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Weight Distribution in Matching Games

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Abstract

Simple games that permit a weight representation such that each winning coalition has a weight of at least 1 and all losing coalitions have a weight of at most α, are called α-roughly weighted games. For a given game the smallest such value of α is called the critical threshold of the game. Freixas and Kurz [1] improved the lower bound on α after initial work of Gvozdeva, Hemaspaandra and Slinko [2] and conjectured that their bound is tight. In this study we give a proof of their conjecture for simple games that have minimal winning coalitions of order 2.

1 Introduction

A *cooperative game* is defined by a finite set N of *players* and a *value function* v, assigning a certain value $v(S) \in \mathbb{R}$ to every subset $S \subseteq N$. Each subset $S \subseteq N$ is interpreted as a coalition of players and the corresponding value $v(S)$ represents the gain which the players $i \in S$ can achieve by cooperating. In the simplest case the value function takes only values 0 and 1. In this case we simply distinguish between *winning* ($v(S) = 1$) or *losing* coalitions ($v(S) = 0$). If, in addition, v is monotone, i.e., supersets of winning coalitions are winning, the game is referred to as a *simple game*.
Specific examples of simple games are so-called *weighted voting games*: Assume that each player $i \in S$ has an associated weight $w_i \geq 0$ and define $v(S) = 1$ if $w(S) := \sum_{i \in S} w_i \leq 1$ and $v(S) = 0$ otherwise. This obviously defines a simple game. Not every simple game can be defined this way. Consider, for example, a set $N = \{1...n\}$ of $n \geq 4$ players and define the winning coalitions to be the sets of the form $\{i, i+1\}$ and supersets thereof. This defines a simple game in which both sets $S_{odd} = \{i \in N \mid i \text{ is odd}\}$ and $S_{even} = \{i \in N \mid i \text{ is even}\}$ are losing. Assume for simplicity that n is even. Then, if our game were a weighted voting game, there were corresponding non-negative weights satisfying $w_i + w_{i+1} \geq 1$ for all winning coalitions $S = \{i, i+1\}$. This implies $w(N) = n/2$ and hence either $w(S_{odd})$ or $w(S_{even})$ must exceed $n/4$. So there are losing coalitions of weight $n/4 > 1$, a contradiction.
Freixas and Kurz [1] have conjectured that for every simple game there exist weights w_i such that all winning coalitions S have weight $w(S) \geq 1$ and all losing coalitions have weight at most $\frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$ where n is the number of players. In this study we will investigate and prove this conjecture for some natural and

interesting subclasses of simple games and for the special case where all minimal winning coalitions have cardinality 2.

2 About simple games

A nice overview of the subject of weighted simple games with references to early work is given by Gvozdeva and Slinko [3] and Taylor and Zwicker [4,5]. We recommend these works to the interested reader who wants to know more about simple games. Here we restrict ourselves to the fundamental definitions and necessary notions to (partially) prove the conjecture of Freixas and Kurz.

Definition 2.1. *Let $P = [n] = \{1, 2, ..., n\}$ be a set of players and let $\emptyset \neq \mathcal{W} \subseteq 2^P$ be a collection of subsets of P that satisfies the monotonicity condition:*

$$\textit{if } X \in \mathcal{W} \textit{ and } X \subseteq Y \subseteq P \textit{ then } Y \in \mathcal{W}.$$

In such case the pair $G = (P, \mathcal{W})$ is called a **simple game** *and the set $\mathcal{W}$ is called the set of* **winning** *coalitions of G. Coalitions that are not in $\mathcal{W}$ are called* **losing**. *A winning coalition is said to be* **minimal** *if every proper subset in it is a losing coalition, so removing any player from such coalition will make it losing. Analogue, a losing coalition is said to be* **maximal** *if every proper superset of it is a winning coalition, i.e. adding any player will make it winning. The set of all losing coalitions is called $\mathcal{L}$.*

Due to the monotonicity property the set $\mathcal{W}$ is completely determined by the collection $\mathcal{W}^{min}$ of all minimal winning coalitions of G. Because $\mathcal{W} \neq \emptyset$ its clear that due to the monotonicity condition $P \in \mathcal{W}$. Furthermore, a game is also fully determined by the collection $\mathcal{L}^{max}$ of maximal losing coalitions. To exclude trivial games we demand $\emptyset \notin \mathcal{W}$.

Definition 2.2. *A simple game $G = (P, \mathcal{W})$ is called a* **weighted majority** *game if there exist nonnegative weights $w_1, ..., w_n$ and a real number q, called quota, such that*

$$\begin{array}{ll} w(S) \geq q & \textit{for all } S \in \mathcal{W} \\ w(S) < q & \textit{for all } S \in \mathcal{L} \end{array}$$

Instead of $\sum_{i \in X} w_i$ we will often write $w(X)$.

Not all simple games are weighted majority games. Moreover, most games[1] are not weighted [5]. Games exist with $w(S) \geq q$ for all $S \in \mathcal{W}$ and $w(S) \leq q$ for all $S \in \mathcal{L}$. These games are called *roughly weighted*. Some games are not even roughly weighted. In those games the lightest winning coalition has a weight that is less than the most heavy losing coalitions. Gvozdeva, Hemaspaandra & Slinko [2] introduced the class of α-roughly weighted games to be able to measure the distance of a game to a (roughly) weighted game. In this class the quota is extended to an interval $[1, \alpha]$ for an $\alpha \in \mathbb{R}_{\geq 1}$, while $w(S) \geq 1$ for all $s \in \mathcal{W}$ and $w(S) \leq \alpha$ for all $S \in \mathcal{L}$ with $\mathcal{L} = 2^p \setminus \mathcal{W}$.

[1] Because this article is just about simple games, we will often omit the word simple when we speak about simple games. So each time we write *game* we mean *simple game* unless we specify otherwise.

Given an α-roughly weighted game $G = (P, \mathcal{W})$ we are looking for a weight function w such that α is as small as possible. The idea behind this is that a game with a smaller α is nearer to a (roughly) weighted game than games with a larger α. This smallest α suitable for a given α-roughly weighted game G is called the *critical threshold-value* $\alpha(G)$ of game G [1]. Finding $\alpha(G)$ can be formulated as a linear program. Because a simple game is fully determined by its collection of minimal winning coalitions $\mathcal{W}^{min}$, this linear program is:

$$\begin{array}{llcll} \alpha(\mathrm{G}) := & \min \alpha & & & \\ \text{subject to} & w(S) & \geq & 1 & \text{for } S \in \mathcal{W}^{min} \\ & w(S) & \leq & \alpha & \text{for } S \in \mathcal{L}^{max} \\ & w_1, .., w_n & \geq & 0 & \end{array}$$

where $\mathcal{L}^{max}$ is the collection of maximal losing coalitions.

