Karisa Laras Widyadari

Master Thesis

Integrated Resources Planning: A Continuous Approach

August 2016

University of Twente

Master Applied Mathematics

Master Thesis

Integrated Resources Planning: A Continuous Approach

Author:
Karisa Laras Widyadari

Assessment Committee:
Prof. Dr. Marc Uetz,
Dr. Ahmad Al Hanbali,
Dr. Peter J.C. Dickinson,
Dr. Jan-Kees van Ommeren

Integrated Resources Planning: A Continuous Approach

Discrete Mathematics and Mathematical Programming
group of the
University of Twente

August 12, 2016

"You have to be odd to be number one."

Dr. Seuss

UNIVERSITY OF TWENTE

Faculty of Electrical Engineering, Mathematics, and Computer Science

Master of Applied Mathematics

Abstract

Integrated Resources Planning:
A Continuous Approach

by Karisa Laras WIDYADARI

Maintenance logistics is an important area that has been receiving wide attention in scientific literature. Efficient maintenance and logistics can help improve industry's and business' efficiency. In this problem, high availability of resources (spare parts and service engineers) are needed to facilitate corrective maintenance. However, these resources call for high investments since these resources are mostly expensive. This leads to an interest in efficient cost savings; minimizing the cost while still maintaining to meet the resources availability requirement. Therefore, an optimal resource availability in the maintenance logistics is needed.

In this master thesis assignment, a service logistics system that involves both spare parts and highly skilled engineers is considered. Highly skilled engineers are considered expensive assets, thus it is necessary to optimally plan the engineers availability. The objective is to determine the required capacity of each resource to minimize the total service costs subject to a specified requirement on resources availability. A constrained optimization model has been the developed for the problem by previous work. In this thesis we formulate the continuous relaxation of the optimization problem. We propose to approach the problem by using logarithmic barrier method. Using this method, we aim to find a good solution on the continuous relaxation problem and improve the performance of the heuristic algorithm from previous work.

Contents

This work is dedicated to my family. Thank you for being the best supporters throughout the years.

Chapter 1

Introduction

Maintenance logistics is an important area that has been receiving wide attention in scientific literature [1]. According to [2], ”In today’s global economies, logistics is a key facilitator of trade, and hence an important factor in rising prosperity and welfare”. An efficient maintenance and logistics system can help to improve industry’s and business’ efficiency.

The importance of maintenance logistics is a result of high investment in assets which require high operational availability. An unplanned downtime of an equipment can be costly. These unplanned downtimes should be avoided. However, if the unplanned downtimes do occur, the duration should be kept as short as possible to keep a low cost. This means that unavailable parts or components which cause the system breakdown are immediately replaced by a ready-to-use parts or components. This avoids repairing a part on site which require too much time.

Therefore, high availability of resources (spare parts and service engineers) are needed to facilitate corrective maintenance. However, these resources call for high investments since these resources are mostly expensive. This leads to an interest in efficient cost savings; minimizing the cost while still maintaining to meet the resources availability requirement. Thus, an optimal resource availability in the maintenance logistics is needed.

In this master thesis assignment, a service logistics system that involves both spare parts and highly skilled engineers is considered. Highly skilled engineers are considered expensive assets, thus it is necessary to optimally plan the engineers’ availability. The objective is to determine the required capacity of each resource to minimize the total service costs subject to a specified requirement on resources availability.

1.1 Motivation and Research Questions

The motivation for this master thesis assignment has been a paper by Rahimi-Ghahroodi, et al. [3]. In the paper, a constrained optimization model has been developed for the problem. However, in this master thesis assignment, we approach the service engineers' queue with a simpler approach, an $M/M/E$ queueing system.

The scope of this assignment is to find an optimization algorithm to improve an existing heuristic method in [3]. Therefore, the general research question for this master thesis assignment is:

"How to find an optimization algorithm that improves the existing heuristic method for the optimization problem in [3]?"

In this master thesis assignment, we aim to find good solutions using the continuous relaxation formulation of the problem in [3]. Another aim of the master thesis assignment is to improve the performance of an existing (greedy) heuristic algorithm in [3].

1.2 Approach

In this master thesis assignment, we formulate the continuous relaxation problem of the optimization problem in [3]. We use the extended definition of the Erlang-loss function to non-integral values by Jaganerman ([4], [5]) to replace the Erlang-loss function that is used in the optimization problem in [3]. Using the Erlang-loss function extension enables applications to nonintegral numbers [5].

In this assignment, we propose to find the solution using the barrier method. With this method, we approach a constrained optimization problem by approximating the problem using an unconstrained optimization problem. By using the barrier method, we approach the solution from inside the feasible region. The method forms a "barrier" that prevents the iterated solutions to go outside the feasible region. The unconstrained problem is formulated using a logarithmic-barrier function. In the end, we will implement the method to a given data used in practice and compare it to the solutions using the heuristic method in [3].

1.3 Structure of the Report

This report has the following structure. An overview of relevant literature is given in Chapter 2. In Chapter 3, we describe the model formulation and state the mathematical model for the problem. Chapter 4 describes the solution approach, in which continuous relaxation problem is introduced and methods are discussed. A complete explanation of how we implement the methodology described to the problem is explained in Chapter 5 . Finally, in Chapter 6, conclusions and recommendations are given. For an overview of the notations used in this report, the reader is referred to the list of notations at the end of this report.

Chapter 2

Related literature

2.1 Maintenance logistics and Integrated resources planning

Maintenance logistics is a widely discussed topic in literature [1, 3]. One important areas in maintenance logistics that is extensively studied is the spare part management, particularly the study of spare parts optimization models [1]. A seminal paper by Sherbrooke [6] marked the start of the comprehensive studies, of which he developed a METRIC (Multi-Echelon Technique for Recoverable Item Control) model. [7] and [8] discuss full overview and techniques to address problems in spare parts inventory control models. Basten and van Houtum [9] give an update overviews of models in literature on spare parts inventory control, focusing on system-oriented perspective.

Integration of spare parts management and service engineers is rarely considered, however each of these areas have been studied separately in some papers [3]. Several papers that considered the integration are by Visser and Howes[10] and Hertz et al.[11]. Visser and Howes investigated a model in a service company to determine the optimum number of maintenance technician to optimize the service company's profit, while Hertz et al. presented a decision support system that can create simulation models of different field service network, by considering both spare parts and manpower management. Both of these works use simulation as their performance analysis. In integrating resources, not only spare parts and service engineers, but service tools are often also considered in repair system. Vliegen discussed the integration between service tools and spare parts planning in her PhD thesis [12]. In her work, she showed that considering the integration of spare parts and service tools lead to more accurate results and saving cost up to 15%, concluding that integration of different resources planning in maintenance logistics have high benefits.

This master thesis assignment was motivated by the work in integrated resource planning in maintenance logistics by Rahimi-Ghahroodi et al. [3]. In this paper, both spare parts management and service engineers (manpower planning) are considered. The model is structured as a queueing model: spare parts queue and service engineers queue. A greedy heuristic procedure is used to optimize the problem. In this master thesis assingment, we use a simpler queueing model for the service engineers: the $M/M/E$ queueing system.

In the model of this master thesis assignment, we assume that the repair calls occur according to Poisson process. This assumption is commonly used in literature. Most of these literature discuss lateral transshipment inventory models. This assumption is done by Axsäter[13], Alfredsson and Verrijdt [14], Kukreja et al. [15], Sherbrooke [16], Kutanoglu [17], Wong, et al. [18], Kranenburg and van Houtum [19]. This assumption is justified because lifetimes of parts are exponential. This is also justified in the case when lifetimes are non-exponential but the set of systems is so large that the merged stream of failure processes of individual technical systems is close to Poisson [18]. Another assumption that our model use is that the repair lead-times are exponential. Alfredsson and Verrijdt [14] used exponential assumption in their work and justified it using sensitivity analysis.

2.2 Extension of Erlang-loss formula to non-integral numbers and its convexity

In this master thesis assignment, we deal with the continuous relaxation of the optimization problem of [3]. The aim of this assignment is to find the lower bound on the solutions. We use the extended definition of Erlang-loss function to non-integral values by Jagerman [4, 5].

In [4], Jagerman developed the properties of the Erlang loss function by extending the scope of application of the Erlang loss function to nonintegral numbers and complex numbers. The extension to the complex plane permits the powerful methods of complex analysis to be applied for obtaining exact, asymptotic, and approximate representations [4]. The convexity proof of the Erlang-loss function (in the domain of the non-negative integers) had been a known result which was proved by Messerli [20]. Jagers and van Doorn then proved the convexity of the Erlang-loss function extension to non-integral numbers in their paper [21].

2.3 The Logarithmic-Barrier method

The Lagrangian relaxation has been used as a method of choice in the related spare part management works by Wong et al. [18], Daskin [22], Diabat [23], Miranda [24], You and Grossmann [25]. Wong et al. [18] analysed a multi-item, continuous review model of two-location inventory sytems for repairable spare parts. In [18], Wong et al. formulated a Lagrange relaxation to find the lower bound on the optimal objective function of the original multi-item problem.

In this master thesis assignment, we are interested to find the lower bound of the optimization problem in [3]. We do this by formulating a continuous relaxation of the optimization problem. In this master thesis assignment, we propose to approach the solution of the continuous optimization problem by reducing it to unconstrained optimzation problem using transformation method. This is done by combining the objective function and the constraints to form a new unconstrained function whose minimum approximate the solution of the constrained problem [26]. This method is motivated by practical considerations, since unconstrained optimization problems are easier to handle [26]. One of the procedures for approximating constrained optimization problems by unconstrained problems is using barrier methods. In this assignment, we use the barrier method to approach the optimization problem. The barrier method is proved to be "as successful for nonlinear programming as for linear programming" and "together with active-set SQP methods, they are currently considered the most powerful algorithms for large-scale nonlinear programming" [27]. In particular, we use the logarithmic-barrier method. This method uses logarithmic function as its *barrier function*.

Chapter 3

Mathematical model

In this chapter, the mathematical formulation of the problem, together with the notation, is presented. First, we give a description of the model in section **??**. Then, section **??** describes the mathematical formulation of the model. An overview of all parameters, variables, and functions used in the model can be found in Appendix.

3.1 Model description

The problem in this master assignment is motivated from the problem of integrated resources planning problem in [3]. In this problem, we consider a service maintenance logistics system that consists of K types of spare parts subject to random failures to store in its local spare parts inventory. Repair calls for spare parts arrive randomly with rate λ. A repair call of type-k requires one unit of type-k spare part. Each type-k repair call arrives with probability p_k with $\sum_k^K p_k = 1$.

The service region has a team of engineers that to do the repair. For each repair call, a service engineer is needed to complete the repair job. In this model, we assume that each engineer has the same service time regardless the type of repair call that are assigned to them. The service time is the time from a repair job is assigned to a service engineer until the repair job is finished. It is exponentially distributed with rate μ.

When there is no available spare part for repair call type-k, the requested spare parts are satisfied by an emergency channel. When it is the case, both spare parts and service engineers are considered to be satisfied by the emergency channel. This means that the local service engineers are not going to do the repair job.

In the case when a spare part is available, a backlogging policy is applied for the service engineers queue. When the spare part is available but there are no service engineer available to take a repair call request, the system waits until a service engineer becomes available. There is no priority between spare part types, thus the backorders are served First Come First Served (FCFS). A maximum waiting time is defined for the total waiting time in the service region.

