Inverse Boundary Problem with the Maxwell Equations

An Alternative Method to Breast Cancer Detection

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Contents

1 Introduction

In the modern age cancer has become a great burden. For the United States alone 1.688.780 are expected to be diagnosed with cancer and 600.920 are expected to die of cancer in the year 2017. The most prevalent diagnosis is expected to be breast cancer with 255.180 new cases. Even though only 41.070 people are expected to die of breast cancer, people are still dieing.[8]

In figure 1 the number of deaths due to cancer can be seen. The number of deaths for most cancers has been falling since the 1990's. For breast cancer on of the things responsible for reducing the number of deaths due to cancer is the advent of regular screening. Since it is usually a bit too late already, if someone falls ill due to a tumor, and since tumors can be spotted before they cause any significant damage, regular screening has been introduced. The regular screening consists of a 2D mammography and clinical examination, which can be followed by a series of other tests in case these were not conclusive or showed that something is not right. In the United States regular screening is recommended by the American Cancer Society annually for persons over 45 years old given that they are expected to live another 10 years. From the age of 55 onward the person can also do a screening biennially. The American Cancer Society also recommends that people between 40 and 44 should have the option to begin annual screening.[1]

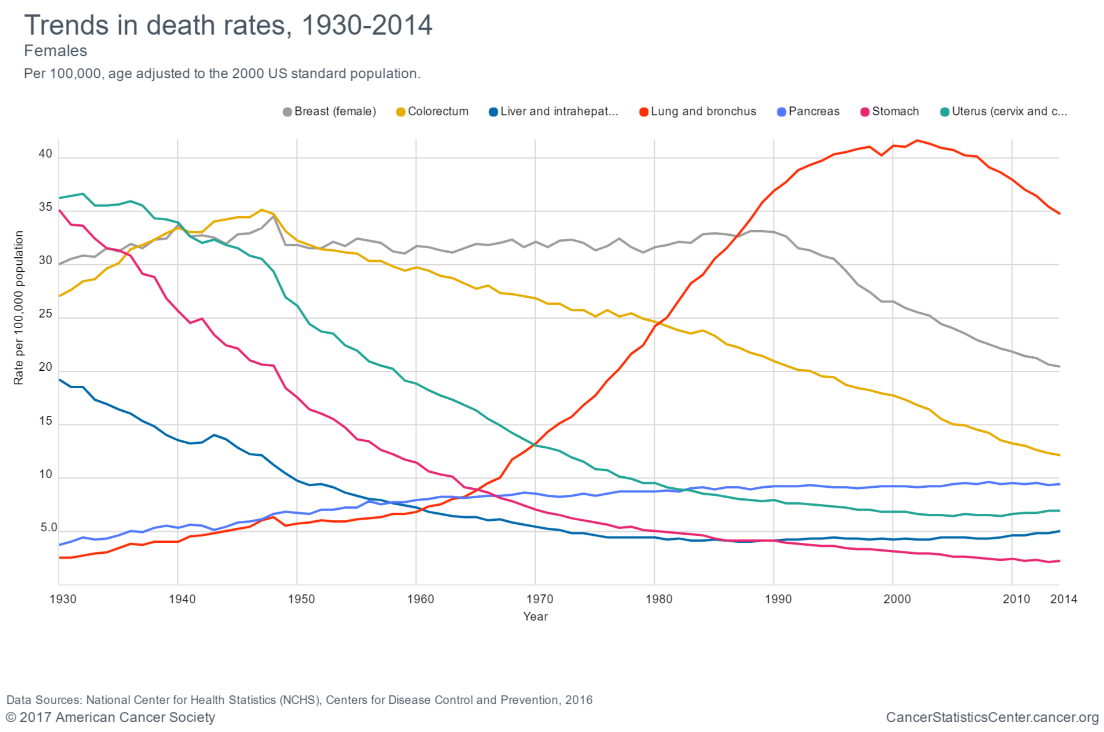

Figure 1: Death rates due to cancer for females from 1930 to 2014 normalized on the US population in 2000

Once screening hints at a possible tumor, further tests are done. The methods currently in use for the diagnosis of breast cancer are

- Mammography
- Clinical Examination
- Ultrasound
- MRI

Mammography

Mammography is a technique that relies on X-rays. The breast is placed in between a for X-ray translucent plate and a film plate. The plates make sure the breast is flattened and still. X-rays are then sent from above through the breast into the film plate to be analyzed. The X-rays will be defracted by things like microcalcifications. Based on the defracted light tumors can be found.

Clinical Examination

Clinical examination is the act of checking upon the breast using touch and sight. A doctor will examine the breast for bumps and bulges and for any damage or hardening of the skin. Based on the types and characteristics of the things found, further tests might be instigated.[9]

Ultrasound

Ultrasound works by acoustic waves similar to how echo location and sonar work. An acoustic gel will be applied onto the breast and a probe will sent acoustic waves through the gel into the breast. Inside the breast the acoustic waves will scatter depending on the tissue the waves encounter. Based on the returning waves through the gel to the probe, the internal structure of the breast can be established.[13]

Magnetic Resonance Imaging

Magnetic Resonance is currently the most costly method in use. A person will be placed inside a MRI-scanner. The scanner will erect a static magnetic field strong enough to align all the hydrogen nuclei in the same direction. After the magnetic field has aligned the nuclei, radio frequency pulses will be sent into the area of interest exciting specific nuclei, depending on the frequency. The area of interest is then observed for photons coming from precessing nuclei. From these photons a mapping can be made.

In figure 2 a table is shown with the performance by the methods described above. The columns in the table mean the following:

- Sensitivity:
  Ratio between the number of malignant tumors detected and the actual number of malignant tumors present
- Specificity:
  Ratio between the number of benign tumors detected and the actual number of benign tumors present
- Positive Predicative Value:
  Ratio between the number of malignant tumors detected and the total positively identified

Modality	Sensitivity	Specificity	Positive Predictive Value	Accuracy
Mammography	67.8% (120/177)	75% (61/81)	85.7% (120/140)	70.2% (181/258)
Mammography and clinical examination	77.4% (137/177)	72% (58/81)	58.6% (137/160)	75.6% (195/258)
Clinical examination	50.3% (89/177)	92% (75/81)	94% (89/95)	63.6% (164/258)
Ultrasound	83.0% (147/177)	34% (28/81)	73.5% (147/200)	67.8% (175/258)
Mammography and Ultrasound	91.5% (162/177)	23% (19/81)	72.3% (162/224)	70.2% (181/258)
Mammography, clinical examination, and Ultrasound	93.2% (165/177)	22% (18/81)	72.4% (165/228)	70.9% (183/258)
MRI	94.4% (167/177)	26% (21/81)	73.6% (167/227)	72.9% (188/258)
Mammography, clinical examination, MRI	99.4% (176/177)	7% (6/81)	70.1% (176/251)	70.5% (182/258)

Figure 2: Methods for detecting breast cancer and their performance[1]

tumors, i.e. the number of malignant tumors and the benign tumors determined to be malignant.

- Accuracy:
  Ratio between the sum of the malignant and benign tumors correctly detected and the total patients

From the table it becomes quite clear that these methods nor combinations of these methods are that good. This has spurred science to look for better options. Currently a few methods are awaiting clinical trials.

- Microwave Imaging
- Electrical Impedance Tomography
- Diffuse optical Tomography
- Microwave Radiometry
- Biopotential Detection
- Biomagnetic Detection

Microwave Imaging

With Microwave Imaging you use microwave radiation of 300 MHz to 1000 MHz to map the breast. This is done by letting a circular antenna of 32 monopoles move up and down while the breast is hung vertically inside it. The antenna and the breast are both submerged into a tank filled with a saline solution. The monopoles send and receive radiation. A map of the breast is made by comparing the sent and received radiation.

This technique has shown that for relatively large tumors, 1 cm across, the conductivity of the malignant tumor, $\sigma_{malignant}$, is much greater, sometimes up to twice as great, than the conductivities and normal cells, $\sigma_{healthy}$ and σ_{benign}.

In a trial is has also been shown that for light in the frequency range 0.5 GHz to 20 GHz the dielectric properties at least for low-adipose(fat) tissue the contrast between the tissue and a malignant tumor is fairly small, since $\epsilon_{malignant} \approx 1.1 * \epsilon_{low-adipose}$.

Electrical Impedance Tomography

Electrical Impedance Tomography(EIT) has a similar thought behind it as Microwave Imaging. With both techniques the difference between the sent and the received determines the structures of the breast. EIT however does the sending and receiving using currents instead of microwave radiation. For EIT to work an array of electrodes will be attached to the breast. An current with a frequency of 100 Hz to 1 MHz will be applied to the electrodes. Due to induced currents it will take some additional energy to maintain the current on the electrodes. Knowing the energy cost to maintain the current in conjunction with the current itself helps to create a map of the interior of the breast. Another way to get the mapping is to look at the resulting voltage differences across the sensors.

In trial this has been shown that the depth to which the mapping could be made accurately is about 3.5 cm to 6 cm depending on the number of electrodes used. Using EIT scientists were able to find tumors of 3 mm to 5 mm in diameter. EIT has however some limitations due to the heterogenity of the skin and difficulty to produce a proper mapping near the nipple.

Diffuse Optical Tomography

In X-ray mammography the X-rays travel through the tissue straight, in Diffuse Optical Tomography near infra-red light is send angled into the breast to bump into the cells. The behavior of the light can then be approximated by the diffusion equation. By retrieving the absorption and scattering coefficients molecules like hemoglobin and water and places where oxygen saturation is high or low, can be found. Since hemoglobin is a indicator for blood and malignant tumors tend to burn so much energy that they start to burn anaerobic, malignant tumors can be found by finding the places where the absorption and scattering coefficients hint at that a lot of hemoglobin is present or that oxygen saturation is low.

In trial this has been shown to work for smaller breast, but for breast with a diameter exceeding the 10 cm making proper measurements gets hard to impossible.

Microwave Radiometry

One of the most notable aspects of tumors is that they grow uncontrollably and burn a lot of energy. This causes the temperature of a malignant tumor to be higher than that of a healthy cell by sometimes up to 3 degrees Celsius. Microwave Radiometry is using exactly this phenomenon to determine were the tumors are. By tuning a radiometer to find hot spots in the tissue the tumors are tracked down.

Biopotential Detection

The uncontrolled growing of the malignant tumors creates, aside from a lot of heat, a ion concentration difference between itself and the surrounding tissue. This concentration gradient gives rise to a potential on the surface of the breast. Biopotential Detection uses this potential on the surface of the breast to determine the places inside the cells which cause the potential and thus locate the tumor.

Biomagnetic Detection

Another oddity of tumors is that malignant tumors tend to have a higher innate magnetic field than benign tumors, 754 fT/$\sqrt{Hz}$ respectively 274 fT/$\sqrt{Hz}$ on average on the frequencies from 2 Hz to 7 Hz. Biomagnetic Detection uses the difference in magnetic fields of the cells to find out whether and where the tumors are hiding.

2 Research question

The above list depicts methods that use frequencies as low as 2 Hz to 1000 MHz, but it seems to skip the 1 MHz to 300 MHz range. In the lower part of that range and partially outside the range, 100 kHz to 10 MHz, ions loose the ability to keep up the changing of the field which causes their polarity to change **??**. Looking at the methods above we can conclude that they tend to say that tumors will have a lower pH and a larger amount of ions compared to the cells surrounding it. Lower Ph and higher ion amounts yield a larger dielectric response in the specific frequency range. If we now combine these thoughts, we might be able to derive a new method using the dielectric response of cells.

In this paper we will focus on creating such a method. In particular if it is possible to build a non-invasive low-risk scanner that uses electromagnetic fields to detect tumors small enough in the 100 kHz to 10 MHz frequency range. With other words we ask ourselves the question:

> If you have the ability to create electromagnetic fields outside of an object, can you theoretically non-invasively *uniquely* determine the electromagnetic properties of the object?

To answer this question we will provide in the next section a strategy for solving it. After we have provided our results, we will provide literature and recommendations for further research.

3 Strategy

The question at hand is an inverse boundary problem. Inverse boundary problems are problems that ask for some properties of a domain, while only knowing data from the boundary. Since our question deals with electromagnetic properties and fields, it seems logical to start with the Maxwell Equations. It is however not directly clear how these should be rewritten to solve our inverse boundary problem.

