July 21, 2017

BACHELOR THESIS

Regularization of 3D dyadic Green's function for flat interfaces

Iris Ottens

Faculty of Science and Technology
Faculty of Electrical Engineering, Mathematics and Computer Science (EEMCS)
Complex Photonics Systems (COPS)
Multiscale Modeling and Simulation(MMS)

Supervised by:
Dr. S. B. Hasan
Prof. Dr. Ir. B. J. Geurts
Prof. Dr. W. L. Vos

UNIVERSITY OF TWENTE.

Abstract

In this report we combine a known regularization method of the 3D dyadic Green's function, with a known method to solving the Green's function in cylindrical coordinates. We considered both the free space Green's function, as well as at the scattering part in two different geometries. These geometries are a semi-infinite absorbing dielectric medium with a planar interface and a absorbing dielectric thin film. No regularization has has been done for these parts, as it was not needed in the situations considered in this study. Most problems in the study have been solved analytically. However, some integrals will have to be solved by numerical means, which will be done in a continuation of this study.

Contents

1 Introduction

Calculating the electric field produced by a current source in and around a dielectric medium, often makes use of an electric Green's functions. The same Green's functions can also directly be used to calculate the local density of states (LDOS) of a dipole, as the projected LDOS in a certain direction is directly dependent on the imaginary part of the Green's function evaluated at the source [1]. This LDOS is of importance in determining the transition rates of atoms in Fermi's Golden rule. This transition rate is

$$\Gamma = \frac{2\pi}{\hbar} \left|\langle f|H'|i\rangle\right| N(\mathbf{r}), \tag{1}$$

where $\hbar$ is Planck's constant, f and i are the final and the initial states respectively, H' is the dipole Hamiltonian and $N_{\mathbf{p}}(\mathbf{r})$ is the density of states (DOS). For local variations the LDOS can be used instead [2].

Calculating the imaginary part of the Green's function in order to get the LDOS does not pose any problems when dealing with non absorbing media, as the imaginary part of the Green's function at the location of the dipole is finite in that case. However, when looking at absorbing media, which all materials except for vacuum are, the dielectric constant is complex, causing the singularity in the real part of the Green's function to also enter into the imaginary part.To solve this problem, we introduce a regularization method that basically assigns a finite size to the point dipole [3].

In this report we will specifically look at problems that have an azimuthal symmetry, such as a thin film or half plane of dielectric material. This is a relevant symmetry to, for example, light emitting diodes (LEDs) [4], which are an important application of this problem. To stay close to the solutions found in literature [5], we will approach the problem in cylindrical coordinates.

We will start by solving the problem for a point dipole in free space, after which the regularization will be introduced. The first step will be calculating the free space Green's function in terms of eigenfunctions of the vector Helmholtz equation, after which the solution will be split in a transverse and longitudinal part. These two parts will then both be regularized. Next the same will be done for two different geometries. The first is a dielectric absorbing semi-infinite medium with a planer interface and a dipole. The second is a thin absorbing dielectric film with a dipole either in or outside the film.

2 Theoretical background

In this section we will derive the LDOS in terms of the imaginary part of the Green's function, corresponding to the electric field with a dipole as source, which sets the background of this work. We start the derivation from the macroscopic Maxwell's equations for harmonically oscillating fields in matter with a $e^{-i\omega t}$ time dependence, where ω is the angular frequency [6].

$$\begin{aligned}\nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\epsilon} \\ \nabla \cdot \mathbf{H}(\mathbf{r}) &= 0 \\ \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mu\mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= \mathbf{J}(\mathbf{r}) - i\omega\epsilon\mathbf{E}(\mathbf{r}).\end{aligned} \tag{2}$$

Here $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are the electric field and the magnetic field respectively, μ and ϵ respectively stand for permeability and permittivity of the considered medium, $\mathbf{J}(\mathbf{r})$ is the current density and $\rho(\mathbf{r})$ is the electric charge density. From these equations and by introducing $k = \omega\sqrt{\mu\epsilon}$ the vector Helmholtz equation can be derived to be

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega) - k^2(\omega)\mathbf{E}(\mathbf{r}, \omega) = i\omega\mu\mathbf{J}(\mathbf{r}, \omega). \tag{3}$$

The Green's function of this equation then has to satisfy

$$\nabla \times \nabla \times \mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega) - k^2\mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega) = \mathcal{I}\delta(\mathbf{r} - \mathbf{r_0}), \tag{4}$$

with $\mathcal{I}$ being the identity matrix and $\mathbf{r_0}$ the location of the source [6]. For a source $\mathbf{J}(\mathbf{r})$, the electric field relates to the Green's function by

$$\mathbf{E}(\mathbf{r}, \omega) = i\omega\mu \int\int\int \mathbf{J}(\mathbf{r}, \omega) \cdot \mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega)dV, \tag{5}$$

with the integral going over all space [6]. For the special case of a Hertzian point dipole the source term is

$$\mathbf{J}(\mathbf{r}, \omega) = -i\omega\mathbf{p}(\omega)\delta(\mathbf{r} - \mathbf{r}_0), \tag{6}$$

with $\mathbf{p}$ the dipole moment. In that case the electric field directly relates to the Green's function as

$$\mathbf{E}(\mathbf{r}) = \omega^2\mu\mathbf{p}(\omega) \cdot \mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega). \tag{7}$$

The density of states (DOS) counts the modes corresponding to a certain ω which is expressed as

$$N(\omega) = \sum_n \delta\left(\omega^2 - \omega_n^2\right), \tag{8}$$

with ω_n the angular frequency corresponding to an eigenvalue of the vector Helmholtz equation. The local density of states (LDOS) is the spatial projection of DOS given by

$$N_e(\mathbf{r}, \omega) = \sum_n |\mathbf{E}_n(\mathbf{r})|^2 \delta\left(\omega^2 - \omega_n^2\right), \tag{9}$$

with $\mathbf{E}_n(\mathbf{r})$ a discrete set of orthonormal eigenmodes of the vector Helmholtz equation satisfying

$$\nabla \times \nabla \times \mathbf{E}_n(\mathbf{r}, \omega_n) - k^2(\omega_n)\mathbf{E}_n(\mathbf{r}, \omega_n) = 0. \tag{10}$$

The Green's function can than be expressed as a superposition of these eigenmodes

$$\mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega) = \sum_n \mathbf{A}_n(\mathbf{r}_0) \otimes \mathbf{E}_n(\mathbf{r}). \tag{11}$$

Filling Eq. (11) in into Eq. (4) and using the orthogonality relations between the eigenmodes gives

$$\left(\omega_n^2 - \omega^2\right) \mathbf{A}_n(\mathbf{r}_0) = c^2 \mathbf{E}_n^* \left(\mathbf{r}_0\right). \tag{12}$$

By taking $\left(\omega_n^2 - \omega^2\right)$ to the other side and using the mathematical identity

$$\lim_{\eta \to 0^+} \frac{1}{x - i\eta} = \mathrm{PV}\left[\frac{1}{x}\right] + i\pi\delta(x), \tag{13}$$

with PV the principle value, we get

$$\mathbf{A}_n(\mathbf{r}_0) = c^2 \mathbf{E}_n^* \left(\mathbf{r}_0\right) \left(\mathrm{PV}\left[\frac{1}{\omega_n^2 - \omega^2}\right] + i\pi\delta\left(\omega_n^2 - \omega^2\right)\right). \tag{14}$$

The Green's function then becomes

$$\mathcal{G}_e(\mathbf{r}, \mathbf{r}_0, \omega) = \sum_n c^2 \mathbf{E}_n^*(\mathbf{r}_0) \otimes \mathbf{E}_n(\mathbf{r}) \times \left(\mathrm{PV}\left[\frac{1}{\omega_n^2 - \omega^2}\right] + i\pi\delta\left(\omega^2 - \omega_n^2\right)\right). \tag{15}$$

When comparing the expression of the Green's function with the previous expression of the LDOS, at Eq. (9), the LDOS can also be expressed as

$$N_e(\mathbf{r}, \omega) = \frac{1}{\pi c^2} \mathrm{Im}\left[\mathrm{Tr}\left\{\mathcal{G}_e(\mathbf{r}, \mathbf{r}, \omega)\right\}\right], \tag{16}$$

where Im stands for the imaginary part and Tr takes the trace of the tensor. The projected LDOS, in the direction of the dipole, can then be rewritten as

$$N_{e,\mathbf{p}}(\mathbf{r}, \omega) = \frac{1}{\pi c^2}\left[\mathbf{p} \cdot \mathrm{Im}\left\{\mathcal{G}_e\left(\mathbf{r}, \mathbf{r}, \omega\right)\right\} \cdot \mathbf{p}\right]. \tag{17}$$

With above it can be concluded that the imaginary part of the Green's function at the location of the source is indeed an observable [1], namely the LDOS of a dipole, which determines the transition rate of the dipole.

3 Free space Green's function

3.1 Eigenfunctions

We will start solving the vector Helmholtz equation for the Green's function, Eq. (2) and (4), for free space

$$\nabla \times \nabla \times \mathcal{G}_{e0}(\mathbf{r}, \mathbf{r}_0) - k^2 \mathcal{G}_{e0}(\mathbf{r}, \mathbf{r}_0) = \mathcal{I}\delta(\mathbf{r} - \mathbf{r_0}), \tag{18}$$

with $k = \omega\sqrt{\mu\epsilon}$, the same as above. In case of absorbing media the permittivity will be complex. For the sake of shortening notation $\mathcal{G}_{e0}$ will be written as $\mathcal{G}_0$ as we will only be discussing the Green's function for the electric field. The dependence on the spatial variables is implied, but not explicitly written in the following for the same reasons.

To find eigenfunctions of the vector Helmholtz equation we use the method first introduced by Hansen in 1935 [7]. This method allows us to construct three kinds of vector eigenfunctions, $\mathbf{M}$, $\mathbf{N}$

and $\mathbf{L}$, by using the eigenfunctions of the scalar wave equation. These eigenfunctions of $\nabla^2\psi+\kappa^2\psi=0$ can be obtained by separation of variables, which, in cylindrical coordinates, gives

$$\psi_{\substack{o\\e}} = J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, e^{ihz}, \tag{19}$$

with eigenvalues

$$\kappa^2 = \lambda^2 + h^2.$$

From now on the subscripts o and e that point at the distinction between the upper and the lower part of the solution, the odd and the even parts respectively, will be omitted unless only one of these parts is considered.

With Eq. (19) we can generate the eigenfunctions of the vector Helmholtz equation mentioned above by employing the method introduced in Ref. [7] and choosing the right piloting vector. The piloting vector is used to give a direction to the eigenfunctions of Eq. (19) and can be chosen in any direction. The direction will be decided based on the geometries that are considered to make calculations easier and is thus chosen to be pointing in the $\hat{z}$ direction. With this choice in pointing vector we gets the following functions, $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{L}$, that can easily be shown to be eigenfunctions of the vector Helmholtz equation

$$\begin{aligned}
\mathbf{M}_{n\lambda}(h) &= \nabla \times (\psi \hat{z}) \qquad (20)\\
&= \nabla \times \left(J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, e^{ihz}\hat{z}\right)\\
&= e^{ihz}\left(\pm \hat{r}\frac{nJ_n(\lambda r)}{r} \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) - \hat{\phi}\frac{\partial}{\partial r}J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\right)\\
&= \frac{1}{\kappa}\nabla \times \mathbf{N}_{n\lambda}(h), \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}_{n\lambda}(h) &= \frac{1}{\kappa}\nabla \times \nabla \times (\psi \hat{z}) \qquad (22)\\
&= \frac{1}{\kappa}\nabla \times \mathbf{M}_{n\lambda}(h) \qquad (23)\\
&= \frac{1}{\kappa}e^{ihz}\Bigg[\hat{r}ih\frac{\partial}{\partial r}J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \pm \hat{\phi}ih\frac{nJ_n(\lambda r)}{r} \begin{matrix}\cos\\ \sin\end{matrix} (n\phi)\\
&+ \hat{z}\left(-\frac{\partial^2}{\partial r^2}J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) + \frac{n^2 J_n(\lambda r)}{r^2} \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\right)\Bigg]\\
&= \frac{1}{\kappa}e^{ihz}\left(\hat{r}ih\frac{\partial}{\partial r}J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \pm \hat{\phi}ih\frac{nJ_n(\lambda r)}{r} \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) + \hat{z}\lambda^2 J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{L}_{n\lambda}(h) &= \nabla(\psi) \qquad (24)\\
&= \nabla\left(J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, e^{ihz}\right)\\
&= e^{ihz}\left(\hat{r}\frac{\partial}{\partial r}J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \pm \hat{\phi}\frac{nJ_n(\lambda r)}{r} \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) + \hat{z}ihJ_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\right).
\end{aligned}$$

The $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{L}$ functions are all orthogonal to each other and to themselves if $n \neq n'$, proof of this can be found in Ref. [6] and in Appendix A. And with themselves they have the following normalization constants

$$\int\int\int \mathrm{d}V \mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda'}(-h') = 2\pi^2\lambda(1+\delta_{0n})\delta(h-h')\delta(\lambda-\lambda'), \tag{25}$$

$$\int\int\int \mathrm{d}V \mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda'}(-h') = 2\pi^2\lambda(1+\delta_{0n})\delta(h-h')\delta(\lambda-\lambda') \tag{26}$$

and

$$\int\int\int \mathrm{d}V \mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda'}(-h') = 2\pi^2(\lambda + \frac{hh'}{\lambda'})(1+\delta_{0n})\delta(h-h')\delta(\lambda-\lambda'), \tag{27}$$

where δ_{0n} denotes the Kronecker delta, being 1 if $n=0$ and 0 otherwise. These constants have been found using the method as can be found in Ref. [6].