In this problem, we are interested to find the optimal spare parts stock levels and the optimal number of service engineers to minimize the average total cost. In this master assignment, the problem is restricted under a maximum average waiting time constraint and occupancy rate constraint for the queue model. The waiting times are caused by emergency shipment and service engineers queue. For each spare part, we consider a holding cost per item per unit. Naturally, the hiring cost of service engineers and emergency cost are also considered in the optimization problem. In section **??**, we present the formulation of the model. For a more detailed description of the model, the reader is recommended to see [3].

3.2 Model Formulation

In this section, we give an explanation of the mathematical formulation of the model. The model in this assignment is motivated by the model in [3]. In section 3.2.1, we present the optimization problem. Following this section, we explain the queueing systems that are involved in the model. In section 3.2.2, we briefly discuss the formulation of the spare parts queue in [3]. In section 3.2.3, we explain the formulation of service engineers' queue with $M/M/E$ queueing system.

3.2.1 Optimization problem

In this section, we introduce the optimization problem for the integrated resources planning in [3]. Later in Chapter 4, we will describe the continuous relaxation formulation for this optimization problem. The optimization problem in [3] is presented as follows:

$$
\begin{aligned}
\min_{\mathbf{S},E} \quad & TC(\mathbf{S},E) = O\cdot E + \sum_{k=1}^{K} H_k \cdot S_k + \sum_{k=1}^{K} C_k^L \cdot \Lambda_k^L(S_k) \\
\text{subject to} \quad & W(\mathbf{S},E) = \frac{\gamma}{\lambda}W^E + W^S \leq W^{max} \\
& OR(\mathbf{S},E) = \frac{\gamma}{E\mu} < 1 \\
& E \geq 0, E \in \mathbb{Z} \\
& \mathbf{S} = (S_1, S_2, ..., S_K) \geq \mathbf{0}, \mathbf{S} \in \mathbb{Z}^K
\end{aligned}
$$

K denotes the number of repair calls type in the problem. The variables are denoted by $\mathbf{S}$ and E. $\mathbf{S}$ denotes a vector of $(S_1, S_2, .., S_K)$, where S_k is the stock levels for spare part type-k, while E denotes the number of service engineers. The components of the optimization problem are explained as follows:

- **The objective function** $TC(\mathbf{S},E)$

  The objective of this problem is to minimize the total cost needed for the repair maintenance system. The three parameters in the objective functions are O as the cost of hiring a service engineer per unit time and H_k denotes the holding cost per item per unit time for spare part k. The last term in the objective function is the emergency shipment costs. C_k^L denotes the cost of emergency shipment for repair call k. We have $\Lambda_k^L(S_k)$ is the emergency shipment rate for repair call type k and it is a function of S_k. A complete list of notations can be found in Appendix.

- **The waiting time constraint** $W(\mathbf{S},E)$

  The first constraint describes the waiting time constraint of the system. The system involves two queueing systems: the spare parts queue and the service engineers queue. The first constraint describes that the waiting times in the service engineers and the waiting time caused by emergency shipment cannot be more than a given maximum waiting time W^{max}.

  W^S denotes the waiting time that is caused by the emergency shipment. When there are spare parts available in the system, they go to the service engineers queue for the repairing job to be done. The fraction of spare parts that go to the service engineers queue is described by $\frac{\gamma}{\lambda}$, where γ is the arrival rate in the service engineers queue and λ is the total arrival rate of repair calls in the system. Therefore, the waiting time in the service engineers queue is $\frac{\gamma}{\lambda}W^E$. A detailed explanation of the model in the spare parts queue and the service engineers queue is given in section 3.2.2 and 3.2.3, respectively.

- **The occupancy rate constraint $OR(\mathbf{S}, E)$**

  The second constraint describes the occupancy rate requirement of an $M/M/c$ queue for the service engineers queue, with c as a number of servers (in this case engineers). The occupancy rate gives the server utilization level of the queue system. It describes a stability condition of the service engineer queue. This requirement gives a restriction on the stability condition that the arrival of repair calls can still be handled by the servers (engineers' service). Violating this constraint means that the service engineers queue will 'explode' because the server cannot handle the rate of incoming repair call arrivals.

In the following sections, we will discuss the mathematical formulation of the queue systems in the model. Section 3.2.2 gives a brief explanation of the spare parts queue of the integrated resources planning problem in [3]. In section 3.2.3, a description of the service engineers queue will be given. A complete overview of all the parameters, variables, and functions used in the optimization problem can be found in Appendix.

3.2.2 Spare parts queue

In this section, we give a brief explanation of the spare parts queue formulation of the integrated resources planning problem (for a detailed description of the model, the reader is recommended to see [3]). In this problem, the spare parts stock replenishment is seen as K "servers" with exponentially distributed replenishment time. In this model, we have exponentially distributed inter-arrival times of repair calls. The queueing system is assumed as an $M/M/c/c$ system. This notation is introduced by Kendall (1953) [28] to denote a queueing model with M denoting exponential interarrival and service time distribution. In this queue system, there are c-server model with Poisson arrival and exponential service time, such that when all the c-channels are busy an arrival leaves the system without waiting for service [29]. This is called a (c-channel) *loss system*. In this problem, the c servers are the stock level of spare part k, S_k. Thus, we have a $M/M/S_k/S_k$ queue.

When the stock of spare part k is unavailable, the repair call will be done completely by an emergency service and the repair call is lost. Thus, the probability of emergency service of a repair call is given by the lost probability of $M/M/S_k/S_k$ queue known as the Erlang-B loss formula. The emergency probability is given by,

$$P_k^L = \frac{\frac{(\rho_{parts}^k)^{S_k}}{S_k!}}{\sum_{i=0}^{S_k} \frac{(\rho_{parts}^k)^i}{i!}} \tag{3.1}$$

And the emergency rate of a repair call for spare part k is given by,

$$\Lambda_k^L(S_k) = \lambda_k P_k^L = \lambda_k \left(\frac{\frac{(\rho_{parts}^k)^{S_k}}{S_k!}}{\sum_{i=0}^{S_k} \frac{(\rho_{parts}^k)^i}{i!}} \right). \tag{3.2}$$

with λ_k denotes the arrival rate of repair call for spare part type k. When the repair calls are replenished by the emergency channel, it is done completely by the emergency service and it must wait until the emergency shipment arrives. This waiting time is important to be considered and is accounted in the accepted waiting times for the repair. For the parts that are supplied by emergency shipment, the average waiting time is equal to $\frac{1}{\nu_k^{em}}$. The average waiting time for emergency channel of all repair calls is the fraction of total repair calls that is satisfied by emergency shipment, times the average waiting time $\frac{1}{\nu_k^{em}}$, given by:

$$W^S = \sum_{i=0}^{K} \frac{p_k P_k^L}{\nu_k^{em}} \tag{3.3}$$

with p_k denotes the arrival call probability for repair call type-k with $\sum_k^K p_k = 1$.

3.2.3 Service engineers queue

We assume exponentially distributed inter-arrival times of repair calls in the service engineers queue. The service times by E number of service engineers are also assumed to be exponentially distributed. This can be described as an $M/M/c$ queue (Kendall (1951) [28], with M denoting exponential interarrival and service time distribution). The number of servers is again denoted by c, of which in this case is the number of engineers. Thus, in this problem the service engineers queue can be described as an $M/M/E$ queue.

In an $M/M/c$ queueing system, the occupancy rate is given by $\rho = \frac{\gamma}{E\mu}$. The occupancy rate provides the server utilization level. To have a stable queue, the occupancy rate is supposed to be smaller than one, which can be mathematically expressed as $\rho = \frac{\gamma}{E\mu} < 1$. This requirement becomes the second contraint in the optimization problem (the importance of this constraint will be thouroughly explained in section 4.3).

The arrival rate of service engineers queue is given by,

$$\gamma = \sum_{k=1}^{K} \gamma_k = \sum_{k=1}^{K} \lambda_k \cdot (1 - P_k^L) = \sum_{k=1}^{K} \lambda - \Lambda_k^L(S_k) \tag{3.4}$$

We have multiple type of spare parts that arrives in the service engineers queue with arrival rate λ. In the service engineers queue, we assume that the spare part will be served by exponentially distributed service times that are the same for any type of spare parts. Therefore, the service rate in the service engineers is the same for all the spare part type, which is given by μ.

When there are no available engineers, the repair calls wait until the first engineer becomes available. In an $M/M/E$ queue, probability of engineers are unavailable can be described by the busy probability for the $M/M/E$ queue [30], which is given by:

$$P^B = \frac{\frac{(E\rho)^E}{E!}}{(1-\rho)\sum_{i=0}^{E-1}\frac{(E\rho)^i}{i!} + \frac{(E\rho)^E}{E!}} \tag{3.5}$$

Simplifying by letting $\sigma = E\rho = \frac{\gamma}{\mu}$, (3.5) becomes:

$$P^B = \frac{\frac{\sigma^E}{E!}}{(1-\rho)\sum_{i=0}^{E-1}\frac{\sigma^i}{i!} + \frac{\sigma^E}{E!}} \tag{3.6}$$

Hence, the average waiting time for the service engineers queue is given by:

$$W^E = \frac{P^B}{E\mu(1-\rho)} = \frac{P^B}{\mu(E-\sigma)} \tag{3.7}$$

Chapter 4

Approach

4.1 Continuous relaxation

In this chapter, we describe how we formulate the continuous relaxation of the problem in [3]. For this purpose, we need to use the Erlang-B extension to nonintegral numbers by Jagerman [4]. We discuss how we apply this to the optimization problem.

4.1.1 The discrete problem

We reintroduce the discrete optimization problem in [3] introduced section 3.2. We call this problem (Z^0):

$$
\begin{aligned}
(Z^0): \quad & \min_{\mathbf{S},E} && TC(\mathbf{S},E) = O \cdot E + \sum_{k=1}^{K} H_k \cdot S_k + \sum_{k=1}^{K} C_k^L \cdot \Lambda_k^L(S_k) \\
& \text{subject to} && W(\mathbf{S},E) = \frac{\gamma}{\lambda} W^E + W^S \leq W^{max} \\
& && OR(\mathbf{S},E) = \frac{\gamma}{E\mu} < 1 \\
& && E \geq 0, E \in \mathbb{Z} \\
& && \mathbf{S} = (S_1, S_2, ..., S_K) \geq \mathbf{0}, \mathbf{S} \in \mathbb{Z}^K
\end{aligned}
$$

In this thesis assignment, we aim to find a good solution using the continous relaxation of the problem. Thus we need to formulate the continuous problem (Z^0). However, as we have described in section 3.2, optimization problem (Z^0) contains the Erlang-B function as the emergency shipment probability P_k^L in (3.1). In the $M/M/c/c$ queueing system, the blocking probability P_b (Erlang-B function) is usually stated in the form [4]:

$$B(x,a) = P_b = \frac{\frac{a^x}{x!}}{\sum_{j=0}^{x} \frac{a^j}{j!}} \tag{4.1}$$

with x denotes the number of servers and a is the occupancy rate or the offered load. As we can see this function contains the factorial and summation which can be difficult to do when we work with non-integral numbers. In the optimization problem (Z^0), we have the blocking probability P_k^L in (3.1) with (S_k) integers. As we want to formulate a continuous relaxation problem of problem (Z^0), we will have (S_k) to be a positive real number. Luckily, Jagerman (1974) [4] has introduced the Erlang-B function extension to non-integral numbers

4.1.2 The relaxed problem

In this section we will discuss how we formulate the continuous relaxation problem and introduce the continuous optimization problem.