To find an answer to our question we look therefore to a possible solution given by Caro, Olaxand and Sola[4]. The idea behind what they wrote is that there are aside from the Maxwell Equations many different equations describing waves and one of the useful ones for inverse boundary problems is the Schrödinger Equation. The usefulness of the Schrödinger Equation for this case can be made intuitive by recalling that the Schrödinger Equation describes how a particle or wave with a certain energy acts in a potential. This potential can be taken to be the properties you are interested in across your domain. In fact Cálderon showed that he could by rewriting the Maxwell Equations transform them into a Schrödinger Equation and solve it to retrieve the conductivity of his domain.

The solution Caro, Olaxand and Sola describe is based on this research by Cálderon. Based on the research of others that tried to improve upon the works of Cálderon, they do however take different assumptions than he does to get more information. We will follow the steps Caro, Olaxand and Sola to rewrite the Maxwell Equations into a Schrödinger Equation. We will then derive an identity which uses integration by parts to get an expression for the potential of the Schrödinger Equation in terms of the boundary values. After that we will use a method given by Ola in [22] to derive the electromagnetic properties from this expression.

To get to their solutions Cálderon, Ola, Caro and others took some similar steps. Below are the steps we are going to take listed.

1. Introduce the Maxwell Equations and write them in matrix form
2. Augment the Maxwell Equations
3. Rescale the augmented Maxwell Equations
4. Rewrite the equations into a Dirac System
5. Transform the Dirac System into a Schrödinger Equation
6. Solve the Schrödinger Equation with pseudo-variables set to zero.
7. Find an expression for the potential in terms of the boundary
8. Derive equations for the material parameters γ and μ using the potential

At the start of each of the steps we will briefly explain what happens in the step and why.

4 The answer

In this chapter we will work out all the steps as described above in the section Strategy. To keep track of where we are in the process we will include the step number.

4.1 Step 1: Introduce Maxwell Equations

The Maxwell Equations are time-dependent. Since every time-dependent signal can be constructed from time harmonic signals, we will use time harmonic signals giving rise to the time harmonic Maxwell Equations. This implies that $\mathbf{F}(\mathbf{x},t) = \mathbf{F}(\mathbf{x})e^{-i\omega t}$ for all vectors $\mathbf{F}$ where ω is the angular frequency, which is taken not be to a resonance frequency. The time harmonic Maxwell Equations are thus given by

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E} &= i\omega \mathbf{B} \\
\boldsymbol{\nabla} \cdot \mathbf{D} &= \boldsymbol{\rho}_f \\
\boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J} - i\omega \mathbf{D} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0
\end{aligned}
$$

In this form the Maxwell Equations are still hard to use. We will need to make some assumptions first. We assume that no free current will be flowing or free charges will float around in the breast tissue, because the breast does not contain muscles or any mayor nerve and therefore at most very weak currents will be flowing. Since biological structures in the breast vary but are not crystalline in nature, we also take the breast tissue to be isotropic, but inhomogeneous. We will furthermore assume that the breast tissue is non-chiral and linear such that

$$
\begin{aligned}
\mathbf{B} &= \mu \boldsymbol{H} \\
\mathbf{D} &= \epsilon \boldsymbol{E} \\
\mathbf{J} &= \sigma \boldsymbol{E} + \boldsymbol{J}_f \\
\gamma &= \epsilon + i\frac{\sigma}{\omega} \\
\boldsymbol{\rho}_f &= 0 \\
\boldsymbol{J}_f &= 0
\end{aligned}
$$

For μ and γ we will also assume that they are at least twice differentiable and that they are near the boundary constants with $\mu = \mu_0$ and $\gamma = \gamma_0$. With these we can rewrite the time harmonic Maxwell Equations to

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E} &= i\omega\mu \boldsymbol{H} \\
\boldsymbol{\nabla} \cdot \gamma \boldsymbol{E} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{H} &= -i\omega\gamma \boldsymbol{E} \\
\boldsymbol{\nabla} \cdot \mu \boldsymbol{H} &= 0
\end{aligned}
$$

We can also write this in matrix form as

$$
\begin{pmatrix} \boldsymbol{\nabla}\times & -i\omega\mu \\ \boldsymbol{\nabla}\cdot(\gamma\star) & 0 \\ i\omega\gamma & \boldsymbol{\nabla}\times \\ 0 & \boldsymbol{\nabla}\cdot(\mu\star) \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \mathbf{0}
$$

Note that in the matrix the $\star$ in operators are placeholders for where the relevant quantities should be inserted.

4.2 Step 2: Augmentation

The matrix form consists of 4 equations and 2 unknowns. This means the unknowns are overdetermined and to solve this we introduce pseudo-variables. These pseudo-variables will ensure that we have the same amount of equations as unknowns. We introduce the time harmonic pseudo-variables Ψ and Φ with

$$\begin{aligned} -i\omega\Phi &= \boldsymbol{\nabla}\cdot\gamma\boldsymbol{E} \\ -i\omega\Psi &= \boldsymbol{\nabla}\cdot\mu\boldsymbol{H} \\ -\frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\gamma}\Phi\right) &= \boldsymbol{\nabla}\times\mathbf{H} + i\omega\gamma\mathbf{E} \\ \frac{1}{\gamma}\boldsymbol{\nabla}\left(\frac{1}{\mu}\Psi\right) &= \boldsymbol{\nabla}\times\mathbf{E} - i\omega\mu\mathbf{H} \end{aligned}$$

This particular choice will prove useful later on. If we now plug these equations in the matrix form we get

$$\begin{pmatrix} i\omega & \boldsymbol{\nabla}\cdot(\gamma\star) & 0 & 0 \\ 0 & 0 & \boldsymbol{\nabla}\cdot(\mu\star) & i\omega \\ \frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\gamma}\right) & i\omega\gamma & \boldsymbol{\nabla}\times & 0 \\ 0 & \boldsymbol{\nabla}\times & -i\omega\mu & -\frac{1}{\gamma}\boldsymbol{\nabla}\left(\frac{1}{\mu}\right) \end{pmatrix}\begin{pmatrix}\Phi \\ \boldsymbol{E} \\ \boldsymbol{H} \\ \Psi\end{pmatrix} = \mathbf{0}$$

We reorder the matrix so it is easier to read.

$$\begin{pmatrix} i\omega & \boldsymbol{\nabla}\cdot(\gamma\star) & 0 & 0 \\ \frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\gamma}\right) & i\omega\gamma & \boldsymbol{\nabla}\times & 0 \\ 0 & \boldsymbol{\nabla}\times & -i\omega\mu & -\frac{1}{\gamma}\boldsymbol{\nabla}\left(\frac{1}{\mu}\right) \\ 0 & 0 & \boldsymbol{\nabla}\cdot(\mu\star) & i\omega \end{pmatrix}\begin{pmatrix}\Phi \\ \boldsymbol{E} \\ \boldsymbol{H} \\ \Psi\end{pmatrix} = \mathbf{0}$$

4.3 Step 3: Rescaling

The current matrix is not neatly transformable to a Schrödinger Equation. There is however a rescaling possible that makes this matrix equation a Dirac System. A Dirac System looks like $(\mathbf{P}+\mathbf{V})\mathbf{X} = \mathbf{0}$ and has the property that $\mathbf{P^2} = \boldsymbol{\Delta}$. This will help us make the Schrödinger Equation. Therefore we will choose to rescale the elliptical Maxwell Equations using

$$\begin{pmatrix} \frac{1}{\gamma\sqrt{\mu}}\Phi \\ \sqrt{\gamma}\boldsymbol{E} \\ \sqrt{\mu}\boldsymbol{H} \\ \frac{1}{\mu\sqrt{\gamma}}\Psi \end{pmatrix} = \begin{pmatrix} X_1 \\ \mathbf{X_2} \\ \mathbf{X_3} \\ X_4 \end{pmatrix} = \mathbf{X}$$

The resulting form of the matrix equation after this rescaling is

$$\begin{pmatrix} i\omega\gamma\sqrt{\mu} & \boldsymbol{\nabla}\cdot\left(\sqrt{\gamma}\star\right) & 0 & 0 \\ \frac{1}{\mu}\boldsymbol{\nabla}\sqrt{\mu} & i\omega\sqrt{\gamma} & \boldsymbol{\nabla}\times\left(\frac{1}{\sqrt{\mu}}\star\right) & 0 \\ 0 & \boldsymbol{\nabla}\times\left(\frac{1}{\sqrt{\gamma}}\star\right) & -i\omega\sqrt{\mu} & -\frac{1}{\gamma}\boldsymbol{\nabla}\sqrt{\gamma} \\ 0 & 0 & \boldsymbol{\nabla}\cdot\left(\sqrt{\mu}\star\right) & i\omega\mu\sqrt{\gamma} \end{pmatrix}\mathbf{X} = \mathbf{0}$$

4.4 Step 4: Making the Dirac System

We mentioned that the matrix equation becomes a Dirac System after rescaling. To see this we need to first separate the matrix in a zeroth order and higher order part. Doing this we get

$$\left(\begin{pmatrix} 0 & \sqrt{\gamma}\boldsymbol{\nabla}\cdot & 0 & 0 \\ \frac{1}{\sqrt{\mu}}\boldsymbol{\nabla} & 0 & \frac{1}{\sqrt{\mu}}\boldsymbol{\nabla}\times & 0 \\ 0 & \frac{1}{\sqrt{\gamma}}\boldsymbol{\nabla}\times & 0 & -\frac{1}{\sqrt{\gamma}}\boldsymbol{\nabla} \\ 0 & 0 & \sqrt{\mu}\boldsymbol{\nabla}\cdot & 0 \end{pmatrix} + \begin{pmatrix} i\omega\gamma\sqrt{\mu} & \boldsymbol{\nabla}\sqrt{\gamma}\cdot & 0 & 0 \\ \frac{1}{\mu}\boldsymbol{\nabla}\sqrt{\mu} & i\omega\sqrt{\gamma} & \boldsymbol{\nabla}\left(\frac{1}{\sqrt{\mu}}\right)\times & 0 \\ 0 & \boldsymbol{\nabla}\left(\frac{1}{\sqrt{\gamma}}\right)\times & -i\omega\sqrt{\mu} & -\frac{1}{\gamma}\boldsymbol{\nabla}\sqrt{\gamma} \\ 0 & 0 & \boldsymbol{\nabla}\sqrt{\mu}\cdot & i\omega\mu\sqrt{\gamma} \end{pmatrix}\right)\mathbf{X} = \mathbf{0}$$

We see that we can multiply the rows of the matrices and clean the leftmost matrix by doing so. We divide each row with the value in front of the operator of the left matrix and multiply the third row with -1 to get

$$\left(\begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \\ \boldsymbol{\nabla} & 0 & \boldsymbol{\nabla}\times & 0 \\ 0 & -\boldsymbol{\nabla}\times & 0 & \boldsymbol{\nabla} \\ 0 & 0 & \boldsymbol{\nabla}\cdot & 0 \end{pmatrix} + \begin{pmatrix} i\omega\sqrt{\gamma}\sqrt{\mu} & \frac{1}{\sqrt{\gamma}}\boldsymbol{\nabla}\sqrt{\gamma}\cdot & 0 & 0 \\ \frac{1}{\sqrt{\mu}}\boldsymbol{\nabla}(\sqrt{\mu}) & i\omega\sqrt{\gamma}\sqrt{\mu} & \sqrt{\mu}\boldsymbol{\nabla}\left(\frac{1}{\sqrt{\mu}}\right)\times & 0 \\ 0 & -\sqrt{\gamma}\boldsymbol{\nabla}\left(\frac{1}{\sqrt{\gamma}}\right)\times & i\omega\sqrt{\mu}\sqrt{\gamma} & \frac{1}{\sqrt{\gamma}}\boldsymbol{\nabla}(\sqrt{\gamma}) \\ 0 & 0 & \frac{1}{\sqrt{\mu}}\boldsymbol{\nabla}\sqrt{\mu}\cdot & i\omega\sqrt{\mu}\sqrt{\gamma} \end{pmatrix}\right)\mathbf{X} = \mathbf{0}$$

By setting a few variable we can simplify this system. Let

$$\begin{aligned} k &= \omega\sqrt{\gamma\mu} \\ \alpha &= \frac{1}{2}\log(\gamma) \\ \beta &= \frac{1}{2}\log(\mu) \end{aligned}$$

