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Empirical study on the existence of Tuned Risk Aversion in option pricing

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Abstract

We combine Tuned Risk Aversion and Conic Finance into a discrete-time option pricing model. The model values bid and ask prices by distorted expectations with non-static risk aversion under a weaker form of consistency. With static risk aversion imposed by strong dynamic consistency, spreads will explode due to the unnecessary build-up of conservatism. With Tuned Risk Aversion we introduce an alternative that is able to produce prices that reflect market quotations while remaining consistent. We show that we are able to capture the complete probability adjustment of implied volatility by distortion, which we believe to be more intuitive. The bridging of Tuned Risk Aversion with Conic Finance provides a very promising outlook into finding a realistic uniform framework for pricing derivatives.

Keywords Tuned Risk Aversion · Conic Finance · Two price valuations · Option pricing · Weak time consistency · Implied distortion · Implied volatility

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Chapter 1

Research design

We follow the research design proposed by Verschuren *et al.* (2010). First, we conceptualize our framework in terms of Research Context (Section 1.1), Research Objective (Section 1.2) and Research Questions (Section 1.3). In the second part of this chapter (Section 1.4) we propose our technical design in order to carry out our study in terms of planning and methodology.

1.1 Problem context

"The purely economic man is indeed close to being a social moron. Economic theory has been much preoccupied with this rational fool." Thaler (2015).

Long since the introduction of Risk-Neutral Pricing by Merton, Black and Scholes in 1973 (Black and Scholes, 1973) & (Merton, 1973), it is well-recognized by practitioners and academics that theoretical prices deviate from prices observed in the market. Because of, for instance: non-deterministic volatility (Heston, 1993), non-lognormally distributed stock-prices (Jackwerth and Rubinstein, 1996) and a required (non-existing) frictionless market (Derman and Taleb, 2005) & (Leland, 1985). But the most important deviation is the fact that when we look at real markets, derivatives always have two prices instead of one.

With Conic Finance, Madan and Schoutens (2010) present a theory that is more in line with how real markets behave by recognizing that prices depend on the direction of trade, opposed to traditional one price frameworks. In a two price framework the difference between prices reflects the cost of holding unhedgeable risk due to market incompleteness and consistent application of risk aversion. In order to price financial contracts, we need to formulate which risk we believe is acceptable to hold. Risk aversion is modeled as a distortion of probabilty distributions on the risk-neutral measure. Although the framework presents a significant breakthrough, the theory still requires some weaker assumptions.

Because Conic Finance is relatively new (Madan and Schoutens, 2010), research on valuation methods within this framework is very limited. Current applications are built upon two assumptions: **(I)** strong dynamic consistency such that positions with identical risk adjusted values in every state at a future date must have the same value today, and **(II)** the reflection principle where buying a contingent claim equals selling its negative.

(I) Strong dynamically consistent valuations require risk aversion to be a constant equal for every step of a valuation. Roorda, Joosten and Schumacher show in several recent articles that strong dynamic consistency potentially leads to the accumulation of conservatism in risk assessment, and theoretical choice preferences that are not in line with observed choice preferences like the Allais and Ellsberg paradoxes (Roorda and Schumacher, 2016) & (Roorda and Joosten, 2017). In order to overcome these limitations they introduce a framework called Tuned Risk Aversion (TRA) that operates under a weaker form of consistency called sequential consistency. Under sequential consistency risk aversion is not necessarily a constant, but can be distributed over time. We develop a model that bridges the theory of Conic Finance with TRA and test whether there exists empirical proof that markets price under sequential consistency.

(II) Conic Finance assumes that bid and ask prices are connected by the assumption that buying a random cash flow equals selling its negative. Under strong dynamic consistency this equals that bid prices today only depend on tomorrows' conditional bid prices. We try to investigate whether this relationship is reflected in market quotations. Perhaps different relationships are possible, for instance dependency on the risk-neutral price.

Besides these weaker assumptions and unexplored territory of modeling best practices, the empirical evidence whether Conic Finance pricing models are able to produce prices close to market quotations is limited. Most of the current literature is based on simulation experiments, for instance Madan and Schoutens (2017) and Bielecki *et al.* (2013). With this thesis we seek to contribute to two theories which we believe to be a good step forward into pushing the field of mathematical finance towards more realistic and intuitive models.

1.2 Research objectives

This thesis is an explorative journey to empirically test dynamically and sequentially consistent Conic Finance models in their ability to produce prices close to market quotations. Research steps can be summarized in the following way:

- **Summarize** the current work done in Conic Finance and time consistency for multi-period valuation models.
- **Develop** a Conic Finance option pricing model that is able to produce bid and ask prices and extend it with the application of Tuned Risk Aversion.
- **Calibrate** the model against market data and see whether we are able to produce prices that are close to market quotations.
- **Investigate** implied distortion patterns to see if there exists a relationship between the amount of distortion needed to produce a quoted bid price and option charachteristics like moneyness and time to maturity that we can capture with the additional flexibility provided by Tuned Risk Aversion.
- **Propose** ideas and research recommendations based on our results. Because of the explorative character of our study, we expect interesting insights that could help further research in both Tuned Risk Aversion and Conic Finance.

During these five steps we try to achieve the following four research objectives:

1. Develop a pricing model that is able to price under both strong and sequential consistency.
2. Test whether the model is able to produce prices close to market quotations.
3. Investigate if there is empirical proof for the reflection principle.
4. See if there exists a functional relationship between distortion and option characteristics which we can capture by the additional flexibility provided by Tuned Risk Aversion.

1.3 Research questions

In order to achieve our research goals we formulated the following main research question:

Does there exist empirical evidence that markets price with Tuned Risk Aversion?

We broke down the main research question into sub-questions in order to answer the main research question incrementally (Remler and van Ryzin, 2015).

1. From classical derivative valuation towards a two price framework

1. What are the fundamental building blocks in valuation theory?
2. How do we adjust these building blocks into a two price model?
3. How do we define Tuned Risk Avesion in a valuation context?
4. How do we define consistency in multi-period frameworks?

The first question provides an overview on the recent work done on two price frameworks and explains how it is related to traditional pricing models. We define several types of consistency and the role of acceptancy of risk.

2. From theory to a pricing model

1. How do we model option prices in Conic Finance?
2. How do we apply Tuned Risk Aversion within Conic Finance?
3. Which assumptions do we need to make in our numerical implementation?
4. How does our model behave with a stylized case?

This question transforms theory into a practical pricing model. We explain the modeling choices made and define the most important building blocks upon which our model is based. At the end we demonstrate how the model behaves in a stylized valuation case.

3. Model calibration and parameter sensitivity

1. Which parameters do we need to calibrate?
2. How can we calibrate these parameters against market quotations?
3. What are the theoretical consequences of the modeling assumptions and calibration sequence?

We calibrate our model against market quotations. In order to do so we have to make modeling choices which potentially have an impact on the generality and validity of our findings.

4. Testing model performance

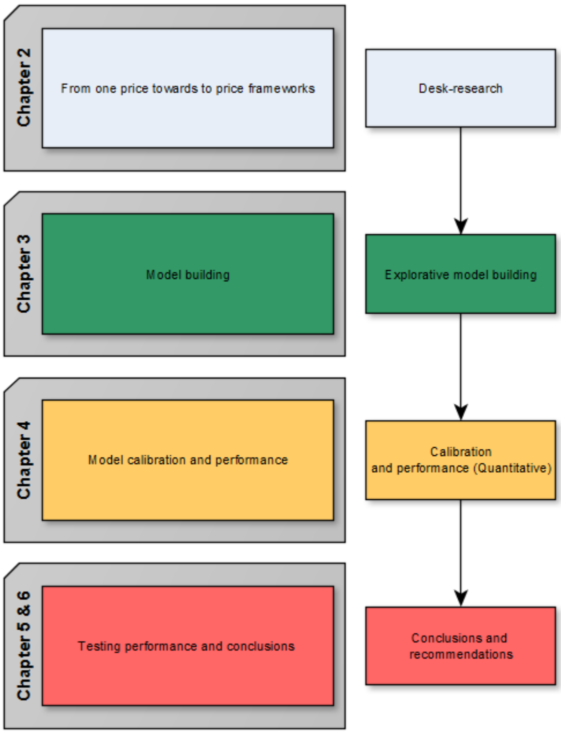
1. What are methods to measure performance of pricing models?
2. Are Conic Finance models, both under sequential- and strong dynamic consistency, able to produce prices close to market quotations?
3. Can we find empirical evidence that bid and ask prices are connected through the reflection principle?
4. Do we find empirical proof that markets price under Tuned Risk Aversion when we look at implied distortion patterns?

We work towards the answering of our central research question. We are going to assess whether there exists empirical evidence that markets price under Tuned Risk Aversion. We also test if we can find empirical proof for the reflection principle.

1.4 Thesis outline and methodology

We use a combination of research methods to fulfill our research objective. In order to grasp the theoretical aspects and foundations of our thesis subject, we start with a literature research and use this freshly gained knowledge to develop an option pricing model both under strong dynamic and sequential consistency. We test the workings of our model on a stylized case and we conduct a quantitative case study on S&P 500 option data to provide an answer to our main research question.

Figure 1.1: Research methods overview



We present our thesis outline and methodology used per chapter in Figure 1.1. In Chapter 2 we present our literature study. In order to test our assumptions in practice, we develop a model which we present in Chapter 3. Subsequently, we calibrate and test this model against market data in Chapter 4 and compare the model with existing pricing methods in Chapter 5. Finally, we present our conclusions and propose recommendations and ideas for further research in Chapter 6.

Chapter 2

From Classical Derivative Valuation Towards a Two Price Framework

For our literature review we used the framework proposed by Webster and Watson (2002). First we start with developing key constructs related to our research questions and use these to find relevant articles. Then we go backwards by reviewing the citations used by the articles identified. Finally we go forward by identifying articles that cite the key articles found in the first two steps. We start this chapter by presenting our conclusions.

2.1 Conclusions and foundations for model development

Conic Finance is a paradigm shift from a theoretical one price risk-neutral world towards a more realistic one. However, the framework is still subject to several assumptions that we believe to be less convincing. Below we present our conclusions.

1. **Recent developments in mathematical finance try to improve risk-neutral pricing frameworks while holding on to the law of one price**. It has long been accepted that the Black, Scholes & Merton (BSM) framework is not fit for finding prices in line with observed prices in the market. Recent developments within the risk-neutral pricing framework focus on building models that adjust one or more of the assumptions within the BSM framework, for instance Heston-models (allow for non-deterministic volatility) or Lévy-models (allow for jumps). The problem is that even with these models we only find one unique 'correct' price while we observe two prices (bid and ask) in the market.

2. **Conic Finance abandons the law of one price by recognizing that risk cannot be eliminated completely and therefore investors need to determine the acceptability of risk.** When we think of the market as being the counterparty in all trades, one way to model bid and ask prices is by taking them as infimum and supremum expectations over a set of probability measures. Usually bid and ask prices are then connected by the equality that buying is the same as selling its negative.

3. **Studies that present empirical results on both Conic Finance models and Tuned Risk Aversion are very limited.** Most Conic Finance studies show evidence based on simulated options prices based on calibrated probability distributions. For instance there is no academic

literature at all that investigates whether there is a distortion relationship between long- and short-term options (which is indirectly assumed by strong dynamic consistency. At the same time, empirical studies that apply Tuned Risk Aversion in a valuation or risk measure context are non-existent sofar.

4. **Acceptability of risk is operationalized by distortion of a probability measure by concave or convex functions which is similar to the foundations of Behavioral Finance.** For bid prices the market determines its price to buy a future non-negative contingent claim and therefore it shifts its perspective to the area of gains. The market therefore applies a concave distortion function and distorts its upward probabilities or payoffs downwards. The use of convexity and concavity is broadly in line with the general idea of Kahneman and Tversky on Cumulative Prospect Theory, however there is no effect of wealth accumulation, and distortion is strictly concave while usually S-shaped in behavioral finance. Our perception is that the existence of two prices is nothing else than consistent risk aversion applied in both selling and buying.

5. **Both traditional and Conic Finance frameworks all require dynamic consistency.** For Conic Finance this leads to static risk aversion and potentially the unnecessary build-up of conservatism, and therefore valuations not in line with real market behavior. An alternative approach is to require a weaker form of consistency which allows for a more flexible distribution of risk aversion, but still provides unique updates of valuations.

2.2 Fundamental building blocks in traditional discrete-time derivative pricing

Traditional asset pricing theory is built around two cornerstones called the First and Second Fundamental Theorem of Asset Pricing (FTAP) (Föllmer *et al.*, 2004):

1. **No-Arbitrage**: A finite time financial market is arbitrage-free if there exists at least one equivalent probability measure[1] $\mathcal{Q}$ on a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that every discounted price process is a martingale.
2. **Complete Markets**: A market is arbitrage-free and complete if this equivalent probability measure $\mathcal{Q}$ is unique. Within this market every derivative can be replicated by the underlying security and the money market account.

These two theorems form the basis of the no-arbitrage pricing framework developed by Merton (1973), Black and Scholes (1973), Harrison and Kreps (1979) and Ross (1978) in which the price of a derivative equals the discounted expected value of its future payoff under the unique equivalent martingale measure.

2.2.1 Traditional pricing models

Within this thesis we focus on the pricing of derivatives within discrete-time tree option pricing frameworks. Therefore we will spend little time on providing an extensive overview of other option pricing models and focus solely on the introduction of tree models. The interested reader is referred to Madan and Schoutens (2016) for an extensive overview of option pricing methods within a two price framework and to Eberlein *et al.* (2014) & Madan *et al.* (2013) for continuous-time option pricing with Variance-Gamma and Sato models.

[1] Also refered to as the risk-neutral measure in literature.

2.2.2 Binomial tree model

The binomial tree model has been introduced by Cox *et al.* (1979) (CRR model) as a no-arbitrage pricing framework to price derivatives in a discrete-time setting and converges to the BSM when we take the limit of the stepsize to zero (Föllmer *et al.*, 2004). The model is arbitrage-free and complete under the following conditions (Föllmer *et al.*, 2004);

1. $u < e^{-r\Delta t} = B(\Delta t)^{-1} < d$ under exponential discounting.
2. There exists a unique equivalent martingale measure $\mathcal{Q}$ under which the probability of an upwards movement[2] equals $p^* = \frac{B-d}{u-d}$.

We follow Duffie (2001) and Shreve (1996) in further explanation of derivative valuation within the CRR-model. Due to market completeness we can replicate a derivative X by a self-financing strategy which we will call φ. The no-arbitrage price $\phi(t)$ of X, and the value of the self-financing replicating strategy coincide and are given by the martingale property expressed in the following equation:

$$\frac{\phi(T)}{B(t)} = \frac{\phi_\varphi(t)}{B(t)} = \mathbf{E}^{\mathcal{Q}}\left[\frac{X(T)}{B(T)}|\mathcal{F}_t\right]. \tag{2.1}$$

We can price the derivative by computing the expectation under the risk-neutral measure $\mathcal{Q}$ or we exploit backward recursive characteristics by pricing the replicating portfolio. Backward recursion exploits that the discounted value of an European derivative is a martingale. The value of a derivative at maturity equals the contingent claim at maturity (2.2) and for intermediate time-steps (2.3)

$$\frac{\phi(T)}{B(T)} = \frac{X(T)}{B(T)}, \tag{2.2}$$

$$\frac{\phi(t)}{B(t)} = \mathbf{E}^{\mathcal{Q}}\left[\frac{\phi(t+1)}{B(t+1)}|\mathcal{F}_t\right]. \tag{2.3}$$

The change in the value of the undiscounted and discounted replicating portfolio equals the price of the derivative where the replicating portfolio consists of the risky underlying security and the risk-free asset. The exact equations that provide the recursivity scheme can be found in Appendix 7.1.

