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Disturbance Decoupling in Graphs

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Abstract

A graph can be interpreted as a discrete time system where information moves from a vertex to the neighbors at each step. Disturbances can change the information at these vertices. Can such a disturbance occurring in one or multiple vertices of a graph be counteracted by controlling another vertex? In this report we study this so-called disturbance decoupling problem, and the recursive algorithm that can solve it for discrete time systems. We rewrite simple graph structures as discrete systems and solve the problem for these basic examples. By observing what happens in the graph in each step of the recursive algorithm we are able to form hypotheses on the relation between disturbance decoupling and the graph structure. Based on these hypotheses we proved theorems that solved the problem for certain types of graphs, like graphs where each vertex has no more than one incoming edge. For other graphs, where vertices may have multiple incoming edges, we have a strong suspicion on how to solve the disturbance decoupling problem.

Contents

1 Introduction

Consider the network in Figure 1, where information is shifted from vertex to vertex via the directed edges in fixed time-steps. In the vertex 3 the information can be read by the observer y and in vertex 2 an outside controller u can change the information. Furthermore, it is possible that in every single vertex some disturbance d occurs that changes the information at that vertex. For example, in Figure 1 there is a disturbance d in vertex 1.

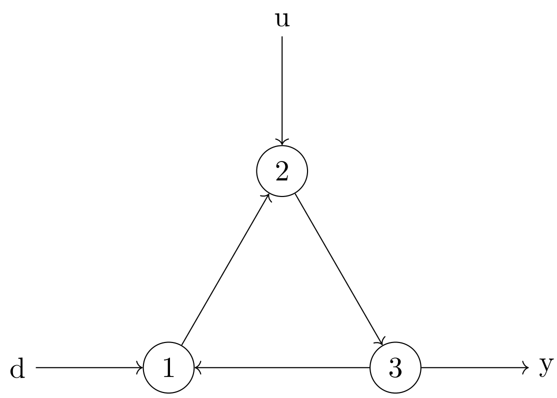


Figure 1: Example of a graph

We want to know if there is a control u such that the disturbance d cannot be observed by y. In other words if the disturbance d can be counteracted by the controller u.

Now suppose we have a similar network with one observer and one controller but with n vertices instead of 3. Again we want to know if the disturbance d acting on one vertex or multiple vertices can be counteracted by choosing the right control u.

Formally stated our research question is:
In a network where information travels with constant speed, which is observed at a single vertex and which can be controlled at another vertex: in what subpart of the network can disturbances acting on one or multiple vertices be counteracted by the controller.

2 Problem statement and preliminaries

In systems theory the problem stated in the introduction is known as the disturbance decoupling problem (DDP) and has been solved by W. Murray Wonham and others using linear algebra, see e.g. [1, Chapter 4]. Therefore it is useful to first show the known solution to this problem for a discrete time system and then use this solution to answer the research question.

In this section we will briefly explain the disturbance decoupling problem for a discrete time system and provide a solution as can be found in literature [1, Section 4.3]. Then we will explain how to rewrite the structure of the graph to a discrete time system. To help the reader we compiled a list of all the symbols and notations used in this report. This can be found in Section 7.

2.1 DDP for a discrete-time system

Consider the discrete time system, described by the set of equations:

$$x(k+1) = Ax(k) + Bu(k) + Sd(k) \tag{2.1}$$
$$y(k) = Cx(k) \tag{2.2}$$

where $x \in \mathscr{X}$, the *state space*, $u \in \mathbb{R}$ and $d \in \mathbb{R}$. Here x is a real vector of dimension n and A, B, S, C are real and constant matrices of the proper sizes.

Solving equation (2.1) for $x(k)$ when $u(k) = Fx(k)$ and $F : \mathscr{X} \to \mathbb{R}$ gives:

$$x(k) = (A+BF)^k x(0) + \sum_{i=1}^{k} (A+BF)^{k-i} Sd(i-1). \tag{2.3}$$

Hence the solution for the output $y(k)$ is:

$$y(k) = C(A+BF)^k x(0) + C\left[\sum_{i=1}^{k} (A+BF)^{k-i} Sd(i-1)\right]. \tag{2.4}$$

Now we can define disturbance decoupling as follows.

Definition 1. *The system* (2.1) (2.2) *is said to be disturbance decoupled if there exists a mapping* $F : \mathscr{X} \to \mathbb{R}$ *such that for each initial state* $x(0) \in \mathscr{X}$ *the output* $y(k)$ *is the same for every disturbance* $d(\cdot)$*. In other words, the forced response* $C\left[\sum_{i=1}^{k} (A+BF)^{k-i} Sd(i-1)\right]$ *must be zero for the sequence* $d(\cdot)$*, and for every* $k = 0, 1, 2, \ldots$*.*

Now setting $\mathscr{S} := \operatorname{im}(S)$, $\mathscr{K} := \ker(C)$ and $\langle A+BF \mid \mathscr{S}\rangle := \mathscr{S} + (A+BF)\mathscr{S} + \ldots + (A+BF)^{n-1}\mathscr{S}$ we arrive at the following result.

Theorem 2. *By Definition 1 we have that the system* (2.1)*,* (2.2) *is disturbance decoupled if and only if there exists a mapping* $F : \mathscr{X} \to \mathbb{R}$ *such that:*

$$\langle A+BF \mid \mathscr{S}\rangle \in \mathscr{K}. \tag{2.5}$$

Proof. The system (2.1),(2.2) is disturbance decoupled, if and only if:

$$C\left[\sum_{i=1}^{k} (A+BF)^{k-i} Sd(i-1)\right] = 0 \text{ for } k = 0,1,2,\ldots \ \forall d(\cdot) \in \mathbb{R}$$
$$\Leftrightarrow \ C\left[\sum_{i=1}^{k} (A+BF)^{k-i} S\mathbb{R}\right] = 0 \text{ for } k = 0,1,2,\ldots$$
$$\Leftrightarrow \ C\left[\sum_{i=1}^{k} (A+BF)^{k-i} \mathscr{S}\right] = 0 \text{ for } k = 0,1,2,\ldots.$$

Because $\mathbb{R}$ is the domain of S and consequently $S\mathbb{R}$ is the image of S.