Inserting these into the system, we get the simplified matrix equation

$$\left(\begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \\ \boldsymbol{\nabla} & 0 & \boldsymbol{\nabla}\times & 0 \\ 0 & -\boldsymbol{\nabla}\times & 0 & \boldsymbol{\nabla} \\ 0 & 0 & \boldsymbol{\nabla}\cdot & 0 \end{pmatrix} + \begin{pmatrix} ik & \boldsymbol{\nabla}\alpha\cdot & 0 & 0 \\ \boldsymbol{\nabla}\beta & ik & -\boldsymbol{\nabla}\beta\times & 0 \\ 0 & \boldsymbol{\nabla}\alpha\times & ik & \boldsymbol{\nabla}\alpha \\ 0 & 0 & \boldsymbol{\nabla}\beta\cdot & ik \end{pmatrix}\right)\mathbf{X} = \mathbf{0}$$

If we now write this matrix equation into the form $(\mathbf{P} + \mathbf{V})\mathbf{X} = \mathbf{0}$ by taking

$$\mathbf{P} = \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \\ \boldsymbol{\nabla} & 0 & \boldsymbol{\nabla}\times & 0 \\ 0 & -\boldsymbol{\nabla}\times & 0 & \boldsymbol{\nabla} \\ 0 & 0 & \boldsymbol{\nabla}\cdot & 0 \end{pmatrix}$$
$$\mathbf{V} = \begin{pmatrix} ik & \boldsymbol{\nabla}\alpha\cdot & 0 & 0 \\ \boldsymbol{\nabla}\beta & ik & -\boldsymbol{\nabla}\beta\times & 0 \\ 0 & \boldsymbol{\nabla}\alpha\times & ik & \boldsymbol{\nabla}\alpha \\ 0 & 0 & \boldsymbol{\nabla}\beta\cdot & ik \end{pmatrix}$$

we see that this is indeed a Dirac System, since

$$\mathbf{P^2} = \begin{pmatrix} \Delta & & & \\ & \boldsymbol{\Delta} & & \\ & & \boldsymbol{\Delta} & \\ & & & \Delta \end{pmatrix}$$

4.5 Step 5: Rewriting the Dirac System to a Schrödinger Equation

A Schrödinger Equation is of the form $(\mathbf{\Delta} + E - \mathbf{Q})\mathbf{Z} = \mathbf{0}$ for some potential $\mathbf{Q}$ and some E. Note that the laplacian and potential do not need to be vector, but can also be scalars. We will use of the property that for a Dirac System $\mathbf{P^2} = \mathbf{\Delta}$ to rewrite the Dirac System into a Schrödinger Equation.

Let us first define $\mathbf{D}$ as

$$\mathbf{D} = \mathbf{P}^2 = \begin{pmatrix} \Delta & & & \\ & \mathbf{\Delta} & & \\ & & \mathbf{\Delta} & \\ & & & \Delta \end{pmatrix}$$

Let us further define the variable $\mathbf{Z}$ such that

$$\mathbf{X} = (\mathbf{P} - \mathbf{V^T})\mathbf{Z}$$

where $\mathbf{X}$, $\mathbf{P}$ and $\mathbf{V}$ are as in the Dirac System and $\mathbf{V^T}$ is de transpose of $\mathbf{V}$. By substituting this variable $\mathbf{Z}$ into the Dirac System we get

$$\begin{aligned} \mathbf{0} = (\mathbf{P} + \mathbf{V})\mathbf{X} &= (\mathbf{P} + \mathbf{V})(\mathbf{P} - \mathbf{V^T})\mathbf{Z} \\ &= \left(\mathbf{P^2} + \mathbf{VP} - \mathbf{PV^T} - \mathbf{VV^T}\right)\mathbf{Z} \\ &= \left(\mathbf{D} + \mathbf{VP} - \mathbf{PV^T} - \mathbf{VV^T}\right)\mathbf{Z} \\ &= \left(\mathbf{D} - \mathbf{Q}\right)\mathbf{Z} \end{aligned}$$

where we have substituted $\mathbf{Q} = -\mathbf{VP} + \mathbf{PV^T} + \mathbf{VV^T}$. In appendix 1 we calculated that this potential $\mathbf{Q}$ has the form[4]

$$\mathbf{Q} = -k^2\mathbf{1}_8 - \begin{pmatrix} \Delta\alpha + \|\mathbf{\nabla}\alpha\|^2 & 2i\mathbf{\nabla}k\cdot & 0 & 0 \\ 2i\mathbf{\nabla}k & 2\mathbf{\nabla}\mathbf{\nabla}\beta\cdot -(\Delta\beta - \|\mathbf{\nabla}\beta\|^2)\mathbf{1_3} & 0 & 0 \\ 0 & 0 & 2\mathbf{\nabla}\mathbf{\nabla}\alpha\cdot -(\Delta\alpha - \|\mathbf{\nabla}\alpha\|^2)\mathbf{1_3} & 2i\mathbf{\nabla}k \\ 0 & 0 & 2i\mathbf{\nabla}k\cdot & \Delta\beta + \|\mathbf{\nabla}\beta\|^2 \end{pmatrix}$$

4.6 Step 6: Solving the Schrödinger Equations

We have the from of the Schrödinger Equation. We will try to rewrite the Schrödinger Equation into a integral equation, so we can solve it for $\mathbf{Z}$. Before we do this we first alter our potential $\mathbf{Q}$ a bit to make it compactly supported. This is a requirement for solving the integral equation [3][2]. The change we want to make to the potential is

$$\tilde{\mathbf{Q}} = \mathbf{Q} + k_0^2\mathbf{1}_8$$

As a result of this $\tilde{\mathbf{Q}} = 0$ for all $\mathbf{x}$ outside of the domain and on the boundary. This gives that $\tilde{\mathbf{Q}}$ is now compactly supported[4]. Incorporating this $\tilde{\mathbf{Q}}$ into the Schrödinger Equation gives us

$$0 = \left(\mathbf{D} - \mathbf{Q}\right)\mathbf{Z} = \left(\mathbf{D} + k_0^2\mathbf{1_8} - \tilde{\mathbf{Q}}\right)\mathbf{Z}$$

or

$$\left(\mathbf{D} + k_0^2 \mathbf{1_8}\right)\mathbf{Z} = \tilde{\mathbf{Q}}\mathbf{Z}$$

Taking the left hand side to be a single operator we can write this as

$$\mathbf{L}\mathbf{Z} = \tilde{\mathbf{Q}}\mathbf{Z}$$

with

$$\mathbf{L} = \mathbf{D} + k_0^2 \mathbf{1}_8$$

For this operator $\mathbf{L}$ we can derive a Green's Function $\mathbf{G}$. We derive the Green's Function in appendix 2. The resulting Green's Function is for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x} = \mathbf{u} - \mathbf{v}$ given by

$$\mathbf{G}(\mathbf{u}, \mathbf{v}) = \mathbf{G}(\mathbf{x}) = \begin{pmatrix} G_1(\mathbf{x}) & & & \\ & \mathbf{G_3}(\mathbf{x}) & & \\ & & \mathbf{G_3}(\mathbf{x}) & \\ & & & G_1(\mathbf{x}) \end{pmatrix}$$

where $G_1(\mathbf{x})$ is the scalar Helmholtz Green's Function and $\mathbf{G_3}(\mathbf{x})$ is the vector Helmholtz Green's Function, which are given by[7][5][6]

$$G_1(\mathbf{x}) = \frac{1}{4\pi}\frac{e^{ik_0 x}}{x}$$
$$\mathbf{G_3}(\mathbf{x}) = (1 + \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot)G_1(\mathbf{x})\mathbf{1}_3$$

By getting the expression for the Green's Function we can retrieve the integral equation we were looking for using Green's theorem:

$$\mathbf{Z}(\mathbf{x}) = \mathbf{Z_0}(\mathbf{x}) + \int_{\boldsymbol{\Omega}} \mathbf{G}(\mathbf{x} - \mathbf{x_0})\tilde{\mathbf{Q}}(\mathbf{x_0})\mathbf{Z}(\mathbf{x_0})\mathbf{d}\mathbf{x_0}$$

where $\mathbf{Z_0}(\mathbf{x})$ is the solution for $\mathbf{L}\mathbf{Z_0}(\mathbf{x}) = 0$. According to Reyesa[3] and Caro[2] solutions exists to this equation for $\mathbf{Z}(\mathbf{x})$. These solutions are given by

$$\mathbf{Z}(\mathbf{x}) = e^{i\boldsymbol{\zeta}\cdot\mathbf{x}}(\mathbf{A} + \mathbf{R})$$

with

$$\zeta > C \sum_{m,n=1}^{8} \left\|\tilde{\mathbf{Q}}_{m,n}\right\|_{L^\infty(B_\rho)}$$
$$\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} = k_0^2$$
$$\mathbf{A} = \mathbf{A}(\boldsymbol{\zeta}) = \frac{1}{\zeta}\begin{pmatrix} \boldsymbol{\zeta}\cdot\mathbf{A_1} \\ k_0\mathbf{A_1} \\ k_0\mathbf{A_2} \\ \boldsymbol{\zeta}\cdot\mathbf{A_2} \end{pmatrix}$$
$$\|\mathbf{R}\|_{\mathbf{L}^2_\delta\dagger} \le \frac{C}{\zeta}\|\mathbf{A}\| \sum_{m,n=1}^{8} \left\|\tilde{\mathbf{Q}}_{m,n}\right\|_{L^\infty(B_\rho)}$$

In this $\mathbf{A_1}$ and $\mathbf{A_2}$ are constant vector fields and the pseudo variables Φ and Ψ vanish. We will later use these parameters, $\mathbf{A_1}$ and $\mathbf{A_2}$, to find an expression for γ and μ. Following from the form of $\mathbf{Z}$ and knowing that the pseudo variables vanish, we get that $\mathbf{X} = (\mathbf{P} - \mathbf{V^T})\mathbf{Z}$ solves $(\mathbf{P} + \mathbf{V})\mathbf{X} = 0$ with

$$\mathbf{X} = \begin{pmatrix} 0 \\ \sqrt{\gamma}\mathbf{E} \\ \sqrt{\mu}\mathbf{H} \\ 0 \end{pmatrix}$$

4.7 Step 7: Expressing the potential

To formulate an expression for the potential we will use an integration by parts identity that we can use on the Dirac operator $\mathbf{P}$. Since the operator $\mathbf{P}$ has gradients, divergences and curls as components we need to derive an integration by parts identity for those first. To be able to give the integration by parts identities we will first introduce another identity for convenience. This identity is for given vectors $\mathbf{u}$ and $\mathbf{w}$ and scalars s and t

$$\langle \mathbf{u}|\mathbf{w}\rangle_{\mathbf{\Omega}} = \int_{\mathbf{\Omega}} \mathbf{w}^*\mathbf{u}dV = \int_{\mathbf{\Omega}} \mathbf{u}\cdot\bar{\mathbf{w}}dV$$
$$\langle s|t\rangle_{\mathbf{\Omega}} = \int_{\mathbf{\Omega}} s\bar{t}dV$$

where $\mathbf{w}^*$ is the conjugate transpose of $\mathbf{w}$, $\bar{\mathbf{w}}$ the complex conjugate of $\mathbf{w}$, $\cdot$ the regular Euclidean inner product and $\mathbf{\Omega}$ is the domain for integration. The integration by parts formula with partial derivatives[21] is given for scalars s and t by

$$\int_{\mathbf{\Omega}} s\frac{\partial t}{\partial x_i}dV = \int_{\partial\mathbf{\Omega}} st\hat{\mathbf{n}}dA - \int_{\mathbf{\Omega}} \frac{\partial s}{\partial x_i}tdV$$

where $\partial\mathbf{\Omega}$ is the boundary of $\mathbf{\Omega}$ and $\hat{\mathbf{n}}$ is the normal unit vector. We can use this general expression to get an integration by parts identity for the gradient, divergence and curl. For the three of them we have

$$\langle \nabla\times\mathbf{u}|\mathbf{w}\rangle_{\mathbf{\Omega}} = \langle \hat{\mathbf{n}}\times\mathbf{u}|\mathbf{w}\rangle_{\partial\mathbf{\Omega}} - \langle \mathbf{u}|\nabla\times\mathbf{w}\rangle_{\mathbf{\Omega}}$$
$$\langle \nabla\cdot\mathbf{u}|\mathbf{w}\rangle_{\mathbf{\Omega}} = \langle \hat{\mathbf{n}}\cdot\mathbf{u}|\mathbf{w}\rangle_{\partial\mathbf{\Omega}} - \langle \mathbf{u}|\nabla\cdot\mathbf{w}\rangle_{\mathbf{\Omega}}$$
$$\langle \nabla s|t\rangle_{\mathbf{\Omega}} = \langle \hat{\mathbf{n}}s|t\rangle_{\partial\mathbf{\Omega}} - \langle s|\nabla t\rangle_{\mathbf{\Omega}}$$

We can combine the three expressions to get an integration by parts identity for the operator $\mathbf{P}$. For vectors $\mathbf{U} = \begin{pmatrix} U_1 & \mathbf{U_2} & \mathbf{U_3} & U_4 \end{pmatrix}^{\mathbf{T}}$ and $\mathbf{W} = \begin{pmatrix} W_1 & \mathbf{W_2} & \mathbf{W_3} & W_4 \end{pmatrix}^{\mathbf{T}}$ we get the integration by parts

$$\langle \mathbf{PU}|\mathbf{W}\rangle_{\mathbf{\Omega}} = \langle \mathbf{P}(\hat{\mathbf{n}})\mathbf{U}|\mathbf{W}\rangle_{\partial\mathbf{\Omega}} - \langle \mathbf{U}|\mathbf{PW}\rangle_{\mathbf{\Omega}}$$

where $\mathbf{P}(\hat{\mathbf{n}})$ is the matrix $\mathbf{P}$ with every ∇ replaced by $\hat{n}$.