3.2 Eigenfunction expansion

To find the solution for the Green's function Eq. (18) will be expanded in the eigenfunctions we determined in the previous section. The right side of Eq. (18), namely the delta function, can be expressed in terms of the eigenfunctions

$$\mathcal{I}\delta(\mathbf{r}-\mathbf{r_0}) = \int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{\mathbf{N}_{n\lambda}(h)\mathbf{a}_{n\lambda}(h) + \mathbf{M}_{n\lambda}(h)\mathbf{b}_{n\lambda}(h) + \mathbf{L}_{n\lambda}(h)\mathbf{c}_{n\lambda}(h)\right\}. \tag{28}$$

Both sides of Eq. (28) will be multiplied by $\mathbf{N}_{n\lambda}(-h)$, $\mathbf{M}_{n\lambda}(-h)$ and $\mathbf{L}_{n\lambda}(-h)$ respectively and integrated over space, which gives the coefficients

$$\begin{aligned} \mathbf{a}_{n\lambda}(h) &= \mathbf{N}_{n\lambda0}(-h) / \int_{-\infty}^{\infty} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\lambda' \int\int\int \mathrm{d}V \mathbf{N}_{n\lambda'}(h')\mathbf{N}_{n\lambda}(-h) \\ &= \mathbf{N}_{n\lambda0}(-h)/(1+\delta_{0n})2\pi^2\lambda, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathbf{b}_{n\lambda}(h) &= \mathbf{M}_{n\lambda0}(-h) / \int_{-\infty}^{\infty} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\lambda' \int\int\int \mathrm{d}V \mathbf{M}_{n\lambda'}(h')\mathbf{M}_{n\lambda}(-h) \\ &= \mathbf{M}_{n\lambda0}(-h)/(1+\delta_{0n})2\pi^2\lambda \end{aligned} \tag{30}$$

and

$$\begin{aligned} \mathbf{c}_{n\lambda}(h) &= \mathbf{L}_{n\lambda0}(-h) / \int_{-\infty}^{\infty} \mathrm{d}h' \int_0^{\infty} \mathrm{d}\lambda' \int\int\int \mathrm{d}V \mathbf{L}_{n\lambda'}(h')\mathbf{L}_{n\lambda}(-h) \\ &= \lambda\mathbf{L}_{n\lambda0}(-h)/(1+\delta_{0n})2\pi^2(\lambda^2+h^2). \end{aligned} \tag{31}$$

When filling in the above mentioned coefficients into Eq. (28) we get

$$\begin{aligned} &\mathcal{I}\delta(\mathbf{r}-\mathbf{r_0}) = \\ &\int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{1}{2\pi^2(1+\delta_{0n})} \left\{\mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda0}(-h)/\lambda + \mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda0}(-h)/\lambda + \right. \\ &\left. \lambda\mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda0}(-h)/\left(\lambda^2+h^2\right)\right\}. \end{aligned} \tag{32}$$

The Green's function can also be expressed as a superposition of the eigenfunctions as follows

$$\mathcal{G}_0 = \int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \mathbf{N}_{n\lambda}(h)\mathbf{A}_{n\lambda}(h) + \mathbf{M}_{n\lambda}(h)\mathbf{B}_{n\lambda}(h) + \mathbf{L}_{n\lambda}(h)\mathbf{C}_{n\lambda}(h) \right\}. \tag{33}$$

Eq. (33), as well as the expansion of the delta function, will now be substituted back into Eq. (18) to get the coefficients. Here the relations between $\mathbf{M}$ and $\mathbf{N}$ are used as well as the fact that the curl of $\mathbf{L}$ is zero. This leads us to

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \big\{ \left(\kappa^2 - k^2\right) \mathbf{N}_{n\lambda}(h)\mathbf{A}_{n\lambda}(h) + \left(\kappa^2 - k^2\right) \mathbf{M}_{n\lambda}(h)\mathbf{B}_{n\lambda}(h) \\ -k^2\mathbf{L}_{n\lambda}(h)\mathbf{C}_{n\lambda}(h) \big\} = \\ \int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{1}{2\pi^2(1+\delta_{0n})} \left\{ \mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda 0}(-h)/\lambda + \mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda 0}(-h)/\lambda \right. \\ \left. +\lambda \mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda 0}(-h)/\left((\lambda + h^2\right) \right\}. \end{aligned} \tag{34}$$

By comparing the coefficients of the eigenfunctions we get

$$\mathbf{A}_{n\lambda}(h) = \mathbf{a}_{n\lambda}(h)/\left(\kappa^2 - k^2\right) = \mathbf{N}_{n\lambda 0}(-h)/\left(2\pi^2\lambda(1+\delta_{0n})\left(\kappa^2 - k^2\right)\right), \tag{35}$$

$$\mathbf{B}_{n\lambda}(h) = \mathbf{b}_{n\lambda}(h)/\left(\kappa^2 - k^2\right) = \mathbf{M}_{n\lambda 0}(-h)/\left(2\pi^2\lambda(1+\delta_{0n})\left(\kappa^2 - k^2\right)\right) \tag{36}$$

and

$$\mathbf{C}_{n\lambda}(h) = \mathbf{c}_{n\lambda}(h)/\left(-k^2\right) = -\lambda\mathbf{L}_{n\lambda 0}(-h)/\left(2\pi^2\left(h^2+\lambda^2\right)(1+\delta_{0n})k^2\right). \tag{37}$$

Filling in the coefficients found above gives a solution of the Green's function in terms of the eigenfunctions

$$\begin{aligned} \mathcal{G}_0 = \int_{-\infty}^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \Bigg\{ \frac{\mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda 0}(-h)}{(1+\delta_{0n})\,2\pi^2\lambda\left(\kappa^2-k^2\right)} + \frac{\mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda 0}(-h)}{(1+\delta_{0n})\,2\pi^2\lambda\left(\kappa^2-k^2\right)} \\ + \frac{-\lambda\mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda 0}(-h)}{(1+\delta_{0n})\,2\pi^2\left(h^2+\lambda^2\right)k^2} \Bigg\}. \end{aligned} \tag{38}$$

3.3 Solving the integrals

We now have the Green's function in terms of the eigenfunctions of the vector Helmholtz equation. This solution, however, still contains some integrals. Of these the integral with respect to h necessary to solve for the application of the boundary conditions. This will be done by making use of contour integration in the complex plane. The function only has simple poles, which makes it easy to use the residue theorem. The details can be found in Appendix B.

The solution, derived in Appendix B, is for $z > z_0$

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda\sqrt{k^2-\lambda^2}} \left[\mathbf{N}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right) \right.\right.$$
$$\left. + \mathbf{M}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right) \right]$$
$$\left. - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\delta(z-z_0)\hat{z}\hat{z}_0 \right\} \tag{39}$$

and for $z < z_0$

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda\sqrt{k^2-\lambda^2}} \left[\mathbf{N}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right) \right.\right.$$
$$\left. + \mathbf{M}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right) \right].$$
$$\left. - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\delta(z-z_0)\hat{z}\hat{z}_0 \right\} \tag{40}$$

We will not analytically solve the integral with respect to λ. When trying a similar approach as with the one for h, three problems can be found. Firstly, the integral goes from zero to infinity instead of from - to + infinity. This can be solved as seen in Ref. [6]. Additionally there is also the problem of the pole at $\lambda = 0$ lying on the contour if the same contour is used as for h. This is also easily to solved by going around it, thus having a contribution of πi instead of $2\pi i$. The last problem is the square root, which is multi valued. For this a branch cut must be created between the branch points, which the contour is not allowed to cross. This might be done in a way similar to what was done in ref. [8]. However the contour integral still remains too challenging for analytical calculations. Therefore, the standard practice of numerically integrating Eqs.(**??**-**??**) is recommended. As this is not the main problem of what we want to solve in this report, the numerical solution will be considered at a later time.

3.4 Transverse and longitudinal parts

As the strength of singularity of the Green's function at the location of the origin is different for the transverse and longitudinal parts of the Green's function [3], thus we will separately look at these parts, $\mathcal{G}_0^T$ and $\mathcal{G}_0^L$ respectively [3]. It is easy to see that the $\mathbf{M}$ and $\mathbf{N}$ eigenfunctions correspond to the transverse part and the $\mathbf{L}$ eigenfunctions to the longitudinal part, as it follows from the definition of these functions that the divergence of the first two is zero while the curl of the last one equals zero. This gives before the integration

$$\mathcal{G}_0^T = \int_{-\infty}^{\infty} \mathrm{d}h \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{\mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\lambda\left(\kappa^2-k^2\right)} + \frac{\mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\lambda\left(\kappa^2-k^2\right)} \right\} \tag{41}$$

and

$$\mathcal{G}_0^L = \int_{-\infty}^{\infty} \mathrm{d}h \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{-\lambda\mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\left(h^2+\lambda^2\right)k^2}. \tag{42}$$

And after the integration to h for $z > z_0$

$$\mathcal{G}_0^T = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{i\mathbf{N}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right) + i\mathbf{M}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right)}{2\left(1+\delta_{0n}\right)\pi\lambda\sqrt{k^2-\lambda^2}} + \mathcal{G}_A^+ \right\} \quad (43)$$

and

$$\mathcal{G}_0^L = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{-\mathbf{L}_{n\lambda}(i\lambda)\mathbf{L}_{n\lambda 0}(-i\lambda)}{2(1+\delta_{0n})\pi k^2} - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \delta(z-z_0)\hat{z}\hat{z}_0 \right\}, \quad (44)$$

with

$$\begin{aligned}\mathcal{G}_A^+ = \frac{\exp[-\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi} \Bigg(& \frac{\partial}{\partial r} J_n(\lambda r) \frac{\partial}{\partial r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \hat{r}\hat{r}_0 \\ & + \frac{n^2}{r r_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix}(n\phi) \begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\, \hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r} J_n(\lambda r) \frac{n}{r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\, \hat{r}\hat{\phi}_0 \\ & \pm \frac{\partial}{\partial r_0} J_n(\lambda r_0) \frac{n}{r} J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0) \begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\, \hat{r}_0\hat{\phi} - \lambda^2 J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \hat{z}\hat{z}_0 \\ & + \lambda \frac{\partial}{\partial r} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \hat{r}\hat{z}_0 - \lambda \frac{\partial}{\partial r_0} J_n(\lambda r_0) J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\, \hat{r}_0\hat{z} \\ \pm i\lambda \frac{n}{r} J_n(\lambda r) J_n(\lambda r_0) & \begin{matrix}\cos\\ \sin\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \hat{\phi}\hat{z}_0 \mp \lambda \frac{n}{r_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\, \hat{\phi}_0\hat{z} \Bigg). \quad (45)\end{aligned}$$

Similarly, for $z < z_0$

$$\mathcal{G}_0^T = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{i\mathbf{N}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right) + i\mathbf{M}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right)}{2\left(1+\delta_{0n}\right)\pi\lambda\sqrt{k^2-\lambda^2}} + \mathcal{G}_A^- \right\} \quad (46)$$

and

$$\mathcal{G}_0^L = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \left\{ \frac{-\mathbf{L}_{n\lambda}(-i\lambda)\mathbf{L}_{n\lambda 0}(i\lambda))}{2(1+\delta_{0n})\pi k^2} - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\, \delta(z-z_0)\hat{z}\hat{z}_0 \right\}, \quad (47)$$

with

$$
\begin{aligned}
\mathcal{G}_A^- = \frac{\exp[\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi}\Bigg(& \frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{r}\hat{r}_0 \\
& + \frac{n^2}{rr_0}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\,\hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r}J_n(\lambda r)\frac{n}{r_0}J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\,\hat{r}\hat{\phi}_0 \\
& \pm \frac{\partial}{\partial r_0}J_n(\lambda r_0)\frac{n}{r}J_n(\lambda r)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\,\hat{r}_0\hat{\phi} - \lambda^2 J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{z}\hat{z}_0 \\
& - \lambda\frac{\partial}{\partial r}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{r}\hat{z}_0 + \lambda\frac{\partial}{\partial r_0}J_n(\lambda r_0)J_n(\lambda r)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\,\hat{r}_0\hat{z} \\
& \mp i\lambda\frac{n}{r}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{\phi}\hat{z}_0 \pm \lambda\frac{n}{r_0}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\,\hat{\phi}_0\hat{z}\Bigg). \qquad (48)
\end{aligned}
$$

Now that the longitudinal and traverse parts of the Green's function have been identified, we can start the regularization process.

3.5 Regularized problem

The Green's function derived in the previous section contains singularities, in which the one of the longitudinal part is the strongest. To solve the problem of this singularity, both parts of the Green's function need to be regularized. This will be done by introducing two low pass filters, by the method introduced in Ref. [3]. Here the physical meaning of these filters have already been described, with the parameter $\frac{1}{\Lambda_L}$ being proportional to the size of the source. There the filters are used in momentum space, which corresponds to the κ domain in our solutions. The singularities are connected to large κ values, which are made to die out by the filters. In physical therms, this grants a finite size to the source, which was originally regarded as a point source, as mentioned above in the meaning of the parameter. The actual filters for the transverse and longitudinal parts respectively are

$$f^T(\Lambda_T,\kappa) = \frac{\Lambda_T^2}{\Lambda_T^2+\kappa^2} \qquad (49)$$

and

$$f^L(\Lambda_L,\kappa) = \frac{\Lambda_L^4}{\Lambda_L^4+\kappa^4}. \qquad (50)$$

As the non integrable singularity of the longitudinal part is stronger than the one in the transverse part, which is integrable, the low pass filter that is used has to be stronger as well. Adding these to the Green's function before the integration to h was done, Eqs. (39) and (40) gives the following regularized Green's functions

$$\tilde{\mathcal{G}}_0^T = \int_{-\infty}^{\infty}\mathrm{d}h\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{\Lambda_T^2}{\Lambda_T^2+\kappa^2}\left\{\frac{\mathbf{N}_{n\lambda}(h)\mathbf{N}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\lambda\,(\kappa^2-k^2)} + \frac{\mathbf{M}_{n\lambda}(h)\mathbf{M}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\lambda\,(\kappa^2-k^2)}\right\} \qquad (51)$$

for the transverse part and

$$\tilde{\mathcal{G}}_0^L = \int_{-\infty}^{\infty}\mathrm{d}h\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{\Lambda_L^4}{\Lambda_L^4+\kappa^4}\frac{-\lambda\mathbf{L}_{n\lambda}(h)\mathbf{L}_{n\lambda 0}(-h)}{(1+\delta_{0n})2\pi^2\,(h^2+\lambda^2)\,k^2} \qquad (52)$$

for the longitudinal part.

The filter adds an additional simple poles at respectively $h = \pm i\sqrt{\Lambda_T^2 + \lambda^2}$ and $h = \pm\sqrt{\pm i\Lambda_L^2 - \lambda^2}$ to the previously calculated integrals. By taking these into account when using the residue theorem, as can be found in Appendix C the following expressions are found. For $z > z_0$

$$\tilde{\mathcal{G}_0^T} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^\infty \frac{\Lambda_T^2}{2(1+\delta_{0n})\pi\lambda\left(k^2+\Lambda_T^2\right)} \left\{ \frac{i\left(\mathbf{M}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right) + \mathbf{N}_{n\lambda}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(-\sqrt{k^2-\lambda^2}\right)\right)}{\sqrt{k^2-\lambda^2}} - \frac{\mathbf{M}_{n\lambda}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right) + \mathbf{N}_{n\lambda}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)}{\sqrt{\Lambda_T^2+\lambda^2}} + \frac{\Lambda_T^2}{\Lambda_T^2+k^2}\mathcal{G}_A^+ \right\} \tag{53}$$

and

$$\tilde{\mathcal{G}}_0^L = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^\infty \left\{ \frac{-\mathbf{L}_{n\lambda}(i\lambda)\mathbf{L}_{n\lambda 0}(-i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{\lambda i\mathbf{L}_{n\lambda}(\sqrt{i\Lambda_L^2-\lambda^2})\mathbf{L}_{n\lambda 0}(-\sqrt{i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{i\Lambda_L^2-\lambda^2}} + \frac{\lambda i\mathbf{L}_{n\lambda}(\sqrt{-i\Lambda_L^2-\lambda^2})\mathbf{L}_{n\lambda 0}(-\sqrt{-i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{-i\Lambda_L^2-\lambda^2}} \right\}. \tag{54}$$

Likewise for $z < z_0$

$$\tilde{\mathcal{G}_0^T} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^\infty \frac{\Lambda_T^2}{2(1+\delta_{0n})\pi\lambda\left(k^2+\Lambda_T^2\right)} \left\{ \frac{i\left(\mathbf{M}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right) + \mathbf{N}_{n\lambda}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(\sqrt{k^2-\lambda^2}\right)\right)}{\sqrt{k^2-\lambda^2}} - \frac{\mathbf{M}_{n\lambda}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{M}_{n\lambda 0}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right) + \mathbf{N}_{n\lambda}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_{n\lambda 0}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)}{\sqrt{\Lambda_T^2+\lambda^2}} + \frac{\Lambda_T^2}{\Lambda_T^2+k^2}\mathcal{G}_A^- \right\} \tag{55}$$

and

$$\tilde{\mathcal{G}}_0^L = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^\infty \left\{ \frac{-\mathbf{L}_{n\lambda}(-i\lambda)\mathbf{L}_{n\lambda 0}(i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{\lambda i\mathbf{L}_{n\lambda}(-\sqrt{i\Lambda_L^2-\lambda^2})\mathbf{L}_{n\lambda 0}(\sqrt{i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{i\Lambda_L^2-\lambda^2}} + \frac{\lambda i\mathbf{L}_{n\lambda}(-\sqrt{-i\Lambda_L^2-\lambda^2})\mathbf{L}_{n\lambda 0}(\sqrt{-i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{-i\Lambda_L^2-\lambda^2}} \right\}. \tag{56}$$

This concludes the regularization of the free space Green's function. In the next chapters some important geometries will be treated.

