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Investigation of methods for the calculation of swash bed shear stress

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Abstract

The swash flow is simulated with a simple hydrodynamic model used as a tool to calculate the bed shear stresses. The Colebrook method and the Swart method are turbulent water flow methods used to simulate the swash bed shear stress. The goal of the thesis is to compared the two methods and see which method is the most representative. To choose the best method it has to be tested on different cases with velocity and beach slope as variables.

Contents

1 Introduction

1.1 Beaches

"A beach is a dynamic environment located where land, sea and air meet. It may be defined as a zone of unconsolidated sediment (i.e., loose materials) deposited by water, wind or glaciers along the coast, between low tide line and the next important landward change in topography or composition." [1]
Beaches provide protection against erosion from the surrounding area into the sea. This can only be achieved if the stability of the beach remains. Sediment transport is the most critical reason that the stability declines. Sediment transport is the movement of solid organic and inorganic particles (sediment) due to the movement of water rushing up and down the beach. This occurs under the influence of hydrodynamic processes such winds, waves and water currents [2]. In this thesis, sediment transport is referred to as cross-shore sediment transport. Cross-shore sediment transport is from the water to the beach and not the whole shoreline (longshore sediment transport). Sediment transport under waves near beaches may result in beach erosion or accretion. This is a problem that occurs on beaches all around the world. It is problematic because erosion can undermine the stability of a beach or a building located on (or close to) a beach. There is also less room for recreation, because erosion reduces beach surface area.

The amount of sediment transported onto and off beaches is influenced by several parameters, namely: the height of the wave, beach slope, the steepness of the waves, water depth and settling velocity of the sediment particles. At one moment in time the slope of the beach reaches its dynamic equilibrium. This means that the same amount of sediments shifts in each direction, however the particle size distribution is not the same and this results in an offshore movement of fine materials and an onshore movement of coarse materials [3].

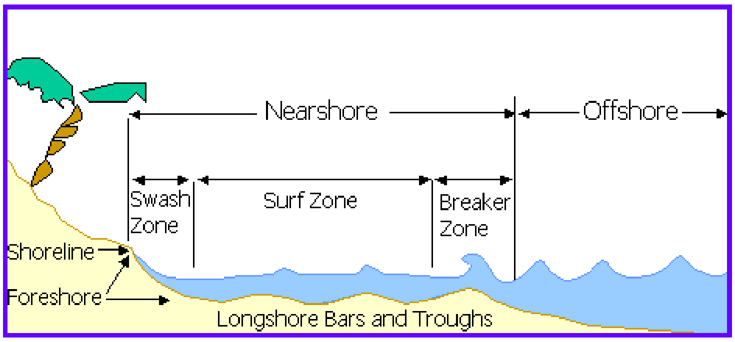

Figure 1: The designation of the beach cross section [4]

1.2 Swash zone

The cross section of a beach, also called near shore, consists of three elements, namely the breaker zone, the surf zone and the swash zone. These three different zones can be found in Figure 1. The breaker zone is the area where the waves that are approaching the shore become unstable and break. The height of the wave depends on the composition of the beach [5]. The surf zone is the area of water after the surf line till the beach at ebb. The surf line is where the wave is affected by the underwater bottom surface which results in the waves becoming breakers [6]. The bed slope is usually shallow between $\tan(1/10) = 0.01 < \beta < \tan(3/10) = 0.03$. The height of the waves are mostly controlled by the water depth [7]. The swash zone, also called foreshore, is the area where the bore is rushed up the beach slope. The beach is alternately covered with water (the swash) and exposed if the water retreats (the backwash [3]).

The swash flow is difficult to simulate correctly, because the swash flow is turbulent, extremely unsteady and an aerated flow that changes rapidly [8]. It is not only hard to simulate swash flows, but is it also a challenge to measure the flow in reality. Even for the most advanced equipment it is a challenge to measure even little changes in for example the water depth [9].

The swash zone is important in the cross-shore sediment transport process for two reasons. First, the motion of the water in the swash zone provides the mechanism of the sediment exchange between the zones of the beach that is exposed to water or air. Offshore and onshore sediment distribution within the swash zone contributes significantly to the accretion and erosion of the beach profile. Secondly, the water motion in the swash zone contributes to a significant part to the longshore sediment transport. This may result in a large part of the total longshore drift, this occurs particularly in steep beaches [9].

1.3 Swash bed shear stress

Swash bed shear stress is the friction between the flow and the bed. Good prediction of swash hydrodynamics and especially swash sediment transport requires accurate estimations of the bed shear stress. However, swash bed shear stress varies in a complex manner with time and cross-shore distance, and to date, there is no generally-accepted method for its calculation. In the absence of an established model, various methods have been used by researchers to estimate the bed shear stress, including methods normally used for steady flow conditions and for estimating bed shear stress under waves [10]. Common difficulties in estimating the bed shear stress in swash flow is that the flow depths are mostly shallow, the water is aerated, unsteady and almost always turbulent [11].

1.4 Research aim

The primary purpose of the research is to review and compare the behaviour of various methods used to estimate swash bed shear stress.

The following research questions provide a framework for the information that should be obtained in the research project:

1. How does swash flow arise?
2. What are the characteristics of swash flow?
3. What is bed shear stress?
4. What are the different methods used to calculate swash bed shear stress?

5. What is the best method to calculate swash bed shear stress for particular swash scenarios?

1.5 Methodology

1.5.1 Swash bed shear stress

The definitions and characteristics of swash bed shear stress were evaluated via a literature study. This includes the estimation of swash bed shear stress for a steady open channel flow and under waves. This will result in a better insight in the sediment transport caused by swash flows. Two turbulent methods that are applied to estimate the swash bed shear stress are the Colebrook and the Swart method [8]. The two methods will be compared which each other.

1.5.2 Hydrodynamic model

For the hydrodynamic model the situation is idealised to create a simpler model for the swash flows. This model is implemented in MATLAB to conduct simulations for calculating the swash hydrodynamics for a range of bore and swash conditions [12]. The input parameters for the model are the characteristics of the wave that will rush onto the beach. The characteristics of the wave are described in the technical note, provide by Thomas O'Donoghue (supervisor from Aberdeen). By making the model an insight is provided in the basics of the swash hydrodynamics. Beside that, it becomes more intuitive how the model reacts if you change variables and what the model represents.

1.5.3 Results interpretation

At last the results needs to be interpreted and this is done by investigation and comparing the behaviour of the best bed shear stress methods for a range of bore and swash flow conditions. The methods will be implemented in MATLAB.

2 Swash bed shear stress

The goal of this chapter is to understand the definition and the characteristics of shear stress and specific bed shear stress. Bed shear stress is the friction force between the flow and the bed per unit of bed area (N/m^2).

2.1 Shear stress

2.1.1 Channel flow

In an open channel flow, the force of moving water against the channel bed creates the shear stress. The equation to calculate shear stress for open channel flow is [13]

$$\tau = \gamma D S_w \tag{1}$$

where τ is the shear stress in N/m^2, γ is the weight density of water in N/m^3, D is the average water depth in m and S_w is the water surface slope in m/m.

2.1.2 Boundary layer

If a flow moves across a stationary surface, the fluid will touch the surface and will create a shear stress. The shear stress at the bottom (the bed) will be higher than on the surface. The reason for this is that the bed is stationary while the water flow is dynamic. Shear stress between the layers of water is less, because flow moves in the same direction. The shear stress varies over the water column and leads to velocity differences over the height y (see Figure 2.1.2). The profile, shown in Figure 2.1.2 does not exist in the beginning, but must be build up gradually from the bottom of the flow till it reaches the surface. The creation of the profile is build up over time. Boundary layer profiles velocity are shown in rivers and channels, because the flow is always in the same direction and the velocity and water depth does not increase or decline quickly.

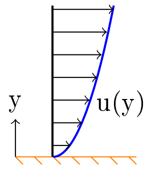

Figure 2: The velocity profile of a boundary layer. The black arrows are the velocity of the flow.