Using the Cayley-Hamilton theorem we know that $(A+BF)^k$ for every $k=0,1,2,\ldots$ is a linear combination of $I,(A+BF),(A+BF)^2,\ldots,(A+BF)^{n-1}$ and so $C\left[\sum_{i=1}^{k}(A+BF)^{k-i}\mathscr{S}\right]=0$ for $k=0,1,2,\ldots$ if and only if:

$$
\begin{aligned}
&C\left[\sum_{i=1}^{k}(A+BF)^{k-i}\mathscr{S}\right]=0 \text{ for } k=0,1,2,\ldots,n-1\\
&\Leftrightarrow C\left\langle A+BF \mid \mathscr{S}\right\rangle=0\\
&\Leftrightarrow \left\langle A+BF \mid \mathscr{S}\right\rangle\in\mathscr{K}.
\end{aligned}
$$

□

2.2 Solution to the DDP in a discrete time system

The goal of this subsection is to find a solution to the disturbance decoupling problem for the system in (2.1) (2.2). In other words we want to find the "largest" subspace $\mathscr{V}\subset\mathscr{X}$, such that if $\mathscr{S}\subset\mathscr{V}$ then the system is disturbance decoupled.

By Theorem 2 we see that the subspace $\langle A+BF\mid\mathscr{S}\rangle$ satisfies two properties. First $\langle A+BF\mid\mathscr{S}\rangle$ is a part of the kernel of C ($\langle A+BF\mid\mathscr{S}\rangle\subset\mathscr{K}$) and second if the state $x(0)\in\langle A+BF\mid\mathscr{S}\rangle$, then $(A+BF)^kx(0)\in\langle A+BF\mid\mathscr{S}\rangle$, which means that $\langle A+BF\mid\mathscr{S}\rangle$ is (A,B)-*invariant.*

Definition 3. *A subspace $\mathscr{V}\subset\mathscr{X}$ is (A,B)-invariant if there exists a linear mapping $F:\mathscr{X}\to\mathbb{R}$ such that*

$$
(A+BF)\mathscr{V}\subset\mathscr{V}. \tag{2.6}
$$

So we are looking for subspaces $\mathscr{V}$ that are (A,B)-invariant and contained in the kernel of C. To accomplish this we construct the class $\mathfrak{J}(A,B;\mathscr{K})$, containing all the (A,B)-invariant subspaces contained in $\mathscr{K}$. We can define $\mathfrak{J}(A,B;\mathscr{K})$ in two ways, namely via the closed loop and the open loop system.

Definition 4. *$\mathfrak{J}(A,B;\mathscr{K})_{cl}$ is the class of subspaces such that $\mathscr{V}\in\mathfrak{J}(A,B;\mathscr{K})_{cl}$ if and only if there is a mapping $F:\mathscr{X}\to\mathbb{R}$ such that for every $x(0)\in\mathscr{V}$ both $(A+BF)^kx(0)\in\mathscr{V}$ and $C(A+BF)^kx(0)=0$ for $k=0,1,2,\ldots$.*

Definition 5. *$\mathfrak{J}(A,B;\mathscr{K})_{ol}$ is the class of subspaces such that $\mathscr{V}\in\mathfrak{J}(A,B;\mathscr{K})_{ol}$ if and only if for every $x(0)\in\mathscr{V}$ there exists a control $u(k)$ for $k=0,1,2,\ldots$ such that $x(k)\in\mathscr{V}$ and $y(k)=Cx(k)=0$ for $k=0,1,2,\ldots$.*

We can show that these two definitions are equivalent by choosing $u(k)=Fx(k)$, see Theorem 28 in Appendix A.

Next we define $\mathscr{V}^*$ as the supremal element of $\mathfrak{J}(A,B;\mathscr{K})$. The *supremal element* $\sup\mathfrak{B}$ of a class $\mathfrak{B}$ is an element in $\mathfrak{B}$ (if it exists) such that if $\mathscr{V}\in\mathfrak{B}$ then $\mathscr{V}\subseteq\sup\mathfrak{B}$. We can prove that the supremal element of $\mathfrak{J}(A,B;\mathscr{K})$ exists (see Appendix A).

Theorem 6. *Let $\mathscr{V}^*$ be the supremal element of $\mathfrak{J}(A,B;\mathscr{K})$. Then $\mathscr{S}\subset\mathscr{V}^*$ if and only if the system* (2.1) (2.2) *is disturbance decoupled.*

Proof. Suppose $\mathscr{S} \subset \mathscr{V}^*$. Since $\mathscr{V}^*$ is (A, B)-invariant there exists $F : \mathscr{X} \to \mathbb{R}$ such that (2.6) holds. Then

$$\langle A + BF \mid \mathscr{S} \rangle \subset \langle A + BF \mid \mathscr{V}^* \rangle \subset \mathscr{V}^* \subset \mathscr{K}.$$

By Theorem 2 this means that the system is disturbance decoupled.

Conversely, suppose the system is disturbance decoupled. Then there exists an $F : \mathscr{X} \to \mathbb{R}$ such that $\langle A + BF \mid \mathscr{S} \rangle \subset \mathscr{K}$ and thus $\langle A + BF \mid \mathscr{S} \rangle \in \mathfrak{J}(A, B; \mathscr{K})$. So:

$$\mathscr{S} \subset \langle A + BF \mid \mathscr{S} \rangle \subset \mathscr{V}^*.$$

$\square$

For this report $\mathscr{V}^* = \sup \mathfrak{J}(A, B; \mathscr{K})$ will be computed as in Wonham [1, Chapter 4.3], see Theorem 7.

Theorem 7. *Let $A : \mathscr{X} \to \mathscr{X}$, $B : \mathbb{R} \to \mathscr{X}$, $\mathscr{B} = \operatorname{im}(B)$ and $\mathscr{K} \subset \mathscr{X}$ with $\dim(\mathscr{K})$ the dimension of $\mathscr{K}$. If we define the sequence $\mathscr{V}^\mu$ as:*

$$\mathscr{V}^0 = \mathscr{K} \tag{2.7}$$

$$\mathscr{V}^\mu = \mathscr{K} \cap A^{-1}\left(\mathscr{B} + \mathscr{V}^{\mu-1}\right), \tag{2.8}$$

then $\mathscr{V}^\mu \subset \mathscr{V}^{\mu-1}$ and for some $m \leq \dim(\mathscr{K})$

$$\mathscr{V}^\mu = \sup \mathfrak{J}(A, B; \mathscr{K}) = \mathscr{V}^* \tag{2.9}$$

for all $\mu \geq m$.

The operation A^{-1} is defined as follows by Wonham [1, Chapter 0.4]. If $\mathscr{V} \subset \mathscr{X}$ then

$$A^{-1}(\mathscr{V}) := \{x \in \mathscr{X} \mid Ax \in \mathscr{V}\}. \tag{2.10}$$

2.3 Constructing the feedback matrix to solve the DDP

Now that we can find $\mathscr{V}^*$ the question still remains how to construct a feedback matrix $F : \mathscr{X} \to \mathbb{R}$ such that $(A + BF)\mathscr{V}^* \subset \mathscr{V}^*$. If F has that property we will call F a *friend* of $\mathscr{V}^*$ and write $F \in \mathbf{F}(\mathscr{V}^*)$. Because $u \in \mathbb{R}$ we can prove $F \in \mathbf{F}(\mathscr{V}^*)$ is either unique or zero depending on B.