4 Green's function for a semi-infinite medium

The first geometry we will consider is a semi-infinite absorbing dielectric medium. For this problem the Green's function can be divided in a part for free space, $\mathcal{G}_0$, as discussed above, and a scattering part,

$\mathcal{G}_s$. The full Green's function is then

$$\mathcal{G}_e = \mathcal{G}_0 + \mathcal{G}_s \tag{57}$$

In this section the scattering part, $\mathcal{G}_s$, will be calculated for the semi-infinite medium. The medium will divide the space into two regions, the first above the medium and the second within it. This can be seen in Fig. (1). These will be respectively called regions 1 and 2, the scattered part of the Green's

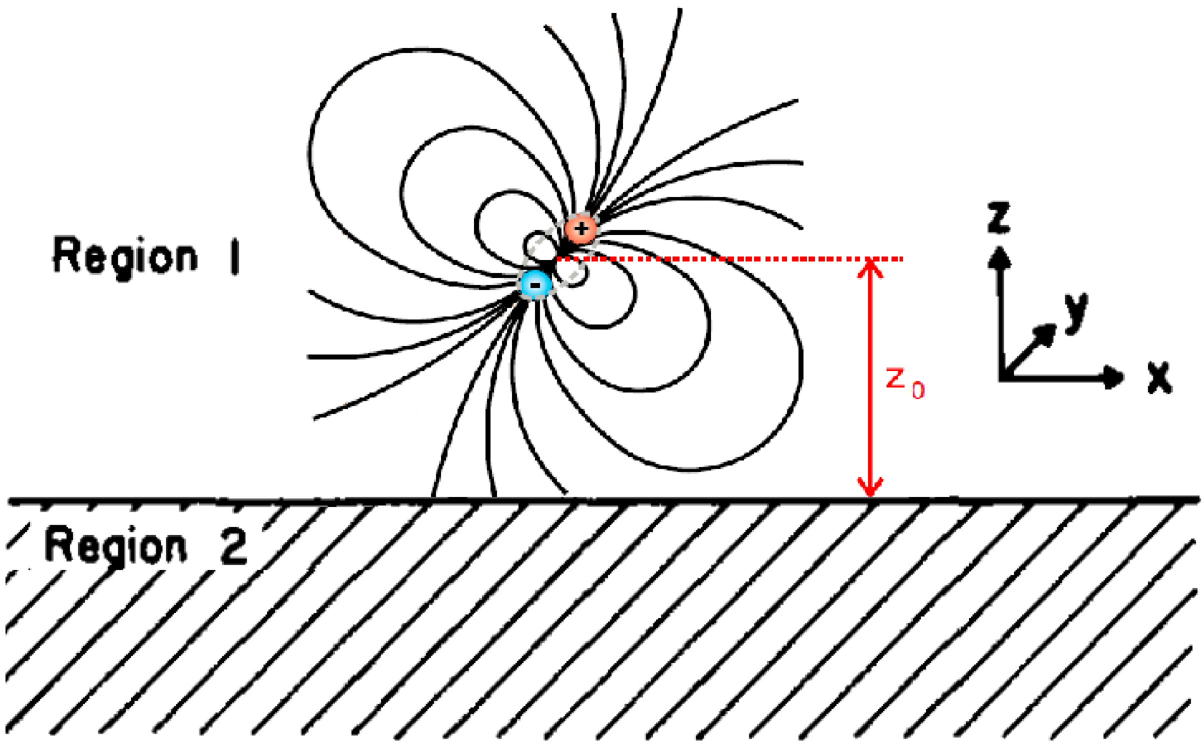

Figure 1: A semi-infinite medium up to $z = 0$ with a dipole above it. The figure is adapted from Ref. [4] and [9].

function is now denoted as $\mathcal{G}_s^{(ij)}$, in which $i, j \in \{1, 2\}$, denoting the region in which the Green's function is evaluated and region in which the dipole is positioned respectively. The boundary between the two regions is at $z = 0$ without loss of generality. This gives the following functions for a dipole placed above the slab, in which $h_i = \sqrt{k_i^2 - \lambda^2}$ for compactness of notation, the free space Green's function is also given in this new notation for $0 < z < z_0$, with z_0 the position of the dipole along the z-axis,

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2(1+\delta_{0n})\pi\lambda h_1} \{\mathbf{M}_{n\lambda}(-h_1)\mathbf{M}_{n\lambda 0}(h_1) + \mathbf{N}_{n\lambda}(-h_1)\mathbf{N}_{n\lambda 0}(h_1)\}. \tag{58}$$

The scattering part above the interface, where the dipole is located is

$$\mathcal{G}_s^{(11)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2(1+\delta_{0n})\pi\lambda h_1} \{a\mathbf{M}_{n\lambda}(h_1)\mathbf{M}_{n\lambda 0}(h_1) + c\mathbf{N}_{n\lambda}(h_1)\mathbf{N}_{n\lambda 0}(h_1)\} \tag{59}$$

and below the interface

$$\mathcal{G}_s^{(21)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2(1+\delta_{0n})\pi\lambda h_1} \{b\mathbf{M}_{n\lambda}(-h_2)\mathbf{M}_{n\lambda 0}(h_1) + d\mathbf{N}_{n\lambda}(-h_2)\mathbf{N}_{n\lambda 0}(h_1)\}. \tag{60}$$

The terms $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ are chosen instead of $\mathbf{M}_{n\lambda 0}\left(-h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(-h_1\right)$ in order to match the free space Green's function at the boundary. The choice between $\mathbf{M}_{n\lambda}\left(h_1\right)$ and $\mathbf{M}_{n\lambda}\left(-h_1\right)$ is made with the concern of wanting the functions to die out when going to $z = \infty$ and $z = -\infty$ respectively, as required by the radiation boundary condition. This is similar to the methods used in Ref. [5] and [6]. The coefficients a, b, c and d are the scattering coefficients that will be determined below.

The scattering coefficients will be found by looking at the boundary conditions for the electric and magnetic fields. The tangential electric field, and thus the tangential component of the Green's function, has to be continues at the boundary. The same goes for the magnetic field, which is proportional to the curl of the electric field, as long as there is no current present across the boundary. Thus employing the boundary conditions, first for the electric field, which is

$$\hat{z} \times \left[\mathcal{G}_0 + \mathcal{G}_s^{(11)}\right] = \hat{z} \times \mathcal{G}_s^{(21)}. \tag{61}$$

By expanding all the terms, the above mentioned equation translates to

$$\begin{aligned}
\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[\left\{\mathbf{M}_{n\lambda}\left(-h_1\right) + a\mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\
\left. + \left\{\mathbf{N}_{n\lambda}\left(-h_1\right) + c\mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right] = \\
\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[b\mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\
\left. + d\mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right].
\end{aligned} \tag{62}$$

If we then compare the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ we get

$$1 + a = b \tag{63}$$

and

$$\frac{h_1}{k_1}\left(-1 + c\right) = -\frac{h_2}{k_2} d. \tag{64}$$

Two more equations are necessary to uniquely determine the coefficients. For this we use the continuity of the tangential magnetic field at the interfaces

$$\frac{1}{\mu_1} \hat{z} \times \nabla \times \left[\mathcal{G}_0 + \mathcal{G}_s^{(11)}\right] = \frac{1}{\mu_2} \hat{z} \times \nabla \times \mathcal{G}_s^{(21)}. \tag{65}$$

When expanding all the terms in Eq. (65) we get

$$\begin{aligned}
\frac{1}{\mu_1} \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \nabla \times \left[\left\{\mathbf{M}_{n\lambda}\left(-h_1\right) + a\mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\
\left. + \left\{\mathbf{N}_{n\lambda}\left(-h_1\right) + c\mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right] = \\
\frac{1}{\mu_2} \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \nabla \times \left[b\mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\
\left. + d\mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right].
\end{aligned} \tag{66}$$

We can then use the relations between $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ to get

$$\frac{k_1}{\mu_1}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\left[\left\{\mathbf{N}_{n\lambda}\left(-h_1\right)+a\mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.$$

$$\left.+\left\{\mathbf{M}_{n\lambda}\left(-h_1\right)+c\mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]=$$

$$\frac{k_2}{\mu_2}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\left[b\mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.$$

$$\left.+d\mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]. \quad (67)$$

From once again comparing the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ we get

$$\frac{h_1}{\mu_1}\left(-1+a\right)=-\frac{h_2}{\mu_2}b \quad (68)$$

and

$$\frac{k_1}{\mu_1}\left(1+c\right)=\frac{k_2}{\mu_2}d. \quad (69)$$

The above mentioned equations then give the following answer, Eq. (70), for the scattering coefficients, thus completing the solution.

$$\begin{cases} a=\frac{h_1\mu_2-h_2\mu_1}{h_1\mu_2-h_2\mu_1} \\ b=\frac{2h_1\mu_2}{h_1\mu_2-h_2\mu_1} \\ c=\frac{k_2^2h_1\mu_1+(1-2\mu_1)k_1^2h_2\mu_2}{k_2^2h_1\mu_1+k_1^2h_2\mu_2} \\ d=\frac{2\mu_1\mu_2k_1k_2h_1}{k_2^2h_1\mu_1+k_1^2h_2\mu_2} \end{cases} \quad (70)$$

Unlike with the free space Green's function, the scattering part does not need to be regularized. Since the distance from the source to the interface is chosen much larger than the size of the source. This choice is made to prevent distortion to the current distribution of the regularized dipole that was found for free-space, which would bring difficulties beyond the scope of this report. Thus the source will be far away from the surface compared to its size, allowing the dipole to still be approximated as a point dipole with regard to scattering, thus the scattering part of the Green's function does not need to be regularized.

5 Green's function for a thin film

Another important geometry to look at, is that of a thin film. As discussed before, this is an important geometry when discussing problems concerning LEDs [4]. The Green's function in the source medium can for this problem once again be divided in a part for free space and a scattering part. In this chapter the scattering part for a thin film will be calculated in a similar way to the previous section. We will look at a thin film that is infinite in the $\hat{r}$ direction. This will divide the space into three regions, the ones above, in and below the thin film, as can be seen in Fig. (2). These will be respectively called regions 1, 2 and 3. The scattered part of the Green's function will now be denoted as $\mathcal{G}_s^{(ij)}$, in which $i,j\in\{1,2,3\}$. Here the first number indicates the region in which the Green's function is evaluated and the second indicates the region in which the dipole is located. This gives the following expressions in which $h_i=\sqrt{k_i^2-\lambda^2}$ for compactness of notation, once again with $i\in\{1,2,3\}$ indicating the region. The same construction method has been used as with the case of the slab. The three different situations that are considered

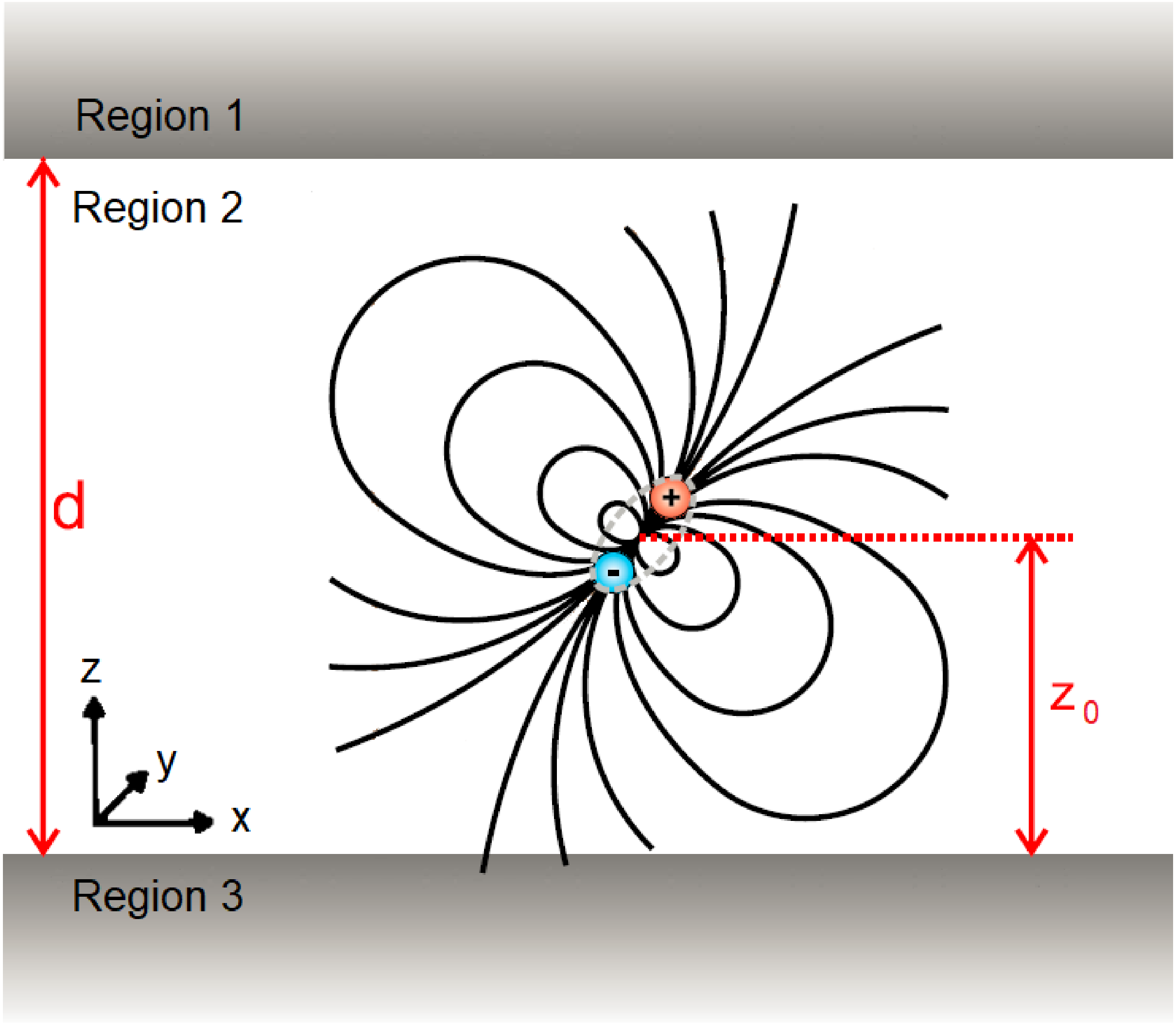

Figure 2: A film, of thickness d, dividing the space into three regions with a dipole inside the film, the figure is adapted from Ref. [4] and [9].

is the dipole being positioned in either region 1, 2 or 3, in which the second is the most important when considering LEDs.

