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Asymptotic Price of Anarchy for affine, symmetric, k-uniform congestion games

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Preface

This thesis is the final result of my research into the asymptotic price of anarchy for affine, symmetric, k-uniform congestion games. The research was done at the chair of Discrete Mathematics and Mathematical Programming of Applied Mathematics at the University of Twente. I would like this opportunity to give thanks to the people who helped me in my research and in writing this thesis.

First off, I would like to thank my supervisors Walter Kern and Marc Uetz. They helped me through many difficult parts of the closing chapter of my studies. They were as supportive as could be hoped for and this thesis would not be finished were it not for the support given by them.

I would also like to thank Jasper de Jong for his earlier work in Game Theory that enhanced my interest into the field and ultimately made me choose this particular topic.

I am very grateful for my parents and my brother for their continuing support, especially when things did not go according to plan. They were there when I had obstacles to overcome, not just in writing this thesis, but during my entire study period. I am glad to have them by my side, now and in the future.

Finally I want to thank all my friends and flatmates who “helped” me with wonderful distractions like board games, movies, drinks and other nerd stuff. Without you, this thesis might have been completed several years earlier, but my student life would have been a lot less interesting.

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Abstract

We consider the class of affine, symmetric, k-affine congestion games and calculate the maximum Price of Anarchy for large number of players. The Price of Anarchy is defined as the ratio between the total cost of a stable equilibrium (Nash Equilbrium) and the total cost of the system's optimum. The Nash Equilibrium is defined as a solution where no player can deviate and thereby lower his individual total cost, while the system's optimum is defined as a solution where the social cost is minimized.

Recent work has shown that the maximum Price of Anarchy for affine, symmetric, k-uniform congestion games lies between $7-4\sqrt{2} \approx 1.343$ and $\frac{28}{13} \approx 2.15$. In this thesis we will improve the upper and lower bound to a constant of ≈ 1.35188.

We do this by calculating both an upper bound via an alternating paths based approach that examines the difference between the equilibrium and the optimal solution, and a lower bound by way of example. The alternating paths compare the system's Nash Equilibrium with the Optimal solution in critical case games, which is a set of games for which the Price of Anarchy is highest. We show that for critical case games and when $N \to \infty$, the price of anarchy can never be higher than ≈ 1.35188. We construct the lower bound by giving an example of a (near) critical case game with a Price of anarchy of ≈ 1.35188, thus proving that critical case games have a Price of Anarchy of at least this value.

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Nomenclature and abbreviations

NE	Nash Equilibrium
OPT	Optimal Solution
PoA	Price of Anarchy
AP	Alternating Path
N	the number of players
R	the set of resources
S_i	set of strategies of player i
k	number of resources chosen by every player
$\mathbb{O}$	$\left(r \in R \vert \overline{d_r} > d_r^*\right)$ ("Overloaded resources")
$\mathbb{O}^>$	the set of resources that appear among $o_{t+1}, \cdots, o_\Delta$
$\mathbb{U}$	$\left(r \in R \vert \overline{d_r} < d_r^*\right)$ ("Underloaded resources")
$\mathbb{U}^>$	the set of resources that appear among $u'_{t+1}, \cdots, u'_\Delta$
$\mathbb{B}$	$\left(r \in R \vert \overline{d_r} = d_r^*\right)$ ("Balanced resources")
B_1	balanced resources used by alternating paths
B_2	balances resources not used by alternating paths
F	set of edges not incident to $\mathbb{O}$
$\mathbb{I}$	$\{i \in N \vert io \in NE$ for any $o \in \mathbb{O}^>\}$ set of player that has at least one connection to a resource $o \in \mathbb{O}^>$
I_1	set of expensive non-alternating players
I_2	set of expensive alternating players
$\mathbb{J}$	$\{j \in N \vert \exists P = (\cdots, j, u) \in \mathcal{P}^>\}$ set of last players on path $P \in \mathcal{P}^>$
J	set of cheap players (choosing only balanced resources in NE)
individual cost	cost of one player
social cost	summation of the cost of all players
$M(i)$	The Action of (or set of resources chosen by) player i

$c_r(x)$	cost of resource r when it is connected to x players
d_r	degree of resource r, or the number of connections it has to players
$\overline{d}_r$	degree of resource r in a Nash Equilibrium
d_r^*	degree of resource r in an Optimum
$C(M)$	Social cost of matching M
c_o^{max}	the highest individual cost of all overloaded resources in NE
$\overline{c}_r$	$c_r(\overline{d_r})$, the cost of a resource in NE
$\overline{c}_r^+$	$c_r(\overline{d_r}+1)$, the opportunity cost of a resource in NE
$\mathcal{C}$	a class of games. We only consider affine, symmetric, k-uniform congestion games in this thesis
P	path $(o, \cdots, u) \in \mathcal{P}$
$\mathcal{P}$	set of all paths P
$\mathcal{P}^>$	the set of paths that correspond to an internal cost increase of $\leq \overline{c_o}/2$
$\mathcal{P}_0^>$	set of paths that start in $o \in \mathbb{O}^>$ with internal cost increase $\leq \overline{c_o}/2$
Δ	Number of paths $\vert\mathcal{P}\vert$
$\Delta^>$	Number of paths $\vert\mathcal{P}^>\vert$
$\Delta_0^>$	Number of paths $\vert\mathcal{P}_0^>\vert$

1 Introduction

Game theory is a branch of mathematics that models conflicting or coinciding goals between rational parties, often called players. Its application is widebut in this thesis the focus will be on economic applications, where we compare the effect of selfish behaviour of individuals with an optimal solution or a best case scenario. Most models use cost functions that are dependant on the choices of all players, so one player's cost can be affected by the choice of another, hence the conflicting goals. A classic example is traffic congestion, where the travel time (cost) is heavily dependant on the route choices of other players.

The effect of selfish behavious, when compared to social optima was illustrated by the well known example of Pigou [3]. The example shows that equilibrium solutions for congestion games with affine (monotonically increasing and non-negative) cost functions can exceed the system's optimum by a factor of $\frac{4}{3}$, see Section 1.2. In traffic Network games, Wardrop [4] introduced the Wardrop Equilibrium where no traffic user can decrease his cost (travel time) by unilaterally deviating. In Wardrop's model, players have access to different resources and demand can be split into arbitrarily small fractions.

An discrete version of the Wardrop equilibrium is the Nash Equilibrium, introduced by Nash [5]. Later it was named after him by Rosenthal [6] where he discussed the class of atomic games. The difference with Wardrop's equilibrium is that in atomic games demand cannot be split into fractions.
The ratio between the social cost of the most expensive Nash Equilibrium and the social cost of an optimal solution (a solution with minimal social cost) has been named the Price of Anarchy by Koutsoupias and Papadimitriou [7].

In Jong [8] and [9] the class of atomic, affine, symmetric, k-uniform congestion games is examined. The corresponding Price of Anarchy for this class of games is bounded between 1.343 and 2.15, which leaves a large gap. Ideally, we would want this gap to be zero, as it gives uncertainty in worst case scenario predictions. For instance, in traffic networks, a government body might want to analyse the worst case congestion scenario during an event of increased network activity. Before this thesis, a worst case scenario would have to work with a total travel time interval, instead of a total travel time value. This thesis will remove this gap by significantly decreasing the upper bound and slightly increasing the lower bound for instances of large number of players, ending up with a Price of Anarchy of ≈ 1.35188 for the class of games described above.

The main results of this thesis can also be found in Kern [10].

1.1 Preliminary Definitions

We start by defining some notions

Definition 1.1. Congestion Game

A congestion game consists of a set of players N, a set of resources R and a set of strategies S_i for each player $i \in N$. For atomic congestion games, a set of strategies represents all possible sets of resources a player can choose, so each strategy $P \in S_i$ is a subset of R. Every resource $r \in R$ has a cost functions $c_r(x_r)$, with x_r representing the number of players that choose this resource.

In this thesis, games are represented as bipartite graphs $G = (N, R, E)$ with players $i \in N$, and resources $r \in R$ as vertices, and actions (when players choose a set of resources) as edges $ir \in E$. A player's action is one element chosen from his set of strategies.

We only consider affine, symmetric, k-uniform congestion games and break this down as follows:

Definition 1.2. k-uniform Game

A k-uniform game is a game in which each player must choose exactly k resources. Therefore, every strategy $P \in S_i$ has size k

Definition 1.3. Symmetric Game

A game that is symmetric has symmetry over the players, so every player has an identical set of strategies and the cost functions for the resources are the same for each player. In non-symmettric games cost functions are $c_r^i(x)$ with $i \in N$.

Definition 1.4. Affine cost functions $c_r(x)$

Finally, affine cost functions are monotonically increasing non-negative functions of the form:

$$c_r(x) = ax + b \quad a, b \geq 0. \tag{1}$$

An affine game is defined as a game with affine cost functions.

An affine, symmetric, k-uniform congestion game combines the definitions given above and is the type of game this thesis is focused on.

Definition 1.5. Degree d_r

The degree of a resource is the number of players it is connected to. If three players choose resource r, then $d_r = 3$.

