definition, to examine whether these levels occur in the solution process when students are solving combinatorial counting problems, and to study the effects of the variation of these levels on the solution strategies students use. Analysis of the data showed students have difficulties transitioning from a lower level to a higher level. Qualitative analysis of students' solutions to combinatorial problems revealed the preference of the use of formulas for some students, while at the same time other students showed more insight by their systematic approach of the problems.


## Introduction

Recommendations to incorporate combinatorics in the school mathematics curriculum date back to the early 1970's (English, 2005). Combinatorial analysis is an appropriate topic in the mathematics curriculum, because it has problems suitable for all grades, it can be used to train students in the concepts of enumeration, making conjectures, generalization and systematic thinking and many applications in different fields can be presented in teaching combinatorial analysis (Kapur, 1970). Combinatorial problems can help children construct meaningful representations, reason mathematically, and abstract and generalize mathematical concepts (Sriraman & English, 2004). Making conjectures, generalizations and systematic thinking are considered to be of more importance in education that prepares children for the future than training them to apply rote learned formulas (Drijvers, 2015). So we might expect that teaching combinatorics has a solid place in the curriculum, founded on well thought-out didactical principles.

In spite of the previous, combinatorics is considered as one of the more difficult topics to learn and teach. Most pedagogic methods used in Dutch mathematics schoolbooks introduce combinatorial subjects separately. For each subject pupils learn procedures and formulas. After a formula for a subject is learned - usually permutations first -, the next subject is taught. To solve combinatorial problems, pupils rely completely on formulas and calculator buttons, but do they understand what they are doing? Apparently not; during tests and exams many students are not able to distinguish between permutations and combinations. Many students seem to have difficulties to detect common structures and cannot identify models of underlying problem types. These abilities of making connections among counting problems are an important factor in solving combinatorial counting problems (Lockwood, 2011).

A main aspect in the process of solving combinatorial counting problems is that students need to recognize certain properties of the contexts of the problems. They must see whether the samples have to be ordered or not and whether elements in the sample are allowed to be repeated or not.

These combinatorial characteristics play an important role in the mathematical thinking process for solving combinatorial counting problems. To be able to assess the role of those *combinatorial characteristics* in the process of combinatorial problem solving we like to build a theoretical framework for the process of combinatorial problem solving. As far as we know such framework does not exist yet.

According to Van Hiele (1986), in the process of learning mathematics, students go through specific levels of thinking. These levels are sequential; students have to pass them to achieve a formal level of mathematics. Initially Van Hiele distinguished five levels of thought in learning geometry (Van Hiele, 1986). Later these levels were generalized to other fields of mathematics (Alberts & Kaenders, 2005). In the process of problem solving we see a shift of attention via some levels that have close similarities with Van Hiele levels (Mason, 2004). In literature we did not find any description of the levels applied to combinatorics. In this study we will show a way to define the Van Hiele levels in the field of combinatorial problem solving and we will indicate the place of the combinatorial characteristics in these levels. Second, we will try to get more insight in how students go through these levels to come to a solution strategy for combinatorial problems.

## Research question

Is it possible to define Van Hiele levels so they apply to combinatorial counting problems, do students go through the defined Van Hiele levels when solving combinatorial counting problems and what role do these levels play in reaching a solution to combinatorial counting problems?

## Theoretical perspectives

### Combinatorics and sensible mathematics

Previous research in the field of combinatorial problem solving mainly focused on the mathematical subject and the problems that occur. This research identified many aspects of problems that students encounter when they are solving combinatorial counting problems.

In combinatorics most problems do not have readily available solution methods, and create much uncertainty for students on how to approach them and what method to employ, because students do not have confidence in their own thinking to unravel the context (Batanero, Navarro-Pelayo, & Godino, 1997; Coenen, Verhoef, & Tall, 2014; Eizenberg & Zaslavsky, 2009). This problem is observed in all types of students with different ages (Lockwood, 2011). Even university students have troubles with identifying problem types and applying the right solution strategies and processes (Godino, Batanero, & Roa, 2005; Le Calvez, Giroire, & Tisseau, 2008). According to English (2005) “a common finding in many of the studies on combinatorics is that students have difficulty identifying related problem structures. As a consequence, students’ ability to transfer their learning to new combinatorial situations is limited.” Recent results on combinatorial reasoning (Coenen et al., 2014) showed that students are not able to solve the thirteen counting problems of Batanero et al. (1997) by classifying them and to relate an appropriate solution strategy for each classification.

In our perspective most research focussed on the flawed procedural understanding of the subject by students, whereas the conceptual understanding stays underexposed. To bring students to ‘sensible mathematics’ it’s important to let them develop their knowledge from concepts building on perceptions and experiences, via visualisations based on descriptions to the procedural use of symbols and formulas.

In order to come to deeper insight, students must derive their own models and strategies in the context in which the problem occurs (Gravemeijer, 1999). When students mathematize contexts, the underlying model, and thus the appropriate strategy, will emerge by perceiving those aspects of the situation that are mathematically important. To give students a better understanding it's better to let them discover '*models of'* the situation first. After the emerging of the model of the situation, this model can transfer into a '*model for'* the solution strategy (Gravemeijer, 1999).

An example of emergent modeling can be found in the research of Batanero et al. (1997), exercise 6. In this exercise, students are given the question how many different ways there are for a grandmother to place four children in two different bedrooms, both with enough room for four children. The addition "*the grandmother can place all the four children in one room, or she can have Alice, Bert and Carol on the first floor and Diana in the upstairs room*" reveals a clue to the solution strategy. It implies to distribute the children and this implicit model suggests considering all decompositions of the number 4. For example, when you place two children on the ground floor and two children upstairs, then there are 6 possibilities to distribute the 4 children - which is a combinatorial problem in itself to solve. If students systematically elaborate all the possible decompositions they can find the correct answer by adding $1 + 4 + 6 + 4 + 1 = 16$. Batanero observed some students solving the problem correctly this way. However, Batanero suggests that the problem should have been solved by a multiplication based on a selection model. Indeed, if we shift our attention to the fact that for each child one room out of two needs to be selected, then the problem is solved quickly: $2 \cdot 2 \cdot 2 \cdot 2 = 16$ possibilities in total. However, the first strategy is based on the model *of* the situation and can be deduced from the context. The second strategy is just a model *for* the mathematical solution procedure and can only be applied after a major shift of attention.