For the dipole in region 1 we get

$$\mathcal{G}_s^{(11)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left\{a_1 \mathbf{M}_{n\lambda}\left(h_1\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + c_1 \mathbf{N}_{n\lambda}\left(h_1\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right\}, \tag{71}$$

$$\mathcal{G}_s^{(31)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left\{b_3 \mathbf{M}_{n\lambda}\left(-h_3\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + d_3 \mathbf{N}_{n\lambda}\left(-h_3\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right\}, \tag{72}$$

$$\mathcal{G}_s^{(21)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left\{a_2 \mathbf{M}_{n\lambda}\left(h_2\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + c_2 \mathbf{N}_{n\lambda}\left(h_2\right)\mathbf{N}_{n\lambda 0}\left(h_1\right) + \right.$$
$$\left. b_2 \mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + d_2 \mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right\}. \tag{73}$$

For this case once more the free space Green's function for $z_1 < z < z_0$, with z_1 the position of the boundary between region 1 and 2 is expressed as

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left\{\mathbf{M}_{n\lambda}\left(-h_1\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + \mathbf{N}_{n\lambda}\left(-h_1\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right\}. \quad (74)$$

Secondly, for the case of the dipole being located in region 2 we have

$$\mathcal{G}_s^{(12)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \left(\mathbf{M}_{n\lambda}\left(h_1\right)\left\{a_1^{+}\mathbf{M}_{n\lambda 0}\left(h_2\right) + a_1^{-}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\} + \mathbf{N}_{n\lambda}\left(h_1\right)\left\{c_1^{+}\mathbf{N}_{n\lambda 0}\left(h_2\right) + c_1^{-}\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right), \quad (75)$$

$$\mathcal{G}_s^{(32)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \left(\mathbf{M}_{n\lambda}\left(-h_3\right)\left\{b_3^{+}\mathbf{M}_{n\lambda 0}\left(h_2\right) + b_3^{-}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\} + \mathbf{N}_{n\lambda}\left(-h_3\right)\left\{d_3^{+}\mathbf{N}_{n\lambda 0}\left(h_2\right) + d_3^{-}\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right), \quad (76)$$

$$\mathcal{G}_s^{(22)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \left(\mathbf{M}_{n\lambda}\left(h_2\right)\left\{a_2^{+}\mathbf{M}_{n\lambda 0}\left(h_2\right) + a_2^{-}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\} + \mathbf{N}_{n\lambda}\left(h_2\right)\left\{c_2^{+}\mathbf{N}_{n\lambda 0}\left(h_2\right) + c_2^{-}\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\} + \mathbf{M}_{n\lambda}\left(-h_2\right)\left\{b_2^{+}\mathbf{M}_{n\lambda 0}\left(h_2\right) + b_2^{-}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\} + \mathbf{N}_{n\lambda}\left(-h_2\right)\left\{d_2^{+}\mathbf{N}_{n\lambda 0}\left(h_2\right) + d_2^{-}\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right). \quad (77)$$

The free space Green's function for $z_0 < z < z_1$ is

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left(\mathbf{M}_{n\lambda}\left(h_2\right)\mathbf{M}_{n\lambda 0}\left(-h_2\right) + \mathbf{N}_{n\lambda}\left(h_2\right)\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right) \quad (78)$$

and for $z_3 < z < z_0$, where z_3 is the position of the boundary between region 2 and 3, Fig. (2),

$$\mathcal{G}_0 = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left(\mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(h_2\right) + \mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(h_2\right)\right). \quad (79)$$

Lastly for the dipole being positioned in region 3 we write

$$\mathcal{G}_s^{(13)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_3} \left\{a_1\mathbf{M}_{n\lambda}\left(h_1\right)\mathbf{M}_{n\lambda 0}\left(-h_3\right) + c_1\mathbf{N}_{n\lambda}\left(h_1\right)\mathbf{N}_{n\lambda 0}\left(-h_3\right)\right\}, \quad (80)$$

$$\mathcal{G}_s^{(33)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_3} \left\{b_3\mathbf{M}_{n\lambda}\left(-h_3\right)\mathbf{M}_{n\lambda 0}\left(-h_3\right) + d_3\mathbf{N}_{n\lambda}\left(-h_3\right)\mathbf{N}_{n\lambda 0}\left(-h_3\right)\right\}, \quad (81)$$

$$\mathcal{G}_s^{(23)} = \int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_3} \left\{a_2\mathbf{M}_{n\lambda}\left(h_2\right)\mathbf{M}_{n\lambda 0}\left(-h_3\right) + c_2\mathbf{N}_{n\lambda}\left(h_2\right)\mathbf{N}_{n\lambda 0}\left(-h_3\right) + b_2\mathbf{M}_{n\lambda}\left(-h_2\right)\mathbf{M}_{n\lambda 0}\left(-h_3\right) + d_2\mathbf{N}_{n\lambda}\left(-h_2\right)\mathbf{N}_{n\lambda 0}\left(-h_3\right)\right\} \quad (82)$$

and the free space Green's function for $z_0 < z < z_3$

$$\mathcal{G}_0 = \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \left\{\mathbf{M}_{n\lambda}\left(h_3\right)\mathbf{M}_{n\lambda 0}\left(-h_3\right) + \mathbf{N}_{n\lambda}\left(h_3\right)\mathbf{N}_{n\lambda 0}\left(-h_3\right)\right\}. \tag{83}$$

To determine the coefficients we will first look at the situation where the dipole is located in the first region. To get the coefficients, we'll employ the boundary conditions on the boundary between region 1 and 2 and between 2 and 3, the same as which has been done with the slab. Firstly at the 1-2 boundary, $z = z_1$

$$\hat{z} \times \left[\mathcal{G}_0 + \mathcal{G}_s^{(11)}\right] = \hat{z} \times \mathcal{G}_s^{(21)}. \tag{84}$$

Substituting the fields into the expression gives

$$\begin{aligned} \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[\left\{\mathbf{M}_{n\lambda}\left(-h_1\right) + a_1 \mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\ \left. + \left\{\mathbf{N}_{n\lambda}\left(-h_1\right) + c_1 \mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right] = \\ \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[\left\{a_2 \mathbf{M}_{n\lambda}\left(h_2\right) + b_2 \mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\ \left. + \left\{c_2 \mathbf{N}_{n\lambda}\left(h_2\right) + d_2 \mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]. \end{aligned} \tag{85}$$

If we then compare the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ we get

$$e^{-ih_1 z_1} + a_1 e^{ih_1 z_1} = a_2 e^{ih_2 z_1} + b_2 e^{-ih_2 z_1} \tag{86}$$

and

$$\frac{h_1}{k_1}\left(-e^{-ih_1 z_1} + c_1 e^{ih_1 z_1}\right) = \frac{h_2}{k_2}\left(c_2 e^{ih_2 z_1} - d_2 e^{-ih_2 z_1}\right). \tag{87}$$

Likewise at the 2-3 boundary, $z = z_3$

$$\hat{z} \times \mathcal{G}_s^{(21)} = \hat{z} \times \mathcal{G}_s^{(31)}. \tag{88}$$

Substituting the fields into the expression gives

$$\begin{aligned} \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[\left\{a_2 \mathbf{M}_{n\lambda}\left(h_2\right) + b_2 \mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right. \\ \left. + \left\{c_2 \mathbf{N}_{n\lambda}\left(h_2\right) + d_2 \mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right] = \\ \int_0^{\infty} \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1} \hat{z} \times \left[b_3 \mathbf{M}_{n\lambda}\left(-h_3\right)\mathbf{M}_{n\lambda 0}\left(h_1\right) + d_3 \mathbf{N}_{n\lambda}\left(-h_3\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]. \end{aligned} \tag{89}$$

Once again we compare the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ we get

$$a_2 e^{ih_2 z_3} + b_2 e^{-ih_2 z_3} = b_3 e^{-ih_3 z_3} \tag{90}$$

and

$$\frac{h_2}{k_2}\left(c_2 e^{ih_2 z_3} - d_2 e^{-ih_2 z_3}\right) = -\frac{h_3}{k_3} d_3 e^{-ih_3 z_3}. \tag{91}$$

Four more equations are necessary to uniquely determine the coefficients. To get these equations we use the continuity of the tangential magnetic field at the interfaces. Once again firstly for the interface between medium 1 and 2

$$\frac{1}{\mu_1}\hat{z}\times\nabla\times\left[\mathcal{G}_0+\mathcal{G}_s^{(11)}\right]=\frac{1}{\mu_2}\hat{z}\times\nabla\times\mathcal{G}_s^{(21)}, \tag{92}$$

which translates to

$$\begin{aligned}\frac{1}{\mu_1}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\nabla\times\left[\left\{\mathbf{M}_{n\lambda}\left(-h_1\right)+a_1\mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.\\ \left.+\left\{\mathbf{N}_{n\lambda}\left(-h_1\right)+c_1\mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]=\\ \frac{1}{\mu_2}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\nabla\times\left[\left\{a_2\mathbf{M}_{n\lambda}\left(h_2\right)+b_2\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.\\ \left.+\left\{c_2\mathbf{N}_{n\lambda}\left(h_2\right)+d_2\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right].\end{aligned} \tag{93}$$

We can then use the relations [Eqs. (21-23)] between $\mathbf{M}$ and $\mathbf{N}$ to get

$$\begin{aligned}\frac{k_1}{\mu_1}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\left[\left\{\mathbf{N}_{n\lambda}\left(-h_1\right)+a_1\mathbf{N}_{n\lambda}\left(h_1\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.\\ \left.+\left\{\mathbf{M}_{n\lambda}\left(-h_1\right)+c_1\mathbf{M}_{n\lambda}\left(h_1\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]=\\ \frac{k_2}{\mu_2}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\left[\left\{a_2\mathbf{N}_{n\lambda}\left(h_2\right)+b_2\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.\\ \left.+\left\{c_2\mathbf{M}_{n\lambda}\left(h_2\right)+d_2\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right].\end{aligned} \tag{94}$$

Once again from comparing the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(h_1\right)$ we get

$$\frac{h_1}{\mu_1}\left(-e^{-ih_1z_1}+a_1e^{ih_1z_1}\right)=\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_1}-b_2e^{-ih_2z_1}\right) \tag{95}$$

and

$$\frac{k_1}{\mu_1}\left(e^{-ih_1z_1}+c_1e^{ih_1z_1}\right)=\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_1}+d_2e^{-ih_2z_1}\right). \tag{96}$$

Similarly, at the interface between medium 2 and 3 we have

$$\frac{1}{\mu_2}\hat{z}\times\nabla\times\mathcal{G}_s^{(21)}=\frac{1}{\mu_3}\hat{z}\times\nabla\times\mathcal{G}_s^{(31)} \tag{97}$$

which gives

$$\begin{aligned}\frac{1}{\mu_2}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\nabla\times\left[\left\{a_2\mathbf{M}_{n\lambda}\left(h_2\right)+b_2\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_1\right)\right.\\ \left.+\left\{c_2\mathbf{N}_{n\lambda}\left(h_2\right)+d_2\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_1\right)\right]=\\ \frac{1}{\mu_3}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_1}\hat{z}\times\nabla\times\left[b_3\mathbf{M}_{n\lambda}\left(-h_3\right)\mathbf{M}_{n\lambda 0}\left(h_1\right)+d_3\mathbf{N}_{n\lambda}\left(-h_3\right)\mathbf{N}_{n\lambda 0}\left(h_1\right)\right].\end{aligned} \tag{98}$$

We can then use the relations between $\mathbf{M}$ and $\mathbf{N}$ again to get

$$\frac{k_2}{\mu_2}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2(1+\delta_{0n})\pi\lambda h_1} \hat{z}\times\nabla\times\left[\left\{a_2\mathbf{N}_{n\lambda}(h_2)+b_2\mathbf{N}_{n\lambda}(-h_2)\right\}\mathbf{M}_{n\lambda 0}(h_1)\right.$$
$$\left. +\left\{c_2\mathbf{M}_{n\lambda}(h_2)+d_2\mathbf{M}_{n\lambda}(-h_2)\right\}\mathbf{N}_{n\lambda 0}(h_1)\right] =$$
$$\frac{k_3}{\mu_3}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2(1+\delta_{0n})\pi\lambda h_1} \hat{z}\times\nabla\times\left[b_3\mathbf{N}_{n\lambda}(-h_3)\mathbf{M}_{n\lambda 0}(h_1)+d_3\mathbf{M}_{n\lambda}(-h_3)\mathbf{N}_{n\lambda 0}(h_1)\right]. \tag{99}$$

When once again comparing the coefficients for $\mathbf{M}_{n\lambda 0}(h_1)$ and $\mathbf{N}_{n\lambda 0}(h_1)$ we get

$$\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_3}-b_2e^{-ih_2z_3}\right)=\frac{h_3}{\mu_3}\left(-b_3e^{-ih_3z_3}\right) \tag{100}$$

and

$$\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_3}+d_2e^{-ih_2z_3}\right)=\frac{k_3}{\mu_3}\left(d_3e^{-ih_3z_3}\right). \tag{101}$$

To get the coefficients we now only need to solve the following set of equations

$$\begin{cases}
a_1e^{ih_1z_1}-a_2e^{ih_2z_1}-b_2e^{-ih_2z_1}=-e^{-ih_1z_1} & [1]\\
\frac{h_1}{k_1}c_1e^{ih_1z_1}-\frac{h_2}{k_2}\left(c_2e^{ih_2z_1}-d_2e^{-ih_2z_1}\right)=\frac{h_1}{k_1}e^{-ih_1z_1} & [2]\\
a_2e^{ih_2z_3}+b_2e^{-ih_2z_3}-b_3e^{-ih_3z_3}=0 & [3]\\
\frac{h_2}{k_2}\left(c_2e^{ih_2z_3}-d_2e^{-ih_2z_3}\right)+\frac{h_3}{k_3}d_3e^{-ih_3z_3}=0 & [4]\\
\frac{h_1}{\mu_1}a_1e^{ih_1z_1}-\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_1}-b_2e^{-ih_2z_1}\right)=\frac{h_1}{\mu_1}e^{-ih_1z_1} & [5]\\
\frac{k_1}{\mu_1}c_1e^{ih_1z_1}-\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_1}+d_2e^{-ih_2z_1}\right)=-\frac{k_1}{\mu_1}e^{-ih_1z_1} & [6]\\
\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_3}-b_2e^{-ih_2z_3}\right)+\frac{h_3}{\mu_3}b_3e^{-ih_3z_3}=0 & [7]\\
\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_3}+d_2e^{-ih_2z_3}\right)-\frac{k_3}{\mu_3}\left(d_3e^{-ih_3z_3}\right)=0 & [8]
\end{cases} \tag{102}$$

In case of the dipole being in the third region, the calculations are symmetric to those of the dipole being in the first region. This gives the following system of equations:

$$\begin{cases}
b_3e^{-ih_3z_3}-a_2e^{ih_2z_3}-b_2e^{-ih_2z_3}=-e^{ih_3z_3} & [1]\\
-\frac{h_3}{k_3}d_3e^{-ih_3z_3}-\frac{h_2}{k_2}\left(c_2e^{ih_2z_3}-d_2e^{-ih_2z_3}\right)=-\frac{h_3}{k_3}e^{ih_3z_3} & [2]\\
a_2e^{ih_2z_1}+b_2e^{-ih_2z_1}-a_1e^{ih_1z_1}=0 & [3]\\
\frac{h_2}{k_2}\left(c_2e^{ih_2z_1}-d_2e^{-ih_2z_1}\right)-\frac{h_1}{k_1}c_1e^{ih_1z_1}=0 & [4]\\
-\frac{h_3}{\mu_3}b_3e^{-ih_3z_3}-\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_3}-b_2e^{-ih_2z_3}\right)=-\frac{h_3}{\mu_3}e^{ih_3z_3} & [5]\\
\frac{k_3}{\mu_3}d_3e^{-ih_3z_3}-\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_3}+d_2e^{-ih_2z_3}\right)=\frac{k_3}{\mu_3}e^{-ih_3z_3} & [6]\\
\frac{h_2}{\mu_2}\left(a_2e^{ih_2z_1}-b_2e^{-ih_2z_1}\right)-\frac{h_1}{\mu_1}\left(a_1e^{ih_1z_1}\right)=0 & [7]\\
\frac{k_2}{\mu_2}\left(c_2e^{ih_2z_1}+d_2e^{-ih_2z_1}\right)-\frac{k_1}{\mu_1}\left(c_1e^{ih_1z_1}\right)=0 & [8]
\end{cases} \tag{103}$$

