Routing Games

Analysis of nonatomic and atomic selfish routing models

Abstract

This work, based on work of Roughgarden (2007), analyzes two models in routing games: the nonatomic and the atomic selfish routing models. Throughout the text, examples are given of both models. Players route a certain amount of traffic through a network consisting of vertices and edges. Each edge has a nondecreasing, nonnegative, and continuous cost function. The network, the amount of traffic and the cost functions together form a nonatomic or an atomic instance $\langle G, r, c \rangle$. For both models, the notion of an equilibrium flow of traffic through the network is defined, as well as the optimal flow. The price of anarchy is defined as the ratio of the cost of the equilibrium flow with the highest total cost and the optimal flow.

A proof is given for the existence of equilibrium flows in nonatomic instances and the fact that every equilibrium flow has equal cost. Furthermore, it is proven that the price of anarchy in affine nonatomic instances (instances with only affine cost functions) is at most $\frac{4}{3}$.

An equilibrium flow in atomic instances need not exist, but it is proven it does when every player routes the same amount of traffic or when all cost functions are affine. Lastly, the price of anarchy in affine weighted atomic instances is proven to be at most $\frac{3+\sqrt{5}}{2}$ and in affine unweighted atomic instances at most $\frac{5}{2}$. 

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1. Introduction

This work is strongly based on Roughgarden (2007), chapter 18 in the book Algorithmic Game Theory by N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani. It follows its examples, definitions, propositions, lemma’s, and theorems. I will refer to a part from Roughgarden by adding [R18.xx] in the title of the corresponding part in this text. This work tries to clarify some definitions and proofs for instance by adding steps in proofs and by adding several calculations to verify some statements made by Roughgarden.

The build-up of this work is as follows: Section 2 deals with the introduction of two models used to analyze noncooperative routing games. First the nonatomic selfish routing model is examined (section 3). The notion of a (Nash) equilibrium is defined (section 3.1) and its existence and uniqueness are analyzed (section 3.4). Next, the price of anarchy is reviewed (section 3.5). A rough upper bound of $p + 1$ is given for any instance with polynomial (of degree $p$) cost functions. This bound is refined using the Pigou bound and it is proven that the price of anarchy in an instance with affine cost functions is at most $\frac{4}{3}$.

Secondly, the atomic selfish routing model is examined (section 4). Again, the notion of an equilibrium flow is defined. Through two examples it is shown that there might be several equilibrium flows with different cost or that an equilibrium flow might not even exist. In section 4.3 it is proven that equilibrium flows exist if the model is either restricted to an unweighted amount of traffic per player (section 4.3.1) or to affine cost functions (section 4.3.2). Lastly, the price of anarchy is studied (section 4.4). It is proven that it is at most $\frac{3 + \sqrt{5}}{2}$ in affine weighted atomic instances and at most $\frac{5}{2}$ in affine unweighted ($R = 1$) atomic instances.

2. Introduction of noncooperative routing games

In noncooperative routing games selfish players route traffic through a network. An example of a network can be seen in Figure 1.

![Network Diagram](image)

**Figure 1.** Example of a multicommodity network.

The network is a directed graph consisting of a certain number of vertices and edges. Throughout this text, vertices will be indicated with uppercase letters and edges with $e_i$ with $i$ an index. $V$ stands for the collection of all vertices while $E$ stands for the collection of edges. Together, the directed graph is denoted by $G = (V, E)$. 
This work focuses on two models, the nonatomic selfish routing model and the atomic selfish routing model. In both models, players are assigned a certain amount of traffic and have to route this amount of traffic from its prescribed source vertex to its sink vertex. The main difference between the two models is that in the nonatomic model, there are infinitely many players each controlling a negligible amount of traffic while in the atomic model, there are finitely many players controlling a nonnegligible amount of traffic. This difference will lead to the first model being of a more continuous nature while the second is of a more discontinuous (discrete) nature.

### 3. Nonatomic selfish routing

In the network exist so-called commodities (numbered from \( i = 1 \) up to \( k \)). To each corresponds a pair of vertices of which one is the source (letter \( s \)) of traffic and the other is the sink (letter \( t \)). For instance, in Figure 1 there are two commodities, \((s_1,t_1)\) and \((s_2,t_2)\). Each of the infinitely many players is identified with one commodity. Note that commodities might have a source or sink vertex in common, but never both.

Looking at commodity 1, there exist three paths that lead from \( s_1 \) to \( t_1 \). The only possible paths for this commodity are \( P_1 = (e_1,e_2) \) (indicated with red in Figure 1), \( P_2 = (e_3,e_5) \) and \( P_3 = (e_3,e_4,e_2) \). Each player of this commodity can choose which path it will take. All possible paths of commodity \( i \) are collected in the set \( \mathcal{P}_i \). So \( \mathcal{P}_1 = (P_1,P_2,P_3) \). Also, \( \mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i \), so \( \mathcal{P} \) is the collection of all paths in the network that are potentially used by players. \( \mathcal{P} \) consists of five paths in the network of Figure 1, i.e. three paths for commodity 1 and two paths for commodity 2.

The routes chosen by the players can be described using a flow \( f \).

#### Definition 1 (Flow)
Flow \( f \) is a vector, indexed by \( \mathcal{P} \) (so \( f \) has as many coordinates as there are paths). It describes the amount of traffic routed over all paths in \( \mathcal{P} \). Furthermore, \( f_e \) is defined as the flow over edge \( e \),

\[
f_e = \sum_{P \in \mathcal{P} \mid e \in P} f_P
\]

For example, \( f_{P_1} \) would indicate the amount of traffic routed over path \( P_1 \) and so \( f = (f_{P_1}, f_{P_2}, f_{P_3}, f_{P_4}, f_{P_5}) \) in the network of Figure 1.

The total amount of traffic commodity \( i \) is indicated by \( r_i \). This means that the sum of the flow over all the paths for commodity \( i \) must equal \( r_i \), i.e. \( \sum_{P \in \mathcal{P}_i} f_P = r_i \).

Not all flows are feasible for some network. Only flows that respect \( \sum_{P \in \mathcal{P}_i} f_P = r_i \) for all \( i \in \{1,...,k\} \) are feasible. This set of feasible flows is convex (this fact is needed in a proof later on).

#### Proof that the set of feasible flows is convex
Take any feasible flow \( f \) and \( \hat{f} \). Then for some \( \lambda \in [0,1] \), \((\lambda f + (1-\lambda)\hat{f})_P \geq 0 \) since \( f_P, \hat{f}_P \geq 0 \) for any \( P \in \mathcal{P}_i \). Furthermore,

\[
\sum_{P \in \mathcal{P}_i} (\lambda f + (1-\lambda)\hat{f})_P = \sum_{P \in \mathcal{P}_i} \lambda f_P + \sum_{P \in \mathcal{P}_i} (1-\lambda)\hat{f}_P = \lambda \sum_{P \in \mathcal{P}_i} f_P + (1-\lambda) \sum_{P \in \mathcal{P}_i} \hat{f}_P
\]

\[
= \lambda r_i + (1-\lambda)\hat{r}_i = r_i \quad \forall i \in \{1,...,k\}
\]

Therefore, \( \lambda f + (1-\lambda)\hat{f} \) is also a feasible flow, making the set of feasible flows convex. \( \blacksquare \)

Every player chooses a path with the lowest cost. This means that every edge in \( E \) needs to have a cost function, \( c_e \). In realistic modeling all cost function are nonnegative, continuous,
and nondecreasing. So \( c_e : \mathbb{R}_+ \to \mathbb{R}_+ \) for any \( e \in E \). Now we have the tools to define the cost of a path \( P \) as a function of a certain flow in the network:

\[
c_P(f) = \sum_{e \in P} c_e(f_e)
\]

In words, \( c_P \) is the cost of using a certain path \( P \) given by adding all costs of all edges that the path contains. Each path adds \( c_P(f) \cdot f_P \) to the total cost of the network, see (1).

Finally, we define the total cost of the network as a function of flow by

(1)

\[
C(f) = \sum_{P \in \mathcal{P}} c_P(f_P)
\]

An alternative manner to express the total cost is by summing the cost over all edges:

(2)

\[
C(f) = \sum_{e \in E} c_e(f_e)f_e
\]

The second definition follows from the first:

\[
C(f) = \sum_{P \in \mathcal{P}} c_P(f_P) = \sum_{P \in \mathcal{P}} \left( \sum_{e \in P} c_e(f_e) \right)f_P
\]

Reversing the order of summation gives

\[
\sum_{P \in \mathcal{P}} \left( \sum_{e \in P} c_e(f_e) \right)f_P = \sum_{e \in E} (c_e(f_e) \cdot \sum_{P \in \mathcal{P}, e \in P} f_P) = \sum_{e \in E} c_e(f_e)f_e.
\]

This objective function \( C \) is called an utilitarian function since it is calculated by summing the players’ costs. The outcome function \( C(f) \) will be important to assess the (in)efficiency of the network.

Finally, a nonatomic selfish routing game, or simply a nonatomic instance, is defined by a triple of the form \((G, r, c)\), with \( G \) the directed graph, \( r \) the amount of traffic of each commodity and \( c \) the cost functions of all edges.

### 3.1 Equilibrium flow and optimal flow

Suppose we have the given network of Figure 1 and each player chooses selfishly the path of minimal cost. One can calculate the costs of all the paths \( c_P \). One player, a negligible part of the traffic, might want to decide to take another path if that other path has a strictly lower cost. If the cost is equal or higher for any alternative path for any player, then a state of equilibrium is reached. Formally:

**Definition 1 (Nonatomic equilibrium flow)** [R18.1] Let \( f \) be a feasible flow for the nonatomic instance \((G, r, c)\). The flow \( f \) is an equilibrium flow if, for every commodity \( i \in \{1, \ldots, k\} \) and every pair \( P, \bar{P} \in \mathcal{P}_i \) of \( s_i - t_i \) paths with \( f_P > 0 \),

\[
c_P(f) \leq c_{\bar{P}}(f)
\]

It says, if both paths \( P \) and \( \bar{P} \) are in use (i.e. \( f_P > 0 \) and \( f_{\bar{P}} > 0 \)), then both \( c_P(f) \leq c_{\bar{P}}(f) \) and \( c_P(f) \leq c_P(f) \) hold, implying \( c_P = c_{\bar{P}} \). Since the costs of both paths are equal, no player will consider to change path. If \( f_{\bar{P}} = 0 \) and \( f_P > 0 \), then the costs might be equal or \( c_P(f) < c_{\bar{P}}(f) \), so no player will be inclined to take the alternative path \( \bar{P} \). So in no case a player will change path and therefore the flow can be called an equilibrium flow.

**Definition 2 (Optimal flow)** Let \( f^* \) be a feasible flow for the nonatomic instance \((G, r, c)\).
The flow $f^*$ is an optimal flow if it is the solution of (*)
\[
\begin{align*}
\min_{f} & \quad C(f) \\
\text{s.t.} & \quad \sum_{P \in \mathcal{P}} f_P = r_i \text{ for all } 1 \leq i \leq k \\
& \quad f_P \geq 0 \text{ for all } P \in \mathcal{P}
\end{align*}
\]

In game theory, one often wants to assess the (in)efficiency of equilibria, that is, one wants to know if the equilibrium solution is optimal as well and if it is not optimal, how inefficient is it then?

Example 1 below shows that the equilibrium flow does not necessarily need to be an optimal flow. To calculate the inefficiency of the equilibrium, one can use the so-called price of anarchy. This is the ratio between the worst objective function value of an equilibrium of the game and that of an optimal outcome:

**Definition 3. (Price of Anarchy)** Let $(G, r, c)$ be a (nonatomic) instance of the selfish routing game and let $f$ and $f^*$ be an equilibrium flow and an optimal flow, respectively. The price of anarchy of the instance $(G, r, c)$ is defined as:
\[
\rho(G, r, c) = \frac{C(f)}{C(f^*)}
\]

The notion of the price of anarchy was first studied by Koutsoupias and Papadimitriou (1999), only it was called ‘coordination ratio’ there. The term ‘price of anarchy’ was first coined by Papadimitriou (2001).