2.2.3 Extension of the CRR-framework

The CRR-model depends on several key assumptions;

1. The purchase price of the risky security equals the selling price of the risky security, i.e. market is frictionless (Shreve, 1996).
2. At any time-step the underlying can only take two possible values in the next period, i.e. underlying cannot keep the same value as the step before. (Shreve, 1996).
3. Logarithmic spacing between nodes, i.e. volatility is constant over the complete times-span (Derman *et al.*, 1996).

Trinomial trees provide an alternative to binomial trees. The extra parameter (mid-node) makes it possible for the price process to remain at the same price for one period (Derman *et al.*, 1996). Schwartz (1977) showed that a trinomial tree approach is equivalent to the explicit finite difference method. The modeling parameters of the trinomial tree can be found in Appendix 7.2 and have originally been presented in Boyle (1986). Another type of models is the class of implied trees. When we plot implied volatilities for traded options, we observe that

[2] $u = e^{\sigma\sqrt{\Delta t}}$ & $d = \frac{1}{u}$.

volatility is not constant and vary with both strike and expiration creating an implied volatility surface (Derman *et al.*, 1996). Implied tree theory was first proposed by Dupire (1994) and extends the CRR/BSM framework in a way that it is consistent with the shape of the smile. Consistency is achieved by extracting an implied evolution for the stock price from market prices of traded options on the underlying security with varying volatility over different time-steps (Derman *et al.*, 1996). An interesting additional feature of implied trees is that once we have a tree that fits the smile, we are able to extract the risk-neutral distribution of future stock prices as implied by the market.

2.3 A two price framework

Traditional valuation models rely, amongst several other weaker assumptions, on the requirement of a frictionless market indicating where buying and selling can be done against the same price (Haug and Taleb, 2011). These theories price by eliminating all risk such that there exist no preference over the required return, and derivatives can be perfectly replicated which leads to the law of one price. Nevertheless, actual markets show two prices (even in the most liquid markets like ATM S&P 500 options); a price for buying (ask) and a price for selling (bid). A lot has been written about the meaning and decomposition of spreads, but the purpose of this thesis is not to replicate this discussion so we will refer to Madan and Schoutens (2010) for an exentisive overview on this topic. In this thesis we build on the standpoint of Madan and Cherny:

'The differences between bid and ask prices can be quite large and may have little connection to processing, inventory, transaction cost or information considerations. The differences instead reflect the very real and substantial cost of holding unhedgeable risk' (Madan and Schoutens, 2010).

The psychological difference that people face when selling or buying is also introduced by Miller and Shapira (2004) and is broadly in line with the work on Cumulative Prospect Theory (CPT) by Tversky and Kahneman (1992) as we explain later on and can also be seen as a protection mechanism in line with the work of Nassim Taleb on antifragility (Taleb, 2013) where the market seeks convexity in buying and concavity in selling.

Our viewpoint is that having two prices for the same financial product is not irrational, but quite logical or even 'rational'. It is being consistent in applying risk aversion when buying *and* selling a future contingent claim. Consider being in the position of selling a lottery where the person on the buy-side has the possibility either to win 500 or 0 euros by equal probability. When we are risk averse, we only sell this lottery against[3] $250+\epsilon$. The other way around, when someone offers this particular lottery, we only want to buy it for $250-\epsilon$, because now again we apply risk aversion.

Conic Finance provides a framework to determine such buying and selling prices. The determination of pricing is always done from the perspective of 'the market' that acts as a counterparty in all trades (Madan, 2010). Fundamental in the theory of Conic Finance is the principle of coherent risk measures and acceptability indices. The concept of one-period static coherent risk measures can be traced back to the early work of Artzner *et al.* (1999). Because the primary focus of this thesis will be on valuation, we will rely on the work of Jobert and Rogers (2008) into defining dynamic valuation measures.[4]

[3] The expected value plus a small amount.

[4] A valuation measure is simply the negative of risk measure explained by Jobert and Rogers (2008) & Artzner *et al.* (2007) & Artzner *et al.* (1999).

2.3.1 Axiomatic introduction of one-period acceptability measures

Let Ω be a finite state set where X(Ω) represents the outcome generated when the state of nature $\omega \in \Omega$ materializes. An acceptability measure is a non-negative number that maps X(Ω) to $\mathcal{R}$. We define $\phi(X)$ as the degree of acceptability associated to a position X. An acceptability measure is called coherent if it statisfies the following four axioms:

A1 *Concavity* $\phi\big(\lambda X + (1-\lambda)Y\big) \geq \lambda\phi(X) + (1-\lambda)\phi(Y)$, $(0 \leq \lambda \leq 1)$,

A2 *Positive homogeneity* $\phi(\lambda X) = \lambda\phi(X)$, $\lambda \geq 0$,

A3 *Monotonicity* $X \leq Y \implies \phi(X) \leq \phi(Y)$,

A4 *Translation invariance* $\phi(X+m) = \phi(X) + m$, $m \in \mathcal{R}$.

The risk adjusted value of a position is then defined by computing $\mathbf{E}^{\mathcal{Q}}[X]$ under each test probability measure $\mathcal{Q} \in \mathcal{M}$. Then we take the minimum of all expectations which corresponds to the following valuation operator (Artzner *et al.*, 2007).

$$\phi(X) = \inf_{\mathcal{Q}\in\mathcal{M}} \mathbf{E}^{\mathcal{Q}}\big[X\big]. \tag{2.4}$$

Recent, a lot of work has been done in extending static one-period frameworks into dynamic ones. In a dynamic framework measurements are done throughout time and adapt to the flow of new information (Bielecki *et al.*, 2017). The majority of these contributions work in an axomiatic framework that requires strong dynamic consistency (sdc). Examples are Riedel (2004), Bielecki *et al.* (2014) and Artzner *et al.* (2007). In Chapter 2.4 we define different types of consistency and introduce the concept of Tuned Risk Aversion (TRA) under a weaker form of consistency.

2.3.2 Pricing based on acceptability

Consider a set of random payoffs X of a derivative paid out at maturity T. The set of random variables X are defined on $(\Omega, \mathcal{F}, P)$. If X is a set of non-negative random variables that can be obtained against zero-initial cost, it should by definition always be acceptable at all levels, because it is in fact an arbitrage opportunity (Cherny and Madan, 2009); an investor obtains a potentially positive cash flow against zero cost wich should be a no-brainer! However investors are also willing, and able, to do non-arbitrage trades. Therefore we need to define the fundamental concept of acceptable risk defined by Artzner *et al.* (1999) & Cherny and Madan (2009) in order to determine when a non-arbitrage trade is acceptable. Like we stated earlier, in traditional theory (under deterministic discounting) the price of a derivative equals:

$$\phi(X) = \mathbf{E}^{\mathcal{Q}}\Big[\frac{X(T)}{B(T)}|\mathcal{F}\Big] = B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\Big[X(T)|\mathcal{F}\Big]. \tag{2.5}$$

Then the market is respectively willing to buy/sell at initial zero cost[5] the following trade (Madan and Schoutens, 2016):

$$Z_{buying} = X - B(T)b, \quad \text{for b} \leq \phi(X), \tag{2.6}$$
$$Z_{selling} = B(T)a - X, \quad \text{for a} \geq \phi(X). \tag{2.7}$$

It is very important to understand that these bid and ask prices are determined from the market' perspective. A bid price is the price the market is willing to pay

[5]If a set future payoffs (X) has a non-zero initial cost we introduce 'b' as price to paid. The difference 'Z' is having zero-initial cost.

to obtain a potential future payoff X. Based on the above equations, a potential candidate for acceptable risk is the set zero-cost cash flows defined as:

$$\mathcal{A}_t^1 = \big\{ Z \mid \phi_t(Z) = B_t(T)^{-1}\mathbf{E}_t^{\mathcal{Q}}\big[Z \mid \mathcal{F}_t\big] \geq 0 \big\}. \tag{2.8}$$

However, market prices for buying and selling are not the risk-neutral ones, but depend on the direction of trade. Therefore we need to consider convex proper subsets of the set of risk of non-negative risk-neutral value that still contain the non-negative random variables as models for the set of potentially acceptable zero-cost cash flows by the market (Madan and Schoutens, 2016). This subset $\mathcal{A}^2$ is modeled as a convex cone and defined by Artzner *et al.* (1999) & Cherny and Madan (2009) under a convex set of probability measures[6] $\mathcal{M}$.

$$Z \in \mathcal{A}^2 \ , \ \ B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\big[Z\big] \geq 0 \ , \ \ \forall \ \mathcal{Q} \in \mathcal{M}. \tag{2.9}$$

When the market then respectively accepts to buy (bid) or to sell (ask) against price 'b' and 'a' we get:

$$B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\big[X - bB(T)^{-1}\big] = B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\big[X\big] - b \geq 0, \tag{2.10}$$

$$B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\big[aB(T)^{-1} - X\big] = a - B(T)^{-1}\mathbf{E}^{\mathcal{Q}}\big[X\big] \geq 0. \tag{2.11}$$

Or in terms of acceptability for $\mathcal{Q} \in \mathcal{M}$:

$$X - bB(T)^{-1} \in \mathcal{A}^2, \tag{2.12}$$

$$aB(T)^{-1} - X \in \mathcal{A}^2. \tag{2.13}$$

An index of acceptability assigns a level of acceptability to a number $X \in \mathcal{R}$ by $\alpha(X)$ with X being acceptable if $\alpha(X) \geq \gamma$ where γ represents a fixed acceptability level. Acceptability indices and families of probability measures are related by the relationship that if α is an index of acceptability, and $\gamma \geq 0$ is a level of acceptability, there exist a set of $\mathcal{M}$ measures such that a random variable $X \in \mathcal{R}$ is acceptable at level γ if and only if it has posive expectation under $\mathcal{Q} \in \mathcal{M}_\gamma$ (Cherny and Madan, 2009). By employing a fixed acceptability index α with a fixed level of acceptability γ we are able to price the residual risk of X. When the market sells X it wants a in return. Therefore the residual cash flow $a - X$ must have an acceptability set by index α above fixed level γ. This minimal price equals the ask price of X (Madan and Schoutens, 2010).

$$bid(X) = B(T)^{-1} \sup_{\mathcal{Q}\in\mathcal{M}} \mathbf{E}^{\mathcal{Q}}\big[X\big] = \inf\big\{b : \alpha(X - b) \geq \gamma\big\}, \tag{2.14}$$

$$ask(X) = B(T)^{-1} \inf_{\mathcal{Q}\in\mathcal{M}} \mathbf{E}^{\mathcal{Q}}\big[X\big] = \sup\big\{a : \alpha(a - X) \geq \gamma\big\}, \tag{2.15}$$

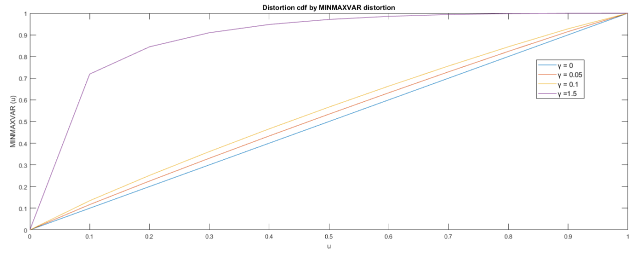
$$\text{ask}(X) = -\text{bid}(-X) \ \ i.e. \text{ Reflection principle.} \tag{2.16}$$

2.3.3 Operationalizing acceptability

When we assume law invariance, the only information we need in order to test acceptability at level γ for a cash flow X, is the distribution function F_X (Madan and Schoutens, 2010). Cherny and Madan (2009) proposed the use of distortion functions in order to operationalize the index of acceptability. We denote the distortion function by $\Psi^\gamma(\cdot)$. The distortion function is increasing and concave on the unit interval $\big[0{,}1\big]$ and zero at zero, and unity at unity for $\gamma \geq 0$. See Figure 2.1 for an example of a probability distortion function under different levels of γ.

[6] The higher the acceptability hurdle, the larger the supporting set $\mathcal{M}$ and the smaller the cone of acceptability (Cherny and Madan, 2009). The widest set is the set of one measure. In this case we have for acceptability the complete half-space.

Figure 2.1: Distortion by MINMAXVAR distortion function for different γ.



When we apply a distortion function on a cumulative distribution function (cdf) it 'distorts' values dependent on γ, the higher γ the more distorted it will be. Due to the concave character, lower quantiles are being reweighted upwards where higher quantiles are reweighted downwards which leads to lower expectations[7]. The acceptability index $\alpha(X)$ is then the largest γ such that the distorted expectation of X remains positive (Madan and Schoutens, 2010):

$$\alpha(X) = \sup\big\{\gamma \geq 0\ ,\ \int_{-\infty}^{\infty} x d\Psi^{\gamma}\big(F_X(x)\big) \geq 0\big\}. \tag{2.17}$$

Now we can define acceptability via distortion functions in the following way: for a random variable X, the family of measures $\mathcal{Q}^{\gamma}$ distorts the cumulative distribution function of X by γ. The value $\alpha(X)$ is the maximal level of distortion such that the distorted expectation defined in Equation 2.17 remains positive. For an overview of possible distortion functions we refer to Cherny and Madan (2009) & Madan and Schoutens (2016).

Under the assumption of law-invariance and comonotone additivity (Kusuoka, 2001), bid and ask prices follow directly from distorted expectations that we will use to operationalize indices of acceptability:

$$rn(X) = B(T)^{-1} \int_{-\infty}^{\infty} x dF_X(x), \tag{2.18}$$

$$bid(X) = B(T)^{-1} \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_X(x)), \tag{2.19}$$

$$ask(X) = -B(T)^{-1} \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_{-X}(x)). \tag{2.20}$$

Due to the concavity of the distortion function, domination of ask prices over bid prices is automatically sustained (Madan, 2010). Now we are gong to apply these definitions on a plain vanilla call option with payout $\big(S-K\big)^{+}$. We price the option against its distorted expectation in case we cannot form a perfect replication strategy. The ask and bid prices of options then become:

$$ask_{\gamma}(C) = \int_{K}^{\infty} \Psi^{\gamma}(1 - F_S(x))dx, \tag{2.21}$$

$$bid_{\gamma}(C) = \int_{K}^{\infty} (1 - \Psi^{\gamma}(F_S(x)))dx. \tag{2.22}$$

This application of distortion functions is in line with the general idea of CPT, where in the area of gains people are risk averse (concave) while in the area of losses people are risk seeking (convex) (Tversky and Kahneman, 1992). For the 'market' acting as a counterparty in all trades, buying a potential future payoff against a bid pricef means the market acts in the area of gains when it needs to assess the possibilities of a future payoff. Therefore the market will adjust the

[7] $\gamma = 0$ gives the standard expectation which is in our case the standard risk-neutral one.

probability of an upward movement downwards and the probability of a downward movement upwards. When the market is selling a future obligation it acts in the area of losses (convex). There it adjusts the upward movement upwards and reduces the probability of a downward movement.