To calculate the shear stress the following equation is used[14]

$$\tau = \mu \cdot \frac{du}{dy} \tag{2}$$

2.2 Bed shear stress methods

The swash flow evolves (uprush, backwash) is time and bed shear stress is and also varies in time created instantly the bed shear stress is calculated (Instantaneous bed shear). The equation for instantaneous bed shear stress is

$$\tau_b = \frac{1}{2}\rho f_b u|u| \tag{3}$$

where ρ ($1000\ kg/m^3$) [15] is the water density, u is the flow velocity and f_b is the bed friction factor [8]. The bed friction factor will be calculated with the different bed shear stress methods.

There is no generally-accepted method for calculating the bed friction factor for the swash zone. Thereby researchers use various methods which include methods that are normally used for steady flow conditions and for estimating bed shear stress under waves. In this research two commonly used methods, namely the Colebrook method and the Swart method, are implemented in a swash flow model and compared. Both methods are applied for turbulent flow and therefore the Reynolds number has to be higher than 2300. The equation for Reynolds number is [8]

$$Re = \frac{uh}{\nu} \tag{4}$$

where u is the flow velocity in m/s, h is the water depth in m and ν is the kinematic viscosity ($1.004 \cdot 10^6\ m^2/s$ [15]).

2.2.1 The Colebrook formula

The Colebrook formula is normally used to estimate the Darcy-Weisbach friction factor pipes flow [16]. The Colebrook formula is

$$\frac{1}{\sqrt{\lambda}} = -2\log_{10}\left(\frac{k_s}{3.7D_h} + \frac{2.51}{Re\sqrt{\lambda}}\right) \tag{5}$$

where $\lambda = 4f_b$ is the friction coefficient, k_s is the roughness height of the bed in m and is given by $k_s = 2.5d_{50}$, where d_{50} is the sediment size in m [8], $D_h = 4h$ is water depth in m and Re is the Reynolds number.

2.2.2 The Swart formula

The Swart formula is used to calculate the friction factor for flow conditions that are oscillatory and wave-driven. Shallow water waves create an elliptical movement for the water particles. The value of the minor axis of the ellipse becomes smaller and smaller as the bed is approached. On the bed itself it is only a backward and forward movement. The Swart formula is

$$f_b = 0.0025\exp\left[5.213\left(\frac{a}{k_s}\right)^{-0.194}\right] \tag{6}$$

where a is the amplitude of the oscillatory flow in m, k_s is the height roughness of the bed in m and is given by $k_s = 2.5d_{50}$ and where d_{50} is the sediment size in m [8]. In order to apply the Swart equation to swash flow, a is defined as

$$a = \frac{\sqrt{2var(u)}T_s}{2\pi} \tag{7}$$

where var(u) is the variance of the velocity time-series $u(t)$ at given cross-shore location x. The total swash period is where the water comes in contact with the beginning of the slope till the moment that all the water went back into the sea/ocean again. T_s is the swash 'period' at a certain x defined as $T_s = t_{end} - t_{ba}$. This period is at a certain is from the moment the water covers the beach at the x-position t_{ba} till it is exposed again t_{end}. The formula calculates a time-invariant f_b at give x, so f_b has a different value for each x and not for each x and t as Colebrook.

2.2.3 Overall comparison

The friction factor is only difference between the two methods when calculating the bed shear stress, see equation (3). To get a good impression of the differences between the two friction factors there is a sensitivity analysis is done. For each method a variable is chosen to review the influence on the friction factor. In the Colebrook method the $\frac{k_s}{Dh}$ is variated and in the Swart method the $\frac{a}{k_s}$. Results of this sensitivity analysis are shown in Figure 3.

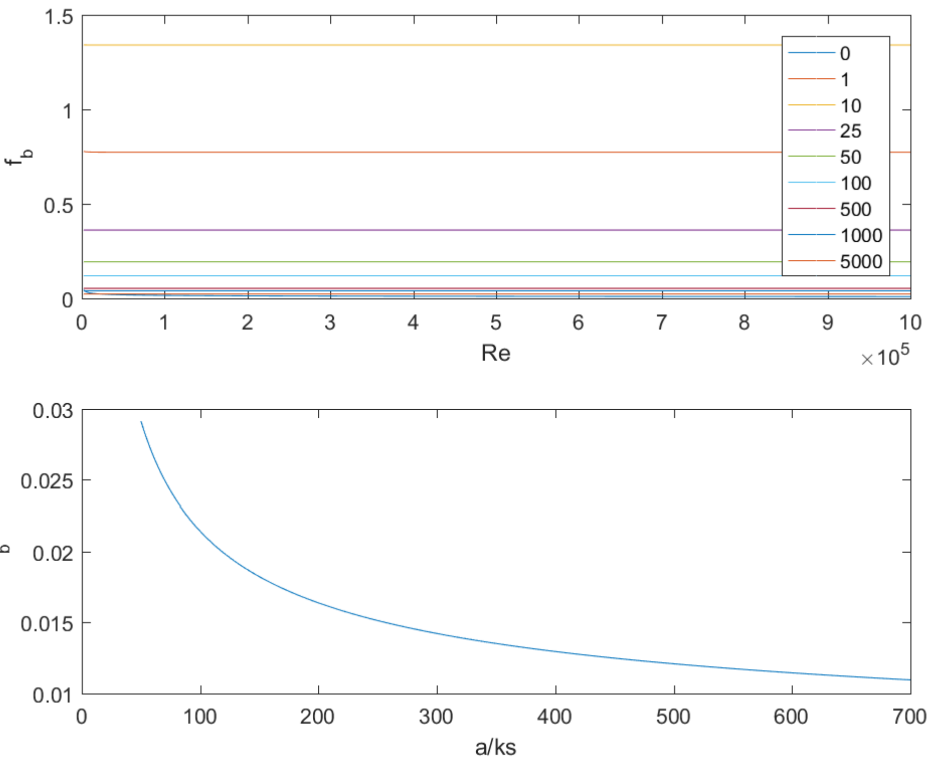

Figure 3: f_b versus Re for various $\frac{k_s}{Dh}$ from the Colebrook Method (top) and f_b versus $\frac{a}{k_s}$ from the Swart Method (bottom).

For the Colebrook method a few lines were chosen to demonstrate the behaviour of the variable. In Figure 3 the Colebrook method has on the x-axis a range from 2300 till 10^6. Interesting is that the lines for $\frac{k_s}{Dh}$ from 0 till 1 the friction factor increases and after the lines that it decreases. It was expected due to the $-2\log_{10}(z)$ function. The value of variable z does not really matter, because the logarithmic function will still increase very suddenly to the top and decrease again very suddenly. In the equation (5) the second term of the equation $\left(\frac{1}{3.7}\frac{k_s}{D_h}\right)$ in the logarithmic function becomes negligible small in relation to first term $\left(\frac{2.51}{Re\sqrt{\lambda}}\right)$. Therefore the lines become straight.

Due to the fact that the Swart Method has a time-invariant friction factor it can be representative by one line. In the beginning the line changes quickly and this descend decreases over time. The quick change in the value of the friction factor is due to the exponential function. The bottom graph in Figure 3 was presented as expected. The formula of Swart does not depend on the Reynolds number only that the value has to be higher than 2300.

3 Swash hydrodynamic model

This chapter presents a simple analytical model for swash flow. The model will be used later to provide swash hydrodynamic conditions for the comparison of the various swash bed shear stress models. The characteristics of the swash flow are determined by the input values of u_o and β. The initial shoreline velocity also called the begin velocity of the bore is u_o, the beach slope is $\beta = \tan(\phi)$ and the ϕ is in radians. A model description was provided by Prof. O'Donoghue (supervisor from Aberdeen) [12] and this is implemented in MATLAB. The MATLAB can be found in Appendix B.