We now have a integration by parts identity that uses the Dirac operator $\mathbf{P}$. To get an expression for the $\mathbf{Q}$ in terms of the boundary we only have to pick two vectors such that all but the boundary terms vanish for a particular identity. We will show that this is possible by taking vectors $\tilde{\mathbf{Z}}_{\mathbf{0}}$, $\mathbf{X}$, $\tilde{\mathbf{X}}$ and $\mathbf{Z}$ subject to

$$(\mathbf{P} + ik_0)\tilde{\mathbf{Z}}_{\mathbf{0}} = \mathbf{0}$$
$$\mathbf{X} = (\mathbf{P} - ik_0 - \tilde{\mathbf{V}}^{\mathbf{T}})\mathbf{Z}$$
$$\tilde{\mathbf{X}} = (\mathbf{P} - ik_0)\mathbf{Z}$$
$$\tilde{\mathbf{Q}}\mathbf{Z} = (\mathbf{D} + k_0^2)\mathbf{Z}$$

where we have set $\tilde{\mathbf{V}} = \mathbf{V} - ik_0$. Reusing the identities we defined earlier we can combine these vectors to get

$$\begin{aligned}
\left\langle \tilde{\mathbf{Q}}\mathbf{Z} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} &= \left\langle (\mathbf{D} + k_0^2)\mathbf{Z} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle (\mathbf{P} + ik_0)(\mathbf{P} - ik_0)\mathbf{Z} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle (\mathbf{P} + ik_0)\tilde{\mathbf{X}} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle \mathbf{P}\tilde{\mathbf{X}} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} + \left\langle ik_0\tilde{\mathbf{X}} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle \mathbf{P}\tilde{\mathbf{X}} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} - \left\langle \tilde{\mathbf{X}} \middle| ik_0\tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle \mathbf{P}\tilde{\mathbf{X}} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} + \left\langle \tilde{\mathbf{X}} \middle| \mathbf{P}\tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\mathbf{\Omega}} \\
&= \left\langle \mathbf{P}(\hat{\mathbf{n}})\mathbf{X} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle_{\partial\mathbf{\Omega}}
\end{aligned}$$

where we have used that $\langle i\mathbf{U}|\mathbf{V}\rangle = -\langle \mathbf{U}|i\mathbf{V}\rangle$ and $\mathbf{X} = \tilde{\mathbf{X}}$ on the boundary of the domain[22].

4.8 Step 8: Deriving material parameters γ and μ

We just derived the expression for an identity with the potential in the domain in terms of the boundary. To find an equation for the material parameters we only need to rewrite the LHS of this identity into an equation for the material parameters. We will take similar choices like Ola did in [22]. Our goal is to cleverly pick our solutions, so that $\left\langle \tilde{\mathbf{Q}}\mathbf{Z} \middle| \tilde{\mathbf{Z}}_\mathbf{0} \right\rangle$ turns into a Fourier transform of $\tilde{\mathbf{Q}}$ so that we can extract the material parameters from that. To this end we let

$$\begin{aligned}
\mathbf{Z} &= e^{i\mathbf{x}\cdot\boldsymbol{\zeta}}\mathbf{A} \\
\tilde{\mathbf{Z}}_\mathbf{0} &= e^{i\mathbf{x}\cdot\tilde{\boldsymbol{\zeta}}}\tilde{\mathbf{A}}
\end{aligned}$$

be solutions to the inhomogeneous respectively homogeneous Schrödinger Equation and have $\tilde{\mathbf{Q}}$ be the potential of the inhomogeneous Schrödinger Equation. Take $\boldsymbol{\xi} \in \mathbb{R}^3$ with $\boldsymbol{\xi} = \begin{pmatrix} \xi & 0 & 0 \end{pmatrix}^\mathbf{T}$ and set for $n > 0$

$$\begin{aligned}
\boldsymbol{\zeta} = \boldsymbol{\zeta}(n) &= \begin{pmatrix} \xi \\ i\sqrt{\frac{\xi}{4+n^2}} \\ \sqrt{n^2 + k^2} \end{pmatrix} \\
\tilde{\boldsymbol{\zeta}} &= \bar{\boldsymbol{\zeta}} - \boldsymbol{\xi}
\end{aligned}$$

Note that $\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = \tilde{\boldsymbol{\zeta}} \cdot \tilde{\boldsymbol{\zeta}} = k_0^2$. Let us furthermore take

$$\tilde{\mathbf{A}} = \frac{1}{\zeta}\left(e^{-i\mathbf{x}\cdot\tilde{\zeta}}\mathbf{P}e^{i\mathbf{x}\cdot\tilde{\zeta}}\right)\mathbf{w}$$

for some vector $\mathbf{w} = \begin{pmatrix} w_1 & \mathbf{w_2} & w_3 & \mathbf{w_4} \end{pmatrix}^{\mathbf{T}}$. This gives us in the limit of $\zeta \to \infty$, if we now choose $\mathbf{w} = \begin{pmatrix} 0 & \mathbf{w_2} & 0 & 0 \end{pmatrix}^{\mathbf{T}}$,

$$\lim_{\zeta\to\infty} \mathbf{A} = \begin{pmatrix} \hat{\boldsymbol{\zeta}} \cdot \mathbf{A_1} \\ 0 \\ 0 \\ \hat{\boldsymbol{\zeta}} \cdot \mathbf{A_2} \end{pmatrix}$$

$$\lim_{\zeta\to\infty} \tilde{\mathbf{A}} = \begin{pmatrix} -i\hat{\boldsymbol{\zeta}} \cdot \mathbf{w_2} \\ 0 \\ 0 \\ i\hat{\boldsymbol{\zeta}} \cdot \mathbf{w_2} \end{pmatrix}$$

where $\hat{\boldsymbol{\zeta}} = \lim_{\zeta\to\infty} \frac{\boldsymbol{\zeta}}{\zeta} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 \end{pmatrix}^{\mathbf{T}}$. Picking $\hat{\boldsymbol{\zeta}} \cdot \mathbf{A_1} = 1$ and $-i\hat{\boldsymbol{\zeta}} \cdot \mathbf{w_2} = 1$ and $\hat{\boldsymbol{\zeta}} \cdot \mathbf{A_2} = 0$ and $i\hat{\boldsymbol{\zeta}} \cdot \mathbf{w_2} = 0$ we can insert these into the identity and get

$$\lim_{\zeta\to\infty} \left\langle \tilde{\mathbf{Q}}\mathbf{Z} \middle| \tilde{\mathbf{Z}}_{\mathbf{0}} \right\rangle = \mathcal{F}\left\{ \tilde{\mathbf{Q}}_{\mathbf{0},\mathbf{0}} \right\}(\boldsymbol{\xi})$$

where $\mathcal{F}\left\{ \mathbf{U}(\mathbf{x}) \right\}(\mathbf{k})$ is the Fourier Transform of $\mathbf{U}(\mathbf{x})$. By making similar choices we get

$$\lim_{\zeta\to\infty} \left\langle \tilde{\mathbf{Q}}\mathbf{Z} \middle| \tilde{\mathbf{Z}}_{\mathbf{0}} \right\rangle = \mathcal{F}\left\{ \tilde{\mathbf{Q}}_{\mathbf{8},\mathbf{8}} \right\}(\boldsymbol{\xi})$$

Once we know the values for $\tilde{\mathbf{Q}}_{\mathbf{0},\mathbf{0}}$ and $\tilde{\mathbf{Q}}_{\mathbf{8},\mathbf{8}}$ we can derive from those the values for α and β and thus γ and μ. We do this by recalling that the equations governing the 0,0 and the 8,8 elements of the potential are respectively

$$\tilde{\mathbf{Q}}_{\mathbf{0},\mathbf{0}} = -\Delta\alpha - \|\boldsymbol{\nabla}\alpha\|^2 - (k^2 - k_0^2)$$
$$\tilde{\mathbf{Q}}_{\mathbf{8},\mathbf{8}} = -\Delta\beta - \|\boldsymbol{\nabla}\beta\|^2 - (k^2 - k_0^2)$$

This is a set of coupled PDE's which can be solved for α and β. From these we can then extract μ and γ.

5 Discussion

In the start of our derivation we have mentioned a few assumptions with brief explanation for the choice. In this section we will elaborate on the choices made. The assumptions we are going to elaborate upon are:

- linear vs non-linear media
- free current or free charge
- isotropic vs anisotropic
- resonance frequencies

5.1 Free charges and currents

The breast consists mostly of fat tissue and milk glands. These tissues do not contain a lot of nerves[14][15]. Since the breast also does not have any mayor nerves going through it, the electric activity inside the breast is not very high[19]. In the introduction however we have seen that there are methods, the biomagnetic detection and biopotential detection, developed that use this not very high electric activity to produce maps of the breast tissue. The effects that these methods use is at most a magnetic field of the order of several picotesla and a voltage of at most 20 milivolts. Since we will be applying a external electric field, it is not unlikely the effects of the external field will be large enough that the natural activity does not matter and thus we have reason to assume that the free charges and free currents are zero. To see whether this is really true further research has to be done.

5.2 (An)isotropic

In the paper we have used thoroughly that we assume the breast tissue to be isotropic. In the paper we have assumed that this is a valid assumption based on the thought that the breast tissue varies in structure but is not crystalline. In research however about the acoustic properties of the breast it has been shown by R. Sinkus and others that the breast tissue is for acoustic purposes anisotropic[16][17]. Furthermore it has been shown that the optical properties for chickens are anisotropic[18]. This hints at that it might be true that breast tissue is also anisotropic. The impact of taking the breast to be isotropic instead of anisotropic however is not know. What we do know is that it makes the math more complicated: In the isotropic case the relation between $\mathbf{D}$ and $\mathbf{E}$ is given by $\mathbf{D} = \epsilon\mathbf{E}$, in the anisotropic case this changes to $\mathbf{D} = \vec{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\mathbf{E}$ with $\vec{\boldsymbol{\epsilon}}$ is a rank 2 tensor[20]. To show why this is more complex we will calculate the divergence for both cases. Let us first take

$$\mathbf{D} = \begin{pmatrix}\mathbf{D_x} & \mathbf{D_y} & \mathbf{D_z}\end{pmatrix}^{\mathbf{T}}$$
$$\mathbf{E} = \begin{pmatrix}\mathbf{E_x} & \mathbf{E_y} & \mathbf{E_z}\end{pmatrix}^{\mathbf{T}}$$
$$\vec{\boldsymbol{\epsilon}} = \begin{pmatrix}\epsilon_{xx} & \epsilon_{yx} & \epsilon_{zx}\\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{zy}\\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz}\end{pmatrix}$$

For the divergence in the isotropic case we get

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \boldsymbol{\nabla}\cdot(\epsilon\mathbf{E}) = \epsilon\boldsymbol{\nabla}\cdot E + \boldsymbol{\nabla}\epsilon\cdot\mathbf{E}$$

Calculating the right hand side yields

$$\epsilon\boldsymbol{\nabla}\cdot E + \boldsymbol{\nabla}\epsilon\cdot\mathbf{E} = \epsilon\frac{\partial\mathbf{E_x}}{\partial\mathbf{x}} + \epsilon\frac{\partial\mathbf{E_y}}{\partial\mathbf{y}} + \epsilon\frac{\partial\mathbf{E_z}}{\partial\mathbf{z}} + \frac{\partial\epsilon}{\partial\mathbf{x}}\mathbf{E_x} + \frac{\partial\epsilon}{\partial\mathbf{y}}\mathbf{E_y} + \frac{\partial\epsilon}{\partial\mathbf{z}}\mathbf{E_z}$$

For the anisotropic case we have for the divergence[23]