Erlang-B formula

In this section, we formulate the continuous relaxation of the problem. For this purpose, we use the extended definition of the Erlang loss function to non-integral values of x from Jagerman (1974) [4]:

$$B(x,a) = \left(a \int_0^\infty e^{-at}(1+t)^x dt \right)^{-1} \tag{4.2}$$

Jagerman (1974) [4] developed an extension of the Erlang-B function to extend the scope of application to nonintegral numbers. This function permits evaluation of the function for nonintegral number x of servers in practical related problems. Using this extension in integral number would be equal to the result of the (discrete) Erlang-B loss formula. A plot of both function for some $a = 0.5$ is given in figure 4.1.

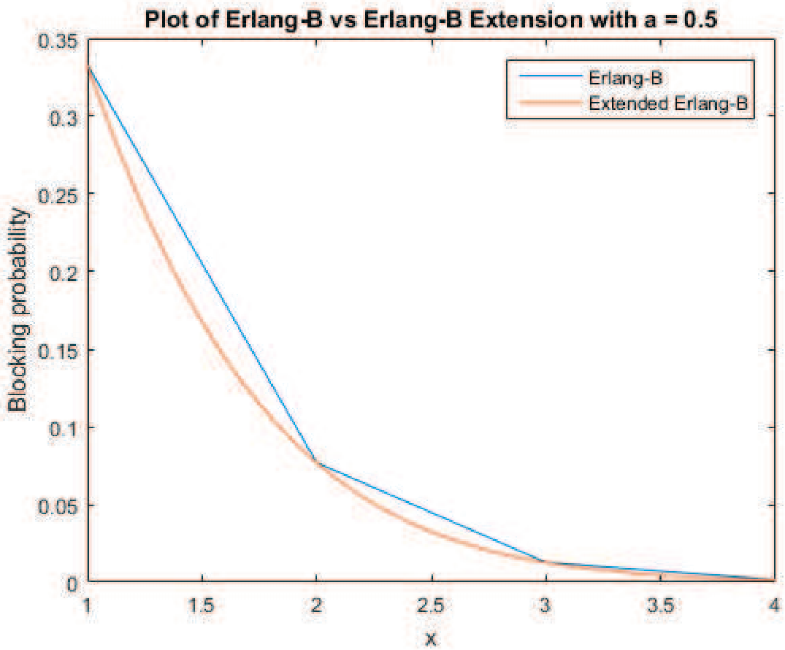


FIGURE 4.1: Plotting of Erlang-B function (4.1) and the extended Erlang-B function (4.2) for some $a = 0.5$

Applying the the Erlang-B extension (4.2) to the emergency probability for $M/M/S_k/S_k$ queue given in (3.1) gives:

$$P_k^L = B(S_k, \rho_{parts}^k) = \left(\rho_{parts}^k \int_0^\infty e^{-\rho_{parts}^k t}(1+t)^{S_k} dt\right)^{-1} \tag{4.3}$$

For the busy probability P^B of the $M/M/E$ given in (3.5), we use the relation the between the busy probability P^B and the blocking probability of $M/M/E/E$ (Erlang-B loss function) [29]:

$$P^B = \frac{\rho B(E-1,\sigma)}{1-\rho+\rho B(E-1,\sigma)} \tag{4.4}$$

With Erlang loss function extension (4.2), this function in turn becomes:

$$P^B = \frac{\rho\left(\sigma \int_0^\infty e^{-\sigma t}(1+t)^{E-1} dt\right)^{-1}}{1-\rho+\rho\left(\sigma \int_0^\infty e^{-\sigma t}(1+t)^{E-1} dt\right)^{-1}} \tag{4.5}$$

Continuous optimization problem

We form a continuous relaxation of the optimization problem. The optimization problem then becomes:

$$
\begin{aligned}
(Z): \quad & \min_{S,E} \quad TC(\mathbf{S},E) = O \cdot E + \sum_{k=1}^{K} H_k \cdot S_k + \sum_{k=1}^{K} C_k^L \cdot (\lambda_k P_k^L) \\
\text{subject to} \quad & W(\mathbf{S},E) = \frac{\gamma}{\lambda}\left(\frac{P^B}{\mu(E-\sigma)}\right) + \sum_{i=0}^{K} \frac{p_k P_k^L}{\nu_k^{em}} \leq W^{max} \\
& OR(\mathbf{S},E) = \frac{\gamma}{E\mu} < 1
\end{aligned}
$$

with P_k^L given in (4.3) and P^B given in (4.5). We did not address the constraints $E \geq 0$ and $\mathbf{S} = (S_1, S_2, ..., S_K) \geq \mathbf{0}$ in this formulation since it is implicitly given by the remaining constraints.

In the following section, we discuss our approach on solving problem (Z).

4.2 Logarithmic-barrier method

One of the well-known methods for solving constrained continuous optimization problems is the barrier method (also known as interior-point methods). The barrier method is a procedure of approximating constrained optimization problems using an unconstrained problem, since an unconstrained problem is considered as an 'easier' problem to solve than a constrained problem [27].

Let us consider a constrained optimization problem P:

$$
(P) \qquad f(x) \quad s.t. \;\; g_i(x) \leq 0, i = 1, ..., m, \quad x \in R^n \tag{4.6}
$$

whose feasible region we denote by:

$$
F := \{x \in R^n \;\;|\;\; g_i(x) \leq 0, i = 1, ..., m\}
$$

The barrier method imposed a large cost on feasible points that are close to the boundary of the feasible region, creating a "barrier" for exiting the feasible region. In this method, we consider solving an approximation of the form:

$$\min r(\theta, x) = f(x) + \frac{1}{\theta} b(x) \tag{4.7}$$

with $b(x)$ as the *barrier function* [31]. $\frac{1}{\theta}$ is the barrier parameter for (4.7). The barrier parameter $\frac{1}{\theta}$, which is also known as penalty parameter.

A barrier function for problem (P) is any continuous function $b(x)$ that is defined on the *interior* of the feasible set F such that $b(x) \to \infty$ as $\lim_x \max_i\{g_i(x)\} \to 0$ [32]. The definition of the barrier function indicates that the closer we get to the a constraint boundary, the larger $r(\theta, x)$ becomes. Hence, the points that are exactly on the boundary are not defined [32].

The basic idea of the barrier method is to start with a feasible point and a relatively small value of θ, which will prevent the algorithm from approaching the boundary. With each iteration, we optimize (4.7) and decrease the parameter value $\frac{1}{\theta}$ monotonically to find a new feasible point. This is done iteratively and at some time, the solution will arrive to a local minimum. We will show in a later subsection that the sequence solution $\{x^k\}$ converge to a local minimum.

4.2.1 Logarithmic-barrier function

In this master thesis assignment, we use the logarithmic barrier function that is defined as:

$$b(x) = -\sum_{i=1}^{m} log(-g_i(x)) \tag{4.8}$$

In the logarithmic barrier method, we formulate the inequality constraints into a logarithmic barrier term (4.8) in the (unconstrained) objective. Specifically, problem P is formulated into the unconstrained problem [31]:

$$B(\theta): \qquad \min f(x) + \frac{1}{\theta}\left(-\sum_{i=1}^{m} log(-g_i(x))\right) \tag{4.9}$$

4.2.2 Convergence

Let the sequence $\{\theta_k\}$ satisfy $\theta_{k+1} > \theta_k$ and $\theta_k \to \infty$ as $k \to \infty$. Let x^k denote the exact solution to P_1 for $\theta = \theta_k$, with θ_k being the θ of the barrier parameter for function $r(\theta, x)$ at iteration k.

The following lemma gives a set of properties that follows directly from the definition of x^k and the definition of the sequence $\{\theta_k\}$.

Lemma 4.1. ***(Barrier Lemma)***

1. $r(\theta_k, x^k) \geq r(\theta_{k+1}, x^{k+1})$
2. $b(x^k) \leq b(x^{k+1})$
3. $f(x^k) \geq f(x^{k+1})$
4. $f(x^*) \leq f(x^k) \leq r(\theta_k, x^k)$

Proof. See [32]. □

As previously discussed, the sequence solution $\{x^k\}$ of the logarithmic barrier method converge to a local minimum. In fact, the next result gives the convergence of the barrier method.

Theorem 4.2. ***(Barrier Convergence Theorem)***
Suppose $f(x)$, $g(x)$, and $b(x)$ are continuous functions. Let $x^k, k = 1, ..., \infty$; be a sequence of solutions of $B(\theta^k)$. Recall that $N_\delta(x)$ denote the ball of radius δ centered at the point x. Suppose there exists an optimal solution x^* of P for which $N(\epsilon, x^*) \cap \{x \in \mathbb{R} | g(x) < 0\} \neq \emptyset$ for every $\epsilon > 0$. Then any limit point $\tilde{x}$ of x^k solves P.

Proof. See [32]. □

4.2.3 Barrier algorithm

In this section we present the barrier algorithm. It is based on solving a sequence of unconstrained minimization problems, using the last point found as the starting point for the next unconstrained minimization problem. The method was first proposed by Fiacco and McCormick in the 1960s and was called *the sequential unconstrained minimization* technique (SUMT) [31]. Today, it is called simply as barrier method.

In the following, we present the Barrier algorithm [31] :

Algorithm 4.1 Barrier algorithm

Given: strictly feasible x, $\theta : \theta^0 > 0, \beta > 1$, tolerance $\epsilon > 0$
Repeat:

1. Compute x^* by minimizing problem $r(\theta, x)$ for θ^k.
2. Update $x = x^*$
3. Stopping criterion. Terminate if $m/\theta < \epsilon$
4. Increase θ. With $\theta = \beta \cdot \theta$

   until stopping criterion is satisfied.

To minimize the problem in Step 1, we use the first-order descent technique, the steepest descent method. We describe the method in detail in the following section.

4.2.4 Descent Techniques

As discussed in the previous sections, to solve the problem (Z), we approximate it by solving an unconstrained optimization problem formulated with barrier method. In this section, we will discuss the method that we use to solve the unconstrained continuous optimization problem:

$$\min f(x) \tag{4.10}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$. It is assumed that there exists an optimal point x^* and f is continuously differentiable. Since f is differentiable, a necessary and sufficient condition for a point x^* to be optimal is:

$$\nabla f(x^*) = 0. \tag{4.11}$$

see [31]. Thus, solving unconstrained minimization (4.10) is the same as finding a solution of (4.11), which is a set of n equations in the n variables $x_1, x_2, ..., x_n$ [31]. In this section we will discuss (first-order) descent method that we use to tackle this problem.

Descent methods

The descent methods produce a minimizing sequence $x^{(k)}, k = 1, ...,,$ where:

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$$

and $t^{(k)} > 0$. In here, $\Delta x^{(k)}$ is a vector in $\mathbb{R}^n$. It is called the *search direction*. The scalar $t^{(k)} > 0$ is called the *step length* [31]. The descent methods find

$$f(x^{(k+1)}) < f(x^{(k)}),$$

except when $x^{(k)}$ is optimal [31]. In our research, we use the *steepest descent method*.