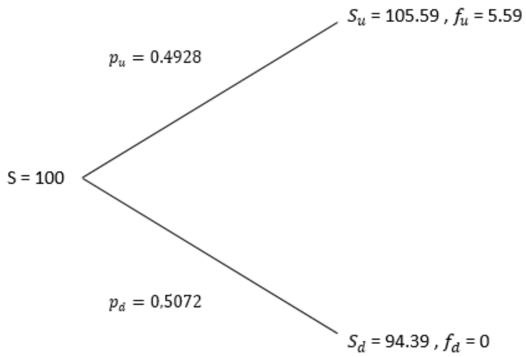
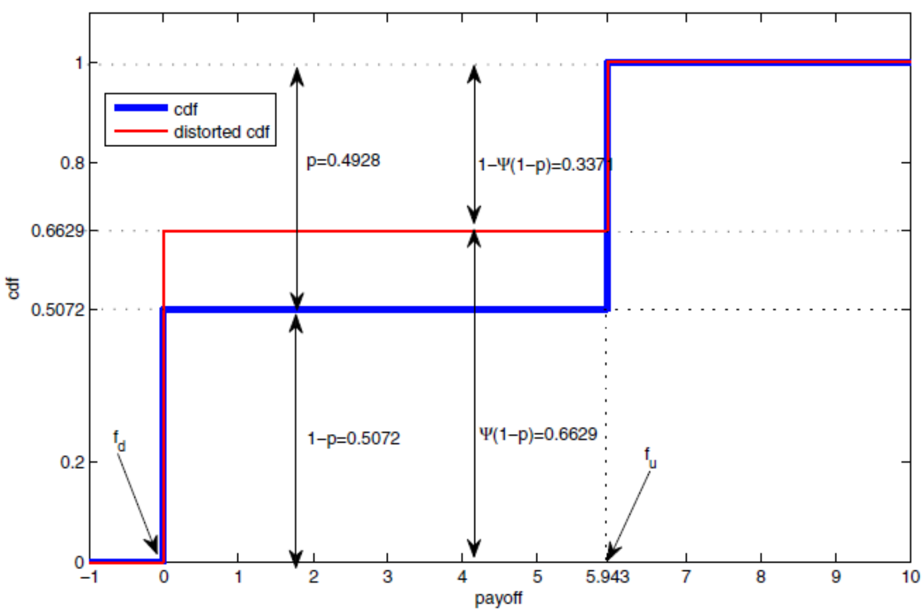


Figure 2.2 presents the distorted cdf of the above one-step binomial tree of an ATM call option where S_i represents the level of the underlying, and f_i the payoff in state i. We see that when the market offers its price (bid) for which it accepts to buy this option, it adjusts the probability of the lowest outcome upwards $\Psi(1-p) = 0.6629$ versus the base probability $(1-p) = 0.5072$, and lowers the upward probability to $1 - \psi(1-p) = 0.3371$ versus the base $p = 0.4928$.

Figure 2.2: Distortion of probabilities for a call option bid price (Madan *et al.*, 2016).



The bridging of CPT and option valuation is actually not something new and already introduced in the following papers: (Versluis and Lehnert, 2010) and (Nardon and Pianca, 2014).

Delta hedging in a two-price framework

Hedging in Conic Finance is completely different than it is in traditional risk-neutral frameworks. In Conic Finance one seeks to design hedges that maximize the concave bid prices for positions held, or minimize the convex ask prices for positions promised for one-step ahead risk (Madan and Schoutens, 2016). The concept of hedging applies to add positions that enhance current market values where risk-neutral finance seek to zero out risk to price derivatives in a risk-neutral setting (Madan *et al.*, 2016). As a result derivatives get five different prices in Conic Finance framework; unhedged (bid/ask), hedged (bid/ask) and the risk-neutral one.

Lets consider a tree setting with a portfolio π consisting of a long position in the derivative, in our case an option C, and a Δ short position (negative) in the underlying S. The idea of hedging in Conic Finance is to find the optimal Δ^* where the bid price is maximized or the ask price is minimized (Madan *et al.*, 2016).

When the stock price moves up/down the portfolio value becomes respectively:

$$\pi_{up} = C_u + \Delta S_u, \tag{2.23}$$

$$\pi_{down} = C_d + \Delta S_d. \tag{2.24}$$

Where the bid and the ask price become:

$$ask_\gamma = B(T)^{-1}\bigg(\Psi^\gamma(p)\pi_{up} + (1 - \Psi^\gamma(p))\pi_{down}\bigg). \tag{2.25}$$

$$bid_\gamma = B(T)^{-1}\bigg((1 - \Psi^\gamma(1-p))\pi_{up} + \Psi^\gamma(1-p)\pi_{down}\bigg). \tag{2.26}$$

2.4 Consistency and Tuned Risk Aversion

When we extend one-period frameworks into multi-period frameworks an important role is played by time consistency. In a dynamic set-up, measurements are done throughout time and adapt to the flow of available information, where the assessment of value should be updated in a consistent way over time (Bielecki *et al.*, 2017). Consistency is important, because we seek preferences that are consistent when we evaluate investment opportunities. We follow the work of Roorda and Schumacher (2013) and Roorda and Schumacher (2016) in order to define consistency and corresponding update rules. For the interested reader we will refer to a very recent and extensive literature study on time consistency by Bielecki *et al.* (2017).

We will define two types of consistency in a dynamic valuation context:

1. Strong dynamic consistency.
2. Sequential consistency.

2.4.1 Consistency

Strong dynamic consistency
Strong dynamic consistency (sdc) is imposed by the strictly monotonicity axiom of preference relations and is closely related to the law of iterated expectations and the Bellman principle (for American-stye options) and enables backward recursive solving (Artzner *et al.*, 2007). It requires that evaluations under a given acceptability measure should not change when the payoffs following a given event in the future are replaced by their evaluations conditional on that same event. It basically means that two positions with identical conditional values in every state at some future date must have the same value today (Roorda and Joosten, 2015) that leads to non-linear valuations that only depend on conditional values. In most of the recent work on multi-period dynamically consistent valuations,

the requirement of sdc is imposed upon the developed frameworks, examples are Artzner *et al.* (2007) and Bielecki *et al.* (2014). At the same time, several studies showed the shortcomings and restrictiveness of sdc (Roorda and Schumacher, 2007) and for an extensive and recent overview we refer to Machina and Viscusi (2014). We define sdc as:

$$\phi_t(X) = \phi_s\big(\phi_t(X)\big), \; s \leq t. \tag{2.27}$$

or equivalently

$$\phi_t(X) = \phi_t(Y) \implies \phi_s(X) = \phi_s(Y), \tag{2.28}$$

$$\phi_t(X) \leq \phi_t(Y) \implies \phi_s(X) \leq \phi_s(Y). \tag{2.29}$$

Under dynamic consistency, valuations are updated per time-step according to a Bayesian updating scheme.[8] In Conic Finance the bid price at an intermediate date is valued by:

$$\phi_s(X) = \inf_{\mathcal{Q}\in\mathcal{M}} \mathbf{E}^{\mathcal{Q}}\big[\phi_t(X) \mid \mathcal{F}_t\big] \; t \geq s. \tag{2.30}$$

Roorda and Schumacher (2007) showed in a rather easy way that sdc automatically imposes sequential consistency.

Sequential consistency
Sequential consistency is introduced by Roorda and Schumacher (2007) and requires that a position cannot be evaluated positively if all conditional evaluations at later stages are negative. It is called sequential to express that values at a given position in a sequence of time instants should not change sign predictably. It combines the notions of 'weak acceptance' and 'weak rejection consistency' into one that can be defined as:

$$\phi_t(X) = 0 \implies \phi_s(X) = 0 \;, \; s \leq t. \tag{2.31}$$

That is basically the combination of the following two conditions that apply directly to respectively acceptance and rejection consistency:

$$\phi_t(X) \geq 0 \implies \phi_s(X) \geq 0, \tag{2.32}$$

$$\phi_t(X) \leq 0 \implies \phi_s(X) \leq 0. \tag{2.33}$$

$$\tag{2.34}$$

As defined by Roorda and Schumacher (2013), an update is sequentially consistent when

$$\phi_t(X) = 0 \implies \phi_s(X) = 0, \tag{2.35}$$

$$\inf \phi_t(X) \leq \phi_s(X) \leq \sup \phi_t(X). \tag{2.36}$$

Example
In order to clarify the difference between sequential and sdc we present an example inspired by example 3.8 presented in Roorda and Schumacher (2013). Consider two nonrecombinding two-step binomial trees with probability 99% for moving to an upward node during all time-steps. X has payoff profile (0, 0, 0, -10) and Y = (0, 0, -10, -10). We take $\hat{\phi}_0$ and $\hat{\phi}_1$ as single-step worst-case operators between time intervals [0,1] and [1,2].

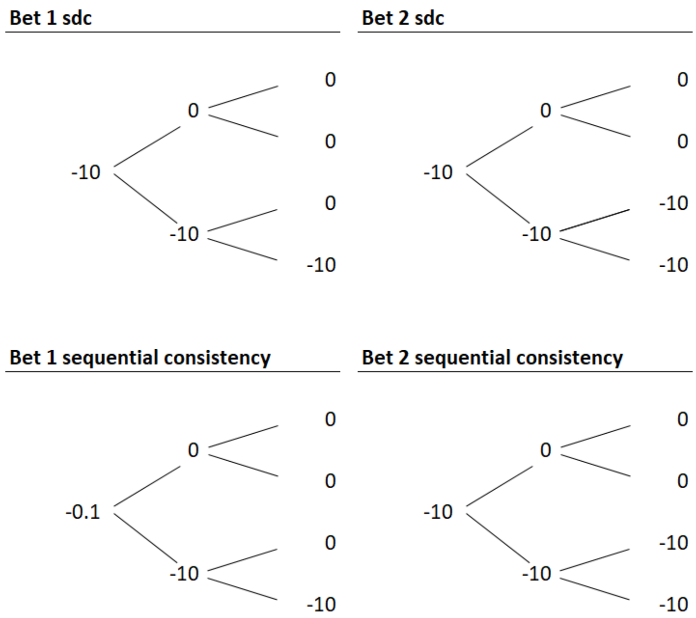
[8] Updates according to Bayes rule.

We define two valuation operators ϕ in the following way (Roorda and Schumacher, 2013):

$$\textbf{sequential} \quad \phi_1 = \hat{\phi}_1 \ , \ \phi_0(X) = \min\left[E_0(\hat{\phi}_1(X)),\ \hat{\phi}_0(E_1(X))\right], \tag{2.37}$$

$$\textbf{sdc} \quad \phi_1 = \hat{\phi}_1 \ , \ \hat{\phi}_0(X) = \hat{\phi}_0\big(\hat{\phi}_1(X)\big). \tag{2.38}$$

Figure 2.3: Valuation under strong dynamic and sequential consistency.



Intuitively we would always favor X over Y if we would be offered the choice. However, when we assess the risk of the bets under sdc, both bets carry the same risk $\hat{\phi}_0(X) = \hat{\phi}_0(Y) = -10$ which appears to be very conservative for X. We can see in Figure 2.3 that the obvious difference in risk is better captured under sequential consistency: $\phi_0(X) = -0.1 > \phi_0(Y) = -10$ with $\hat{\phi}_1(X) = \hat{\phi}_1(Y) = -10$.

2.4.2 Risk aversion in option pricing

Dynamic risk aversion

A valuation measure ϕ_t is risk averse in the valuation of a future payoff X, at time t, with respect to some measure $\mathcal{Q}$, when $\phi_t(X) \leq \mathbf{E}_t^{\mathcal{Q}}[X]$ holds. The difference between $\mathbf{E}_t^{\mathcal{Q}}(X) - \phi_t(X)$ is called the risk margin (Föllmer *et al.*, 2004).

A dynamic valuation ϕ_t exhibits consistent risk aversion with respect to a measure $\mathcal{Q}$ under the assumption that the supermartingale property[9] holds; $\forall t\ \phi_s \leq E_s^{\mathcal{Q}} \phi_t$ (Detlefsen and Scandolo, 2005), which is in line with the consistent risk aversion argument made in Roorda and Schumacher (2007) that requires that average risk premiums at given level of information could not exceed the risk premium that is required without the new information. Risk premiums should therefore on average decrease by the obtaining of new information.

In all current Conic Finance applications, risk aversion is parameterized by γ as a rectangular set requiring it to be fixed for every time-step. This rectangularity of γ is imposed by the strong dynamic consistency requirement. We, however,

[9] Threshold function $\theta_t^{min}(Q) = 0$.

believe that imposing sdc could be too restrictive and unrealistic in practice.

For instance, Roorda, Joosten and Schumacher showed in several recent articles that requiring sdc comes at a price. Roorda and Schumacher (2007) show that it could lead to the accumulation of conservatism and therefore to overly conservative measures of risk that could potentially also result in valuations of derivatives that are too conservative, like we showed in Figure 2.3. At the same time, sdc leads to normative choice preferences that are not in line with descriptive experimental choice preferences showed by Roorda and Joosten (2015). In order to overcome these shortcomings they introduced a concept called TRA where we are allowed to use more flexible distributions of risk and therefore allow non-rectangular tuning-sets under sequential consistency. We also suspect that the additional flexibility of TRA enables us to build functional relationships between distortion and option characteristics like time to maturity and moneyness.

2.4.3 Introduction of Tuned Risk Aversion

In TRA the outcomes of valuation functions are compared under patterns of risk aversion with a risk aversion parameter for each single period. The crux, explained in Roorda and Schumacher (2016), is that single-period dynamic consistency does not dictate long-term features of the valuation. Because opposed to sdc, it is not a requirement to keep the level of risk aversion the same during every time-step. The key is that valuations become set-recursive where risk aversion is stored in an auxiliary vector. The method of storing additional parameters in option valuation is similar to the forward shooting grid method used for the pricing of path-dependent options like lookback options. The maximum value conditioned on time is stored and updated in an auxiliary vector. This method is defined in Bormetti *et al.* (2004) and Barraquand and Pudet (1996). We still value recursively, but risk aversion is not longer a constant over all time-steps.

We define Ψ_t^γ as the one-step conditional valuation corresponding to MINMAX(γ) for $\gamma \in \mathcal{N}$. When we set $\gamma = 0$, we price under the original non-distorted measure, usually the risk-neutral one i.e., $\Psi_t^0(X) = \mathbf{E}_t^{\mathcal{Q}}[X]$. Under sdc, γ is equal during every time-step (rectangular tuning-set) and the backward-recursive valuation becomes:

$$\phi(X) = \big\{\Psi_t^\gamma\big(..\big(\Psi_{T-1}^\gamma(X)\big)...\big) \mid \sum_{i=((T-1)-t)}^{t} \gamma_i = n\gamma\big\}. \tag{2.39}$$

Under sdc multi-step valuations are just the sum of single-step valuations in terms of risk-aversion. In order to be truly sdc the risk aversion budget needs to increase linearly with an increase in time. For instance when we value options by equal step-size (20 steps every week) risk aversion budgets under sdc become:

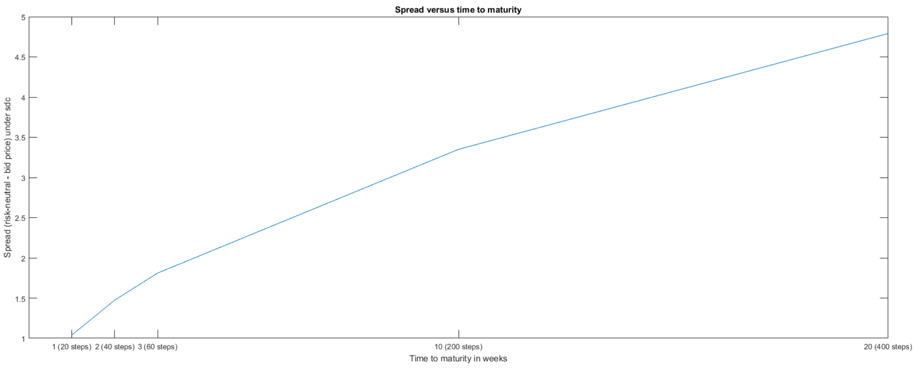
$$\sum_{i=1}^{20} \gamma_i \leq n * \gamma \qquad \textbf{one-week}, \tag{2.40}$$

$$\sum_{i=1}^{40} \gamma_i \leq 2 * n * \gamma \qquad \textbf{two-week}, \tag{2.41}$$

$$\sum_{i=1}^{T*20} \gamma_i \leq T * n * \gamma \qquad \textbf{T-week}. \tag{2.42}$$

Under sdc it is highly unlikely that the one-week option γ equals the T-week option γ, because this will make spreads (risk-neutral versus bid price) blow up due to the increase in γ.[10] In Figure 2.4 we plotted the spread development of an ATM call option with different time to maturities, constant step-size (0.35 days) and a constant $\gamma = 0.15$. Due to the blow-up behavior we will never be able to find a functional relationship between γ and time to maturity, while at the same time remaining strong dynamically consistent.