Lastly we will consider the case of the dipole being in the second region. The coefficients for this case will be calculated by the same method as before by applying the boundary conditions and the continuity of the tangential magnetic field at the interfaces. Firstly, the boundary condition at the 1-2 boundary, $z=z_1$ is

$$\hat{z}\times\left[\mathcal{G}_0+\mathcal{G}_s^{(22)}\right]=\hat{z}\times\mathcal{G}_s^{(12)}. \tag{104}$$

Substituting the fields from Eqs. (75), (77) and (78) into the expression gives

$$\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \hat{z} \times \left[\left\{\mathbf{M}_{n\lambda}\left(h_2\right)+a_2^- \mathbf{M}_{n\lambda}\left(h_2\right)+b_2^- \mathbf{M}_{n\lambda}\left(-h_2\right)\right\} \mathbf{M}_{n\lambda 0}\left(-h_2\right)\right.$$
$$+\left\{a_2^+ \mathbf{M}_{n\lambda}\left(h_2\right)+b_2^+ \mathbf{M}_{n\lambda}\left(-h_2\right)\right\} \mathbf{M}_{n\lambda 0}\left(h_2\right)+$$
$$\left.\left\{\mathbf{N}_{n\lambda}\left(h_2\right)+c_2^- \mathbf{N}_{n\lambda}\left(h_2\right)+d_2^- \mathbf{N}_{n\lambda}\left(-h_2\right)\right\} \mathbf{N}_{n\lambda 0}\left(-h_2\right)+\left\{c_2^+ \mathbf{N}_{n\lambda}\left(h_2\right)+d_2^+ \mathbf{N}_{n\lambda}\left(-h_2\right)\right\} \mathbf{N}_{n\lambda 0}\left(h_2\right)\right]=$$
$$\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \hat{z} \times \left[\mathbf{M}_{n\lambda}\left(h_1\right)\left\{a_1^+ \mathbf{M}_{n\lambda 0}\left(h_2\right)+a_1^- \mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}\right.$$
$$\left.+\mathbf{N}_{n\lambda}\left(h_1\right)\left\{c_1^+ \mathbf{N}_{n\lambda 0}\left(h_2\right)+c_1^- \mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right]. \tag{105}$$

If we then compare the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$, $\mathbf{N}_{n\lambda 0}\left(h_1\right)$, $\mathbf{M}_{n\lambda 0}\left(-h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(-h_1\right)$ we get

$$a_2^+ e^{ih_2 z_1}+b_2^+ e^{-ih_2 z_1}=a_1^+ e^{ih_1 z_1}, \tag{106}$$

$$\frac{h_2}{k_2}\left(c_2^+ e^{ih_2 z_1}-d_2^+ e^{-ih_2 z_1}\right)=\frac{h_1}{k_1}\left(c_1^+ e^{ih_1 z_1}\right), \tag{107}$$

$$e^{ih_2 z_1}+a_2^- e^{ih_2 z_1}+b_2^- e^{-ih_2 z_1}=a_1^- e^{ih_1 z_1} \tag{108}$$

and

$$\frac{h_2}{k_2}\left(e^{ih_2 z_1}+c_2^- e^{ih_2 z_1}-d_2^- e^{-ih_2 z_1}\right)=\frac{h_1}{k_1}\left(c_1^- e^{ih_1 z_1}\right). \tag{109}$$

And at the 2-3 boundary, $z=z_3$,

$$\hat{z} \times \left[\mathcal{G}_0+\mathcal{G}_s^{(22)}\right]=\hat{z} \times \mathcal{G}_s^{(32)}. \tag{110}$$

Substituting the fields from Eqs. (76), (77) and (79) into the expression gives

$$\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \hat{z} \times \left[\left\{a_2^- \mathbf{M}_{n\lambda}\left(h_2\right)+b_2^- \mathbf{M}_{n\lambda}\left(-h_2\right)\right\} \mathbf{M}_{n\lambda 0}\left(-h_2\right)\right.$$
$$+\left\{\mathbf{M}_{n\lambda}\left(-h_2\right)+a_2^+ \mathbf{M}_{n\lambda}\left(h_2\right)+b_2^+ \mathbf{M}_{n\lambda}\left(-h_2\right)\right\} \mathbf{M}_{n\lambda 0}\left(h_2\right)+$$
$$\left.\left\{c_2^- \mathbf{N}_{n\lambda}\left(h_2\right)+d_2^- \mathbf{N}_{n\lambda}\left(-h_2\right)\right\} \mathbf{N}_{n\lambda 0}\left(-h_2\right)+\left\{\mathbf{N}_{n\lambda}\left(-h_2\right)+c_2^+ \mathbf{N}_{n\lambda}\left(h_2\right)+d_2^+ \mathbf{N}_{n\lambda}\left(-h_2\right)\right\} \mathbf{N}_{n\lambda 0}\left(h_2\right)\right]=$$
$$\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2} \hat{z} \times \left[\mathbf{M}_{n\lambda}\left(-h_3\right)\left\{b_3^+ \mathbf{M}_{n\lambda 0}\left(h_2\right)+b_3^- \mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}\right.$$
$$\left.+\mathbf{N}_{n\lambda}\left(-h_3\right)\left\{d_3^+ \mathbf{N}_{n\lambda 0}\left(h_2\right)+d_3^- \mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right]. \tag{111}$$

Once again we compare the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$, $\mathbf{N}_{n\lambda 0}\left(h_1\right)$, $\mathbf{M}_{n\lambda 0}\left(-h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(-h_1\right)$ we get

$$e^{-ih_2 z_3}+a_2^+ e^{ih_2 z_3}+b_2^+ e^{-ih_2 z_3}=b_3^+ e^{-ih_3 z_3}, \tag{112}$$

$$\frac{h_2}{k_2}\left(-e^{-ih_2 z_3}+c_2^+ e^{ih_2 z_3}-d_2^+ e^{-ih_2 z_3}\right)=\frac{h_3}{k_3}\left(-d_3^+ e^{-ih_3 z_3}\right), \tag{113}$$

$$a_2^- e^{ih_2 z_3}+b_2^- e^{-ih_2 z_3}=b_3^- e^{-ih_3 z_3} \tag{114}$$

and

$$\frac{h_2}{k_2}\left(c_2^- e^{ih_2 z_3} - d_2^- e^{-ih_2 z_3}\right) = \frac{h_3}{k_3}\left(-d_3^- e^{-ih_3 z_3}\right). \tag{115}$$

Next we'll consider the continuity of the tangential magnetic field at the interfaces. Once again firstly for the interface between 1 and 2

$$\frac{1}{\mu_2}\hat{z}\times\nabla\times\left[\mathcal{G}_0 + \mathcal{G}_s^{(22)}\right] = \frac{1}{\mu_1}\hat{z}\times\nabla\times\mathcal{G}_s^{(12)}. \tag{116}$$

Which translates to

$$\begin{aligned}
&\frac{1}{\mu_1}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\left[\left\{\mathbf{M}_{n\lambda}\left(h_2\right) + a_2^-\mathbf{M}_{n\lambda}\left(h_2\right) + b_2^-\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right.\\
&\qquad + \left\{a_2^+\mathbf{M}_{n\lambda}\left(h_2\right) + b_2^+\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_2\right) + \\
&\left.\left\{\mathbf{N}_{n\lambda}\left(h_2\right) + c_2^-\mathbf{N}_{n\lambda}\left(h_2\right) + d_2^-\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(-h_2\right)\left\{+c_2^+\mathbf{N}_{n\lambda}\left(h_2\right) + d_2^+\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_2\right)\right] = \\
&\frac{1}{\mu_2}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\left[\mathbf{M}_{n\lambda}\left(h_1\right)\left\{a_1^+\mathbf{M}_{n\lambda 0}\left(h_2\right) + a_1^-\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}\right.\\
&\qquad\left. + \mathbf{N}_{n\lambda}\left(h_1\right)\left\{c_1^+\mathbf{N}_{n\lambda 0}\left(h_2\right) + c_1^-\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right].
\end{aligned} \tag{117}$$

We again use the relations between $\mathbf{M}$ and $\mathbf{N}$ to get

$$\begin{aligned}
&\frac{k_1}{\mu_1}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\left[\left\{\mathbf{N}_{n\lambda}\left(h_2\right) + a_2^-\mathbf{N}_{n\lambda}\left(h_2\right) + b_2^-\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right.\\
&\qquad + \left\{a_2^+\mathbf{N}_{n\lambda}\left(h_2\right) + b_2^+\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_2\right) + \\
&\left.\left\{\mathbf{M}_{n\lambda}\left(h_2\right) + c_2^-\mathbf{M}_{n\lambda}\left(h_2\right) + d_2^-\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(-h_2\right) + \left\{c_2^+\mathbf{M}_{n\lambda}\left(h_2\right) + d_2^+\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_2\right)\right] = \\
&\frac{k_2}{\mu_2}\int_0^\infty \mathrm{d}\lambda \sum_{n=0}^{\infty} \frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\left[\mathbf{N}_{n\lambda}\left(h_1\right)\left\{a_1^+\mathbf{M}_{n\lambda 0}\left(h_2\right) + a_1^-\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}\right.\\
&\qquad\left. + \mathbf{M}_{n\lambda}\left(h_1\right)\left\{c_1^+\mathbf{N}_{n\lambda 0}\left(h_2\right) + c_1^-\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\right].
\end{aligned} \tag{118}$$

Once again from comparing the coefficients for $\mathbf{M}_{n\lambda 0}\left(h_1\right)$, $\mathbf{N}_{n\lambda 0}\left(h_1\right)$, $\mathbf{M}_{n\lambda 0}\left(-h_1\right)$ and $\mathbf{N}_{n\lambda 0}\left(-h_1\right)$ we get

$$\frac{h_2}{\mu_2}\left(a_2^+ e^{ih_2 z_1} - b_2^+ e^{-ih_2 z_1}\right) = \frac{h_1}{\mu_1}a_1^+ e^{ih_1 z_1}, \tag{119}$$

$$\frac{k_2}{\mu_2}\left(c_2^+ e^{ih_2 z_1} + d_2^+ e^{-ih_2 z_1}\right) = \frac{k_1}{\mu_1}\left(c_1^+ e^{ih_1 z_1}\right), \tag{120}$$

$$\frac{h_2}{\mu_2}\left(e^{ih_2 z_1} + a_2^- e^{ih_2 z_1} - b_2^- e^{-ih_2 z_1}\right) = \frac{h_1}{\mu_1}a_1^- e^{ih_1 z_1} \tag{121}$$

and

$$\frac{k_2}{\mu_2}\left(e^{ih_2 z_1} + c_2^- e^{ih_2 z_1} + d_2^- e^{-ih_2 z_1}\right) = \frac{k_1}{\mu_1}\left(c_1^- e^{ih_1 z_1}\right). \tag{122}$$

Similarly, at the interface between region 2 and 3 we apply

$$\frac{1}{\mu_2}\hat{z}\times\nabla\times\left[\mathcal{G}_0 + \mathcal{G}_s^{(22)}\right] = \frac{1}{\mu_3}\hat{z}\times\nabla\times\mathcal{G}_s^{(32)}. \tag{123}$$

When filling in Eqs. (76), (77) and (79) we get

$$\frac{1}{\mu_2}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\Big[\left\{a_2^-\mathbf{M}_{n\lambda}\left(h_2\right)+b_2^-\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(-h_2\right)$$
$$+\left\{\mathbf{M}_{n\lambda}\left(-h_2\right)+a_2^+\mathbf{M}_{n\lambda}\left(h_2\right)+b_2^+\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_2\right)$$
$$+\left\{c_2^-\mathbf{N}_{n\lambda}\left(h_2\right)+d_2^-\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(-h_2\right)+\left\{\mathbf{N}_{n\lambda}\left(-h_2\right)+c_2^+\mathbf{N}_{n\lambda}\left(h_2\right)+d_2^+\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_2\right)\Big]=$$
$$\frac{1}{\mu_3}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\Big[\mathbf{M}_{n\lambda}\left(-h_3\right)\left\{b_3^+\mathbf{M}_{n\lambda 0}\left(h_2\right)+b_3^-\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}$$
$$+\mathbf{N}_{n\lambda}\left(-h_3\right)\left\{d_3^+\mathbf{N}_{n\lambda 0}\left(h_2\right)+d_3^-\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\Big] \quad (124)$$

and after using the relations between $\mathbf{M}$ and $\mathbf{N}$

$$\frac{k_2}{\mu_2}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\Big[\left\{a_2^-\mathbf{N}_{n\lambda}\left(h_2\right)+b_2^-\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(-h_2\right)$$
$$+\left\{\mathbf{N}_{n\lambda}\left(-h_2\right)+a_2^+\mathbf{N}_{n\lambda}\left(h_2\right)+b_2^+\mathbf{N}_{n\lambda}\left(-h_2\right)\right\}\mathbf{M}_{n\lambda 0}\left(h_2\right)+\left\{c_2^-\mathbf{M}_{n\lambda}\left(h_2\right)+d_2^-\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(-h_2\right)$$
$$+\left\{\mathbf{M}_{n\lambda}\left(-h_2\right)+c_2^+\mathbf{M}_{n\lambda}\left(h_2\right)+d_2^+\mathbf{M}_{n\lambda}\left(-h_2\right)\right\}\mathbf{N}_{n\lambda 0}\left(h_2\right)\Big]=$$
$$\frac{k_3}{\mu_3}\int_0^{\infty}\mathrm{d}\lambda\sum_{n=0}^{\infty}\frac{i}{2\left(1+\delta_{0n}\right)\pi\lambda h_2}\hat{z}\times\nabla\times\Big[\mathbf{N}_{n\lambda}\left(-h_3\right)\left\{b_3^+\mathbf{M}_{n\lambda 0}\left(h_2\right)+b_3^-\mathbf{M}_{n\lambda 0}\left(-h_2\right)\right\}$$
$$+\mathbf{M}_{n\lambda}\left(-h_3\right)\left\{d_3^+\mathbf{N}_{n\lambda 0}\left(h_2\right)+d_3^-\mathbf{N}_{n\lambda 0}\left(-h_2\right)\right\}\Big]. \quad (125)$$