Steepest descent method

The function $f(x)$ can be approximated by its linear expansion, the first-order Taylor approximation of $\hat{f}(x + v)$ around x:

$$\hat{f}(x + v) \approx \hat{f}(x + v) = f(x) + \nabla f(x)^T v.$$

if v small i.e. if $\|v\| = 1$ [33]. Note that if the approximation in the above expression is good, then we want to choose v such that the inner product $\nabla f(x)^T v$ is as small as possible. v is normalized so that $\|v\| = 1$. Among all direction v with norm $\|v\| = 1$, the direction:

$$\tilde{v} = -\frac{\nabla f(x)}{\|\nabla f(x)\|}$$

makes the smallest inner product with gradient $\nabla f(x)$ [33]. It follows from the following inequalities:

$$\nabla f(x)^T v \geq -\|\nabla f(x)^T\| \|v\| = \nabla f(x)^T \left(-\frac{\nabla f(x)}{\|\nabla f(x)\|} \right) = -\nabla f(x)^T \tilde{v}$$

The unnormalized direction $\bar{v} = -\nabla f(x)$ is called the *direction of steepest descent* at the point x [33]. Let us note that $\bar{v} = -\nabla f(x)$ is a descent direction as long as $\nabla f(x) \neq 0$, simply observe that $\bar{v}^T \nabla f(x) = -(\nabla f(x))^T \nabla f(x) < 0$ as long as $\nabla f(x) \neq 0$ [33].

The steepest descent algorithm uses the steepest descent direction as its search direction. The following is the simple steepest descent method algorithm [31, 33]:

Algorithm 4.2 Steepest descent algorithm

Given x^0, set $k = 0$
Repeat

- **Step 1.** $v^k = -\nabla f(x^k)$. If $v^k = 0$, then stop.
- **Step 2.** Choose step length $t^{(k)}$ with *line search*
- **Step 3.** Update $x^{(k+1)} = x^{(k)} + t^{(k)} \cdot v^{(k)}$

until stopping criterion is satisfied.

From Step 1 and the fact that $v^k = -\nabla f(x^k)$ is a descent direction, we have that $f(x^{(k+1)}) < f(x^{(k)})$ [33].

4.3 Constraint functions

In this section, we present the analysis on the convexity of the optimization problem. We start by introducing the optimization problem in the case where there is only one type of spare part in section 4.3.1 and then in section 4.3.2, we investigate the convexity properties of the constraint functions.

4.3.1 Single-item problem

We implement our approach by first considering the case in which there is only one type of spare part. In this way, we can see the problem in the simple case (single stock type) and analyze the problem before going into multi-item item case. We present the optimization problem in the single-item case:

$$
\begin{aligned}
(Z^1): \quad & \min_{S,E} \quad TC(S,E) = O \cdot E + H \cdot S + C^L \left(\lambda \cdot P^L \right) \\
\text{subject to} \quad & W(S,E) = \frac{\gamma}{\lambda} W^E + W^S \leq W^{max} \\
& OR(S,E) = \frac{\gamma}{E\mu} < 1
\end{aligned}
$$

4.3.2 Convexity properties

In this section, we investigate the convexity properties of the functions in the optimization problem of the single-item case to give a good idea of the optimization problem's properties.

We first approach this by analyzing the constraints and the feasible region of the optimization problem to give an idea of the problem before we prove the optimization problem's convexity.

We investigate the properties of the constraint function $W(S, E)$ to give a good idea of the constraint functions. We start by plotting the average waiting time of emergency shipment and the average waiting time of service engineer to see how the function behaves. For this purpose, we use several sample parameters listed as follows:

λ (Total arrival rate) = 0.5
ν (Replenishment rate of sparepart) = 0.4
ρ_{parts} (Occupancy rate of spare parts queue) = 1.2
ν^{em} (rate of emergency shipment) = 10
μ (Service rate in service engineers queue) = 3
W^{max} (Maximum accepted average waiting time) = 0.002

We see in in Figure 4.2 of the average time for emergency shipment W^S, it is decreasing as the number of the spare parts increase. This is because when there are spare parts available, the repair calls would be replenished by the system's stock parts instead of the emergency channel since the emergency shipment requires a more expensive cost. This results in the decrease of the waiting time in emergency channel as the spare part stocks increase.

In contrast with the average waiting time for the emergency shipment, the average waiting time for the service engineers W^E increase as the stock levels for the spare parts increase. This happens because when there are more spare parts available in the system, there are more repair calls that go to the service engineers. Figure 4.3 gives the plot with the number of service engineer equals to 1.

For this case, we also plot the feasible regions of waiting time constraint and occupancy rate constraint. This figures can be found in figure 4.4. Looking at these figure, we can see that the second constraint is redundant, since the feasible region of the problem can be defined by the waiting time constraint alone. Therefore, the feasible region of the optimization problem in this case is given by figure 4.4a.

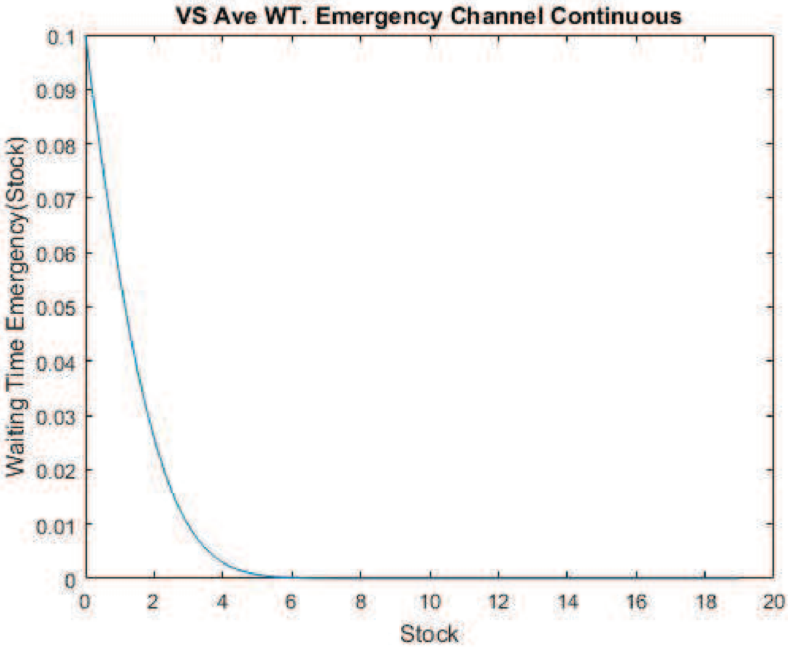


FIGURE 4.2: Average waiting time for emergency shipment

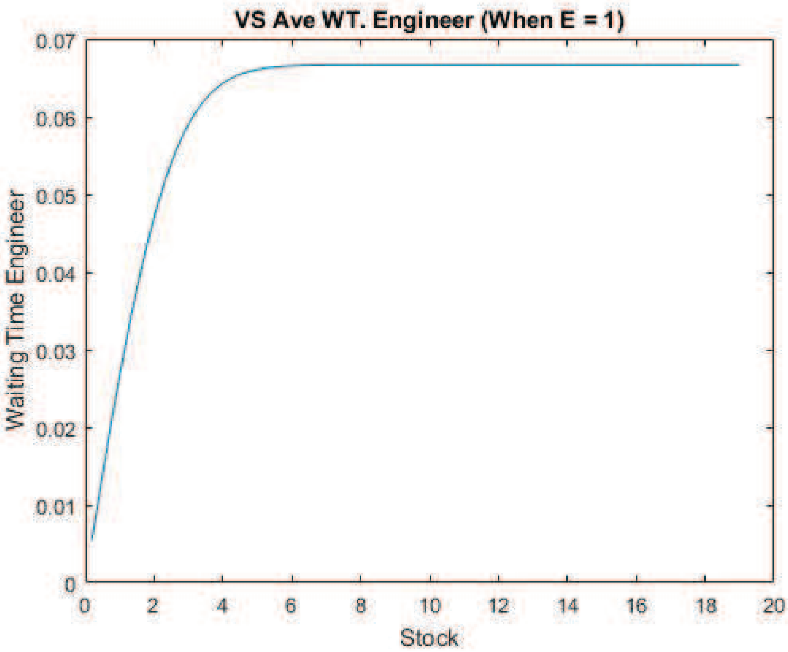
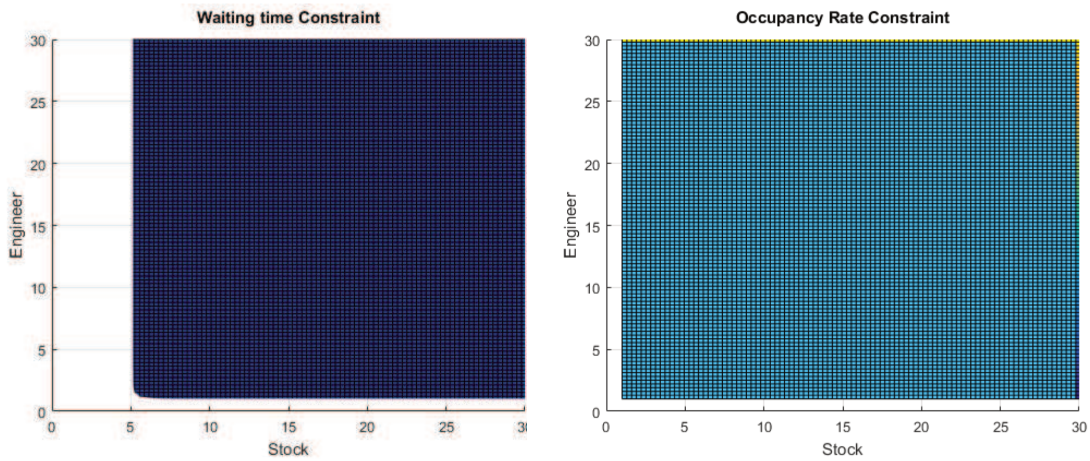


FIGURE 4.3: Average waiting time for service engineers queue with service engineer = 1



(A) Waiting time Constraint (B) Occupancy rate constraint

FIGURE 4.4: Feasible region of the case with $\lambda = 0.5$, $\nu = 0.4$, $\nu^{em} = 10$, $\mu = 3$, $W^{max} = 0.002$. The feasible region is indicated by the dark blue color on figure 4.4a.

A question arises: what is the importance of the occupancy rate constraint? Since the feasible region defined by this constraint seems redundant. The following example shed some light on this question. We plot the constraint functions using different parameters. In this case, the arrival rate is much higher than the replenishment rate of spare parts. The parameters for this plotting are listed as follows:

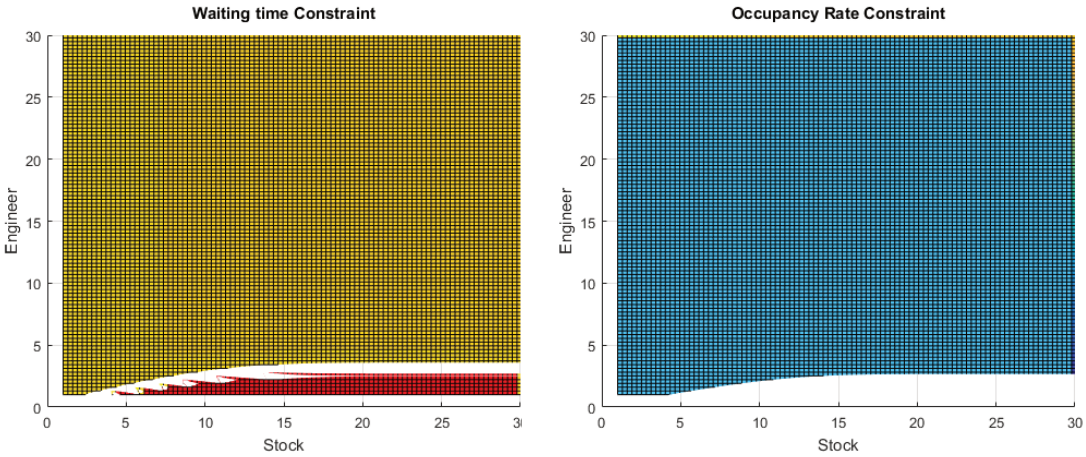
λ (Total arrival rate) $= 4$
ν (Replenishment rate of sparepart) $= 0.4$
ρ_{parts} (Occupancy rate of spare parts queue) $= 10$
ν^{em} (rate of emergency shipment) $= 4$
μ (Service rate in service engineers queue) $= 1.5$
W^{max} (Maximum accepted average waiting time) $= 0.4$

Figure 4.5 gives regions defined by waiting time constraint and occupancy rate constraint. In figure 4.5a, we see that there is a fragment (indicated by red color) in the feasible region for the waiting time constraint. Consider the region defined by the occupancy rate constraint given in figure 4.5b. This figure suggests that the second constraint in the problem will leave out the fragment in figure 4.5a.