We will show later on that in a nonatomic instance, any equilibrium flow $f$ has the same outcome value, $C(f)$. Therefore, in a nonatomic instance, any equilibrium flow can be used to calculate the price of anarchy. In an atomic instance, however, there might exist multiple equilibria with different total cost. Then, the price of anarchy is calculated using ‘the worst’ outcome value.

3.2 Example 1. Pigou’s example
Consider the network in Figure 2a. It was analyzed by Pigou (1920), the first person to study nonatomic selfish routing (Roughgarden, 2007).

There is only one commodity, from $s$ to $t$, with two possible paths. The upper edge, $e_1$, has constant cost of 1. The lower edge, $e_2$, has increasing costs as more players take this route. Take $r = 1$. This means that there is one unit of traffic, representing a very large population of players. $f_{e_1} + f_{e_2} = 1$ and $f_{e_1}, f_{e_2} \geq 0$. Table 1 shows various results.
If the traffic over the lower edge is less than 1, then that edge will be more favorable than the upper edge for any player. This strategy is the dominant strategy. But as every player chooses this strategy, the price of the lower path is just as high as the price of the upper path. By Definition 1, this is an equilibrium flow (see the first row in Table 1).

The second row shows another feasible flow, i.e. all the players choose the upper edge. This is no equilibrium flow since $c_{e_1}(1) = 1$ and $c_{e_2}(0) = 0$ and so $c_{e_1} \leq c_{e_2}$ for $f_{e_1} > 0$ does not hold. In this situation, any player would benefit from changing path since its cost will go from 1 to $x$ with $x \in [0, 1)$.

The third and last row shows the cost of the optimal flow, which is $\frac{3}{4}$.

### Calculation of the optimal flow

Consider Pigou’s network and $f_{e_1} = x$ being the amount of traffic that chooses the upper edge and $f_{e_2} = (1 - x)$ the amount of traffic that chooses the lower edge. $x \in [0, 1)$. The total cost function is

$$C(x) = c_{e_1}(f_{e_1}) \cdot f_{e_1} + c_{e_2}(f_{e_2}) \cdot f_{e_2} = 1 \cdot x + (1 - x) \cdot (1 - x) = x + 1 - 2x + x^2 = 1 - x + x^2$$

Simple calculus shows that $x = \frac{1}{2}$ minimizes $C$

$$C'(x) = -1 + 2x = 0 \implies x = f_{e_1} = \frac{1}{2}$$

$$f_{e_2} = 1 - f_{e_1} = 1 - \frac{1}{2} = \frac{1}{2}.$$ Therefore, the optimal flow is $f = (\frac{1}{2}, \frac{1}{2})$ and the cost of the optimal flow is

$$C\left(\frac{1}{2}, \frac{1}{2}\right) = 1 \cdot \frac{1}{2} + \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

The price of anarchy can now be calculated: $\rho = \frac{C(0,1)}{C(\frac{3}{4})} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$. In Appendix A, two variants of the Pigou example are treated.

### 3.3 Example 2. Braess’s Paradox
Another example of a nonatomic instance is the network of Braess’s Paradox. Braess (1968) was the first to discover that adding a supposedly beneficial edge to a network need not necessarily decrease the cost of an equilibrium flow. The current example uses other cost functions than the original example (according to Roughgarden (2007), the current variant was noted by L. Schulman (1999) in a personal communication between Roughgarden and Schulman).

See Figure 3a for the network and the cost functions. There is only one commodity, \( s - t \), and two paths that players of this commodity can choose. Call the upper path \( P_1 \) and the lower path \( P_2 \). Assume there is one unit of traffic, i.e. \( r = 1 \). The cost of \( P_1 \) and \( P_2 \), \( c_{P_1} \) and \( c_{P_2} \) respectively, is both \( 1 + x \). (Note that this manner of writing the cost of a path, i.e. just summing the cost functions of the edges, is not always possible: if for example in Figure 3b, we change the cost function of edge \( s - w \), \( c_{e_{s-w}}(x) = x \), then \( c_{P_2}(x) \neq 2x \), since \( f_{e_{s-w}} \) is not necessarily equal to \( f_{e_{w-t}} \).)

If all traffic chooses \( P_1 \) in Figure 3a, the total cost of the flow is
\[
C(f) = (c_{e_1}(f_{e_1}) + c_{e_2}(f_{e_2})) \cdot f_{P_1} + (c_{e_3}(f_{e_3}) + c_{e_4}(f_{e_4})) \cdot f_{P_2} =
\]
\[
C(1,0) = (1 + 1) \cdot 1 + (1 + 0) \cdot 0 = 2
\]

Players might reconsider their choice since deviating to \( P_2 \) would decrease their cost \( (c_{P_2}(1,0) = 1 < 2 = c_{P_1}(1,0)) \). Players keep deviating from \( P_1 \) to \( P_2 \) until \( c_{P_1} = c_{P_2} \) (and vice versa). It is obvious that the equilibrium flow splits the traffic in half. At that point, both paths have equal costs \( (c_{P_1}(1/2,1/2) = c_{P_2}(1/2,1/2) = 1/2 + 1 = 3/2) \) so no player will benefit from changing its path. The total cost is \( C(1/2,1/2) = 3/2 + 3/2 = 3 \).

In this network, the optimal flow coincides with the equilibrium flow. Therefore, the price of anarchy is equal to 1. This means that even though the players will exhibit selfish behavior, the overall cost will be minimal.

**Calculation of the optimal flow in the network of Figure 3a:**

\( f_{P_1} = x \) is the amount of traffic choosing \( P_1 \), \( f_{P_2} = 1 - x \) chooses \( P_2 \).

\[
C(x) = (x + 1) \cdot x + (1 + 1 - x) \cdot (1 - x) = x + x^2 + 2 - 3x + x^2 = 2 - 2x + 2x^2
\]

Calculating the minimum:

\[
C'(x) = -2 + 4x = 0 \iff f_{P_1} = x = 1/2, \text{ and } f_{P_2} = 1 - f_{P_1} = 1 - 1/2 = 1/2
\]

Optimal flow: \( f = (1/2, 1/2) \).

Consider now Figure 3b. This is the original network with one added edge, \( e_5 \), with zero cost. The previous equilibrium does not hold. The new path, \( s - v - w - t \), \( P_3 \), has lower cost than either \( P_1 \) or \( P_2 \). Defining the cost of the third path, \( P_3 \), as function of the amount of traffic taking the other two paths, we can calculate the equilibrium flow as follows:

**Calculation of the equilibrium flow in the network of Figure 3b:**

The flow over \( P_1 \) and \( P_2 \) will be equal since the edge cost functions are equal, call it \( f_{P_1} = f_{P_2} = x, f_{P_3} = 1 - 2x \).

\[
c_{P_1}(x, x, 1 - 2x) = c_{P_2}(x, x, 1 - 2x) = (x + 1 - 2x) + 1 = 2 - x
\]

\[
c_{P_3}(x, x, 1 - 2x) = (x + 1 - 2x) + 0 + (x + 1 - 2x) = 2 - 2x
\]

In an equilibrium flow, \( c_{P_1} = c_{P_3} \) (if \( f_{P_1}, f_{P_2} > 0 \) so

\[
2 - x = 2 - 2x \iff -x = -2x \iff x = 0
\]

So \( f = (0,0,1) \) is the equilibrium flow. The equilibrium flow routes all the traffic over \( P_3 \).
\[
C(0,0,1) = c_{p_1}(0,0,1) \cdot 0 + c_{p_2}(0,0,1) \cdot 0 + c_{p_3}(0,0,1) \cdot 1
= (1 + 1) \cdot 0 + (1 + 1) \cdot 0 + (1 + 0 + 1) \cdot 1 = 2
\]

The optimal flow can be calculated as follows:

**Calculation of the optimal flow in the network of Figure 3b:**

The cost function (using equation (2)) is as follows:

\[
C(f) = (x_1 + x_3)(x_1 + x_3) + 1 \cdot x_1 + 1 \cdot x_2 + (x_2 + x_3)(x_2 + x_3)
\]

where \( x_1 \) is \( f_{p_1} \), \( x_2 = f_{p_2} \), \( x_3 = f_{p_3} \), the first term is the cost of edge \( s - v \), the second term of edge \( v - t \), the third of \( s - w \) and the fourth of \( w - t \). The edge \( v - w \) has zero cost, therefore it is not added.

\[
C(f) = x_1 + x_2 + x_1^2 + x_2^2 + 2x_1^2 + 2x_1x_2 + 2x_2x_3
\]

\[
\nabla^2 C(f) = 
\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}
\]

which has eigenvalues \( \lambda_1 = 0, \lambda_2 = 4, \lambda_3 = 4 \), making \( \nabla^2 C(f) \) a positive semidefinite matrix for any (feasible) \( f \). This implies that \( C(f) \) is convex. Also, \( x_1 + x_2 + x_3 = 1 \) is a convex function. So

\[
\min_{f \in \mathbb{R}_+^3} x_1 + x_2 + x_1^2 + x_2^2 + 2x_1^2 + 2x_1x_2 + 2x_2x_3 \text{ s.t. } x_1 + x_2 + x_3 = 1
\]

is a convex program. Rewrite the equality constraint:

\[
g_1(f) = x_1 + x_2 + x_3 - 1 \leq 0 \\
g_2(f) = 1 - x_1 - x_2 - x_3 \leq 0
\]

The KKT-condition reads:

\[
\begin{pmatrix}
1 + 2x_1 + 2x_3 \\
1 + 2x_2 + 2x_3 \\
x_3 + x_1 + 2x_2
\end{pmatrix}

= -y_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - y_2 \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
\]

The top two equations read:

\[
1 + 2x_1 + 2x_3 = -y_1 + y_2 \\
1 + 2x_2 + 2x_3 = -y_1 + y_2
\]

implies \( x_1 = x_2 \)

The third equation can be rewritten as

\[
4x_3 + 4x_1 = -y_1 + y_2 \Rightarrow 4x_3 + 4x_1 = 1 + 2x_1 + 2x_3 \Leftrightarrow 2x_3 + 2x_1 = 1
\]

The equality-constraint can be written as \( 1 = 2x_3 + 2x_1 \). Together:

\[
1 = 2x_3 + 2x_1 \text{ if } x_1 = 0, \text{ then the system is insolvable, so } x_1 > 0
\]

It is only solvable if \( x_3 = 0 \) and \( 1 = 2x_1 \Leftrightarrow x_1 = x_2 = \frac{1}{2} \). Solution to the minimization problem:

\[
f^* = \left( \frac{1}{2}, \frac{1}{2}, 0 \right).
\]

\[
C \left( \frac{1}{2}, \frac{1}{2}, 0 \right) = \left( \frac{1}{2} + 1 \right) \cdot \frac{1}{2} + \left( \frac{1}{2} + 1 \right) \cdot \frac{1}{2} + \left( \frac{1}{2} + 0 + \frac{1}{2} \right) \cdot 0 = \frac{3}{2} + \frac{3}{2} \cdot \frac{1}{2} + 0 = \frac{3}{2}
\]

The price of anarchy is \( \rho = \frac{C(f)}{C(f^*)} = \frac{C(0,0,1)}{C(\frac{1}{2}, \frac{1}{2}, 0)} = \frac{2}{\frac{3}{2}} = \frac{4}{3} \).

**3.4 Existence and uniqueness of equilibrium flows**

Any nonatomic instance admits an equilibrium. Furthermore, any equilibrium flow in a nonatomic instance is essentially unique, which means that there might be several equilibria in an instance but they all have the same total cost. This part is going to prove these claims.