By once again comparing the coefficients of the different $\mathbf{M}_0$ and $\mathbf{N}_0$ functions we get

$$\frac{h_2}{\mu_2}\left(-e^{-ih_2z_3}+a_2^+e^{ih_2z_3}-b_2^+e^{-ih_2z_3}\right)=-\frac{h_3}{\mu_3}b_3^+e^{-ih_3z_3}, \quad (126)$$

$$\frac{k_2}{\mu_2}\left(e^{-ih_2z_3}+c_2^+e^{ih_2z_3}+d_2^+e^{-ih_2z_3}\right)=\frac{k_3}{\mu_3}d_3^+e^{-ih_3z_3}, \quad (127)$$

$$\frac{h_2}{\mu_2}\left(a_2^-e^{ih_2z_3}-b_2^-e^{-ih_2z_3}\right)=-\frac{h_3}{\mu_3}b_3^-e^{-ih_3z_3} \quad (128)$$

and

$$\frac{k_2}{\mu_2}\left(c_2^-e^{ih_2z_3}+d_2^-e^{-ih_2z_3}\right)=\frac{k_3}{\mu_3}d_3^-e^{-ih_3z_3}. \quad (129)$$

For the dipole in the second region we get a double set of equations, one for the + and one for the - coefficients, as seen below

$$\begin{cases}
a_2^+e^{ih_2z_1}+b_2^+e^{-ih_2z_1}-a_1^+e^{ih_1z_1}=0 & [1]\\
\frac{h_2}{k_2}\left(c_2^+e^{ih_2z_1}-d_2^+e^{-ih_2z_1}\right)-\frac{h_1}{k_1}\left(c_1^+e^{ih_1z_1}\right)=0 & [2]\\
a_2^+e^{ih_2z_3}+b_2^+e^{-ih_2z_3}-b_3^+e^{-ih_3z_3}=-e^{-ih_2z_3} & [3]\\
\frac{h_2}{k_2}\left(c_2^+e^{ih_2z_3}-d_2^+e^{-ih_2z_3}\right)-\frac{h_3}{k_3}\left(-d_3^+e^{-ih_3z_3}\right)=\frac{h_2}{k_2}e^{-ih_2z_3} & [4]\\
\frac{h_2}{\mu_2}\left(a_2^+e^{ih_2z_1}-b_2^+e^{-ih_2z_1}\right)-\frac{h_1}{\mu_1}a_1^+e^{ih_1z_1}=0 & [5]\\
\frac{k_2}{\mu_2}\left(c_2^+e^{ih_2z_1}+d_2^+e^{-ih_2z_1}\right)-\frac{k_1}{\mu_1}\left(c_1^+e^{ih_1z_1}\right)=0 & [6]\\
\frac{h_2}{\mu_2}\left(a_2^+e^{ih_2z_3}-b_2^+e^{-ih_2z_3}\right)+\frac{h_3}{\mu_3}b_3^+e^{-ih_3z_3}=\frac{h_2}{\mu_2}e^{-ih_2z_3} & [7]\\
\frac{k_2}{\mu_2}\left(c_2^+e^{ih_2z_3}+d_2^+e^{-ih_2z_3}\right)-\frac{k_3}{\mu_3}d_3^+e^{-ih_3z_3}=-\frac{k_2}{\mu_2}e^{-ih_2z_3} & [8]
\end{cases} \quad (130)$$

$$
\begin{cases}
a_2^- e^{ih_2z_1} + b_2^- e^{-ih_2z_1} - a_1^- e^{ih_1z_1} = -e^{ih_2z_1} & [1] \\
\frac{h_2}{k_2}\left(c_2^- e^{ih_2z_1} - d_2^- e^{-ih_2z_1}\right) - \frac{h_1}{k_1}\left(c_1^- e^{ih_1z_1}\right) = -\frac{h_2}{k_2} e^{ih_2z_1} & [2] \\
a_2^- e^{ih_2z_3} + b_2^- e^{-ih_2z_3} - b_3^- e^{-ih_3z_3} = 0 & [3] \\
\frac{h_2}{k_2}\left(c_2^- e^{ih_2z_3} - d_2^- e^{-ih_2z_3}\right) - \frac{h_3}{k_3}\left(-d_3^- e^{-ih_3z_3}\right) = 0 & [4] \\
\frac{h_2}{\mu_2}\left(a_2^- e^{ih_2z_1} - b_2^- e^{-ih_2z_1}\right) - \frac{h_1}{\mu_1} a_1^- e^{ih_1z_1} = -\frac{h_2}{\mu_2} e^{ih_2z_1} & [5] \\
\frac{k_2}{\mu_2}\left(c_2^- e^{ih_2z_1} + d_2^- e^{-ih_2z_1}\right) - \frac{k_1}{\mu_1}\left(c_1^- e^{ih_1z_1}\right) = -\frac{k_2}{\mu_2} e^{ih_2z_1} & [6] \\
\frac{h_2}{\mu_2}\left(a_2^- e^{ih_2z_3} - b_2^- e^{-ih_2z_3}\right) + \frac{h_3}{\mu_3} b_3^- e^{-ih_3z_3} = 0 & [7] \\
\frac{k_2}{\mu_2}\left(c_2^- e^{ih_2z_3} + d_2^- e^{-ih_2z_3}\right) - \frac{k_3}{\mu_3} d_3^- e^{-ih_3z_3} = 0 & [8]
\end{cases}
\tag{131}
$$

With the sets of equations of Eqs. (102), (103 and (131) we have a solution for the scattered part of the Green's function for a thin film. The sets of equations can be solved with Mathematica. Similar to the case of the semi-infinite medium, this scattered part does not need to be regularized as the dipole is chosen to be much further away from the boundary compared to its size.

6 Conclusion/Outlook

We have used a known regularization method to make the Green's function finite at the location of a dipole, by giving the dipole a finite size [3], to regularize the known solutions of the Green's function in cylindrical coordinates [6]. The solution used can be expanded to describe a stack of multiple thin films, as in [5]. In that case the Green's function can accurately describe the LDOS that is connected to the transmissions rate of light in LEDs [4]. For actual use the integrals and summations that are still present in the solution will need to be solved numerically. This is also necessary to compare the regularized solution with known solutions. This will thus be done in a continuation of this work

Acknowledgements

I would like to thank my daily supervisor, Shakeeb bin Hasan, as well as my other supervisors, Bernard Geurts and Willem Vos, for their help and feedback during my assignment. I would also like to thank my fellow bachelor students and all other members of COPS, for useful discussions and a nice work environment.

References

[1] A. Lagendijk, and B. A. van Tiggelen, Phys. Rep. **270**, 143 (2015).

[2] D.J. Griffiths, *Introduction to Quantum Mechanics* (Cambridge University Press, Cambridge, 2016).

[3] P. de Vries, D. van Coevorden, and A. Lagendijk, Rev. Mod. Phys. **70**, 447 (1998).

[4] T. D. Schmidt, B. J. Scholz, C. Mayr, and W. Brütting, IEEE J. Sel. Topics in Quantum Electron. **19**, 7800412 (2013).

[5] K. Celebi, T.D. Heidel, and M. A. Baldo, Opt. Express **15**, 1762 (2007).

[6] C.-T. Tai, *Dyadic Green Functions in Electromagnetic Theory* (IEEE PRESS, New York, 1993).

[7] W. W. Hansen, Phys. Rev. **47**, 139 (1935).

[8] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949).

[9] R. R. Chance, A. Prock, and R. Silbey, Adv. Chem. Phys. **37**, 1 (1978).

Appendices

A Orthogonality relations

In Ref. [6] the orthogonality relations for the $\mathbf{M}$ and $\mathbf{N}$ functions have been provided. In this section the same will be done for the $\mathbf{L}_{n\lambda}(h)$ functions. The relationships that hold are

$$\int\int\int \mathrm{d}V \mathbf{L}_{n\lambda}(h)\mathbf{L}_{n'\lambda'}(-h') = \begin{cases} 0 & n \neq n' \\ 2\pi^2(\lambda + \frac{hh'}{\lambda'})(1+\delta_{0n})\delta(h-h')\delta(\lambda-\lambda') & n = n' \end{cases}, \tag{132}$$

$$\int\int\int \mathrm{d}V \mathbf{L}_{n\lambda}(h)\mathbf{M}_{n'\lambda'}(-h') = 0 \tag{133}$$

and

$$\int\int\int \mathrm{d}V \mathbf{L}_{n\lambda}(h)\mathbf{N}_{n'\lambda'}(-h') = 0. \tag{134}$$

Because of the trigonometric functions, all of eigenfunctions are orthogonal if $n \neq n'$. For the other cases, the proofs follow a similar manner as the once found in the book of Tai. As an example

$$I = \int\int\int \mathrm{d}V \mathbf{L}_{n\lambda o}(h)\mathbf{M}_{n'\lambda' e}(-h') \tag{135}$$

will be calculated below.

$$\begin{aligned}
I =& \int_0^{2\pi} d\phi \int_0^{\infty} rdr \int_{-\infty}^{\infty} dz \mathbf{L}_{n\lambda o}(h)\mathbf{M}_{n'\lambda' e}(-h') \\
=& -(1+\delta_{0n})\,\pi \int_0^{\infty} rdr \int_{-\infty}^{\infty} dz e^{i(h-h')z} \frac{n}{r}\left(J_n(\lambda' r)\frac{\partial}{\partial r}J_n(\lambda r) + J_n(\lambda r)\frac{\partial}{\partial r}J_n(\lambda' r)\right) \\
=& -2n\pi^2(1+\delta_{0n})\,\delta(h-h') \int_0^{\infty} dr \left(J_n(\lambda' r)\frac{\partial}{\partial r}J_n(\lambda r) + J_n(\lambda r)\frac{\partial}{\partial r}J_n(\lambda' r)\right) \\
=& -2n\pi^2(1+\delta_{0n})\,\delta(h-h')\left[J_n(\lambda r)J_n(\lambda' r)\right]_0^{\infty} \\
=& 0
\end{aligned} \tag{136}$$

B Solving parts of the integrals

In this section and the next all parts of the integral of Eqs. (38), (GT), (GL), (GTR) and (GLR) will be solved. To shorten the notation, implicit n and λ dependence will be omitted as the integrals only involve h.

B.1 First integral

We will first solve the integral, which is a part of Eqs. (38) and (39),

$$I_1 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{\mathbf{M}(h)\mathbf{M}_0(-h)}{(1+\delta_{0n})2\pi^2\lambda\left(\kappa^2 - k^2\right)}. \tag{137}$$

When filling in the definition of $\mathbf{M}$ according to Eq. (20) and shortening the notation by collecting the therms that are independent of h into the coefficient f_1 we get

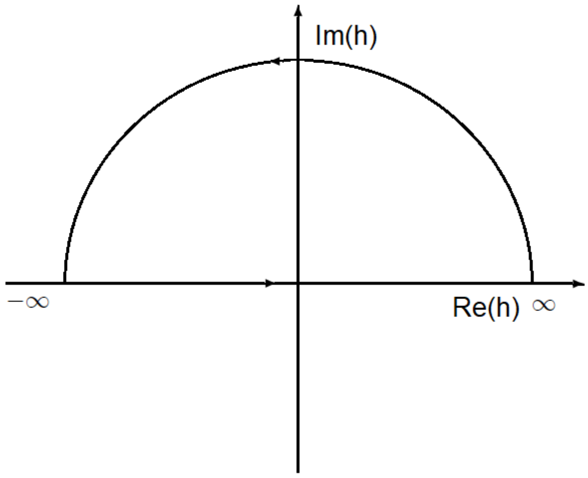

Figure 3: Contour following the complete real axis and than making half a circle over the upper half of the complex h-plane

$$I_1 = \frac{1}{(1+\delta_{0n})2\pi^2\lambda}\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp\left[ih\left(z-z_0\right)\right]}{(\lambda^2+h^2-k^2)}\left(\frac{n^2}{rr_0}J_n(\lambda r)J_n(\lambda r_0)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\,\hat{r}\hat{r}_0\right.$$
$$+\frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{\phi}\hat{\phi}_0$$
$$\left.\mp\frac{n}{r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{r}\hat{\phi}_0\mp\frac{n}{r_0}J_n(\lambda r_0)\frac{\partial}{\partial r}J_n(\lambda r)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\,\hat{r}_0\hat{\phi}\right)$$
$$=\frac{f_1}{(1+\delta_0)2\pi^2\lambda}\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp\left[ih\left(z-z_0\right)\right]}{(\lambda^2+h^2-k^2)}. \quad (138)$$

Then we rewrite the equation to highlight the poles as follows

$$I_1 = \frac{f_1}{(1+\delta_{0n})2\pi^2\lambda}\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp\left[ih\left(z-z_0\right)\right]}{\left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)}. \quad (139)$$

The above integral can be solved by the contour integration approach in complex plane. For $z > z_0$ we integrate over the upper half of the complex plane, as can be seen in Fig. (3). Identifying the poles at $h = \pm\sqrt{k^2-\lambda^2}$, with $k \in \mathbb{C}$, and then using the residue theorem we get for $z > z_0$

$$I_1 = \frac{f_1}{(1+\delta_{0n})2\pi^2\lambda}2\pi i\frac{\exp\left[i\sqrt{k^2-\lambda^2}\left(z-z_0\right)\right]}{2\sqrt{k^2-\lambda^2}}, \quad (140)$$

$$I_1 = \frac{f_1}{(1+\delta_{0n})\pi\lambda}i\frac{\exp\left[i\sqrt{k^2-\lambda^2}\left(z-z_0\right)\right]}{2\sqrt{k^2-\lambda^2}} \quad (141)$$

and finally after working out the coefficients

$$I_1 = \frac{i\mathbf{M}(\sqrt{k^2-\lambda^2})\mathbf{M}_0(-\sqrt{k^2-\lambda^2})}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}}. \quad (142)$$

We can repeat the procedure for $z < z_0$ by carrying out the integration in the lower half of the complex plane as seen in fig. (4) . This gives

$$I_1 = \frac{f_1}{(1+\delta_{0n})\pi\lambda}i\frac{\exp\left[-i\sqrt{k^2-\lambda^2}\left(z-z_0\right)\right]}{2\sqrt{k^2-\lambda^2}} \quad (143)$$

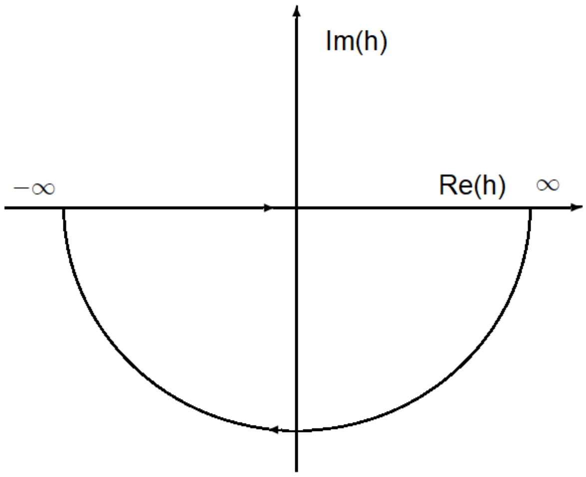

Figure 4: Contour following the complete real axis and then making half a circle over the lower half of the complex h-plane

or after once more working out the coefficients.

$$I_1 = \frac{i\mathbf{M}(-\sqrt{k^2-\lambda^2})\mathbf{M}_0(\sqrt{k^2-\lambda^2})}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}}. \tag{144}$$

B.2 Second integral

Continuing with the second integral, which is a part of Eqs. (38) and (39),

$$I_2 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{\mathbf{N}(h)\mathbf{N}_0(-h)}{(1+\delta_{0n})2\pi^2\lambda\,(\lambda^2+h^2-k^2)}. \tag{145}$$