The model describes the motion of the bore up and down the beach. In Figure 4 are two blue lines displayed and they represent the positions of the front bore. x_s is defined as the position of the bore front, i.e. places where the blue line (= front bore) meets the beach slope. The front bore moves in time, so x_s moves in time up and down the beach slope. The highest point of x_s, the maximum run-up is represented by t_{max} or $x_{s_{max}}$. t_{max} is the moment in time and $x_{s_{max}}$ is the distance where x_s is at the highest point. The model is explained in more

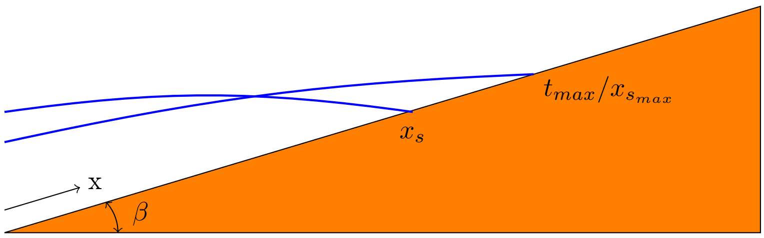

Figure 4: Definition sketch

detail in Appendix A. The bore driven wave moves up and down the beach. The movement can be described by the equation:

$$x_s(t) = u_o t - \frac{1}{2} g t^2 \sin(\beta) \tag{8}$$

"Shen and Meyer" [17] show that the water depth near shoreline can be calculated with:

$$h(x,t) = \frac{(x - x_s)^2}{A g t^2} \tag{9}$$

Only in Shen and Meyer [17] it is assumed that $A = 9$, but instead it will be left an A to calibrate the model accordingly.

From equation (9) the maximum run-up time can be calculated:

$$t_{max} = \frac{u_o}{g\sin(\beta)} \tag{10}$$

Combining equation (8) and equation (10) gives the maximum run-up position:

$$x_{s_{max}} = \frac{u_o^2}{2g\sin(\beta)} \tag{11}$$

To calculate the volume of water on the slope above position x, h is integrated over the interval x to x_s. The steps between the two functions are explained in Appendix A.2.

$$V_{ax}(t) = \int_x^{x_s} h\ dx \tag{12}$$

$$V_{ax}(t) = \frac{1}{Agt^2}\left(-\frac{1}{3}x^3 + \frac{1}{3}x_s^3 + x^2 x_s - x x_s^2 \right) \tag{13}$$

The volume flux at location x can be calculated via the derivative of V_{ax} with respect to t. The steps between the two functions are also explained in Appendix A.2

$$\frac{d}{dt}V_{ax}(t) = \frac{1}{Ag}\left(\frac{dA}{dt} + \frac{dB}{dt} + \frac{dC}{dt} + \frac{dD}{dt} \right) \tag{14}$$

$$V_{ax} = \frac{1}{Ag}\left(\frac{2}{3}\frac{1}{t^3}\left(x - x_s \right)^3 + \frac{x_s'}{t^2}\left(x_s - x \right)^3 \right) \tag{15}$$

The depth average velocity is defined as the volume flux divided by the depth. The steps between the two functions can be found explained in Appendix A.3. The depth average velocity given by

$$u = \frac{1}{h} \cdot \frac{dV_{ax}}{dt} \tag{16}$$

$$u = \frac{u_o}{3} \cdot \left(1 - \frac{2gt\sin(\beta)}{u_o} + \frac{2}{u_o} \cdot \frac{x}{t} \right) \tag{17}$$

To simulate the right water depths the A in the equation (9) will modify, changing this constant does not have an effect on the velocity. The derivation of the equations can be found in Appendix A. To find the correct A to make the model representative, there is chosen to look at previous laboratory experiments in Aberdeen [8]. The reference with an initial velocity of $2.5 m/s$ and the angle of the slope is $1 : 10$. In experiments the water depth was at $x = 0$ approximately $0.3m$. This means that if $A = 2$ is used to correct water depth will be simulated. Also another equation is used to demonstrate that the water depths are valid for this particular value. Namely the equation for shallow water waves for the water depth at the beginning of the swash flow:

$$u = \sqrt{2gh} \tag{18}$$

$$u^2 = 2 \cdot 10 \cdot h \tag{19}$$

$$h = \frac{u^2}{2 \cdot 10} \tag{20}$$

$$h(2.5) = \frac{6.25}{20} = 0.3125m \tag{21}$$

In Figure 5 the water depth and velocity of the wave is shown as a function over time at 5 cross-shore locations on the beach. A line begins when the shoreline (x_s) is the same or greater than the cross-shore position and ends when shoreline is again smaller that cross-shore position.

The water depth declines rapidly in the beginning see in Figure 5. The top graph shows that maximum value of the blue line is $\frac{u_o^2}{2g} = 0.44m$ while the red line is only at $0.15m$. The calculations are shown in Appendix A.4. At positions $x > 0$ water depth grows slowly, reaches a maximum and decays even more slowly back to zero.

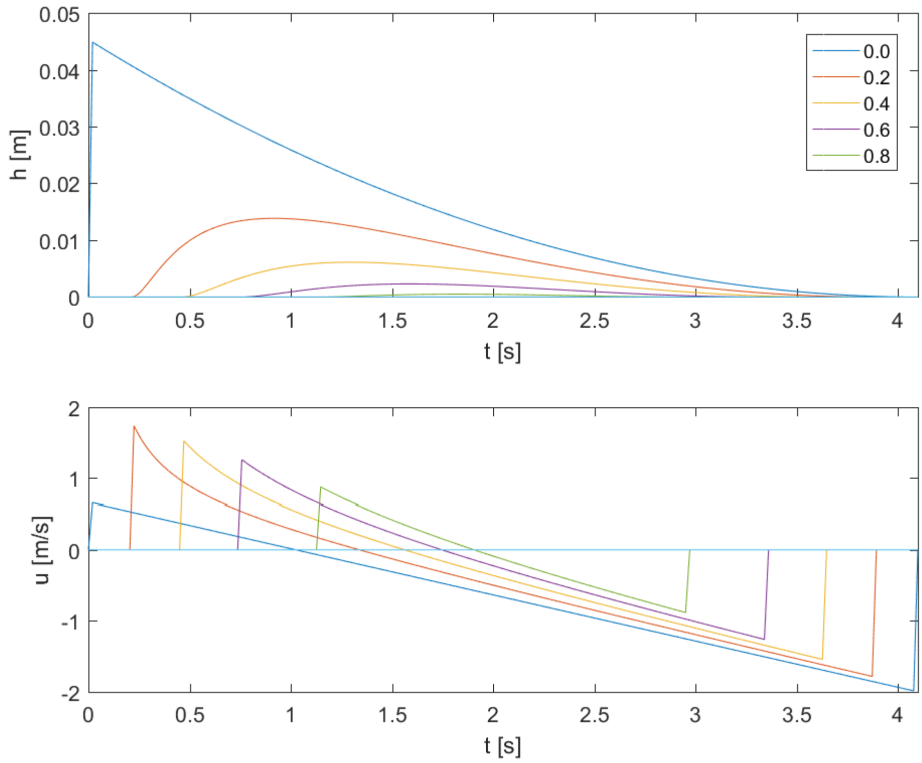

Figure 5: Model output of $h(t)$ (top) and $u(t)$ (bottom) for 5 cross-shore positions, ($x = 0.0, 0.2, 0.4, 0.6, 0.8m$); $u_o = 2m/s$ and $\tan\beta = 0.1$.

The maximum velocity is highest for the red line ($x = 0.2m$) and at each position further up the slope the maximum velocity declines slowly. The blue line ($x = 0m$) is different from the rest, because it is the first moment that the water wave makes contact with the slope and creates a lot of friction which results in a loss of velocity. The rest of the lines have for that line the same value for the minimum and the maximum velocity. On a lower x-position the velocity is already becoming negative (backwash), while the front of the bore is moving forward (uprush). The velocity becomes already negative at locations lower down the beach before the backwash begins. This means that part of the water is already pulling back when the shoreline is still moving forward. The flow is therefore divergent, which agrees with measurement of swash from laboratory studies (e.g. [8]). The velocity at the end of the backwash should not go from the maximum till zero almost instantaneously. From the results of the experiments the line decends with an arc to zero (e.g. [8]).