Definition 1.6. k-matching M and Action $M(i)$

Let M be a set of edges between players and resources, $M \subseteq N \times R$. Let $M(i) = \{r|ir \in M\}$ and $M(r) = \{i|ir \in M\}$, meaning that Action $M(i)$ denotes the set of resources player i is connected to, and $M(r)$ denotes the set of players resource r is connected to. M is k-matching if $\forall\ i \in N : |M(i)| = k$. In other words, M is k-matching if every player is connected to exactly k resources.
When we say that one player changes his action unilaterally, we mean that this player changes his action while all other players do not.

We define an action profile $\mathcal{A}$ as a set of actions $\mathcal{A} = (M(1), \cdots, M(n))$, one for each player.
Let $d_r^M = |M(r)|$ denote the degree of resource r in matching M.

Definition 1.7. Opportunity Cost $c_r(x + 1)$

Opportunity Cost is a term used in this thesis for the cost a resource will have if it is chosen by one extra player. It is useful for comparing optimal and suboptimal solutions and to determine whether a player would want to change his action.

Definition 1.8. Nash Equilibrium

Consider a k-matching M and a player $i \in N$. Suppose $R^i = R \setminus M(i)$ is the set of resources not chosen by player i. A k-matching M is a Nash Equilibrium (NE) if for all $ir \in M$ the following holds:

$$\forall\ s \in R^i \quad c_s(d_s + 1) \geq c_r(d_r) \tag{2}$$

This means that no player has the option to change his action and thereby reduce his individual cost. For all the resources he has not chosen, the opportunity cost is never lower than the cost of any of his chosen resources.
The degree of resource $r \in R$ in a given NE is denoted as $\overline{d}_r$.

Definition 1.9. Social Cost

If the cost of an edge $ir \in M$ is $C(ir) = c_r(d_r^M)$, then the social cost is the summation of the cost of all edges: $C(M) = \sum_{ir \in M} c_r(d_r^M)$, or in other terms: $C(M) = \sum_{r \in R} d_r^M c_r(d_r^M)$.
Note that this summation means some resources are counted multiple times, this is intended because their costs should be counted as many times as they are chosen by players.

Definition 1.10. Optimal Solution

We define a k-matching M as an Optimal solution (OPT) if it has minimal social cost, e.g. no other k-matching has lower social cost than M. Note that there can be several different OPTs. Also note that OPT does not need to be a NE and vice versa, although they might be.
The degree of resource $r \in R$ in a given OPT is denoted as d_r^*.

Definition 1.11. Price of Anarchy

The Price of Anarchy (PoA) is the ratio between $C(NE)$ and $C(OPT)$, where NE has highest cost, in other words, for any game I, the PoA is

$$PoA(I) = \max_{M^{NE} \in M^{NE}(I)} \frac{C(M^{NE})}{C(M^{OPT})} \tag{3}$$

where $M^{NE}(I)$ denotes the set of all NE of game I, $C(M^{NE})$ is the social cost of the corresponding M^{NE} and $C(M^{OPT})$ is the social cost of any M^{OPT}.
In later chapters, we abbreviate $C(NE) = C(M^{NE})$ and $C(OPT) = C(M^{OPT})$.

The PoA of a class of games $\mathcal{C}$ is defined as

$$PoA(\mathcal{C}) = \sup_{I \in \mathcal{C}} PoA(I) \tag{4}$$

In the rest of this thesis, when we write PoA, we mean $PoA(\mathcal{C})$, where $\mathcal{C}$ is the class of affine, symmetric, k-uniform congestion games.

1.2 Pigou's Example

Consider an example of Pigou [3]. Suppose there are two people (players) wanting to cross a river as fast as possible. There are two options (resources) available to them: Option A is to take a bridge which is some distance away, this route will take either player 20 minutes regardless of the other player's action. In other words, this route has a constant costfunction $c_A(x) = 20$.

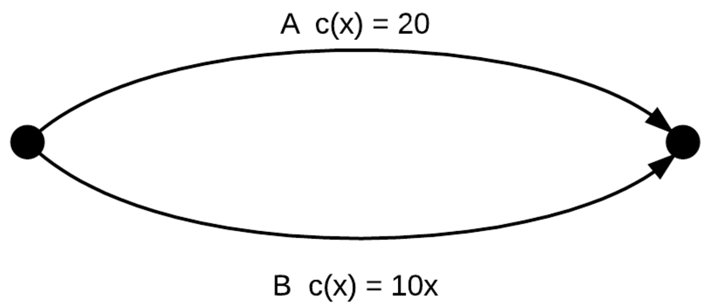

Figure 1: Routing game from example 1

Option B is to take a ferry directly across which is being pulled by the ferryman. As more people take the ferry it becomes heavier and harder to pull to the other side, making it more time-expensive. If only one player chooses the ferry, it will take him 10 minutes, but if both choose the ferry, it will take them both 20 minutes. In other words, option B has variable costfunction $c_B(x) = 10x$. See Figure 1.

Note that this game is atomic since demand cannot be split into fractions, e.g. players either choose route A or B, it is affine because cost functions are monotonically increasing and non-negative, it is symmetric because both players have acces to the same resources and cost functions are the same for both players, and it is 1-uniform as both players choose exactly one resource.

There are four different scenarios available in which this example can unfold, as illustrated in Table 1.

Table 1: Action profiles for the example

Action Profile	individual cost	social cost	OPT or NE
(A,A)	(20,20)	40	neither
(A,B)	(20,10)	30	OPT and NE
(B,A)	(10,20)	30	OPT and NE
(B,B)	(20,20)	40	NE

First, observe that (A, B) and (B, A) yield the same results because of symmetry. Let's consider (A, B) henceforth for simplicity's sake and disregard

(B, A). This profile is both OPT and NE. Consider definitions 1.8 and 1.10, the social cost is minimal and neither player can unilaterally change theiraction to reduce his individual cost. (B, B) is NE because of the same reasoning, neither player can reduce his individual cost by changing from B to A, because the cost of A is always 20.
This example has $PoA = \frac{C(B,B)}{C(A,B)} = \frac{4}{3}$. One might think that $PoA = 1$ because the optimum is also a NE. However this is not the case because the PoA always uses the NE with highest cost.

1.3 Research Question

In this thesis we will answer the following question:

> What is the maximum Price of Anarchy for affine, symmetric, k-uniform congestion games when the number of players is large ($N \to \infty$)?

The point of this question is to find the most extreme cases, where the PoA is highest, and to show this value is the most extreme value.

1.4 Outline

Page ix contains a Nomenclature and abbreviations for terms used in the thesis that may prove useful when reading the main body. Definitions and notions are always explained in the text as they are introduced, but a short description is given here as well.

The main body of this thesis is split into two parts; bounding the PoA from above and bounding it from below.
Chapter 2 proves that $PoA \leq 1.35188\cdots$ for our class of games. This is done via an alternating paths based approach and by making assumptions for critical case games.
Chapter 3 gives an example of a game with many players and $PoA \approx 1.35188$, thus proving a lower bound for critical case games.
Chapter 4 concludes with our findings and makes some recommendations for future research.
Finally, the bibliography follows in the appendix

2 Constructing the Upper bound for the PoA

It is the goal of this section to provide an upper bound of $1.35188\cdots$ to the PoA for the case $|N| \to \infty$.

Theorem 2.1. $PoA \leq 1.35188\cdots$ for $|N| \to \infty$

Because PoA compares $C(NE)$ with $C(OPT)$, it would be useful to find general expressions for both social costs. Unfortunately this is difficult to do, as we want a PoA for an entire class of games, not just one specific game. So, in order to prove this theorem, we start by comparing the difference in social cost between NE and OPT in Section 2.1. We do this by introducing a method that lets us switch actions so that we move from OPT to NE. We call this the switching process. We define the difference in social cost between OPT and NE in two parts, an "internal increase" and an "external increase". At the end of this section, we have an expression for $C(NE) - C(OPT)$.
Section 2.2 shows an upper bound for the external increase in social cost. We do not use this bound in the expression for $C(NE) - C(OPT)$, but instead use it as a part of proof in other sections.
Section 2.3 analyzes properties of games with the highest possible PoA which we call critical case games. This lets us improve the bound on the difference in social cost.
Section 2.4 improves the upper bound for the internal increase in social cost, which in turn improves the bound for $C(NE) - C(OPT)$.
Section 2.5 introduces the relative social cost increase $\frac{C(NE)-C(OPT)}{C(NE)}$ and constructs an upper bound for it, which we eventually use to calculate the PoA in Section 2.6.