Emergent modeling helps students to develop a relational network of knowledge. They are not just provided with separate mathematical instruments to solve separate problems but they see more mathematical relationships (including differences) among diverse situations. Two advantages in relational mathematics over instrumental mathematics are that it's easier to remember and it's more adaptable to new tasks (Skemp, 1976). Mathematical thinking develops in the child as perceptions are recognized and described using language and as actions become coherent operations to achieve a specific mathematical purpose (Tall, 2012). So in order to solve combinatorial problems students must be able to interpret contexts, see appropriate structures and models, and generalize these structures to more formal solution procedures. Solving combinatorial problems must be more than applying rote learned tricks (Timmer & Verhoef, 2014). When students solve combinatorial counting problems, their solution strategies should be based on mathematical thinking.

## Mathematical thinking levels

The long-term development of mathematical thinking is consequently more subtle than the addition of new experiences to a fixed knowledge structure. It is a continual reconstruction of mental connections that evolve to build increasingly sophisticated knowledge structures over time (Tall, 2015). Every mathematical subject, like geometry, arithmetic, algebra etc. has its own development over time. In geometry, van Hiele has traced cognitive development through increasingly sophisticated succession of levels (Tall, 2004). A nice summary of the Van Hiele levels is given by Zsombori and András (2013), who adapted the levels to teaching probability and based their description on the work of Usiskin (1982). (In most original studies, the levels are numbered from 0 to 4, we choose to number from 1 to 5).

- Level 1: Intuitive, also called the level of visualization or the level of global recognition. At this level geometric objects are recognized based on their appearance and are connected with the use of common language. For example a square is not recognized as a rectangle, etc.
- Level 2: Analysis (and description). At this level objects are recognized due to their properties, but properties are not organized hierarchically. Usually the relations between different objects and categories are not emphasized.
- Level 3: Abstraction (and informal deduction). At this level properties are organized into a hierarchy, relations between different objects, properties and categories are recognized. The argumentation on this level often depends on perception, there exists some kind of reasoning based on motivated steps, but in general complex and formal proofs are not yet constructed.
- Level 4: Deduction (formal deduction). At this level the formal (complete and correct) proofs are used and constructed.
- Level 5: Rigor. This is the level where mathematicians work, where the objects are constructed by axiomatic systems (and are independent of their realizations). This last level is only reached at university level.

In the process of problem solving we see a shift of attention via some levels that have close similarities with Van Hiele levels (Mason, 2004). These levels are: - being aware of the whole situation, - focus on details and awareness of relations or similarities, - focus on properties as attributes that objects might satisfy and - focus on reasoning solely on the basis of properties (Mason, 2004). Typically all levels are sequential and if students want to develop mathematical understanding, they have to go through all the levels. So when students really understand mathematical subjects, they have built their knowledge through different levels of notions.

Tall (2013) distinguishes three worlds of mathematics. The first is the conceptual-embodied world which is based on perceptions of and reflections on properties of objects. The second world is the operational-symbolic world that grows out of the embodied world through – physical - action of the learner into mathematical procedures. The third world is the axiomatic-formal world based on formal definitions and proof. Tall (2012) terms the first four Van Hiele levels in the following four successive levels: recognition of basic concepts, description of observed properties, definition of concepts and deduction in the form of proof. In the process of emergent modeling Gravemeijer (1999) distinguishes four levels of activity that - we think - have close relations to the four Van Hiele levels as described by Tall (2012). These four levels of activity are:

- Activity in the task setting involves interpretations and solutions that depend on understanding how to act in the problem setting (often out-of-school settings).
- Referential activity involves *models of* that refer to activity in the setting described in instructional activities.
- General activity involves *models for* that facilitate a focus on interpretations and solutions independent of situation-specific imagery.

- The activity of formal mathematical reasoning is no longer dependent on the support of *models for* to achieve mathematical activity.

All previously mentioned level classifications basically revere to the same processes of learning to think mathematically and thus to solve mathematical problems. We choose to use the Van Hiele levels to describe the process of mathematical problem solving.

### Mathematical thinking attention

In mathematical thinking, attention shifts between holistic encompassing, discerning distinctions (stress and ignoring foregrounding and backgrounding), recognizing relationships amongst discerned features, perceiving properties that objects or elements may possess, and deducing formal definitions and axioms (Mason, 2004).

A main aspect in thinking mathematically is the power to generalize. It's important to become aware of structural relations in order to generalize (Mason, Burton, & Stacey, 2010). It is important to look at several instances to discover the similarities. By stressing these similarities and ignoring differences one may come to generalizations. Given the nature of counting problems, being able to determine similarities among problems, problem types, situations and techniques is a vital aspect of being a successful counter. (Lockwood, 2011). People naturally seek similarities and recognize repetition and patterns (Mason, 2004). Recognition of patterns is an essential facility for mathematics, including patterns in shape and numbers (Tall, 2008). It is important that during the solution process, the attention is focused on those aspects which are important for the mathematical approach to a problem. It seems important for students to focus on the right details.

Batanero et al. (1997) gives a nice overview of pedagogical perspectives on combinatorial reasoning and problem solving based on the work of Dubois (1984). Dubois identified selection, distribution and partition models, each leading to specific solution strategies. The difficulty levels of combinatorial problems highly depend on these implicit models: - selection, - distribution, - partitioning, and on the nature of elements that are combined, such as letters, numbers, people and objects (Batanero et al., 1997). The implicit models in the exercise do have a big influence on students' ability to solve the question. But we believe that it is not just the implicit model that causes the problems. The way in which students look at the context and their focus on certain aspects may complicate or simplify the solution based on implicit models. Students can have difficulties with the interpretation of the situation outlined in the question and some (mis)guiding examples can lead them to complicated solution strategies. For exactly this reason this research places the exercises only in a selection context and the formulation of the question is kept as simple and clear as possible. In the process of solving no discussion must arise about the interpretation of the question. The students must not be distracted by ambiguities in the formulation but must be able to focus their attention on the combinatorial characteristics repetition and ordering.

## Method

### Participants

In this study we observed three groups of secondary school students. In the first group, a 4$^{th}$ year VWO group, were 5 boys and 9 girls aged 15/16 with a basic knowledge on tree diagrams and counting North-East lattice paths. The second group, a 3 VWO group, consisted of 7 boys and 8 girls aged 14/15 with no prior education in counting problems, tree diagrams or probability. In the third group, another 3 VWO group, there were 7 boys and 14 girls aged 14/15; these students had learned how to draw a tree diagram and how to calculate basic probabilities.