After filling in the definition for $\mathbf{N}$ from Eq. (22) and collecting the prefactors into the coefficients f_1, f_2 and f_3 we get

$$
\begin{aligned}
I_2 = &\frac{1}{(1+\delta_{0n})2\pi^2\lambda}\Bigg(\Bigg(\Bigg(\frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{r}\hat{r}_0 \\
&+\frac{n^2}{rr_0}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\,\hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r}J_n(\lambda r)\frac{n}{r_0}J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\,\hat{r}\hat{\phi}_0 \\
&\pm\frac{\partial}{\partial r_0}J_n(\lambda r_0)\frac{n}{r}J_n(\lambda r)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\,\hat{r}_0\hat{\phi}\Bigg)\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h^2}{\kappa^2\left(\lambda^2+h^2-k^2\right)} \\
&+\Bigg(\lambda^4 J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{z}\hat{z}_0\Bigg)\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]}{\kappa^2\left(\lambda^2+h^2-k^2\right)}\Bigg) \\
&+\Bigg(i\lambda^2\frac{\partial}{\partial r}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{r}\hat{z}_0 \\
&-i\lambda^2\frac{\partial}{\partial r_0}J_n(\lambda r_0)J_n(\lambda r)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\,\hat{r}_0\hat{z} \pm i\lambda^2\frac{n}{r}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi_0)\,\hat{\phi}\hat{z}_0 \\
&\mp i\lambda^2\frac{n}{r_0}J_n(\lambda r)J_n(\lambda r_0)\ \begin{matrix}\cos\\ \sin\end{matrix}\ (n\phi_0)\ \begin{matrix}\sin\\ \cos\end{matrix}\ (n\phi)\,\hat{\phi}_0\hat{z}\Bigg)\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h}{\kappa^2\left(\lambda^2+h^2-k^2\right)} \\
&=\frac{1}{(1+\delta_{0n})2\pi^2\lambda}\Bigg(f_1\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h^2}{\kappa^2\left(\lambda^2+h^2-k^2\right)}+f_2\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]}{\kappa^2\left(\lambda^2+h^2-k^2\right)} \\
&+f_3\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h}{\kappa^2\left(\lambda^2+h^2-k^2\right)}\Bigg).
\end{aligned}
\tag{146}
$$

After rewriting to highlight the poles

$$
\begin{aligned}
I_2 = &\frac{1}{(1+\delta_{0n})2\pi^2\lambda}\Bigg(f_1\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h^2}{\kappa^2\left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)} \\
&+f_2\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]}{\kappa^2\left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)}+f_3\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h}{\kappa^2\left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)}\Bigg).
\end{aligned}
\tag{147}
$$

The above integral can be solved similar to the first integral. Identifying the poles at $h=\pm\sqrt{k^2-\lambda^2}$ and $h=\pm i\lambda$ we get for $z>z_0$

$$
\begin{aligned}
I_2 = \frac{i}{(1+\delta_{0n})\pi\lambda}\Bigg(&f_1\left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)](k^2-\lambda^2)}{k^2\left(2\sqrt{k^2-\lambda^2}\right)}+\frac{\exp[-\lambda(z-z_0)]\lambda^2}{2i\lambda k^2}\right)+ \\
&f_2\left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)]}{k^2\left(2\sqrt{k^2-\lambda^2}\right)}+\frac{\exp[-\lambda(z-z_0)]}{-2i\lambda k^2}\right)+ \\
&f_3\left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)]\sqrt{k^2-\lambda^2}}{k^2\left(2\sqrt{k^2-\lambda^2}\right)}+\frac{i\lambda\exp[-\lambda(z-z_0)]}{-2i\lambda k^2}\right)\Bigg),
\end{aligned}
\tag{148}
$$

or

$$
I_2 = \frac{i\mathbf{N}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(-\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}}+f_1\frac{\exp[-\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi}-f_2\frac{\exp[-\lambda(z-z_0)]}{2\lambda^2k^2(1+\delta_{0n})\pi}-f_3\frac{i\exp[-\lambda(z-z_0)]}{2\lambda k^2(1+\delta_{0n})\pi},
\tag{149}
$$

with

$$
\begin{aligned}
\mathcal{G}_A^{+} = \frac{\exp[-\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi}\Bigg(&\frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{r}\hat{r}_0 \\
&+\frac{n^2}{rr_0}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\,\hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r}J_n(\lambda r)\frac{n}{r_0}J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\,\hat{r}\hat{\phi}_0 \\
&\pm \frac{\partial}{\partial r_0}J_n(\lambda r_0)\frac{n}{r}J_n(\lambda r)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\,\hat{r}_0\hat{\phi} - \lambda^2 J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{z}\hat{z}_0 \\
&+\lambda\frac{\partial}{\partial r}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{r}\hat{z}_0 - \lambda\frac{\partial}{\partial r_0}J_n(\lambda r_0)J_n(\lambda r)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\,\hat{r}_0\hat{z} \\
\pm i\lambda\frac{n}{r}J_n(\lambda r)J_n(\lambda r_0)&\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{\phi}\hat{z}_0 \mp \lambda\frac{n}{r_0}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\,\hat{\phi}_0\hat{z}\Bigg) \qquad (150)
\end{aligned}
$$

this gives

$$
\begin{aligned}
I_2 &= \frac{i\mathbf{N}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(-\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}} + \frac{\mathbf{L}(i\lambda)\mathbf{L}_0(-i\lambda)}{2(1+\delta_{0n})\pi k^2} && (151)\\
&= \frac{i\mathbf{N}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(-\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}} + \mathcal{G}_A^{+}. && (152)
\end{aligned}
$$

Repeating for $z < z_0$by integrating over the lover half of the complex plane

$$
\begin{aligned}
I_2 = \frac{i\mathbf{N}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}} &+ f_1\frac{\exp[\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi} \\
&- f_2\frac{\exp[\lambda(z-z_0)]}{2\lambda^2k^2(1+\delta_{0n})\pi} + f_3\frac{i\exp[\lambda(z-z_0)]}{2\lambda k^2(1+\delta_{0n})\pi}, \qquad (153)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{G}_A^{-} = \frac{\exp[\lambda(z-z_0)]}{2k^2(1+\delta_{0n})\pi}\Bigg(&\frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{r}\hat{r}_0 \\
&+\frac{n^2}{rr_0}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\,\hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r}J_n(\lambda r)\frac{n}{r_0}J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\,\hat{r}\hat{\phi}_0 \\
&\pm \frac{\partial}{\partial r_0}J_n(\lambda r_0)\frac{n}{r}J_n(\lambda r)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\,\hat{r}_0\hat{\phi} - \lambda^2 J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{z}\hat{z}_0 \\
&-\lambda\frac{\partial}{\partial r}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{r}\hat{z}_0 + \lambda\frac{\partial}{\partial r_0}J_n(\lambda r_0)J_n(\lambda r)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\,\hat{r}_0\hat{z} \\
\mp i\lambda\frac{n}{r}J_n(\lambda r)J_n(\lambda r_0)&\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi_0)\,\hat{\phi}\hat{z}_0 \pm \lambda\frac{n}{r_0}J_n(\lambda r)J_n(\lambda r_0)\genfrac{}{}{0pt}{}{\cos}{\sin}(n\phi_0)\genfrac{}{}{0pt}{}{\sin}{\cos}(n\phi)\,\hat{\phi}_0\hat{z}\Bigg) \qquad (154)
\end{aligned}
$$

makes

$$
I_2 = \frac{i\mathbf{N}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}} + \frac{\mathbf{L}(-i\lambda)\mathbf{L}_0(i\lambda)}{2(1+\delta_{0n})\pi k^2} \qquad (155)
$$

or

$$
I_2 = \frac{i\mathbf{N}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}} + \mathcal{G}_A^{-} \qquad (156)
$$

B.3 Third integral

Finally, the third and last of the integrals involving the non-regularized Green's functions, which is a part of Eqs. (38) and (40)

$$I_3 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{-\lambda \mathbf{L}(h)\mathbf{L}_0(-h)}{(1+\delta_{0n})2\pi^2 \left(\lambda^2+h^2\right) k^2}. \tag{157}$$

Substituting the definition of $\mathbf{L}$ from Eq. (24) gives us

$$\begin{aligned}
I_3 = \frac{-\lambda}{k^2(1+\delta_{0n})2\pi^2} \Bigg(\Bigg(& \frac{\partial}{\partial r} J_n(\lambda r) \frac{\partial}{\partial r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}\hat{r}_0 \\
+ \frac{n^2 J_n(\lambda r) J_n(\lambda r_0)}{r r_0} & \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0)\, \hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r} J_n(\lambda r) \frac{n}{r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0)\, \hat{r}\hat{\phi}_0 \\
& \pm \frac{\partial}{\partial r_0} J_n(\lambda r_0) \frac{n}{r} J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi)\, \hat{r}_0\hat{\phi} \Bigg) \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]}{(h^2+\lambda^2)} \\
& + \left(J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{z}\hat{z}_0 \right) \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h^2}{(h^2+\lambda^2)} \\
+ \Bigg(-i \frac{\partial}{\partial r} J_n(\lambda r) J(\lambda r_0) & \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}\hat{z}_0 + i \frac{\partial}{\partial r_0} J_n(\lambda r_0) J(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}_0\hat{z} \\
\mp \frac{in}{r} J_n(\lambda r) J_n(\lambda r_0) & \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{\phi}\hat{z}_0 \pm \frac{in}{r_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, \hat{\phi}_0\hat{z} \Bigg) \\
& \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h}{(h^2+\lambda^2)} \Bigg) = \\
\frac{-\lambda}{k^2(1+\delta_{0n})2\pi^2} \Bigg(f_1 \int_{-\infty}^{\infty} & \mathrm{d}h \frac{\exp[ih(z-z_0)]}{(h^2+\lambda^2)} + f_2 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h^2}{(h^2+\lambda^2)} + f_3 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h}{(h^2+\lambda^2)} \Bigg).
\end{aligned} \tag{158}$$

The above integral can be solved similar to the previous integrals. Identifying the poles at $h = \pm i\lambda$ we get for $z > z_0$

$$I_3 = \frac{-\lambda i}{k^2(1+\delta_{0n})\pi} \left(f_1 \frac{\exp[-\lambda(z-z_0)]}{2i\lambda} + f_2 \left(\frac{\exp[-\lambda(z-z_0)]\lambda^2}{2i\lambda} + \frac{2\pi}{2\pi i}\delta(z-z_0) \right) + f_3 \frac{\exp[-\lambda(z-z_0)]i\lambda}{2i\lambda} \right), \tag{159}$$

$$I_3 = \frac{-\mathbf{L}(i\lambda)\mathbf{L}_0(-i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{-\lambda f_2}{k^2(1+\delta_{0n})\pi}\delta(z-z_0), \tag{160}$$

which comes down to

$$I_3 = \frac{-\mathbf{L}(i\lambda)\mathbf{L}_0(-i\lambda)}{2(1+\delta_{0n})\pi k^2} - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{z}\hat{z}_0 \delta(z-z_0). \tag{161}$$

Repeating for $z < z_0$ by integrating over the lower half of the complex plane

$$I_3 = \frac{-\mathbf{L}(-i\lambda)\mathbf{L}_0(i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{-2\lambda i f_2}{k^2(1+\delta_{0n})}\delta(z-z_0) \tag{162}$$

gives

$$I_3 = \frac{-\mathbf{L}(-i\lambda)\mathbf{L}_0(i\lambda)}{2(1+\delta_{0n})\pi k^2} - \frac{\lambda}{k^2(1+\delta_{0n})\pi} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{z}\hat{z}_0 \delta(z-z_0). \tag{163}$$

C Solutions of the integrals involving the regularization

The integrals from the previous section will be solved again for the additional poles caused by the regularization. These are thus parts of the integrals of Eqs. (51) and (52). To shorten the notation, implicit n and λ dependence will be omitted as the integrals will only involve h.

C.1 First integral

Once more starting from the first part of the integral of Eq. (51)

$$I_1 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{\Lambda_T^2}{(\Lambda_T^2+\kappa^2)} \frac{\mathbf{M}(h)\mathbf{M}_0(-h)}{(1+\delta_{0n})2\pi^2\lambda\,(\kappa^2-k^2)}. \tag{164}$$

Substituting the definition of $\mathbf{M}$ from Eq. (20) and then shortening notation gives

$$\begin{aligned} I_1 = \frac{\Lambda_T^2}{(1+\delta_{0n})2\pi^2\lambda} &\left(\frac{n^2}{rr_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0)\, \hat{r}\hat{r}_0 \right. \\ &+ \frac{\partial}{\partial r} J_n(\lambda r) \frac{\partial}{\partial r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{\phi}\hat{\phi}_0 \mp \frac{n}{r} J_n(\lambda r) \frac{\partial}{\partial r_0} J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}\hat{\phi}_0 \\ &\left. \mp \frac{n}{r_0} J_n(\lambda r_0) \frac{\partial}{\partial r} J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, \hat{r}_0\hat{\phi} \right) \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp\left[ih\,(z-z_0)\right]}{(\lambda^2+h^2-k^2)\,(\Lambda_T^2+\kappa^2)} \\ &= \frac{f_1}{(1+\delta_0)2\pi^2\lambda} \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp\left[ih\,(z-z_0)\right]}{(\lambda^2+h^2-k^2)\,(\Lambda_T^2+\kappa^2)}, \end{aligned} \tag{165}$$

or in different notation to highlight the poles

$$I_1 = \frac{f_1}{(1+\delta_{0n})2\pi^2\lambda} \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp\left[ih\,(z-z_0)\right]}{\left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)\left(h+i\sqrt{\Lambda_T^2+\lambda^2}\right)\left(h-i\sqrt{\Lambda_T^2+\lambda^2}\right)}. \tag{166}$$

The above integral can be solved by the contour integration approach in complex plane. Identifying the poles at $h=\pm\sqrt{k^2-\lambda^2}$ and $h=\pm i\sqrt{\Lambda_T^2+\lambda^2}$ and then using the residue theorem we get for $z>z_0$

$$\begin{aligned} I_1 = \frac{f_1}{(1+\delta_{0n})2\pi^2\lambda} 2\pi i &\left(\frac{\exp\left[i\sqrt{k^2-\lambda^2}\,(z-z_0)\right]}{2\sqrt{k^2-\lambda^2}\left(\sqrt{k^2-\lambda^2}+i\sqrt{\Lambda_T^2+\lambda^2}\right)\left(\sqrt{k^2-\lambda^2}-i\sqrt{\Lambda_T^2+\lambda^2}\right)} + \right. \\ &\left. \frac{\exp\left[-\sqrt{\Lambda_T^2+\lambda^2}\,(z-z_0)\right]}{\left(i\sqrt{\Lambda_T^2+\lambda^2}-\sqrt{k^2-\lambda^2}\right)\left(i\sqrt{\Lambda_T^2+\lambda^2}+\sqrt{k^2-\lambda^2}\right)\left(2i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right), \end{aligned} \tag{167}$$

which, when worked out, becomes

$$I_1 = \frac{f_1}{(1+\delta_{0n})\pi\lambda} i \left(\frac{\exp\left[i\sqrt{k^2-\lambda^2}\,(z-z_0)\right]}{2\sqrt{k^2-\lambda^2}\,(k^2+\Lambda_T^2)} - \frac{\exp\left[-\sqrt{\Lambda_T^2+\lambda^2}\,(z-z_0)\right]}{(\Lambda_T^2+k^2)\left(2i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right) \tag{168}$$

or, when writing out f_1 again

$$I_1 = \frac{\Lambda_T^2}{2(1+\delta_{0n})\pi\lambda\,(k^2+\Lambda_T^2)} \left(\frac{i\mathbf{M}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{M}_0\left(-\sqrt{k^2-\lambda^2}\right)}{\sqrt{k^2-\lambda^2}} - \frac{\mathbf{M}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{M}_0\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)}{\sqrt{\Lambda_T^2+\lambda^2}} \right). \tag{169}$$