2.1 The Switching Process

The general method of comparing social cost between NE and OPT is by examining the graph with edges $e \in NE \cup OPT$ and paths $P \in \mathcal{P}$ that when followed transform OPT into NE. To analyze these paths, the following definitions are helpful:

Definition 2.1. $E := NE \cup OPT$
$\mathbb{O} := \left(r \in R | \overline{d_r} > d_r^*\right)$ ("Overloaded resources")
$\mathbb{U} := \left(r \in R | \overline{d_r} < d_r^*\right)$ ("Underloaded resources")
$\mathbb{B} := \left(r \in R | \overline{d_r} = d_r^*\right)$ ("Balanced resources")

E is the set of edges that are in either NE, OPT or both. $\mathbb{O}$, $\mathbb{U}$, and $\mathbb{B}$ are sets of resources which have either higher cost in the NE than in OPT, lower

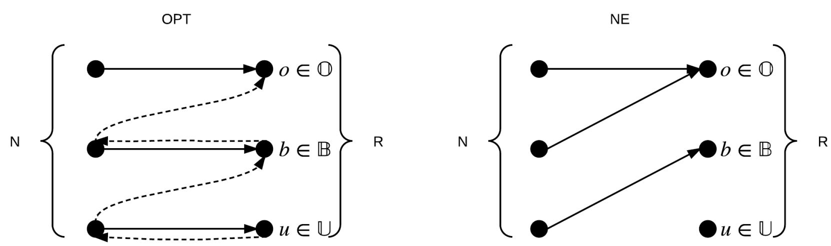

Figure 2: Switching from OPT to NE over path P. Full lines represent connections in either OPT or NE, while dotted lines represent a sinle path P

cost in the NE than in OPT, or equal cost, respectively. Together, they form the entire set of resources $\mathbb{O} \cup \mathbb{U} \cup \mathbb{B} = R$.

The set $NE \oplus OPT = (NE \setminus OPT) \cup (OPT \setminus NE)$ consists of alternating paths successively, with paths running from resource to player to resource a number of times, starting at a resource $o \in \mathbb{O}$ and ending at a resource $u \in \mathbb{U}$. Let $\mathcal{P}$ be the corresponding set of paths. Passing from OPT to NE can be interpreted as switching OPT to NE along each path $P = (o, i, \cdots, j, u) \in \mathcal{P}$.

See Figure 2 for a graphical representation of switching from OPT to NE over one path P.

After each switch has been made and we moved from OPT to NE, each path $P \in \mathcal{P}$ will have caused both an "internal increase" where his own cost may have changed and an "external increase" where the cost of other players may have changed. Resources on paths other than the start and end will have the same degree as before the switch, so the contribution to the social cost of these resources remains the same. However, resources on the start and end of each path have their degree decreased and increased prespectively. After the switch, the internal increase for each path is equal to the cost of new resource o in NE minus the cost of old resource u in OPT: $c_o(\overline{d}_o) - c_u(d_u^*)$. Since $d_u^* \geq \overline{d}_u + 1$ for $u \in \mathbb{U}$, the internal increase is bounded from above by $c_o(\overline{d}_o) - c_u(\overline{d}_u + 1)$.

The external increase is counted not per path but per resource $o \in \mathbb{O}$. Every connection $io \in OPT$ experiences an increase in cost equal to $c_o(\overline{d}_o) - c_o(d_o^*)$, as extra players are now connected to the same resource after the switch. Therefore, every resource $o \in \mathbb{O}$ will have an increase in cost equal to $c_o(\overline{d}_o) - c_o(d_o^*)$ times the number of players that were connected to o in OPT, or d_o^*.

This results in a difference in social cost equal to $d_o^*(c_o(\overline{d}_o) - c_o(d_o^*))$ for each resource $o \in \mathbb{O}$.

Any path that is an alternating cycle $C \in \mathcal{P}$ that arises during the switching process may be eliminated by passing from OPT to $OPT' := OPT \oplus C$ without affecting the social cost and constructing a new set of paths from $OPT' \oplus NE$. We now have an upper bound for the difference in social cost

$$c(NE) - c(OPT) \leq \sum_{P=(o,\cdots,u)\in\mathcal{P}} c_o(\overline{d}_o) - c_u(\overline{d}_u + 1) + \sum_{o\in\mathbb{O}} d_o^*(c_o(\overline{d}_o) - c_o(d_o^*)) \tag{5}$$

which we will improve in the following sections.

2.2 Bounding the External Increase

In this section we show the upper bound for the external increase in social cost can be improved. While we do not apply this improvement immediately in Inequality (5), we still make use of the lemma later in this chapter.

Lemma 2.1. for any $o \in \mathbb{O}$, the external increase is bounded by $d_o^*(\overline{d}_o - d_o^*) \leq \frac{1}{4}\overline{d}_o^2$

Consider the function $f(x) = x(y-x)$, its maximum is determined by taking the derivative $f'(x) = -2x + y$ and solving for zero, giving us $x = \frac{1}{2}y$. It is easy to see that this value gives a maximum value for the function. Now we simply input this into our function: $f(\frac{1}{2}y) = \frac{1}{4}y^2$. □

2.3 Critical Case Games

We define the class of games with maximal PoA as critical case games. Because the PoA is maximized, these games should have high $C(NE)$ and low $C(OPT)$. As we try to find an upper bound for the maximal PoA, we can assume that any game used in calculating this bound is a critical case game.

The following Lemma is helpful:

Lemma 2.2. For critical case games, cost functions of overloaded resources are linear, meaning we can substitute

$$c_o(x) = c_o x. \tag{6}$$

Proof: Suppose the converse is true: $\exists\ o \in \mathbb{O} : c_o(x)\ non-linear$ (but still affine), e.g. $c_o(x) = ax + b$, $a, b > 0$. We will prove that we can replace $c_o(x)$ with a linear $\widetilde{c_o}(x)$ without changing $C(NE)$, while reducing $C(OPT)$.

If we replace $c_o(x)$ by $\widetilde{c_o}(x) = \frac{c_o(\overline{d}_o)}{\overline{d}_o}x$, indeed, the social cost of NE remains identical:

$$\widetilde{c_o}(\overline{d}_o) = \frac{c_o(\overline{d}_o)}{\overline{d}_o}\overline{d}_o = c_o(\overline{d}_o). \tag{7}$$

This result also means that the opportunity cost $\widetilde{c_o}(\overline{d}_o+1)$ is higher than $c_o(\overline{d}_o+1)$, so the alteration will cause no player to change his action in NE. This proves that $C(NE)$ remains the same.

In order to prove that $C(OPT)$ is lower when $\widetilde{c_o}(x)$ is used we will show that $\widetilde{c_o}(d_o^*) - c_o(d_o^*) < 0$. First we examine $\widetilde{c_o}(d_o^*)$ more closely, using $c(x) = ax + b$

$$\tilde{c_o}(d_o^*) = \frac{c_o(\overline{d}_o)}{\overline{d}_o}d_o^* = \frac{a\overline{d}_o + b}{\overline{d}_o}d_o^* = ad_o^* + b\frac{d_o^*}{\overline{d}_o}. \tag{8}$$

Subtract $c_o(d_o^*)$ and we get:

$$\begin{aligned} \tilde{c_o}(d_o^*) - c_o(d_o^*) &= ad_o^* + b\frac{d_o^*}{\overline{d}_o} - (ad_o^* + b) && (9) \\ &= b(\frac{d_o^*}{\overline{d}_o} - 1) < 0 && (10) \end{aligned}$$

The last inequality follows from $\overline{d}_o > d_o^*$ from Def. 2.1 and $b > 0$.

We have now shown that when we replace $c_o(x)$ with $\widetilde{c_o}$ we enlarge the PoA. This contradicts the assumption of a critical case game, thereby proving the lemma. □

We combine Lemma 2.2 with Inequality (5) for a better upper bound for the difference in social cost:

$$c(NE) - c(OPT) \leq \sum_{P=(o,\cdots,u)\in\mathcal{P}} c_o\overline{d}_o - c_u(\overline{d}_u + 1) + \sum_{o\in\mathbb{O}} d_o^*(\overline{d}_o - d_o^*)c_o. \tag{11}$$

2.4 Bounding the Internal Increase

In this section, we improve the bound for the internal increase. We show that paths either contribute $c_o(\overline{d}_o)/2$ to the internal increase, contribute a maximum of $c_o(\overline{d}_o)/2$ when combined with another path, or solely contribute negatively (in which case we disregard them). To do this, we will distinguish between four different sets of paths in 2.4.1. The first contributes positively, the second requires the third to contribute positively and the fourth contributes negatively. Then we use this to improve the bound on the internal increase in Section 2.4.2.

The following definitions prove helpful.

Definition 2.2. $c_0^{max} = \max_{o\in\mathbb{O}} c_o\overline{d}_o$

Definition 2.3. $\overline{c}_r = c_r(\overline{d}_r)$ and $\overline{c}_r^+ = c_r(\overline{d}_r + 1)$

So $\overline{c}_r$ is the cost of resource $r \in R$ in NE, and $\overline{c}_r^+$ is the opportunity cost of resource $r \in R$ in NE.

Inequality (11) now simplifies to

$$c(NE) - c(OPT) \leq \sum_{P=(o,\cdots,u)\in\mathcal{P}} \overline{c}_o - \overline{c}_u^+ + \sum_{o\in\mathbb{O}} d_o^*(\overline{d}_o - d_o^*)c_o. \tag{12}$$

Lemma 2.3. For critical case games, all resources $r \in R$ have $\overline{d}_r < |N|$.