Recall Definition 2. Flow \( f^* \) is optimal if it is feasible and minimizes

\[
C(f) = \sum_{e \in E} c_e(f_e)f_e
\]
\(c_e(f_e)f_e\) is the contribution to the total cost of the flow of edge \(e\). Call the derivative of this contribution ‘the marginal cost function’, i.e.
\[
c^*_e(f_e) = (c_e(f_e)f_e)' = c_e(f_e) + f_e \cdot c'_e(f_e)
\]
This marginal cost function indicates how much the contribution to the total cost increases as more traffic uses the edge. \(c^*_e\) is always positive as \(c_e(f_e)f_e\) is always increasing.

Let \(c^*_P(f) = \sum_{e \in P} c^*_e(f_e)\) define the sum of the marginal costs of the edges in the path \(P\) with respect to the flow \(f\). If we are dealing with an optimal flow, the sum over all \(c_P(f) \cdot f_P\) is minimized. This means that the marginal cost function over two (or more) paths that are in use must be equal. If they are not, then a player can switch to the path where the marginal cost function is lower and thereby decreasing the total cost (i.e. if one unit of traffic will induce 5 extra units to the total cost on \(P_1\) and 4 extra units on \(P_2\), changing from \(P_1\) tot \(P_2\) will decrease the total cost by 1). An unused path must always have an equal or a higher marginal cost for the same reasons. More formally:

**Proposition 1 (Characterization of optimal flows) [R18.9]** Let \((G, r, c)\) be a nonatomic instance such that, for every edge \(e\), the function \(c_e(f_e)f_e\) is convex and continuously differentiable. Let \(c^*_e\) denote the marginal cost function of the edge \(e\). Then \(f^*\) is an optimal flow for \((G, r, c)\) if and only if, for every commodity \(i \in \{1, \ldots, k\}\) and every pair \(P, \hat{P} \in \mathcal{P}_i\) of \(s_i - t_i\) paths with \(f^*_P > 0\),
\[
c^*_P(f^*) \leq c^*_\hat{P}(f^*)
\]

**Proof:** It has already been proven before that the feasible set of flows is convex. It is assumed that \(c_e(f_e)f_e\) is convex, and the sum of convex functions is in turn convex (i.e. \(C(f) = \sum_{e \in E} c_e(f_e)f_e\) is convex). Consider the following convex program
\[
\begin{align*}
\min_{\mathcal{P}} C(f) &= \sum_{e \in E} c_e(f_e)f_e \\
\sum_{P \in \mathcal{P}} f_P &= r_i \text{ for all } 1 \leq i \leq k \\
f_P &\geq 0 \text{ for all } P \in \mathcal{P} \\
f &\in \mathbb{R}^{\vert \mathcal{P} \vert}
\end{align*}
\]
Flow \(f\) is optimal if and only if it satisfies the KKT-conditions. Denote the objective function and constraints of the program as follows:
\[
C(f) = \sum_{e \in E} c_e(f_e)f_e \\
g_P(f) = -f_P \leq 0 \text{ for all } P \in \mathcal{P} \\
h_i(f) = (\sum_{P \in \mathcal{P}_i} f_P) - r_i = 0, \text{ for all } 1 \leq i \leq k
\]
So \(f\) is optimal if and only if there exist multipliers \(\lambda_P, \lambda_i\) such that
\[
\nabla C(f) = -\sum_{P \in \mathcal{P}} \lambda_P \nabla g_P(f) - \sum_{i=1}^{k} \mu_i \nabla h_i(f)
\]
Suppose that \(f\) is optimal, then there exist \(\lambda_P, \mu_i\) that satisfy the KKT-conditions. Considering the conditions per path:
\[
\begin{align*}
(\nabla C(f))_P &= \sum_{e \in P} (c_e(f_e)f_e)' = \sum_{e \in P} c^*_e(f_e) = c^*_P(f) \\
(\nabla g_P(f))_P &= -1 \\
(\nabla h_i(f))_P &= \begin{cases} 1 & \text{if } P \in \mathcal{P}_i \\ 0 & \text{if } P \notin \mathcal{P}_i \end{cases}
\end{align*}
\]
\(\sum_{P \in \mathcal{P}_i} f_P\) is the sum of all flows over commodity \(i\), e.g. if \(\mathcal{P}_i = \{P_1, P_2, P_3\}\), then

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Corollary 1 (Equivalence of equilibrium and optimal flows) [R18.10] Let \((G, r, c)\) be a nonatomic instance such that, for every edge \(e\), the function \(c_e(f_e)\) is convex and continuously differentiable. Let \(c_e^*\) denote the marginal cost function of the edge \(e\). Then \(f^*\) is an optimal flow for \((G, r, c)\) if and only if it is an equilibrium flow for \((G, r, c^*)\).

(First proof given by Beckman, 1956)

To proof the existence of equilibrium flows in nonatomic instances, one can use Corollary 1. We are looking for some function \(h_e\) for every edge such that its derivative is \(c_e\). The only assumption on \(h_e\) would be that \(h_e(f_e)\) is convex. An obvious candidate is \(h_e(f_e) = \int_0^{f_e} c_e(x)dx\). Since \(c_e\) is continuous and nondecreasing for every edge \(e\), every function \(h_e\) is both continuously differentiable and convex (\(h'_e = c_e\) for every \(e \in E\)).

Next we are going to define the potential function of a nonatomic instance. The idea of the potential function is that the global minima of this function are precisely the equilibrium flows of the nonatomic instance. Define the potential function \(\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx\).

Since this is a sum of convex functions, the potential function is also convex. By using Proposition 1, one can see that if we replace \(f_e \cdot c_e(f_e)\) with \(h_e(f_e)\) for every \(e \in E\), then \(h'_e(f_e) = c_e(f_e)\). In Corollary 1, \(h_e\) plays the role of \(c_e(f_e)\).

Formally:
Proposition 2 (Potential function for equilibrium flows) [R18.11] Let \((G, r, c)\) be a nonatomic instance. A flow feasible for \((G, r, c)\) is an equilibrium flow if and only if it is a global minimum of the corresponding potential function \(\Phi\) given by

\[
\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx
\]

(First proof given by Beckman, 1956)

In appendix B one can find the calculations of the equilibrium flow using the potential function for the two examples of sections 3.2 and 3.3.

Now we have all tools to prove the following theorem:

Theorem 1 (Existence and uniqueness of equilibrium flows) [R18.8] Let \((G, r, c)\) be a nonatomic instance.

(a) The instance \((G, r, c)\) admits at least one equilibrium flow.

(b) If \(f\) and \(\tilde{f}\) are equilibrium flows for \((G, r, c)\), then \(c_e(f_e) = c_e(\tilde{f}_e)\) for every edge \(e\).

Proof: The set of feasible flows is bounded (i.e. any \(f_P\) is bounded by 0 and \(r_i\)) and closed (since any \(f_P \in [0, r_i]\)) and therefore compact. It is a subset of \(\mathbb{R}^{|E|}\). Since edge cost functions are continuous, the \(h_e\) functions are continuous and therefore the potential function \(\Phi\) is continuous as well. Weierstrass’s Theorem tells us that a continuous function on a compact set always admits at least one minimum on that set. So, the potential function on the compact set of feasible flows admits at least one minimum. By Proposition 2, this minimum is an equilibrium flow for the atomic instance \((G, r, c)\). Thus, part (a) is proven.

We know that the potential function is convex. This means that any minimum of the function is a global minimum and that any point on the line between any minima also is a global minimum. Suppose \(f\) and \(\tilde{f}\) are equilibrium flows for \((G, r, c)\), then they are global minima for \(\Phi(f)\). Then for any \(\lambda \in [0,1]\), \(\lambda f + (1 - \lambda)\tilde{f}\) is also a global minimum of \(\Phi(f)\).

\[
\Phi(\lambda f + (1 - \lambda)\tilde{f}) = \lambda \Phi(f) + (1 - \lambda)\Phi(\tilde{f})
\]

since \(\Phi\) is constant between \(f\) and \(\tilde{f}\). The sum of \(|E|\) convex summands can only be linear if every summand is linear too (If a function is strictly convex, then its derivative is not constant but increasing over the entire interval and the sum of two increasing functions is also increasing. Therefore the sum of two strictly convex functions can never be a constant function). It is linear between \(f_e\) and \(\tilde{f}_e\). If an integral over \(c_e\) is linear between \(f_e\) and \(\tilde{f}_e\), then \(c_e\) is constant between \(f_e\) and \(\tilde{f}_e\), meaning \(c_e(f_e) = c_e(\tilde{f}_e)\) for every edge \(e\), proving part (b).

(Proof adapted from Roughgarden, 2007, first proof given by Beckman, 1956)

If \(c_e(f_e) = c_e(\tilde{f}_e)\) for every edge \(e\), then \(c_P(f) = c_P(\tilde{f})\) for \(f, \tilde{f}\) two equilibrium flows. If the flows are distinct, then \(f_e \neq \tilde{f}_e\) for at least two edges. \(c_e(f_e) = c_e(\tilde{f}_e)\) can only hold if \(c_e\) is constant between \(f_e\) and \(\tilde{f}_e\). Therefore, there is only one unique equilibrium flow if the cost functions are strictly increasing. We can have two distinct equilibrium flows if there is at least one edge cost function that is not strictly increasing.

Furthermore, since \(c_e(f_e) = c_e(\tilde{f}_e)\) holds for every edge \(e\), also \(c_P(f) = c_P(\tilde{f})\) holds for every path \(P\). This implies that any path in use, either in \(f\) or in \(\tilde{f}\), has the same cost, say \(L_i\). Since \(\sum_{e \in P} f_e = r_i = \sum_{e \in P} \tilde{f}_e\), we know that
\[ C(f) = \sum_{p \in P} c_p(f) f_p = \sum_{i=1}^{k} \sum_{p \in P_i} L_i \cdot f_p = \sum_{i=1}^{k} L_i \cdot \sum_{p \in P_i} f_p = \sum_{i=1}^{k} L_i \cdot \sum_{p \in P_i} f_p = \sum_{i=1}^{k} L_i \cdot f_i = C(\tilde{f}) \]

Therefore, for any two or more (distinct) equilibrium flows, \( C(f) = C(\tilde{f}) \).

### 3.5 The price of anarchy

This section gives a short analysis of the price of anarchy in nonatomic instances. The price of anarchy (\( \rho \)) is defined as the ratio of the objective value of an equilibrium flow and the objective value of the optimal flow (see Definition 2). The price of anarchy in the two examples above was \( \frac{4}{3} \). In this section I will show that in any nonatomic instance with only affine cost functions, the price of anarchy is at most \( \frac{4}{3} \).

For any nonatomic instance, the potential function gives a rough upper bound on the price of anarchy.

**Theorem 2 (Potential function upper bound) [R18.16]** Let \( (G, r, c) \) be a nonatomic instance, and suppose that \( x \cdot c_e(x) \leq \gamma \cdot \int_0^x c_e(y) dy \) for all \( e \in E \) and \( x \geq 0 \). Then the price of anarchy of \( (G, r, c) \) is at most \( \gamma \).

**Proof:** Let \( f \) be an equilibrium flow and \( f^* \) an optimal flow for \( (G, r, c) \).

\[ C(f) \leq \gamma \cdot \Phi(f) \leq \gamma \cdot \Phi(f^*) \leq \gamma \cdot C(f^*) \]

The first inequality follows from the hypothesis (summing over all \( e \) does not change the inequality sign), the second from the fact that \( f \) is a global minimizer of \( \Phi \) according to Proposition 2, and the third follows from the next observation: Since \( c_e \) is nonnegative and nondecreasing, \( c_e(f_e) f_e \) will always be at least as big as \( \int_0^{f_e} c_e(x) dx \), \( \int_0^{f_e} c_e(x) dx \leq c_e(f_e) f_e \) (equality holds when \( c_e \) is constant between 0 and \( f_e \))

\[ C(f) \leq \gamma \cdot C(f^*) \iff \rho(G, r, c) = \frac{C(f)}{C(f^*)} \leq \gamma. \]

(Proof adapted from Roughgarden, 2007, first proof by Roughgarden and Tardos, 2002)

Note that this upper bound \( \gamma \) equals 2 when all cost functions are linear.