We can repeat the procedure for $z < z_0$ by carrying out the integration in the lower half of the complex plane. This gives

$$I_1 = \frac{f_1}{(1+\delta_{0n})\pi\lambda} i \left(\frac{\exp\left[-i\sqrt{k^2-\lambda^2}\,(z-z_0)\right]}{2\sqrt{k^2-\lambda^2}\,(k^2+\Lambda_T^2)} - \frac{\exp\left[\sqrt{\Lambda_T^2+\lambda^2}\,(z-z_0)\right]}{(\Lambda_T^2+k^2)\left(2i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right), \tag{170}$$

which becomes

$$I_1 = \frac{\Lambda_T^2}{2(1+\delta_{0n})\pi\lambda\,(k^2+\Lambda_T^2)} \left(\frac{i\mathbf{M}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{M}_0\left(\sqrt{k^2-\lambda^2}\right)}{\sqrt{k^2-\lambda^2}} - \frac{\mathbf{M}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{M}_0\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)}{\sqrt{\Lambda_T^2+\lambda^2}} \right). \tag{171}$$

C.2 Second integral

Continuing with the second part of the integral of Eq. (51),

$$I_2 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{\Lambda_T^2}{(\Lambda_T^2+\kappa^2)} \frac{\mathbf{N}(h)\mathbf{N}_0(-h)}{(1+\delta_{0n})2\pi^2\lambda\,(\lambda^2+h^2-k^2)} \tag{172}$$

or when writing out the definition of $\mathbf{N}$ of Eq. (22) and then shortening notation

$$\begin{aligned}
I_2 = \frac{\Lambda_T^2}{(1+\delta_{0n})2\pi^2\lambda} & \Bigg(\Bigg(\frac{\partial}{\partial r} J_n(\lambda r) \frac{\partial}{\partial r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}\hat{r}_0 \\
& + \frac{n^2}{r r_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0)\, \hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r} J_n(\lambda r) \frac{n}{r_0} J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0)\, \hat{r}\hat{\phi}_0 \\
& \pm \frac{\partial}{\partial r_0} J_n(\lambda r_0) \frac{n}{r} J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi)\, \hat{r}_0\hat{\phi} \Bigg) \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h^2}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} \\
& + \left(\lambda^4 J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}\, n\phi \begin{matrix}\sin\\ \cos\end{matrix}\, n\phi_0 \hat{z}\hat{z}_0 \right) \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} \\
& + \Bigg(i\lambda^2 \frac{\partial}{\partial r} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{r}\hat{z}_0 - i\lambda^2 \frac{\partial}{\partial r_0} J_n(\lambda r_0) J_n(\lambda r) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, \hat{r}_0\hat{z} \\
& \pm i\lambda^2 \frac{n}{r} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi_0)\, \hat{\phi}\hat{z}_0 \mp i\lambda^2 \frac{n}{r_0} J_n(\lambda r) J_n(\lambda r_0) \begin{matrix}\cos\\ \sin\end{matrix} (n\phi_0) \begin{matrix}\sin\\ \cos\end{matrix} (n\phi)\, \hat{\phi}_0\hat{z} \Bigg) \\
& \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} \Bigg) \\
= \frac{\Lambda_T^2}{(1+\delta_{0n})2\pi^2\lambda} & \Bigg(f_1 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h^2}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} + f_2 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} + \\
& f_3 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h}{\kappa^2(\lambda^2+h^2-k^2)(\Lambda_T^2+\kappa^2)} \Bigg).
\end{aligned} \tag{173}$$

After changing notation to highlight the poles

$$I_2 = \frac{\Lambda_T^2}{(1+\delta_{0n})2\pi^2\lambda}$$
$$\left(f_1 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h^2}{\kappa^2 \left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)\left(h+i\sqrt{\Lambda_T^2+\lambda^2}\right)\left(h-i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right.$$
$$+ f_2 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]}{\kappa^2 \left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)\left(h+i\sqrt{\Lambda_T^2+\lambda^2}\right)\left(h-i\sqrt{\Lambda_T^2+\lambda^2}\right)}$$
$$\left. + f_3 \int_{-\infty}^{\infty} \mathrm{d}h \frac{\exp[ih(z-z_0)]h}{\kappa^2 \left(h-\sqrt{k^2-\lambda^2}\right)\left(h+\sqrt{k^2-\lambda^2}\right)\left(h+i\sqrt{\Lambda_T^2+\lambda^2}\right)\left(h-i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right). \quad (174)$$

The above integral can be solved similar to the first integrals. Identifying the poles at $h = \pm\sqrt{k^2-\lambda^2}$, $h = \pm i\lambda$ and $h = \pm i\sqrt{\Lambda_T^2+\lambda^2}$ we get for $z > z_0$

$$I_2 = \frac{i\Lambda_T^2}{(1+\delta_{0n})\pi\lambda} \left(f_1 \left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)](k^2-\lambda^2)}{k^2\left(2\sqrt{k^2-\lambda^2}\right)(k^2+\Lambda_T^2)} + \frac{\exp[-\lambda(z-z_0)]\lambda^2}{2i\lambda k^2\Lambda_T^2} \right.\right.$$
$$\left. + \frac{-\exp[-\sqrt{\Lambda_T^2+\lambda^2}(z-z_0)]\left(\Lambda_T^2+\lambda^2\right)}{\Lambda_T^2\left(\Lambda_T^2+k^2\right)2i\sqrt{\Lambda_T^2+\lambda^2}} \right) +$$
$$f_2 \left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)]}{k^2\left(2\sqrt{k^2-\lambda^2}\right)(k^2+\Lambda_T^2)} + \frac{\exp[-\lambda(z-z_0)]}{-2i\lambda k^2\Lambda_T^2} + \frac{\exp[-\sqrt{\Lambda_T^2+\lambda^2}(z-z_0)]}{\Lambda_T^2\left(\Lambda_T^2+k^2\right)\left(2i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right)$$
$$\left. f_3 \left(\frac{\exp[i\sqrt{k^2-\lambda^2}(z-z_0)]\sqrt{k^2-\lambda^2}}{k^2\left(2\sqrt{k^2-\lambda^2}\right)(k^2+\Lambda_T^2)} + \frac{i\lambda\exp[-\lambda(z-z_0)]}{-2i\lambda k^2\Lambda_T^2} + \frac{i\sqrt{\Lambda_T^2+\lambda^2}\exp[-\sqrt{\Lambda_T^2+\lambda^2}(z-z_0)]}{\Lambda_T^2\left(\Lambda_T^2+k^2\right)\left(2i\sqrt{\Lambda_T^2+\lambda^2}\right)} \right) \right), \quad (175)$$

which, after filling in f_1, f_2 and f_3, becomes

$$I_2 = \frac{i\Lambda_T^2\mathbf{N}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(-\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}\left(k^2+\Lambda_T^2\right)} - \frac{\Lambda_T^2\mathbf{N}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_0\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\left(\Lambda_T^2+k^2\right)\sqrt{\Lambda_T^2+\lambda^2}}$$
$$+ \frac{\mathbf{L}(i\lambda)\mathbf{L}_0(-i\lambda)}{2(1+\delta_{0n})\pi k^2}$$
$$= \frac{i\Lambda_T^2\mathbf{N}\left(\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(-\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}\left(k^2+\Lambda_T^2\right)} - \frac{\Lambda_T^2\mathbf{N}\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_0\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\left(\Lambda_T^2+k^2\right)\sqrt{\Lambda_T^2+\lambda^2}}$$
$$+ \mathcal{G}_A^+. \quad (176)$$

Repeating for $z < z_0$ by integrating over the lover half of the complex plane

$$I_2 = \frac{i\Lambda_T^2 \mathbf{N}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}\left(k^2+\Lambda_T^2\right)} - \frac{\Lambda_T^2\mathbf{N}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_0\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\left(\Lambda_T^2+k^2\right)\sqrt{\Lambda_T^2+\lambda^2}} + \frac{\mathbf{L}(-i\lambda)\mathbf{L}_0(i\lambda)}{2(1+\delta_{0n})\pi k^2}$$

$$= \frac{i\Lambda_T^2 \mathbf{N}\left(-\sqrt{k^2-\lambda^2}\right)\mathbf{N}_0\left(\sqrt{k^2-\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\sqrt{k^2-\lambda^2}\left(k^2+\Lambda_T^2\right)} - \frac{\Lambda_T^2\mathbf{N}\left(-i\sqrt{\Lambda_T^2+\lambda^2}\right)\mathbf{N}_0\left(i\sqrt{\Lambda_T^2+\lambda^2}\right)}{2(1+\delta_{0n})\pi\lambda\left(\Lambda_T^2+k^2\right)\sqrt{\Lambda_T^2+\lambda^2}} + \mathcal{G}_A^-. \quad (177)$$

C.3 Third integral

Finally, the last integral from Eq. (52),

$$I_3 = \int_{-\infty}^{\infty} \mathrm{d}h \frac{\Lambda_L^4}{\Lambda_L^4+\kappa^4}\frac{-\lambda\mathbf{L}(h)\mathbf{L}_0(-h)}{(1+\delta_{0n})2\pi^2\left(\lambda^2+h^2\right)k^2} \quad (178)$$

or, when writing out the definition of $\mathbf{L}$ from Eq. (24) and shortening notation

$$\begin{aligned}
I_3 = \frac{-\lambda\Lambda_L^4}{k^2(1+\delta_{0n})2\pi^2}\Bigg\{ \Bigg(& \frac{\partial}{\partial r}J_n(\lambda r)\frac{\partial}{\partial r_0}J_n(\lambda r_0) \begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{r}\hat{r}_0 \\
& + \frac{n^2 J_n(\lambda r)J_n(\lambda r_0)}{r r_0}\begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\,\hat{\phi}\hat{\phi}_0 \pm \frac{\partial}{\partial r}J_n(\lambda r)\frac{n}{r_0}J_n(\lambda r_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\,\hat{r}\hat{\phi}_0 \\
& \pm \frac{\partial}{\partial r_0}J_n(\lambda r_0)\frac{n}{r}J_n(\lambda r)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\,\hat{r}_0\hat{\phi}\Bigg)\int \mathrm{d}h\frac{\exp[ih(z-z_0)]}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)} \\
& + \left(J_n(\lambda r)J_n(\lambda r_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{z}\hat{z}_0\right)\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h^2}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)}\Bigg) \\
& + \Bigg(-i\frac{\partial}{\partial r}J_n(\lambda r)J(\lambda r_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{r}\hat{z}_0 + i\frac{\partial}{\partial r_0}J_n(\lambda r_0)J(\lambda r)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{r}_0\hat{z} \\
& \mp \frac{in}{r}J_n(\lambda r)J_n(\lambda r_0)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi_0)\,\hat{\phi}\hat{z}_0 \pm \frac{in}{r_0}J_n(\lambda r)J_n(\lambda r_0)\begin{matrix}\cos\\ \sin\end{matrix}(n\phi_0)\begin{matrix}\sin\\ \cos\end{matrix}(n\phi)\,\hat{\phi}_0\hat{z}\Bigg) \\
& \int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)}\Bigg\} = \\
& \frac{-\lambda\Lambda_L^4}{k^2(1+\delta_{0n})2\pi^2}\left(f_1\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)} + f_2\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h^2}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)}\right. \\
& \left. + f_3\int_{-\infty}^{\infty}\mathrm{d}h\frac{\exp[ih(z-z_0)]h}{\left(h^2+\lambda^2\right)\left(\Lambda_L^4+\kappa^4\right)}\right). \quad (179)
\end{aligned}$$

The above integral can be solved similar to the previous integrals. Identifying the poles at $h = \pm i\lambda$ and $h = \pm\sqrt{\pm i\Lambda_L^2 - \lambda^2}$ we get for $z > z_0$

$$
\begin{aligned}
I_3 = \frac{-\lambda i \Lambda_L^4}{k^2(1+\delta_{0n})\pi} \Bigg\{ & f_1 \left(\frac{\exp[-\lambda(z-z_0)]}{2i\lambda\Lambda_L^4} - \frac{\exp[i\sqrt{i\Lambda_L^2-\lambda^2}(z-z_0)]}{4\Lambda_L^4\sqrt{i\Lambda_L^2-\lambda^2}} - \frac{\exp[i\sqrt{-i\Lambda_L^2-\lambda^2}(z-z_0)]}{4\Lambda_L^4\sqrt{-i\Lambda_L^2-\lambda^2}} \right) + \\
& f_2 \left(\frac{\exp[-\lambda(z-z_0)]\lambda^2}{2i\lambda\Lambda_L^4} - \frac{\exp[i\sqrt{i\Lambda_L^2-\lambda^2}(z-z_0)]\left(i\Lambda_L^2-\lambda^2\right)}{4\Lambda_L^4\sqrt{i\Lambda_L^2-\lambda^2}} \right. \\
& \left. - \frac{\exp[i\sqrt{-i\Lambda_L^2-\lambda^2}(z-z_0)]\left(-i\Lambda_L^2-\lambda^2\right)}{4\Lambda_L^4\sqrt{-i\Lambda_L^2-\lambda^2}} \right) + \\
& f_3 \left(\frac{\exp[-\lambda(z-z_0)]i\lambda}{2i\lambda\Lambda_L^4} - \frac{\exp[i\sqrt{i\Lambda_L^2-\lambda^2}(z-z_0)]\sqrt{i\Lambda_L^2-\lambda^2}}{4\Lambda_L^4\sqrt{i\Lambda_L^2-\lambda^2}} \right. \\
& \left. - \frac{\exp[i\sqrt{-i\Lambda_L^2-\lambda^2}(z-z_0)]\sqrt{-i\Lambda_L^2-\lambda^2}}{4\Lambda_L^4\sqrt{-i\Lambda_L^2-\lambda^2}} \right) \Bigg\}, \quad (180)
\end{aligned}
$$

which, when once more working out the coefficients, becomes

$$
I_3 = \frac{-\mathbf{L}(i\lambda)\mathbf{L}_0(-i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{\lambda i \mathbf{L}(\sqrt{i\Lambda_L^2-\lambda^2})\mathbf{L}_0(-\sqrt{i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{i\Lambda_L^2-\lambda^2}} + \frac{\lambda i \mathbf{L}(\sqrt{-i\Lambda_L^2-\lambda^2})\mathbf{L}_0(-\sqrt{-i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{-i\Lambda_L^2-\lambda^2}}. \quad (181)
$$

Repeating for $z < z_0$ by integrating over the lower half of the complex plane gives

$$
I_3 = \frac{-\mathbf{L}(-i\lambda)\mathbf{L}_0(i\lambda)}{2(1+\delta_{0n})\pi k^2} + \frac{\lambda i \mathbf{L}(-\sqrt{i\Lambda_L^2-\lambda^2})\mathbf{L}_0(\sqrt{i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{i\Lambda_L^2-\lambda^2}} + \frac{\lambda i \mathbf{L}(-\sqrt{-i\Lambda_L^2-\lambda^2})\mathbf{L}_0(\sqrt{-i\Lambda_L^2-\lambda^2})}{4(1+\delta_{0n})\pi k^2\sqrt{-i\Lambda_L^2-\lambda^2}}. \quad (182)
$$