## Research instruments

### Lesson study

The study is conducted in the form of a lesson study. Lesson study has its origin in Japanese mathematics education. The lesson study approach involves the design of the research lesson as part of an extended sequence of lessons to teach a particular topic, the implementation of the research lesson, followed by evaluation and analysis, then refining of the lesson. Observation of the research lesson by colleagues and other interested persons is an essential part of this approach. Having several pairs of eyes looking at the classroom activity gives a more comprehensive view of different aspects (Verhoef & Tall, 2011). The focus is on observing the students, not the teacher. Based on findings the next lesson is redesigned.

### Group work

In the process of mathematical problem solving cooperation is an important factor. The presence of others is a stimulating factor in the impulse to express and clarify own thinking as well as to connect it to the thinking of others (Mason et al., 2010). So, in order to be able to observe the mathematical thoughts of the students, we will stimulate them to express their thoughts. This is achieved by making them work together in groups of three or four. By observing their expressions we can determine their Van Hiele level of thinking, the Van Hiele levels at which they interact and in what way students go through different Van Hiele levels to reach a solution. To be able to assess the awareness of the important aspects we ask the students to categorize the given problems.

For the reliability, each group was audio or video taped and had an observer. The observer made field notes. Students had to write down their solutions and answers. Video tapes or audio recordings were used to transcribe student remarks. If a remark - while transcribing the video or audio - was unclear, the field notes and students work were consulted to determine the final transcript.

### Van Hiele applied to combinatorial problems

In this study we investigate how levels of thinking describe secondary school student's thinking processes on combinatorial problem solving. We will define and use the Van Hiele levels to distinguish certain levels in the solution process in combinatorial problem solving. We will try to get an insight in how Van Hiele levels appear in the process of emergent modeling, in the transition from relational knowledge building to instrumental use of strategies, from the conceptual-embodied world to the operational-symbolic world.

We think that the first two Van Hiele levels are easily applicable to combinatorics. At the first level, the intuitive level of visualization, students (try to) understand the question, (try to) find some examples of samples that are in play to get a 'feel' of the possible outcomes. In the second level, that of analysis and description, students see relevant aspects in the context that are important for the problem: students see combinatorial characteristics such as repetition and ordering and are aware of the implicit model. The third level, of definition of concepts, can be seen as creating a structure that reveals the sequence of choice, the way of arranging etc. that makes sure that all possibilities are accounted for in the solution. This can be done by drawing tree diagrams, drawing a grid to count

North-East lattice paths or systematically enumerating possibilities. The fourth level is, of course, the level of formal calculation which is deduced from the concepts at level three. We don't need the fifth level, as we are going to investigate the applicability of the Van Hiele levels at combinatorial problem solving at secondary school.

We described the Van Hiele levels with appropriate characterizations of combinatorial reasoning as shown in table 1.

**Table 1 Van Hiele levels in combinatorics**

| Level | description |
|---|---|
| **1** | Visual level; students use concrete drawings or random enumerations to find some samples. They are trying to get "the whole picture" to understand what sort of samples are to be produced. |
| **2** | Descriptive level; students are trying to find combinatorial characteristics in the problems, using terms like repetition, order, not to be used again, double, etc. |
| **3** | Informal deduction; students use the combinatorial characteristics to investigate the structure of the given problem using schematic drawings, tree diagrams, or systematically writing down all possibilities. |
| **4** | Formal theoretical; using formulas and procedures to calculate the number of possibilities, based on recognition, experience or insight. |

For example, imagine that students are working on the problem 'ice cream top 3' (see appendix A) and trying to figure out how many different top 3's one could make out of 6 flavours. At the first level students may call some triples of flavours like banana-strawberry-chocolate, vanilla-banana-cerise, banana-chocolate-banana, and so on. At the second level, they may evaluate some of these triples as incorrect, because they notice from the context that flavours are not to be repeated, so banana-chocolate-banana is not a possible top 3, or they may wonder whether strawberry-banana-chocolate is different from banana-chocolate-strawberry or not. At the third level students could use a systematic enumeration like the odometer strategy (English, 1991), or use tree-diagrams to represent all possibilities. At the fourth level they may deduce from the enumeration or the tree diagram the formal calculation $6 \times 5 \times 4$ as the solution to the problem.

## Procedure

### Context

The study consisted of three observed classroom lessons, all in pre-university secondary education. The data was collected during March and April 2014. The first two lessons took 50 minutes each and the last lesson 45 minutes.

### Interventions

During the first lesson nine different problems were used, all in a context of selection with the basic combinatorial operations: arrangements with and without repetition and combinations without repetition (see Appendix A). Students were asked to solve the problems and, after solving, to seek similarities among the problems. During the discussion after the lesson the observers stated they noticed that the problems were categorized based only on the type of formal calculations. Students didn't mention the basic combinatorial characteristics repetition and order. By reducing the amount of contexts in which the problems occurred, the expectation was that students would distinguish the combinatorial characteristics. So during the second lesson the amount of contexts was reduced to two: choosing ice cream flavours and choosing books (see Appendix B). The assignment of solving and categorizing the problems remained the same for the second lesson. Students could focus their mathematical thinking attention on the combinatorial characteristics repetition and order.

During the second lesson students did eventually recognize the combinatorial characteristics. This recognition was based on formal calculations. So in these cases the characteristics were not the basis for the solution strategy, but the solution strategy was the basis of the recognition of the characteristics. For efficient problem solving one should first see properties and base a solution strategy on these properties. So to see if students are able to see the right properties to base their strategies on, we decided to change the order of solving and categorization.

### Data processing and analysis

Taking all transcripts from every lesson, all remarks made by students were assessed. First all three researchers individually assessed the remarks. After that the researchers discussed their assessments until total agreement about every remark was reached. Remarks that could not be related to mathematics were ignored and every single mathematical remark has been discussed to categorize its Van Hiele level and to determine the correctness of the remark. In total 541 remarks were categorized.

All data were entered and processed as a table in Excel. Every categorized remark was provided with columns for codes for the lesson number, the problem number and the group number. Also columns were added for the changes of levels and the changes in correctness. For every two consecutive remarks on the same problem, made by a single group of students, the change in quality and change of level were calculated.