**An upper bound on \( \rho(G, r, c) \) is 2 for an affine nonatomic instance**

We are looking for a \( \gamma \) as small as possible for which \( x \cdot c_e(x) \leq \gamma \cdot \int_0^x c_e(y) dy \) holds for any \( c_e(x) = a_e x + b_e \), and \( a_e, b_e \geq 0 \).

\[ x \cdot c_e(x) \leq \gamma \cdot \int_0^x c_e(y) dy \Rightarrow a_e x^2 + b_e x \leq \gamma \left( \frac{1}{2} a_e x^2 + b_e x \right) \]

This is true for any \( a_e, b_e \geq 0 \) for \( \gamma = 2 \) (an upper bound for \( \gamma \)).

Set \( b_e = 0 \), then a lower bound on \( \gamma \) is \( \gamma = 2 \). Together, \( \gamma = 2 \) is a tight bound for the price of anarchy in an atomic instance with only linear cost functions.

In fact, for any polynomial cost function, \( \gamma = p + 1 \) where \( p \) is the degree of the polynomial.

**Corollary 2 (Potential function bound for polynomials) [R18.17]** If \( (G, r, c) \) is a
nonatomic instance with polynomial cost functions with nonnegative coefficients and degree at most \( p \), then the price of anarchy of \((G, r, c)\) is at most \( p + 1 \).

**Proof:** According to Theorem 2, if we can find a \( \gamma \) such that \( x \cdot c_e(x) \leq \gamma \int_0^x c_e(y) \, dy \) for all \( e \), then that \( \gamma \) is the upper bound on the price of anarchy.

\[
c_e(x) = a_p x^p + a_{p-1} x^{p-1} + \cdots + a_1 x + a_0
\]

\[
x \cdot c_e(x) = a_p x^{p+1} + a_{p-1} x^p + \cdots + a_0 x
\]

\[
\leq \gamma \int_0^x c_e(y) \, dy = \gamma \left( \frac{1}{p + 1} a_p x^{p+1} + \frac{1}{p} a_{p-1} x^p + \cdots + a_0 x \right)
\]

This inequality holds for any polynomial if \( \gamma = p + 1 \):

\[
a_p x^{p+1} + a_{p-1} x^p + \cdots + a_0 x \leq a_p x^{p+1} + \frac{p + 1}{p} a_{p-1} x^p + \cdots + (p + 1) a_0 x
\]

is true since

\[
a_p x^{p+1} = a_p x^{p+1}, a_{p-1} x^p \leq \frac{p + 1}{p} a_{p-1} x^p, \ldots, a_0 x \leq (p + 1) a_0 x
\]

Therefore, \( \gamma = p + 1 \) is an upper bound on \( \gamma \).

Set all coefficients \( a_j, j = 0, ..., p - 1 \), equal to zero. Then

\[
a_p x^{p+1} \leq \gamma \frac{1}{p + 1} a_p x^{p+1}
\]

This is true if \( \gamma \) at least equals \( p + 1 \), making \( \gamma = p + 1 \) an lower bound on \( \gamma \).

Together \( \gamma = p + 1 \) is a tight bound for the price of anarchy of \((G, r, c)\).

The potential function already gives a good upper bound estimate, but it is not optimal yet. In the next part, I will show that there is a tight bound on the price of anarchy that is even lower than \( \gamma \).

The idea is as follows: first we formalize a lower bound by using Pigou-like networks (i.e. a network with two vertices and two edges, one commodity with \( r \) units of traffic and cost functions \( c_1(x) = c(r), c_2(x) = c(x) \)). Then we establish an upper bound on the price of anarchy in general multicommodity flow networks. This bound was called the anarchy value by Roughgarden (2003) but later he renamed it to the Pigou bound (Roughgarden, 2007).

**Definition 4 (Pigou bound) \([R18.18]\):** Let \( \mathcal{C} \) be a nonempty set of cost functions. The Pigou bound \( \alpha(\mathcal{C}) \) for \( \mathcal{C} \) is

\[
\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x)c(r)}
\]

with the understanding that \( \frac{0}{0} = 1 \).

The fraction is the price of anarchy: the numerator is the cost of the equilibrium flow in a Pigou-like network, where all the traffic is routed over the lower edge. The denominator is the cost of the optimal flow: \( x \) units are routed over the lower edge and \( r - x \) units are routed over the upper edge. The next proposition shows that this Pigou bound is a lower bound on the price of anarchy.

**Proposition 3 (Lower bound on price of anarchy) \([R18.19]\):** Let \( \mathcal{C} \) be a set of cost functions that contains all of the constant cost functions. Then the price of anarchy in nonatomic instances with cost functions in \( \mathcal{C} \) can be arbitrarily close to \( \alpha(\mathcal{C}) \).
Theorem 3 (Tightness of the Pigou bound) [R18.21] Let $\mathcal{C}$ be a set of cost functions and $\alpha(\mathcal{C})$ the Pigou bound for $\mathcal{C}$. If $(G, r, c)$ is a nonatomic instance with cost functions in $\mathcal{C}$, then the price of anarchy of $(G, r, c)$ is at most $\alpha(\mathcal{C})$. 

Proof: Fix a choice of $c \in \mathcal{C}$ and $x, r \geq 0$. Since $c$ is nondecreasing,

$$\sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} = 1 \text{ if } x \geq r$$

If $x = r$, then

$$\frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} = r \left( \frac{r}{x} \right) = 1$$

and if $x > r$, then

$$\frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} = \frac{r \cdot c(r)}{x \cdot c(x) - c(r)}$$

for the nonatomic instance $(G, r, c)$, where $\mathcal{C}$ is nondecreasing, and if $x \cdot (c(x) - c(r)) \geq 0$ so the expression is at most 1. Therefore, we can assume that $x \leq r$.

Let $\mathcal{G}$ be a Pigou-like network. The lower edge has cost function $c_2(x) = c(x)$ and the upper edge has cost function $c_1(x) = c(r)$, $c \in \mathcal{C}$. Set the traffic rate to $r$. The equilibrium flow routes all the traffic over the lower edge, yielding cost $r \cdot c(r)$. Any feasible flow routes $x$ units of traffic on the lower edge and $r - x$ units on the upper edge. This yields cost $x \cdot c(x) + (r - x) c(r)$. By varying $x$, one can find the $x$ for which $x \cdot c(x) + (r - x) c(r)$ is approximately minimal. Doing this for any $c \in \mathcal{C}$, one ends up with the price of anarchy being at least $\alpha(\mathcal{C})$, or $\rho \geq \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} = \alpha(\mathcal{C})$. 

(Proof adapted from Roughgarden, 2007)
Proof: Let $f^*$ and $f$ be optimal and equilibrium flows, respectively. Then

$$C(f^*) = \sum_{e \in E} f^*_e c_e(f^*_e) = \sum_{e \in E} \left( f^*_e c_e(f^*_e) + (f_e - f^*_e)c_e(f_e) \right) \frac{c_e(f_e)}{c_e(f^*_e)} f_e + (f^*_e - f_e)c_e(f_e)$$

The fraction resembles the calculation of the total cost for a Pigou-like network with $x = f^*_e$ and $r = f_e$ (the fraction is raised to the power $-1$). The Pigou bound is the supremum of the fraction over all cost functions in $C$ and all $x, r \geq 0$, therefore

$$\frac{f^*_e c_e(f^*_e) + (f_e - f^*_e)c_e(f_e)}{c_e(f_e)f_e} \geq \frac{1}{\alpha(C)}$$

for each edge $e$.

$$C(f^*) \geq \frac{1}{\alpha(C)} \sum_{e \in E} c_e(f_e)f_e + \sum_{e \in E} (f^*_e - f_e)c_e(f_e) \geq \frac{1}{\alpha(C)} \sum_{e \in E} c_e(f_e)f_e = \frac{C(f)}{\alpha(C)}$$

The last inequality follows from Proposition 4 that basically says: $\sum_{e \in E} (f^*_e - f_e)c_e(f_e) \geq 0$. Therefore,

$$\frac{C(f^*)}{\alpha(C)} \Rightarrow \rho = \frac{C(f)}{\alpha(C)} \leq \alpha(C)$$

Therefore, $\alpha(C)$ is an upper bound on the price of anarchy of $(G, r, c)$ with $c \in C$. ■

(Proof adapted from Roughgarden, 2007, first proof by Roughgarden, 2003)

Theorem 3 implies that the price of anarchy on any nonatomic instance is maximized by Pigou-like networks, no matter what network $G$ is considered (Roughgarden, 2003). The only restrictions are on the cost functions, and not on the network size, the network structure, nor the number of commodities. It is also remarkable that ‘the worst possible ratio’ for a certain instance with cost functions in $C$ can always be achieved with a network with only two parallel links (Roughgarden, 2003).

As a last example, we will show that the price of anarchy is at most $\frac{4}{3}$ for nonatomic instances with affine cost functions.

**Theorem 4 (The price of anarchy in affine nonatomic instances)** If $(G, r, c)$ is a nonatomic instance with affine cost functions, then the price of anarchy of $(G, r, c)$ is at most $\frac{4}{3}$.

Proof: Theorem 3 says that the price of anarchy of a nonatomic instance is at most $\alpha(C)$. So for linear cost functions, we need to find $\alpha(C)$ with $C$ the set of all possible affine cost functions.

$$\alpha(C) = \sup_{c \in C} \sup_{x \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x)c(r)}$$

with $c(r) = ar + b, c(x) = ax + b$, with $a, b \geq 0$ for any edge cost function. To maximize $\frac{r \cdot c(r)}{x \cdot c(x) + (r - x)c(r)}$ over $x$, we need to minimize the following expression

$$x \cdot c(x) + (r - x)c(r) = ax^2 + bx + ar^2 + br - arx - bx = ax^2 + ar^2 + br - arx$$

Differentiating with respect to $x$ and equate to 0 yield

$$2ax - ar = 0 \iff x = \frac{r}{2}$$

$$\alpha(C) = \sup_{c \in C} \frac{r \cdot c(r)}{\frac{1}{2} r \cdot c\left(\frac{1}{2} r\right) + \left(\frac{1}{2} r\right)c(r)} = \sup_{c \in C} \frac{ar^2 + br}{\frac{1}{2} ar^2 + \frac{1}{2} br + \frac{1}{2} ar^2 + \frac{1}{2} br} = \sup_{c \in C} \frac{3}{4} ar + b$$

If $b$ is increasing, the ratio is decreasing. Therefore, choose $b$ as low as possible, i.e. $b = 0$. 

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4. Atomic selfish routing

Most ingredients for atomic selfish routing are the same as for nonatomic selfish routing: There is a network $G = (V, E)$ with a finite number of commodities $i = \{1, \ldots, k\}$, a positive amount of traffic per commodity $r_i$ and each edge has a nondecreasing, continuous, nonnegative cost function $c_e : \mathbb{R}_+ \to \mathbb{R}_+$. We also denote an atomic instance by $(G, r, c)$. The main difference are the players: in an atomic instance, there are finitely many players, each assigned to one commodity, and each player directs its amount of traffic over just one path. This is the reason why sometimes this model is also called selfish unsplittable routing (e.g. see Fotakis, Kontogiannis, and Spirakis, 2005). There are $k$ players, each assigned to one of the $k$ commodities.