4 Results

In this chapter the results of both hydrodynamic models will be discussed. First, the method of bed shear stress calculations is explained. Secondly, the swash flow is evaluated via a sensitivity analysis. In the third and fourth part, the swash bed shear stress is calculated for two different sediment sizes, respectively $d_{50} = 1.3mm$ and $8.4mm$. These sediment sizes are chosen, because the first one represents a sand beach and the second one represents a gravel beach. The MATLAB code is shown in Appendix C

To receive accurate results the bed shear stress methods are tested for four different beach types with the parameters shown in Table 1. There are two different input variables that can be changed. Namely, the initial velocity and the beach slope.

Table 1: Beach type input parameters for swash bed shear stress calculations

	Initial velocity [m/s]	Beach slope [rad]
Case A	1	1:20
Case B	1	1:10
Case C	2.5	1:20
Case D	2.5	1:10

The bed shear stress is calculated with the model output at three different locations, respectively $x = [0.25 \cdot x_{s_{max}}, 0.5 \cdot x_{s_{max}}, 0.75 \cdot x_{s_{max}}]$. These three locations were chosen, because they are evenly spread over the slope. The calculations are preformed for both the Colebrook and the Swart method.

4.1 Hydrodynamic cases

To gather a better understanding on the swash bed shear stresses for the four cases, the computed swash flow for the four cases has been studied (see Figure 6).

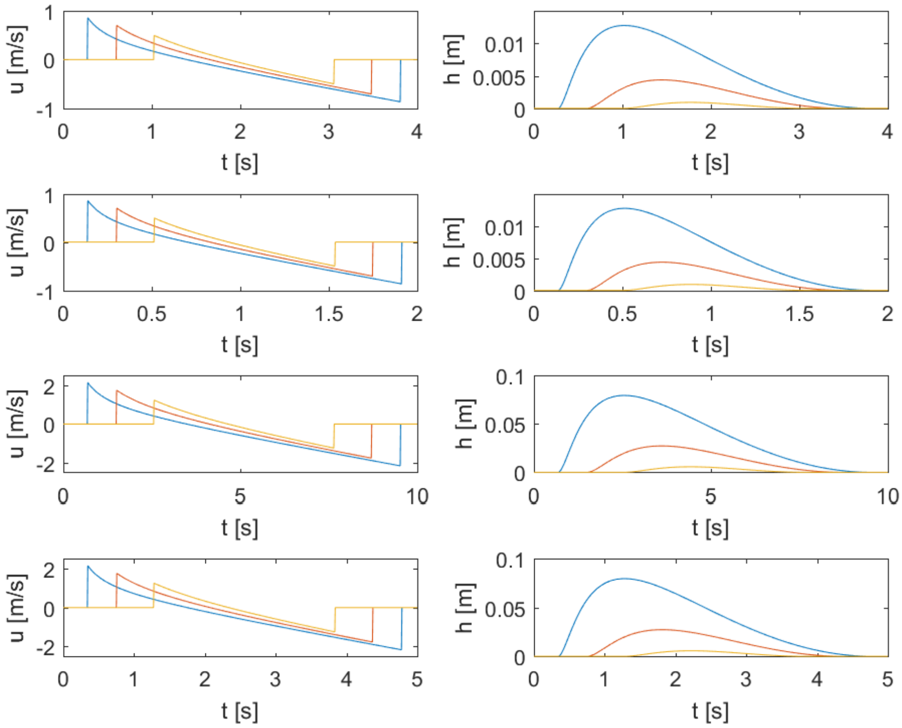

Figure 6: Model output for the hydrodynamic model: Case A (first row), Case B (second row), Case C (third row) and Case D (fourth row). Water velocity (first column) and water depth (second column). Test location on the slope $0.25 \cdot x_{s_{max}}$ (blue line), $0.5 \cdot x_{s_{max}}$ (red line), $0.75 \cdot x_{s_{max}}$ (yellow line).

The initial velocity ensures that magnitude of the value of the water depth and velocity remains the same. In Figure 6 it can be seen that the first two cases have the same y-axis and the last two cases also. While the beach slope determines the length of the swash period. By beach slope of 1 : 20 is twice as long as the beach slope of 1 : 10. Thereby case A is twice as long is space as case B and also case C in twice as long in space as case D.

4.2 Compare bed shear stress for $d_{50} = 1.3mm$

The sediment size for all the graphs in this section will be 1.3mm. In the Figures 7 and 8 the y-axis is adapted so that most of the irrelevant data from the Colebrook method is not shown in the graphs. The reason for this is explained in Appendix D.

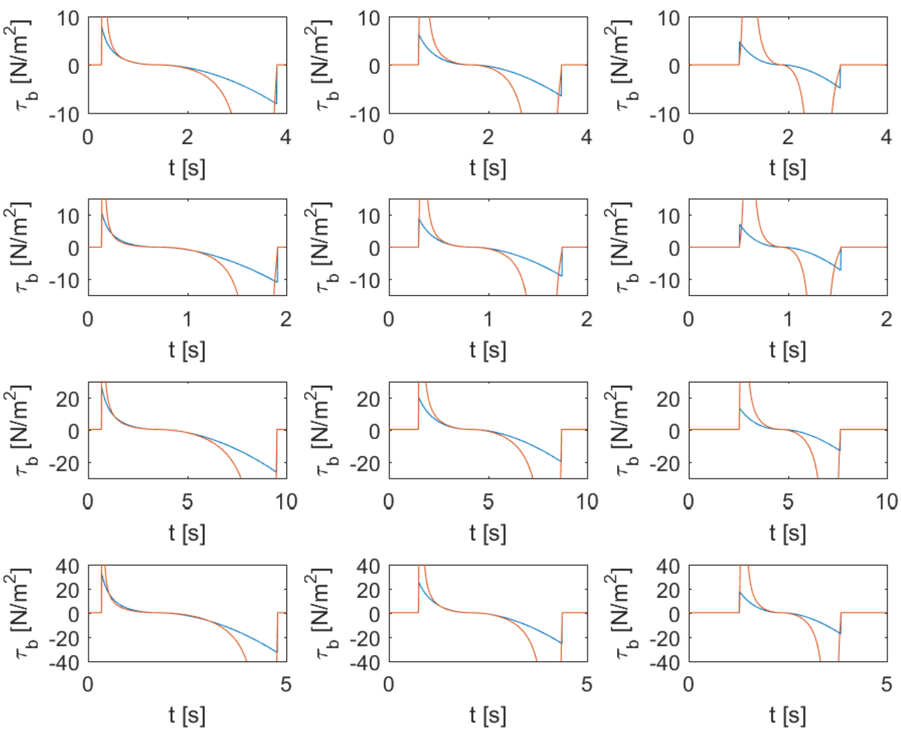

Figure 7: Model output for swash bed shear stresses with $D_{50} = 1.3mm$: Case A (first row), Case B (second row), Case C (third row) and Case D (fourth row). Test location on the slope $0.25 \cdot x_{s_{max}}$ (first column), $0.5 \cdot x_{s_{max}}$ (second column), $0.75 \cdot x_{s_{max}}$ (third column). Colours of the method the Colebrook method (red) and the Swart method (blue).

In general the swash period remained the same if you compare Figure 7 with the Figure 6. Furthermore, the peaks for the slope of the beach for the Swart method become smaller and for the Colebrook method the peaks become higher. The Colebrook and Swart method do not really overlap, but at x-position $0.25 \cdot x_{s_{max}}$ the two methods have to biggest part where the lines are almost the same compared to the rest of the x-positions. The best correspondence is case D with x-position $0.25 \cdot x_{s_{max}}$.