In this LP $\alpha(G)$ denotes the minimum weight of the maximum weighted losing coalition of a game $G(P, \mathcal{W})$ while the weight of the minimum weighted winning coalition is 1. This formulation is a slight modification of the formulation by Freixas and Kurz [1]. They demand $\alpha \geq 1$ as a consequence of the definition of α-roughly weighted games by Gvozdeva et al. However, if we omit the constraint $\alpha \geq 1$ in the LP we will find the weight of the maximum weighted losing coalition for any type of weighted game, while the minimum weighted winning coalition has a weight 1. Notice that the LP will yield $\alpha \geq 0$ because $w(S) \geq 0$ for all $S \subseteq P$. In the case of a weighted majority game we will find $0 \leq \alpha < 1$, in the case of a roughly weighted game we will find $\alpha = 1$ and for α-roughly weighted games we will find $\alpha > 1$. The reason why we allow $\alpha \geq 0$ instead of $\alpha \geq 1$ is that in the rest of this study we will consider $\alpha(G)$ not only for games with $\alpha(G) \geq 1$ but for all games, so we need $\alpha(G) \geq 0$.
Obviously the size of the class of α-roughly weighted games varies with α. A larger value of α will capture more games, and a smaller α will capture less. So a very natural question to ask is whether a smallest α exists, such that all games are in the class of α-roughly weighted games. It is clear that such an α doesn't exist in general, but for games G_n with the same size n this α depends on n. So we are looking for a function $\alpha : \mathbb{N} \to \mathbb{R}$ such that $\max_{G_n} \alpha(G_n) \leq \alpha(n)$ for all n and $\alpha(n)$ is minimal with this property.

3 Known bounds on the critical threshold

Gvozdeva, Hemaspaandra and Slinko [2] gave a lower bound for $\max_{G_n} \alpha(G_n)$ for $n \geq 4$ by considering games with $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint minimal winning coalitions of two players and proved $\max_{G_n} \alpha(G_n) \geq \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor$. Freixas and Kurz [1] improved this bound a little for specific odd games by using duality in linear programming and proved $\max_{G_n} \alpha(G_n) \geq \frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$. They showed this bound by considering odd games G where all players are in $n-1$ minimal winning coalitions $\{i, i+1\}$. They found a feasible solution for the dual of the LP and deduced $\alpha(G) \geq \frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$ for this type of games. For games G with an even number of players they followed Gvozdeva, Hemaspaandra and Slinko [2] to show $\alpha(G) = \frac{n}{4}$

which equals $\frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor$ when n is even (see Proposition 4.3 on page 4). So by considering these games it's clear that $\max_{G_n}\alpha(G_n) \geq \frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor$. Freixas and Kurz [1] also conjectured that this bound is tight, so $\max_{G_n}\alpha(G_n) = \alpha(n) = \frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor$ for games with four or more players.

Games with $n \leq 4$ are (roughly) weighted [3], so $\max_{G_n}\alpha(G_n) \leq 1$ for $n \in \{1,2,3,4\}$. For $n \leq 4$ it's easy to check, by considering all possibilities for W and choosing an appropriate weight distribution w that $\alpha(G_1) = 0 = \alpha(1)$, $\alpha(G_2) = 1/2 = \alpha(2)$, $\alpha(G_3) = 2/3 = \alpha(3)$ and $\alpha(G_4) = 1 = \alpha(4)$.
So the conjecture can be relaxed to $\alpha(G_n) \leq \alpha(n) = \frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor$ for all $G \in \mathcal{G}_n$ with $n \geq 1$.

We will get back to this after the following preliminaries.

4 Preliminaries

Because α-roughly weightedness was only defined for games with $n \geq 4$ players with $\alpha \geq 1$ by Gvozdeva and Slinko [3] we like to state the next definition, which is a slight modification of the original definition.

Definition 4.1. *A simple game $G(P,W)$ is called* **$\alpha(n)$-roughly weighted** *if there are weights $w_1,\dots,w_n \in \mathbb{R}_{\geq 0}$ fulfilling*

$$\begin{array}{ll} w(S) \geq 1 & \text{for all } S \in \mathcal{W} \\ w(S) \leq \alpha(n) & \text{for all } S \in \mathcal{L} \text{ with } \mathcal{L} = 2^P \setminus \mathcal{W} \end{array}$$

with $n = |P|$.

Now we state some properties related to fractions and to $\alpha(n)$.
Proposition 4.1. *for $a_1, b_1, a_2, b_2 \in \mathbb{N}$*

$$\text{if } \tfrac{a_1}{b_1} \leq \tfrac{a_2}{b_2} \text{ then } \tfrac{a_1}{b_1} \leq \tfrac{a_1+a_2}{b_1+b_2} \leq \tfrac{a_2}{b_2}$$

Proof. First notice that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$ is equivalent to $a_1b_2 \leq a_2b_1$ for $a_1, b_1, a_2, b_2 \in \mathbb{N}$. Now $\frac{a_1}{b_1} = \frac{a_1(b_1+b_2)}{b_1(b_1+b_2)} = \frac{a_1b_1+a_1b_2}{b_1(b_1+b_2)} \leq \frac{a_1b_1+a_2b_1}{b_1(b_1+b_2)} = \frac{a_1+a_2}{b_1+b_2}$. The second inequality follows by a similar argument. □

Proposition 4.2. *for $a_1, b_1, a_2, b_2 \in \mathbb{N}$*

$$\text{if } \tfrac{a_1}{b_1} \leq \tfrac{a_2}{b_2} \text{ then } \tfrac{a_1}{a_1+b_1} \leq \tfrac{a_2}{a_2+b_2}$$

Proof. $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \Leftrightarrow \frac{b_1}{a_1} \geq \frac{b_2}{a_2} \Leftrightarrow \frac{b_1}{a_1}+1 \geq \frac{b_2}{a_2}+1 \Leftrightarrow \frac{b_1+a_1}{a_1} \geq \frac{b_2+a_2}{a_2} \Leftrightarrow \frac{a_1}{a_1+b_1} \leq \frac{a_2}{a_2+b_2}$. □