Proof: Suppose the converse is true, $\exists\, r \in R,\ \ \overline{d}_r = |N|$. Obviously, $r \notin \mathbb{U}$, because in NE these are underloaded. So let $r \in \mathbb{B}\cup\mathbb{O}$. Consider each scenario:

If $r = b \in \mathbb{B}$, then $\overline{d}_b = d_b^* = |N|$. We can remove b from the game and replace k with $k-1$, meaning we remove a resource that all players are always connected to and all connections to it. This reduces both $C(NE)$ and $C(OPT)$ with $\overline{d}_b^2\overline{c}_b$, so the price of anarchy of this new game equals $\frac{C(NE)-\overline{d}_b^2\overline{c}_b}{C(OPT)-\overline{d}_b^2\overline{c}_b}$ which is greater than $\frac{C(NE)}{C(OPT)}$. This contradicts criticality of the original game, so r cannot be in $\mathbb{B}$.

If $r = o \in \mathbb{O}$, then there must be at least one path $P = (o, i, \cdots, j, u) \in \mathcal{P}$. If the corresponding internal increase is strictly positive $\overline{c}_o - \overline{c}_u^+ > 0$, then $jo \notin NE$ because of Definition 1.8, contradicting $\overline{d}_r = |N|$. So we can assume that for all paths $P = (o, \cdots, u) \in \mathcal{P}$, the internal increase is $\overline{c}_o - \overline{c}_u^+ \leq 0$. Now we construct k-matching M by following all paths $P \in \mathcal{P}$ starting in o with $\overline{c}_o - \overline{c}_u^+ \leq 0$, e.g. $M = OPT \oplus \mathcal{P}$. The difference in social cost between M and OPT is $C(M) - C(OPT) = \overline{c}_o - \overline{c}_u^+ + d_o^*(\overline{d}_o - d_o^*)c_o \leq d_o^*(\overline{d}_o - d_o^*)c_o \leq \frac{1}{4}\overline{d}_o^2 c_o$, using Lemma 2.1 for the last step. Because we followed every path from o, the degree of o in M is $d_o^M = d_o^* = |N|$. Now we remove resource o again from both NE and M and replace k with $k-1$, thus creating $\hat{NE}$ and $\hat{M}$ with social costs $C(\hat{NE}) = C(NE) - \overline{d}_o^2 c_o$ and $C(\hat{M}) = C(M) - \overline{d}_o^2 c_o \leq C(OPT) - \frac{3}{4}\overline{d}_o^2 c_o$ respectively. Also, because we are observing critical case games, $C(NE) = PoA\, C(OPT)$, with $PoA > \frac{4}{3}$ (see Chapter 3). We conclude by calculating the price of anarchy in the new instance:

$$\begin{aligned}
P\hat{o}A &= \frac{C(\hat{NE})}{C(\hat{M})} && (13)\\
&\geq \frac{C(NE) - \overline{d}_o^2 c_o}{C(OPT) - \frac{3}{4}\overline{d}_o^2 c_o} && (14)\\
&= \frac{PoA\, C(OPT) - \overline{d}_o^2 c_o}{C(OPT) - \frac{3}{4}\overline{d}_o^2 c_o} && (15)\\
&> PoA \frac{C(OPT) - \frac{3}{4}\overline{d}_o^2 c_o}{C(OPT) - \frac{3}{4}\overline{d}_o^2 c_o} = PoA && (16)
\end{aligned}$$

meaning the PoA for the new game is strictly greater than the PoA for the original, contradicting criticality for the original. □

Lemma 2.4. For critical case games, every $r \in R$ with $\overline{d}_r > 0$ satisfies $\overline{c}_r^+ \geq \frac{1}{2} c_o^{max}$

Proof: Suppose the converse is true, i.e. $\exists\ r \in R$ with $\overline{d}_r > 0$ such that $\overline{c}_r^+ < \frac{1}{2} c_o^{max}$. Consider the set of resources cheaper than r's opportunity cost

$$R^- = \{r^- \in R : \overline{c}_{r^-} \leq \overline{c}_r^+\}. \tag{17}$$

Because $\overline{c}_r \leq \overline{c}_r^+$, r must be in R^-, so the size of R^- is $|R^-| \geq 1$. Now, for each $r^- \in R^-$, we have

$$\overline{c}_{r^-}^+ \leq 2\overline{c}_{r^-} \leq 2\overline{c}_r^+ < c_o^{max} \tag{18}$$

where the first inequality follows from the game being affine and the third is the inverse assumption we try to disprove.

Because $\overline{c}_{r^-}^+ \leq c_0^{max}$, all players who choose o^{max} also choose all $r^- \in R^-$, which means $|R^-| \leq k-1$. Now, as all players choose exactly k resources, they all choose at least one resoure $\tilde{r} \in R \setminus R^-$. By definition of R^-, $\overline{c}_{\tilde{r}} > \overline{c}_r^+$. Again, by Definition 1.8, this implies that all players choose r, so $\overline{d}_r = |N|$, which contradicts Lemma 2.3. □

2.4.1 Ordering the Alternating Paths

Consider paths $P = (o, \cdots, u) \in \mathcal{P}$ and let $\Delta = |\mathcal{P}|$ the number of alternating non-cyclic paths. We introduce two ways of ordering the paths.

For $1 \leq t \leq \Delta$, let o_t be ordered in cost ascending order: $\overline{c}_{o_1} \leq \cdots \leq \overline{c}_{o_\Delta}$ and let $P_t = (o_t, \cdots, u_t)$ be the corresponding path for o_t.

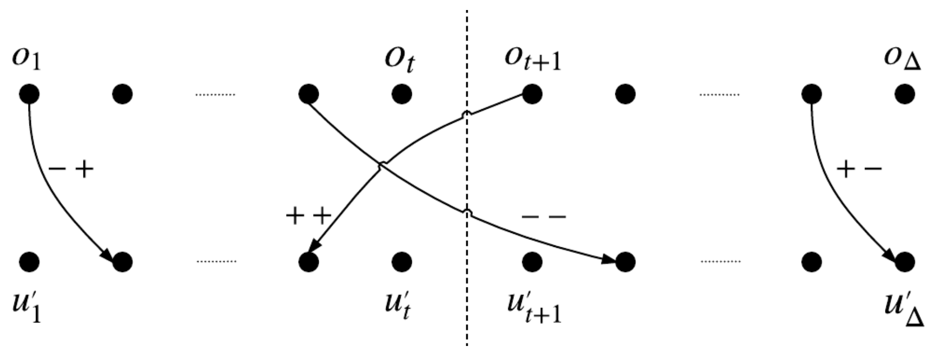

Figure 3: Possible alternating paths in $\mathcal{P}^{+-}$, $\mathcal{P}^{++}$, $\mathcal{P}^{-+}$ and $\mathcal{P}^{--}$

For $1 \le t \le \Delta$, let u'_t be ordered in opportunity cost descending order: $\bar{c}^+_{u'_1} \ge \cdots \ge \bar{c}^+_{u'_\Delta}$ and let $P'_t = (o'_t, \cdots, u'_t)$ be the corresponding path for u'_t.

Note that because o_t are ordered based on paths, an o_t might refer to a resource that is later referred to again by o_{t+k}, with $k > 0$. The same holds for u_t.

When $\bar{c}_{o_\Delta} \le \bar{c}^+_{u'_\Delta}$, the most expensive o_t is lower in cost than the u'_t with lowest opportunity cost, so every internal increase is negative, $\bar{c}_o - \bar{c}^+_u \le \bar{c}_\Delta - \bar{c}^+_{u'_\Delta} \le 0$. However, when $\bar{c}_{o_\Delta} > \bar{c}^+_{u'_\Delta}$, there exist paths with $\bar{c}_o - \bar{c}^+_u > 0$.

More specifically, suppose $t \ge 0$ is the first index for which $\bar{c}_{o_{t+1}} > \bar{c}^+_{u'_{t+1}}$ (meaning t is also the last index for which $\bar{c}_{o_t} \le \bar{c}^+_{u'_t}$). We then let

$$\mathcal{P}^{+-} := \{P_{t+1}, \cdots, P_\Delta\} \cap \{P'_{t+1}, \cdots, P'_\Delta\} \tag{19}$$

be the set of paths that begin in $o \in \{o_{t+1}, \cdots, o_\Delta\}$ and end in $u \in \{u'_{t+1}, \cdots, u'_\Delta\}$. The notation $\mathcal{P}^{+-}$ is chosen to represent the resources o that have higher cost than $\bar{c}_{o_t}$ (with $+$) and the resources u with lower cost than $\bar{c}^+_{u'_t}$ (with $-$).
Each of these paths has internal cost bound of $\bar{c}_o - \bar{c}^+_u > 0$, but also, $\bar{c}_o - \bar{c}^+_u \le \bar{c}_o - c_0^{max}/2 \le \bar{c}_o/2$.