To see the reason why this is important, let us consider again the waiting time constraint and the occupancy rate constraint defined by:

$$W(S,E) = \frac{\gamma}{\lambda} \frac{P^B}{\mu(E-\sigma)} + \frac{1}{\nu^{em}} \cdot P^L \leq W^{max} \tag{4.12}$$

$$OR(S,E) = \frac{\gamma}{E\mu} < 1 \tag{4.13}$$



(A) Waiting time Constraint

(B) Occupancy rate constraint

FIGURE 4.5: Feasible region with of occupancy rate constraint $\lambda = 4$, $\nu = 0.4$, $\mu = 1.5$, and $W^{max} = 0.4$. The yellow region on figure 4.5a indicates the feasible region of the problem.

The occupancy rate (4.13) constraint can be rewritten as:

$$OR(S, E) = E - \sigma > 0 \tag{4.14}$$

We see that the occupancy rate constraint is no other than the denominator for the waiting time in service engineer queue in the waiting time constraint. The occupancy rate constraint (4.14) avoids the denominator to become zero of which will make $\frac{P^B}{\mu(E-\sigma)}$ becomes infinity, and thus violating the constraint. This constraint also avoids the waiting time becoming negative. Violating constraint (4.14) means that we have the condition that $\frac{\gamma}{E\mu} > 1$ of which violates the stability requirement of a $M/M/c$ queue. This stability requirement is needed as violating this requirement means that the queue will 'explode' because the server cannot handle the rate of incoming arrivals.

To see this behaviour in detail, we plot the average waiting time of the service engineer's queue and compare it as the number of engineers increase. We see in figure 4.6, for the case of 1 engineer (blue line) the waiting time ascend sharply before dropping sharply to a negative number between 4-5 stocks. This also happens to the case with 2 engineers (red line) with the number of stock between 9-10. These sharp ascends happen when the denominator is approaching zero. The sharp drops indicates that the denominator then starts to become negative, resulting in negative waiting time. Thus, the occupancy rate constraint eliminates the condition when the service engineer's queue is unstable ($\frac{\gamma}{E\mu} > 1$) and when the waiting time becomes negative, as indicated in figure 4.6.

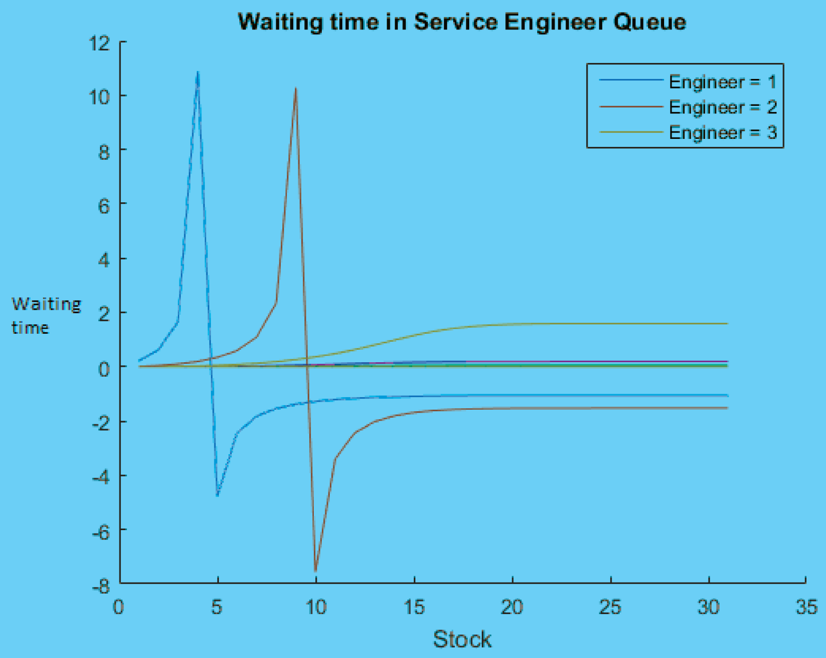


FIGURE 4.6: Average waiting time for service engineer queue with $\lambda = 4$, $\nu = 0.4$, $\nu^{em} = 4$, $\mu = 1.5$, $W^{max} = 0.4$

4.4 Multi-item problem

In this section, we discuss the multi-item formulation of the problem. We start this section by a proof on the non-convexity of the optimization problem. We then proceed on implementing the logarithmic barrier method discussed in section 4 to the multi-item optimization problem.

4.4.1 Convexity of the optimization problem

Following the discussion of the functions in section 4.3 for the single-item problem, in this section we proceed with the discussion of the convexity of the optimization problem for the multi-item case. We present a proof on the convexity of the optimization problem (Z):

Theorem 4.3.

1. *The objective function $TC(\boldsymbol{S}, E)$ is a convex function.*
2. *The waiting time constraint $W(\boldsymbol{S}, E)$ in general is not a convex function.*
3. *The occupancy rate constraint in general is not a convex function.*

4. *The optimization problem Z in general is not a convex optimization problem.*

Proof. 1. For the objective function of optimization problem P, there are three parts in the equation. For the first and second parts of the objective function, we see that they are linear functions, which are both convex and concave functions. The last part of the objective function includes the emergency probability, which is an Erlang-B function. As a known result by Messerli [20] and Jagers and van Doorn [21], the Erlang-B function is a decreasing and convex function. So the emergency rate $\lambda^L(S)$ is decreasing and convex function in the number of stock level S. Thus, using summation rule of convex function, we have a convex objective function.

2. We will prove that the waiting time constraint is a non-convex function.

Suppose for the sake of contradiction, $W(\bar{z})$ is convex for all $\bar{z} = (\mathbf{S}, E)$. Then for all $\bar{x}, \bar{y}$, the points in the feasible region $\mathfrak{F}$ defined by function $W(\bar{z})$ and $0 \leq \varphi \leq 1$, we have:

$$W(\varphi\bar{x} + (1-\varphi)\bar{y}) \leq \varphi W(\bar{x}) + (1-\varphi)W(\bar{y})$$

Consider the case where $K = 1$ and the feasible region defined by the following parameters: Arrival rate $\lambda = 4$, replenishment rate of spare part $\nu = 0.4$, engineer's service rate $\mu = 1.5$, and maximum waiting time $W^{max} = 0.4$ which is shown in figure 4.5a. Consider also two feasible points $\bar{x} = (20, 1)$ and $\bar{y} = (20, 6)$ with $\varphi = 0.5$ We have:

$$W(\varphi\bar{x} + (1-\varphi)\bar{y}) = 0.6959 > 0.5018 = \varphi W(\bar{x}) + (1-\varphi)W(\bar{y})$$

A contradiction.

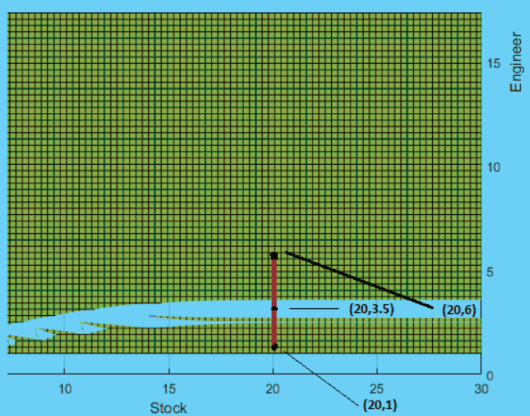


FIGURE 4.7: Convex combination of points $\bar{x}$ and $\bar{y}$ in picture

Moreover, consider the convex combination of points $\bar{x}$ an $\bar{y}$, $\varphi\bar{x} + (1-\varphi)\bar{y}$ is not in $\mathfrak{F}$, as $0.6959 > W^{max}$. Figure 4.7 shows the convex combination in the region defined by the waiting time constraint (figure 4.5a).

Figure 4.5a shows the region defined by $W(\bar{z})$ with $\lambda = 4$, $\nu = 0.4$, $\mu = 1.5$, and $W^{max} = 0.4$.

Thus, $W(\bar{z})$ in general is not a convex function.

3. We will prove that the occupancy rate constraint is a non-convex function.

The occupancy rate function can be rewritten as:

Suppose for the sake of contradiction, $OR(\bar{z})$ is convex for all $\bar{z} = (\mathbf{S}, E)$. Then for all $\bar{p}, \bar{q}$, the points in the feasible region $\mathfrak{G}$ defined by function $OR(\bar{z})$ and $0 \leq \varphi \leq 1$, we have:

$$OR(\varphi\bar{p} + (1-\varphi)\bar{q}) \leq \varphi OR(\bar{p}) + (1-\varphi)OR(\bar{q})$$

Consider the case where $K = 1$ and the region defined by the following parameters: Arrival rate $\lambda = 4$, replenishment rate of spare part $\nu = 0.4$, engineer's service rate $\mu = 1.5$ which is shown in figure 4.5b. Consider two feasible points $\bar{p} = (8, 2)$ and $\bar{q} = (29, 3)$ with $\varphi = 0.5$. We have:

$$OR(\varphi\bar{p} + (1-\varphi)\bar{q}) = 1.0611 > 0.8856 = \varphi OR(\bar{p}) + (1-\varphi)OR(\bar{q})$$

A contradiction.

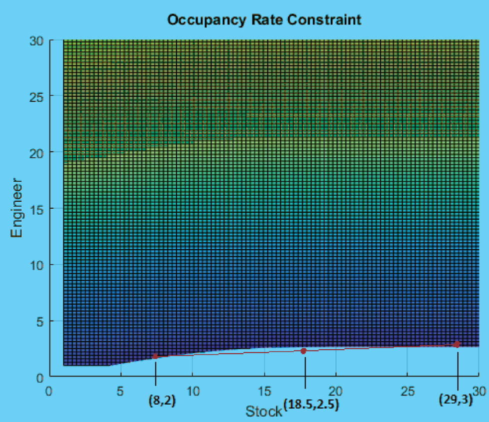


FIGURE 4.8: Convex combination of points $\bar{p}$ and $\bar{q}$ in picture

Moreover, consider the convex combination of points $\bar{p}$ and $\bar{q}$, $\varphi\bar{p} + (1-\varphi)\bar{q}$ is not in $\mathfrak{G}$, as $1.0611 > 1$. Figure 4.8 shows the convex combination in the region defined by the occupancy rate constraint (figure 4.5b).

Thus, $OR(\bar{z})$ in general is not a convex function.

4. Suppose for the sake of contradiction, problem (Z) is a convex optimization problem. Then, the feasible set of the problem is convex.