4.3 Compare bed shear stress for $d_{50} = 8.4mm$

The sediment size for all the graphs in this section will be 8.4mm. In general the same commentary applies to the bed shear stresses with d_{50} = 1.3mm. The swash period remain the same if you compare Figure 8 with Figure 6. Furthermore, the peaks for the slope of the beach for the Swart method become smaller and for the Colebrook method the peaks become higher. The Colebrook and the Swart method do not really overlap, but at x-position $0.25 \cdot x_{s_{max}}$ the two methods have to biggest part where the lines are almost the same compared to the rest of the x-positions. The best correspondence is case D with x-position $0.25 \cdot x_{s_{max}}$.

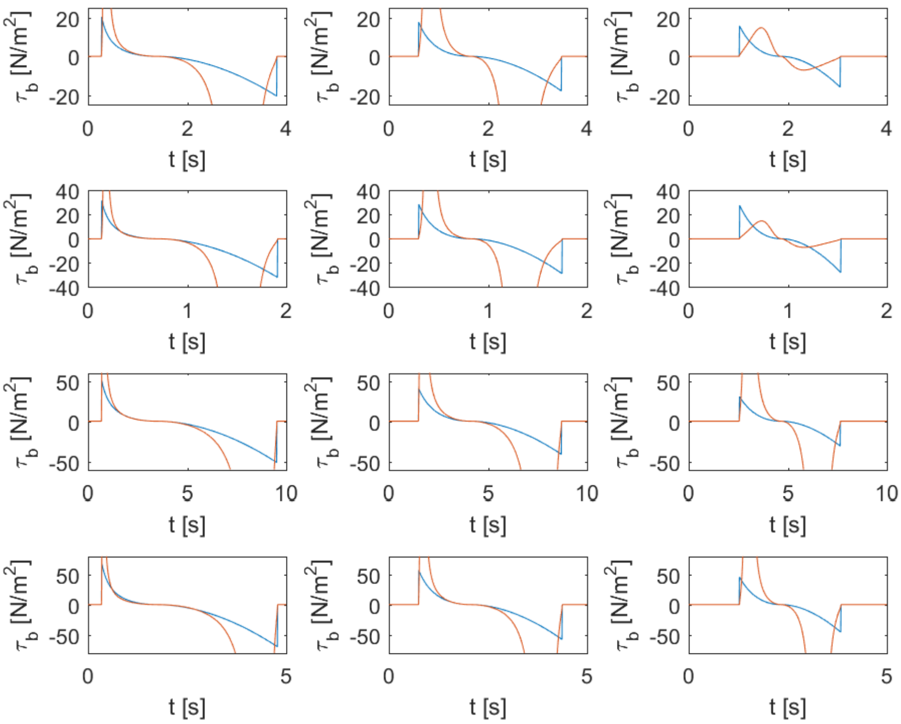

Figure 8: Model output for the hydrodynamic model: Case A (first row), Case B (second row), Case C (third row) and Case D (fourth row). Test location on the slope $0.25 \cdot x_{s_{max}}$ (first column), $0.5 \cdot x_{s_{max}}$ (second column), $0.75 \cdot x_{s_{max}}$ (third column). Colours of the method the Colebrook method (red) and the Swart method (blue).

For the cases A and B with x-position $0.75 \cdot x_{s_{max}}$ the maximum shear stresses of the Colebrook method are visible in the graphs. It was discovered that for these two cases the Reynolds numbers are too low to be a turbulent flow (see Figure 9). For the cases A and B with x-position $0.5 \cdot x_{s_{max}}$ for a large part of the swash flow Re $<$ 2300. The methods only work for turbulent flow, so the methods should not be calculating the swash bed shear stresses. Since the flow is not turbulent the Cases A and B with sediment size d_{50} = 8.4mm should be dismissed.

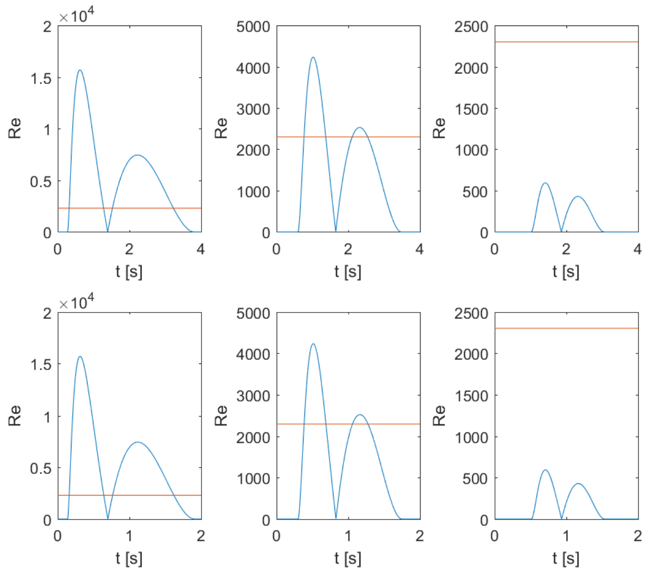

Figure 9: Model output for Reynolds number: Case A (top row) and Case B (bottom row). Test location on the slope $0.25 \cdot x_{s_{max}}$ (first column), $0.5 \cdot x_{s_{max}}$ (second column), $0.75 \cdot x_{s_{max}}$ (third column). The blue line indicates the Reynolds number over time and the red line is an indication when the flow becomes turbulent.

4.4 Overall comparison

In general for all calculations the tops are smaller for the Swart method and the slope of the Colebrook function is steeper. The Colebrook and the Swart method do not really have an overlap, but x-position $0.25 \cdot x_{s_{max}}$ corresponds better than the rest of the x-positions. The best correspondence is case D with x-position $0.25 \cdot x_{s_{max}}$.

The swash bed shear stresses of d_{50} with 8.4mm are higher than the bed shear stresses of d_{50} with 1.3mm. This is expected, because the bigger grain size results in a bigger contacting surface area of the bed with the flow. Thereby the friction force also becomes bigger. The Swart method is better than the Colebrook method due to the big spikes which are found by the Colebrook method.

5 Conclusion

Swash flow is a turbulent layer of water that washes up upon the beach after an incoming wave has broken. The action can cause a movement of beach materials up and down the beach, which results in the cross-shore sediment exchange. The swash flow consists of two phases: the uprush which covers the beach with water and the backwash which exposes the beach of water. The greatest velocity for the uprush is at the start of swash period and then decreases in velocity. Whereas the backwash increases in velocity and at the end the velocity is at its maximum. The characteristics of the swash flow is that the flow is turbulent, extremely unsteady, aerated and changes rapidly.

The swash hydrodynamic model becomes more representative if the variable A is changed from 9 to 2, because looking at papers from previous laboratory experiments at the University of Aberdeen (e.g. [8]) the value of the water depth has a similar value and the velocity remains unchanged.

Bed shear stress is the friction between the flow and the bed. In this thesis the swash bed shear stress has been calculated with two different methods. Namely the Colebrook and the Swart method. Both are normally used for a different situation, but there is not a generally-accepted method to calculate the swash bed shear stress. Therefore these turbulent applied methods are applied on the hydrodynamic model to investigate if they can be applied for the swash flow.

The Swart method is a better method to simulate the swash bed shear stress than the Colebrook method. By zooming in on the graphs for the bed shear stresses, you can actually see how much of the data of the Colebrook method are not relevant see in Appendix D. And by the Colebrook method the bed shear stress at the end of the swash period is not always maximum. The Swart method simulates that in all the 24 graphs see Figures 7 and 8.

6 Discussion

In this study a simple swash flow model was chosen, because it was only a tool available to calculate the swash bed shear stresses. Not much time was available for verificating if the model was correct. Nevertheless a lot of time had to be spend on the model since it was difficult to understand how it was built exactly and on top of that there were a few changes necessary to make the model more representative.

The two methods used to calculate the swash bed shear stresses are normally used for oscillatory driven waves or flows in pipes. It is questionable if the methods should be even used to simulate the bed shear stress of the swash zone. Both of the formulas are adapted a bit to make it more logically for the swash zone.