An example of the categorization of student's remarks and quality changes is given in table 2. The students are solving the bookstore question: There is a top ten thrillers, all books are in a certain bookstore, with plenty copies available. The owner of the store records the sales from the top ten most popular books for the first four customers. How many different lists with the choices of customer 1 up to 4 are possible?

The correct argument would be that, when a customer buys a book, all ten books can be selected. This means that the owner could have $10 \times 10 \times 10 \times 10$ different lists.

Table 2 coding of students remarks at levels and quality

| | Level | correctness<br>1=right/0=wrong |
|---|---|---|
| **now that one,… the books…, bookstore first** | | |
| **J: 10x9x8x7?** | 4 | 0 |
| **L: reads exercise out loud and draws ten books in a row** | 1 | 1 |
| **S: If you choose this one, there are only nine left** | 2 | 0 |
| **J: so 10x9x8x7** | 4 | 0 |
| **S: would it be right?** | | |

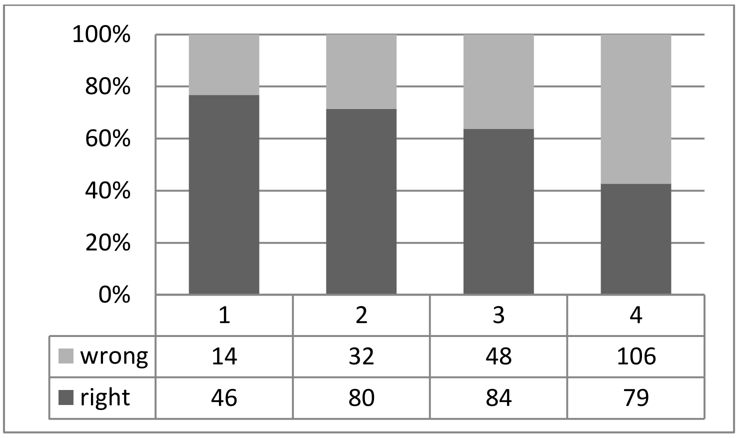
The first remark (by J) shows an example of an incorrect remark on level 4 followed by a good action at level 1. This represents a level change of -3 and a quality change of +1. The next remark (by S) is wrong on level 2. So now we see a level change of +1 and quality change of -1. The next formal calculation is a remark on level 4. The calculation, even though correct looking at S' remark, is not appropriate for the given problem, so it's labelled wrong and therefore represents a transition of +2 with quality change of 0. For all student remarks and actions on the same problem both the level change and the quality change were determined this way.

# Quantitative Results

## Correctness of the level definition

Indications for the accuracy of our definition of the Van Hiele levels follow from the fact that at lower levels (relatively) more correct comments are made than at higher levels (see figure 1). The reverse is also true: at higher levels (relatively) more incorrect comments are made. So if a student makes a comment at a lower level, the greater the probability that the comment is correct. The table in figure 1 gives the absolute number of categorized correct and incorrect remarks per level, with all three lessons combined. The diagram in figure 1 shows the percentage of correct and false statements per level. So for example: the students made 60 statements in the visual level, level 1, of which 46 were right, and 14 were wrong. Resulting in 76,7% right and 23,3% wrong statements.

Figure 1 Ratio of correct and false remarks per level




These results indicate that the defined Van Hiele levels (level 1 to level 4) show a hierarchy of cognitive levels rising in difficulty. This shows the Van Hiele levels are applicable to combinatorial problems.

## Correct and incorrect remarks within levels per lesson

In the graphs in figure 2 to figure 4 we see the absolute number - split in right and wrong - of students remarks during the separate lessons at the y-axis, and the Van Hiele levels at the x-axis.

The graph in figure 2 shows that students in lesson 1 mainly reasoned in level 3 and 4, while the students of lesson 2 mainly reasoned in level 2 and 3. It is not possible to attribute this difference solely to the interventions between the lessons. The students who attended lesson 1 were a year ahead compared to the students of lesson 2 and 3, and had a basic knowledge of tree diagrams, lattice paths and formal calculations. The students of the third lesson had learned tree diagrams and the permutational multiplication, but according to the observations they did not use the tree diagrams as a part of the solution strategy.