Similarly, let

$$\mathcal{P}^{++} := \{P_{t+1}, \cdots, P_\Delta\} \cap \{P'_1, \cdots, P'_t\} \tag{20}$$

$$\mathcal{P}^{-+} := \{P_1, \cdots, P_t\} \cap \{P'_1, \cdots, P'_t\} \tag{21}$$

$$\mathcal{P}^{--} := \{P_1, \cdots, P_t\} \cap \{P'_{t+1}, \cdots, P'_\Delta\} \tag{22}$$

be the three sets of remaining combinations between o and u'. Figure 3 may be helpful in visualizing what these paths look like. Between $(o_1, \cdots, o_\Delta)$ and

$(u'_1, \cdots, u'_\Delta)$ are a number of players and resources that are part of the path but omitted for graphical viewability. Note that every o_t is only connected to one u'_t and vice versa by this notation.
To provide some intuition into the deconstruction of the paths, consider the following lemma.

Lemma 2.5. Every $P \in \mathcal{P}^{+-}$ has length ≥ 4

PROOF: If $P \in \mathcal{P}^{+-}$ has length 2, P would look like $P = (o, i, u)$ with $io \in NE \setminus OPT$ and $iu \in OPT \setminus NE$, then NE implies that $\overline{c}_u^+ \geq \overline{c}_o$. This is a contradiction of the definition of $\mathcal{P}^{+-}$. $\square$

2.4.2 Improving the Upper Bound on the Internal Increase

In order to improve the upper bound on the internal increase from Inequality (12), we need the following definitions:

Definition 2.4. $\mathbb{O}^> \subseteq \mathbb{O}$ denotes the set of resources $o_{t+1}, \cdots, o_\Delta$
$\mathbb{U}^< \subseteq \mathbb{U}$ denotes the set of resources $u'_{t+1}, \cdots, u'_\Delta$
$\mathcal{P}^> = \mathcal{P}^{+-} \cup \mathcal{P}^{++}$
$\mathcal{P}_o^>$ denotes the set of paths in $\mathcal{P}^>$ that end in a given $o \in \mathbb{O}^>$
$\Delta_o^> = |\mathcal{P}_o^>|$ is the number of paths that end in o

We try to prove the following lemma by considering the four different sets of paths constructed in the previous section and their contribution to the internal increase.

Lemma 2.6. $\sum_{P=(o,\cdots,u)\in\mathcal{P}} \overline{c}_o - \overline{c}_u^+ \leq \sum_{o\in\mathbb{O}^>} \Delta_o^> \overline{c}_o/2$

PROOF: The proof is split in three parts, one for paths $P \in \mathcal{P}^{-+}$, one for $P \in \mathcal{P}^{+-}$, and one for $P \in \mathcal{P}^{--} \cup \mathcal{P}^{++}$. Each part introduces different os and us, so one should be careful not to confuse the resources when reading.

We observe $\mathcal{P}^{-+}$ first. These paths connect $o \in \{o_1, \cdots, o_t\}$ to $u \in \{u'_1, \cdots, u'_t\}$ which are low cost o and high opportunity cost u'. They have internal cost bounds of $\overline{c}_o - \overline{c}_u^+ \leq \overline{c}_{o_t} - \overline{c}_{u'_t}^+ \leq 0$, because t is the last index for which $\overline{c}_{o_t} \leq \overline{c}_{u'_t}^+$.

The paths $P \in \mathcal{P}^{+-}$ connect $o \in \{o_{t+1}, \cdots, o_\Delta\}$ to $u \in \{u'_{t+1}, \cdots, u'_\Delta\}$ which are high cost o and low opportunity cost u'. We apply Lemma 2.4 to calculate their internal cost bound $\overline{c}_o - \overline{c}_u^+ \geq \overline{c}_o - \frac{1}{2}c_0^{max} \leq \frac{1}{2}\overline{c}_o$.

Consider the sets $\mathcal{P}^{++}$ and $\mathcal{P}^{--}$. We can say that any path $P = (o, \cdots, u) \in \mathcal{P}^{++}$ has internal cost bound $\overline{c}_o - \overline{c}^+_u \leq \overline{c}_o - \overline{c}^+_{u'_t}$ and any path $P = (o', \cdots, u') \in \mathcal{P}^{--}$ has internal cost bound $\overline{c}_{o'} - \overline{c}^+_{u'} \leq \overline{c}_{u'_t} - \overline{c}^+_{u'} \leq \overline{c}^+_{u'_t} - \overline{c}^+_{u'}$.

We can deduce from Fig 3 that $|\mathcal{P}^{++}| = |\mathcal{P}^{--}|$, because for every path $P \in \mathcal{P}^{++}$ that crosses the vertical line, exactly one path from $P \in \mathcal{P}^{--}$ should also cross it. So for any path $P \in \mathcal{P}^{++}$ we assign one "matching" path $P \in \mathcal{P}^{--}$. If we add these paths together we get a combined internal cost bound of $\overline{c}_o - \overline{c}^+_u + (\overline{c}^+_{0'} - \overline{c}^+_{u'}) \leq \overline{c}_o - \overline{c}^+_{u'_t} + (\overline{c}^+_{u'_t} - \overline{c}^+_{u'}) = \overline{c}_o - \overline{c}^+_{u'} \leq \overline{c}_o/2$. In the last inequality we apply Lemma 2.4 again.

We conclude the proof by adjusting the objective function by splitting the paths into their subsets and adjusting their internal cost to the new bounds:

$$\sum_{P \in \mathcal{P}} \overline{c}_o - \overline{c}^+_u \leq \sum_{P \in \mathcal{P}^{+-}} \overline{c}_o - \overline{c}^+_u + \sum_{P \in \mathcal{P}^{++} \cup \mathcal{P}^{--}} \overline{c}_o - \overline{c}^+_u + \sum_{P \in \mathcal{P}^{-+}} \overline{c}_o - \overline{c}^+_u \tag{23}$$

$$\leq \sum_{P \in \mathcal{P}^{+-}} \frac{1}{2}\overline{c}_o + \sum_{P \in \mathcal{P}^{++}} \frac{1}{2}\overline{c}_o + \sum_{P \in \mathcal{P}^{-+}} 0. \tag{24}$$

Observe that every path $P \in \mathcal{P}^{+-} \cup \mathcal{P}^{++}$ has an end in $o \in \mathbb{O}^>$. This means that we have shown that for every path that ends in $o \in \mathbb{O}^>$ (where needed combined with a "matching" path ending in $o \in \mathbb{O} \setminus \mathbb{O}^>$), the internal cost is bounded by $\overline{c}_o - \overline{c}^+_u \leq \frac{1}{2}\overline{c}_o$. Since there are $\Delta^>_o$ paths in $\mathcal{P}^>_o$ per $o \in \mathbb{O}^>$,

$$\sum_{P \in \mathcal{P}^{+-}} \frac{1}{2}\overline{c}_o + \sum_{P \in \mathcal{P}^{++}} \frac{1}{2}\overline{c}_o = \sum_{o \in \mathbb{O}^>} \Delta^>_o \frac{1}{2}\overline{c}_o \tag{25}$$

□

We use Lemma 2.6 to update Inequality (12):

$$c(NE) - c(OPT) \leq \sum_{o \in \mathbb{O}^>} \Delta^>_o \overline{c}_o/2 + \sum_{o \in \mathbb{O}} d^*_o(\overline{d}_o - d^*_o)c_o. \tag{26}$$

2.5 The Relative Social Cost Increase

We seek an upper bound for the relative social cost increase $\frac{c(NE)-c(OPT)}{c(NE)}$, so we need a lower bound for $c(NE)$. We do this by constructing a lower bound for the number of edges in NE in Section 2.5.1. A lower bound for the social cost in NE is then calculated in Section 2.5.2, after which we can calculate the relative social cost increase in Section 2.5.3.

2.5.1 A lower bound for $|NE|$

First we split the number of edges in NE as follows:

$$|NE| \geq \sum_{o \in \mathbb{O}} \overline{d}_o + |F| \tag{27}$$

$$\geq \sum_{o \in \mathbb{O}^>} \overline{d}_o + |F| \tag{28}$$

so that F is the set of edges not incident to any $o \in \mathbb{O}$. What we will prove is that the cost of each of these edges is bounded by $\frac{1}{2}c_0^{max}$ either by itself or in combination with some other edge not yet accounted for.

Let $\mathbb{I} := \{i \in N | \exists\ io \in NE \text{ for any } o \in \mathbb{O}^>\}$ be the set of players that has at least one connection to a resource $o \in \mathbb{O}^>$, and $\mathbb{J} := \{j \in N | \exists P = (\cdots, j, u) \in \mathcal{P}^>\}$ be the set of first players on paths in $\mathcal{P}^>$.