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{D} &= \boldsymbol{\nabla} \cdot (\vec{\boldsymbol{\epsilon}} \cdot \mathbf{E}) \\ &= \vec{\boldsymbol{\epsilon}} : \boldsymbol{\nabla}\mathbf{E} + (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{\epsilon}}) \cdot \mathbf{E}\end{aligned}$$

The right part of the right hand side is

$$\begin{aligned}(\boldsymbol{\nabla} \cdot \vec{\boldsymbol{\epsilon}}) \cdot \mathbf{E} =& \begin{pmatrix} \frac{\partial \epsilon_{xx}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yx}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zx}}{\partial \mathbf{z}} \\ \frac{\partial \epsilon_{xy}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yy}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zy}}{\partial \mathbf{z}} \\ \frac{\partial \epsilon_{xz}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yz}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zz}}{\partial \mathbf{z}} \end{pmatrix} \cdot \mathbf{E} \\ =& \left(\frac{\partial \epsilon_{xx}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yx}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zx}}{\partial \mathbf{z}}\right)\mathbf{E_x} + \\ & \left(\frac{\partial \epsilon_{xy}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yy}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zy}}{\partial \mathbf{z}}\right)\mathbf{E_y} + \\ & \left(\frac{\partial \epsilon_{xz}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yz}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zz}}{\partial \mathbf{z}}\right)\mathbf{E_z}\end{aligned}$$

The left part of the right hand side is slightly more complicated. First we need to calculate the gradient of the vector **E**.

$$\boldsymbol{\nabla}\mathbf{E} = \begin{pmatrix} \frac{\partial E_x}{\partial x} & \frac{\partial E_y}{\partial x} & \frac{\partial E_z}{\partial x} \\ \frac{\partial E_x}{\partial y} & \frac{\partial E_y}{\partial y} & \frac{\partial E_z}{\partial y} \\ \frac{\partial E_x}{\partial z} & \frac{\partial E_y}{\partial z} & \frac{\partial E_z}{\partial z} \end{pmatrix}$$

Now we can calculate the double colon product. This is by definition[24] equal to

$$\begin{aligned}\vec{\boldsymbol{\epsilon}} : \boldsymbol{\nabla}\mathbf{E} =& \frac{\partial \mathbf{E_x}}{\partial \mathbf{x}}\epsilon_{xx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{x}}\epsilon_{xy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{x}}\epsilon_{xz} + \\ & \frac{\partial \mathbf{E_x}}{\partial \mathbf{y}}\epsilon_{yx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{y}}\epsilon_{yy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{y}}\epsilon_{yz} + \\ & \frac{\partial \mathbf{E_x}}{\partial \mathbf{z}}\epsilon_{zx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{z}}\epsilon_{zy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{z}}\epsilon_{zz}\end{aligned}$$

Joining these expression we get

$$\begin{aligned}\vec{\boldsymbol{\epsilon}} : \boldsymbol{\nabla}\mathbf{E} + (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{\epsilon}}) \cdot \mathbf{E} =& \frac{\partial \mathbf{E_x}}{\partial \mathbf{x}}\epsilon_{xx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{x}}\epsilon_{xy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{x}}\epsilon_{xz} + \\ & \frac{\partial \mathbf{E_x}}{\partial \mathbf{y}}\epsilon_{yx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{y}}\epsilon_{yy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{y}}\epsilon_{yz} + \\ & \frac{\partial \mathbf{E_x}}{\partial \mathbf{z}}\epsilon_{zx} + \frac{\partial \mathbf{E_y}}{\partial \mathbf{z}}\epsilon_{zy} + \frac{\partial \mathbf{E_z}}{\partial \mathbf{z}}\epsilon_{zz} + \\ & \left(\frac{\partial \epsilon_{xx}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yx}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zx}}{\partial \mathbf{z}}\right)\mathbf{E_x} + \\ & \left(\frac{\partial \epsilon_{xy}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yy}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zy}}{\partial \mathbf{z}}\right)\mathbf{E_y} + \\ & \left(\frac{\partial \epsilon_{xz}}{\partial \mathbf{x}} + \frac{\partial \epsilon_{yz}}{\partial \mathbf{y}} + \frac{\partial \epsilon_{zz}}{\partial \mathbf{z}}\right)\mathbf{E_z}\end{aligned}$$

Comparing both cases clearly shows that the anisotropic case is more complex than the isotropic case. Note that the anisotropic case can be reduced to the isotropic case by taking $\epsilon_{ij} = \epsilon$ for $i = j$ and $\epsilon_{ij} = 0$ otherwise. As a result of this complexity is it not straightforward to say what the influence would be of taking the breast tissue to be anisotropic and whether that would yield more more accurate results.

5.3 Linear media

The polarization and magnetization of materials is dependent on the strength of the electric field respectively the magnetic field. This dependence is material specific. For dielectrics the dependence is represented by a hysteresis curve. The general shape for the magnetic hysteresis curve for a dielectric is shown in figure 3. The dotted line in the figure represents the relation between **B** and **H** when starting from a state with no previous magnetic field applied. The line ends in saturated state and follows the top purple line under application of a negative field. If a not too strong magnetic field is applied the dotted line will never reach the saturation level and will stay approximately linear in nature. For all intent and purposes we assumed that the new method we wanted to derive will not reach the strength in its fields to reach saturation and thus we are able to state that we are only dealing with linear media.

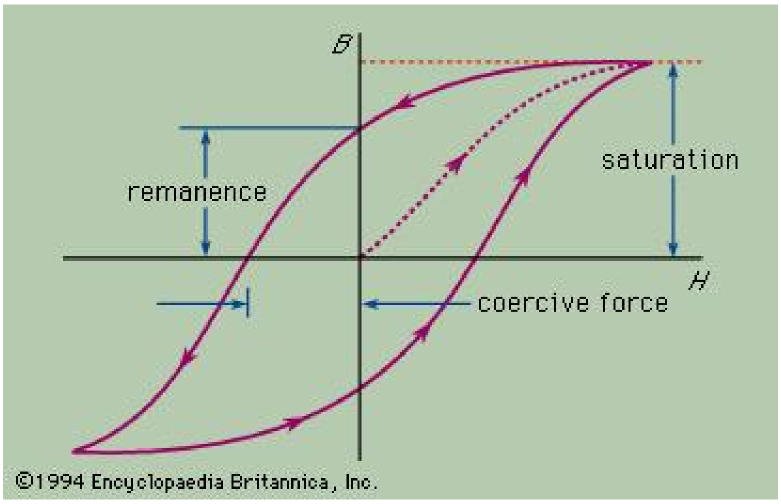

Figure 3: General shape of a hysteresis curve[25]

5.4 Resonance frequencies

According to the Lorentz Dispersion Model we may assume when looking at the electron in a molecule or an atom that there is some elastic force keeping it in place[27]. By modeling the whole as a mass-spring system with as input the Lorentz Force term $\mathbf{F} = -e\mathbf{E_0}e^{-i\omega t}$ we get

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + m\gamma\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + m\omega_0^2\mathbf{x} = -e\mathbf{E_0}e^{-i\omega t}$$

By guessing a solution $\mathbf{x}(t) = \mathbf{x_0}e^{-i\omega t}$ we can solve this equation for $\mathbf{x_0}$ in the steady state.

$$\mathbf{x_0} = \frac{1}{m}\frac{-e\mathbf{E_0}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

From this we can calculate the dipole moment $\mathbf{p}$ and the polarization $\mathbf{P}$, if we assume that we have for each molecule 1 dipole and N molecules for each unit volume. First the dipole moment is given by

$$\mathbf{p} = -e\mathbf{x} = \frac{1}{m}\frac{e^2\mathbf{E_0}}{\omega_0^2 - \omega^2 - i\gamma\omega}e^{-i\omega t}$$

and the polarization is equal to

$$\mathbf{P} = N\mathbf{p} = \frac{1}{m}\frac{Ne^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\mathbf{E_0}e^{-i\omega t}$$

but we also now that $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ and thus we can express the permittivity ϵ.

$$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon\left(1 + \frac{1}{m}\frac{Ne^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)$$

We can also derive a similar expression for the permeability μ. In figure 4 both of them are plotted against the relative frequency $\frac{\omega}{\omega_0}$. If we now consider the refractive index n we can clearly see from its relation to the permittivity and permeability that near the resonance frequency the refractive index peeks greatly[26].

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

According to Griffiths[28] this peak in refractive index not only signals a peek in dispersion, but also has a accompanying maximum in absorption. The reason given for this is that the amplitude for electron is large and the damping mechanism dissipates energy according to that amplitude. Knowing these mechanisms Caro states that is not possible to find a unique solution when the angular frequency is a resonance frequency[4]. Since we want a unique solution, it is therefore necessary to pick our angular frequencies so that they are not resonance frequencies. In practice this may be hard to avoid, since resonance frequencies are hard to determine and likely unknown[22]. This is however beyond the scope of this paper.

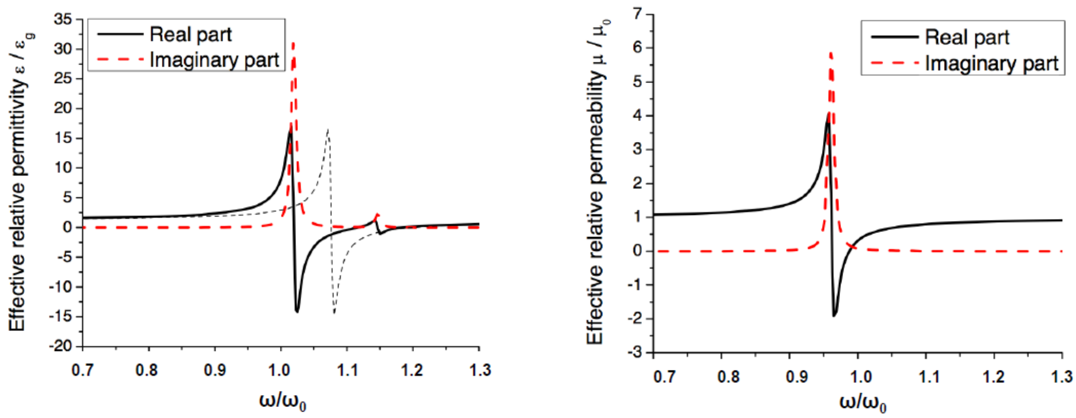

Figure 4: Relative permittivity(left) and permeability(right) plotted against the relative angular frequency[26].

6 Conclusion

In this paper we started with an overview of the current mammography methods. We noted that a lot of new methods focus on electromagnetic fields, yet seem to skip a part of the frequency spectrum. We used this missing part of the spectrum to look for a design of a new method for breast cancer detection using electromagnetic fields. To achieve this we formulated the research question

> If you have the ability to create electromagnetic fields outside of an object, can you theoretically non-invasively *uniquely* determine the electromagnetic properties of the object?

We reformulated this into inverse boundary problem which we have show we can solve. By starting from the Maxwell Equations we have shown that we can derive an expression for a potential describing the object in our domain and express this in values on the boundary thereof. Under suitable choices for the applied fields we can then solve the expression for the potential and thus retrieve a mapping of the electromagnetic properties of the object in our domain.

7 Recommendation for further research

Throughout this paper we have been looking for a method for detecting breast cancer using only measurement data taken on boundary. We however have in practice that we cannot measure the entire boundary. We thus have to concoct a plan to make up for the unattainable measurement data.

The first option is to reduce the math to 2D so that we do have the full boundary. The downside of this is that we only have 2D data, thus we need to make multiple layers that we can apply too or we need to make the scanner movable so that is scans the breast layer by layer. Stacking the information gathered for either layer method will then give us an image of the full breast. Both these methods however sound really similar to the procedure followed by Microwave Imaging[1](1). The math however is different to the math we have used in this paper. In this paper we have dealt with a electromagnetic inverse boundary problem, in Microwave Imaging an electromagnetic inverse scattering problem is solved[29]. Both approaches start of with the Maxwell Equations and end with an expression for the values of the domain in terms of values on the boundary. According to Ola the equation $\frac{1}{2}\mathbf{t}e_{sc} = \mathbf{t}e_i + D_k\Lambda\mathbf{t}e_{sc} - K_k\mathbf{t}e_{sc}$[22], where e_{sc} is a rescaled scattered eletric field for the incoming field e_i, D_k,K_k and Λ are based on the boundary values and $\mathbf{t}$ is the trace, $\hat{\mathbf{n}}\times$, is solved and in [30] you can read that this leads to an end equation consisting of an integral over time with terms depending on some weights and an input combined with a time delay. Their method is also limited by frequency, since the higher the frequency the better the image, but if too high the tissue losses start to disrupt the image. You can thus see that even though the approaches seem similar they are not so similar after all. The reduction to 2D might thus be worth researching.