Flow is again denoted by $f$. Since the traffic is unsplittable and multiple players might be assigned the same source-sink pair, the flow per path has an extra index: $f^{(i)}_P$ indicates the amount of traffic that player $i$ directs over path $P$, so $f_P = \sum_{i=1}^k f^{(i)}_P$. This vector is again nonnegative. A flow is feasible if for each player $i$, $f^{(i)}_P$ equals $r_i$ for one path $P \in \mathcal{P}_i$ and 0 for $\tilde{P} \in \mathcal{P}_i \setminus P$.

The cost per path is the same as in a nonatomic instance: $c_P(f) = \sum_{e \in P} c_e(f_e)$ where $f_e = \sum_{P \in \mathcal{P}_i \setminus P} f^{(i)}_P$. Also the total cost, the outcome of the game, is the same: $C(f) = \sum_{P \in \mathcal{P}_i} c_P(f)$ or equivalently $C(f) = \sum_{e \in E} c_e(f_e)$.

Suppose there is a certain flow in a network. This flow is an equilibrium flow if no player can decrease its cost by deviating from its chosen path. Formally:

**Definition 5 (Atomic equilibrium flow)** [R18.5] Let $f$ be a feasible flow for the atomic instance $(G, r, c)$. The flow $f$ is an equilibrium flow if, for every player, $i \in \{1, \ldots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f^{(i)}_{\tilde{P}} > 0$,

$$c_P(f) \leq c_{\tilde{P}}(\tilde{f})$$

where $\tilde{f}$ is the flow identical to $f$ except that $\tilde{f}^{(i)}_{\tilde{P}} = 0$ and $f^{(i)}_{\tilde{P}} = r_i$.

The last line tells us that if only one player deviates from $P$ to $\tilde{P}$, the flow changes from $f$ to $\tilde{f}$ and the only change is that all the traffic $r_i$ is now directed over $\tilde{P}$ instead of $P$. The definition also tells us that in an equilibrium flow, a player has always chosen a path that has either equal or lower cost than when he would direct the traffic over another path.

In a nonatomic instance equilibria always existed and had equal costs; in an atomic instance, however, equilibria might not exist and if there exist multiple ones, the costs might be different. Both cases are explored in the following two examples.

4.1 Example 3. AAE example

This example is named after its discoverers: Awerbuch, Azar, and Epstein (2005). Consider the network in Figure 4. It is a bidirected graph with affine edge cost functions. There are four players, each routing one unit of traffic (i.e. $r_i = 1$) from $s_i$ to $t_i$. Each player can either choose a one-step path or a two-step path. Since players can never choose the same path, $f_P = f^{(i)}_P$ for every $i$. Some results are shown in Table 2.
Table 2. Results for the AAE example

<table>
<thead>
<tr>
<th>Player</th>
<th>Cost per edge, $c_e$</th>
<th>Cost per player $i$ $(c_p, \text{ whichever } P i \text{ might choose})$</th>
<th>Total cost</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$c_{uv}(x)$</td>
<td>$c_{vu}(x)$</td>
<td>$c_{uw}(x)$</td>
</tr>
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<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>2</td>
<td>2</td>
</tr>
<tr>
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<td>0</td>
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<tr>
<td>2211</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2121</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2112</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2221</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2122</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2212</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2222</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

*1 means player $i$ takes the one-step path, 2 means he takes the two-step path. So for example, 1221 means that player 1 and 4 take the one-step path and player 2 and 3 the two-step path.

Example of calculations of one row in Table 2

Take a look at the row where player 1 and 3 take the one-step path and player 2 and 4 take the two-step path: 1212 (colored row). Player 1, 2 and 3 take edge $u - v$, $e_{u-v}$, its cost is $c_{uv}(3) = 3$. $e_{w-u}$ (player 4) costs $c_{wu}(1) = 0$, $e_{v-w}$ is taken by player 2 and 3, and costs $c_{vw}(2) = 2$. All other edges are not used and cost 0.

The costs per player are calculated as follows:

$$C_i(f) = c_p(f) \cdot f_p, \text{ with } c_p(f) = \sum_{e \in P} c_e(f_e)$$

Player 1: $C_1(f) = c_{uv}(3) \cdot f_{uv} = 3 \cdot 1 = 3$.

Player 2: $C_2(f) = (c_{uw}(3) + c_{vw}(2)) \cdot f_{uw} = (3 + 2) \cdot 1 = 5$.

Player 3: $C_3(f) = c_{vw}(2) \cdot f_{vw} = 2 \cdot 1 = 2$.

Player 4: $C_4(f) = (c_{wu}(1) + c_{uv}(3)) \cdot f_{uw} = (0 + 3) \cdot 1 = 3$. 
The total cost of the network is $3 + 5 + 2 + 3 = 13$.

Obviously, every player choosing the one-step path, (1111) is both an equilibrium flow and an optimal flow. It is an equilibrium flow since if any player would deviate to the two-step path (e.g. player 1 deviating (2111) or player 3 (121)) the cost for that path increases (in the first example, the cost of player 1 increases from 1 to 4. In the second example: player 3 increases its cost from 1 to 2).

Also, (2222) is an equilibrium flow. To check this, look up (2222), (2122), (2212), and (2221) and verify that the costs of the player that deviates from the two-step path to the one-step path are either equal or higher (as it says in Definition 4). However, its total cost is higher than the total cost of the first mentioned equilibrium flow.

For all flows, except for (1111) or (2222), one can make a sort of diagram for flows that would be a better choice for a certain player. Check for instance (2112). If player 1 changes path, he will decrease his cost from 3 to 2. If next player 4 changes path, he will decrease his cost from 2 to 1, and we end up in the optimal/equilibrium flow (1111). Any flow will end up at (1111), none end up at (2222).

This example has shown that in an atomic instance, there might exist several equilibria with different total cost.

The price of anarchy is $\rho = \frac{C(f)}{C(f^*)} = \frac{10}{4} = \frac{5}{2}$. Note that the ‘worst outcome value’ equilibrium flow is chosen to calculate the price of stability (see the discussion at the end of Section 3.1).

Later on, I will prove that in an unweighted atomic instance with affine cost functions, the price of anarchy is at most $\frac{5}{2}$.

4.2 Example 4. Existence of equilibrium flows

This example is created by Goemans, Mirrokni, and Vetta (2005). Consider the network shown in Figure 5. There are two players, both with source $s$ and sink $t$. Their traffic amounts are $r_1 = 1$ and $r_2 = 2$. Denote the following: $P_1: s - t$, $P_2: s - v - t$, $P_3: s - w - t$, $P_4: s - v - w - t$. Some results are shown in Table 3.

**Example of calculations:** Suppose player 1 takes path 2 and player 2 path 4 (colored cell in Table 3).

**Cost player 1:**
\[
C_1(f) = (c_{sv}(f_{sv}) + c_{vt}(f_{vt})) \cdot f_{P_2} = (3 \cdot 3^2 + (1^2 + 44)) \cdot 1 = 27 + 45 = 72
\]

**Cost player 2:**
\[
C_2(f) = (c_{sv}(f_{sv}) + c_{vw}(f_{vw}) + c_{wt}(f_{wt})) \cdot f_{P_4} = (3 \cdot 3^2 + 6 \cdot 2^2 + 13 \cdot 2) \cdot 2 = 154
\]
Table 3. Results for the atomic instance as shown in Figure 5 and described in the text

<table>
<thead>
<tr>
<th>Player 1 chooses $P_i$</th>
<th>Player 2 chooses $P_j$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>141*</td>
<td>282</td>
</tr>
<tr>
<td>$P_2$</td>
<td>48</td>
<td>188</td>
</tr>
<tr>
<td>$P_3$</td>
<td>47</td>
<td>188</td>
</tr>
<tr>
<td>$P_4$</td>
<td>22</td>
<td>188</td>
</tr>
</tbody>
</table>

*The left value is the cost of player 1, the right value the cost of player 2.

There is no equilibrium flow in this atomic instance:
- If player 2 chooses $P_1$ or $P_2$, the unique best response of player 1 is to choose $P_4$ (see yellow marking).
- If player 2 chooses $P_3$ or $P_4$, the unique best response of player 1 is to choose $P_1$ (also yellow marking).
- If player 1 chooses path $P_1$, the unique best response of player 2 is to choose $P_2$ (green marking).
- If player 1 chooses path $P_4$, the unique best response of player 2 is to choose path $P_3$ (green marking).

Because of these four observations, there is no possible equilibrium flow. From any cell in the table, either player 1 or player 2 can decrease his cost by deviating from his chosen path. The possible deviations are indicated with arrows. The fact that the arrows are pointing in a circle at the middle top row and middle bottom row, and the fact that from any other cell the arrow points eventually to one of those four cells, imply that an equilibrium state is not possible.

There are two optimal flows: $f^* = (1, 2, 0, 0)$ and $f^* = (0, 2, 1, 0)$. $C(f^*) = C(\tilde{f}^*) = 167$.

4.3 Existence of equilibria

Why is there no equilibrium flow in Example 4 but does Example 3 admit two equilibria? The difference lies in the weights assigned to the players. You could call Example 3 an affine unweighted instance while Example 4 is a non-affine weighted instance. ‘Unweighted’ means that every player has the same amount of traffic to direct, i.e. $r_i = R$ for all $i = \{1, \ldots, k\}$. We will see that an atomic instance will admit an equilibrium flow if

1) All players have the same weight ($r_i$ is identical for all players) or
2) All cost functions are affine ($c_e(x) = a_e x + b_e$, $a_e, b_e \geq 0$ for all edges).

4.3.1 Existence of equilibria: unweighted atomic instance

In this section, we prove Theorem 5 that says that any unweighted atomic instance admits at least one equilibrium flow. Formally:

**Theorem 5 (Equilibrium flows in unweighted atomic instances) [R18.12]** Let $(G, r, c)$ be an atomic instance in which every traffic amount $r_i$ is equal to a common positive value $R$. Then $(G, r, c)$ admits at least one equilibrium flow.

**Proof:** Recall the potential function for a nonatomic instance. By discretizing it, one can prove Theorem 5. Assume for simplicity that $R = 1$. Set

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

for every feasible flow $f$.
Note how the integral part (of the continuous potential function) is substituted by a Riemann-sum with steps of 1. To prove the existence of an equilibrium flow, we prove that any global minimum of $\Phi(f)$ is an equilibrium flow.

First, since there are a finite number of players and a finite number of paths to choose, the set of all feasible flows ($\mathcal{F}$) is finite. This means that the range of $\Phi$ is finite and thus a minimal value exists. Call the flow of minimal value $f$.

Now we prove that $f$ is an equilibrium flow in $(G, r, c)$. Assume to the contrary that $f$ is not an equilibrium flow. Then player $i$ could strictly decrease its cost by deviating from path $P$ to path $\tilde{P}$ (strictly: according to the definition, it is an equilibrium flow if all other paths have higher or equal cost. For a flow to be a nonequilibrium flow, at least one alternative path must have strictly lower cost). So we assume that there is a flow $\tilde{f}$ where one player chooses an alternative path $\tilde{P}$ with lower cost. This means:

$$c_p(f) > c_p(\tilde{f}) \iff 0 > c_p(\tilde{f}) - c_p(f)$$

The edges of $\tilde{P}$ are either in $P$ or they are not. $P$ and $\tilde{P}$ might have some edges in common, the cost of those edges does not change as player $i$ changes to path $\tilde{P}$. This means that only the edges that they have not in common should be considered. First: When player $i$ deviates, he transfers his one unit from path $P$ to path $\tilde{P}$. This means that edges that are now used extra have flow $f_e + 1$ (and edges that are not anymore used $f_e - 1$). Expressing the above-mentioned inequality in $f$ yields:

$$0 > c_p(\tilde{f}) - c_p(f) = \sum_{e \in P \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$$

The first sum is the cost all edges that are in $\tilde{P}$ but not in $P$, the second sum is the cost of all edges that are in $P$ but not in $\tilde{P}$.