Lemma 2.7. $\mathbb{I} \cap \mathbb{J} = \emptyset = (\mathbb{J} \times \mathbb{O}^>) \cap NE$

PROOF: Suppose that $\mathbb{I} \cap \mathbb{J} \neq \emptyset$, e.g. there is at least one player in both $\mathbb{I}$ and $\mathbb{J}$. Let $i \in \mathbb{I} \cap \mathbb{J}$, then there exists an $o \in O$ for which $io \in NE$ and an $u \in \mathbb{U}^<$ for which $iu \notin NE$. According to Definition 1.8: $\overline{c}_u^+ \geq \overline{c}_o$, which is in direct contradiction to the definitions of $\mathbb{O}^>$ and $\mathbb{U}^<$. $(\mathbb{J} \times \mathbb{O}^>) \cap NE = \emptyset$ follows similarly. $\square$

Lemma 2.8. $|F| \geq \frac{\sum_{o \in \mathbb{O}^>} \overline{d}_o}{k - |\mathbb{U}^<|} |\mathbb{U}^<| + \frac{k}{|\mathbb{U}^<|} \sum_{o \in \mathbb{O}^>} \Delta_o^>$

PROOF: First we notice that all $i \in \mathbb{I}$ are connected to all $\mathbb{U}^<$. This is because $\overline{c}_o > \overline{c}_u^+$ for all $o \in \mathbb{O}^>$ and for all $u \in \mathbb{U}^<$. So any player that is connected to an $o \in \mathbb{O}^>$ must also be connected to all $u \in \mathbb{U}^<$, meaning that any player i has $k - |\mathbb{U}^<|$ edges left to connect to $\mathbb{O}^>$. We also know that overloaded resources are only connected to $\mathbb{I}$, so there are $\sum_{o \in \mathbb{O}^>} \overline{d}_o$ edges from $\mathbb{O}^>$ to $\mathbb{I}$, of which every i can receive at most $k - |\mathbb{U}^<|$. This gives us a lower bound for $|\mathbb{I}|$:

$$|\mathbb{I}| \geq \frac{\sum_{o \in \mathbb{O}^>} \overline{d}_o}{k - |\mathbb{U}^<|} \tag{29}$$

Following the same reasoning, there are $|\Delta^>|$ alternating paths, of which each $j \in \mathbb{J}$ can only receive $|\mathbb{U}^<|$:

$$|\mathbb{J}| \geq \frac{\sum_{o \in \mathbb{O}^>} \Delta_0^>}{|\mathbb{U}^<|}$$

We declared F to be all edges not incident to $\mathbb{O}$. That means that it at least entails all edges from $\mathbb{U}^{<}$ to $\mathbb{I}$ and from $\mathbb{B}$ to $\mathbb{J}$. Every $i \in \mathbb{I}$ is connected to all $r \in \mathbb{U}^{<}$, and each $j \in \mathbb{J}$ is not connected to $\mathbb{O}^{>}$. Therefore we can state

$$|F| \geq |\mathbb{I}||\mathbb{U}^{<}| + |\mathbb{J}|k \tag{30}$$

$$\geq \frac{\sum\limits_{o\in\mathbb{O}^{>}} \overline{d}_o}{k - |\mathbb{U}^{<}|}|\mathbb{U}^{<}| + \frac{k}{|\mathbb{U}^{<}|}\sum_{o\in\mathbb{O}^{>}} \Delta_0^{>} \tag{31}$$

thus proving the lemma. □

2.5.2 A Lower Bound for the Social Cost in NE

The following lemma is useful when calculating the social cost in a given NE:

Lemma 2.9. For critical case games, if $|N| \to \infty$ then $|\mathbb{I}| \to \infty$.

Proof: Suppose $\sum\limits_{o\in\mathbb{O}^{>}} \overline{d}_o \leq \epsilon \sum\limits_{r\in R} \overline{d}_r$ for some small $\epsilon > 0$, then we deduce from (26)

$$C(NE) - C(OPT) \leq \sum_{o\in\mathbb{O}^{>}} \Delta_o^{>}\overline{c}_o/2 + \sum_{o\in\mathbb{O}} d_o^*(\overline{d}_o - d_o^*)c_o \tag{32}$$

$$\leq \sum_{o\in\mathbb{O}^{>}} \overline{d}_o\overline{c}_o/2 + \sum_{o\in\mathbb{O}} \frac{1}{4}\overline{d}_o^2 c_o \tag{33}$$

$$\leq \epsilon\sum_{r\in R} \overline{d}_r\overline{c}_o/2 + \frac{1}{4}C(NE) \tag{34}$$

$$\leq \epsilon\sum_{r\in R} \overline{d}_r 2\overline{c}_r + \frac{1}{4}C(NE) \tag{35}$$

so if ϵ is small enough, $C(NE) - C(OPT) \leq \frac{1}{4}C(NE)$ which corresponds to a $PoA = \frac{4}{3}$. This contradicts the lower bound generated in Chapter 3, so we may assume that $\sum\limits_{o\in\mathbb{O}^{>}} \overline{d}_o > \epsilon \sum\limits_{r\in R} \overline{d}_r$. We combine Inequality (29) and $\sum\limits_{r\in R} \overline{d}_r = k|N|$ for

$$\mathbb{I} \geq \epsilon\frac{\sum\limits_{r\in R} \overline{d}_r}{k - |\mathbb{U}^{<}|} \tag{36}$$

$$\geq \epsilon\frac{k|N|}{k - |\mathbb{U}^{<}|} \tag{37}$$

and because $N \to \infty$ we can assume that $\mathbb{I} \to \infty$. □

A lower bound on the social cost of the edges in F is given by the following Lemma.

Lemma 2.10. $c(F) \geq \frac{c_0^{max}}{2} \frac{\sum_{o \in \mathbb{O}^>} \overline{d}_o}{k - |\mathbb{U}^<|} |\mathbb{U}^<| + \frac{c_0^{max}}{2} \frac{k}{|\mathbb{U}^<|} \sum_{o \in \mathbb{O}^>} \Delta_0^>$

PROOF: The proof of this Lemma is split in two parts. First, we analyse the cost of players $i \in \mathbb{I}$ connected to $\mathbb{U}^<$. By combining the affinity of cost functions, the fact that every $i \in \mathbb{I}$ is connected to every $u \in \mathbb{U}^<$, and Lemma 2.4:

$$c_u(\overline{d}_u) \geq \frac{\overline{d}_u}{\overline{d}_u + 1} \overline{c}_r^+ \tag{38}$$

$$= \frac{|\mathbb{I}|}{|\mathbb{I}| + 1} \overline{c}_r^+ \tag{39}$$

$$\geq \frac{|\mathbb{I}|}{|\mathbb{I}| + 1} \frac{1}{2} c_0^{max} \to \frac{1}{2} c_0^{max}. \tag{40}$$

For the last step we apply Lemma 2.9.
We know that there are $|\mathbb{I}||\mathbb{U}^<|$ edges with this cost, so we can combine Lemma 2.8 and Inequality (29) with Inequality (40) to get the first part of what we need to prove: $\frac{c_0^{max}}{2} \frac{\sum_{o \in \mathbb{O}^>} \overline{d}_o}{k - |\mathbb{U}^<|} |\mathbb{U}^<|$.

The second part is the cost of players $j \in \mathbb{J}$ choosing resources $b \in \mathbb{B}$. We will show that we can equate these costs to be no greater than $\frac{1}{2} c_0^{max}$. For an arbitrary edge $jb \in NE$, pick any $o \in \mathbb{O}^>$ with $c_o = c_0^{max}$ and any $i \in \mathbb{I}$ with $io \in NE$. Now observe two scenarios:

If $ib \notin NE$, then the opportunity cost of b should be bounded by $\overline{c}_r^+ \geq c_0^{max}$, otherwise player i would have chosen resource b instead of o in NE. Because of this bound and the fact that functions are affine we get $\overline{c}_b \geq \frac{1}{2} c_0^{max}$.
On the other hand, if $ib \in NE$, then $\overline{d}_b \geq 2$, since both players i and j are connected. From this we can deduce $\overline{c}_b \geq \frac{c_0^{max}}{3}$ since $\overline{c}_b \geq \frac{2}{3} \overline{c}_b^+$ and Lemma 2.4. Now, in the calculation of $c(NE)$ we do not count edges from $i \in \mathbb{I}$ to $b \in \mathbb{B}$, so we can add to the cost of each edge $jb \in NE$ a corresponding edge cost from $ib \in NE$, resulting in an accounted cost of at least $\frac{2}{3} c_0^{max}$. Our desired equation requires a lower bound of only $\frac{1}{2} c_0^{max}$, which we have now reached.

Now each edge cost is lower bounded by $\frac{1}{2} c_0^{max}$, combined with Lemma 2.8 this concludes the proof. $\square$

We conclude this section by calculating the lower bound for the social cost of NE.