Figure 2.4: Spread (risk-neutral price - bid price) for different times to maturity under constant $\gamma = 0.15$ and budget equation 2.39.



Under TRA we can distribute γ over periods of time. The restrictions on how to do this are defined by tuning-sets. A possible tuning-set is to limit the total amount of γ we can distribute over the complete time interval while at the same time cap the amount of γ we can apply during single-steps. In order to produce bid prices we distribute risk aversion such that we minimize the conditional expected outcome as follows:

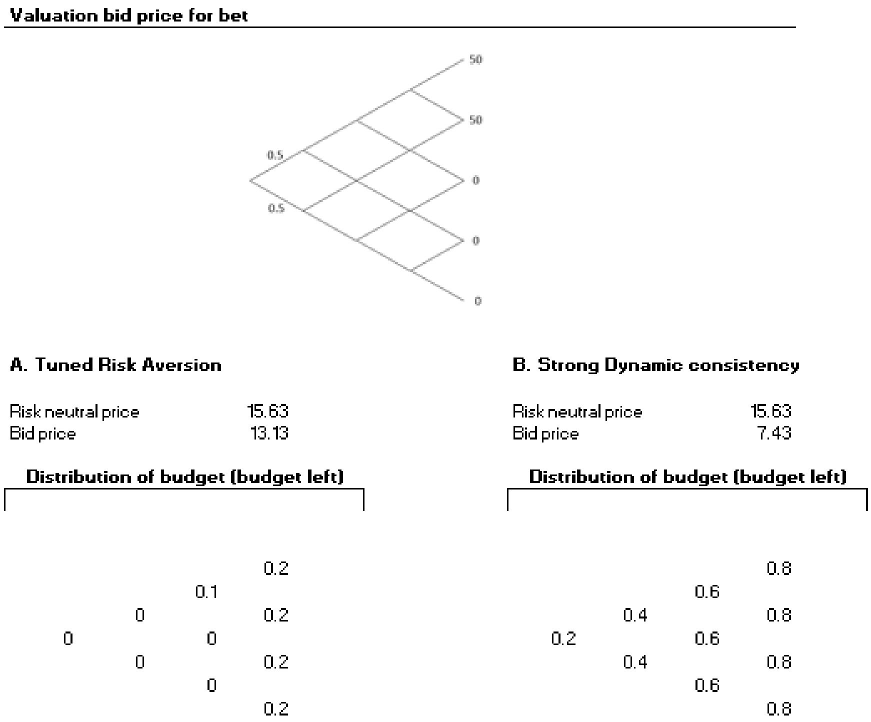
$$\phi(X) = \inf \left\{ \Psi_t^{\gamma_t} \left(.. \left(\Psi_{T-1}^{\gamma_{T-1}}(X) \right) ... \right) \mid \sum_{i=((T-1)-t)}^{t} \gamma_i \leq \Gamma, \quad \gamma_i \leq \Gamma_{step} \right\}. \tag{2.43}$$

The major advantage of TRA over sdc is that we can model different budget dependencies and therefore have a way to solve the blow up behavior we face under sdc. Therefore TRA provides a promising framework in order to build a uniform valuation framework in Conic Finance. Below we provide an example that shows the difference between sdc and TRA. Lets assume we are offered to buy a derivative with the following payoff profile (50, 50, 0, 0, 0) and equal probability of moving up and down. In Figure 2.5 we show the difference between TRA and sdc in producing a bid price for this derivative.

Under sdc risk aversion is set equal during every time-step $\gamma = 0.2$ where under TRA we can distribute the $\gamma = 0.2$ over the different time-steps. In order to come up with a price, we need to determine the residual risk we feel comfortable to hold $(X - bid)$ by setting a price which makes $\alpha(X - bid) \geq \gamma$. Because we are buying, we act in the area of gains and therefore we decrease the probability of an upward movement by applying risk aversion there where it has the most impact, because in Conic Finance a bid price is the infimum of a set of expected values under a set of probability measures.

[10] Spreads for longer-term options are bigger due to the higher level of uncertainty, but they do not grow linearly with time.

Figure 2.5: TRA versus sdc for $\gamma = 0.2$. Distribution of budgets shows the budget of risk aversion left to allocate during every step.



Because applying risk aversion has no effect on zero-outcomes, the algorithm distributes risk aversion in the top nodes of the tree where we face the possibility of earning 50 and therefore lowering the probability to end up in that state. We see that after the first backward-recursive step we are left with only $\gamma = 0.1$. All other budget is already spent in previous state. For ask prices we see an opposite distribution where risk aversion is applied low in the tree, because we face the risk of losing 50 and we want set a price that reflects this residual risk that we hold $\alpha(ask - X) \geq \gamma$.

Chapter 3

From Theory to Practice

In this chapter we describe the workings of our model based on the theory presented in the previous chapter. We start by explaining the building blocks and assumptions followed by its dynamics and sensitivities. The exact numerical description of the model can be found in Appendix 7.3. All versions of the algorithm are modeled in Matlab.

3.1 Outline Model and building blocks

Within a two price framework bid and ask prices represent respectively the infimum and supremum of distorted expectations over the set of supporting measures (Madan and Schoutens, 2010). In line with the work of Madan *et al.* (2016), we start with the risk-neutral measure as base probability distribution. We model acceptability[1] with (concave) distribution functions for bid prices proposed by Cherny and Madan (2009), and bid and ask prices become expectations under convave distortion (Kusuoka, 2001):

$$\text{bid} = \sum_{i=1}^{n} \Psi^{\gamma}(p_i) f_i. \tag{3.1}$$

Alternatively we may use an entropic risk-measure (Föllmer *et al.*, 2004):

$$\text{bid} = -\frac{1}{\gamma} \log \big(\sum_{i=1}^{n} p_i e^{-\gamma f_i} \big). \tag{3.2}$$

Within this thesis we follow the first method, because we want to test the difference between the earlier work done in Conic Finance compared to our new approach.

3.1.1 Methods of distortion

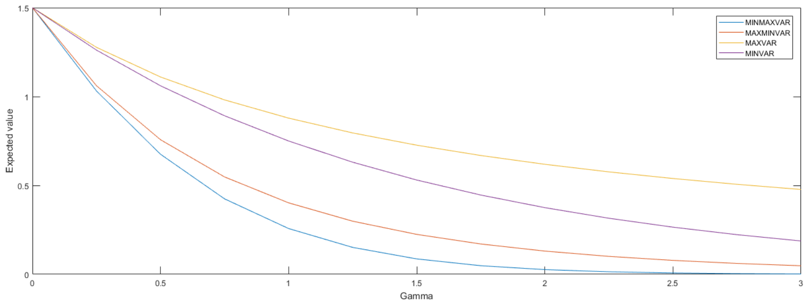
In our model we use the proposed distortion functions presented by Cherny and Madan (2009). In Table 3.1 we present an overview of possible concave distortion functions which are graphically presented in Figure 3.1.

Table 3.1: Overview distortion functions $\mid \gamma \geq 0$

Distortion	$\Psi(p)$
MINVAR	$1-(1-p)^{1+\gamma}$
MAXVAR	$p^{\frac{1}{1+\gamma}}$
MAXMINVAR	$(1-(1-p)^{1+\gamma})^{\frac{1}{1+\gamma}}$
MINMAXVAR	$1-\big(1-p^{\frac{1}{1+\gamma}}\big)^{1+\gamma}$

[1]Under the assumptions of Comonotone additivity and law invariance.

Figure 3.1: Graphical representation of [3 0] payoff under distortion for several levels of gamma.



3.1.2 Tuning-sets

By imposing a weaker form of consistency we allow backward recursion with flexible risk aversion for every time-step. Because this approach is completely new, there are no suggestions on the shape tuning-sets should or could have whatsoever. Within this thesis we test two different sequential consistent tuning-sets as alternatives for strong dynamic consistency. We define γ_t as the amount of risk aversion we apply during a time-step.[2] Tuning-sets are calibrated against $\overline{\gamma}$ where $\overline{\gamma}$ caps γ_t that can be applied during a single period of time (Equation 3.3, 3.5 and 3.7), and determines the amount of risk aversion that can be applied over all time-steps together (Equation 3.4, 3.6 and 3.8). See Figure 3.2 for a graphical representation of risk distribution under all three scenarios.

$$\delta = 1 \; (\textbf{sdc}) \qquad \gamma_t \leq \overline{\gamma}, \tag{3.3}$$

$$\sum_{0}^{N*T} \gamma_t \leq N*T*\overline{\gamma}, \tag{3.4}$$

$$\delta = \frac{1}{2} \; (\textbf{TRA1}) \qquad \gamma_t \leq \overline{\gamma}, \tag{3.5}$$

$$\sum_{0}^{N*T} \gamma_t \leq \sqrt{N*T}*\overline{\gamma}, \tag{3.6}$$

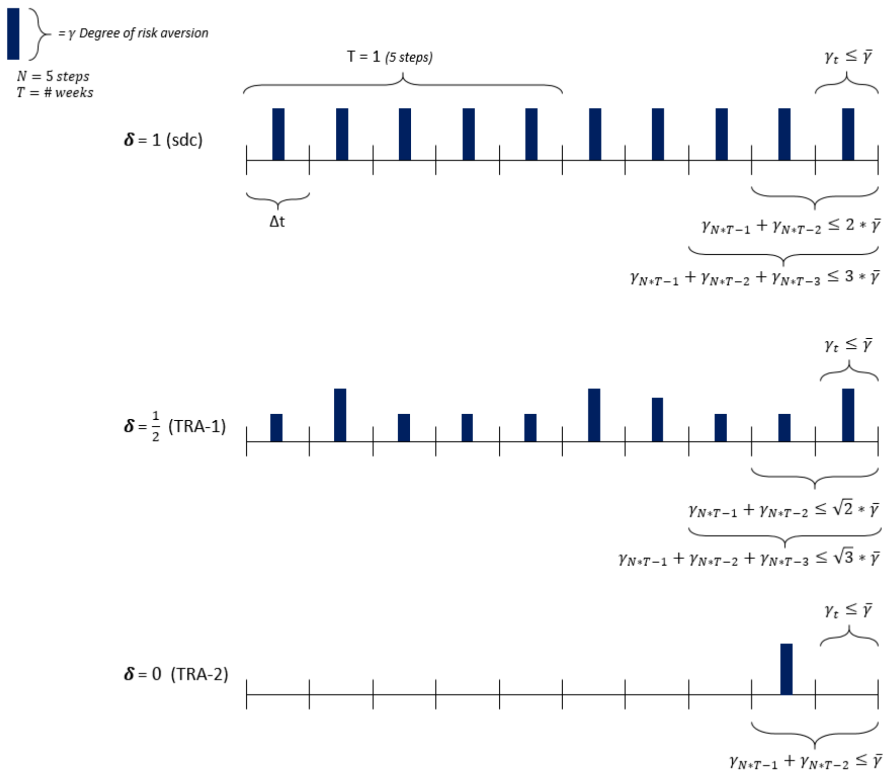
$$\delta = 0 \; (\textbf{TRA2}) \qquad \gamma_t \leq \overline{\gamma}, \tag{3.7}$$

$$\sum_{0}^{N*T} \gamma_t \leq \overline{\gamma}. \tag{3.8}$$

Under **strong dynamic consistency** ($\delta = 1$ sdc) we are allowed to apply $\overline{\gamma}$ (or lower) during every time-step where the sum of risk aversion applied during all time-steps needs to be below the total budget of risk aversion which is simply the total number of steps multiplied by $\overline{\gamma}$. One can easily understand that this leads to applying $\overline{\gamma}$ (the maximum allowed amount) every time-step when we model bid prices as the infimum of a conditional expectation. Under **TRA-1** (**$\delta = 0.5$**) single-step risk aversion is still capped at $\overline{\gamma}$, but the multistep budget constraint is more restrictive, see Equation 3.6. This limits the amount of risk aversion that can be applied and prevents the blow-up of spreads when we extend time. **TRA-2** (**$\delta = 0$**) is even more restrictive where the total amount of γ that can be distributed is equal to the single-step maximum $\overline{\gamma}$, but we are still free to distribute this small amount of γ over all time-steps or apply it all at once as in ($\delta = 0$).

[2]N equals the total number steps during a single time-period (a week), and n equals the total number of steps during the complete time period (N*T).

Figure 3.2: Illustration of strong dynamic consistency and Tuned Risk Aversion.



We can see that the single-step maximum is the same under all three tuning-sets, but budget functions are different. This again is the difference between TRA and strong dynamic consistency; single-period dynamic consistency does not dictate long-term features of valuations. The tuning-sets under TRA ($\delta \in \{0, 0.5\}$) are more restrictive in the total amount of risk aversion that can be applied over time-periods. Therefore by equal $\overline{\gamma}$ for all three tuning-sets, we face the following price ranking where $\delta = 1$ provides the lowest bid price (b) and $\delta = 0$ the highest:

$$b_{\delta=0} < b_{\delta=0.5} < b_{\delta=1}. \tag{3.9}$$

3.2 Binomial and trinomial modeling

Most of the recent work in Mathematical Finance aims at improving the BSM framework while still working under the law of one price. A good example is the work on Heston models where volatility is no longer deterministic (or constant), but modeled as a correlated stochastic process. We start with simple intuitive binomial and trinomial tree models where we seek empirical fitting with the additional flexibility provided by TRA. The most important reason is that trees can be solved backwardly and they are perfectly suited to distribute risk over different time-step positions.

The binomial tree is simple and intuitive, but quite rigid while it allows underlyings only to jump up or down. Binomial trees converge to the BSM model when increments are taken infinitesimal. It is therefore a discrete approximiation of geometric Brownian motion. While the binomial tree framework is a complete-market model, the trinomial tree framework is an incomplete market model (Madan *et al.*, 2016) and therefore no unique equivalent martingale measure exist. Every time-step the underlying can move up, down or to a middle state which creates an extra flexibility parameter to fit risk-neutral transition probabilities (Derman *et al.*, 1996). Trinomial trees converge faster than binomial trees to the risk-neutral valuation $E^Q\left[S\right] = Se^{rT}$ which increases computational speed (Rubinstein, 2000).

3.3 Value enhancement by hedging

Conic hedging aimes at value enhancement of bid and ask prices offered by the market for derivatives in stead of eliminating exposure completely. Within Conic Finance hedging aimes at maximizing the bid price of a future contingent claim as a functional of the risk held Madan *et al.* (2016) while at the same time minimizing the ask price as a functional of the risk promised. Madan *et al.* (2016) and Madan and Schoutens (2016) show that under the assumption of a frictionless hedging instrument and no short-selling restrictions, delta hedging does narrow bid and ask spreads, but there is no evidence for pricing improvement.