Proposition 4.3. *For even $a \in \mathbb{N}$*

$$\left\lfloor\frac{a^2}{4}\right\rfloor = \frac{a^2}{4}$$

Proof. Suppose $a = 2k$ with $k \in \{1, 2, 3, \ldots\}$. Then $\left\lfloor \frac{a^2}{4} \right\rfloor = \left\lfloor \frac{4k^2}{4} \right\rfloor = \lfloor k^2 \rfloor = k^2 = \frac{4k^2}{4} = \frac{a^2}{4}$. $\square$

Proposition 4.4. *For odd $a \in \mathbb{N}$*

$$\left\lfloor \frac{a^2}{4} \right\rfloor = \frac{a^2 - 1}{4}$$

Proof. Suppose $a = 2k + 1$ with $k \in \{0, 1, 2, \ldots\}$. Then $\left\lfloor \frac{a^2}{4} \right\rfloor = \left\lfloor \frac{(2k+1)^2}{4} \right\rfloor = \left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k = \frac{4k^2+4k+1-1}{4} = \frac{(2k+1)^2-1}{4} = \frac{a^2-1}{4}$. $\square$

Proposition 4.5. *For $a \in \mathbb{N}$*

$$\frac{a-1}{a} \le \frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor \le \frac{a}{4}$$

Proof. First notice that $a \neq 0$, so $\frac{1}{a}$ exists. For odd $a \geq 3$ the first inequality in the proposition follows because $\frac{a+1}{4} \geq 1$ so, $\frac{a-1}{a} \le \frac{a-1}{a} \cdot \frac{a+1}{4} = \frac{1}{a} \cdot \frac{a^2-1}{4} = \frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor$ and for $a = 1$ by the simple substitution $\frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor = \frac{1}{1} \left\lfloor \frac{1^2}{4} \right\rfloor = 0 = \frac{a-1}{a}$. The second inequality in the proposition follows by $\frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor = \frac{1}{a} \cdot \frac{a^2-1}{4} < \frac{1}{a} \cdot \frac{a^2}{4} = \frac{a}{4}$ for odd $a \geq 1$.
For even a we see that $\frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor = \frac{1}{a} \cdot \frac{a^2}{4} = \frac{a}{4}$. This proves the second inequality of the proposition for even a. The first inequality in the proposition holds for even a because $0 \le (a-2)^2 = a^2 - 4a + 4 = a^2 - 4(a-1)$. So $4(a-1) \le a^2$ which yields $\frac{a-1}{a} \le \frac{a}{4} = \frac{1}{a} \left\lfloor \frac{a^2}{4} \right\rfloor$. $\square$

Lemma 4.1. *Any simple game $G(N, \mathcal{W})$ with just one minimal winning coalition X is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$.*

Proof. If G contains dummies, we set their weights to 0. We set $w(i) = \frac{1}{|X|}$ for all $i \in X$, so $w(X) = 1$. Now the maximum weighted losing coalition L^{max} can contain at most $|X| - 1$ players from X and some or all dummies. Since $|X| \le n$ it follows that $\alpha(G) \le w(L^{max}) = \frac{|X|-1}{|X|} \le \frac{n-1}{n} \le \frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$. $\square$

Lemma 4.2. *Let $G_1(N_1, \mathcal{W}_1)$ and $G_2(N_2, \mathcal{W}_2)$ be two disjoint simple games with $N_1 \cap N_2 = \emptyset$, $n_1 = |N_1|$, $n_2 = |N_2|$. If G_1 is $\alpha(n_1)$-roughly weighted and G_2 is $\alpha(n_2)$-roughly weighted, then the joined game $G(N, \mathcal{W})$ with $N = N_1 \cup N_2$ and $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ is $\alpha(n)$-roughly weighted with $n = n_1 + n_2$ and $\alpha(n) = \frac{1}{n} \left\lfloor \frac{n^2}{4} \right\rfloor$.*

Proof. For all $X \in \mathcal{W}^{min}$ it is obvious that $X \in \mathcal{W}_1$ or $X \in \mathcal{W}_2$, so it clear that $w(X) = 1$. Because $N_1 \cap N_2 = \emptyset$ also the maximum weighted losing coalitions $L_1 \subset N_1$ and $L_2 \subset N_2$ are disjoint. So its clear that in the joint game G the maximum weighted losing coalition $L_G = L_1 \cup L_2$. So $\alpha(G) = w(L_G) = w(L_1) + w(L_2)$.
If n is even, then $\alpha(n) = \frac{1}{n_1+n_2} \left\lfloor \frac{(n_1+n_2)^2}{4} \right\rfloor = \frac{(n_1+n_2)}{4} = \frac{n_1}{4} + \frac{n_2}{4} \ge \frac{1}{n_1} \left\lfloor \frac{n_1^2}{4} \right\rfloor +$

$\frac{1}{n_2}\left\lfloor\frac{n_2^2}{4}\right\rfloor = \alpha(n_1)+\alpha(n_2) \geq w(L_1)+w(L_2) = w(L_G) = \alpha(G)$.
If n is odd, then, w.o.l.g. we may assume that n_1 is even and n_2 is odd, so $\alpha(n) = \frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor = \frac{n^2-1}{4n} = \frac{n_1+n_2}{4} - \frac{1}{4(n_1+n_2)} > \frac{n_1+n_2}{4} - \frac{1}{4n_2} = \frac{n_1}{4} + \frac{n_2^2-1}{4n_2} = \frac{1}{n_1}\left\lfloor\frac{n_1^2}{4}\right\rfloor + \frac{1}{n_2}\left\lfloor\frac{n_2^2}{4}\right\rfloor = \alpha(n_1)+\alpha(n_2) \geq w(L_1)+w(L_2) = w(L_G) = \alpha(G)$. □

Lemma 4.3. *Let $G(N,\mathcal{W})$ be an $\alpha(n)$-roughly weighted game for $\alpha(n) = \frac{1}{n}\left\lfloor\frac{n^2}{4}\right\rfloor$. Now let $S \subset N$ with $w(S) \geq 1$. Then the game $G^*(N,\mathcal{W}^*)$ with $\mathcal{W}^* = \mathcal{W} \cup \{S\}$ is $\alpha(n)$-roughly weighted.*