Lemma 2.11. $c(NE) \geq \sum_{o\in\mathbb{O}} \overline{d}_o^2 c_o + \left(\frac{\mathbb{U}^<}{k-|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \overline{d}_o + \frac{k}{|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \Delta_0^> \right) \frac{c_0^{max}}{2}$

Proof: The lower bound follows from combining Inequality (28) with Lemma 2.10. □

2.5.3 Constructing a Bound on the Relative Cost Increase

We now use Inequality (26) and combine it with Lemma 2.11 to obtain the following inequality:

$$\frac{C(NE)-C(OPT)}{C(NE)} \leq \frac{\sum_{o\in\mathbb{O}^>} \frac{\Delta_o^> c_o}{2} + \sum_{o\in\mathbb{O}} d_o^*(\overline{d}_o - d_o^*)c_o}{\sum_{o\in\mathbb{O}} \overline{d}_o^2 c_o + \left(\frac{\mathbb{U}^<}{k-|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \overline{d}_o + \frac{k}{|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \Delta_0^> \right) \frac{c_0^{max}}{2}}. \tag{41}$$

Lemma 2.1 states that $d_o^*(\overline{d}_o - d_o^*) \leq \frac{1}{4}\overline{d}_o^2$. So for any $o \in \mathbb{O}$ if we remove $d_o^*(\overline{d}_o - d_o^*)c_o$ from the numerator and $\overline{d}_o^2 co$ from the denominator we increase the fraction as a whole. This means we can remove any $o \in \mathbb{O} \setminus \mathbb{O}^>$ from Inequality (41) in both sums where they appear. Also, we can replace c_0^{max} in the denominator with its smaller $\overline{c}_o$, since it will only increase the fraction as a whole:

$$\frac{C(NE)-C(OPT)}{C(NE)} \leq \frac{\sum_{o\in\mathbb{O}^>} \frac{\Delta_o^> \overline{c}_o}{2} + \sum_{o\in\mathbb{O}^>} d_o^*(\overline{d}_o - d_o^*)c_o}{\sum_{o\in\mathbb{O}^>} \overline{d}_o^2 c_o + \left(\frac{\mathbb{U}^<}{k-|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \overline{d}_o \frac{\overline{c}_o}{2} + \frac{k}{|\mathbb{U}^<|} \sum_{o\in\mathbb{O}^>} \Delta_0^> \frac{\overline{c}_o}{2} \right)}. \tag{42}$$

The fraction of sums is less than or equal to the maximum of the individual fractions, $\frac{\sum a_i}{\sum b_i} \leq max(\frac{a_i}{b_i})$. Therefore, we may bound the fraction of sums with the maximum fraction for $o \in \mathbb{O}^>$, after which we divide by c_o:

$$\frac{C(NE)-C(OPT)}{C(NE)} \leq \max_{o\in\mathbb{O}^>} \frac{\Delta_o^> \frac{\overline{c}_o}{2} + d_o^*(\overline{d}_o - d_o^*)c_o}{\overline{d}_o^2 c_o + \left(\frac{\mathbb{U}^<}{k-|\mathbb{U}^<|} \overline{d}_o \frac{\overline{c}_o}{2} + \frac{k}{|\mathbb{U}^<|} \Delta_0^> \frac{\overline{c}_o}{2} \right)}. \tag{43}$$

$$\leq \max_{o\in\mathbb{O}^>} \frac{\Delta_o^> \frac{\overline{d}_o}{2} + d_o^*(\overline{d}_o - d_o^*)}{\overline{d}_o^2 + \left(\frac{\mathbb{U}^<}{k-|\mathbb{U}^<|} \frac{\overline{d}_o^2}{2} + \frac{k}{|\mathbb{U}^<|} \Delta_0^> \frac{\overline{d}_o}{2} \right)}. \tag{44}$$

Now we fix any $o \in \mathbb{O}^{>}$ for which the maximum is obtained. Since Lemma 2.1 suggests that the maximum is obtained when $\Delta_o^{>}$ is as large as possible, we assume $\Delta_o^{>} = \overline{d}_o - d_o^*$. Below, we also replace $\overline{d}_o = \beta d_o^*$ (and thus $\Delta_o^{>} = (\beta - 1)d_o^*$) and $k = \alpha|\mathbb{U}^{<}|$. Now in a few steps Inequality (44) becomes

$$\frac{C(NE) - C(OPT)}{C(NE)} \leq \frac{(\beta-1)d_o^* \frac{\beta d_o^*}{2} + d_o^*(\beta d_o^* - d_o^*)}{\beta^2 d_o^{*2} + \left(\frac{|\mathbb{U}^{<}|}{k-|\mathbb{U}^{<}|}\beta^2 \frac{d_o^{*2}}{2} + \frac{k}{|\mathbb{U}^{<}|}(\beta-1)\beta\frac{d_o^{*2}}{2}\right)} \tag{45}$$

$$\leq \frac{(\beta-1)d_o^* \frac{\beta d_o^*}{2} + d_o^*(\beta d_o^* - d_o^*)}{\beta^2 d_o^{*2} + \left(\frac{1}{\alpha-1}\beta^2 \frac{d_o^{*2}}{2} + \alpha(\beta-1)\beta\frac{d_o^{*2}}{2}\right)} \tag{46}$$

$$\leq \frac{\frac{1}{2}\beta^2 + \frac{1}{2}\beta - 1}{\beta^2 + \left(\frac{1}{\alpha-1}\beta^2\frac{1}{2} + \alpha(\beta-1)\beta\frac{1}{2}\right)} \tag{47}$$

where the last inequality is ensured by dividing over d_o^{*2}.

2.6 Results

The right hand side of Inequality (47) is a function with variables α and β, which we can maximize with wolframalpha for instance, yielding

$$\frac{C(NE) - C(OPT)}{C(NE)} \leq 0.260292\cdots \tag{48}$$

for $\alpha \approx 2.3$ and $\beta \approx 2.5$. Now we can easily calculate the required bound for the PoA:

$$\frac{C(NE) - C(OPT)}{C(NE)} \leq 0.260292\cdots \tag{49}$$

$$\frac{C(NE)}{C(OPT)} \leq \frac{1}{1 - 0.260292\cdots} = 1.35188\cdots. \tag{50}$$

This finishes the proof of Theorem 2.1. $\square$

3 Constructing the Lower Bound for the PoA

In this section we show the method used for constructing a game with maximal PoA. Then, an example is given to illustrate the result.

Theorem 3.1. $PoA \geq 1.35188$

PROOF: The proof is given by providing an example with $PoA = 1.35118$. After the example, an explanation of its construction follows. In either case, we need to set a few definitions first. These are similar to the sets from the previous chapter, but as this is an example and not a class of games, their notation is slightly different.

The set of players is partitioned into three distinct sets.

- I_1 expensive non-alternating players
- I_2 expensive alternating players
- J cheap players

The set of resources is partitioned into four distinct sets.

- O overloaded resources, with $c_o(x) = x$
- U underloaded resources, with $c_u(x) = \frac{1}{2}|U|$
- B_1 balanced resources used by alternating paths, with $c_b(x) = \frac{1}{2}|U|x$
- B_2 balances resources not used by alternating paths, with $c_b(x) = \frac{1}{2}|U|x$

Each player, regardless of type chooses k resources because of uniformity.

Each balanced resource is ever only chosen by one player, which means that because of its cost function $c_b(x) = \frac{1}{2}|U|x$ with $x = 1$, the cost of these resources are lowest in the game. Thus, given the rules described above, cheap players have the lowest possible cost per player (in NE).

Expensive players are separated in two subcategories: alternating and non-alternating expensive players. The non-alternating players do not change their action when moving from OPT to NE, while the alternating players do. Both types of expensive players choose overloaded and underloaded resources in the NE. When moving from OPT to NE, the alternating expensive players change their resources so that they only have underloaded and balanced resources.

Resources are split into three categories, overloaded resources O, underloaded resources U and balanced resources B.
The overloaded resources are chosen by more players in the NE than in the OPT. Their cost is equal to the number of players that chose the resource.
The underloaded resources are chosen by less players in the NE than in the OPT. Their costfunction is constant.
The balanced resources are split into two subcategories themselves, those that are used by Alternating Paths (APs), and those that are not. In NE there is no visible difference, but in OPT some balanced resources (B_1) are now connected to different players. The others (B_2) are still connected to the same player as in NE. Both have variable resources and are only ever chosen by one player in any given situation.

The goal is to construct a game with high social cost in NE but a low social cost in OPT. The following section describe an example of such a game.

3.1 Example of a (near) critical Game

Consider a game with $I = 50$ expensive players and $J = 39$ cheap players. Set the number of resources as follows: $|U| = 50$ underloaded resources with costfunction $c_u(x) = 25$, $|O| = 65$ overloaded resources with costfunction $c_o(x) = x$, $|B_1| = 1950$ balanced resources used by APs with costfunction $c_b(x) = 25x$ and $|B_2| = 2535$ balanced resources not used by APs with cost function $c_b(x) = 25x$. Each player chooses $k = 115$ resources.
We try to find a situation in which social costs are as high as possible and no player can reduce his own cost by unilaterally changing his action.

3.1.1 Examining a Nash Equilibrium

First, consider the cheap players. They choose only balanced resources B_1 and B_2. Every balanced resource is only chosen by one cheap player. We distribute it so that every cheap player chooses exactly 50 resources from B_1 and 65 from B_2. All balanced resources are now chosen by exactly one player.
There are two types of expensive players, but in NE they behave the same. Each expensive player chooses all resources from U and O.