   Consider the case where $K = 1$ and the region defined by the following parameters: Arrival rate $\lambda = 4$, replenishment rate of spare part $\nu = 0.4$, engineer's service rate $\mu = 1.5$. Consider two feasible points $\bar{a} = (5.5, 2.2)$ and $\bar{b} = (29, 4)$ with $\varphi = 0.5$. The convex combination of these points is given by:

$$\varphi\bar{a} + (1-\varphi)\bar{b}) = 0.5 \cdot (5.5, 2.2) + 0.5 \cdot (29, 4) = (17.25, 3.1)$$

   is not in the feasible set of problem (Z). A contradiction.

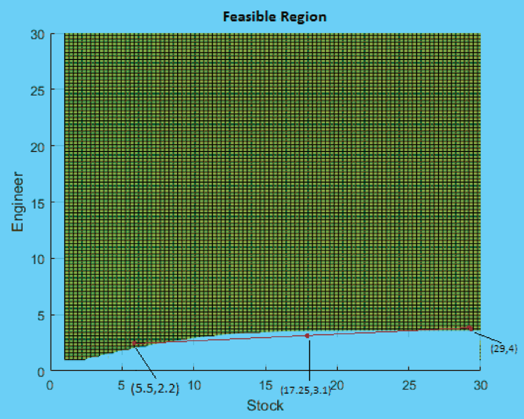


FIGURE 4.9: Convex combination of points $\bar{a}$ an $\bar{b}$ in picture

   Moreover, the constraints in the optimization problem (Z) are generally non-convex problems. Thus, the optimization problem (Z) is not a convex optimization problem.

□

4.4.2 Logarithmic barrier formulation

We formulate problem (Z) using the logarithmic barrier method. We call the resulting unconstrained problem as problem $(\bar{Z})$:

$$(\bar{Z}): \quad \min_{S,E} TC(\mathbf{S}, E) - \frac{1}{\theta}\left(log(W^{max} - W(\mathbf{S}, E)) + log(1 - OR(\mathbf{S}, E))\right) \tag{4.15}$$

We will implement the logarithmic barrier formulation (4.15) with the case of 2 spare parts items and 5 spare parts items and see how the solutions achieved with this method compared to the the previous work in [3]. This will be discussed in detail in the following chapter.

Chapter 5

Implementation and results

This section deals with the implementation of the logarithmic barrier method to problem (Z). First we present the barrier method algorithm that we use. To analyze whether the algorithm does as what it is expected, we implement it first to a case with single spare part type. Then we implement the logarithmic barrier method to the case with 2 spare part types and 5 spare part types and see how the solutions achieved with this method compared to the previous work in [3].

5.1 Algorithm implementation

In this section we discuss how the algorithm is implemented. We explain the implementation into two subsections, the barrier algorithm implementation and the steepest descent method implementation.

5.1.1 Barrier algorithm implementation

In this section, we explain the choice of barrier parameter θ, parameter β that we use in the barrier algorithm.

Choice of parameter β

The choice of parameter β involves a trade-off in the number of inner and outer iterations required [31]. Choosing a small β would make each outer iteration θ increases by a small factor, making the previous iterated solution for the Newton steps a very good starting point, which in turn make number of Newton steps needed to compute the next iteration small [31]. We have the opposite situation when β parameter is large. Since after outer iteration θ would increase for a large amount, the current iterated solution would probably not a good approximation for the next iteration [31]. Thus, we can expect more inner Newton steps iteration.

This trade-off in the choice of parameter has been confirmed in both practice and theory [31]. In practice, small values of β (near one) results in many outer iterations with just a few steps for each outer iteration. For β large, from around 3 to 100, the two effects nearly cancel, so the total number of Newton steps remains approximately constant [31]. This means that the choice of β is not critical and values around 10 to 20 works well [31]. In our algorithm we choose the β values to be 15.

Choice of parameter θ^0

The choice of θ^0in the algorithm also requires a trade-off in the iteration. Choosing a large θ^0 would make the first outer iteration require too many iterations. When θ^0 is small, the algorithm will require extra outer iterations, and possibly too many inner iterations in the first step [31].

The algorithm 4.1 is a $m/t-$suboptimal algorithm with m as the number of constraints and a guaranteed accuracy ϵ [31]. In this method we take $\theta^0 = m/\epsilon$ and solve the problem using Newton method.

5.1.2 Steepest descent implementation

We use the first-order gradient method, the steepest descent method, to minimize problem $(\bar{Z})$ in step 1. A discussion of the steepest descent method has been given in section 4.2.4. In this section we describe the search direction and the step length that we use in the steepest descent method in our algorithm.

Search direction

The direction of locally steepest descent is given by:

$$s^k = -(A^k)^{-1}\nabla M(x^k)$$

for an objective function $M(x^k)$ at point x^k, with A^k a positive definite matrix [26].

For our algorithm, the search direction that we use is defined as:

$$s^k = -\nabla \bar{Z}((\mathbf{S}, E)^k) \tag{5.1}$$

by choosing $A^k = I$, the $n \times n$ matrix. Since, $I^{-1} = I$, we obtain 5.1.

Step length

For computing the descent step length δ^k, we use the nonoptimal step alternative. In the nonoptimal step, we find δ^k which satisfies:

$$M(x^k + \delta^k s^k) \leq M(x^k) \tag{5.2}$$

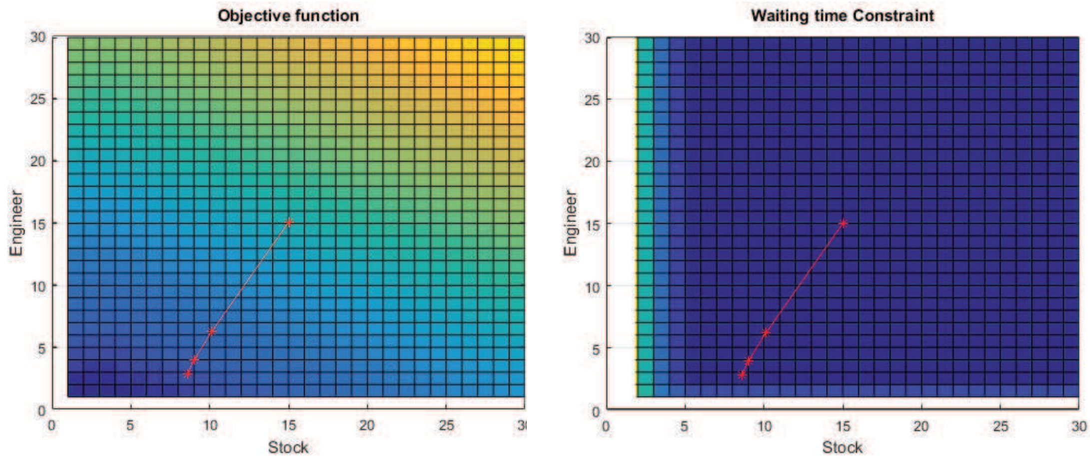
By using the nonoptimal step, it increases the computational efficiency since more computation would be needed to find δ^k that satisfy $M(x^k + \delta^k s^k) = \min_\delta M(x^k + \delta^k s^k)$ [26]. However, in our algorithm, we use the optimal step with the last barrier parameter. For optimization along the line $(x^k + \delta^k s^k)$, we use the sequential halving search [26]. This procedure is similar to a procedure for finding the real roots of an algebraic or transcendental function [26]. In this method, the interval to be searched is halved and the portion wherein the objective functions are smallest is retained [26].

5.2 Single-item case

In this section, we explain the implementation to the case of single spare part type. The purpose is to have a good idea on how the algorithm works. We present two different case for the implementation. The parameters that we use for this implementation is described as follows:

Parameters	Case1	Case2
Arrival Rate of Repair calls (λ)	4	0.5
Rate of spare part replenishment (ν)	0.4	0.4
Rate of Engineer's service time (μ)	3	12
Rate of Emergency shipment (ν^{em})	10	10
Cost of Engineer (O)	700	900
Holding Cost of Spare part (H)	500	500
Cost of Emergency Shipment (C^L)	5000	5000
Maximum Waiting time (W^{max})	0.05	0.03

TABLE 5.1: Parameters of Case 1 and 2 for Single spare part case



(A) Objective value (B) Feasible region

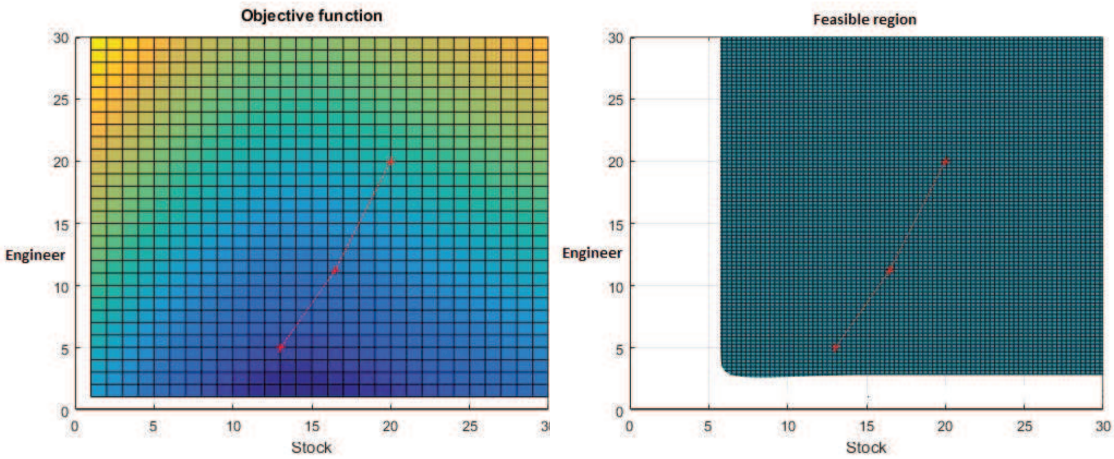
FIGURE 5.1: Objective value and feasible region for case 2. The darker the color in figure 5.2a, the smaller the objective value.

We present region plotting of the objective value and the feasible region for both cases. In the pictures, we see there are different colors on the region. These colors represents the value difference of the plotted function. The darker the color the smaller the value is.

We see from the two sets of figures, for both case 1 (Figure 5.1) and case 2 (Figure 5.2), the iterated solutions go towards the minimum since they go towards the darkest region of the objective value. For case 1, we see in figure 5.1b, there are lighter colored area. These indicates that the value of the waiting time is near the maximum waiting time, even though these are still feasible. We can also see from Figure 5.1b, how the barrier "effect" comes into play as the iterated solutions (indicated by red dots) refused to go near lighter colored area. For case 2, the iterated solutions also go towards the minimum and the solution is feasible.

From both of the cases that we present, the iterated solutions from the algorithm is working as it is expected. This means that iterated solutions go towards the minimum and barrier prevents the iterated solutions to go outside the feasible region. However, we should note that from these two example that the iterated solutions may not produce the lowest total cost as the barrier may prevent in doing so.

In the following section, we implement the algorithm to multi-item case.



(A) Objective value (B) Feasible region

FIGURE 5.2: Objective value and feasible region for case 2. The darker the color in figure 5.2a, the smaller the objective value.

5.3 Multi-item case

In this section we implement the barrier algorithm to the multi-item case and present the results. In particular, we implement the algorithm to 2 spare parts case and 5 spare parts case. As the aim of the assignment is to provide an improvement of the solutions in the previous work in [3], we implement the solutions from the barrier algorithm as the initial solutions for the heuristic algorithm in [3]. To make both problems comparable, the heuristics algorithm is adjusted so that the formulation assumes Poisson process as in our model. We will call the solutions as the *Barrier-Heuristic solutions*. Then, we will compare the barrier-heuristic solutions with the original heuristic solutions and see whether there is an improvement in the solutions achieved.