Theorem 8. *Let $\mathscr{V} \in \mathfrak{J}(A, B; \mathscr{K})$. Then*

(a) $B \notin \mathscr{V} \Rightarrow F$ is unique on V;

(b) $B \in \mathscr{V} \Rightarrow F = 0$.

Proof. (a) Suppose $F_1, F_2 \in \mathbf{F}(\mathscr{V})$. Then

$$(A + BF_1)\mathscr{V} \subset \mathscr{V},$$
$$(A + BF_2)\mathscr{V} \subset \mathscr{V}.$$

That means that for every $\nu \in \mathscr{V}$ we have

$$(A + BF_1)\nu \subset \mathscr{V} = \nu_1 \in \mathscr{V}, \tag{2.11}$$
$$(A + BF_2)\nu \subset \mathscr{V} = \nu_2 \in \mathscr{V}. \tag{2.12}$$

Because the subspace $\mathscr{V}$ is closed under addition we can subtract (2.11) en (2.12) and find

$$\begin{aligned}
&(A + BF_1)\nu - (A + BF_2)\nu = \nu_1 - \nu_2 \in \mathscr{V} \\
&\Leftrightarrow BF_1\nu - BF_2\nu \in \mathscr{V} \\
&\Leftrightarrow B(F_1 - F_2)\nu \in \mathscr{V}.
\end{aligned}$$

Because $B \notin \mathscr{V}$, $Bu \in \mathscr{V}$ if and only if $u = 0$ and thus

$$\begin{aligned}
&\forall \nu \in \mathscr{V} : \ (F_1 - F_2)\nu = 0 \\
&\Rightarrow \forall \nu \in \mathscr{V} : \ F_1\nu = F_2\nu.
\end{aligned}$$

Hence F is unique on $\mathscr{V}$.

(b) We know that $\mathscr{V}$ is (A, B)-invariant so by Lemma 29 in Appendix A $A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$. Since $B \in \mathscr{V}$ we have

$$\begin{aligned}
&\forall u \in \mathbb{R} : Bu \in \mathscr{V} \\
&\Rightarrow \mathscr{B} \subset \mathscr{V}.
\end{aligned} \tag{2.13}$$

We know $A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$, so by 2.13 we have $A\mathscr{V} \subset \mathscr{V}$. This means that

$$(A + BF)\mathscr{V} \subset \mathscr{V} \text{ for } F = 0.$$

□

Now following Wonham [1, Chapter 4.2] we can construct the unique F on $\mathscr{V}^*$ as follows.

Definition 9. *Let $\{\nu_1, \nu_2, \ldots, \nu_m\}$ be a basis for $\mathscr{V}^*$ and choose $u_i \in \mathbb{R}$ such that $A\nu_i = \omega_i - Bu_i$ (these exist by Lemma 29 in Appendix A). We define $F_0 : \mathscr{V}^* \to \mathbb{R}$ by*

$$F_0\nu_i = u_i \text{ for } i \in \{1, 2, \ldots m\} \text{ and } u_i \in \mathbb{R}. \tag{2.14}$$

Then F_0 is unique on $\mathscr{V}^*$ and any linear extension F of F_0 to $\mathscr{X}$ is a friend of $\mathscr{V}^*$.

2.4 Network as a discrete time system

Now that we found a solution to the disturbance decoupling problem for a discrete time system the question remains how to solve the disturbance decoupling problem in a network such as in Figure 1. We only consider graphs with dynamics satisfying the following assumption.

Assumption 10. *Graph $G = (V, E)$ is directed and*

1. *G has n vertices.*
2. *Only one vertex v_y can be observed.*

3. *Only one vertex v_u can be controlled.*
4. *The controlled vertex and the observed vertex do not coincide.*
5. *The edge (i, j) means that (a part of) the information that was in x_i on time k is in x_j on time $k + 1$.*

To solve the disturbance decoupling problems in these kinds of graphs we use the structure of the graph to construct the matrices A, B and C.

We start with the construction of A. In a discrete time system the element A_{ji} tells you how much of the information that was in x_i on time k is in x_j on time $k+1$, just like the edge (i, j). Therefore it makes sense to define the matrix A based on the out-degree of the vertices in the graph. The *out-degree*, d_v^{out}, of vertex v is the number of edges that end in v. Now matrix A is defined as the $n \times n$-matrix such that:

$$\begin{cases} A_{ji} = f(d_i^{out}) & \text{if } (i,j) \in E \\ A_{ji} = 0 & \text{if } (i,j) \notin E \end{cases}. \tag{2.15}$$

Here f is some function on the out-degree, such that $f(1) = 1$ and $f(k) > 0$ for $k > 1$. You can see that the value of A_{ij} is dependent on the out-degree of vertex i. That means that if the information in i splits along multiple edges, only a part of the original information travels along each edge.

The construction of matrices B and C is fortunately a lot easier. Since we observe only at the vertex with index y, matrix C is a row vector where the y-th entry equals 1 and the other $n-1$ entries are zero. Similarly, matrix B is a column-vector where the u-th entry equals 1 and the other $n-1$ entries are zero. The 4th item of Assumption 10 implies that B never equals C^T, the transpose of C.

For example the matrices, A, B and C in the graph of Figure 1 will be:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

To use the algorithm in Theorem 7 we also need to define the subspaces $\mathscr{K}$ and $\mathscr{B}$.

$\mathscr{K}$ is the kernel of C. So $\mathscr{K}$ contains all the states $x \in \mathscr{X}$ for which $Cx = 0$. $Cx = 0$ if and only if the y-th entry of the state x is 0. Then for $\mathscr{K}$ we can find

$$\mathscr{K} = \text{Span}\,\{e_i \mid i \neq y\}. \tag{2.16}$$

In a similar manner we can find $\mathscr{B}$. $\mathscr{B}$ is the image of B. That means $\mathscr{B}$ is B times the domain of B, which is $\mathbb{R}$. Since the u-th entry is the only non-zero entry of B we find

$$\mathscr{B} = B\mathbb{R} = \text{Span}\,\{e_u\}. \tag{2.17}$$

3 Disturbance Decoupling in basic graphs

In this section Theorem 7 is used to compute $\mathscr{V}^*$ for several basic graphs. These graphs all satisfy Assumption 10. Based on our observations during the computations made on these examples we will form some hypotheses that will be studied in later sections.

To make the subspaces $\mathscr{V}^\mu$ visible in the graph, we will introduce the vertex sets D^μ and O^μ.

Definition 11. *D^μ is a set of vertices ($D^\mu \subset V$) such that*

$$\{e_i \mid i \in D^\mu\}$$

forms a basis for $\mathscr{V}^\mu$. In particular $\mathscr{V}^* = \mathrm{Span}\,\{e_i \mid i \in D^*\}$.