Consider the cost $c_r(x)$ and opprotunity cost $c_r(x+1)$ of each resource. The balanced resources all have cost $c_b(x) = 25$ and opportunity cost $c_b(2) = 50$. The overloaded resources have cost $c_o(50) = 50$ and opportunity cost $c_o(51) = 51$.

The underloaded resources have cost and opportunity cost $c_u(50) = c_u(51) = 25$.

The Nash condition from Definition 1.8 is true for all cheap players, since their costs are lowest.
This condition is also true for all expensive players, as their opportunity costs are $c_b(2) = 50$, while their current cost functions are $c_u(50) = 25$ and $c_o(50) = 50$. So the balanced resources have opportunity cost equal to the current cost of the overloaded resources, which is not enough to warrant unilateral change.
Therefore we conclude that we are in a NE.

In the current example, social cost is determined by calculating the cost of each player and summing over all players:

$$
\begin{aligned}
C(NE) &= k * c_b(1)|J| + |O| * c_o(|I|)|I| + |U| * c_u(|I|)|I| && (51)\\
&= 115 * 25 * 39 + 65 * 50 * 50 + 50 * 25 * 50 && (52)\\
&= 337125 && (53)
\end{aligned}
$$

3.1.2 Examining the Optimum

In the OPT we try to achieve the lowest possible social cost. With the parameters as given above, we make the following changes to the situation in the NE:
Each cheap player changes his chosen resources from B_1 to U. Their cost will remain the same but it will make room for expensive players to choose resources in B_1 as their opportunity costs are now lower.
Each cheap player is now connected to each resource in U and to 65 resources in B_2. Each resource in B_2 is connected to only one player.

$I_2 = 30$ Expensive alternating players disconnect all from O and connect to B_1. There are 30 players and they each have 65 resourcces to connect to. Still, each resource in B_1 should only be chosen by one player, but the game is constructed so that the number of resources needed $= 30 * 65 = 1950$ is exactly the number provided.
$I_1 = 20$ Expensive non-alternating players do not change resources, but their individual cost still changes because of the other expensive players that do change.

Now consider the cost of the described situation. Cheap players still have a cost of 25 per resource. Expensive alternating players are now in essence cheap players, since all their costs are 25 as well, while the expensive non-alternating players are even cheaper than the rest, because the overloaded resources are now 20 per resource. The social cost is:

$$
\begin{aligned}
C(OPT) =& |B_2| * c_b(1) + |B_1| * c_b(1) + |O| * c_o(20) * 20 + && (54)\\
& |U| * c_u (|J| + |I|) * (|J| + |I|) && (55)\\
=& 2535 * 25 + 1950 * 25 + 50 * 25 * 89 + 65 * 20 * 20 && (56)\\
=& 249375 && (57)
\end{aligned}
$$

3.1.3 Graphical representation

Figure 4 gives a graphical representation. Black dots on the left represent Players, while similar dots on the right represent resources. A player chooses a resource if there is a connected line between the two. Full lines represent connections in NE while dotted lines represent connections in OPT. Obviously, there are many more players and resources than can be shown in any graphical representation, which is why the players and resources should be extrapolated along the vertical dotted lines, along with their connections.
As can be seen, all expensive players are connected to all overloaded and underloaded resources in NE, with only the alternating expensive players changing their action from overloaded to balanced, thus participating in APs.

3.2 Results

The PoA can be calculated from Eq. (53) and Eq. (57)

$$PoA = \frac{c(NE)}{c(OPT)} = \frac{337125}{249375} \approx 1.35188. \tag{58}$$

and concludes the proof of Theorem 3.1. □

As we see, this example yields a PoA equal to the upper bound provided in chapter 2 up to 5 decimal points. As the proof of the upper bound was beholden to instances where the number of players tended to infinity, there is still a possibility for a gap between our two bounds. However, both bounds are close enough to each other to make this gap irrelevant.

Still, instances might occur where the Price of Anarchy is higher than the upper bound when the number of players is small. We did not find any such games, but it might be interesting for future research to investigate this matter. One could either find such instances or (preferably) prove the upper bound for any number of players.

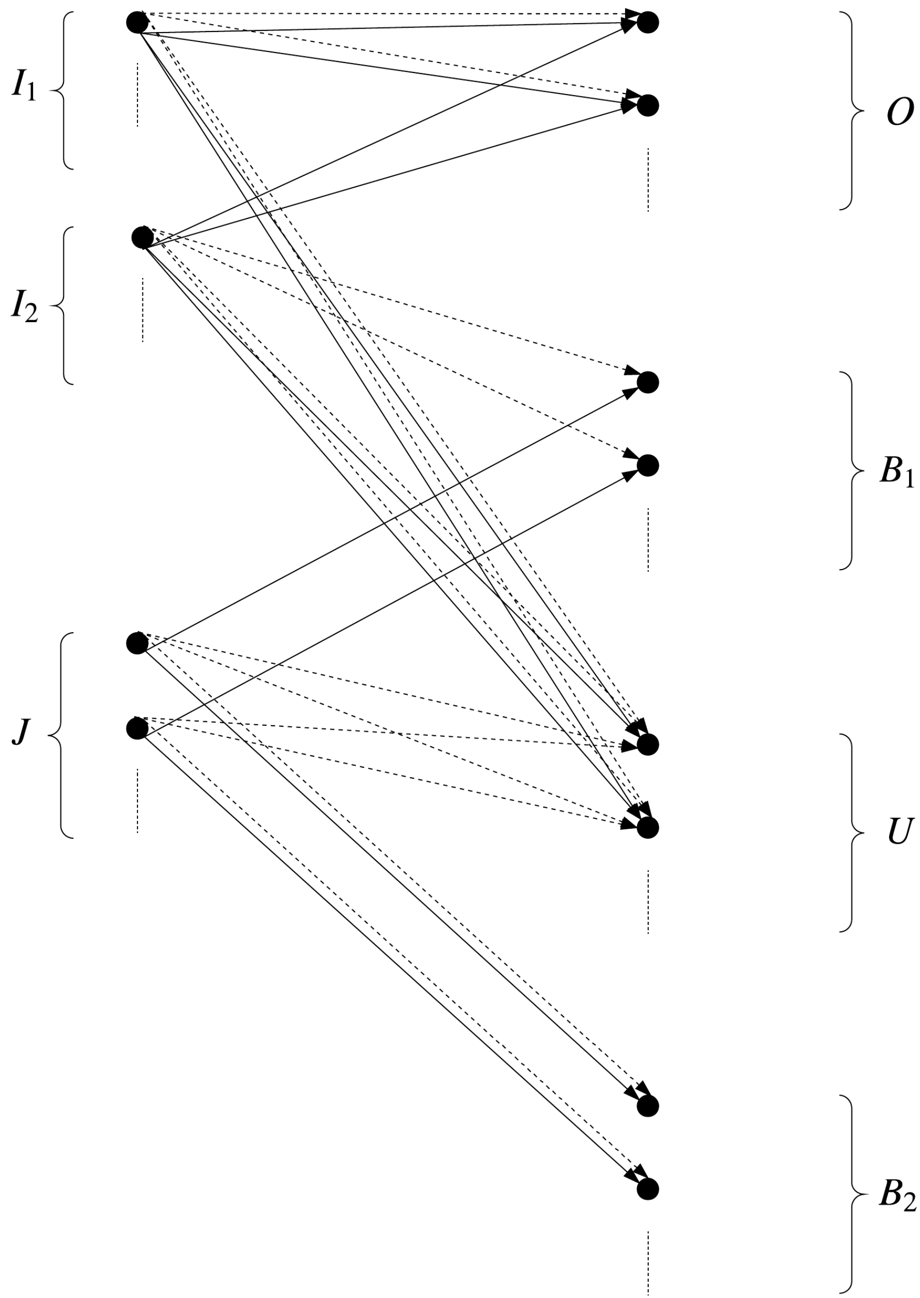

Figure 4: graphical representation of the example. Edges in NE have full lines and edges in OPT have dotted lines

4 Conclusions

4.1 Theoretical results

This thesis shows that the Price of Anarchy for affine, symmetric, k-uniform congestion games is $1.35188\cdots$ in critical cases for a large number of players. This conclusion was reached in two parts.
Chapter 2 proved a tight upper bound for the Price of Anarchy of $1.35188\cdots$.
This result was achieved by assuming the number of players went to infinity.
Chapter 3 provided an example of a game with a Price of Anarchy of ≈ 1.35188, thus proving a lower bound. The example had 89 players, which is sufficiently large.

4.2 Recommendations

While writing this thesis, the following recommendations for future research are made by the author.

4.2.1 Price of Anarchy for any number of players

It would be interesting to find a general expression for the Price of Anarchy that either depends on the number of players or does not require the number of players to reach infinity. We set out to write this thesis with that goal in mind but were unable to come to this expression.

4.2.2 Price of Stability

The sole purpose of this thesis was to examine the Price of Anarchy to analyse worst case scenarios. An interesting research topic might be to find an expression for the Price of Stability for the same class of games. While the Price of Anarchy divides the highest possible NE by the OPT, the Price of Stability divides the lowest possible NE by the OPT.

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