Proof. Note that $S \neq \emptyset$ because $w(S) \geq 1$. If $S \in \mathcal{W}$ the lemma is clear because $G^* = G$ so $\alpha(G^*) = \alpha(G) \leq \alpha(n)$. Now let $L^* \in \mathcal{W}^*$ be a maximum weighted losing coalition in G^* so $X \nsubseteq L^*$ for all $X \in \mathcal{W}^*$. Because $\mathcal{W}^* \supseteq \mathcal{W}$ this means that $X \nsubseteq L^*$ for all $X \in \mathcal{W}$, so L^* is a maximum weighted losing coalition for G. So $\alpha(G^*) = w(L^*) \leq \alpha(G) \leq \alpha(n)$. □

5 Games and graphs

In the previous sections we considered games from a set-theoretical point of view. Another, very much related viewpoint is graph-theoretical. A simple game can be seen as a hypergraph, where the players are the vertices and the coalitions are the hyperedges. So a weighted simple game is now a hypergraph with weighted vertices. The weights of the hyperedges are the weights of the coalitions. Because simple weighted games are fully determined by their minimal winning coalitions, a simple game can be represented by the hypergraph with the minimal winning coalitions as hyperedges. A coalition is winning if it contains any hyperedge as a (not necessarily proper) subset.
A special type of games are the games with a collection of minimal winning coalitions that all have cardinality 2. We call these games *matching games*. In this type of game all minimal winning coalitions are pairs of players. These coalitions may intersect. The hypergraph representation of this type of game is a simple graph, where the vertices are the players and the edges are the minimal winning coalitions. With $G(V,E)$ we denote the corresponding graph to the game $G(N,\mathcal{W})$. Although the notation of both is very similar, no confusion will occur and we will use both notations.
For matching games $G(N,\mathcal{W})$ any maximal weighted losing coalition is an independent set in the corresponding graph $G(V,E)$. However, a maximum weighted losing coalition doesn't need to be a maximum independent set. Suppose, for instance, that we have a game on $n \geq 3$ players represented by a star $K_{1,n-1}$. Suppose that player 1 is the center of the star. Then, by giving the center of the star a weight of $w_1 = 1$ and all other players a weight of $w_i = 0$ ($i \in \{2,\ldots,n\}$), the center of the star is a maximal weighted losing coalition $L_1 = \{1\}$ with $w(L_1) = 1$. The center is a maximal independent set, but not a maximum independent set. In fact, we have two maximal losing coalitions L_1 with weight $w(L_1) = 1$, and $L_2 = \{2,\ldots,n\}$ with weight $w(L_2) = 0$. In order to keep the weight of the maximum weighted losing coalition as small as possible, we can decide to distribute the weights more equally. By setting $w_i = \frac{1}{n}$ for $i \in \{2,\ldots,n\}$ and $w_1 = 1 - \frac{1}{n}$ we can create a situation where both maximal losing coalitions

L_1 and L_2 have a weight of $w(L_1) = w(L_2) = \frac{n-1}{n}$ and the winning coalitions still have a weight 1. Notice that this weight distribution respects the bound $\alpha(G) \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ for any star with (even with $n \geq 2$) by Proposition 4.5.

6 Matchings and games

Another way of looking at maximum weighted losing coalitions in matching games is by considering matchings. Any losing coalition can contain at most one player per minimal winning coalition. So any maximum weighted losing coalition can not have two adjacent vertices. This means we can pick one vertex per edge at most in a losing coalition. So any maximum weighted losing coalition, can contain at most half of the players of a maximum matching plus all players that are no part of this maximum matching.

Lemma 6.1. *Every matching game $G(N, \mathcal{W})$ with a perfect matching is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$.*

Proof. Suppose the game has $n = 2m$ players ($m \in \mathbb{N}$). In the maximum matching there are m edges, corresponding with m minimal winning coalitions. All players are matched because the matching is perfect. In any losing coalition there are at most m players. If we would have more than m players, there must be at least two players of the same minimal winning coalition in the losing coalition, which is a contradiction. Now we can give all edges $e \in M$ a weight $w(e) = 1$ by giving all players i a weight $w_i = \frac{1}{2}$. Now the weight of the maximum weighted losing coalition L^{max} is $\alpha(G) = w(L^{max}) = m \cdot \frac{1}{2} = \frac{n}{4} = \alpha(n)$. □

Definition 6.1. *A graph G is called* **factor-critical** *if deleting any vertex from G will result in a graph with a perfect matching.*

Lemma 6.2. *Every matching game $G(N, \mathcal{W})$ that can be represented by a factor-critical graph $G(V, E)$ is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ by the weight distribution $w_u = \frac{1}{2}$ for all $u \in V$.*

Proof. Observe that a factor-critical graph has an odd order, because deleting any vertex will leave a graph with a perfect matching M, which must have an even order M. Because there is at least one winning coalition of 2 players, and the number of players is odd, it's clear that $n = |N| \geq 3$. Notice that any minimal winning coalition W^{min} has a weight $w(W^{min}) = 1$.
First notice that it is impossible that all players are in a maximal losing coalition L^{max}. So there is at least one player $u \in V$ such that $u \notin L^{max}$. Because G is factor critical, $G' = G - u$ contains a perfect matching. Now by Lemma we know that this graph G' can have at most half of its vertices to be chosen in any maximal losing coalition. So in G there can be at most $\frac{|G'|}{2} = \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$ vertices which are in the maximum losing coalition. Now we give all players $i \in N$ the proposed weight $w_i = \frac{1}{2}$, so $\alpha(G) = w(L^{max}) = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$. □

Lemma 6.3. *Any game represented by a biregular graph $G(A, B; E)$ is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ where $n = |A| + |B|$.*