Please note that this definition is only valid if $\mathscr{V}^*$ is indeed the span of standard unit vectors.

Definition 12. *O^μ is the complement of D^μ in V.*

In this section we only briefly discuss the calculation of $\mathscr{V}^*$. For a more precise computation of $\mathscr{V}^*$, we refer the reader to Appendix B.

3.1 Directed line graph

A *directed line graph* $\vec{L}_n$ is a graph $G = (V, E)$ whose vertices can be listed in the order $\{v_1, v_2, \ldots, v_n\}$ such that: $E = \{(v_i, v_j) \in V \times V : j - i = 1\}$ (Fagnani & Frasca [3, Chapter 2])

Now suppose we have a line-graph where vertex y is observed, see Figure 2. Note that in this figure y is both the output and the index of the observed vertex.

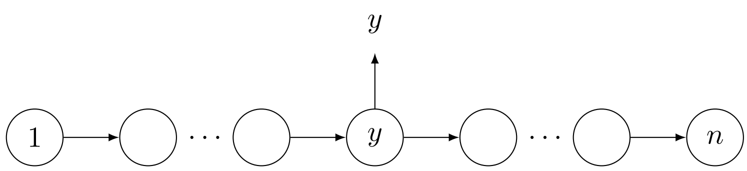


Figure 2: A directed linegraph $\vec{L}_n$ where one vertex is observed

As in Section 2.4 the structure of the graph is used to construct the matrices A, B, and C.

Since there is no splitting of information we have $d_i^{out} = 1$ and thus

$$\begin{cases} A_{ji} = 1 & \text{if } (i,j) \in E \\ A_{ji} = 0 & \text{if } (i,j) \notin E \end{cases} \tag{3.1}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

C is a row-vector where the y-th entry is 1 and the other $n-1$ entries are zero.

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \underbrace{1}_{y} & 0 & \cdots & 0 \end{bmatrix}.$$

There are two spots to put the controller that might result in a fundamentally different $\mathscr{V}^*$. The controller can either be on the right, u_R, or on the left, u_L, of the observer. Therefore we also have two different B matrices, depending on which vertex is controlled.

$$B_{\text{control right}} = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{0}_{y} & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T.$$

$$B_{\text{control left}} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \underbrace{0}_{y} & \cdots & 0 \end{bmatrix}^T.$$

In Appendix B we used Theorem 7 to compute $\mathscr{V}^*$ for both situations. The results are shown in Figures 3 and 4. There the vertices in set D^* are white while the vertices in set O^* are black. We chose this color-coding because only disturbances occuring in the open, white, vertices and not in the closed, black, vertices can be decoupled.

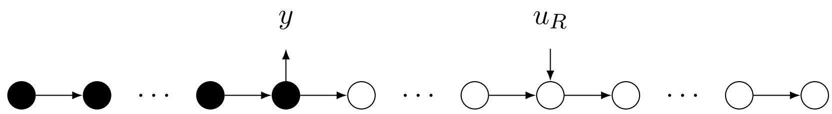


Figure 3: $\mathscr{V}^*$ in a directed linegraph $\vec{L}_n$ where controller u_R is to the right of observer y.

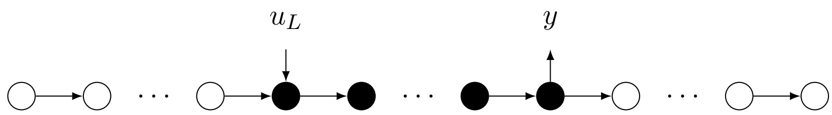


Figure 4: $\mathscr{V}^*$ in a directed linegraph $\vec{L}_n$ where controller u_L is to the left of observer y.

3.2 Directed circle graph

A *directed circle graph* $\vec{C}_n$ is a graph $G = (V, E)$ whose vertices can be listed in the order $\{v_1, v_2, \ldots, v_n\}$ such that: $E = \{(v_i, v_j) \in V \times V : j - i = 1 \bmod n\}$ (Fagnani & Frasca [3, Chapter 2]).

Now suppose that we have a directed circle graph where only the vertex n is observed and a different vertex u can be controlled, see Figure 5.

As before we used the structure of the graph to find the matrices A, B and C.

To compute matrix A we used Definition 3.1, because there is no splitting of information, just as in Section 3.1 and we find

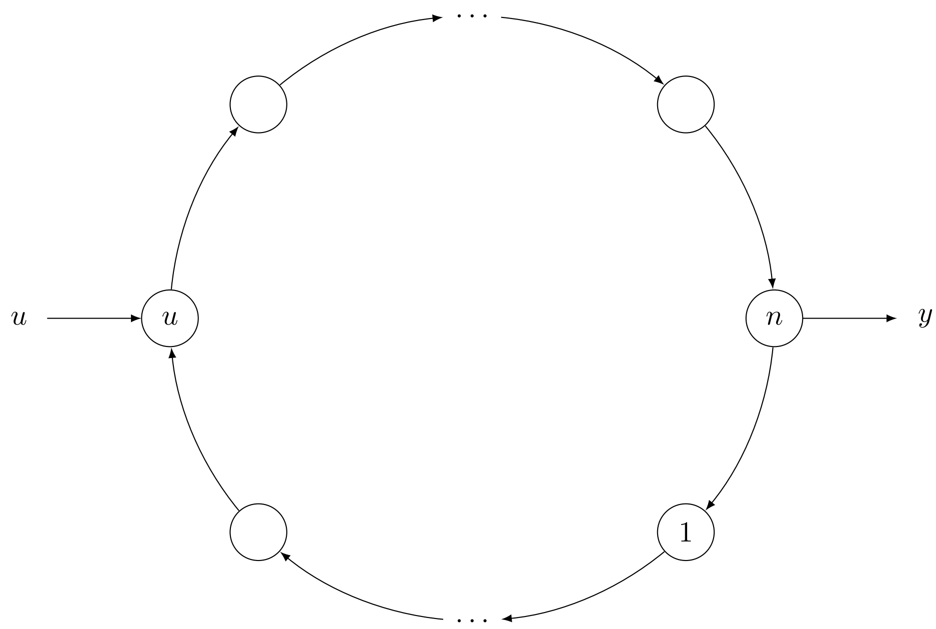


Figure 5: A directed circle graph $\vec{C}_n$ with one observed and one controlled vertex

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & \underbrace{1}_{u} & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

With these matrices A, B and C, we compute $\mathscr{V}^*$ using Theorem 7 (see Appendix B). In Figure 6 you can see what O^* and D^* look like. Again the white vertices are elements of D^* and the black vertices are elements of O^*.