Another method is the highlighted by Caro, Ola and Salo in [31] and [4]. They assume that there exists some form of symmetry in what is to be measured. First they show that in the case that the unmeasurable part is a flat plane, you can still derive the information you need for fully describing the domain by means of a reflection argument. After showing that it is possible to do this when the unknown part is a flat plane, they also show that for a sphere by doing a Kelvin transformation. This transformation bends space is such a way that the unmeasurable part which was a sphere is now a flat plane and with that the original reflection argument can again be applied. Since this only transforms and extends the set that we needed to measure anyway, this seems not to hard to implement and worth researching.

Appendices

Appendix A

Deriviations

1 Schrödinger potential Q

The current form of the potential has a nice diagonal structure. For computing the elements of $\mathbf{Q}$ we will however opt for using a slightly other format. This format rewrites $\mathbf{Q}$ into block elements, which make it easier to apply and calculated the needed transposes. In the next subsection we will therefore rewrite our potential in diagonal form $\mathbf{Q}$ to our potential in block form $\tilde{\mathbf{Q}}$. Note that this $\tilde{\mathbf{Q}}$ is not the compactly supported $\tilde{\mathbf{Q}} = ik_0^2\mathbf{1_8} + \mathbf{Q}$. Following that subsection we will derive some useful properties of the block form. Once we have done that we will combine these results to derive the elements of $\tilde{\mathbf{Q}}$ and, since knowing the block form implies we also know the diagonal form, of $\mathbf{Q}$.

1.1 Diagonal to block form

We know

$$
\begin{aligned}
\mathbf{P} &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \\ \boldsymbol{\nabla} & 0 & \boldsymbol{\nabla}\times & 0 \\ 0 & -\boldsymbol{\nabla}\times & 0 & \boldsymbol{\nabla} \\ 0 & 0 & \boldsymbol{\nabla}\cdot & 0 \end{pmatrix} \\
\mathbf{V} &= \begin{pmatrix} ik & \boldsymbol{\nabla}\alpha\cdot & 0 & 0 \\ \boldsymbol{\nabla}\beta & ik & -\boldsymbol{\nabla}\beta\times & 0 \\ 0 & \boldsymbol{\nabla}\alpha\times & ik & \boldsymbol{\nabla}\alpha \\ 0 & 0 & \boldsymbol{\nabla}\beta\cdot & ik \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} X_1 \\ \mathbf{X_2} \\ \mathbf{X_3} \\ X_4 \end{pmatrix} \\
\mathbf{Q} &= -\mathbf{VP} + \mathbf{PV^T} + \mathbf{VV^T}
\end{aligned}
$$

For rewriting the potential we start by rewriting our $\mathbf{X}$ to $\tilde{\mathbf{X}}$.

$$
\mathbf{X} = \begin{pmatrix} X_1 \\ \mathbf{X_2} \\ \mathbf{X_3} \\ X_4 \end{pmatrix} \rightarrow \tilde{\mathbf{X}} = \begin{pmatrix} X_1 \\ \mathbf{X_3} \\ X_4 \\ \mathbf{X_2} \end{pmatrix}
$$

If we now use this $\tilde{\mathbf{X}}$ we can change our Dirac System into a block form. To achieve this we first switch the columns of the Dirac System so that we get

$$(\mathbf{P}+\mathbf{V})\mathbf{X} \to \left(\begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \boldsymbol{\nabla}\times & 0 & 0 \\ 0 & 0 & \boldsymbol{\nabla} & -\boldsymbol{\nabla}\times \\ 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \end{pmatrix} + \begin{pmatrix} ik & 0 & 0 & \boldsymbol{\nabla}\alpha\cdot \\ \boldsymbol{\nabla}\beta & -\boldsymbol{\nabla}\beta\times & 0 & ik \\ 0 & ik & \boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\times \\ 0 & \boldsymbol{\nabla}\beta\cdot & ik & 0 \end{pmatrix}\right)\tilde{\mathbf{X}}$$

If we now reorder the rows we will get the block form of the Dirac System.

$$\left(\begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\nabla}\cdot \\ 0 & 0 & \boldsymbol{\nabla} & -\boldsymbol{\nabla}\times \\ 0 & \boldsymbol{\nabla}\cdot & 0 & 0 \\ \boldsymbol{\nabla} & \boldsymbol{\nabla}\times & 0 & 0 \end{pmatrix} + \begin{pmatrix} ik & 0 & 0 & \boldsymbol{\nabla}\alpha\cdot \\ 0 & ik & \boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\times \\ 0 & \boldsymbol{\nabla}\beta\cdot & ik & 0 \\ \boldsymbol{\nabla}\beta & -\boldsymbol{\nabla}\beta\times & 0 & ik \end{pmatrix}\right)\tilde{\mathbf{X}}$$

The block form of the Dirac System can be expressed more simply once we introduce the matrices

$$\begin{aligned} \mathbf{P}^{\pm} = \mathbf{P}^{\pm}(\boldsymbol{\nabla}) &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \pm\boldsymbol{\nabla}\times \end{pmatrix} \\ \mathbf{A}^{\pm} &= \mathbf{P}^{\pm}(\boldsymbol{\nabla}\alpha) \\ \mathbf{B}^{\pm} &= \mathbf{P}^{\pm}(\boldsymbol{\nabla}\beta) \end{aligned}$$

By doing this we can rewrite our Dirac System in diagonal form $(\mathbf{P}+\mathbf{V})\mathbf{X}$ to our Dirac System in block form $(\tilde{\mathbf{P}}_{\mp}+\tilde{\mathbf{V}})\tilde{\mathbf{X}}$ by letting

$$\begin{aligned} \tilde{\mathbf{P}}_{\mp} &= \begin{pmatrix} 0 & \mathbf{P}^{-} \\ \mathbf{P}^{+} & 0 \end{pmatrix} \\ \tilde{\mathbf{V}} &= \begin{pmatrix} ik\mathbf{1_4} & \mathbf{A}^{+} \\ \mathbf{B}^{-} & ik\mathbf{1_4} \end{pmatrix} \end{aligned}$$

By applying the same steps we used to reorder the Dirac System from diagonal to block form we can rewrite our potential $\mathbf{Q}$ to $\tilde{\mathbf{Q}}$. We start of by transforming $\mathbf{Z}$ to $\tilde{\mathbf{Z}}$.

$$\mathbf{Z} = \begin{pmatrix} Z_1 \\ \mathbf{Z_2} \\ \mathbf{Z_3} \\ Z_4 \end{pmatrix} \to \tilde{\mathbf{Z}} = \begin{pmatrix} Z_1 \\ \mathbf{Z_3} \\ Z_4 \\ \mathbf{Z_2} \end{pmatrix}$$

As a result we get that our potential $\mathbf{Q}$ gets replaced by $\tilde{\mathbf{Q}}$. $\tilde{\mathbf{Q}}$ is then given by

$$\tilde{\mathbf{Q}} = -\tilde{\mathbf{V}}\tilde{\mathbf{P}}_{\mp} + \tilde{\mathbf{P}}_{\mp}\tilde{\mathbf{V}}^{\mathbf{T}} + \tilde{\mathbf{V}}\tilde{\mathbf{V}}^{\mathbf{T}}$$

1.2 Properties of the block form Dirac System

The primary property that we were looking for by writing the Dirac System in block form shows itself once we recall that we have for block matrices $\mathbf{U}$ with elements $\mathbf{U_{n,m}}$ for the transpose that

$$\tilde{\mathbf{U}}^{\mathbf{T}} = \begin{pmatrix} \mathbf{U_{0,0}} & \mathbf{U_{0,1}} \\ \mathbf{U_{1,0}} & \mathbf{U_{1,1}} \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} \mathbf{U_{0,0}^T} & \mathbf{U_{1,0}^T} \\ \mathbf{U_{0,1}^T} & \mathbf{U_{1,1}^T} \end{pmatrix}$$

As a result of this we get for the differential operators

$$\begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \boldsymbol{\nabla}\times \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} 0 & \boldsymbol{\nabla}^{\mathbf{T}} \\ \boldsymbol{\nabla}\cdot^{\mathbf{T}} & \boldsymbol{\nabla}\times^{\mathbf{T}} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & -\boldsymbol{\nabla}\times \end{pmatrix}$$

This latter statement becomes clear when we express the differential operator in vector form using partial differentials. This gives

$$\boldsymbol{\nabla}\cdot^{\mathbf{T}} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} = \boldsymbol{\nabla}$$

$$\boldsymbol{\nabla}\times^{\mathbf{T}} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \end{pmatrix} = -\boldsymbol{\nabla}\times$$

This behavior under the transpose gives us

$$\begin{aligned} (\tilde{\mathbf{P}}^{\pm})^{\mathbf{T}} &= \tilde{\mathbf{P}}^{\mp} \\ \tilde{\mathbf{P}}_{\mp}^{\mathbf{T}} &= \tilde{\mathbf{P}}_{\mp} \\ \tilde{\mathbf{V}}^{\mathbf{T}} &= \begin{pmatrix} ik & \mathbf{B}^{+} \\ \mathbf{A}^{-} & ik \end{pmatrix} \end{aligned}$$

Secondary properties for the block form follow from the definition of k, α, β and the chain rule. First of all we have by the chain rule that

$$\begin{aligned} \boldsymbol{\nabla}(ku) &= \boldsymbol{\nabla}(k)u + k\boldsymbol{\nabla}u \\ \boldsymbol{\nabla}\cdot(k\mathbf{U}) &= \boldsymbol{\nabla}k\cdot\mathbf{U} + k\boldsymbol{\nabla}\cdot\mathbf{U} \\ \boldsymbol{\nabla}\times(k\mathbf{U}) &= \boldsymbol{\nabla}k\times\mathbf{U} + k\boldsymbol{\nabla}\times\mathbf{U} \end{aligned}$$

where $\mathbf{U}$ is an arbitrary vector and u an arbitrary scalar[32]. As a result of this we get that

$$\tilde{\mathbf{P}}_{\mp}(\boldsymbol{\nabla}(k\tilde{\mathbf{Z}})) = \tilde{\mathbf{P}}_{\mp}(\boldsymbol{\nabla}k)\tilde{\mathbf{Z}} + k\tilde{\mathbf{P}}_{\mp}\tilde{\mathbf{Z}}$$

Furthermore recall that

$$\begin{aligned} k &= \omega\sqrt{\mu\gamma} \\ \alpha &= \frac{1}{2}\log(\gamma) \\ \beta &= \frac{1}{2}\log(\mu) \end{aligned}$$

Applying the chain rule gives

$$
\begin{aligned}
\frac{1}{\omega}\boldsymbol{\nabla}k &= \boldsymbol{\nabla}(\sqrt{\mu\gamma}) \\
&= \sqrt{\gamma}\boldsymbol{\nabla}(\sqrt{\mu}) + \sqrt{\mu}\boldsymbol{\nabla}(\sqrt{\gamma}) \\
&= \frac{\sqrt{\gamma}}{2\sqrt{\mu}}\boldsymbol{\nabla}\mu + \frac{\sqrt{\mu}}{2\sqrt{\gamma}}\boldsymbol{\nabla}\gamma \\
&= \sqrt{\gamma\mu}\frac{1}{2}\boldsymbol{\nabla}(\log\mu) + \sqrt{\mu\gamma}\frac{1}{2}\boldsymbol{\nabla}(\log\gamma) \\
&= \sqrt{\gamma\mu}\boldsymbol{\nabla}(\frac{1}{2}\log\mu) + \sqrt{\mu\gamma}\boldsymbol{\nabla}(\frac{1}{2}\log\gamma) \\
&= \sqrt{\gamma\mu}\boldsymbol{\nabla}\beta + \sqrt{\mu\gamma}\boldsymbol{\nabla}\alpha \\
&= \frac{k}{\omega}(\boldsymbol{\nabla}\beta + \boldsymbol{\nabla}\alpha)
\end{aligned}
$$

Combining this with $\mathbf{A}^{\pm}$ and $\mathbf{B}^{\pm}$ we get

$$
k\mathbf{A}^{\pm} + k\mathbf{B}^{\pm} = \tilde{\mathbf{P}}^{\pm}(\boldsymbol{\nabla}k)
$$

1.3 Calculating $\tilde{\mathbf{Q}}$

Now that we have rewritten our potential in block form and we have derived some properties, we can commence with deriving the expressions for the elements of our potential. We will do this computing each part of the potential separately and joining all pieces afterwards. For easy of writing we will drop all the tildes in next part of this section.