We hedge by adding a position in the future contract on the underlying. We therefore not only price the derivative but a package. By borrowing the price and paying it the next step with interest, we ensure initial zero-cost while the hedge position is self-financing (Madan *et al.*, 2016). The main idea is to seek a Δ^* position where the bid price is maximized or the ask price is minimized. The delta position in the future has close to zero contribution to the mid-package payoff due to the minor movement of the underlying in this case.

3.4 Stylized example

Below we present the results of our model under a stylized case which is an extension of the example used in Madan *et al.* (2016). We value a plain vanilla ATM two-month call option with the following parameters:

Table 3.2: Modeling parameters

$\sigma = 0.2$	$r_f = 0.01$	n = 4	N = 2	$\Delta t = 0.0417$	$T = 2$	$S_0 = 100$

We set $\overline{\gamma} = 0.2$. Under strong dynamic consistency (sdc) we therefore apply $\gamma_t = \overline{\gamma}$ during every time-step. Under TRA the total budget depends on the tuning-set.

$$\textbf{TRA-1} = \qquad \gamma_t \leq 0.2, \tag{3.10}$$

$$\sum_0^4 \gamma_t \leq \sqrt{4} * 0.2, \tag{3.11}$$

$$\textbf{TRA-2} = \qquad \gamma_t \leq 0.2, \tag{3.12}$$

$$\sum_0^4 \gamma_t \leq 0.2. \tag{3.13}$$

Based on Table 3.3 and Figure 3.3, we can draw some conclusions. All tuning-sets are very sensitive to changes in $\overline{\gamma}$ and bid prices show a convex decreasing profile when we increase $\overline{\gamma}$. As expected sdc has the strongest dependency on $\overline{\gamma}$. We also found that by applying hedging in trinomial trees, bid prices become very close to risk-neutral valuations when we decrease step-size, see Figure 3.4. This introduces a problem when we move from discrete to continuous pricing models and complicates pricing of options with significant spreads.

Table 3.3: Results valuation of stylized example $\overline{\gamma} = 0.2$.

1. Binomial tree $\overline{\gamma} = 0.2$	
Risk neutral	3.14
Sdc bid	1.39
TRA-1 bid	2.11
TRA-2 bid	2.59

				117.74
			113.03	
		108.51		108.51
	104.17		104.17	
100.00		100.00		100.00
	96.00		96.00	
		92.16		92.16
			88.47	
				84.93

up	1.04	**p_u**	0.49
down	0.96	**p_d**	0.51

1. Trinomial tree $\overline{\gamma} = 0.2$	
Risk neutral	3.24
Sdc bid	1.30
TRA-1 bid	2.09
TRA-2 bid	2.61

				125.98
			118.91	118.91
		112.24	112.24	112.24
	105.94	105.94	105.94	105.94
100.00	100.00	100.00	100.00	100.00
	94.39	94.39	94.39	94.39
		89.09	89.09	89.09
			84.10	84.10
				79.38

up	1.06	**p_u**	0.25
mid	1.00	**p_m**	0.50
down	0.94	**p_d**	0.25

Figure 3.3: Sensitivity of tuning-sets to $\overline{\gamma}$ (20 steps).

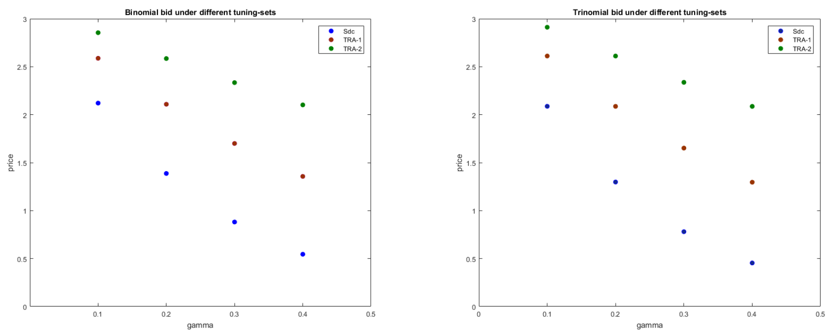
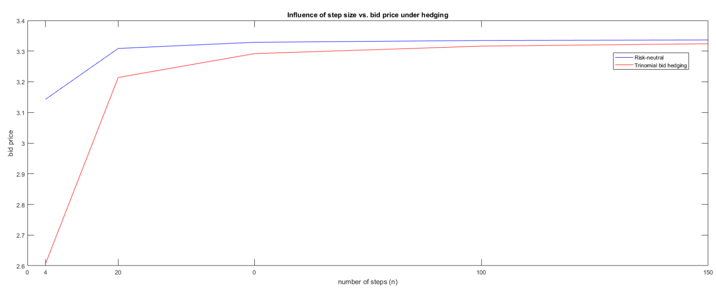


Figure 3.4: Sensitivity of pricing under hedging to step-size.



Chapter 4

Model Calibration and Results

In this chapter we calibrate our model against market quotations. We start with a broad description of our dataset after which we explain the calibration methods we used. We show whether Conic Finance models in general are able to produce prices close to market quotations, and we test whether there exists emperical evidence for the reflection principle. Subsequently we look at implied distortion patterns to see if there exists evidence that markets apply Tuned Risk Aversion in pricing options and investigate whether we can improve our original model. Below we present our different calibration steps:

Step 1 Calibrate implied volatility σ_I of each individual option against quoted mid prices by trying different σ' as input parameter for our algorithm. We calibrate both the binomial and trinomial case under $\overline{\gamma} = 0$ which results in the same price for all three tuning-sets. This way our valuation algorithm produces the exact mid price when we use $\{\sigma_I, \overline{\gamma} = 0\}$ as input parameters.

Step 2 Calibrate implied distortion $\overline{\gamma}$ against bid prices where we use the calibrated σ_I of each individual option found in **Step 1**. Within this step we try different $\overline{\gamma}$' to match a quoted bid price such that $\overline{\gamma}_I$ provides the model bid price that is closest to the quoted bid price. Each $\overline{\gamma}$ is calibrated under all three tuning-sets, such that we have three unique $\overline{\gamma}_I$ for each specific option.

Step 3 Use each calibrated $\overline{\gamma}_I$-bid as input variable to produce ask prices such that we are able to test for existence of the reflection principle. Due to our modeling setup (first calibrating against mid prices) we indirectly assume that ask prices are not only dependent on bid prices, but also on the mid price which has no theoretical foundation, but is a modeling consequence.

Step 4 Plot patterns of $\overline{\gamma}_I$ against moneyness and time to maturity to see whether TRA provides a way to model functional relationships between distortion and option characteristics. A functional relationship between time and distortion would enable us to price a large set of options by simply calibrating distortion against a smaller set of options, for instance one-week maturity options. Another potential interesting relationship is the one between implied volatility and distortion. Coming up with models that are capable of producing volatilities that match the exact implied volatility surface appears to be very complicated. More important, it is intuitively very odd that options with the exact same underlying apply different volatilities to that same underlying. Perhaps it is just a way to work with an imperfect model that became the general standard in the financial industry. Distortion potentially provides the key to solve this problem by keeping volatility a constant and adjust probability distributions solely by distortion.

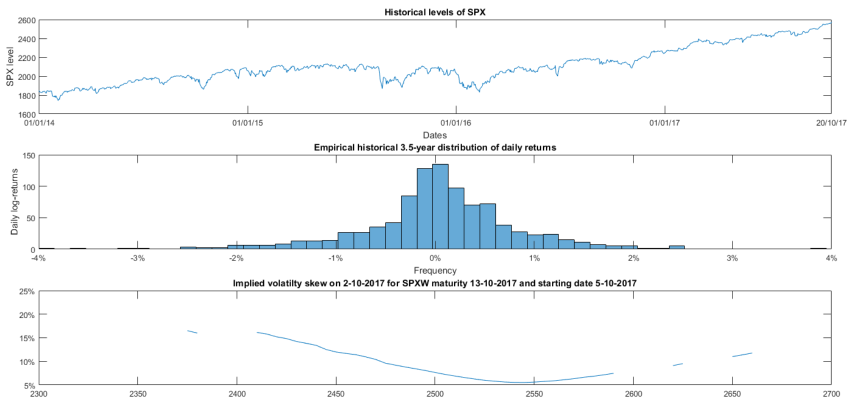
Our modeling setup is based on five important simplifying assumptions:

1. Discrete approximation of lognormal distribution for the underlying.
2. Relatively small number of steps during every period of time (20 per week).
3. Discretization of γ by a linearly spaced vector of 20-steps.
4. Constant risk-free rate for all maturities of 1% which equals the annual risk-free rate in the US (USA Department of Treasury, 2017).
5. Algorithm keeps track on complete sum of used γ budget used and available, but not of intermediate sub-budgets. For instance we know the γ-budget used between [T-5,T], but not on sub-interval [T-5, T-2] which potentially leads to inconsistent budget allocations.

4.1 Dataset and calibration parameters

The SPX is one of the most liquid financial instruments and its traded in a almost frictionless way (spread less than 1%). We use a dataset of historical option prices on the SPX obtained via CBOE Options Exchange (2017). The dataset contains historical end-of-day option chains between June 2017 - October 2017. Index levels of the S&P 500 are obtained via Yahoo Finance (2017). Because of the explorative character of this study, and the computational heaviness of the model, our test-case only contained 219 options (October 2017).

Figure 4.1: Overview dataset CBOE Options Exchange (2017). Historical level S&P 500 (a), historical distribution of daily returns S&P 500 (b), and plot on implied volatility of call options on the S&P 500 (c).



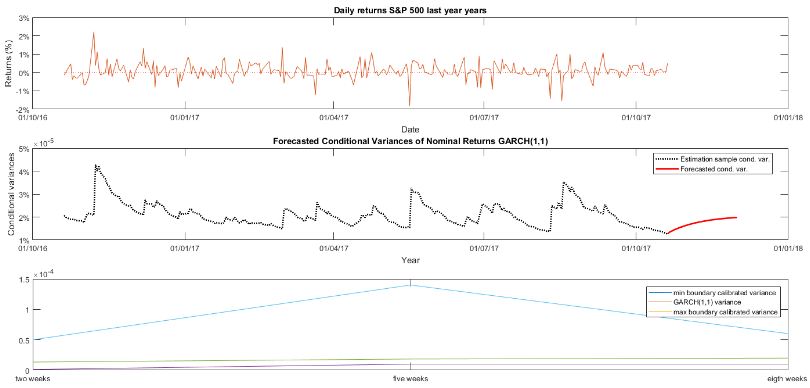
We need to calibrate both volatility (σ) and distortion ($\overline{\gamma}$) in order to test our model performance. Volatility determines the tree parameters; probability distribution for underlying $\{p_u, p_m, p_d\}$ and the change in the level of the underlying $\{S_u, S_m, S_d\}$, where distortion ($\overline{\gamma}$) adjusts the base probability distribution.

4.2 Calibration methodology

4.2.1 Volatility

We calibrated volatility against mid quotes in order to ensure that the risk-neutral case ($\gamma = 0$) delivers the mid price which is necessary for our distortion approach. By first calibrating against mid prices we follow the work of Madan *et al.* (2016). Our approach differs in the fact that we calibrate σ for a lognormal distribution (underlying assumption for trees) where Madan *et al.* (2016) calibrate a fmidvariance gamma process. Alternatively we could calibrate σ by econometric models like ARCH and GARCH to forecast volatility based on historical return data (Cumby *et al.*, 1993) and (Noh *et al.*, 1994). The problem with this approach is that we will never be able to fit prices due to the existence of the implied volatility surface, because GARCH produces single unique variance forecasts, see Appendix 7.4 for GARCH parameters. In Figure 4.2.c. we show the difference between the GARCH forecasted variance and our calibrated implied variance.

Figure 4.2: Historical daily returns (a), historical and forecasted day σ^2 (b) and bandwith calibrated implied variance from our model versus GARCH(1,1) forecast (c).



A second alternative is to use the markets' forecast of volatility by using the implied volatility calculated from quoted market prices. Research on the forecast quality of implied volatility is however quite mixed (Fleming, 1998) and this would result in multiple mid quotes outside the bid-ask spread. A third alternative is to model volatility as a stochastic process (Heston, 1993). This approach is however quite contradictive in our case, because by using a tree model we indirectly assume volatility to be deterministic. We used the Nelder-Mead calibration scheme presented in Olsson and Nelson (1975) and Kienitz and Wetterau (2013). This method is widely applied for non-linear optimization problems. The advantage is that the method is based on simplex optimization rather than derivatives of an objective function. This is necessary due to the fact that we don't price by a closed form formula, but by a recursive algorithm.

4.2.2 Distortion

Recently several articles have appeared on the calibration of distortion parameters. Madan and Schoutens (2010) and Madan *et al.* (2016) start by calibrating a risk-neutral variance-gamma distribution by historical stock returns and then calibrate γ by bid and ask prices of options on that same underlying index where each option type in terms of maturity and moneyness is calibrated individually. The difference with our setting is that they work in continuous time and calibrate closed forms for bid and ask prices, see Madan and Schoutens (2010) for proof of Equations 4.1 and 4.2[1]:

$$b_\gamma(C) = \int_K^\infty \big(1 - \Psi^\gamma(F_S(x))\big)dx, \tag{4.1}$$

$$a_\gamma(C) = \int_K^\infty \Psi^\gamma\big(1 - F_S(x)\big)dx. \tag{4.2}$$

An alternative approach is proposed by Bannör and Scherer (2014). An extra feature of this method is the ability to obtain the shape of a market-implied distortion function. They use a piecewise linear approximation of a distortion function by using market quotes of option prices. Then the bid ask price calibration problem corresponds to minimizing the dinstance between model bid-ask prices and quoted market prices. The differences between the results obtained by the two methods are minor according to Bannör and Scherer (2014).

We combine both methods of Bannör and Scherer (2014) & Madan and Schoutens (2017) and execute them with a Nelder-Mead simplex calibration scheme. First we calibrate volatility against the mid quote ($\gamma = 0$ case) in order to make sure our bid-ask spread entails the mid quote. Subsequently we calibrate $\overline{\gamma}$ for each individual option against bid prices where we use the implied volatility calibrated in the step before as volatility input parameter in our algorithm. The difference with the approach used by Madan and Schoutens (2010) is that we work with a lognormally distributed underlying imposed by the tree-framework, instead of a variance gamma distribution. Then we calibrate our distortion budget against bid quotations and test the reflection principle by applying the same amount of distortion for producing ask prices. We look for evidence of TRA by plotting calibrated implied distortion profiles against moneyness and time to maturity.

4.3 Empirical results

We measure the goodness of fit by the Average Absolute Error (EEA) (Kienitz and Wetterau, 2013) of the model price compared to the quoted market price and the EEA as percentage of the absolute spread[2], see Equation 4.3 and 4.4.

$$\text{AAE} = \frac{1}{N}\sum_{i=1}^{N} |P_i^{market} - P_i^{model}(x)|, \tag{4.3}$$

$$\text{AAE as \% of spread} = \frac{1}{N}\sum_{i=1}^{N} \big(\frac{|P_i^{marketbid} - P_i^{modelbid}(x)|}{|P_i^{marketbid} - P_i^{marketmid}|}\big). \tag{4.4}$$

where $x = (x_1..x_n)^T$ represents the vector with option parameters belonging to option i. We calibrate σ against mid prices and we calibrate $\overline{\gamma}$ against bid prices. We test for the reflection principle by using the implied calibrated distortion found through bid prices to produce the corresponding ask price. Within this chapter we left out the results after hedging, because hedging made the algorithm inflexible for small step-sizes and pricing performance decreases when spreads become significant.