On the other hand, consider $\Phi$. The new edges in $\tilde{P}$ (i.e. $e \in \tilde{P} \setminus P$) add to $\Phi(f)$ one extra term, $c_e(f_e + 1)$ (the +1 stems from $R = 1$). The edges that are not anymore in $\tilde{P}$, i.e. $e \in P \setminus \tilde{P}$, subtract a term of $\Phi(f)$, namely $c_e(f_e)$ (since $f_e$ is now one less. The new sum for this edge is $\sum_{i=1}^{f_e - 1} c_e(i)$). The sum for common edges remains the same, i.e. $\tilde{f}_e = f_e \Rightarrow \sum_{i=1}^{f_e} c_e(i) = \sum_{i=1}^{f_e} c_e(i)$. Thus:

$$\Phi(\tilde{f}) - \Phi(f) = \sum_{e \in P \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$$

The right-hand term of (5) is the same as the right-hand term in (4). Therefore,

$$0 > \Phi(\tilde{f}) - \Phi(f)$$

This contradicts the fact that $f$ is a global minimum of $\Phi$ (i.e. $0 < \Phi(\tilde{f}) - \Phi(f)$).

Therefore, $f$ is an equilibrium flow of $(G, r, c)$. ■

(Proof adapted from Roughgarden, 2007, first proof by Rosental, 1973)

In the proof, $R = 1$ for simplicity. The proof works for any $R$.

**Proof of Theorem 5** with $\tau_i = R \forall i$

Again, set the potential function $\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$ for every feasible flow $f$. Since the set of feasible flows is finite, there exists a global minimum of $\Phi$. Call this flow $f$. We claim that $f$ is an equilibrium flow for $(G, r, c)$.

Consider the case where player $i$ deviates from $P$ to $\tilde{P}$, then the change in $\Phi$ will be as follows:
The potential function
\[ \Phi(f) - \Phi(f') = \sum_{e \in P \setminus \tilde{P}} \left( \sum_{i=1}^{f_{e}+R} c_{e}(i) - \sum_{i=1}^{f_{e}} c_{e}(i) \right) + \sum_{e \in P \setminus \tilde{P}} \left( \sum_{i=1}^{f_{e}-R} c_{e}(i) - \sum_{i=1}^{f_{e}} c_{e}(i) \right) \]
\[ = \sum_{e \in P \setminus \tilde{P}} \sum_{i=f_{e}+1}^{f_{e}+R} c_{e}(i) - \sum_{e \in P \setminus \tilde{P}} \sum_{i=f_{e}-R+1}^{f_{e}} c_{e}(i) \]
\[ = \sum_{e \in P \setminus \tilde{P}} \sum_{i=1}^{R} c_{e}(f_{e} + i) - \sum_{e \in P \setminus \tilde{P}} \sum_{i=1}^{R} c_{e}(f_{e} - R + i) \]

On the other hand, the change in the cost of a player equals
\[ c_{p}(f') - c_{p}(f) = \sum_{e \in \tilde{P}} c_{e}(f_{e} + R) - \sum_{e \in \tilde{P}} c_{e}(f_{e}) \]

We want to prove that \( \Phi(f') - \Phi(f) = c_{p}(f') - c_{p}(f) \) (for then, \( f \) is indeed the global minimum of \( \Phi \) if and only if \( f' \) is an equilibrium flow). With this goal in mind, consider the case where player \( i \) does not transfer his flow all at once, but one unit at a time. Define \( f^{(n)} \) as the flow where player \( i \) routes \( n \) units over \( \tilde{P} \) and \( R - n \) units over \( P \). We get \( R \) flows:
\[ f = f^{(0)} , f^{(1)} , ..., f^{(R-1)} , f^{(R)} = f' \]

For instance, \( c_{p}(f^{(1)}) - c_{p}(f^{(0)}) \) is the change in cost if player \( i \) routes one unit of traffic over path \( \tilde{P} \) and \( R - 1 \) over \( P \). Therefore, the change in total cost can be expressed as:
\[ c_{p}(f') - c_{p}(f) = \sum_{n=1}^{R} (c_{p}(f^{(n)}) - c_{p}(f^{(n-1)})) \]

Each summand is equal to
\[ c_{p}(f^{(n)}) - c_{p}(f^{(n-1)}) = \sum_{e \in \tilde{P}} c_{e}(f_{e} + n) - \sum_{e \in \tilde{P}} c_{e}(f_{e} - n + 1) \]

Plugging this result in (6) and changing the order of summation:
\[ c_{p}(f') - c_{p}(f) = \sum_{e \in \tilde{P}} \sum_{n=1}^{R} c_{e}(f_{e} + n) - \sum_{e \in \tilde{P}} \sum_{n=1}^{R} c_{e}(f_{e} - n + 1) \]
\[ = \sum_{e \in \tilde{P}} \sum_{i=1}^{R} c_{e}(f_{e} + i) - \sum_{e \in \tilde{P}} \sum_{i=1}^{R} c_{e}(f_{e} - R + i) = \Phi(f') - \Phi(f) \]

(Proof adapted from Olsthoorn, 2012)

4.3.2 Existence of equilibria: affine cost functions

**Theorem 6 (Equilibrium flows with affine cost functions) [R18.15]** Let \((G, r, c)\) be an atomic instance with affine cost functions. Then \((G, r, c)\) admits at least one equilibrium flow.

**Proof:** Use the following potential function:
\[ \Phi(f) = C(f) + W(f) = \sum_{e \in E} c_{e}(f_{e})f_{e} + \sum_{i \in S_{e}} c_{e}(r_{i})r_{i} \]

where \( S_{e} \) is the set of players that choose a path that contains edge \( e \). The set of feasible \( f \) is again finite since the number of players is finite, as is the number of paths the players can choose from. Therefore, there exist an \( f \) that is a global minimum of the potential function.
Φ. We prove that this flow \( f \) is an equilibrium flow for \((G, r, c)\).

Assume to the contrary that \( f \) is not an equilibrium flow. Then player \( i \) can change from path \( P \) to path \( \tilde{P} \) and thereby decrease his cost:

\[
0 > c_p(\tilde{f}) - c_p(f)
\]

On the other hand, consider \( \Phi(\tilde{f}) - \Phi(f) = C(\tilde{f}) - C(f) + W(\tilde{f}) - W(f) \).

For the next calculations, remember that

- If \( e \in P \cap \tilde{P} \) or \( e \notin P \cup \tilde{P} \Rightarrow \tilde{f}_e = f_e \) and \( S_e(f) = S_e(\tilde{f}) \),
- If \( e \in P \setminus \tilde{P} \Rightarrow \tilde{f}_e = f_e - r_i \) and \( S_e(\tilde{f}) \cup \{i\} = S_e(f) \),
- If \( e \in \tilde{P} \setminus P \Rightarrow \tilde{f}_e = f_e + r_i \) and \( S_e(\tilde{f}) = S_e \cup \{i\} \).

\[
C(\tilde{f}) - C(f) = \sum_{e \in E} \left( c_e(\tilde{f}_e)(\tilde{f}_e - c_e(f_e)f_e) \right)
\]

\[
= \sum_{e \in P \setminus \tilde{P}} \left( a_e f_e + r_i \right) \left( f_e + r_i \right) + b_e \left( f_e + r_i \right) - a_e f_e^2 - b_e f_e \]

\[
+ \sum_{e \in E \setminus P} \left( a_e \left( f_e - r_i \right) \left( f_e - r_i \right) + b_e \left( f_e - r_i \right) - a_e f_e^2 - b_e f_e \right)
\]

\[
= \sum_{e \in E \setminus \tilde{P}} \left( a_e f_e^2 + a_e r_i^2 + 2 a_e r_i f_e + b_e f_e + b_e r_i - a_e f_e^2 - b_e f_e \right)
\]

\[
+ \sum_{e \in E \setminus \tilde{P}} \left( a_e f_e^2 + a_e r_i^2 - 2 a_e f_e r_i + b_e f_e - b_e r_i - a_e f_e^2 - b_e f_e \right)
\]

\[
= \sum_{e \in E \setminus \tilde{P}} \left( 2 a_e r_i^2 + 2 a_e r_i f_e + 2 b_e r_i - a_e r_i^2 - b_e r_i \right)
\]

\[
+ \sum_{e \in E \setminus \tilde{P}} \left( a_e r_i^2 - 2 a_e f_e r_i - 2 b_e r_i + b_e r_i \right)
\]

\[
= 2r_i \left( \sum_{e \in E \setminus \tilde{P}} \left( a_e \left( f_e + r_i \right) + b_e \right) - \sum_{e \in E \setminus \tilde{P}} \left( a_e f_e + b_e \right) \right)
\]

\[
- \left( \sum_{e \in P \setminus \tilde{P}} \left( a_e r_i^2 + b_e r_i \right) - \sum_{e \in E \setminus \tilde{P}} \left( a_e r_i^2 + b_e r_i \right) \right)
\]

\[
= 2r_i \left( \sum_{e \in E \setminus \tilde{P}} c_e(\tilde{f}_e) - \sum_{e \in E \setminus \tilde{P}} c_e(f_e) \right) - \left( \sum_{e \in E \setminus \tilde{P}} c_e(r_i) r_i - \sum_{e \in E \setminus \tilde{P}} c_e(r_i) r_i \right)
\]

\[
W(\tilde{f}) - W(f) = \sum_{e \in E} \left( \sum_{e \in S_e(\tilde{f})} c_e(r_i) r_i - \sum_{e \in S_e(f)} c_e(r_i) r_i \right) = \sum_{e \in E \setminus \tilde{P}} c_e(r_i) r_i - \sum_{e \in E \setminus \tilde{P}} c_e(r_i) r_i
\]

Combining \( C(\tilde{f}) - C(f) \) and \( W(\tilde{f}) - W(f) \) yields
\[
\Phi(\tilde{f}) - \Phi(f) = 2r_i \left( \sum_{e \in P_i} c_e(\tilde{f}_e) - \sum_{e \not\in P_i} c_e(f_e) \right) - \left( \sum_{e \in P_i} c_e(r_i) - \sum_{e \not\in P_i} c_e(r_i) \right)
+ \sum_{e \in P_i} c_e(r_i) r_i - \sum_{e \not\in P_i} c_e(r_i) r_i
= 2r_i \left( \sum_{e \in P_i} c_e(\tilde{f}_e) - \sum_{e \not\in P_i} c_e(f_e) \right)
\]

According to the inequality of (8), \(0 > 2r_i(c_P(\tilde{f}) - c_P(f))\) since \(r_i > 0\). But this contradicts the fact that \(f\) is a global minimum of \(\Phi\). Therefore, \(f\) is an equilibrium flow of \((G, r, c)\).

(Proof adapted from Fotakis et al., 2005)

### 4.4 The price of anarchy

Recall from Example 4 that an atomic instance need not admit an equilibrium flow, in which case the price of anarchy does not exist. Also, in Example 3 we have seen that an atomic instance may admit multiple equilibrium flows with different total cost. Because of these observations, obtaining a bound on the price of anarchy is a little complicated.

In this section we are looking to find a tight bound on the price of anarchy of atomic instances with affine cost functions.

We begin with a lemma that follows immediately from Definition 5, in this case in an instance with affine cost functions.