Figure 2 Number of right and wrong remarks at each level in lesson 1

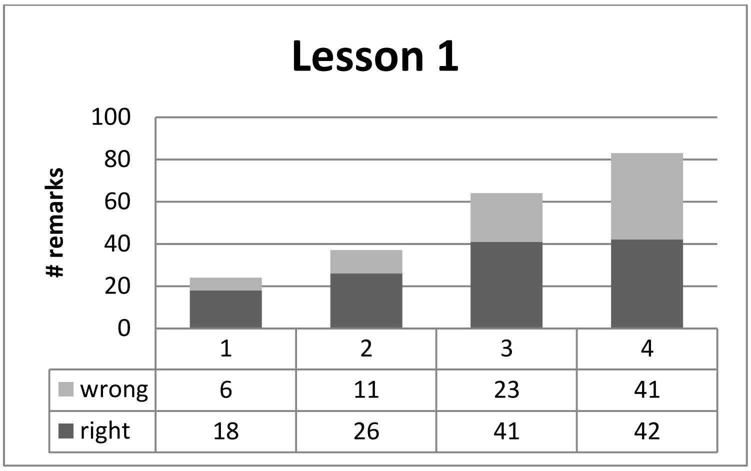


Figure 3 Number of right and wrong remarks at each level in lesson 2

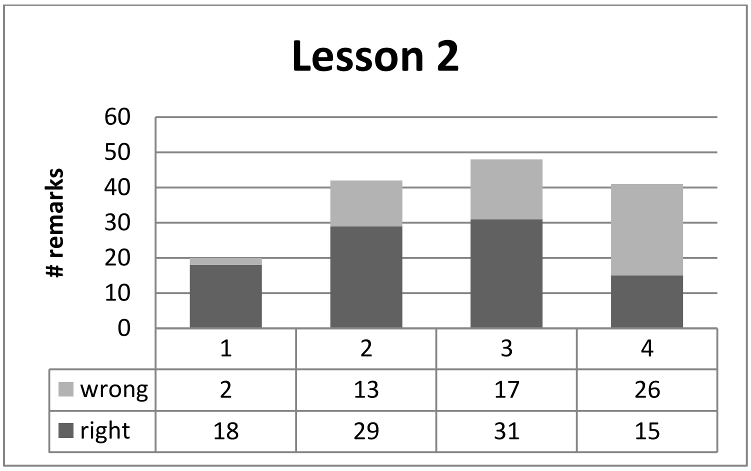
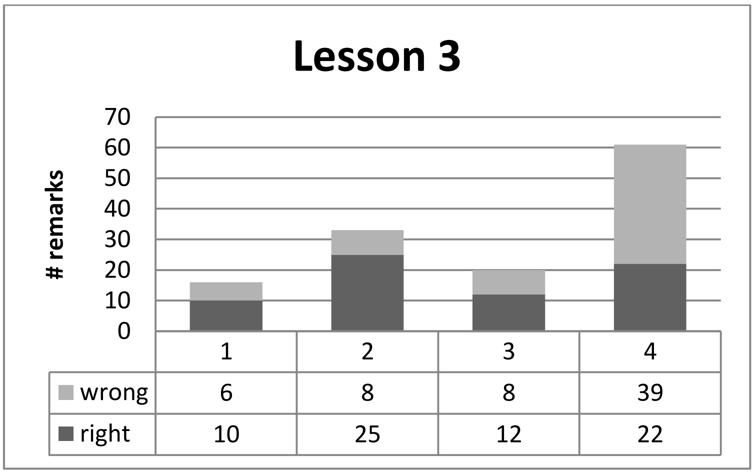



Figure 4 Number of right and wrong remarks at each level in lesson 3

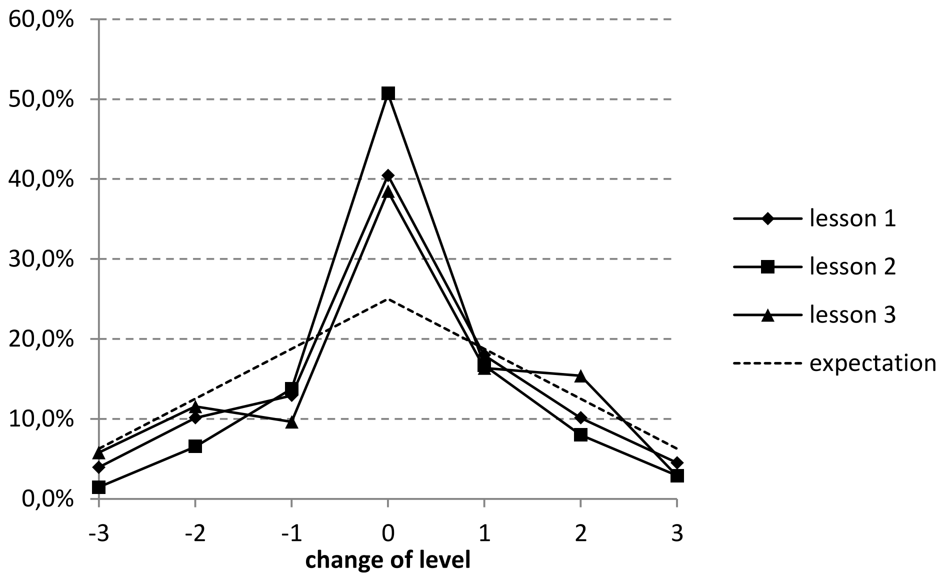



During lesson 2 (figure 3) the students had the highest number of remarks in level 2, 3 and 4, whereas the students participating in lesson 3 (figure 4) had the highest number of remarks in level 4, remotely followed by level 2. This might mean that the students who were first asked to just solve the problems explore more. Or perhaps this result can be attributed to the fact that the students in lesson 3 already had basic knowledge on tree diagrams, therefor assuming they know what to do, and immediately resorting to formal calculations, without exploring the problem. When looking at the number of incorrect remarks, the students in lesson 3 demonstrate not to be aware of the subtle differences between the given problems, and are just using the same -formal- method of solving for different problems.

## Number of level changes

Figure 5 shows that during the interaction students mainly respond at the same level as the comment that was made immediately before. 40 to 50 % of the comments is at the same level as the previous one. The larger the (absolute) level change, the less frequent it occurs. The dotted line in figure 5 represents the expected change of level if remarks were made at a random level, independent of previous remark levels. Indeed, if the occurrence of levels was uniformly distributed, the ratio of the number of the possible level differences (-3 up to +3) between two consecutive remarks would be 1, 2, 3, 4, 3, 2 and 1 out of 16.The results show that more than statistically expected, remarks have been made on the same level as the preceding remark. Remarks that differ in level from the previous one occur less frequent than in a uniform distribution.

Figure 5 Relative number of change in Van Hiele levels in consecutive remarks




One explanation could be that the comments of students match with what is said, because they listen to each other. They react to what was said and it is natural that a reaction is at the same Van Hiele level.

Another explanation may be that students lack the ability to use different levels, so they are not able to make these level changes. The observations, however, show that comments made by students are at all levels, contradicting this possible explanation.

The irregularities in the graph (figure 5) for lesson 3 within the level changes -1 and +2, may not be significant because of the small amount of data. Table 3 shows the absolute and relative numbers regarding the transitions and the separate lessons.