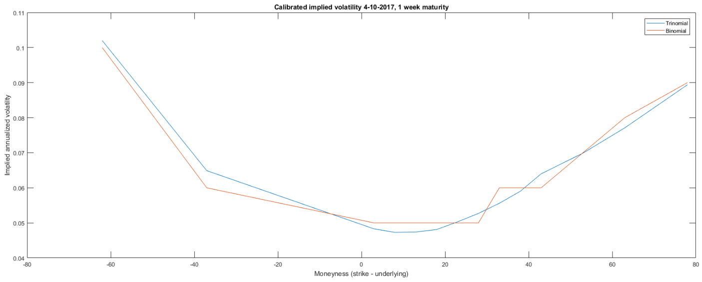
[1] F_S represent the distribution of the underlying S.

[2] Spread is defined as the difference between the bid and ask price.

4.3.1 Implied volatility

In Figure 4.3 we show the calibrated implied volatility skew which corresponds to implied volatility skews we would expect to find for call options (Hull, 2012).

Figure 4.3: Calibrated Implied volatility.



The option prices produced by volatility calibration are very close to mid quotes with marginal fitting error, which shows that our models are able to produce prices that are close to quoted mid prices of SPX options. Calibrated implied volatilities for the trinomial tree setting show a smoother pattern than the binomial setting, due to the fact the we used a relative small number of modeling steps during every time period and trinomial trees converge faster to stable pricing profiles. When we decrease step-size we will see that implied volatilites under trinomial and binomial become roughly the same.

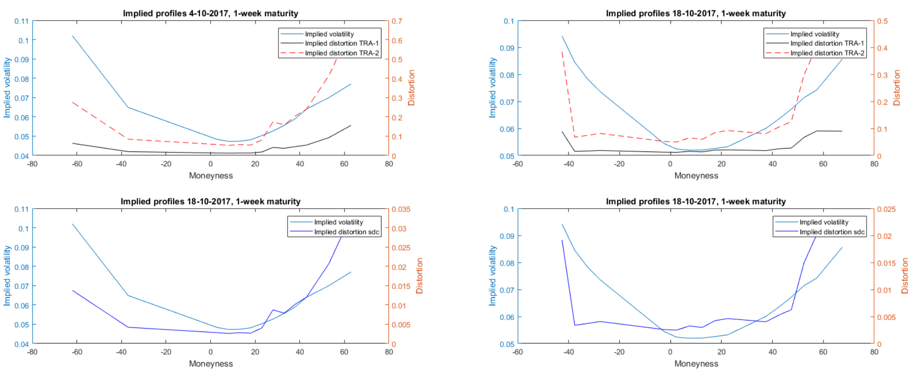
4.3.2 Implied distortion

Conic Finance models are able to produce bid prices that empirically fit market prices when the right amount of distortion is applied, both under strong and sequential consistency, see Table 7.2 in Appendix 7.6. The difference between TRA and sdc is that a wider set of distortion budget dependencies is allowed and therefore risk aversion is not necessary a constant over all time-steps. This enables us to model different relationships between $\overline{\gamma}$ and time to maturity and prevent spreads from blowing up as under sdc.

Moneyness

When we look at moneyness versus implied volatility in Figure 4.4, we observe patterns similar to the implied volatility skew. The implied volatility exists, because traders consider that the lognormal distribution understates probabilities of extreme events (Hull, 2012), which we can consider as a type of risk aversion that we can potentially capture with distortion, see Chapter 4.4.

Figure 4.4: Calibrated implied distortion ($\overline{\gamma}$) versus moneyness (Trinomial).

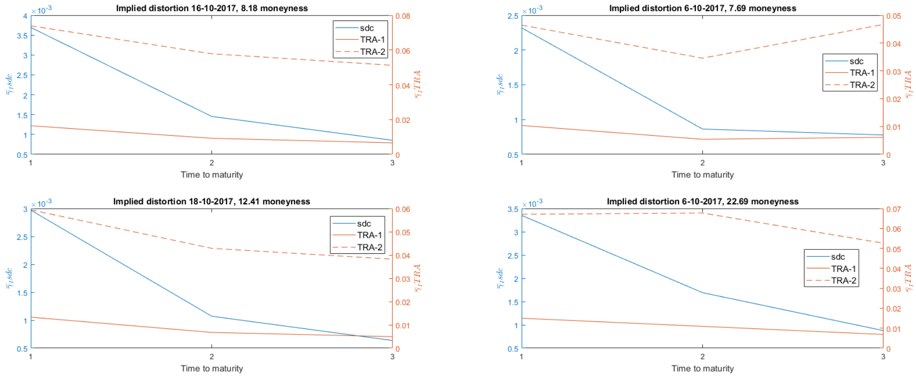


TRA-2 and sdc show similar profiles, because for one-week options TRA-2 is just a scaled version of sdc where the single-step $\overline{\gamma}$ of sdc is the maximum risk aversion that can be distributed over all time-steps for TRA-2. TRA-1 shows a different more stable pattern, because the limit is a squaroot sum of all $\overline{\gamma}$'s.

Time to maturity

In Figure 4.5 we plotted the calibrated implied distortion ($\overline{\gamma}_I$) for options with equal moneyness, but different time to maturities.

Figure 4.5: Calibrated implied distortion ($\overline{\gamma}_I$) versus time to maturity (trinomial).



Implied distortion $\overline{\gamma}_I$ under sdc shows a sharply decreasing profile in order to match longer-term quoted bid prices. If we kept $\overline{\gamma}_I$ constant we would end up with bid prices significantly below the quoted bid price due to the blow up behavior described in Chapter 2.4.3. Under TRA we observe flatter profiles for $\overline{\gamma}_I$. This seems promising in order to find a uniform functional relationship between time and distortion such that we prevent spreads from blowing up. For further research purposes it would be interesting to test a wider set of budget equations and time dependencies.

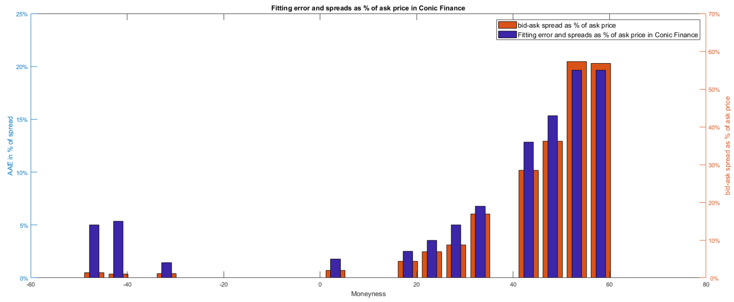
4.3.3 Reflection principle

One fundamental modeling choice in current Conic Finance frameworks is the reflection principle, which assumes that bid and ask prices are connected by the equality that buying is the same as selling its negative:

$$bid(X) = -ask(-X). \tag{4.5}$$

We test if there exists empirical proof for this assumption by applying each individual calibrated amount of bid price distortion ($\overline{\gamma}_I$) to produce its corresponding ask price and compare this ask price with the ask price quotation provided by the market. An important assumption within our modeling setup is that our bid price depends on the mid price (because we first calibrated the mid price σ) and therefore ask prices not only depend on bid prices imposed by the reflection principle, but also on the mid price as a modeling consequence. In Appendix 7.6 we present the performance of our model in fitting ask prices. It shows that the reflection principle holds for close-to ATM options, but fitting performance decreases significantly for OTM options, see Figure 4.6. A possible explanation is that spreads are not necessarily symmetric around mid quotes in Conic Finance (Madan and Schoutens, 2017). This is indeed not the case when we look at implied spreads as % of the model ask price versus moneyness. In Figure 4.6 we see a similar increase of spreads compared to the increase in fitting error for different moneyness.

Figure 4.6: Bid-ask spread as % of ask price (orange) versus fitting error ask price (blue) for different moneyness. TRA-1, 2-10-2017 for a one-week option valued by a trinomial tree.



We found no statistical evidence[3] that models under TRA produce better ask prices (under calibrated bid distortion) than the sdc one, nor a difference between binomial and trinomial models.

4.4 Constant implied volatility setup

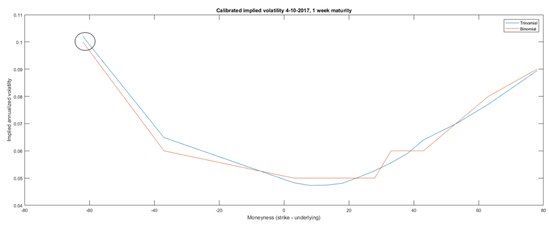
Based on our findings on the implied distortion ($\overline{\gamma}_I$) versus implied volatility (σ_I), we want to test another modeling setup in which we keep implied volatility constant and try to fit the complete adjustment of the probability distribution with our $\overline{\gamma}_I$ parameter. In order to do so we undertake the following steps:

Step 1 We take **one** volatility level that we thereafter use for all strikes and maturities which is the implied volatility of the farthest ITM option, see Figure 4.7. We could also take the σ of the farthest OTM options, as long as we ensure that all other mid prices modeled with this specific σ_I are above their bid price which is a necessity for our algorithm.

Step 2 Calibrate $\overline{\gamma}$ for each individual option with the one unique σ_I that we found in **Step 1**.

Step 3 Test if we could fit the total adjustment on the probability distribution with $\overline{\gamma}$ and investigate if our ask price fitting performance increases.

Figure 4.7: Implied volatility of farthest ITM option illustrated by black circle.

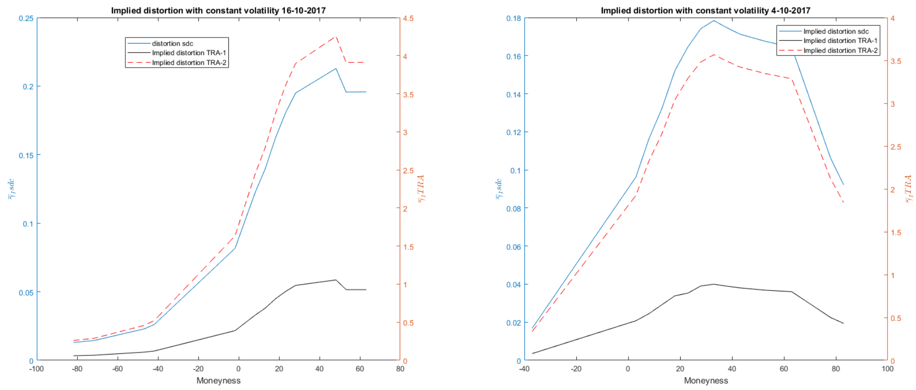


4.4.1 Results

We plotted the results of two different options on two separate days in Figure 4.8. It shows that we are able to capture the complete adjustment of the underlying probability distribution necessary to fit bid prices by only changing $\overline{\gamma}$ and keeping σ constant. We believe implied distortion to be a much more intuitive adjustment parameter, because it represents risk aversion which is likely to be different for options with different moneyness, where it theoretically impossible that one underlying has multiple volatilities.

[3] 2-sample t-test at 99% significance level.

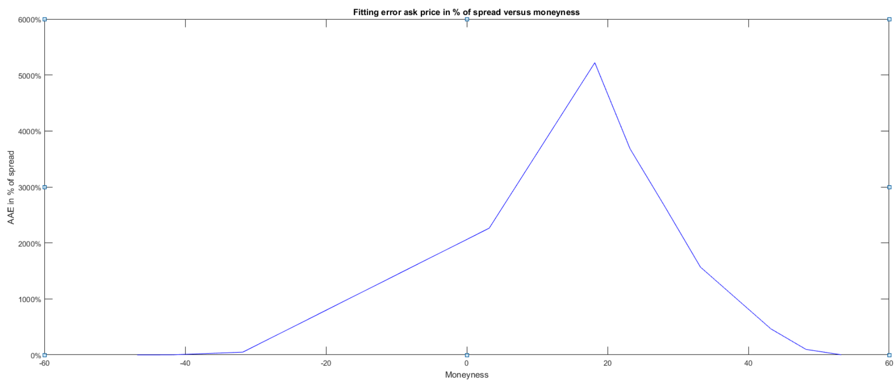
Figure 4.8: Implied distortion ($\overline{\gamma}_I$) after constant volatility calibration. Left y-axis is SDC, right y-axis is TRA.



Again we see that TRA-2 (square-root budget) shows the most stable pattern where TRA-1 and sdc show more divergent profiles with significant changes in distortion around ATM strikes. Within this setup ask pricing performance decreases dramatically, because of two reasons:

- Implied distortion levels increase around ATM strikes in order to adjust the relatively high mid price into a bid price. But because the mid price is already relatively close to the quoted ask prices at these points, the amount of distortion calibrated against bid prices is far too high to adjust that same mid price into an ask price and therefore ask prices explode compared to quoted ones, see Figure 4.9.
- For options with low implied volatilities, especially around ATM strikes, mid prices lay sometimes above ask prices. This happens because we used a relatively high implied volatility (the one for the farthest ITM option) that delivers a disproportionate high mid price for options with low implied volatilities.

Figure 4.9: Fitting error ask price in % of spread versus moneyness (sdc).



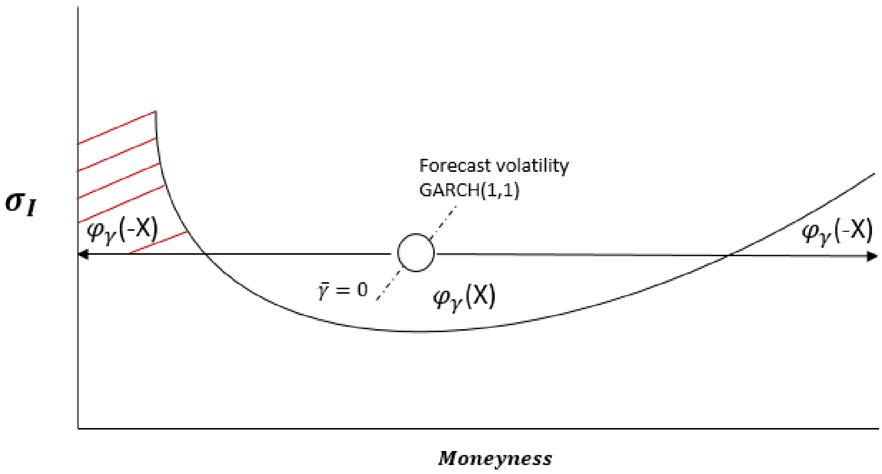
4.5 Alternative use of Reflection Principle

In the previous section we showed that we are able to capture the complete adjustment of the probability measure by changing $\overline{\gamma}$ instead of σ, however the reflection principle makes it impossible to produce ask prices with the calibrated bid $\overline{\gamma}$. In order to solve this problem we introduce an alternative use of the reflection principle.