In general, we compare 3 different solutions in this section: the barrier algorithm solutions, original heuristic solutions, and the barrier-heuristic solutions.

In this section, we present the results of the implementation. We implement this to 150 different cases for to achieve solutions for 3 methods: the barrier algorithm, original heuristic algorithm, and the barrier-heuristic algorithm.

Barrier algorithm solutions vs Original heuristics solutions

First, we implement the barrier algorithm to solve the continuous optimization problem $(\bar{Z})$ to the case of 2 items and 5 items of spare parts. In the 2 spare parts case, the barrier algorithm did not perform as well as the simple case. The algorithm produce solutions that is not close to the solutions produced by the original heuristic solutions.

The average difference of the total cost is 23105,22. It seems that the algorithm does not work as well in more variables as it is in the simple case. For the 5 spare parts case, the difference of the total cost is 41594,655. The high difference is caused by some variables that are far too large compared to the heuristics solutions. A possible explanation is that this happens because of the nonconvexity of the problem or that the barrier comes into play. Based on these results, in both 2-item and 5-item case, the barrier algorithm never produce better solutions compared to the original heuristics.

To solve one case, the barrier algorithm takes on average 40.778 sec for 2-item case and 105.167 sec for 5-item case. The results from the barrier algorithm can be found in table 5.2. Figure 5.3 and 5.4 represent the comparation over the achieved result for 2 spare parts case and 5 spare parts case, respectively.

Barrier algorithm	**2-item case**	**5-item case**
Average difference of total cost to original heuristic	23105.22	41594.655
Average computational time to solve 1 case	40.778 sec	105.167 sec

TABLE 5.2: Average difference of Barrier algorithm solutions compared to original heuristic solutions

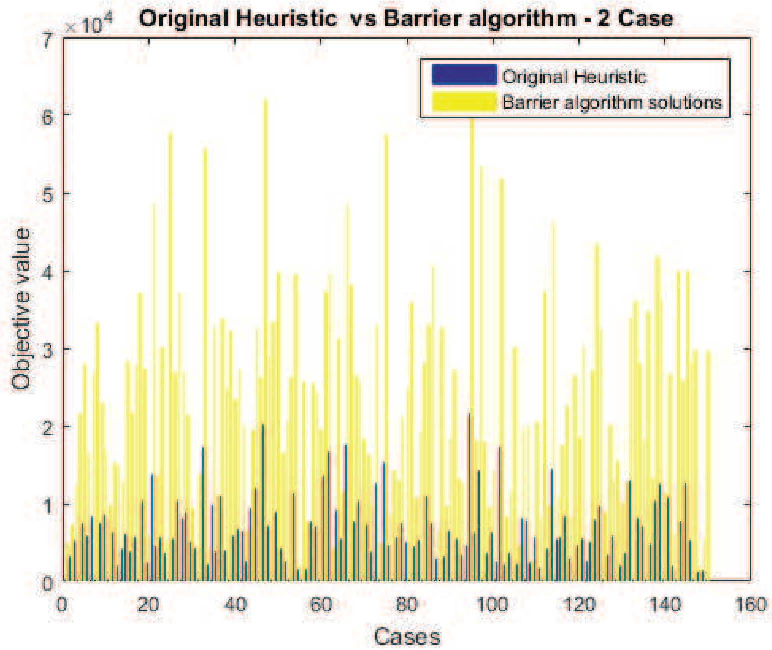


FIGURE 5.3: Comparison on original heuristic algorithm solutions and barrier algorithm solutions in 2 items case

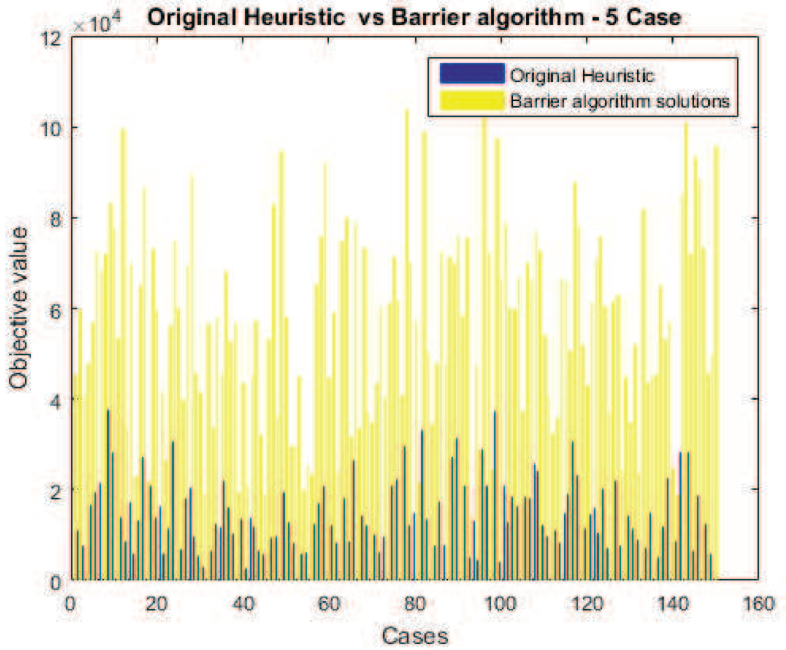


FIGURE 5.4: Comparison on original heuristic algorithm solutions and barrier algorithm solutions in 5 items case

Barrier-heuristics solutions vs Original heuristics solutions

To see whether the achieved solutions on the barrier algorithm make an improvement on the original heuristic algorithm, we apply the barrier algorithm solutions to be the starting points for the heuristic algorithm. Since we solved the solutions of a continuous relaxation problem, we round the solutions to the nearest integer. This is done so that it can be applied to the heuristics algorithm that solves the discrete optimization problem. We applied this for 150 different data for both 2 and 5 items.

The computational time for the barrier-heuristics seems to perform less well than the heuristic algorithm, although the difference is relatively small. We did the test by running the algorithm to solve 150 cases. For the 2-item case, the original heuristic algorithm takes on average 0.732 seconds, while the barrier-heuristic takes 1.250 seconds. For the 5-item case, the original heuristic algorithm takes around 1.5757 seconds while the barrier-heuristics takes 7.3940 seconds. Table 5.3 shows the computational time comparison between the two methods.

Average computational time	2-item case	5-item case
Original heuristics	0,732 sec	1.5757 sec
Barrier-heuristics	1.250 sec	7.3940 sec

Table 5.3: Average computational time of Barrier-heuristics vs Original heuristics in seconds

Although the barrier-heuristics takes more computational time, some solutions produce better solutions than the original heuristics method. In 7 out of 150 cases, the barrier-heuristics produce better solutions than the original heuristic algorithm, with a difference of 727,339. However, there are 2 cases where the original heuristics produce better solutions than the barrier-heuristic solutions. For the rest of the cases, both methods give the same solutions. This might be the case that the solutions are already optimal. Table 5.4 presents the results of the comparation for 2-item case. A pie chart of the results produce by the two methods is given in figure 5.5.

2-items	**Barrier-heuristics**	**Original heuristics**
Number of better solutions	7	2
Percentage of better solutions	4.67%	1.33%
Average difference of better solutions	727,339	905,877

Table 5.4: Results of Barrier-Heuristic solutions compared to original heuristic solutions in 2-item case

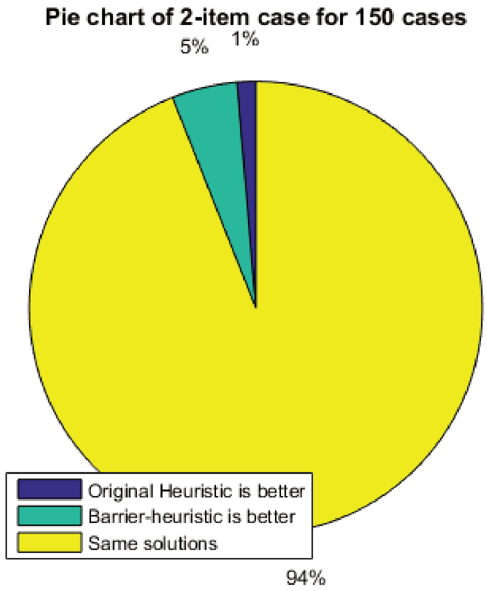


FIGURE 5.5: Pie chart on original heuristics solutions and barrier-heuristic solutions in 2-item case over 150 cases

Similar results are also found when we compare both methods on the 5-item case. In 10 out of 150 cases, the barrier-heuristics give better solutions than the original heuristic method. Although, we also found that in 6 cases, the original heuristics give better solutions than the barrier-heuristics solutions. For the rest of the problem, the solutions from both method is the same. Again, this might be the case that the solutions are already optimal. The results are given in table 5.5. A pie chart on the solutions achieved by the two methods can be seen in figure 5.6.

5-items	**Barrier-heuristics**	**Original heuristics**
Number of better solutions	10	6
Percentage of better solutions	6.67%	4%
Average difference of better solutions	1236,839	2987,719

TABLE 5.5: Results of Barrier-Heuristic solutions compared to original heuristic solutions in 5-item case

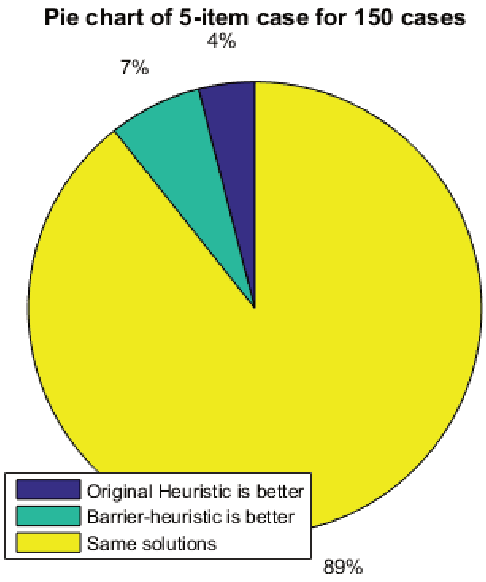


FIGURE 5.6: Pie chart on original heuristics solutions and barrier-heuristic solutions in 5-item case over 150 cases

We have presented the results of using the barrier algorithm to approach good solutions on the optimization problem. On the following chapter, we state the conclusion of this research and recommendations for further research.

Chapter 6

Conclusions and further research

This section will describe the conclusions that can be drawn from the results of this research and the recommendations for improvements and further research.

6.1 Conclusions

In the following, we consider the research question and answer this question by drawing the conclusions from this research. The research question is:

"How to find an optimization algorithm that improves the existing heuristic method for the optimization problem in [3]?"

In this research, we propose to do a continuous relaxation of the original discrete problem in [3] and find a good solution on the problem. We use the barrier method to approach the solutions of continuous relaxation problem. The barrier method approach the solutions of the problem by approximating the constrained problem using an unconstrained problem, because an unconstrained problem is easier to solve. It approaches the solution from inside the feasible region, which guarantees the achieved solutions to be feasible.

However, from the results of the barrier algorithm in Chapter 5, it is shown that the achieved solutions do not approach a good solution in the case of 2-item and 5-item spare parts. The solutions are feasible, however, they are not better compared to the heuristic solutions. It might be the case that the barrier "effect" comes into play or the fact that the optimization problem is generally a non-convex problem. In conclusion, the barrier algorithm still needs to be improved to be able to achieve better solutions.