Table 3 Level changes per lesson

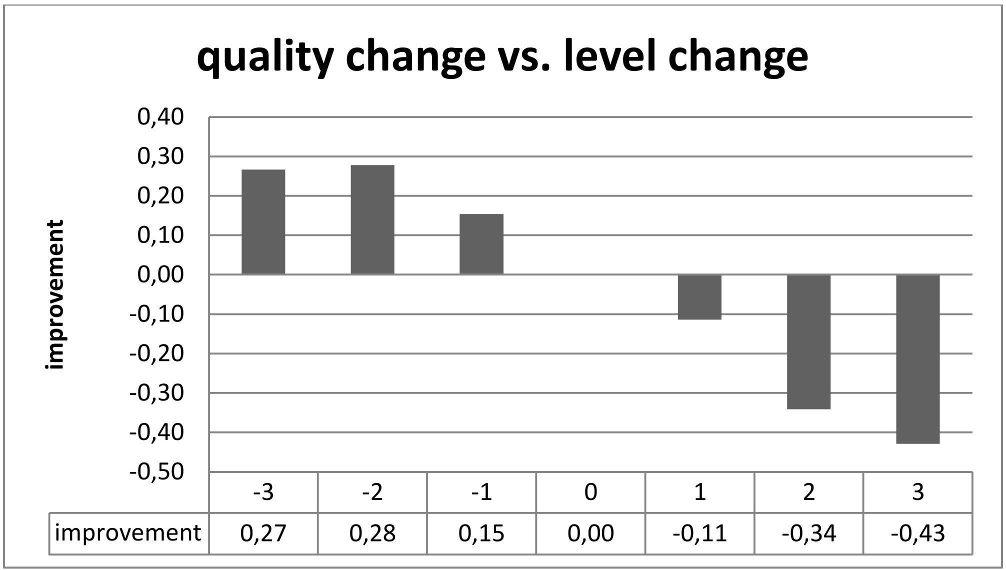
| **Level transition** | **-3** | **-2** | **-1** | **0** | **1** | **2** | **3** | **total** |
|---|---|---|---|---|---|---|---|---|
| **Lesson 1** | 7 | 18 | 23 | 72 | 32 | 18 | 8 | 178 |
| | 3,9% | 10,1% | 12,9% | 40,4% | 18,0% | 10,1% | 4,5% | 100,0% |
| **Lesson 2** | 2 | 9 | 19 | 70 | 23 | 11 | 4 | 138 |
| | 1,4% | 6,5% | 13,8% | 50,7% | 16,7% | 8,0% | 2,9% | 100,0% |
| **Lesson 3** | 6 | 12 | 10 | 40 | 17 | 16 | 3 | 104 |
| | 5,8% | 11,5% | 9,6% | 38,5% | 16,3% | 15,4% | 2,9% | 100,0% |

Both figure 5 and table 3 show that the different tasks, as described in the section procedure, given during the three lessons do not have any significant influence on the number and distribution of the level changes.

## Quality change vs. level change

Figure 6 shows the average of the changes in quality of all interactions from all three lessons.

Figure 6 improvement of remarks related to change of level




The average quality change was calculated per level change. These average quality changes, with a minimum of -1 and a maximum of +1, are shown on the vertical axis, the level changes are on the horizontal axis (see figure 6). Looking at the data of all three classes it appears to be that the transition to a higher level frequently has a worsening as result, while the transition to a lower level generally indicates an improvement of the correctness of the following remark.

## Locating mistakes

In total there were 420 consecutive remarks, meaning two following remarks about the same problem by the same group. These consecutive remarks are distributed over the possible level changes as shown in table 4.

Table 4 Number of consecutive remarks per level change

| from level | to level 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **1** | 17 | 18 | 11 | 15 |
| **2** | 11 | 46 | 20 | 34 |
| **3** | 11 | 14 | 53 | 34 |
| **4** | 15 | 28 | 27 | 66 |

Looking at consecutive remarks which have a quality change of -1, meaning in the interaction and solution process the group makes a mistake after a correct previous remark, we see that relatively many mistakes are located in the transition from level 1 to 3. There were 11 of such changes of which 6 were wrong, resulting in a ratio of 0.55. All ratios are listed in table 5.

Table 5 Ratio of number of quality change -1 and total number of occurrence of level change

| from level | to level 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0,24 | 0,22 | 0,55 | 0,40 |
| 2 | 0,09 | 0,15 | 0,15 | 0,35 |
| 3 | 0,00 | 0,21 | 0,21 | 0,12 |
| 4 | 0,20 | 0,14 | 0,04 | 0,15 |

The ratio of 0.55 when transitioning from level 1 to level 3 is the highest in the table. This might indicate that students make relatively more mistakes when they skip level 2, so when they fail to identify the properties of combinatorial counting problems in the process of solving them.

# Qualitative results

## Procedures and formula

During lesson 1, one group started with the problem of t-shirts: going on a holiday, you want to take three of your ten shirts. How many options are there to take three? In this group one student had had private tuition and had been taught to calculate numbers of combinations with the calculator function ‘nCr’. Immediately this student, C, took his calculator - working at level 4 - and found the right answer 120. Student A mentioned that the problem could be solved by using a tree diagram, which would be an action at level 3. Student A was convinced of the correctness of 10x9x8, which she deduced from the tree diagram, and challenged student C to explain how it is possible that the formula for combinations gives the answer of 120 possibilities and 10x9x8 = 720 possibilities. Student C couldn’t explain why both answers are not the same. Student C mentioned that this problem maybe had to be solved by counting lattice paths. Student A drew an x-y grid (see figure 7), tried to calculate the numbers without writing them down, but she made a mistake calculating and didn’t find the correct answer of 120 possibilities. Now student A is convinced of her solution of 720 possibilities.

Figure 7 An x-y grid about choosing 3 out of 10

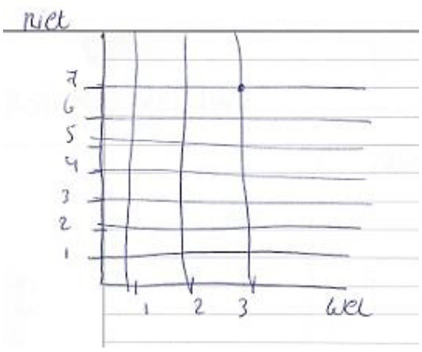
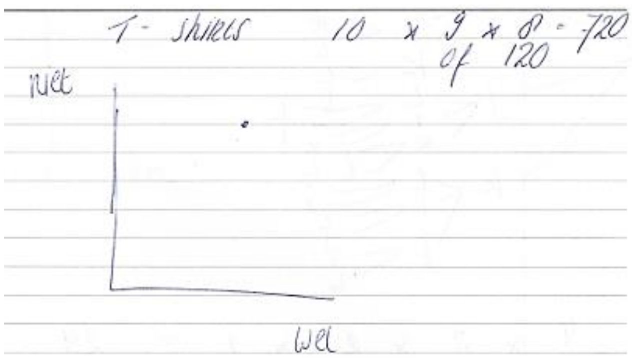