7 Recommendations

In this chapter recommendations given on how a next study can be done better or expand in the focus.

- The hydrodynamic model is not accurate enough. For example the water depth declines quickly in the swash period when in reality it should decline more slowly. In further studies it would be better to look if a more representative model can be used or developed to simulate the swash flow.
- The slope is one straight line. In an actual beach there will be at least erosion and deposition due to sediment transport, which results into a non-uniform slope. In the next study there can be experimented with the shape of the beach slope. For example simulating a flat or less steep section.
- There is also the possibility that the beach is composed of different materials. Which can mean that a section of the beach is strengthened/reinforced with concrete or simply that a beach is composed of both gravel and sand. It could be possible to review different compositions of materials in the beach slope.
- The method from Clarke and Dodd in the paper "Modeling flow in and above a porous beach"[18] could be interesting to include in the study to obtain a broader view of swash bed shear stress. It can also be investigated in the literature if more methods are available that possibly could be used.

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A Details of the swash hydrodynamic model

The bore driven wave move up and down the beach. The movement can be described by the equation

$$x_s(t) = u_o t - \frac{1}{2} g t^2 \sin(\beta)$$

"Shen and Meyer" show that the water depth near shoreline can be calculated with:

$$h(x,t) = \frac{(x - x_s)^2}{Agt^2}$$

To calculate the maximum run-up derivative x_s to equation should be rewritten such that t is on the left hand side.Then the maximum run-up position is

$$t_{max} = \frac{u_o}{g \sin(\beta)}$$

Implementing t_{max} in the equation x_s gives the maximum run-up position

$$x_{s_{max}} = \frac{u_o^2}{2g \sin(\beta)}$$

A.1 Water depth calculations

The water depth near shoreline is

$$h(x,t) = \frac{(x - x_s)^2}{Agt^2}$$

Non-dimensionalising of the model:

$$\bar{x} = \frac{x}{x_{s_{max}}}, \bar{x_s} = \frac{x_s}{x_{s_{max}}}, \bar{t} = \frac{t}{t_{max}}$$

$$\therefore h(x,t) = \frac{(x_{s_{max}}\bar{x} - x_{s_{max}}\bar{x}_s)^2}{Agt_{max}^2 \bar{t}^2}$$

$$\therefore h(x,t) = \frac{x_{s_{max}}^2}{t_{max}^2} \cdot \frac{(\bar{x} - \bar{x}_s)^2}{Ag\bar{t}^2}$$

Simplify $\frac{x_{s_{max}}}{t_{max}}$ from $\frac{x_{s_{max}}^2}{t_{max}^2}$

$$\frac{x_{s_{max}}}{t_{max}} = \frac{u_o^2}{2g \sin \beta} \cdot \frac{g \sin \beta}{u_o} = \frac{u_o}{2}$$

Fill in

$$\therefore h(x,t) = \frac{{u_o}^2}{4} \cdot \frac{(\bar{x} - \bar{x}_s)^2}{Ag\bar{t}^2}$$

Which we can write as

$$h(x,t) = \frac{u_o^2}{4g} \cdot \bar{h}\big(\bar{x}, \bar{t}\big)$$

A.2 Volume calculations

At time t, volume of water on slope shoreword of x:

$$\begin{aligned}
V_{ax}(t) &= \int_x^{x_s} h dx \\
h &= \frac{(x - x_s)^2}{Agt^2} = \frac{x^2 + x_s^2 - 2xx_s}{Agt^2} \\
&= \frac{x^2 + u_o^2 t^2 + \frac{1}{4} g^2 t^4 \sin^2\beta - u_o g t^3 \sin\beta - (2xu_o t - xgt^2 \sin\beta}{Agt^2} \\
\int h &= \frac{1}{Agt^2}\left(\frac{1}{3}x^3 + u_o^2 t^2 x + \frac{1}{4} g^2 t^4 \sin^2\beta x - u_o g t^3 \sin\beta x - x^2 u_o t + \frac{1}{2} x^2 g t^2 \sin\beta\right) \\
or \int_x^{x_s} h &= \frac{1}{Agt^2}\Big[1/3x^3 + xx_s^2 - x^2 x_s\Big]_x^{x_s} = \frac{1}{Agt^2}\left(\left(\frac{1}{3}x_s^3 + x_s^3 - x_s^3\right) - \left(\frac{1}{3}x^3 + xx_s^2 + x^2 x_s\right)\right) \\
V_{ax}(t) &= \frac{1}{Agt^2}\left(-\frac{1}{3}x^3 + \frac{1}{3}x_s^3 + x^2 x_s - xx_s^2\right)
\end{aligned}$$

Volume flux is $\frac{d}{dt}V_{ax}$

Write Vax as:

$$\begin{aligned}
V_{ax}(t) &= \frac{1}{9A} \cdot \left(-\frac{1}{3}\frac{x^3}{t^2} + \frac{1}{3}\left(\frac{x_s^3}{t^2}\right) + \frac{x^2 x_s}{t^2} - \frac{xx_s^2}{t^2}\right) \\
B &= \frac{x^3}{t^2} \\
C &= \frac{x_s^3}{t^2} \\
D &= \frac{x^2 x_s}{t^2} \\
E &= \frac{xx_s^2}{t^2}
\end{aligned}$$

$$\frac{dB}{dt} = -\frac{x^3}{3} \cdot \frac{-2}{t^2} = \frac{\frac{2}{3}x^3}{t^3}$$
$$\frac{dC}{dt} = \frac{x_s^3}{3} \cdot \frac{-2}{t^3} + \frac{1}{3} \cdot 3x_s^2 x_s' \cdot \frac{1}{t^2} = -\frac{2x_s^3}{3} \cdot \frac{1}{t^3} + \frac{1}{t^2} \cdot x_s^2 x_s'$$
$$\frac{dD}{dt} = x^2 x_s \cdot \frac{-2}{t^3} + \frac{1}{t^2} \cdot x^2 x_s'$$
$$\frac{dE}{dt} = x x_s^2 - \frac{2}{t^3} + \frac{1}{t^2} x \cdot 2x_s x_s'$$
$$V_{ax} = \frac{1}{Ag}\left(\frac{dA}{dt} + \frac{dB}{dt} + \frac{dC}{dt} + \frac{dD}{dt}\right)$$
$$V_{ax} = \frac{1}{Ag}\left(\frac{1}{t^3} \cdot \frac{2}{3}x^3 - \frac{1}{t^3} \cdot \frac{2x_s^3}{3} + \frac{1}{t^2} x_s^2 x_s' - \frac{1}{t^3} 2x^2 x_s + \frac{1}{t^3} 2x x_s^2 - \frac{1}{t^2} 2x x_s x_s'\right)$$
$$V_{ax} = \frac{1}{Ag}\left(\frac{1}{t^3}\left(\frac{2}{3}x^3 - \frac{2}{3}x_s^3 - 2x^2 x_s + 2x x_s^2\right) + \frac{1}{t^2}\left(x_s^2 x_s' + x^2 x_s' - 2x x_s x_s'\right)\right)$$
$$V_{ax} = \frac{1}{Ag}\left(\frac{2}{3}\frac{1}{t^3}\left(x^3 - x_s^3 - 3x^2 x_s + 3x x_s^2\right) + \frac{x_s'}{t^2}\left(x_s^2 + x^2 - 2x x_s\right)\right)$$
$$V_{ax} = \frac{1}{Ag}\left(\frac{2}{3}\frac{1}{t^3}\left(x - x_s\right)^3 + \frac{x_s'}{t^2}\left(x_s - x\right)^3\right)$$