We will start by computing the first two matrix products of $\mathbf{Q}$.

$$
\begin{aligned}
\mathbf{P}_{\mp}\mathbf{V}^{\mathbf{T}} &= \begin{pmatrix} 0 & \mathbf{P}^{-} \\ \mathbf{P}^{+} & 0 \end{pmatrix} \begin{pmatrix} ik\mathbf{1_4} & \mathbf{B}^{+} \\ \mathbf{A}^{-} & ik\mathbf{1_4} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{P}^{-}\mathbf{A}^{-} & i\mathbf{P}^{-}(\boldsymbol{\nabla}(k\mathbf{1_4}\star)) \\ i\mathbf{P}^{+}(\boldsymbol{\nabla}(k\mathbf{1_4}\star)) & \mathbf{P}^{+}\mathbf{B}^{+} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{P}^{-}\mathbf{A}^{-} & 0 \\ 0 & \mathbf{P}^{+}\mathbf{B}^{+} \end{pmatrix} + i\mathbf{P}_{\mp}(\boldsymbol{\nabla}(k\mathbf{1_8}\star))
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}\mathbf{P}_{\mp} &= \begin{pmatrix} ik\mathbf{1_4} & \mathbf{A}^{+} \\ \mathbf{B}^{-} & ik\mathbf{1_4} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{P}^{-} \\ \mathbf{P}^{+} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{P}^{+} & ik\mathbf{1_4}\mathbf{P}^{-} \\ ik\mathbf{1_4}\mathbf{P}^{+} & \mathbf{B}^{-}\mathbf{P}^{-} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{P}^{+} & 0 \\ 0 & \mathbf{B}^{-}\mathbf{P}^{-} \end{pmatrix} + ik\mathbf{1_8}\mathbf{P}_{\mp}
\end{aligned}
$$

Joining them gives

$$
\begin{aligned}
\mathbf{V}\mathbf{P}_{\mp} - \mathbf{P}_{\mp}\mathbf{V}^{\mathbf{T}} &= \begin{pmatrix} \mathbf{A}^{+}\mathbf{P}^{+} - \mathbf{P}^{-}\mathbf{A}^{-} & 0 \\ 0 & \mathbf{B}^{-}\mathbf{P}^{-} - \mathbf{P}^{+}\mathbf{B}^{+} \end{pmatrix} + ik\mathbf{1_8}\mathbf{P}_{\mp} - i\mathbf{P}_{\mp}(\boldsymbol{\nabla}(k\mathbf{1_8}\star)) \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{P}^{+} - \mathbf{P}^{-}\mathbf{A}^{-} & 0 \\ 0 & \mathbf{B}^{-}\mathbf{P}^{-} - \mathbf{P}^{+}\mathbf{B}^{+} \end{pmatrix} - i\mathbf{P}_{\mp}(\boldsymbol{\nabla}k)
\end{aligned}
$$

where we have used that $\mathbf{P}_\mp(\boldsymbol{\nabla}(k\mathbf{Z})) = \mathbf{P}_\mp(\boldsymbol{\nabla}k)\mathbf{Z} + k\mathbf{1_8}\mathbf{P}_\mp\mathbf{Z}$. This leaves us with the calculation of 4 further matrix products. We will start by computing the top two matrix products. The first of these is by direct computation

$$\begin{aligned}\mathbf{A}^+\mathbf{P}^+ &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\alpha\cdot \\ \boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\times \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \boldsymbol{\nabla}\times \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\nabla}\alpha\cdot\boldsymbol{\nabla}\star & \boldsymbol{\nabla}\alpha\cdot(\boldsymbol{\nabla}\times\star) \\ \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\star & \boldsymbol{\nabla}\alpha\boldsymbol{\nabla}\cdot+\boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\times\star) \end{pmatrix}\end{aligned}$$

The second is however a little more tricky. To solve this one we first recall the vector calculus identities

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{U}\times\mathbf{V}) &= \mathbf{U}(\boldsymbol{\nabla}\cdot\mathbf{V}) - \mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{U} + (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{V} - (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{U} \\ \boldsymbol{\nabla}(\mathbf{U}\cdot\mathbf{V}) &= (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{V} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{U} + \mathbf{U}\times\boldsymbol{\nabla}\times\mathbf{V} + \mathbf{V}\times\boldsymbol{\nabla}\times\mathbf{U}\end{aligned}$$

for vectors $\mathbf{U}$ and $\mathbf{V}$. Using these we get

$$\begin{aligned}\mathbf{P}^-\mathbf{A}^- &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & -\boldsymbol{\nabla}\times \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\nabla}\alpha\cdot \\ \boldsymbol{\nabla}\alpha & -\boldsymbol{\nabla}\alpha\times \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\alpha\star) & -\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\alpha\times\star) \\ -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}(\boldsymbol{\nabla}\alpha\cdot\star)+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\alpha\times\star) \end{pmatrix} \\ &= \begin{pmatrix} \Delta\alpha+\boldsymbol{\nabla}\star\cdot\boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\cdot(\boldsymbol{\nabla}\times\star) \\ 0 & \boldsymbol{\nabla}(\boldsymbol{\nabla}\alpha\cdot\star)+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\alpha\times\star) \end{pmatrix} \\ &= \begin{pmatrix} \Delta\alpha+\boldsymbol{\nabla}\star\cdot\boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\cdot(\boldsymbol{\nabla}\times\star) \\ 0 & 2(\star\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\alpha+\boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\times\star)+\boldsymbol{\nabla}\alpha\boldsymbol{\nabla}\cdot-\Delta\alpha\mathbf{1_3} \end{pmatrix} \\ &= \begin{pmatrix} \Delta\alpha+\boldsymbol{\nabla}\star\cdot\boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha\cdot(\boldsymbol{\nabla}\times\star) \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot+\boldsymbol{\nabla}\alpha\boldsymbol{\nabla}\cdot+\boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\times\star)-\Delta\alpha\mathbf{1_3} \end{pmatrix}\end{aligned}$$

Joining the computed matrix products gives

$$\mathbf{A}^+\mathbf{P}^+ - \mathbf{P}^-\mathbf{A}^- = -\begin{pmatrix} \Delta\alpha & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot-\Delta\alpha\mathbf{1_3} \end{pmatrix}$$

For the second pair of matrix products we can repeat the steps of the previous pair to get

$$\begin{aligned}\mathbf{B}^-\mathbf{P}^- &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\beta\cdot \\ \boldsymbol{\nabla}\beta & -\boldsymbol{\nabla}\beta\times \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & -\boldsymbol{\nabla}\times \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\star & -\boldsymbol{\nabla}\beta\cdot(\boldsymbol{\nabla}\times\star) \\ -\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\star & \boldsymbol{\nabla}\beta\boldsymbol{\nabla}\cdot+\boldsymbol{\nabla}\beta\times(\boldsymbol{\nabla}\times\star) \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{P}^+\mathbf{B}^+ &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\cdot \\ \boldsymbol{\nabla} & \boldsymbol{\nabla}\times \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\nabla}\beta\cdot \\ \boldsymbol{\nabla}\beta & \boldsymbol{\nabla}\beta\times \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\beta\star) & \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\beta\times\star) \\ \boldsymbol{\nabla}\times\boldsymbol{\nabla}\beta & \boldsymbol{\nabla}(\boldsymbol{\nabla}\beta\cdot\star)+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\beta\times\star) \end{pmatrix} \\ &= \begin{pmatrix} \Delta\beta+\boldsymbol{\nabla}\star\cdot\boldsymbol{\nabla}\beta & -\boldsymbol{\nabla}\beta\cdot(\boldsymbol{\nabla}\times\star) \\ 0 & \boldsymbol{\nabla}(\boldsymbol{\nabla}\beta\cdot\star)+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\beta\times\star) \end{pmatrix} \\ &= \begin{pmatrix} \Delta\beta+\boldsymbol{\nabla}\star\cdot\boldsymbol{\nabla}\beta & -\boldsymbol{\nabla}\beta\cdot(\boldsymbol{\nabla}\times\star) \\ 0 & \boldsymbol{\nabla}\beta\boldsymbol{\nabla}\cdot+\boldsymbol{\nabla}\beta\times(\boldsymbol{\nabla}\times\star)+2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\cdot-\Delta\beta\mathbf{1_3} \end{pmatrix}\end{aligned}$$

and thus the second joint matrix becomes

$$\mathbf{B}^{-}\mathbf{P}^{-} - \mathbf{P}^{+}\mathbf{B}^{+} = -\begin{pmatrix} \Delta\beta & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\,\cdot - \Delta\beta\mathbf{1_3} \end{pmatrix}$$

Now we have all terms of the first part of our potential and this is given by

$$\mathbf{VP} - \mathbf{PV^T} = -\begin{pmatrix} \begin{pmatrix} \Delta\alpha & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\,\cdot - \Delta\alpha\mathbf{1_3} \end{pmatrix} & \\ & \begin{pmatrix} \Delta\beta & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\,\cdot - \Delta\beta\mathbf{1_3} \end{pmatrix} \end{pmatrix} - i\mathbf{P}_{\mp}(\boldsymbol{\nabla}k)$$

What remains is the final part of our potential.

$$\begin{aligned}
\mathbf{VV^T} &= \begin{pmatrix} ik\mathbf{1_4} & \mathbf{A}^{+} \\ \mathbf{B}^{-} & ik\mathbf{1_4} \end{pmatrix} \begin{pmatrix} ik\mathbf{1_4} & \mathbf{B}^{+} \\ \mathbf{A}^{-} & ik\mathbf{1_4} \end{pmatrix} \\
&= \begin{pmatrix} -k^2\mathbf{1_4} + \mathbf{A}^{+}\mathbf{A}^{-} & ik\mathbf{B}^{+} + \mathbf{A}^{+}(ik) \\ \mathbf{B}^{-}(ik) + ik\mathbf{A}^{-} & -k^2\mathbf{1_4} + \mathbf{B}^{-}\mathbf{B}^{+} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{A}^{-} & ik\mathbf{B}^{+} + \mathbf{A}^{+}(ik) \\ \mathbf{B}^{-}(ik) + ik\mathbf{A}^{-} & \mathbf{B}^{-}\mathbf{B}^{+} \end{pmatrix} - k^2\mathbf{1}_8 \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{A}^{-} & ik\mathbf{B}^{+} + ik\mathbf{A}^{+} \\ ik\mathbf{B}^{-} + ik\mathbf{A}^{-} & \mathbf{B}^{-}\mathbf{B}^{+} \end{pmatrix} - k^2\mathbf{1}_8 \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{A}^{-} & i\mathbf{P}^{+}(\boldsymbol{\nabla}k) \\ i\mathbf{P}^{-}(\boldsymbol{\nabla}k) & \mathbf{B}^{-}\mathbf{B}^{+} \end{pmatrix} - k^2\mathbf{1}_8 \\
&= \begin{pmatrix} \mathbf{A}^{+}\mathbf{A}^{-} & 0 \\ 0 & \mathbf{B}^{-}\mathbf{B}^{+} \end{pmatrix} - k^2\mathbf{1}_8 + i\mathbf{P}_{\pm}(\boldsymbol{\nabla}k)
\end{aligned}$$