Figure 8 Students’ solution to choosing 3 out of 10

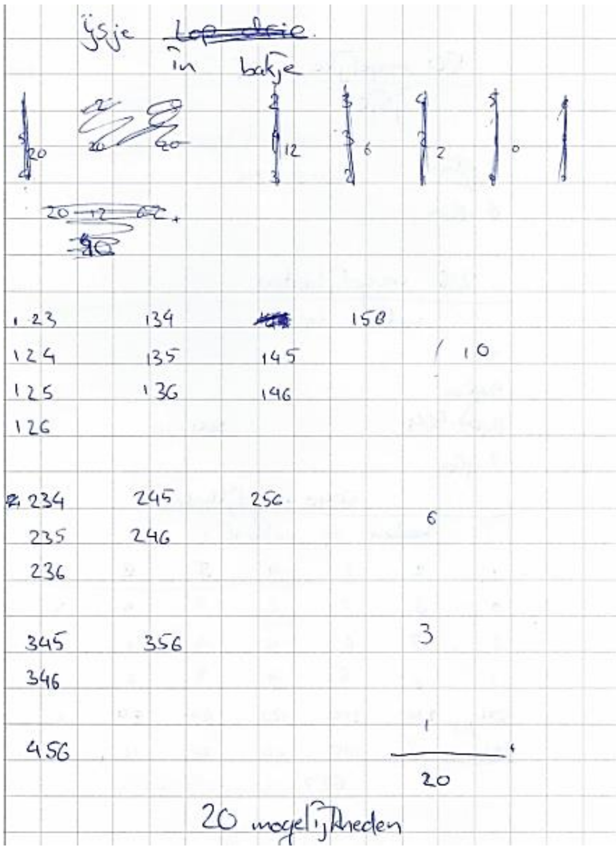



After the students had solved the other problems, they returned to the t-shirt problem. They acknowledged that taking the three t-shirts white, blue and red is similar to taking blue, white and red, but they could not translate this notion to an appropriate solution strategy. Student C mentioned that probably there are double sets of shirts. According to student A all the problems can be solved with a tree diagram. Immediately after this remark, student C drew an x-y grid and mentioned, without any explanation, that you only have the choice between yes or no. The final answer of the group is in figure 8. Not being able to agree on a solution, they decided to write down both answers.

## Construction of a systematic method

During lesson 2, in a particular group, one student E had the lead. First he tried to solve the problem: Ice-cream top three. On top of the page he writes down: ice-cream top three. Student E starts to investigate the problem. He involves repeating the selection of six different ice-cream flavours (numbered 1, 2, 3, 4, 5 and 6). Student E starts with writing down a 1 (see figure 9).

Figure 9 students' solution to ice-cream in cup




Underneath 1 the student C writes down 5 and 4, because for scoop 2 and 3, there are 5 x 4 = 20 possibilities left. Next, student C writes down 2. Again he thinks that there are 20 possibilities left for scoop 2 and 3. He repeats this for 3.Then student E hesitates. He crosses out some notes (the second and third small columns.) He thinks that if 2 is on top, there are only 4 x 3 = 12 possibilities left. He continues with 3 on top (3x2= 6 possibilities), 4 on top (2x1= 2 possibilities), etc. Again student E hesitates. The reason for this hesitation is not clear, but he doesn't trust the solutions. Student E decides to systematically write down all the possibilities (see figure 9.) In the columns with 1, student E makes one mistake, namely 144, but he realizes his mistake immediately. Student E continues till he has formed all the combinations in a structured way. Student E tries to explain the structure to the other students. The other students challenge him why for example 132 is not in the columns. Student C tries to explain the structure in the number of possibilities: 4,3,2,1 - 3,2,1 - 2,1 - 1.

Suddenly, student E realises that the structure and combinations belong to the solution of ice-cream cup and not to the ice-cream top three. He crosses the top of the page and replaces this with ice-cream in cup. It is unclear how he concluded this.

In this example we see that student E first uses a mostly formal approach. After choosing the first flavour on top, he calculates the number of possibilities for that one flavour with a multiplication. This multiplication is based on a recognition of 'one less left' and can be characterized as an action on the 4th Van Hiele level. He repeats this for flavour 2 and 3. In fact, up here, the calculation could be proceeded in a correct way but he changes his mind. He decreases both factors in the next multiplications – for flavour 2 - with 1. Student E didn't express his thoughts about this, but probably again a sort of 'one less each time'-idea made him do it this way. The decrease is built on a wrong interpretation of the situation that, after flavour 1 is put in the first place, this flavour is not to be chosen in any other top 3. So, however the combinatorial characteristics order and repetition seem to be considered by the student, which is an action on level 2, the calculation on the forth level is wrong. Student E doubts himself, and after systematically writing down all possibilities (level 3) based on the combinatorial characteristics (level 2) the student reaches insight in what he was doing. His insight is that deep that he can interpret his formal calculation as wrong for the problem and even better, he was able to match the solution to another – but the right! - question.

## Conclusion

In this study we introduced a way to categorize students' remarks while solving combinatorial counting problems. Our first goal was to find a way to identify different levels of thinking in relation to combinatorics. We showed that the Van Hiele levels can be defined to apply to combinatorics and that these levels can be used to classify remarks of students while they are working in groups on combinatorial counting problems.

The analysis of the data shows that students do not go to higher or lower thinking levels as often as one might expect. They do not reach the highest -formal level- by using the lower levels as a justifying level. Neither did we find any indication that students use the different levels to build their solution on the combinatorial characteristics of the two contexts (ice-cream and books), nor that they use different levels to verify their reasoning. We did however see that students make more mistakes when transitioning to a higher level and make fewer mistakes when they go to a lower level. Skipping a level causes even more mistakes.

Interaction might be an obstructing factor for students to use different levels of thinking while working together. Talking about a solution does not stimulate students to make a step to a higher level, but seems to block them to make this step because they often react at the same level as the level at which their fellow students are speaking. In this study, it seems that working in groups does not guarantee that students discover models and use emergent models to come to a solution on a formal level.

Also we tried to investigate the capability of students to recognize combinatorial characteristics in a set of problems. Earlier studies, e.g. by Batanero et al. (1997), indicated that students have difficulties in doing so. By decreasing the number of implicit models, and by decreasing the number of different contexts, we tried to help students to focus on the appropriate combinatorial characteristics to determine similarities and differences among counting problems. During all three lessons students were able to see the difference between a problem with and without repetition but the ordering property was not explicitly named even though some groups did use this property to find the correct solution.

The qualitative results show that a student who has learned the formal approach, in this case the use of the calculator button nCr as a trick for combinations, relies completely on his calculator, while not being able to explain how the result is derived. The student who says that you can solve every problem by using a tree diagram is right for the counting problems in secondary education, but does not see that there are many the same samples in the tree. This shows us that applying a trick, formula or a procedure thoughtlessly might give the correct answer, but that the student is not able to identify the underlying structure or even to assess whether the chosen solution method is applicable.