Conic Finance is built upon the assumption that bid and ask prices exist due to the existence of unhedgeable risk. In order to assess this risk, we apply constant risk aversion both for selling and buying. However, in Conic Finance the amount of risk aversion is the same for bid and ask by the reflection principle. We believe there is room to deviate from this principle. The level of risk aversion is not necessary the same for buying and selling. It is widely accepted in normative Behavioral Finance theories that the perception of losing (negative contingent claim/selling) is different than for winning (positive contingent claim/buying). We therefore propose that bid and ask prices could have their own level of risk aversion ($\overline{\gamma}$). We introduce a different setup where we price with a forecasted underlying volatility (again the same for all strikes) provided by GARCH(1,1). We already know that this volatility will produce mid prices that are too low for ITM options, but too high for ATM options. The price point $\overline{\gamma} = 0$ is the point where the forecasted volatility provides the exact (or closest to) bid-price. We fit prices around this point by applying the reflection principle for the points where we are not able price with normal valuation operators; $\phi_\gamma(X)$ (bid) or $\phi_\gamma(-X)$ (ask). For instance on the points in the red-marked area in Figure 4.10 we cannot price with $\phi_\gamma(X)$ and therefore we price with $\phi_\gamma(-X)$. We define the points priced by the reflection principle in the following way:

$$\textbf{bid} \quad \phi_{-\gamma}(X) = \phi_\gamma(-X), \tag{4.6}$$
$$\textbf{ask} \quad \phi_{-\gamma}(X) = \phi_\gamma(X). \tag{4.7}$$

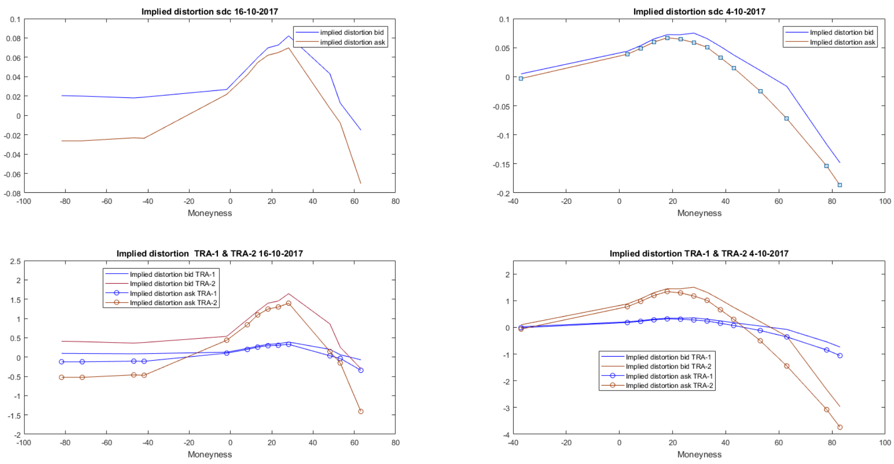
Figure 4.10: Bid and ask price operators with constant forecasted volatility. The graph shows which valuation operator is used in different areas of the quoted implied volatility.



4.5.1 Results

Figure 4.11 shows the results of the bid and ask price $\overline{\gamma}$ after calibration where we applied the reflection principle, not to find the complementary ask price, but to find the bid/ask prices that we can't price with their standard valuation operator. We see that we are able to price both bid and ask prices with a forecasted σ and capture the complete adjustment of the probability measure by $\overline{\gamma}$. We observe that as we expected, bid and ask prices don't have the same amount of risk aversion. We also see that the spread between the $\overline{\gamma}$'s for bid and ask widens when we move away from the ATM strikes. Probably because valuation uncertainty increases for OTM and ITM options. Again we observe the most stable pattern under TRA-1.

Figure 4.11: Graphical representation of $\overline{\gamma}$ for bid and ask prices under new setup. $\overline{\gamma}$'s-ask are multiplied by -1 for illustrative purposes.



Chapter 5

Conclusion

This chapter provides an answer to the sub-research questions posed in our research design. The answering of our main research question accompanied with main insights gained during this research project follows logically.

From classical derivative valuation towards a two price framework

Traditional valuation models are built on the fundamental theories of asset pricing in which the pricing of financial products is based on the law of one price. However, when we look at real financial markets we observe two prices, one for selling and one for buying. We believe this to be logical when people apply risk aversion consistently in both selling and buying. Conic Finance provides a framework that aims at producing these two prices such that bid-ask spreads reflect the cost of unhedgeable risk. Financial contracts are then priced in terms of a marketed cone which is operationalized by acceptability indices under the modeling assumption that buying equals selling the negative (reflection principle), and strong dynamic consistency. Strong dynamic consistency potentially leads to the build-up of conservatism and introduces complications when we want to build functional relationships between time to maturity and risk aversion. Tuned Risk Aversion provides a framework that still provides unique updates of prices in multi-period valuations, but is more flexible and able to produce results more in line with experimental choice behavior. The extra flexibility of Tuned Risk Aversion also provides the possibility to build other more feasible functional relationships between risk aversion and time to maturity.

From theory to a pricing model

Bid and ask prices are respectively the infimum and supremum of distorted expectations in Conic Finance, where distortion depends on the degree of risk aversion. We developed a model that is able to price plain vanilla options both in a binomial and trinomial tree setting. The pricing algorithm has two possible tuning-sets under Tuned Risk Aversion with different time and risk aversion dependencies. Strong dynamic consistency has been modeled by keeping the amount of distortion constant during every time-step. The model has been tested on a stylized case where we showed the difference between tuning-sets under Tuned Risk Aversion and strong dynamic consistency in pricing a two-month call option.

Model calibration and pricing performance

We calibrated our model against a dataset of call options on the S&P 500 (SPX) in October-2017 by the following steps:

> **Step 1** Calibrate implied volatility (σ_I) of each individual option against quoted mid-prices by testing different σ' as input parameter for our algorithm. Calibration executed through a Nelder-Mead simplex calibration scheme.

Step 2 Calibrate implied distortion ($\overline{\gamma}$) against bid prices where we use the calibrated σ_I of each individual option found in **Step 1** also through a Nelder-Mead simplex calibration scheme.

Step 3 Use each calibrated $\overline{\gamma}_I$-bid as input-variable to produce ask prices such that we are able to test for existence of the reflection principle.

Step 4 Plot patterns of $\overline{\gamma}_I$ against moneyness and time to maturity to see if TRA provides a way to model functional relationships between distortion and option characteristics and to see if markets price with TRA.

Our model is able to produce bid prices close to market prices, both under strong dynamic consistency and TRA. By applying the calibrated amount of bid-distortion, we were able to match ask prices around ATM strikes, but fitting performance decreased for OTM options. This can be explained by the fact that in Conic Finance bid-ask spreads are not necessarily symmetric around mid prices, which we indirectly assumed in our calibration setup.

We looked at patterns of implied distortion and found profiles similar to the implied volatility skew. We tested two alternative modeling setups where we kept volatility constant and captured the complete probability distribution adjustment, normally provided by changing σ of the underlying, by solely applying distortion $\overline{\gamma}$. Within this modeling setup we were able to fit all bid and ask prices of different strikes in a model with a single volatility for the underlying. Also implied distortion patterns under TRA showed a significantly flatter profile compared to sdc and therefore TRA provides a way to model a functional relationship between time to maturity and distortion, while at the same remain consistent and prevent spreads from blowing up. TRA provides a very promising framework into finding a uniform model for derivative pricing that is able to price with a single volatility for the underlying and a feasible way to relate risk aversion and time to maturity.

5.1 Main insights

1. Spreads under strong dynamic consistency blow up if we increase time to maturity and therefore it is impossible to model a truly consistent relationship between time to maturity and distortion that matches market prices.

2. Conic Finance models are able to produce bid and ask prices close to market quotations, both under sequential and strong dynamic consistency by applying the right amount of distortion.

3. Implied distortion patterns show that with TRA, we are able to solve the blow up behavior that we face under sdc which provides promising results for finding a functional relationship between distortion and time to maturity.

4. The probability adjustment (implied volatility skew), normally provided by different volatilities for the underlying, can be completely captured by distortion $\overline{\gamma}$. We are able to price all strikes of an option chain by a single forecasted volatility.

5. Market quotations for ATM options provide evidence for the reflection principle as modeling relationship between bid and ask prices. However, we believe that for OTM and ITM options, bid and ask prices could have different levels of risk aversion.

Chapter 6

Further Research and Model Improvement

Empirical studies on both Conic Finance and Tuned Risk Aversion are scarce and theoretical ones that try to combine these two theories even non-existent. To our knowledge this study is the first attempt to bridge Tuned Risk Aversion and the field of derivative pricing, especially Conic Finance. We can say that the study was an explorative journey and therefore many new ideas and improvements came to our minds during the process. This final chapter captures all these recommendations and ideas for further research.

1. **Flexibility** Allow for a wider range of tuning-sets and distortion functions. Because of the explorative character of this study we restricted ourselves to two types of tuning-sets and one distortion function MINMAX (the most conservative one). The model can be extended by trying different distortion functions, for instance: MAXMIN, MAXVAR and MINVAR (Cherny and Madan, 2009). We can also model different tuning-sets where we calibrate both δ and $\overline{\gamma}$ (Equation 6.1) or use different single-step restrictions for risk aversion (Equation 6.2):

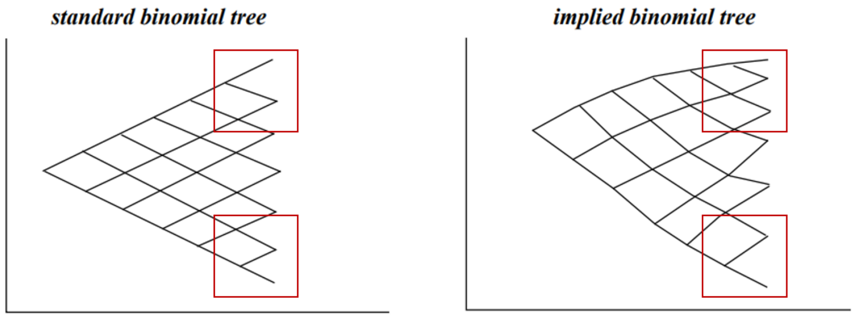
$$\sum_{0}^{N*T} \gamma_t \leq (N*T)^{\delta} * \overline{\gamma} \quad | \quad \gamma_t \leq \overline{\gamma}, \tag{6.1}$$

$$\sum_{0}^{N*T} \gamma_t \leq (N*T)^{\delta} * \overline{\gamma} \quad | \quad \gamma_t \leq \sqrt{\overline{\gamma}}. \tag{6.2}$$

2. **Connection with Behavorial Finance** An extension would be to incorporate S-shaped distortion instead of strictly concave which is more in line with general theories in Behavioral Finance. By applying this type of distortion functions we need to be careful if no-arbitrage conditions still hold (Madan *et al.*, 2016). Another potential extra research dimension would be to incorporate the accumulation of wealth within the TRA framework. We would expect utility of 'extra gains' for highly ITM options to be less than the gains obtained by for instance ATM options. In other words, the accumulation of wealth plays a role in the risk aversion of investors. Consider a binomial pricing framework as a compound choice lottery where in every time-step an investor has the hypothetical choice of holding his position, or to close/sell it. The choice to hold his position means that he continues the 'gamble' at least one more period of time. When we are high in the tree the investor gained a lot, so his reference point changes and utility of gains diminishes in line with Cumulative Prospect Theory (Tversky and Kahneman, 1992). Therefore risk aversion should differ when we are somewhere high in the tree compared to when we are somewhere low.

Especially for derivatives with intermediate payoffs like swaps this would be an interesting point for further research.

Figure 6.1: Indicative risk aversion in normal and implied binomial tree.



3. **Functional relationship distortion** Use the insights from implied distortions patterns under TRA to find a functional relationship between time (tuning-set) and moneyness. Especially the relationship between distortion and implied volatility looks very promising which could be the key to develop a model that is able to price options with different strikes, but the same underlying volatility. A first and easy attempt could be to look at the development of historical implied distortion levels to see if this changes over time and whether it is forecastable.

4. **Hedging improvement** The hedging strategy in this thesis is the one proposed by Madan *et al.* (2016), but whether this approach is optimal remains questionable, especially in the presence of transaction cost for the underlying. Therefore we propose to improve the model with more advanced hedging strategies for trinomial trees presented in: Schulmerich and Trautmann (2003), Huang and Guo (2013) and Föllmer and Leukert (2000).

5. **Validity** Develop a model that does not require 'knowing' the exact mid quote parameterization. Even by using a single σ, like we tested in Chapter 4.4 and 4.5, we still use the mid price as starting point for distortion. Modeling the implied volatility surface is a constantly developing field. Especially the article Carr and Wu (2016) looks very promising and is very close to approaches used by practitioners (Vega-Vanna-Volga method).

6. **Generality** Test the model on more complicated derivatives like structured products. A first attempt has already been made by Madan and Schoutens (2012). Also the dataset we used was very limited both in type of options, maturity (less than two months) and dates (only one). In order to increase generality we need to test the model on larger datasets.

6.1 Model Improvement

1. **Finite difference** Extend the model towards a finite-difference model where the underlying is modeled on a grid. This way we would be able to price options with other underlying processes like: variance-gamma or Lèvy, but are still able to distribute risk aversion every time-step. The application of Conic Finance in grid methods is further explained in Madan and Schoutens (2016).

2. **Time dependency** Within our model we searched for time dependencies in terms of week-maturities and calibrated $\overline{\gamma}$ budgets at weeks. We would suggest to decrease this to day-dependency which potentially increases precision and allow us to calibrate against a wider set of options. This modeling-setup is already possible in our current algorithm.

3. **Detailed consistency** Improve the algorithm such that also intermediate budgets are stored, see introduction Chapter 4 for explanation of this modeling assumption. This way we can ensure that we are completely consistent over all time-steps.

4. **Calibration (I)** Test whether there is a significant improvement when volatility is calibrated against the volume weighted-average mid quote instead of the standard mid quote.

5. **Calibration (II)** Only use OTM options of put and calls and apply the put-call parity to price ITM options, because they are more competitively priced (Aït-Sahalia *et al.*, 2001).

6. **Precision (I)** Increase granularity of possible γ values the model can assign every time-step. This can lead to better pricing performance and better implied distortion patterns due to the fact that risk aversion can be applied more precisely.

7. **Precision (II)** We took the risk-free rate as constant for all maturities and used γ as calibration parameter to fit the lognormal distribution such that our model produced the exact mid-price. It would be better to use the exact risk-free term-structure, for instance the EONIA in Europe (ECB, 2014). For equity derivatives it is not necessary to model stochastic interest rates, because the main driver of value is the equity underlying opposed to, for instance, interest rate derivatives (Brigo and Mercurio, 2006).

8. **Model Performance** The current model is programmed in Matlab which is a relatively slow programming language compared to C++ or Python. Especially for calibration purposes performance can be increased by recoding it to a faster programming language.

Bibliography

Aït-Sahalia, Y., Wang, Y., and Yared, F. (2001). Do option markets correctly price the probabilities of movement of the underlying asset? *Journal of Econometrics*, 102(1):67–110.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance*, 9(3):203–228.

Artzner, P., Delbaen, F., Eber, J., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and Bellman's principle. *Annals of Operations Research*, 152(1):5–22.

Bannör, K. F. and Scherer, M. (2014). On the calibration of distortion risk measures to bid-ask prices. *Quantitative Finance*, 14(7):1217–1228.

Barraquand, J. and Pudet, T. (1996). Pricing of American Path Dependent Contingent Claims. *Mathematical Finance*, 6(1):17–51.

Bielecki, T., Cialenco, I., and Pitera, M. (2017). A survey of time consistency of dynamic risk measures and dynamic performance measures in discrete time: LM-measure perspective. *Probability, Uncertainty and Quantitative Risk*, 2.1(1):3.

Bielecki, T., Cialenco, I., and Zhang, Z. (2014). Dynamic coherent acceptability indices and their applications to finance. *Mathematical Finance*, 24(3):411–441.

Bielecki, T. R., Cialenco, I., Iyigunler, I., and Rodriguez, R. (2013). Dynamic Conic Finance: Pricing and Hedging in Market Models With Transaction Costs Via Dynamic Coherent Acceptability Indices. *International Journal of Theoretical and Applied Finance*, 16(01):1350002, 36.

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654.

Bormetti, G., Montagna, G., Moreni, N., and Nicrosini, O. (2004). Pricing Exotic Options in a Path Integral Approach. *Quantitative Finance*, 6(1):55–66.

Boyle, P. (1986). Option Valuation Using a Three-Jump Process. *International Options Journal*, 3:7–12.