3.3 Graphs where vertices have multiple outgoing edges

In the previous examples every vertex had only one incoming and one outgoing edge. It might be interesting to see what happens if one or more vertices have multiple outgoing edges. As an example we chose the graph $\vec{G}_{\text{split}}$ of Figure 7. In that graph there is a vertex l with two outgoing edges, $(l, l+1)$ and $(l, m+1)$.

Since there the information splits in vertex l the definition of A needs to be slightly alterated. We will assume that half of the information in l travels to $l+1$ and the other half to $m+1$.

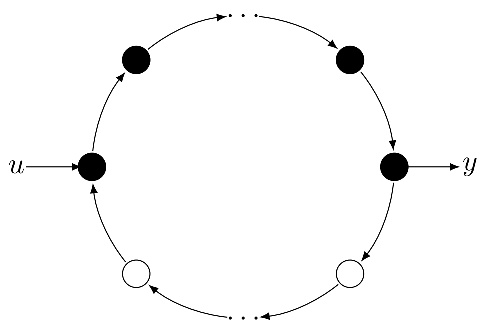


Figure 6: $\mathscr{V}^*$ in a directed circle graph $\vec{C}_n$ with one observed and one controlled vertex

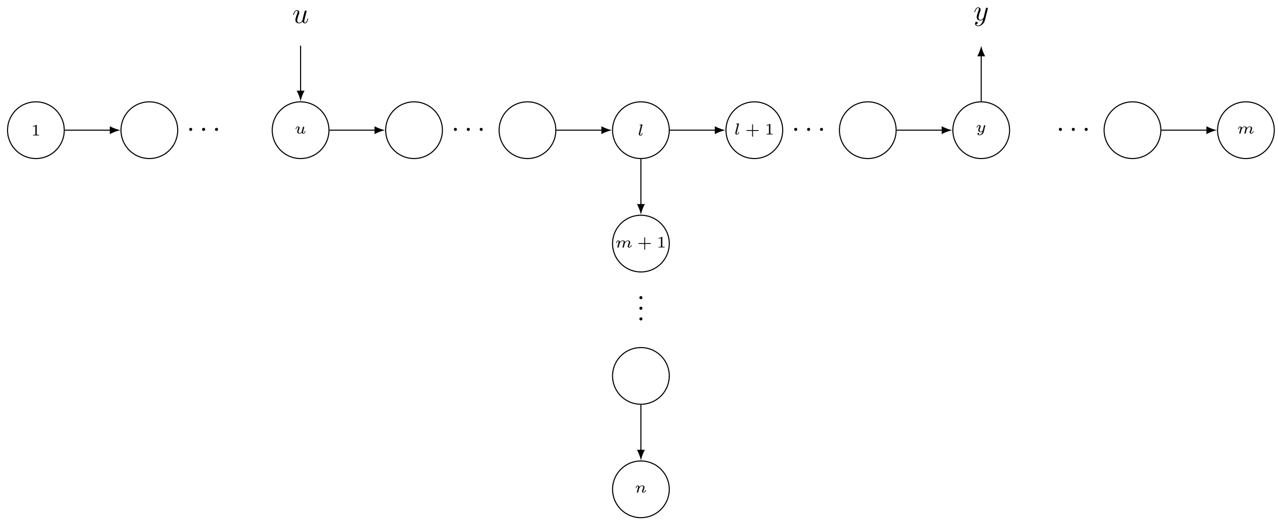


Figure 7: Graph $\vec{G}_{\text{split}}$ where vertex l has two outgoing edges, vertex u is controlled and vertex y is observed.

Thus we define A as follows:

$$\begin{cases} A_{ji} = \frac{1}{2} & \text{if } i = l \text{ and } \ (l,j) \in E \\ A_{ji} = 1 & \text{if } i \neq l \text{ and } \ (i,j) \in E \\ A_{ji} = 0 & \text{elsewhere} \end{cases} \tag{3.2}$$

Using equation (3.2) the matrix A is as follows.

$$A = \begin{bmatrix}
0 & 0 & 0 & & & & & \cdots & & & & 0 & 0 \\
1 & 0 & 0 & & & & & \cdots & & & & 0 & 0 \\
0 & 1 & 0 & & & & & \cdots & & & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & & & & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & & & & \cdots & & & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & & & \cdots & & & 0 & 0 \\
 & & & & 0 & 1 & 0 & & & & & & \\
\vdots & \vdots & & & & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\
 & & & & & & 0 & 1 & 0 & & & & \\
0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & & & & & & & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & & & & & \cdots & & & & 0 & 1 & 0
\end{bmatrix}.$$

Matrix B is a column-vector where only the u-th entry, between the first and the l-th entry, is non-zero and equal to 1.

$$B = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{u} & 0 & \cdots & 0 & \underbrace{0}_{l} & 0 & \cdots & 0 \end{bmatrix}^T.$$

Matrix C is just as in Section 3.1 a row-vector where the y-th entry, between the l-th and m-th entry, is 1 and the other $n-1$ entries are zero.

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \underbrace{0}_{l} & 0 & \cdots & 0 & \underbrace{1}_{y} & 0 & \cdots & 0 & \underbrace{0}_{m} & 0 & \cdots & 0 \end{bmatrix}.$$

For these matrices $\mathscr{V}^*$ is computed in Appendix B. The results can be seen in Figure 8 where the elements of D^* are white and the elements of O^* black.

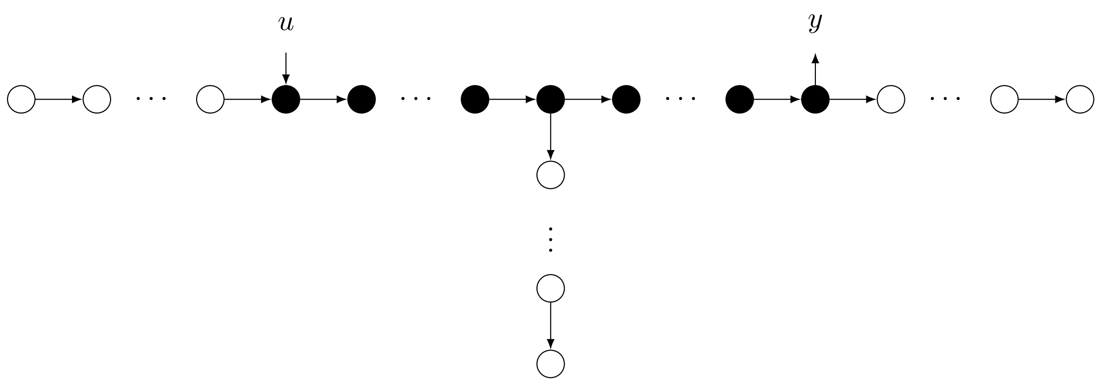


Figure 8: $\mathscr{V}^*$ in graph $\vec{G}_{\text{split}}$.