Proof. W.l.o.g. we assume $|A| \leq |B|$. Let $0 < \lambda = \frac{|A|}{n} \leq \frac{1}{2}$ be the fraction of the number of players that are in A, so $|A| = \lambda n$ and $|B| = (1-\lambda)n$. Any maximal losing coalition L will contain a number $l_A = |L \cap A|$ players in A and a number $l_B = |L \cap B|$ players in B. Those l_A vertices in A are incident to $deg(A) \cdot l_A$ edges that are incident to $\frac{deg(A)}{deg(B)} \cdot l_A$ vertices in B. Its clear that in the biregular graph $deg(A) \cdot |A| = deg(B) \cdot |B|$, so l_A vertices in A cover $\frac{deg(A)}{deg(B)} \cdot l_A = \frac{|B|}{|A|} \cdot l_A = \frac{1-\lambda}{\lambda} \cdot l_A$ vertices in B.
Assume L contains a fraction ρ of the players in A, so $l_A = \rho|A|$ ($0 \leq \rho \leq 1$ such that $\rho|A| \in \mathbb{N} \cup \{0\}$). Then there are $\frac{1-\lambda}{\lambda} \cdot l_A$ vertices in B that are connected with these vertices in A. Because no losing coalition can contain two players from a minimal winning coalition, these vertices can not be present in the maximal losing coalition. So its clear that $l_B \leq |B| - \frac{1-\lambda}{\lambda} l_A = (1-\lambda)n - \frac{1-\lambda}{\lambda}\rho\lambda n = (1-\lambda)n - (1-\lambda)\rho n = (1-\rho)(1-\lambda)n$.
If we chose for each player $a \in A$ a weight $w_a = 1 - \lambda$ and for each player $b \in B$ a weight $w_b = \lambda$ then the weights of all edges, which are the minimal winning coalitions, are 1. Any maximal losing coalition L will have a weight $w(L) \leq w_a l_A + w_b l_B \leq (1-\lambda)\rho\lambda n + \lambda(1-\lambda)(1-\rho)n = \lambda(1-\lambda)n(\rho + 1 - \rho) = \lambda(1-\lambda)n$.
By simple calculus we know that $P(\lambda) = \lambda(1-\lambda)n$ has a maximum value of $\frac{n}{4}$ for $\lambda = \frac{1}{2}$. But since $\lambda = \frac{|A|}{n}$ the value $\lambda = \frac{1}{2}$ can only occur when n is even. The maximum weighted losing coalition L^{max} will have a weight $w(L^{max}) = \frac{n}{4}$ when $\lambda = \frac{1}{2}$, so when $|A| = |B|$. When $\lambda < \frac{1}{2}$, so when $|A| < |B|$ the weight of the maximum weighted losing coalition $w(L^{max}) < \frac{n}{4}$. This proves $\alpha(G) = w(L^{max}) < \frac{n}{4} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ when n is even. If n is odd, the maximum weight is reached for $|A| = \left\lfloor \frac{n}{2} \right\rfloor$, so $\lambda = \left\lfloor \frac{n}{2} \right\rfloor / n$. This yields $w(L^{max}) = \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{n}(1 - \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{n})n = \frac{n-1}{2}(1 - \frac{n-1}{2} \cdot \frac{1}{n}) = \frac{n-1}{2} \cdot \frac{n+1}{2n} = \frac{1}{n} \cdot \frac{n^2-1}{4} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$. So for every game G represented by a biregular graph choosing the proposed λ-weight distribution, will guarantee that $\alpha(G) \leq \alpha(n)$. □

Lemma 6.4. *Any game represented by a bipartite graph $G(A, B; E)$ which contains a Hamilton path, is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ where $n = |A| + |B|$.*

Proof. W.l.o.g. we may assume $|A| \leq |B|$. If $|A| = |B|$, its clear that G has a perfect matching, so the lemma follows from Lemma 6.1. Suppose $|A| < |B|$. Then the Hamiltonpath starts in B, and the vertices in the path are alternately in A and B, with the last vertex in B, so $|B| = |A| + 1$. Now we know that n is odd and $|A| = \frac{n-1}{2}$ and $|B| = \frac{n+1}{2}$.
Its clear that if we take $\lambda|A|$ vertices from A in any maximal losing coalition, this coalition can contain at most $|B| - (\lambda|A| + 1) = (1-\lambda)|A|$ vertices in B.
Now we give weights $w_a = \frac{|B|}{n}$ for $a \in A$ and $w_b = \frac{|A|}{n}$ for $b \in B$.
The weight of any minimal winning coalition $\{a, b\}$ with $a \in A$ and $b \in B$ is $w_a + w_b = \frac{|B|}{n} + \frac{|A|}{n} = 1$.
The weight of a maximal losing coalition L^{max} is bounded by $\alpha(G) = w(L^{max}) \leq \lambda|A| \cdot w_a + (1-\lambda)|A| \cdot w_b < \lambda|A| \cdot w_a + (1-\lambda)|A| \cdot w_a = |A| \cdot w_a = \frac{n-1}{2} \cdot \frac{n+1}{2n} =$

$\frac{1}{n} \cdot \frac{n^2-1}{4} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor = \alpha(n)$ where the second last equality holds because of Propostion 4.4 since n is odd. □

Lemma 6.5. *Any game represented by a bipartite graph G(A,B;E) with a matching of A into B is $\alpha(n)$-roughly weighted with $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ where $n = |A| + |B|$.*

Proof. Since a matching of A into B exists its clear that $|A| \leq |B|$ and, by Halls condition, $|S| \leq |N(S)|$ for $S \subseteq A$. Let σ_S be the ratio of the number of vertices in S and the number of neighbors of S in B, so $\sigma_S = \frac{|S|}{|N(S)|}$. Now we decompose G in the following way: Let $A_1 \subseteq A$ be a largest subset of A among the subsets $S \subseteq A$ with the largest σ_S. If $A_1 \neq A$ then remove A_1 from A to get A' and remove $N(A_1)$ from B to get B' and iterate this procedure on the remaining subgraph $G'(A', B'; E')$ to find $A_2, A_3, \ldots, A_k$. In this way we partition A into $\{A_1, \ldots, A_k\}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_i A_i = A$ with $i, j \in \{1, \ldots, k\}$. Next let $B_1 = N(A_1)$ and $B_i = N(A_i) \setminus \bigcup_{j=1}^{i-1} B_j$ for $i \in \{2, \ldots, k\}$. So $\{B_1, \ldots, B_k\}$ is a partition of B.