Brigo, D. and Mercurio, F. (2006). *Interest rate models-theory and practice: with smile, inflation and credit.* Springer Science and Business Media, third edition.

Carr, P. and Wu, L. (2016). Analyzing volatility risk and risk premium in option contracts. *Journal of Financial Economics*, 120(1):1–20.

CBOE Options Exchange (2017). CBOE optian chain on S&P500. $https://datashop.cboe.com/option-quotes-end-of-day-with-calcs$.

Cherny, A. and Madan, D. (2009). New measures for performance evaluation. In *Review of Financial Studies*, volume 22, pages 2571–2606.

Cox, J. C., Ross, S. A., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263.

Cumby, R., Figlewski, S., and Hasbrook, J. (1993). Forecasting Volatilities and Correlations with EGARCH Models. *Journal of Derivatives*, 1(2):51–63.

Derman, E., Kani, I., and Chriss, N. (1996). Implied Trinomial Trees of the Volatility Smile. *The Journal of Derivatives*, 3(4):7–22.

Derman, E. and Taleb, N. (2005). The illusions of dynamic replication. *Quantitative Finance*, 5(4):323–326.

Detlefsen, K. and Scandolo, G. (2005). Conditional and dynamic convex risk measures. *Finance and Stochastics*, 9(4):539–561.

Duffie, D. (2001). *Dynamic Asset Pricing Theory*. Princeton University Press, third edition.

Dupire, B. (1994). Pricing and hedging with smiles. *Mathematics of derivative securities*, 1(1):103–111.

Eberlein, E., Madan, D., Pistorius, M., Schoutens, W., and Yor, M. (2014). Two price economies in continuous time. *Annals of Finance*, 10(1):71–100.

ECB (2014). Euro Area Risk-Free Interest Rates: Measurement Issues , Recent Developments and Relevance to Monetary Policy. Technical Report 7 (July).

Fleming, J. (1998). The quality of market volatility forecasts implied by S&P 100 index option prices. *Journal of Empirical Finance*, 5(4):317–345.

Föllmer, H. and Leukert, P. (2000). Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 4(2):117–146.

Föllmer, H., Schied, A., and Lyons, T. J. (2004). *Stochastic finance. an introduction in discrete time*, volume 26. Walter de Gruyter.

Harrison, J. and Kreps, D. (1979). Martingales and arbitrage in multi-period security markets. *Journal of Economic Theory*, 20(3):381–408.

Haug, E. G. and Taleb, N. N. (2011). Option traders use (very) sophisticated heuristics, never the Black-Scholes-Merton formula. *Journal of Economic Behavior and Organization*, 77(2):97–106.

Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6(2):327–343.

Huang, S. F. and Guo, M. (2013). An optimal multi-step quadratic risk-adjusted hedging strategy. *Journal of the Korean Statistical Society*, 42(1):37–49.

Hull, J. (2012). *Options, Futures and other Derivatives*. Pearson, Toronto, eighth edition.

Jackwerth, J. and Rubinstein, M. (1996). Recovering Probability Distributions from Option Prices. *The Journal of Finance*, 51(5):1611–1631.

Jobert, A. and Rogers, L. C. G. (2008). Valuations and dynamic convex risk measures. *Mathematical Finance*, 18(1):1–22.

Kienitz, J. and Wetterau, D. (2013). *Financial Modelling: Theory, Implementation and Practice*. John Wiley & Sons, first edition.

Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in Mathematical Economics*, 3:83–95.

Leland, H. (1985). Option Pricing and Replication with Transactions Costs. *The Journal of Finance*, 40(5):1283–1301.

Machina, M. and Viscusi, W. (2014). *Economics of risk and uncertainty*. Elsevier, Amsterdam, first edition.

Madan, D. (2010). Pricing and hedging basket options to prespecified levels of acceptability. *Quantitative Finance*, 10(6):607–615.

Madan, D., Pistorius, M., and Schoutens, W. (2013). The valuation of structured products using Markov chain models. *Quantitative Finance*, 13(1):125–136.

Madan, D., Pistorius, M., and Schoutens, W. (2016). Dynamic conic hedging for competitiveness. *Mathematics and Financial Economics*, 10(4):405–439.

Madan, D. and Schoutens, W. (2010). Markets As a Counterparty: an Introduction To Conic Finance. *International Journal of Theoretical and Applied Finance*, 13(08):1149–1177.

Madan, D. and Schoutens, W. (2016). *Applied Conic Finance*. Camebridge University Press, Clays, first edition.

Madan, D. and Schoutens, W. (2017). Conic Option Pricing (Robert H. Smith School of Business).

Madan, D. B. and Schoutens, W. (2012). Structured products equilibria in conic two price markets. *Mathematics and Financial Economics*, 6(1):37–57.

Merton, R. (1973). Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183.

Miller, K. D. and Shapira, Z. (2004). An Empirical Test of Heuristics and Biases Affecting Real Option Valuation. *Strategic Management Journal*, 25(3):269–284.

Nardon, M. and Pianca, P. (2014). A behavioural approach to the pricing of European options. In *Mathematical and Statistical Methods for Actuarial Sciences and Finance*, pages 219–230. Springer.

Nelson, D. B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica: Journal of the Econometric Society*, 59(2):347–370.

Noh, J., Engle, F., and Kane, A. (1994). Forecasting Volatility and Option Prices on the S&P 500 index. *Journal of Derivatives*, 2(1):17–30.

Olsson, D. M. and Nelson, L. S. (1975). The Nelder-Mead simplex procedure for function minimization. *Technometrics*, 17(1):45–51.

Remler, D. and van Ryzin, G. (2015). *Research Methods in Practice: Strategies for Description and Causation*. Sage, Thousand Oaks, second edition.

Riedel, F. (2004). Dynamic coherent risk measures. *Stochastic Processes and their Applications*, 112(2):185–200.

Roorda, B. and Joosten, R. (2015). Dynamically Consistent Non-Expected Utility Preferences with Tuned Risk Aversion (University of Twente).

Roorda, B. and Joosten, R. (2017). Consistent Preferences with Fixed Point Updates (University of Twente).

Roorda, B. and Schumacher, J. (2013). Membership conditions for consistent families of monetary valuations. *Statistics & risk modeling*, 30(3):255–280.

Roorda, B. and Schumacher, J. (2016). Weakly time consistent concave valuations and their dual representations. *Finance and Stochastics*, 20(1):123–151.

Roorda, B. and Schumacher, J. M. (2007). Time consistency conditions for acceptability measures, with an application to Tail Value at Risk. *Insurance: Mathematics and Economics*, 40(2):209–230.

Ross, S. (1978). A simple approach to the valuation of risky streams. *Journal of business*, 51(3):453–475.

Rubinstein, M. (2000). On the Relation Between Binomial and Trinomial Option Pricing Models. *The Journal of Derivatives*, 8(2):47–50.

Schulmerich, M. and Trautmann, S. (2003). Local expected shortfall-hedging in discrete time. In *European Finance Review*, volume 7, pages 75–102.

Schwartz, E. (1977). The valuation of Warrants: Implementing a new approach. *Journal of Financial Economics*, 4(1):79–93.

Shreve, S. E. (1996). *Stochastic Calculus and Finance*. Springer Finance, first edition.

Taleb, N. N. (2013). Philosophy: 'Antifragility' as a mathematical idea. *Nature*, 494(7438):430.

Thaler, R. H. (2015). *Misbehaving: The Making of Behavioral Economics*, volume 57. W. W. Norton & Company, sixth edition.

Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323.

USA Department of Treasury (2017). Daily treasury yield curve T-bills.

Verschuren, P., Doorewaard, H., and Mellion, M. (2010). *Designing a Research Project*. Eleven International Publishing, The Hague, second edition.

Versluis, C. and Lehnert, T. (2010). A Cumulative Prospect Theory Approach to Option Pricing (Luxembourg School of Finance).

Webster, J. and Watson, R. (2002). Analyzing the Past to Prepare for the Future: Writing a Literature Review. *MIS Quarterly*, 26(2):13–23.

Yahoo Finance (2017). Yahoo Finance historical option data S&P500. $https://finance.yahoo.com/quote/5ESPX/$.

Chapter 7

Appendix

7.1 Binomial tree set-up

Backward recursion is determined by the following four equations:

$$\tilde{\phi}(T) = X(T) \tag{7.1}$$

$$\tilde{\phi}(t) = \mathbf{E}^{\mathcal{Q}}\Big[\tilde{\phi}(t+1)|\mathcal{F}_t\Big] \tag{7.2}$$

$$\varphi_2(t) = \frac{\mathrm{Cov}^{\mathcal{Q}}\Big[\tilde{\phi}(t+1), \Delta\tilde{S}(t)|\mathcal{F}_t\Big]}{\mathrm{Var}^{\mathcal{Q}}\Big[\Delta\tilde{S}(t)|\mathcal{F}_t\Big]} \tag{7.3}$$

$$\varphi_1(t) = \tilde{\phi}(t) - \varphi_t\tilde{S}(t) \tag{7.4}$$

We now observe that $\varphi_2(t)$ is the regression coefficient of the value of the derivative and the underlying, also called delta. For an option in the binomial framework, Equation 7.3 reduces to the following simplified well-known expression of delta:

$$\varphi_2(t) = \frac{\phi(t+1, S(t)u) - \phi(t+1, S(t)d)}{S(t)u - S(t)d} \tag{7.5}$$

In our implementation we followed the setup presented by Cox *et al.* (1979)

$$S_u = Se^{\sigma\sqrt{(\Delta t)}} \tag{7.6}$$

$$S_d = Se^{-\sigma\sqrt{(\Delta t)}} \tag{7.7}$$

$$p_u = \frac{e^{r\Delta t} - S_d}{S_u - S_d} \tag{7.8}$$

$$p_d = 1 - p_u \tag{7.9}$$

7.2 Trinomial tree set-up

Presented in Boyle (1986)

$$S_u = Se^{\sigma\sqrt{(2\Delta t)}} \tag{7.10}$$

$$S_d = Se^{-\sigma\sqrt{(2\Delta t)}} \tag{7.11}$$

$$S_m = S \tag{7.12}$$

$$p_u = (\frac{e^{r\Delta t\frac{1}{2}} - e^{-\sigma\sqrt{\Delta t\frac{1}{2}}}}{e^{\sigma\Delta t\frac{1}{2}} - e^{-\sigma\sqrt{\Delta t\frac{1}{2}}}})^2 \tag{7.13}$$

$$p_m = 2/3 \tag{7.14}$$

$$p_d == (\frac{e^{\sigma\sqrt{\Delta t\frac{1}{2}}} - e^{-r\Delta t\frac{1}{2}}}{e^{\sigma\Delta t\frac{1}{2}} - e^{-\sigma\sqrt{\Delta t\frac{1}{2}}}})^2 \tag{7.15}$$

$$\tag{7.16}$$

7.3 Description Algorithm

7.3.1 Procedure

SDC
Under SDC the valuation model applies the same amount of gamma during every time-step. The starting point is the risk-neutral probability distribution with equal distortion in terms of gamma during every step. The option is valued backward recursively just like we do in normal traditional one price tree frameworks.

TRA
Every time-step the model calculates the intermediate value of an option by calculating all the possible valuations under all possible gamma budget applications. Then the model evaluates the possible values and takes the **worst** value, because bid prices are the infimum of expected values. For ask prices we choose the highest price (supremum). When we found the worst intermediate value (in case of bid), we store the corresponding gamma budget and store the prices for **all** possible gamma budgets used. The next time step we evaluate the optimal gamma budget used in the previous step against all possible gamma budgets used in the current step to see whether the old gamma budget allocation is still optimal. In case two gamma budget applications lead to the same pricing, the algorithm choses the **lowest** gamma application in the current step (which means the highest in te previous step). Values that are produced by gamma budget allocations that violate the budget equations are automatically ignored by the algorithm.

7.4 GARCH

The GARCH(1,1) model is an extension of the ARCH(1) model where the forecast is based on a long run average variance rate and recent observations of returns. The model is mean reverting and therefore more in line with general behavior of volatility compared to methods like exponentially weighted moving average models. GARCH(1,1) is based on the following equation:

$$\sigma_t^2 = \omega + \alpha_1\epsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \tag{7.17}$$

$$\mu_t = \frac{S_t}{S_{t-1}} - 1 \tag{7.18}$$

We estimated ω, ϵ and β via the maximum likelihood method which chooses values that maximize the likelihood of data occuring (Nelson, 1991).

7.4.1 GARCH parameters used for constant volatility forecaste in Chapter 4.5

Table 7.1: Garch(1,1) parameters

```
GARCH(1,1) Conditional Variance Model:
----------------------------------------
Conditional Probability Distribution: Gaussian

                                  Standard          t
 Parameter       Value             Error         Statistic
-----------   -----------      ------------    -----------
 Constant     1.03558e-06     1.03442e-06        1.00112
 GARCH{1}             0.9       0.0180713        49.8026
  ARCH{1}            0.05       0.0185897        2.68966
   Offset     0.000742788     0.000258811           2.87
```

7.5 Nelder-Mead calibration method

The Nelder-Mead is a direct search method that aims at minimizing a given problem of an objective funtion. The method does not rely on derivatives of the objective function which is in our case impossible, because pricing is done through an algorithm. The method starts from a simplex S_0 and tries iteratively to find the optimal solution (minimization problem), where each iteration step is indexed by k identifies a vertex v^k_{min} determined by:

$$v^k_{min} = arg\ min_{x \in \{v^k_0..v^k_n\}} f(x) \tag{7.19}$$

$$f(x) = abs(bid - price.m) \tag{7.20}$$

$$f(x) = abs(wmad - price.m) \tag{7.21}$$

'wmad' equals the mid quote against which the volatility parameter is calibrated (Chapter 4.2.1) and *'bid'* equals the bid price against which the distortion parameter is calibrated. Price.m stands for the pricing algorithm. v^k_{min} represents the vertex where the function takes the minimum. In Matlab the search method is already precoded by *fminsearch*. The functions stops when the maximum absolute difference between two tries is less compared to a pre-set tolerance and the maximum distance of any of the vertex subjects compared to the best vertex is small enough (Kienitz and Wetterau, 2013).

7.6 Model performance standard setup

Table 7.2: Pricing performance model under standard setup

	Binomial			Trinomial		
Bid price fit	*SDC*	*TRA-1*	*TRA-2*	*SDC*	*TRA-1*	*TRA-2*
AAE (*100)						
Max	0.11	0.08	0.07	0.08	0.26	0.15
Min	0.00	0.00	0.00	0.00	0.00	0.00
Average	0.00	0.00	0.00	0.00	0.02	0.01
Median	0.00	0.00	0.00	0.00	0.00	0.00
% AAE of spread						
Max	1.10%	0.22%	0.03%	0.38%	0.26%	0.04%
Min	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Average	0.02%	0.01%	0.01%	0.03%	0.03%	0.01%
Median	0.01%	0.01%	0.00%	0.01%	0.01%	0.01%
Reflection principle	*SDC*	*TRA-1*	*TRA-2*	*SDC*	*TRA-1*	*TRA-2*
AAE						
Max	0.10	0.09	0.10	0.11	0.27	0.11
Min	0.00	0.00	0.00	0.00	0.00	0.00
Average	0.01	0.01	0.01	0.01	0.01	0.01
Median	0.00	0.00	0.00	0.00	0.00	0.00
% AAE of spread						
Max	41.41%	36.94%	39.80%	43.10%	40.84%	42.11%
Min	0.05%	0.01%	0.02%	0.07%	0.07%	0.08%
Average	4.79%	4.30%	4.33%	4.86%	5.84%	4.80%
Median	1.53%	1.45%	1.48%	1.55%	1.70%	1.54%