**Lemma 1 (Equilibrium condition) [R18.24]** Let \((G, r, c)\) be an atomic instance in which each edge \(e\) has an affine cost function \(c_e(x) = a_e x + b_e\) with \(a_e, b_e \geq 0\). Let \(f\) and \(f^*\) be equilibrium and optimal flows, respectively, for \((G, r, c)\). Let player \(i\) use the path \(P_i\) in \(f\) and the path \(P_i^*\) in \(f^*\). Then

\[
\sum_{e \in E} (a_e f_e + b_e) \leq \sum_{e \in E} (a_e (f_e + r_i) + b_e)
\]

**Proof:** Follows from the definition of an equilibrium flow in an atomic instance. \(f_e^* = f_e\) if \(e \in P_i \cap P_i^*\) and \(f_e^* = f_e + r_i\) if \(e \in P_i^* \setminus P_i\). This lemma is true not only for \(f^*\) being the optimal flow, but for any feasible flow.

(Proof adapted from Roughgarden, 2007)

The inequality of Lemma 1 holds for each player. With these inequalities, we can derive Lemma 2:

**Lemma 2 (Equilibrium inequality) [R18.25]** With the same assumptions and notations as in Lemma 1,

\[
C(f) \leq C(f^*) + \sum_{e \in E} a_e f_e f_e^*
\]

**Proof:** For each player \(i\), multiply the inequality of Lemma 1 by \(r_i\):

\[
r_i \sum_{e \in E_i} (a_e f_e + b_e) \leq r_i \sum_{e \in E_i} (a_e (f_e + r_i) + b_e)
\]

Summing the \(k\) inequality of (9) (one inequality for each player), we obtain


\[
\sum_{i=1}^{k} r_i \left( \sum_{e \in P_i} (a_e f_e + b_e) \right) = C(f) \leq \sum_{i=1}^{k} r_i \left( \sum_{e \in P_i^*} (a_e (f_e + r_i) + b_e) \right)
\]

On the right-hand, \( r_i \leq f_e^* \) for any edge \( e \in P_i^* \). If only player \( i \) routed his traffic over edge \( e \), then \( r_i = f_e^* \). If more players use this edge then \( r_i < f_e^* \). Therefore:

\[
\sum_{i=1}^{k} r_i \left( \sum_{e \in P_i^*} (a_e (f_e + f_e^*) + b_e) \right) = \sum_{e \in E} (a_e (f_e + f_e^*) + b_e) f_e^*
\]

The equality follows by reversing the order of summation (since we sum over all edges in \( P_i^* \), we multiply each sum by \( f_e^* \)).

\[
C(f) \leq \sum_{e \in E} (a_e f_e + f_e^*) + b_e f_e^* = \sum_{e \in E} (a_e f_e^* + b_e) f_e^* + \sum_{e \in E} a_e f_e f_e^* = C(f^*) + \sum_{e \in E} a_e f_e f_e^*
\]

The last equality finished the proof. \( \blacksquare \)

(Proof adapted from Roughgarden, 2007)

One could view \( \sum_{e \in E} a_e f_e f_e^* \) as a sort of ‘error’ term: how much do the costs of any equilibrium flow and the optimal flow differ? This error term can be rewritten in terms of \( C(f) \) and \( C(f^*) \) which helps us bounding the price of anarchy.

**Theorem 7 (The price of anarchy in affine weighted atomic instances)** [R18.23] If \((G, r, c)\) is an atomic instance with affine cost functions, then the price of anarchy of \((G, r, c)\) is at most \( \frac{3 + \sqrt{5}}{2} \approx 2.618 \).

**Proof:** Let \( f \) and \( f^* \) denote equilibrium and optimal flow, respectively, for the atomic instance \((G, r, c)\). Assume that edge \( e \) has cost function \( c_e(x) = a_e x + b_e \) for \( a_e, b_e \geq 0 \).

The Cauchy-Schwarz inequality states that for all vectors \( x \) and \( y \) in an inner product space it is true that \( \langle x, y \rangle^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle \rangle \Rightarrow \langle x, y \rangle \leq ||x|| \cdot ||y|| \). Apply the Cauchy-Schwarz inequality to the vectors \( \{\sqrt{a_e f_e}\}_{e \in E} \) and \( \{\sqrt{a_e f_e^*}\}_{e \in E} \) to obtain

\[
\sum_{e \in E} \sqrt{a_e f_e} \cdot \sqrt{a_e f_e^*} = \sum_{e \in E} a_e f_e f_e^* \leq \left( \sum_{e \in E} (\sqrt{a_e f_e})^2 \right) \cdot \left( \sum_{e \in E} (\sqrt{a_e f_e^*})^2 \right)
\]

\[
= \sum_{e \in E} a_e f_e^2 \cdot \sum_{e \in E} (a_e f_e^*)^2 \leq \sum_{e \in E} (a_e f_e^2 + b_e f_e^*) \cdot \sum_{e \in E} (a_e (f_e^*)^2 + b_e f_e^*)
\]

The first inequality is the Cauchy-Schwarz inequality; the second inequality is true because \( b_e f_e, b_e f_e^* \geq 0 \).

Combining this inequality with the inequality of Lemma 2, dividing through by \( C(f^*) \), and rearranging gives:

\[
C(f) \leq \sqrt{C(f^*)} \cdot \sqrt{C(f^*)} \Rightarrow \frac{C(f)}{C(f^*)} \leq 1 + \sqrt{\frac{C(f)}{C(f^*)}} \cdot \sqrt{C(f^*)} \Rightarrow \frac{C(f)}{C(f^*)} \leq 1 + \frac{C(f)}{C(f^*)} \cdot \frac{C(f^*)}{(C(f^*))^2}
\]

\[
= 1 + \frac{C(f)}{C(f^*)} \Rightarrow \frac{C(f)}{C(f^*)} - 1 \leq \frac{C(f)}{C(f^*)}
\]

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Squaring both sides gives
\[ \left( \frac{C(f)}{C(f^*)} \right)^2 + 1 - 2 \frac{C(f)}{C(f^*)} \leq \frac{C(f)}{C(f^*)} \Rightarrow \left( \frac{C(f)}{C(f^*)} \right)^2 + 1 - 3 \frac{C(f)}{C(f^*)} \leq 0 \]

Replace \( \frac{C(f)}{C(f^*)} \) by \( x \) and solve for \( x \):
\[ x^2 - 3x + 1 \leq 0 \Rightarrow x_1 = \frac{3 + \sqrt{5}}{2}, x_2 = \frac{3 - \sqrt{5}}{2} < 0 \]

Since \( \frac{C(f)}{C(f^*)} \geq 1 \), \( x_2 \) is not a solution. Therefore, \( \rho = \frac{C(f)}{C(f^*)} \leq \frac{3 + \sqrt{5}}{2} \approx 2.618 \) as claimed. ■

Awerbuch et al. (2005) showed that by changing the amount of traffic in their example (Section 4.1), the upper bound on the price of anarchy of \( \rho = \frac{3 + \sqrt{5}}{2} \) can be achieved. Player 1 and 2 route then each \( r_1 = r_2 = \frac{1 + \sqrt{5}}{2} \) and \( r_3 = r_4 = 1 \) remain the same.

The price of anarchy for unweighted affine atomic instances is slightly lower, \( \frac{5}{2} \).

**Theorem 8 (The price of anarchy in affine unweighted atomic instances)** If \((G, r, c)\) is an unweighted \((r_i = 1 \ \forall i)\) atomic instance with affine cost functions, then the price of anarchy of \((G, r, c)\) is at most \( \frac{5}{2} \).

**Proof:** We start with the inequality of Lemma 1:
\[ \sum_{e \in P_i} (a_e f_e + b_e) \leq \sum_{e \in P_i} (a_e (f_e + r_i) + b_e) \]

Multiply both sides by \( r_i \)
\[ \sum_{e \in P_i} (a_e f_e + b_e) \cdot r_i \leq \sum_{e \in P_i} (a_e (f_e + r_i) + b_e) \cdot r_i = \sum_{e \in P_i} (a_e f_e + b_e) \cdot r_i + a_e r_i^2 \]

We sum all the above inequalities for all \( i \) and get
\[ \sum_{i=1}^{k} \sum_{e \in P_i} (a_e f_e + b_e) \cdot r_i \leq \sum_{i=1}^{k} \sum_{e \in P_i} (a_e f_e + b_e) \cdot r_i + a_e r_i^2 \]

Define \( J(e) \) as the set of indices of players choosing a path \( P \) containing edge \( e \). Summing over all edges in \( E \) yields
\[ \sum_{e \in E} \sum_{i \in J(e)} (a_e f_e + b_e) r_i \leq \sum_{e \in E} \sum_{i \in J^c(e)} (a_e f_e + b_e) \cdot r_i + a_e r_i^2 \]

We can define \( f_e \) in a new way:
\[ \sum_{i \in J(e)} r_i = f_e, \sum_{i \in J^c(e)} r_i = f_e^* \]

Substituting (11) in (10), and using the fact that \( r_i = r_i^2 = 1 \), yields
\[ \sum_{e \in E} (a_e f_e + b_e) f_e \leq \sum_{e \in E} ((a_e f_e + b_e) f_e^* + a_e f_e^*) = \sum_{e \in E} (a_e f_e f_e^* + a_e f_e^* + b_e f_e^*) \]

The proof requires Lemma 3 whose proof can be found in Appendix C.

**Lemma 3.** Let \( i \geq 0, j \geq 0 \) be integers. Then
1. \[ ij = \frac{1}{3} j^2 + \frac{2}{3} i^2 - \frac{1}{3} (j - \frac{3}{2} i)^2 \]
2. \[ \frac{9}{8} i^2 + \frac{3}{2} i - \frac{1}{2} (j - \frac{3}{2} i)^2 \leq \frac{3}{2} i^2 \]
Applying property 1 of Lemma 3 to the right-hand side of the inequality, we get

\[
\sum_{e \in E} (a_e f_e + b_e) f_e \leq \sum_{e \in E} \left( a_e \left( \frac{1}{3} f_e^2 + \frac{3}{4} f_e^{*2} - \frac{1}{3} \left( f_e - \frac{3}{2} f_e^{*} \right)^2 + f_e^{*} \right) + b_e f_e^{*} \right)
\]

Multiplying both sides with factor \(\frac{3}{2}\) yields

\[
\sum_{e \in E} \left( \frac{3}{2} a_e f_e + \frac{3}{2} b_e \right) f_e \leq \sum_{e \in E} \left( a_e \left( \frac{1}{2} f_e^2 + \frac{9}{8} f_e^{*2} - \frac{1}{2} \left( f_e - \frac{3}{2} f_e^{*} \right)^2 + \frac{3}{2} f_e^{*} \right) + \frac{3}{2} b_e f_e^{*} \right)
\]

Subtracting \(\sum_{e \in E} \frac{1}{2} a_e f_e^2\) on both sides yields

\[
\sum_{e \in E} \left( a_e f_e + \frac{3}{2} b_e \right) f_e \leq \sum_{e \in E} \left( a_e \left( \frac{9}{8} f_e^{*2} - \frac{1}{2} \left( f_e - \frac{3}{2} f_e^{*} \right)^2 + \frac{3}{2} f_e^{*} \right) + \frac{3}{2} b_e f_e^{*} \right)
\]

Applying property 2 of Lemma 3 to the first term of the right-hand side of the inequality, we obtain

\[
C(f) = \sum_{e \in E} (a_e f_e + b_e) f_e \leq \sum_{e \in E} \left( \frac{5}{2} a_e f_e^{*2} + \frac{3}{2} b_e f_e^{*} \right) = \sum_{e \in E} \frac{5}{2} \left( a_e f_e^{*2} + \frac{3}{5} b_e f_e^{*} \right)
\]

\[
\leq \sum_{e \in E} \frac{5}{2} \left( a_e f_e^{*2} + b_e f_e^{*} \right) = \frac{5}{2} C(f^{*}) \Rightarrow \rho = \frac{C(f)}{C(f^{*})} \leq \frac{5}{2}
\]

(Proof adapted from Awerbuch et al., 2005)
5. References


Appendix A

Two variants of the Pigou example are considered (see Figure 2). The first is if we take \( r \) arbitrarily (but \( r > 0 \)). Then
\[
C(x) = 1 \cdot x + (r - x)(r - x) = x + r^2 - 2rx + x^2 = r^2 + (1 - 2r)x + x^2
\]
\[
C'(x) = 1 - 2r + 2x = 0 \iff f_{e_1} = x = r - \frac{1}{2}
\]
This tells us that the optimal flow for traffic \( r \leq \frac{1}{2} \) will always route all the traffic over the lower edge. If \( r > \frac{1}{2} \), then \( f_{e_1} = x = r - \frac{1}{2} \) will give us the amount of traffic that will take the upper edge. So for \( r > \frac{1}{2} \), the amount over the lower edge always will be \( f_{e_2} = \frac{1}{2} \).