A.3 Velocity calculations

Depth-average velocity is defined by the volume flux divided by depth:

$$\begin{aligned}
u &= \frac{1}{h} \cdot \frac{dV_{ax}}{dt} \\
&= \frac{Agt^2}{Ag} \cdot \frac{1}{(x - x_s)^2} \cdot \left(\frac{2}{3} \cdot \frac{1}{t^3}(x - x_s)^3 + \frac{x_s'}{t^2} \cdot (x_s - x)^2\right) \\
&= \frac{t^2}{(x - x_s)^2} \cdot (\frac{2}{3} \cdot \frac{1}{t^3}(x - x_s)^3 + \frac{x_s'}{t^2} \cdot (x_s - x)^2)
\end{aligned}$$
$$u = \frac{2}{3} \cdot \frac{1}{t} \cdot (x - x_s) + x_s'$$
$$u = \frac{2}{3} \cdot \frac{1}{t} \cdot (x - u_o t + \frac{1}{2} g t^2 \sin\beta) + u_o - g t \sin\beta$$
$$u = \frac{2}{3} \cdot \frac{x}{t} - \frac{2}{3} u_o + \frac{1}{3} g t \sin\beta + u_o - g t \sin\beta$$
$$u = \frac{2}{3} \cdot \frac{x}{t} + \frac{1}{3} u_o - \frac{2}{3} g t \sin\beta$$
$$u = \frac{2}{3} \cdot \left(\frac{1}{2} u_o - g t \sin\beta + \frac{x}{t}\right)$$
$$u = \frac{2}{3} \cdot \frac{u_o}{2}\left(1 - \frac{2gt\sin\beta}{u_o} + \frac{2}{u_o}\frac{x}{t}\right)$$
$$u = u_o \frac{1}{3}\left(1 - \frac{2gt\sin\beta}{u_o} + \frac{2}{u_o} \cdot \frac{x}{t}\right)$$

Note that (by making it dimensionless)

$$\frac{2gt\sin\beta}{u_o} = 2\bar{t} \quad \text{and} \quad \frac{2}{u_o}\cdot\frac{x}{t} = \frac{\bar{x}}{\bar{t}}$$

$$\therefore u = u_o\frac{1}{3}\left(1 - 2\bar{t} + \frac{\bar{x}}{\bar{t}}\right)$$

Define $\bar{u} = \frac{u}{u_o}$, which results in

$$\bar{u} = \frac{1}{3}\left(1 - 2\bar{t} + \frac{\bar{x}}{\bar{t}}\right)$$

A.4 Initial conditions calculations

The water height at the moment (0,0) for the dimensional model. At $x = 0$:

$$\begin{aligned}
h(x,t) &= \frac{(x - x_s)^2}{Agt^2} \\
h(0,t) &= \frac{(-x_s)^2}{Agt^2} \\
&= \frac{(u_o t)^2 + (\frac{1}{2}gt^2\sin\beta)^2 - u_o g t^3 \sin\beta}{Agt^2} \\
&= \frac{1}{Agt^2}\cdot\left(u_o^2 t^2 + \frac{1}{4}g^2t^4 sin^2(\beta) - u_o g t^3\sin\beta\right) \\
h(0,t) &= \frac{1}{Ag}\left(u_o^2 + \frac{1}{4}g^2t^2 sin^2(\beta) - u_o g t\sin\beta\right) \\
&= \frac{1}{Ag}\left(u_o - \frac{1}{2}gt\sin\beta\right)^2
\end{aligned}$$

At $t = 0$:

$$h(0,0) = \frac{u_o^2}{Ag}$$

$\frac{u_o^2}{Ag}$ is the maximum water depth anywhere and at any time.

The water velocity at the moment (0,0) for the dimensional model.

$$u = u_o\frac{1}{3}\left(1 - \frac{2gt\sin\beta}{u_o} + \frac{2}{u_o}\cdot\frac{x}{t}\right)$$

At $x = 0$:

$$u(0,t) = \frac{u_o}{3}\cdot\left(1 - \frac{2gt\sin\beta}{u_o}\right)$$

So at the moment $x = 0$ and at $t = 0$:

$$u(0,0) = \frac{u_o}{3}$$

Calculations for the non-dimensional model

The water height at the moment (0,0) for the non-dimensional model:

$$\therefore \bar{h}(0,0) = \left(\frac{u_o^2}{Ag} / \frac{u_o^2}{4g}\right) = \frac{4}{A} = 0.444$$

The water velocity at the moment (0,0) for the non-dimensional model:

$$u(0,0) = \frac{u_o}{3} / u_o = \frac{1}{3}$$

B MATLAB script: Dimensional model

```
clear all

%Input variables
uo=2;
beta=atan(1/10);
g=9.81;
A = 2;

tmax=uo/g/sin(beta); %Time of the swash period
xsmax=uo*uo/(2*g*sin(beta)); %Maximum run-up

t=0:tmax/100:2*tmax; %Time step
x=0:xsmax/5:xsmax; %5 different measure points

for i=1:length(x)
    for j=1:length(t)
        %Position of x in time
        xs=(uo*t(j))-(0.5*g*t(j)*t(j)*sin(beta));
        if x(i)<xs
                %Water depth
            h(i,j)=((x(i)-xs)^2)/(A*g*t(j)^2);
                %Water veolcity
            u(i,j)=(uo/3)*(1-(2*g*t(j)*sin(beta)/uo)+...
                        (2/uo)*(x(i)/t(j)));
        else
            h(i,j)=0.0;
            u(i,j)=0.0;
        end
    end
    figure(1)
    subplot(2,1,1)
    plot(t,h(i,:))
    legend('0.0','0.2','0.4','0.6','0.8')
    hold on
    subplot(2,1,2)
    plot(t,u(i,:))
    hold on
end
```

```
figure(1)
subplot(2,1,1)
axis([0 2*tmax 0 0.05]);
xlabel('t [s]')
ylabel('h [m]')
hold off
subplot(2,1,2)
axis([0 2*tmax -uo uo]);
xlabel('t [s]')
ylabel('u [m/s]')
hold off
```