We have to be sure that at any time in the partitioning $N(A_i) \neq \emptyset$ to be sure B_i exists. For A_1 this is obvious. For $i > 1$ suppose that $N(A_i) = B_i = \emptyset$ and $B_{i-1} \neq \emptyset$. Then $\sigma_{A_{i-1} \cup A_i} = \frac{|A_{i-1} \cup A_i|}{|B_{i-1} \cup B_i|} = \frac{|A_{i-1}| + |A_i|}{|B_{i-1}|} > \frac{|A_{i-1}|}{|B_{i-1}|} = \sigma_{S_{i-1}}$ which is a contradiction to the construction of the partitioning of G because $S_{i-1} \cup S_i$ should have been chosen in the partition instead of S_{i-1}.

Also notice that $\sigma_{A_i} > \sigma_{A_{i+1}}$. Suppose for the contrary, that $\sigma_{A_i} \leq \sigma_{A_{i+1}}$, so $\frac{|A_i|}{|B_i|} \leq \frac{|A_{i+1}|}{|B_{i+1}|}$. But then, by Proposition 4.1, $\sigma_{A_i} = \frac{|A_i|}{|B_i|} \leq \frac{|A_i| + |A_{i+1}|}{|B_i| + |B_{i+1}|} = \frac{|A_i \cup A_{i+1}|}{|B_i \cup B_{i+1}|} = \sigma_{A_i \cup A_{i+1}}$. This again is a contradiction to the construction of the partitioning of G because now $A_i \cup A_{i+1}$ should have been chosen in the partition instead of A_i.
So now we have a partition of the graph G into subgraphs $G_i(A_i, B_i; E_i)$ and $\sigma_i = \frac{|A_i|}{|B_i|}$ where $\sigma_i > \sigma_j$ for $i < j$.

Let $n_i = |A_i| + |B_i|$ and let λ_i be the fraction of the vertices of G_i that are in A_i. So $|A_i| = \lambda_i n_i$ and $|B_i| = (1 - \lambda_i) n_i$. Now suppose there is a fraction ρ_i of the vertices of $|A_i|$ in a maximal losing coalition. Let A'_i be the set of these vertices, so $|A'_i| = \rho_i \lambda_i n_i$. By definition of the partition of A its clear that $\frac{|A'_i|}{|N(A'_i)|} \leq \frac{|A_i|}{|N(A_i)|} = \frac{|A_i|}{|B_i|}$. Indeed, $\frac{|A'_i|}{|N(A'_i)|} > \frac{|A_i|}{|N(A_i)|}$ would contradict the choice of A_i (since the maximality of σ_i). So $|N(A'_i)| \geq \frac{|A'_i|}{|A_i|}|B_i| = \rho_i |B_i|$. This means that $\rho_i \lambda_i n_i$ vertices in A_i cover at least ρ_i of the vertices in $|B_i|$. So, a maximal losing coalition that contains $\rho_i \lambda_i n_i$ vertices in A_i can contain at most $(1 - \rho_i)|B_i|$ of the vertices of B_i.

Now chose $w_a = 1 - \lambda_i$ for $a \in A_i$ and $w_b = \lambda_i$ for $b \in B_i$. Then the weight of a maximum weighted losing coalition L^{max} will be bounded by $w(L^{max}) \leq w_a \rho_i \lambda_i n_i + w_b (1 - \rho_i)(1 - \lambda_i) n_i = \lambda_i (1 - \lambda_i) n_i$. This yields, as shown already in the biregular case in Lemma 6.3, that $\alpha(G_i) \leq \frac{1}{n_i}\left\lfloor \frac{n_i^2}{4} \right\rfloor$.

The rest of the proof follows from Lemma 4.2 and Lemma 4.3. Because all G_i are $\alpha(n_i)$-roughly weighted, the union $\bigcup_i G_i$ is $\alpha(n)$-roughly weighted by Lemma 4.2. Now we have to add edges $e(a,b) \in E(G)$ with $a \in A_i$ and $b \in B_j$ $(i \neq j)$ to add the coalitions that are not in any G_i. Observe that for $i < j$ no edges $e(a,b)$ exist with $a \in A_i$ and $b \in B_j$ due to the definition of the partition of G. In the graph $G(A,B;E)$ edges $e(a,b)$ can exist with $a \in A_j$ and $b \in B_i$ for $i < j$. By definition of the construction of the partition of G we know that for $i < j$ it holds that $\frac{|A_i|}{|B_i|} > \frac{|A_j|}{|B_j|}$. So by Proposition 4.2 we know $\lambda_i = \frac{|A_i|}{|A_i|+|B_i|} > \frac{|A_j|}{|A_j|+|B_j|} = \lambda_j$ and thus $w(e) = w_a + w_b = (1-\lambda_j) + \lambda_i > 1$. So the winning coalition represented by e suffices the minimum weight demanded by the definition of $\alpha(n)$-roughly weighted games and can be added to the game $\bigcup_i G_i$ by Lemma 4.3 without violating the bound. This can be done for all minimal winning coalitions $e(u,v)$ with $u \in A_j$ and $v \in B_i$ for $i < j$. □

7 Proving the conjecture

Now we come to the prove of the conjecture of Freixas and Kurz for games with two-player minimal winning coalitions. Assume the game $G(N, \mathcal{W})$ is such a matching game. Note that all players that are not part of a minimal winning coalition are dummies. So we assume all players are in at least one minimal winning coalition. We construct a graph G, where the vertices are the players, and edges are the minimal winning pairs. As seen before the weighted losing coalition can contain at most half of the players of a maximum matching plus all players that are no part of this maximum matching. To investigate the structure of the matching we use the following decomposition of graphs.

Definition 7.1. *The* **Gallai-Edmonds decomposition** *of a graph $G(V,E)$ is the partition $D \cup A \cup C$ of $V(G)$ given by*

- $D = \{v \in V(G) \mid$ *some maximum matching in G fails to match v*$\}$
- $A = \{u \in V(G) - D \mid u$ *is adjacent to a vertex in D*$\}$
- $C = V(G) - D - A$

When a graph is decomposed according to the Gallai-Edmonds decomposition we know some special properties of the sets in the decomposition:

Theorem 7.1. ***(Gallai-Edmonds Structure Theorem)****. Let A, C, D be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $D_1, \ldots, D_k$ be the components of $G[D]$. If M is a maximum matching in G, then the following properties hold:*

a. M covers C and matches A into distinct components of $G[D]$.

b. Each D_i is factor critical and has a near-perfect matching in M.

c. If $\emptyset \neq S \subseteq A$, then $N(S)$ intersects at least $|S|+1$ of $D_1, \ldots, D_k$.