3.4 Graphs where vertices have multiple incoming edges

It can also happen that a graph has multiple incoming edges, see for example Figure 9 with graph $\vec{G}_{\text{incoming}}$. In that graph there is a vertex l with two incoming edges, $(l-1, l)$ and (n, l).

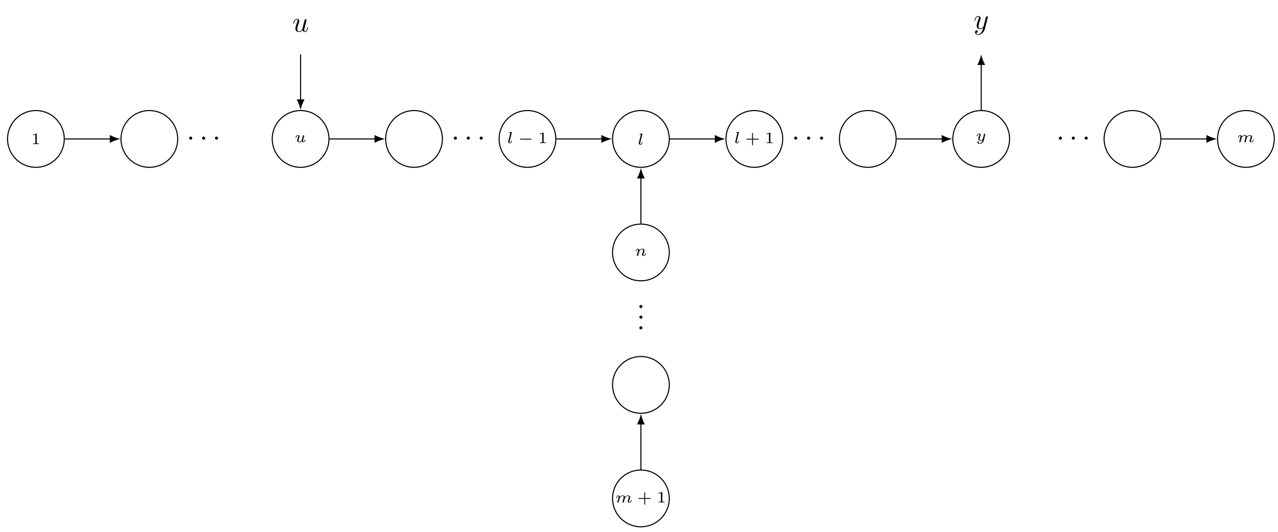


Figure 9: Graph $\vec{G}_{\text{incoming}}$ where vertex l has two incoming edges, vertex u is controlled and vertex y is observed.

In this graph there is no splitting of information and we can use the definition for matrix A in (3.1) to construct the following matrix. This matrix is slightly different from the matrix A in Section 3.1 because row $l+1$ has a second non-zero entry ($(A)_{l+1,n} = 1$) and row $m+1$ consists entirely of zeros.

$$A = \left[\begin{array}{cccccccccccccc}
0 & & & & & & & & & & \cdots & & 0 & 0 \\
1 & 0 & & & & & & & & & \cdots & & 0 & 0 \\
0 & 1 & 0 & & & & & & & & \cdots & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & & & & & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & & & & & & \cdots & & 0 & 0 \\
0 & & \cdots & 0 & 1 & 0 & & & & & \cdots & & 0 & 1 \\
0 & & \cdots & & 0 & 1 & 0 & & & & \cdots & & 0 & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & & & & & \vdots & \vdots \\
0 & & \cdots & & & & 0 & 1 & 0 & & \cdots & & 0 & 0 \\
0 & & \cdots & & & & & 0 & 0 & 0 & \cdots & & 0 & 0 \\
0 & & \cdots & & & & & & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & & \cdots & & & & & & & & 0 & 1 & 0 & 0 \\
0 & & \cdots & & & & & & & & & 0 & 1 & 0
\end{array}\right]$$

Matrix B is still a column-vector where only the u-th entry, between the first and the l-th entry, is non-zero and equal to 1.

$$B = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{u} & 0 & \cdots & 0 & \underbrace{0}_{l} & 0 & \cdots & 0 \end{bmatrix}^T$$

Matrix C is a row-vector where the y-th entry, between the m-th and n-th entry, is 1 and the other $n-1$ entries are zero.

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \underbrace{0}_{m} & 0 & \cdots & 0 & \underbrace{1}_{y} & 0 & \cdots & 0 \end{bmatrix}.$$

The computation for $\mathscr{V}^*$ with these matrices can be found in Appendix B again.

However in this case $\mathscr{V}^*$ is not the span of unit vectors, so we do not have the vertex sets D^* and O^*. Instead, we introduce two new vertex sets D_μ and O_μ.

Definition 13. *Let $\mathscr{E}_\mu$ be the largest span of unit vectors, such that $\mathscr{E}_\mu \subset \mathscr{V}^\mu$. Then $i \in D_\mu$ if and only if $e_i \in \mathscr{E}_\mu$. In particular $\mathscr{E}_*$ is the largest span of unit vectors such that $\mathscr{E}_* \subset \mathscr{V}^*$.*

Definition 14. *O_μ is the complement of D_μ in $\mathscr{V}$.*

In Figure 10 we visualized D_* and O_* by coloring the vertices in D_* white and the vertices in O_* black. Here u' is a vertex such that $d(u', l) = d(u, l)$. Here $d(i, j)$ is defined as *the distance*, the length of shortest path between vertex i and j. That means that the path from u' to l has the same length as the path from u to l. If however there is no such u', because $d(m+1, l) < d(u, l)$ the entire path $(m+1, n)$ belongs to O_* and is subsequently colored black.