The second qualitative result shows a student who truly investigates what he is doing. After finding a solution he questions his own result, and tries to confirm it using a different method. Even though he never learned the odometer strategy, after some trying, he applies it flawlessly. He shows us that a gifted student (he was top of his class) can find his own appropriate solution strategy, with only a very small amount of guidance from the teacher.

## Discussion

Although the interventions between the three lessons didn't show significant differences and the three different groups of students are not quite comparable, we think that the results of the data-analysis indicate that Van Hiele levels can be defined for combinatorial problem solving. We hoped to see differences in the use Van Hiele levels as a result of the interventions. However, practical reasons prevented us from observing in comparable student groups. We decided to leave the idea of comparing three different groups because of the previously two mentioned reasons. In order to be able to assess the correctness of the definition of the Van Hiele levels in the field of combinatorial problem solving we looked at the data with all three lessons combined. In future studies it is advisable to use comparable groups.

In retrospect the bookstore problem appeared to be more difficult than we initially thought. Most questions to the teacher were about this problem. However most groups finally found the correct answer, we think the context of this problem is more complex than that of the other problems. Although it is a selection context, the subject that selects - the customer - is not the subject who counts - the store owner. Moreover there are four different selectors. These aspects may have been confusing for the students.

In mathematical reasoning students don't automatically go through the van Hiele levels and the sequence of the levels does not guarantee a correct solution process. Teachers should be aware of the fact that students easily make mistakes when they go to a higher level in their solution process. Guidance by the teacher seems important. Yet we think that education focused on relational understanding is of much more value than instrumental instruction. Students are more capable of verifying their strategies and justifying their reasoning when education is built on their informal approach (Eizenberg & Zaslavsky, 2009). We believe that the Van Hiele levels can play an important role in this sort of education. Exploration at lower levels can help students to develop a relational network of knowledge and come to sensible mathematics. There seems to be an important role for the teacher; students need guidance to reach a higher level. We think future research should investigate how education in combinatorics could be matched with the Van Hiele levels and what type of guidance is most effective.

In all studies on mathematics where students had to solve problems, we found the problems to be posed by the teacher or researcher. The attributes and properties of the contexts and models were constructed by the teachers and researchers. The main job for the students is to calculate the right number of different possibilities. In this study we followed this habit. But perhaps it might be more valuable to let students discover which aspects in problems are important when solving combinatorial problems. So perhaps we must not ask them how many different ice-creams we can make when we are allowed to choose three different flavours out of six. We have to make them discover what is important to know when we want to solve the problem of the ice cream shop owner: how many different ice creams can I sell. Natural questions arise: how many boules can I choose, can I have a flavour twice, do I want a specific order of the boules in my coupe and so on. From the answers to these questions a student should conclude they induce different problems with their own solution strategies. So far we mainly focused on the similarities among counting problems. Similarities were seen as an important factor in recognizing implicit models. But also differences might be helpful in discovering the scope of implicit models and solution strategies.

An idea for future research follows from the second qualitative result. The student hesitates several times, to continue solving the problem successfully. Interviewing students while solving the problem can give insight into their reasoning, which might help to understand the way students think. Understanding their thinking process helps to develop an adequate teaching strategy for combinatorics.

So, even though this study did not directly reveal which way of teaching combinatorics is best, this must not be an argument to keep the current way of teaching combinatorics unchanged. Goldin (2010) repeats DeBellis & Rosenstein (2004) in citing Gardiner (1991, p 49-50) in cautioning: “If instead discrete mathematics is introduced in the schools as a set of facts to be memorized and strategies to be applied routinely . . . [its qualities] as an arena for problem solving, reasoning, and experimentation are of course destroyed.”

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# Appendix A

## Used problems in lesson 1

Anneke, Brenda, Charlotte, Dirk, Emma, Frits, Gijs and Harm are playing cards and keeping score. After finishing the game, they determine who was in first, second and third place. How many possibilities are there to create this ranking? (category: order and without repetition)

Four Teams (ABCD) are playing a competition, afterwards the ranking is announced, for example ABCD or CDBA. Two teams cannot be equal. How many different results are possible? (category: order and without repetition)

Karel ended up with four letters laying Scrabble: L,I,J,K. How many different rows can he form using these four letters? (The rows do not have to form actual words) (category: order and without repetition)

You are given a multiple choice test, consisting of 15 questions. Each question has four possible answers, of which one will be correct. How many different ways are there of completing the test when giving random answers? (category: order and with repetition)

You are buying an ice-cream cone with three scoops. The store offers 10 flavours. How many cones can you make if you are allowed a flavour more than once? (category: order and with repetition)

You throw three dice, one red, one white, one green. How many different possibilities are there? (throwing 1 with red and 2 with green gives a different result compared to throwing 1 with green and 2 with red.) (category: order and with repetition)

Going on holiday you want to take three books, from a list of ten. How many options are there to take three books? (category: no-order and without repetition)

Anne, Bert, Chantal, Dik and Evert offer to organise the next party at school. However, only three pupils are needed to organise the party. How many different groups of three can you form starting out with these five pupils? (category: no-order and without repetition)

Going on holiday, you want to take three of your ten shirts, how many options are there to take three? (category: no-order and without repetition)

# Appendix B

## Used problems in lesson 2&3

Ice-cream top three
You are buying three scoops of ice-cream, you can choose from 6 flavours. You have to rank your top three favourite flavours. How many different top three lists are possible? (category: order and without repetition)

Ice-cream cup
You are buying three scoops of ice-cream in a cup, you can choose from 6 flavours. How many different cups can you make, if you are allowed to choose a flavour only once? (category: no-order and without repetition)

Ice-cream cone
You are buying an ice-cream cone with three scoops of ice-cream, you can choose from 6 flavours. How many different cones can make, if you can choose a flavour as many times as you like? (category: order and with repetition)

Books top four
Your teachers asks you to write down your top four favourite books from the list of ten you had to read. How many different lists can the teacher get from his pupils? (category: order and without repetition)

Books on holiday
During the holiday you want to take four books from the list of ten, you have to read for school. How many different possibilities are there to bring four books on holiday? (category: no-order and without repetition)

Bookstore
There is a top ten thrillers, all books are in a certain bookstore, with plenty copies available. The owner of the store records the sales from the top ten for the first four customers. How many different lists with customer 1 up to 4 are possible? (category: order and with repetition)