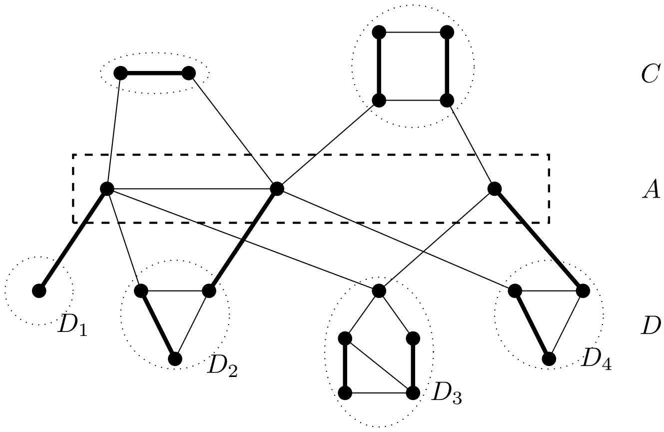

Figure 1: Gallai-Edmonds Decomposition of graph representing an arbitrary matching game. Thick lines indicate a maximum matching.

Proof. See [6] □

So now let the graph $G(V, E)$ represent the matching game $G(N, \mathcal{W})$ with $n = |N| \geq 4$. The minimal winning coalitions may intersect, but w.o.l.g. we may assume that all players are in at least one minimal winning coalition. We will show that we can chose a weight function $w : P \to \mathbb{R}_{\geq 0}$ such that $\alpha(G) \leq \frac{1}{n}\lfloor\frac{n^2}{4}\rfloor$. In order to get an upper bound on the weight of a maximum weighted losing coalition with respect to a weight function, we fix a maximum matching M and decompose G according to the Gallai-Edmonds structure. All components in $C_i \subseteq C$ contain a perfect matching, so by Lemma 6.1 they are $\alpha(n)$-roughly weighted for $n = |C_i|$ by giving all players a weight of 1/2. By Lemma 4.2 it's clear that C is $\alpha(n)$-roughly weighted for $n = |C|$.
In order to prove that $A \cup D$ is $\alpha(n)$-roughly weighted, we will decompose this game into three parts. The first is a bipartite game, consisting of the players in A, the singletons in D and the players of D that are matched into A by M. The second part is the game consisting of the odd components D_i without the matched players and the third part are the odd components that are not connected to A via the matching M.
Notice that due to the Gallai-Edmonds structure theorem we know that in the bipartite graph the number of players in A is less than or equal to the number of matched players in D plus the number of singletons. This will give a weight distribution such that the players in A have a weight greater than or equal to 1/2 and the players in D a weight less than or equal to 1/2. A weight less than 1/2 is good for the singletons, but for the matched players in the odd components we can not allow this. Because these components are factor critical, we would like a weight of 1/2 for all players in those components. We will show that this is possible.
What remains is to add coalitions / edges that are not in any of the components. This are coalitions in A, coalitions that connect players in C with a player in A and coalitions that connect a player in A with an unmatched player in D. But these coalitions can be formed, because the weight of those players is at least

1/2. So, after this outline of the proof of the conjecture of Freixas and Kurz, we state the conjecture formally and proof it.

Theorem 7.2. *For any simple game $G(N, \mathcal{W})$ with $n = |N| \geq 4$ and $|X| = 2$ for all $X \in \mathcal{W}^{min}$, we have $\alpha(G) \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$.*

Proof. Let M be a maximum matching in the graph G. Any maximal losing coalition in G can contain at most half of the vertices in M plus the unmatched vertices. Now decompose the graph representation of the game according to the Gallai-Edmonds decomposition (see Figure 1 for an example of such a game). We set $w_c = 1/2$ for all $c \in C$, so the games represented by these components respect the bound due to Lemma 4. Also set $w_d = 1/2$ for all $d \in D_i$ with $|D_i| \geq 3$.

Now chose in each odd component D_i with $|D_i| \geq 3$ the vertex $d_i \in M \cap D_i$. D' is the union of all these vertices plus the singletons in D. Note that due to properties 1 and 3 of the Gallai-Edmonds structure theorem there can be components D_i for which such a vertex d_i does not exist.

We construct the bigraph $G'(A, D'; E')$, representing the game G'. We only add the edges / coalitions between vertices in A and D' to get a bipartite graph. Notice that in this bipartite graph $|A| \leq |D'|$ and that A is matched by M. We apply Lemma 6.5 to show that G' is $\alpha(n)$-roughly weighted with $n = |V(G')|$ and to achieve a weight distribution $w_a = \frac{|D'|}{|A|+|D'|} \geq \frac{1}{2}$ for $a \in A$ and $w_d = \frac{|A|}{|A|+|D'|} \leq \frac{1}{2}$ for $d \in D'$. Now the minimal winning coalitions represented by an edge $e(a_1, a_2)$ with $a_1, a_2 \in A$ will have a weight $w(e) \geq 1$, so by Lemma 4.3 we can add the desired coalitions in A to the game. Also coalitions represented by $e(a, c)$ with $a \in A, c \in C$ can be added to the game without violating the bound because $w(a, c) = w_a + w_c \geq 1$.

However, the weight distribution in the bipartite graph G' changed the weights of the vertices $d_i \in D'$ into a weight less than $\frac{1}{2}$ while we would like to have all the players in the odd components D_i with $|D_i| \geq 3$ to have a weight of at least $\frac{1}{2}$ to fulfill the demand on the minimal weight of winning coalitions (in D_i). We claim that we can give those players that have a weight less than $\frac{1}{2}$ a weight $\frac{1}{2}$ without making the maximum weighted losing coalition L^{max} in the total game too heavy. We will prove this claim inductively by increasing the weights of all those players $d_i \in D'$ one by one.

Consider an arbitrary D_i for which $D_i \cap M \neq \emptyset$. Let $D_i^* = D_i \setminus \{d_i\}$ and consider the game $G = G' \cup D_i^*$. Notice that $d_i \in G'$ and $d_i \notin D_i^*$. The maximum number of vertices in D_i^* that can be chosen at the same time in any maximal losing coalition is $\frac{m_i}{2}$ with $m_i = |V(D_i)| - 1 = |V(D_i^*)|$. Now let $w_{d_i} = \frac{1}{2}$ and consider two cases.