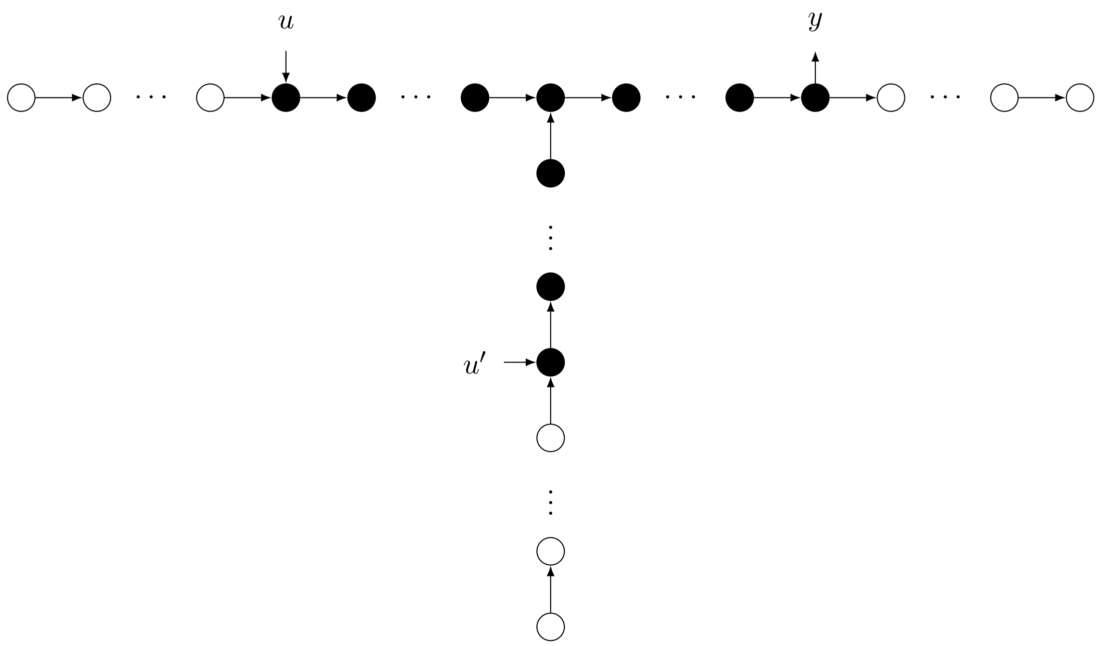


Figure 10: $\mathscr{V}^*$ in graph $\vec{G}_{\text{incoming}}$.

3.5 Observations made during the computation

While computing $\mathscr{V}^*$ for these examples we made several observations in the graph, see also Appendix B. We give a summary of these observations.

Firstly, all the vertices that cannot reach vertex y are always contained in D^μ. Secondly, only the vertex in D^μ that has an outgoing edge to O^μ is added to $O^{\mu+1}$. Finally, in every example where vertices have no more than one incoming edge only the path from u to y, if it exists, is a part of O^*.

However, if there is a vertex with more than one incoming edge, $\mathscr{V}^*$ is not the span of unit vectors. In that case, we saw that all the vertices that cannot reach vertex y are contained in D_μ. Also if there exists a vertex u' such that $d(u', y)$ is equal to $d(u, y)$ then only the shortest paths from u to y and u' to y are contained in O_*.

4 Relationship between disturbance decoupling and the graph structure

In this section we will prove or disprove several theorems regarding $\mathscr{V}^*$ of graphs that satify Assumption 10.

4.1 Unobservable subspace

The *unobservable subspace* of a system is the subspace $\mathscr{N}$ containing all inital states x where the uncontrolled output y is zero at every time $k = 0, 1, 2, 3, \ldots$. (Meinsma [2])

$$\mathscr{N} = \left\{ x(0) \in \mathscr{X} \mid CA^k x(0) = 0 \ \forall \ k = 0, 1, 2, 3, \ldots \right\} \tag{4.1}$$

It is immediately clear that $\mathscr{N} \subset \mathscr{K}$. Now we prove that $\mathscr{N}$ is also an (A, B)-invariant subspace.

Lemma 15. *The unobservable subspace $\mathscr{N}$ is (A, B)-invariant.*

Proof. Let $x(0)$ be an arbitrary element of $\mathscr{N}$. By definition of $\mathscr{N}$ we have

$$CA^k x(0) = 0 \ \forall k = 0, 1, 2, \ldots. \tag{4.2}$$

Now choose $F = 0$, then $(A + BF)x(0) = Ax(0)$ and we have

$$\forall k = 0, 1, 2, \ldots : \ CA^k Ax(0) = CA^{k+1} x(0) = 0$$

according to (4.2). This means that $Ax(0) \in \mathscr{N}$.

Since $x(0)$ was arbitrary we have found an $F : \mathscr{X} \to \mathbb{R}$ $(F = 0)$ such that for every $x(0) \in \mathscr{N}$, $(A + BF)x(0) \in \mathscr{N}$. So $\mathscr{N}$ is (A, B)-invariant. □

Now it is easy to prove that the unobservable subspace is contained in $\mathscr{V}^*$.

Theorem 16.

$$\mathscr{N} \subset \mathscr{V}^*.$$

Proof. $\mathscr{N}$ is a subspace that is contained in $\mathscr{K}$ and by Lemma 15 $\mathscr{N}$ is (A, B)-invariant. Since $\mathscr{V}^*$ is the supremal element of the class with these two properties we have $\mathscr{N} \subset \mathscr{V}^*$. □

In a graph we can visualize something similar. Therefore we make a definition about the subset of "non-observable" vertices (V_N) in a graph.

Definition 17. *Let G=(V,E) be a graph satisfying Assumption 10 and let y be the index of the observed vertex. $V_N \subset V$ is the subset of vertices such that $v \in V_N$ if and only if there is no path from v to y.*

Theorem 18. *Let $G = (V, E)$ be a graph satisfying Assumption 10 where the observer has index y and $\mathscr{V}^* = \mathrm{Span}\{e_i \mid i \in D^*\}$, then*

$$V_N \subset D^*.$$

Proof. By Lemma 32 in Appendix A we know $(A_{yi})^k = 0$ for every $i \in V_N$ for $k = 0, 1, 2, \ldots$. Then for $i \in V_N$

$$\forall k = 0,1,2,\ldots \; CA^k e_i = C \begin{bmatrix} (A^k)_{1i} \\ (A^k)_{2i} \\ \vdots \\ (A^k)_{yi} \\ \vdots \\ (A^k)_{ni} \end{bmatrix} = (A^k)_{yi} = 0$$

and thus $e_i \in \mathscr{N}$ for every $i \in V_N$.

Because a subspace is closed under addition and scalar multiplication that means

$$\mathrm{Span}\{e_i \mid i \in V_N\} \subset \mathscr{N}.$$

Theorem 16 implies

$$\begin{aligned} &\mathrm{Span}\{e_i \mid i \in V_N\} \subset \mathscr{N} \subset \mathscr{V}^* = \mathrm{Span}\{e_i \mid i \in D^*\} \\ &\Rightarrow \mathrm{Span}\{e_i \mid i \in V_N\} \subset \mathrm{Span}\{e_i \mid i \in D^*\} \\ &\Rightarrow V_N \subset D^*. \end{aligned}$$

□

4.2 Out-neighbours

In the graph $G = (V, E)$, satisfying Assumption 10, v is an *out-neighbour* of w if $(w, v) \in E$, in which case $v \in N^{out}(w)$. In that case w is an *in-neighbour* of v and $w \in N^{in}(v)$

$$N^{out}(w) = \{v \in V \mid (w, v) \in E\}, \tag{4.3}$$

$$N^{in}(v) = \{w \in V \mid (w, v) \in E\}. \tag{4.4}$$

As seen in the previous examples there seems to be a relation between the out-neighbours of vertices and the $\mathscr{V}^*$-algorithm. To prove this we will use the properties of the A-matrix.