For the contained matrix we have that

$$\begin{aligned}
\mathbf{A}^{\pm}\mathbf{A}^{\mp} &= \begin{pmatrix} 0 & \boldsymbol{\nabla}\alpha\cdot \\ \boldsymbol{\nabla}\alpha & \pm\boldsymbol{\nabla}\alpha\times \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\nabla}\alpha\cdot \\ \boldsymbol{\nabla}\alpha & \mp\boldsymbol{\nabla}\alpha\times \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\nabla}\alpha\cdot\boldsymbol{\nabla}\alpha & \mp\boldsymbol{\nabla}\alpha\cdot(\boldsymbol{\nabla}\alpha\times\star) \\ \pm\boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha & \boldsymbol{\nabla}\alpha(\boldsymbol{\nabla}\alpha\cdot\star) - \boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\alpha\times\star) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\nabla}\alpha\cdot\boldsymbol{\nabla}\alpha & 0 \\ 0 & \boldsymbol{\nabla}\alpha(\boldsymbol{\nabla}\alpha\cdot\star) - \boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\alpha\times\star) \end{pmatrix} \\
&= \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 & 0 \\ 0 & \boldsymbol{\nabla}\alpha(\boldsymbol{\nabla}\alpha\cdot\star) - \boldsymbol{\nabla}\alpha\times(\boldsymbol{\nabla}\alpha\times\star) \end{pmatrix} \\
&= \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 & 0 \\ 0 & \boldsymbol{\nabla}\alpha(\boldsymbol{\nabla}\alpha\cdot\star) + (\boldsymbol{\nabla}\alpha\times\star)\times\boldsymbol{\nabla}\alpha \end{pmatrix} \\
&= \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 & 0 \\ 0 & \boldsymbol{\nabla}\alpha(\boldsymbol{\nabla}\alpha\cdot\star) - (\boldsymbol{\nabla}\alpha\cdot\star)\boldsymbol{\nabla}\alpha + \|\boldsymbol{\nabla}\alpha\|^2\mathbf{1_3} \end{pmatrix} \\
&= \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 & 0 \\ 0 & \|\boldsymbol{\nabla}\alpha\|^2\mathbf{1_3} \end{pmatrix}
\end{aligned}$$

where $(\boldsymbol{\nabla}\alpha\times\star)\times\boldsymbol{\nabla}\alpha = \|\boldsymbol{\nabla}\alpha\|^2 - (\boldsymbol{\nabla}\alpha\cdot\star)\boldsymbol{\nabla}\alpha$ by the vector triple product[32], and similarly

$$\mathbf{B}^{\pm}\mathbf{B}^{\mp} = \begin{pmatrix} \|\boldsymbol{\nabla}\beta\|^2 & 0 \\ 0 & \|\boldsymbol{\nabla}\beta\|^2\mathbf{1_3} \end{pmatrix}$$

Combining gives

$$\mathbf{V}\mathbf{V}^{\mathbf{T}} = \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 \mathbf{1_4} & 0 \\ 0 & \|\boldsymbol{\nabla}\beta\|^2 \mathbf{1_4} \end{pmatrix} - k^2 \mathbf{1}_8 + i\mathbf{P}_{\pm}(\boldsymbol{\nabla}k)$$

With this we calculated the last term of our potential and what remains is joining the results together. We then get

$$\begin{aligned}
\mathbf{VP} - \mathbf{PV}^{\mathbf{T}} - \mathbf{VV}^{\mathbf{T}} &= -\begin{pmatrix} \begin{pmatrix} \Delta\alpha & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot - \Delta\alpha\mathbf{1_3} \end{pmatrix} & \\ & \begin{pmatrix} \Delta\beta & 0 \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\cdot - \Delta\beta\mathbf{1_3} \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \|\boldsymbol{\nabla}\alpha\|^2 \mathbf{1}_4 & \\ & \|\boldsymbol{\nabla}\beta\|^2 \mathbf{1}_4 \end{pmatrix} \\
&\quad + k^2 - i\Big(\mathbf{P}_{\pm}(\boldsymbol{\nabla}k) + \mathbf{P}_{\mp}(\boldsymbol{\nabla}k)\Big) \\
&= -\begin{pmatrix} \Delta\alpha + \|\boldsymbol{\nabla}\alpha\|^2 & & & \\ & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot - (\Delta\alpha - \|\boldsymbol{\nabla}\alpha\|^2)\mathbf{1_3} & & \\ & & \Delta\beta + \|\boldsymbol{\nabla}\beta\|^2 & \\ & & & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot - (\Delta\beta - \|\boldsymbol{\nabla}\beta\|^2)\mathbf{1_3} \end{pmatrix} \\
&\quad + k^2\mathbf{1}_8 - i\Big(\mathbf{P}_{\pm}(\boldsymbol{\nabla}k) + \mathbf{P}_{\mp}(\boldsymbol{\nabla}k)\Big) \\
&= k^2\mathbf{1}_8 - \begin{pmatrix} \Delta\alpha + \|\boldsymbol{\nabla}\alpha\|^2 & 0 & 0 & 2i\boldsymbol{\nabla}k\cdot \\ 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\cdot - (\Delta\beta - \|\boldsymbol{\nabla}\beta\|^2)\mathbf{1_3} & 2i\boldsymbol{\nabla}k & 0 \\ 0 & 2i\boldsymbol{\nabla}k\cdot & \Delta\beta + \|\boldsymbol{\nabla}\beta\|^2 & 0 \\ 2i\boldsymbol{\nabla}k & 0 & 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot - (\Delta\alpha - \|\boldsymbol{\nabla}\alpha \end{pmatrix}
\end{aligned}$$

Therefore the diagonal form our potential in diagonal form is

$$\mathbf{Q} = -k^2\mathbf{1}_8 - \begin{pmatrix} \Delta\alpha + \|\boldsymbol{\nabla}\alpha\|^2 & 2i\boldsymbol{\nabla}k\cdot & 0 & 0 \\ 2i\boldsymbol{\nabla}k & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\beta\cdot - (\Delta\beta - \|\boldsymbol{\nabla}\beta\|^2)\mathbf{1_3} & 0 & 0 \\ 0 & 0 & 2\boldsymbol{\nabla}\boldsymbol{\nabla}\alpha\cdot - (\Delta\alpha - \|\boldsymbol{\nabla}\alpha\|^2)\mathbf{1_3} & 2i\boldsymbol{\nabla}k \\ 0 & 0 & 2i\boldsymbol{\nabla}k\cdot & \Delta\beta + \|\boldsymbol{\nabla}\beta\|^2 \end{pmatrix}$$

2 Greens Function

A Greens Function $\mathbf{G}(\mathbf{x},\mathbf{y})$ is a function which for some operator $\mathbf{L}(\mathbf{x})$ has that $\mathbf{L}(\mathbf{x})\mathbf{G}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$. In our case we have the equation

$$\mathbf{LZ} = (\mathbf{D} + k_0^2)\mathbf{Z} = \mathbf{QZ}$$

We therefore want our Greens function to satisfy

$$(\mathbf{D} + k_0^2)\mathbf{G}(\mathbf{x},\mathbf{y}) = \left(\begin{pmatrix} \Delta & & & \\ & \mathbf{\Delta} & & \\ & & \mathbf{\Delta} & \\ & & & \Delta \end{pmatrix} + k_0^2\right)\mathbf{G}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

If we look at this matrix equation we can see that we can separate this into two equations

$$(\Delta + k_0^2)G_1(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$
$$(\mathbf{\Delta} + k_0^2)\mathbf{G_3}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

These equations are the same as the equations for the scalar and vector Helmholtz Greens functions.

2.1 Scalar Greens function

In this section we will seek to find an expression for the scalar Helmholtz Greens function. We start of with

$$(\Delta + k_0^2)G_1(\mathbf{x}) = \delta(\mathbf{x})$$

In the Fourier domain this transformation[10] is given by

$$(-k^2 + k_0^2)G_1(\mathbf{k}) = 1$$
$$G_1(\mathbf{k}) = \frac{1}{-k^2 + k_0^2}$$

If we now take the inverse Fourier transform we will get the desired expression for $G_1(\mathbf{x})$. We will solve the inverse Fourier transform in spherical coördinates. First we write the inverse Fourier transform into an indefinite integral

$$\begin{aligned}
G_1(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int G_1(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin(\theta)d\theta \int_0^{\infty} k^2 \frac{e^{ikx\cos(\theta)}}{k_0^2 - k^2} \\
&= \frac{1}{(2\pi)^2}\int_0^{\pi} \sin(\theta) \int_0^{\infty} k^2 \frac{e^{ikx\cos(\theta)}}{k_0^2 - k^2} dk d\theta \\
&= \frac{1}{(2\pi)^2}\int_{-1}^{1}\int_0^{\infty} k^2 \frac{e^{ikxz}}{k_0^2 - k^2} dk dz \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty} \frac{2}{ikx} k^2 \frac{e^{ikxz}}{k_0^2 - k^2} dk \\
&= \frac{1}{ix(2\pi)^2}\int_{-\infty}^{\infty} k \frac{e^{ikx}}{k_0^2 - k^2} dk
\end{aligned}$$

where we have used that the integral is even and $z = \cos(\theta)$. This integral has clearly poles at $k = -k_0$ and $k = k_0$, thus by Jordan's lemma[11] we have

$$\begin{aligned}\int_{-\infty}^{\infty} k\frac{e^{ikx}}{k_0^2 - k^2}dk &= \int_{-\infty}^{\infty} gdk = 2\pi i(\text{Res}[g, -k] + \text{Res}[g, k]) \\ \text{Res}[g, -k] &= \frac{k}{2k}e^{ikx} \\ \text{Res}[g, k] &= \frac{k}{2k}e^{ikx}\end{aligned}$$

where $g = k\frac{e^{ikx}}{k_0^2 - k^2}$. With these results we can complete our inverse Fourier transform to get

$$\begin{aligned}G_1(\mathbf{x}) &= \frac{1}{ix(2\pi)^2}2\pi i(\frac{k}{2k}e^{ikx} + \frac{k}{2k}e^{ikx}) \\ &= \frac{1}{4\pi}\frac{e^{ikx}}{x}\end{aligned}$$

2.2 Vector Greens function

For the Vector Greens function we will start by noting that[12]

$$\boldsymbol{\Delta} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$$

Starting with the operator $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$ we will show it is possible to write $\mathbf{G_3}(\mathbf{x})$ in terms of $G_1(\mathbf{x})$. Let $\mathbf{G_3}(\mathbf{x})$ be the solution to

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{G_3}(\mathbf{x}) - k_0^2\mathbf{G_3}(\mathbf{x}) = -\delta(\mathbf{x})\mathbf{1}_3$$

We now take the divergence of both sides of the equations.

$$\begin{aligned}\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{G_3}(\mathbf{x}) - k_0^2\mathbf{G_3}(\mathbf{x})\big) &= -\boldsymbol{\nabla}\cdot\delta(\mathbf{x})\mathbf{1}_3 \\ -k_0^2\boldsymbol{\nabla}\cdot\mathbf{G_3}(\mathbf{x}) &= -\boldsymbol{\nabla}\delta(\mathbf{x})\end{aligned}$$

Taking the gradient of the resulting equation gives us the other term of our laplacian.

$$\begin{aligned}-\boldsymbol{\nabla}\big(k_0^2\boldsymbol{\nabla}\cdot\mathbf{G_3}(\mathbf{x})\big) &= -\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\mathbf{x}) \\ \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{G_3}(\mathbf{x}) &= \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\mathbf{x})\end{aligned}$$

We can now recombine the two terms of the laplacian to get

$$\begin{aligned}(\boldsymbol{\Delta} + k_0^2)\mathbf{G_3}(\mathbf{x}) &= \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{G_3}(\mathbf{x}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{G_3}(\mathbf{x}) + k_0^2\mathbf{G_3}(\mathbf{x}) \\ &= \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\mathbf{x}) + \delta(\mathbf{x})\mathbf{1}_3 \\ &= (\mathbf{1}_3 + \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla})\delta(\mathbf{x})\end{aligned}$$

Combining this expression with the scalar version we get

$$\mathbf{G_3}(\mathbf{x}) = (\mathbf{1}_3 + \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla})G_1(\mathbf{x})$$

2.3 Joint Greens function

In the start of this section we stated that

$$\left(\begin{pmatrix}\Delta & & & \\ & \boldsymbol{\Delta} & & \\ & & \boldsymbol{\Delta} & \\ & & & \Delta\end{pmatrix} + k_0^2\right)\mathbf{G}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

In the previous sections we have concluded that

$$G_1(\mathbf{x}) = \frac{1}{4\pi}\frac{e^{ikx}}{x}$$
$$\mathbf{G_3}(\mathbf{x}) = (\mathbf{1}_3 - \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla})G_1(\mathbf{x})$$

solve

$$(\Delta + k_0^2)G_1(\mathbf{x}) = \delta(\mathbf{x})$$
$$(\boldsymbol{\Delta} + k_0^2)\mathbf{G_3}(\mathbf{x}) = \delta(\mathbf{x})$$

If we let

$$G_1(\mathbf{x},\mathbf{y}) = G_1(\mathbf{x}-\mathbf{y})$$
$$\mathbf{G_3}(\mathbf{x},\mathbf{y}) = \mathbf{G_3}(\mathbf{x}-\mathbf{y})$$

we have that these also solve the original problem and thus the solution we were looking for is

$$\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{G}(\mathbf{x}-\mathbf{y}) = \begin{pmatrix}1 & & & \\ & \mathbf{1}_3 + \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla} & & \\ & & \mathbf{1}_3 + \frac{1}{k_0^2}\boldsymbol{\nabla}\boldsymbol{\nabla} & \\ & & & 1\end{pmatrix}G_1(\mathbf{x}-\mathbf{y})$$

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