In the first case, when $d_i \notin L^{max}$, we have $w(L^{max}_{G'}) \leq \alpha(G') \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$ with $n = |V(G')|$ and $w(L^{max}_{D_i^*}) = \frac{1}{2} \cdot \frac{m_i}{2}$. So both games G' and D_i^* respect the bound. Since $G' \cap D_i^* = \emptyset$ we know by Lemma 4.2 that the game represented by $G' \cup D_i^*$ is $\alpha(n)$-roughly weighted with $n = |V(G' \cup D_i^*)|$.

In the second case, when $d_i \in L^{max}$, we see that $w(L^{max}_{G'}) \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor + \frac{1}{2}$ with $n = |V(G')|$ because we increased the weight of d_i from $w_{d_i} \leq \frac{1}{2}$ to $w_{d_i} = \frac{1}{2}$.

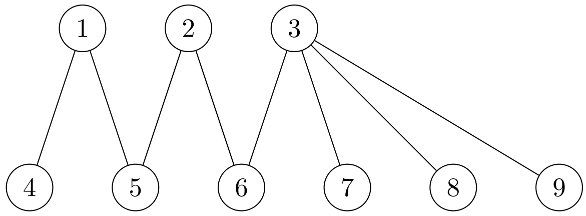

Figure 2: Bipartite graph $G(A, B; E)$ representing a two player minimal wining coalition game with $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8, 9\}$ and a collection of minimal winning coalitions $\mathcal{W} = \{\{1, 4\}, \{1, 5\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{3, 9\}\}$.

The number of players in D_i that can be chosen in L^{max} is $\frac{m_i}{2}$. So because d_i is chosen in L^{max} and $d_i \in D_i = D_i^* \cup \{d_i\}$ we can chose from D_i^* at most $\frac{m_i}{2} - 1$ players in L^{max}.
Now $w(L^{max}) \leq w(L^{max}_{G'}) + w(L^{max}_{D_i^*}) \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor + \frac{1}{2} + \frac{1}{2}\left(\frac{m_i}{2} - 1\right) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor + \frac{1}{2}\frac{m_i}{2}$ for $n = |V(G')|$. Since m_i is even, this yields $w(L^{max}) \leq \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor + \frac{m_i}{4} \leq \frac{1}{n+m_i}\left\lfloor \frac{(n+m_i)^2}{4} \right\rfloor$. So G respects the bound.
Now we add all the minimal winning coalitions $\{d_i, d_i^*\}$ with $d_i^* \in D_i^*$ in order to construct the odd component D_i. This can be done because $w(d_i) = w(d_i^*) = 1/2$ so we can apply Lemma 4.3.
We repeat the inductive step until all matched D_i are in G'. We also add the components D_i for which $D_i \cap M = \emptyset$ to the game. This can be done due to Lemma 6.2 and Lemma 4.2.

Now we apply Lemma 4.3 again to add the remaining minimal winning coalitions $\{a, d\}$ with $a \in A$ and $d \in D \setminus D'$, which can be done because $w_a \geq 1/2$ and $w_d = 1/2$ so $w_a + w_d \geq 1$. Finally, again by Lemma 4.3, we can add the minimal winning coalitions in A because $w_a \geq 1/2$ for all players $a \in A$. □

8 Discussion and conclusion

The decomposition of a matching game that we used in the proof of the conjecture, guarantees that the maximum weighted losing coalition has a weight below $\alpha(n) = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$. However, the weight of this maximum weighted losing coalition is not the minimum possible weight. Let $G(N, \mathcal{W})$ be a simple game with $N = [9]$ and the collection of minimal winning coalitions $\mathcal{W} = \{\{1, 4\}, \{1, 5\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{3, 9\}\}$. This game can be represented by a bipartite graph $G(A, B; E)$ with $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8, 9\}$, see Figure 2.

Notice that A can be matched into B, so we can apply Lemma 6.5. The construction of a partition according to Lemma 6.5 will yield $A_1 = \{1, 2\}, B_1 = \{4, 5, 6\}$ with $w_1 = w_2 = 3/5$ and $w_4 = w_5 = w_6 = 2/5$ and $A_2 = \{3\}$ and $B_2 = \{7, 8, 9\}$ with $w_3 = 3/4$ and $w_7 = w_8 = w_9 = 1/4$. Now A and B are maximum weighted losing coalitions with weight $w(A) = w(B) = 39/20$. This weight respects the bound $\alpha(8) = 2$. However, if we choose weights $w_1 = w_2 = w_4 = w_5 = w_6 = 1/2$, $w_3 = 7/8$ and $w_7 = w_8 = w_9 = 1/8$ still A and B are maximum weighted losing coalitions, but $w(A) = w(B) = 15/8$ which is slightly better than the total weight for a maximum weighted losing coalition of 39/20 that we got as the result of the partitioning according to Lemma 6.5. So a decomposition of the game like in the proof of the conjecture of Freixas and Kurz gives a maximum weighted losing coalition that respects the bound but doesn't yield a weight distribution that makes the maximum weighted losing coalition as light as possible.

In this study we presented a way of proofing the conjecture of Freixas and Kurz for games $G(N, \mathcal{W})$ that have minimal winning coalition which all have order 2. A proof for the case where all minimal winning coalitions have an order of at least 4 is easy. Giving all players in a game a weight of $\frac{1}{4}$ will make $w(S) \geq 1$ for all $S \in \mathcal{W}$. Now any maximum weighted losing coalition $L \in \mathcal{L}$ can contain at most $n-1$ players from the grand coalition N, so for all $S \in \mathcal{L}$ it's clear that $w(S) \leq \frac{1}{4}(n-1)$. For even n this yields $\frac{1}{4}(n-1) = \frac{n-1}{4} < \frac{n}{4} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$. For odd n we see that $\frac{n-1}{4} < \frac{n(n-1)}{4n} < \frac{n^2-1}{4n} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor$.
A natural follow-up would be to investigate games with minimal winning coalitions which all have order 3 and to investigate games with minimal winning coalitions of various orders. Moreover, we suppose that the critical threshold on games with n players is 'bad' for games with small minimal winning coalitions and the bound is only reached in games with minimal coalitions of order 2. So another interesting question is how the critical threshold depends on the size of the smallest minimal winning coalition in relation to the largest losing coalition. This would tell us more about the cost of stability in coalition games than just the value of $\alpha(n)$.

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