C MATLAB script: Methods

```
clear all

%Input values
uo1 = 1; %initial velocity (m/s)
uo2 = 2.5; %initial velocity (m/s)
beta1 = atan(1/20);%slope of the beach
beta2 = atan(1/10);%slope of the beach
g = 9.81; %gravity of Earth
d50 = 8.4/1000; %sediment size (m)
nu = 1.004e-6; %ken. viscosity of water at 20 degrees (m^2/s)
rho = 1000; %Water density at 20 degrees (kg/m^3)
ks = 2.5*d50;
A = 2;
B = 0.75; %X-position

%Case A
dt = 400;
tmax1 = uo1/(g*sin(beta1));
xsmax1=(uo1*uo1)/(2*g*sin(beta1));
t1=0:tmax1/dt:2*tmax1;
x1= B*xsmax1 ;
for j=1:length(t1)
    xs1=(uo1*t1(j))-(0.5*g*t1(j)*t1(j)*sin(beta1));
    if x1<xs1
        h1(j)=((x1-xs1)^2)/(A*g*t1(j)^2);
        u1(j)=(uo1/3)*(1-(2*g*t1(j)*sin(beta1)/uo1)+(2/uo1)...
                *(x1/t1(j)));
        Dh1(j)=4*h1(j);
        K(j) = ks/Dh1(j);
        R(j) = ((abs(u1(j)))*Dh1(j))/nu; % Reynolds number
        lambda1(j) = Colebrook(R(j),K(j));
        fwC1(j) = lambda1(j)/4;
    else
        h1(j)=0.0;
        u1(j)=0.0;
        fwC1(j)=0.0;
        R(j)=0.0;
    end
```

```
end
%Case B
tmax2 = uo1/(g*sin(beta2));
xsmax2 =(uo1*uo1)/(2*g*sin(beta2));
t2 =0:tmax2/dt:2*tmax2;
x2= B*xsmax2 ;
for j=1:length(t2)
    xs2=(uo1*t2(j))-(0.5*g*t2(j)*t2(j)*sin(beta2));
    if x2<xs2
        h2(j)=((x2-xs2)^2)/(A*g*t2(j)^2);
        u2(j)=(uo1/3)*(1-(2*g*t2(j)*sin(beta2)/uo1)+(2/uo1)...
                *(x2/t2(j)));
        Dh2(j)=4*h2(j);
        K(j) = ks/Dh2(j);
        R(j) = ((abs(u2(j)))*Dh2(j))/nu; % Reynolds number
        lambda2(j) = Colebrook(R(j),K(j));
        fwC2(j) = lambda2(j)/4;
    else
        h2(j)=0.0;
        u2(j)=0.0;
        fwC2(j)=0.0;
        R(j)=0.0;
    end
end
%Case C
tmax3 = uo2/(g*sin(beta1));
xsmax3 =(uo2*uo2)/(2*g*sin(beta1));
t3 =0:tmax3/dt:2*tmax3;
x3 = B*xsmax3;
for j=1:length(t3)
    xs3=(uo2*t3(j))-(0.5*g*t3(j)*t3(j)*sin(beta1));
    if x3<xs3
        h3(j)=((x3-xs3)^2)/(A*g*t3(j)^2);
        u3(j)=(uo2/3)*(1-(2*g*t3(j)*sin(beta1)/uo2)+(2/uo2)...
            *(x3/t3(j)));
        Dh3(j)=4*h3(j);
        K(j) = ks/Dh3(j);
        R(j) = ((abs(u3(j)))*Dh3(j))/nu; % Reynolds number
        lambda3(j) = Colebrook(R(j),K(j));
        fwC3(j) = lambda3(j)/4;
    else
        h3(j)=0.0;
```

```
        u3(j)=0.0;
        fwC3(j)=0.0;
        R(j)=0.0;
    end
end
%Case D
tmax4 = uo2/(g*sin(beta2));
xsmax4=(uo2*uo2)/(2*g*sin(beta2));
t4= 0:tmax4/dt:2*tmax4;
x4= B*xsmax4;
for j=1:length(t4)
    xs4=(uo2*t4(j))-(0.5*g*t4(j)*t4(j)*sin(beta2));
    if x4<xs4
        h4(j)=((x4-xs4)^2)/(A*g*t4(j)^2);
        u4(j)=(uo2/3)*(1-(2*g*t4(j)*sin(beta2)/uo2)+(2/uo2)...
                *(x4/t4(j)));
        Dh4(j)=4*h4(j);
        K(j) = ks/Dh4(j);
        R(j) = ((abs(u4(j)))*Dh4(j))/nu; % Reynolds number
        lambda4(j) = Colebrook(R(j),K(j));
        fwC4(j) = lambda4(j)/4;
    else
        h4(j)=0.0;
        u4(j)=0.0;
        fwC4(j)=0.0;
                R(j)=0.0;
    end
end

%Swart formula
ind1=find(u1~=0);
ind2=find(u2~=0);
ind3=find(u3~=0);
ind4=find(u4~=0);
t11=t1(ind1(1));
t12=t2(ind2(1));
t13=t3(ind3(1));
t14=t4(ind4(1));
t21=t1(ind1(end));
t22=t2(ind2(end));
t23=t3(ind3(end));
t24=t4(ind4(end));
```

```
T1=t21-t11;
T2=t22-t12;
T3=t23-t13;
T4=t24-t14;
fwS1= SwashFwSwart(ks,u1,T1); %Case A
fwS2= SwashFwSwart(ks,u2,T2); %Case B
fwS3= SwashFwSwart(ks,u3,T3); %Case C
fwS4= SwashFwSwart(ks,u4,T4); %Case D

%Instantaneous bed shear stress
tb11 = 0.5*rho*fwS1*u1.*abs(u1); %Swart Case A
tb12 = 0.5*rho*fwS2*u2.*abs(u2); %Swart Case B
tb13 = 0.5*rho*fwS3*u3.*abs(u3); %Swart Case C
tb14 = 0.5*rho*fwS4*u4.*abs(u4); %Swart Case D
tb21 = 0.5*rho*fwC1.*u1.*abs(u1); %Colebrook-White Case A
tb22 = 0.5*rho*fwC2.*u2.*abs(u2); %Colebrook-White Case B
tb23 = 0.5*rho*fwC3.*u3.*abs(u3); %Colebrook-White Case C
tb24 = 0.5*rho*fwC4.*u4.*abs(u4); %Colebrook-White Case D

figure(1)
subplot(4,3,3)
plot(t1,tb11,t1,tb21) %Case A
ylabel('\tau_b')
xlabel('t')
axis([0 4 -25 25])

subplot(4,3,6)
plot(t2,tb12,t2,tb22) %Case B
ylabel('\tau_b')
xlabel('t')
axis([0 2 -40 40])

subplot(4,3,9)
plot(t3,tb13,t3,tb23) %Case C
ylabel('\tau_b')
xlabel('t')
axis([0 10 -60 60])

subplot(4,3,12)
plot(t4,tb14,t4,tb24) %Case D
ylabel('\tau_b')
xlabel('t')
```

```
axis([0 5 -80 80])

%Graph Reynolds number
y = 2300
y1=(t2./t2)*y;
figure(2)
subplot(2,3,6)
plot(t2,R,t2,y1)
ylabel('Re')
xlabel('t')
axis([0 2 0 2500])
```

D Problem in the Colebrook Method

There are spikes in the graphs of the Colebrook Method in Figure 10. The top of the spikes are in the thousands while the order of magnitude has to be between zero and 0.1. So this means that the spikes cannot be representative to describe the behaviour of the bed shear stress. The reason for this is that if $\frac{k_s}{3.7Dh} + \frac{2.51}{Re\sqrt{\lambda}} = 1$ then the limit goes to infinity. In the Figure 10 the spikes do not go to infinity. This only occurs when the time step is just too big and is not exactly one. To make it exactly one the time step has to decrease till the tops go the infinity.

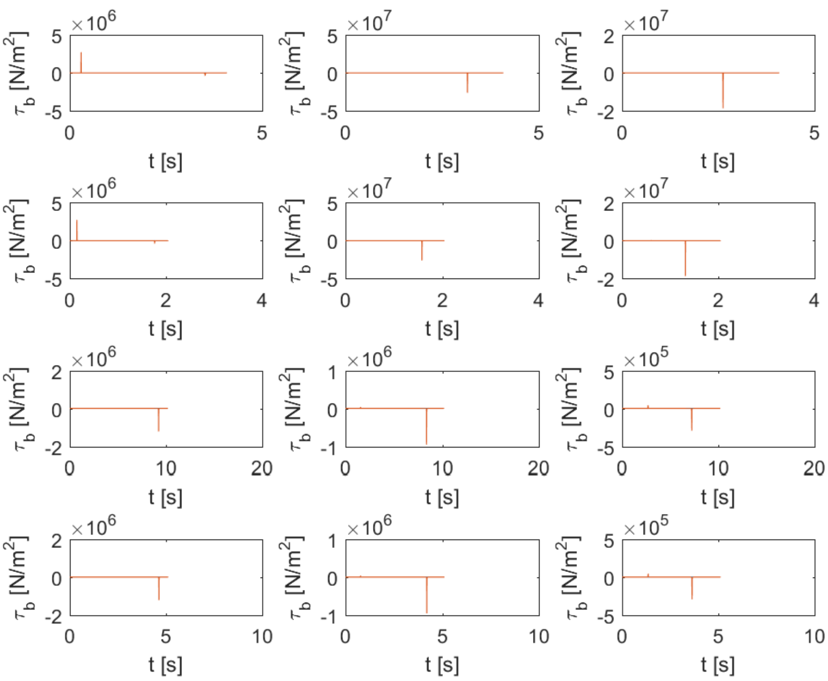

Figure 10: Model output for the hydrodynamic model for the Colebrook method ($d_{50} = 1.3mm$): Case A (first row), Case B (second row), Case C (third row) and Case D (fourth row). Test location on the slope $0.25 \cdot x_{s_{max}}$ (first column), $0.5 \cdot x_{s_{max}}$ (second column), $0.75 \cdot x_{s_{max}}$ (third column).