Consider now the Pigou example with nonlinear cost, see Figure 2b, and \( r = 1 \) The lower edge has now a nonlinear cost function: \( C_e(x) = x^p \). Again, the equilibrium flow will be that all the traffic is routed over the lower path, making the total cost \( C_{p_2}(1) = 1 \) (\( = C_{p_1}(0) \), so Definition 1 holds). The optimal flow can again be calculated by the following:

\[
f_{e_2} = x \text{ is now the amount of traffic choosing the lower edge and } f_{e_1} = 1 - x \text{ the amount that chooses the upper edge, } x \in [0,1].
\]
\[
C(x) = 1 \cdot (1 - x) + x^p \cdot x = 1 - x + x^{p+1}, \text{ minimum: } C'(x) = -1 + (p + 1)x^p = 0 \iff \frac{1}{p+1} = x^p \iff x = \frac{1}{\sqrt[p+1]{p+1}}. \text{ For } p = 1, \text{ this equals indeed } x = \frac{1}{2}. \text{ If } p \to \infty, \text{ then}
\]
\[
\lim_{p \to \infty} \frac{1}{\sqrt[p+1]{p+1}} = \lim_{p \to \infty} (p + 1)^{-\frac{1}{p}} = \lim_{p \to \infty} e^{-\frac{1}{p} \ln(p+1)}.
\]

Apply the Limit Chain Rule: \( \lim_{p \to \infty} -\frac{1}{p} \ln(p + 1) = -\infty \). Apply L'Hôpital: \( \lim_{p \to \infty} \frac{1}{\sqrt[p+1]{p+1}} = \lim_{p \to \infty} -\frac{1}{p+1} = 0. \) So \( \lim_{u \to 0} e^u = 1 \) So \( \lim_{p \to \infty} \frac{1}{\sqrt[p+1]{p+1}} = 1. \)

This implies that as \( p \to \infty \), the optimal flow will have an increasing percentage of traffic over the lower edge. As for the price of anarchy, that will go to infinity as \( p \to \infty \).

\[
\text{The cost of the network with the equilibrium flow is } C \left( \frac{1}{\sqrt[p+1]{p+1}} \right) = 1 - p \frac{1}{\sqrt[p+1]{p+1}} + (p + 1) \frac{1}{\sqrt[p+1]{p+1}}^p + 1. 
\]

The price of anarchy, \( \rho = \frac{1}{\sqrt[p+1]{p+1}} \). The price of anarchy goes to infinity if \( p \to \infty \).

\[
\lim_{p \to \infty} \frac{1}{1 - \frac{1}{\sqrt[p+1]{p+1}}^p} = \frac{1}{1 - 1 - \frac{1}{\infty}} = \frac{1}{0} = \infty.
\]

The price of anarchy grows with a certain rate towards infinity.

\[
\rho = \left[ 1 - p \sqrt[p+1]{p+1} + (p + 1) \frac{1}{p+1} \right]^{-1}. \text{ Since}
\]
\[
\rho = 1 - p \sqrt[p+1]{p+1} + (p + 1) \frac{1}{p+1} = (p + 1)^{-\frac{1}{p+1}} + (p + 1)^{-\frac{1}{p+1}} = (p + 1)^{-\frac{1}{p+1}} \cdot (p + 1)
\]

we have \( \rho = \left[ 1 - p \sqrt[p+1]{p+1} + (p + 1) \frac{1}{p+1} \right]^{-1} = \left[ 1 - p \cdot (p + 1)^{-\frac{1}{p+1}} \right]^{-1} \)
Appendix B

Using the potential function to find the optimal flow of a nonatomic instance.

Pigou’s example

\[ c_1(x) = 1 + x \cdot 0 = 1 \] and \[ c_2(x) = x + x \cdot 1 = 2x. \] \( x = \frac{1}{2} \) will make the marginal cost of the two paths equal, making \( f = \left( \frac{1}{2}, \frac{1}{2} \right) \) an equilibrium flow. Furthermore,

\[ \Phi(f) = \int_0^{f_{e_1}} 1 \, dx + \int_0^{f_{e_2}} x \, dx = f_{e_1} + \frac{1}{2} f_{e_2}^2 \]

With the understanding that \( f_{e_1} + f_{e_2} = 1 \Rightarrow f_{e_1} = 1 - f_{e_2} \), we can rewrite the expression:

\[ \Phi'(f) = 1 - f_{e_2} + \frac{1}{2} f_{e_2}^2 \]

which means that \( f = (0, 1) \) is the flow minimizing \( \Phi \) and thus the equilibrium flow of the nonatomic instance.

Braess’s paradox

(Considering the network in Figure 3b):

\( P_1 \) consists of edges \( e_1 \) and \( e_2 \), \( P_2 \) of edges \( e_3 \) and \( e_4 \) and \( P_3 \) of edges \( e_1, e_5 \) and \( e_4 \).

Table 3. Marginal cost functions in Braess’s paradox

<table>
<thead>
<tr>
<th>Edge</th>
<th>( c(x) )</th>
<th>( c^*(x) )</th>
<th>( h_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 )</td>
<td>( c(x) = x )</td>
<td>( x + x \cdot 1 = 2x )</td>
<td>( \int_0^{f_{e_1}} x , dx = \frac{1}{2} f_{e_1}^2 )</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>( c(x) = 1 )</td>
<td>( 1 + x \cdot 0 = 1 )</td>
<td>( \int_0^{f_{e_2}} 1 , dx = f_{e_2} )</td>
</tr>
<tr>
<td>( e_3 )</td>
<td>( c(x) = 1 )</td>
<td>( 1 + x \cdot 0 = 1 )</td>
<td>( \int_0^{f_{e_3}} 1 , dx = f_{e_3} )</td>
</tr>
<tr>
<td>( e_4 )</td>
<td>( c(x) = x )</td>
<td>( x + x \cdot 1 = 2x )</td>
<td>( \int_0^{f_{e_5}} x , dx = \frac{1}{2} f_{e_4}^2 )</td>
</tr>
<tr>
<td>( e_5 )</td>
<td>( c(x) = 0 )</td>
<td>( 0 + x \cdot 0 = 0 )</td>
<td>( \int_0^{f_{e_5}} 0 , dx = 0 )</td>
</tr>
</tbody>
</table>

- \( c_{P_1}^*(f) = 2(f_{P_1} + f_{P_3}) + 1 \)
- \( c_{P_2}^*(f) = 1 + 2(f_{P_2} + f_{P_3}) \)
- \( c_{P_3}^*(f) = 2(f_{P_1} + f_{P_3}) + 2(f_{P_2} + f_{P_3}) \)

\( c_{P_4}^*(x) = c_{P_5}^*(x) = c_{P_6}^*(x) \iff 2(f_{P_1} + f_{P_3}) + 1 = 1 + 2(f_{P_2} + f_{P_3}) \iff f_{P_1} = f_{P_2} \)

Also: \( 2(f_{P_1} + f_{P_3}) + 1 = 4(f_{P_1} + f_{P_3}) \iff 1 = 2(f_{P_1} + f_{P_3}) \)

We know \( f_{P_3} = 1 - 2f_{P_1} \), therefore

\[ 1 = 2(f_{P_1} + f_{P_3}) \iff 1 = 2(f_{P_1} + 1 - 2f_{P_1}) \iff 2f_{P_1} = 1 \iff \]
\[ f_{P_1} = f_{P_2} = \frac{1}{2} \implies f = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \]

is the optimal flow in the network.

As for the equilibrium flow:
\[
\Phi(f) = \frac{1}{2} f_{e_1}^2 + f_{e_2} + f_{e_3} + \frac{1}{2} f_{e_4}^2 + 0
\]
\[= \frac{1}{2} (f_{P_1} + f_{P_3})^2 + (f_{P_2} + f_{P_3}) + (f_{P_1} + f_{P_3}) + \frac{1}{2} (f_{P_2} + f_{P_3})^2
\]
\[= \frac{1}{2} f_{P_1}^2 + \frac{1}{2} f_{P_2}^2 + f_{P_3}^2 + f_{P_1} f_{P_3} + f_{P_2} f_{P_3} + f_{P_1} + f_{P_2} + 2 f_{P_3}
\]
\[\nabla^2 \Phi(f) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \]
has eigenvalues \( \lambda_1 = 0, \lambda_2 = \lambda_3 = 3 \) so it is positive semidefinite

for any \( f \), making \( \Phi(f) \) a convex function. KKT-condition:
\[
\begin{pmatrix} f_{P_1} + f_{P_3} + 1 \\ f_{P_2} + f_{P_3} + 1 \\ f_{P_1} + f_{P_2} + 2 \end{pmatrix}
= -y_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - y_2 \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \implies f_{P_1} = f_{P_2} \] (see top two equations).

Last equation: \( 2 f_{P_1} + 2 = -y_1 + y_2 \), and the equality constraint reads: \( f_{P_3} = 1 - 2 f_{P_1} \).

Rewriting top equation:
\[
f_{P_1} + 1 - 2f_{P_1} + 1 = -y_1 + y_2 \iff 2 - f_{P_1} = 2f_{P_1} + 2 \iff f_{P_1} = f_{P_2} = 0 \text{ and } f_{P_3} = 1
\]
The equilibrium flow: \( f = (0,0,1) \).
Appendix C

Lemma 3. Let $i \geq 0, j \geq 0$ be integers. Then

1. $ij = \frac{1}{3}j^2 + \frac{3}{4}i^2 - \frac{1}{3}(j - \frac{3}{2}i)^2$
2. $\frac{9}{8}i^2 + \frac{3}{2}i - \frac{1}{2}(j - \frac{3}{2}i)^2 \leq \frac{5}{2}i^2$

Proof:

1. $\frac{1}{3}j^2 + \frac{3}{4}i^2 - \frac{1}{3}(j - \frac{3}{2}i)^2 = \frac{1}{3}j^2 + \frac{3}{4}i^2 - \frac{1}{3}j^2 - \frac{3}{4}i^2 + ij = ij$
2. Note that $\frac{9}{8}i^2 + \frac{3}{2}i \leq \frac{5}{2}i^2$ is true for any integer $i \geq 2$ and $i = 0$ (it is true for scalar $i \geq 1\frac{1}{11}$).
   $-\frac{1}{2}(j - \frac{3}{2}i)^2$ is not positive for any combination of $i$ and $j$, therefore the inequality is true for any $j$ and $i = 0$ or $i \geq 2$. Now we only have to prove it is true for $i = 1$ and any $j \geq 0$.
   For $i = 1$, the inequality is
   \[
   \frac{21}{8} - \frac{1}{2}(j - \frac{3}{2})^2 \leq \frac{5}{2} \implies \frac{21}{8} - \frac{1}{2}j^2 + \frac{3}{2}j - \frac{9}{8} \leq \frac{3}{2} - j^2 + \frac{3}{2}j \leq \frac{5}{2} \implies -j^2 + \frac{3}{2}j - 1 \leq 0
   \]
   This inequality holds for any (integer) $j \geq 0$.