We apply the solutions of the barrier algorithm to the original heuristic algorithm and compare the results to the original heuristic solutions. We call the solutions as the barrier-heuristic solutions. Time-wise, the original heuristic is still faster than the barrier-heuristics algorithm. For the solution comparation, although the original heuristics still produce better solutions than the barrier-heuristic solutions in some cases, there are also cases where the barrier-heuristic solutions have better results. In the cases where the barrier-heuristic give better solutions, it might be the case that the solutions of the barrier algorithm are better starting points than the one from the original heuristics. For the rest of the cases, the barrier-heuristics solutions have the same solutions as the original heuristics solutions. This might be the case that the solutions are already optimal.

Based on these results, we conclude that although the solutions from the barrier algorithm still do not give better solutions than the original heuristic solutions, these solutions do give some improvements when they are used as the starting points for the original heuristics algorithm. However, we should also note that there are cases where the original heuristics give better solutions. We also note that the barrier algorithm still needs to be improved to give better solutions and computational time. Consequently, this may also improve on the computational time for the barrier-heuristics method. In the following section, some recommendations for further research is presented.

6.2 Further Research

In this section, recommendations for further research based on the research is discussed.

6.2.1 Second-order Descent method: Newton's method

For the current research, the barrier algorithm is implemented using the steepest descent method. Due to time limitation for conducting the research, we did not extend our discussion to the second-order descent method. For further research, it would be good to apply the second-order descent method, the Newton's method, for the barrier algorithm. The Newton's method uses the second-order derivatives of the objective function. Thus, in some cases, it can give faster convergence than the gradient descent because it takes less iterations to the local minimum. Using the Newton's method may improve the computational time of the barrier algorithm.

6.2.2 Merit function

As we have an optimization problem that is generally a non-convex problem, it may be good to look into how to tackle this problem specifically within the algorithm. We recommend to use merit functions in the algorithm. Using the merit functions is to determine whether a step is productive and should be accepted. This may improve the achieved solutions of the barrier algorithm.

6.2.3 Starting points feasibility

The barrier algorithm requires feasible point as its starting point. There might be cases where these points are not known. For further research, we suggest to take this into account by extending the starting point to infeasible starting points. This can be done by doing a preliminary phase where a feasible point is computed before executing the barrier algorithm. This may further improve the barrier algorithm.

Appendices

Notations

Parameters

- K : Number of spare parts.
- O : The cost of hiring a service engineer per unit time
- H_k : The holding cost per item per unit time for spare part k
- C_k^L : The cost of emergency shipment for repair call k
- λ : Total repair call arrivals.
- p_k : Probability that the repair call needs type-k spare part.
- $\lambda_k = p_k \cdot \lambda$: Arrival rate of repair call type-k.
- ν_k : The replenishment rate of spare part type k.
- ν_k^{em} : The replenishment rate of for emergency shipment for spare part type-k.
- $\rho_k^{parts} = \frac{\lambda_k}{\nu_k}$: The offered load in spare parts queue.
- μ : The service rate of service engineer.

Variables

- S_k : Non-negative variable that indicates the stock level of spare part type-k.
- E : Non-negative variable that indicates the number of service engineer.

Functions

- The emergency rate of a repair call for spare part k (Λ_k^L):

$$\Lambda_k^L(S_k) = \lambda_k P_k^L = \lambda_k \left(\frac{\frac{(\rho_{parts}^k)^{S_k}}{S_k!}}{\sum_{i=0}^{S_k} \frac{(\rho_{parts}^k)^i}{i!}} \right).$$

- Arrival rate in service engineer's queue γ:

$$\gamma = \sum_{k=1}^{K} \gamma_k = \sum_{k=1}^{K} \lambda_k \cdot (1 - P_k^L) = \sum_{k=1}^{K} \lambda - \Lambda_k^L(S_k)$$

- The offered load in service engineers queue:

$$\sigma = \frac{\gamma}{\mu}$$

- The probability of emergency shipment:

$$P_k^L = \frac{\frac{(\rho_{parts}^k)^{S_k}}{S_k!}}{\sum_{i=0}^{S_k} \frac{(\rho_{parts}^k)^i}{i!}}$$

- Average waiting time of emergency shipment (W^S):

$$W^S = \sum_{i=0}^{K} \frac{p_k P_k^L}{\nu_k^{em}}$$

- Average waiting time in service engineer's queue (W^E):

$$W^E = \frac{P^B}{E\mu(1-\rho)} = \frac{P^B}{\mu(E-\sigma)}$$

- Total Cost (the objective function):

$$TC(\mathbf{S}, E) = O \cdot E + \sum_{k=1}^{K} H_k \cdot S_k + \sum_{k=1}^{K} C_k^L \cdot \Lambda_k^L(S_k)$$

- Waiting time constraint:

$$W(\mathbf{S}, E) = \frac{\gamma}{\lambda} W^E + W^S \leq W^{max}$$

- Occupancy rate constraint:

$$OR(\mathbf{S}, E) = \frac{\gamma}{E\mu} < 1$$

Acknowledgements

In this section of my master's thesis I would like to thank and express my gratitude to those who helped me to complete this work, finalize my studies and stood by me throughout my studies here in The Netherlands.

First and foremost, I would like to my parents and brother. Without their prayers and endless support, I would not be able to overcome the challenges that I have faced. I hope we meet again soon.

I would also like to thank my grandfather, Waloejo DS. He has been the biggest supporter on my academic journey so far and have always believed in me, even when I don't.

Agung, for being one of my closest friend throughout my study in The Netherlands.

Luna, I'm glad we met and have become best friends ever since we met on both the Bandids band and the orchestra.

Lulu, for always listening to my rants and stories. I hope I have been doing the same.

Thank you to Mas Rully, Mas Riswan, Citra, Bayu, (and again, Luna). Thank you for letting me be a member of The Bandids band. It was an honor to be in the band and represent Enschede with you guys. Our journey to get the title of "The best Indonesian student band in The Netherlands" have been one of the most memorable memories in my study. And I do know for sure that this is one of the reason that keep me sane throughout my study here in Twente.

Tanja and Marieke, for letting me to be in the same group with them in the first semester in 2014/2015 academic year. Now, they are one of my closest friends in the applied mathematics program and I'm grateful for that.

Eefje Schut, for her companion throughout the process of doing my master thesis.

Peter, for being my daily supervisor. Thank you for your advice and patience with me.

Sajjad, for his advice and patience on explaining his work to me.

Thank you to the assessment committee members, Marc Uetz, Ahmad Al Hanbali, and Jan-Kees van Ommeren for their time to assess my master thesis.

Bibliography

[1] Janne Huiskonen. Maintenance spare parts logistics: Special characteristics and strategic choices. *International journal of production economics*, 71(1):125–133, 2001.

[2] Henk Zijm. Trends and challenges in logistics and supply chain management. *ERCIM news*, 2016(105):3, 2016.

[3] Sajjad Rahimi-Ghahroodi, Ahmad Al Hanbali, Henk Zijm, Jan-Kees W. van Ommeren, and Andrei Sleptchenko. Integrated field service engineers and spare parts planning in maintenance logistics. *under review*.

[4] David L Jagerman. Some properties of the erlang loss function. *Bell System Technical Journal*, 53(3):525–551, 1974.

[5] David L Jagerman. Methods in traffic calculations. *AT&T Bell Laboratories Technical Journal*, 63(7):1283–1310, 1984.

[6] Craig C Sherbrooke. Metric: A multi-echelon technique for recoverable item control. *Operations Research*, 16(1):122–141, 1968.

[7] Craig C Sherbrooke. *Optimal inventory modeling of systems: multi-echelon techniques*, volume 72. Springer Science & Business Media, 2006.

[8] John A Muckstadt. *Analysis and algorithms for service parts supply chains*. Springer Science & Business Media, 2004.

[9] RJI Basten and GJ Van Houtum. System-oriented inventory models for spare parts. *Surveys in operations research and management science*, 19(1):34–55, 2014.

[10] JK Visser and G Howes. A simulation technique for optimising maintenance teams for a service company. *South African Journal of Industrial Engineering*, 18(2):169, 2007.

[11] Philipp Hertz, Sergio Cavalieri, Gandolf R Finke, Aldo Duchi, and Paul Schönsleben. A simulation-based decision support system for industrial field service network planning. *Simulation*, page 0037549713512685, 2013.

[12] IMH Vliegen. *Integrated planning for service tools and spare parts for capital goods.* PhD thesis, 2009.

[13] Sven Axsäter. Modelling emergency lateral transshipments in inventory systems. *Management Science*, 36(11):1329–1338, 1990.

[14] Patrik Alfredsson and Jos Verrijdt. Modeling emergency supply flexibility in a two-echelon inventory system. *Management science*, 45(10):1416–1431, 1999.

[15] Anil Kukreja, Charles P Schmidt, and David M Miller. Stocking decisions for low-usage items in a multilocation inventory system. *Management Science*, 47(10): 1371–1383, 2001.

[16] Craig C Sherbrooke. *Optimal inventory modeling of systems: multi-echelon techniques*, volume 72. Springer Science & Business Media, 2006.

[17] Erhan Kutanoglu and Divi Lohiya. Integrated inventory and transportation mode selection: A service parts logistics system. *Transportation Research Part E: Logistics and Transportation Review*, 44(5):665–683, 2008.

[18] Hartanto Wong, Geert-Jan van Houtum, Dirk Cattrysse, and Dirk Van Oudheusden. Multi-item spare parts systems with lateral transshipments and waiting time constraints. *European Journal of Operational Research*, 171(3):1071–1093, 2006.

[19] AA Kranenburg and GJ Van Houtum. A new partial pooling structure for spare parts networks. *European Journal of Operational Research*, 199(3):908–921, 2009.

[20] EJ Messerli. Proof of a convexity property of the erlang b formula. *Bell System Technical Journal*, 51(4):951–953, 1972.

[21] AA Jagers and Erik A Van Doorn. On the continued erlang loss function. *Operations Research Letters*, 5(1):43–46, 1986.

[22] Mark S Daskin, Collette R Coullard, and Zuo-Jun Max Shen. An inventory-location model: Formulation, solution algorithm and computational results. *Annals of operations research*, 110(1-4):83–106, 2002.

[23] Ali Diabat, Tarek Abdallah, and Andreas Henschel. A closed-loop location-inventory problem with spare parts consideration. *Computers & Operations Research*, 54:245–256, 2015.

[24] Pablo A Miranda and Rodrigo A Garrido. Incorporating inventory control decisions into a strategic distribution network design model with stochastic demand. *Transportation Research Part E: Logistics and Transportation Review*, 40(3):183–207, 2004.

[25] Fengqi You and Ignacio E Grossmann. Mixed-integer nonlinear programming models and algorithms for large-scale supply chain design with stochastic inventory management. *Industrial & Engineering Chemistry Research*, 47(20):7802–7817, 2008.

[26] S-L-S Jacoby, J-S Kowalik, and J-T Pizzo. Iterative methods for nonlinear optimization problems. 1972.

[27] Jorge Nocedal and Stephen Wright. *Numerical optimization*. Springer Science & Business Media, 2006.

[28] David G Kendall. Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded markov chain. *The Annals of Mathematical Statistics*, pages 338–354, 1953.

[29] Jyotiprasad Medhi. *Stochastic models in queueing theory*. Academic Press, 2002.

[30] Henk C Tijms. *A first course in stochastic models*. John Wiley and Sons, 2003.

[31] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[32] Robert M Freund. Penalty and barrier methods for constrained optimization. *Lecture Notes, Massachusetts Institute of Technology*, 2004.

[33] Robert M Freund. The steepest descent algorithm for unconstrained optimization and a bisection line-search method. *Journal of Massachusetts Institute of Technology. United States of america*, 2004.