Lemma 19. *Let $G = (V, E)$ be a graph satisfying Assumption 10 with matrix A as in (2.15) and let e_w be a standard unit vector. Then for all w in V, Ae_w is a column-vector where the v-th entry is non-zero if and only if v is an out-neighbour of w.*

Proof. By matrix multiplication, we know:

$$Ae_w = \begin{bmatrix} A_{1w} \\ A_{2w} \\ \vdots \\ A_{vw} \\ \vdots \\ A_{nw} \end{bmatrix}. \tag{4.5}$$

By definition of matrix A in (2.15) we know $A_{vw} = 0$ if and only if (w, v) is not in E. So Ae_w is a column-vector where the v-th entry is zero if and only if v is not an out-neighbour of w. The converse of this statement completes the proof. $\square$

Using this property of Ae_w we can use out-neighbours to determine the next step in the $\mathscr{V}^*$-algorithm.

Theorem 20. *Let $G = (V, E)$ be a graph satisfying Assumption 10 and let vertex w be in V, such that w is not the observed vertex. Now if $\mathscr{V}^\mu$ is the subspace after μ iterations of the $\mathscr{V}^*$-algorithm, $\mathscr{V}^\mu = \operatorname{Span}\{e_i \mid i \in D^\mu\}$ and $\mathscr{B} \subset \mathscr{V}^\mu$, then*

$$N^{out}(w) \subset D^\mu \Leftrightarrow w \in D^{\mu+1}.$$

Proof. Suppose $N^{out}(w) \subset D^\mu$ then by the definition of D^μ

$$\operatorname{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \subset \operatorname{Span}\{e_i \mid i \in D^\mu\} = \mathscr{V}^\mu.$$

By Lemma 19 Ae_w is a column vector where the i-th entry is non-zero if and only if i is an out-neighbour of w. That means

$$\begin{aligned} &Ae_w \in \operatorname{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \\ &\Rightarrow Ae_w \in \mathscr{V}^\mu. \end{aligned} \tag{4.6}$$

Since w is not the observed vertex we have $e_w \in \mathscr{K} \subset \mathscr{X}$ and by (A.3) we get

$$\begin{aligned} &e_w \in A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &\Rightarrow e_w \in \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &\Rightarrow e_w \in \mathscr{V}^{\mu+1} = \operatorname{Span}\left\{e_i \mid i \in D^{\mu+1}\right\}. \end{aligned}$$

Finally, by the definition of $D^{\mu+1}$ we have

$$w \in D^{\mu+1}.$$

Conversely, suppose $w \in D^{\mu+1}$. Then by definition of $D^{\mu+1}$ and Theorem 7

$$\begin{aligned} &e_w \in \mathscr{V}^{\mu+1} \\ &\Rightarrow e_w \in \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu+1} + \mathscr{B}) \\ &\Rightarrow e_w \in A^{-1}(\mathscr{V}^\mu) \\ &\Rightarrow Ae_w \in \mathscr{V}^\mu. \end{aligned}$$

Now Ae_w is a column-vector where the i-th entry is non-zero if and only if i is an out-neighbour of w. That means that

$$\begin{aligned}&Ae_w = \sum_{i \in N^{out}(w)} a_i e_i \in \mathscr{V}^\mu \text{ with } a_i \in \mathbb{R} \setminus \{0\} \\ &\Rightarrow e_i \in \mathscr{V}^\mu \ \forall i \in N^{out}(w) \\ &\Rightarrow \text{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \subset \mathscr{V}^\mu = \text{Span}\left\{e_j \mid j \in D^\mu\right\} \\ &\Rightarrow N^{out}(w) \subset D^\mu.\end{aligned}$$

□

This is an interesting result but it doesn't tell us how a controlled vertex will influence the $\mathscr{V}^*$. The next result will tell us something about that.

Theorem 21. *Suppose $\mathscr{V}^\mu$ is the subspace after μ iterations of the $\mathscr{V}^*$-algorithm. Then, if*

$$\begin{aligned}\mathscr{V}^\mu + \mathscr{B} &= \mathscr{V}^{\mu-1} \\ \Rightarrow \mathscr{V}^\mu &= \mathscr{V}^*.\end{aligned} \tag{4.7}$$

Proof. We know

$$\begin{aligned}\mathscr{V}^{\mu+1} &= \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &= \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu-1}).\end{aligned}$$

Now $\mathscr{V}^\mu + \mathscr{B} = \mathscr{V}^{\mu-1}$ also means $\mathscr{B} \subset \mathscr{V}^{\mu-1}$ and:

$$\begin{aligned}\mathscr{V}^{\mu+1} &= \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu-1} + \mathscr{B}) \\ &= \mathscr{V}^\mu.\end{aligned}$$

So $\mathscr{V}^\mu = \mathscr{V}^*$ proving the result. □

In a graph we can visualize this as follows

Theorem 22. *Suppose $\mathscr{V}^\mu$ is the subspace after μ iterations of the $\mathscr{V}^*$-algorithm and u is the index of the controlled vertex. If $D^\mu + u = D^{\mu-1}$ then $D^\mu = D^*$.*

Proof. If $D^\mu + u = D^{\mu-1}$ we know

$$\begin{aligned}\mathscr{V}^\mu + \mathscr{B} &= \text{Span}\left\{e_i \mid i \in D^\mu\right\} + \text{Span}\left\{e_u\right\} \\ &= \text{Span}\left\{e_i \mid i \in D^\mu + u\right\} \\ &= \text{Span}\left\{e_i \mid i \in D^{\mu-1}\right\} = \mathscr{V}^\mu.\end{aligned}$$

So by Theorem 21, $\mathscr{V}^\mu = \mathscr{V}^*$ and $D^\mu = D^*$. □

5 Solving the DDP based on the graph structure

In this section we will prove how to find D^* in a graph satisfying Assumption 10 where the vertices have a maximum in-degree of 1. For an arbitrary graph satisfying Assumption 10 we only show how to find D_* in the case that an extra assumption is fulfilled.

5.1 Graphs where the vertices have a maximum in-degree of 1

In this paragraph we will restrict ourselves to graphs where each vertex has an in-degree of 1 or less ($d_v^{in} \leq 1$). As we have seen in section 4 some theories rely on the fact that $\mathscr{B} \subset \mathscr{N}$. Therefore it might be interesting to see how we can solve the DDP in graphs where the controlled vertex u lies in V_N. Since in that case $\mathscr{B} = \text{Span}\{e_u\} \subset \text{Span}\{e_i \mid i \in V_N\}$.

Theorem 23. *Suppose $G = (V, E)$ is a graph with the given properties, u is the index of the controlled vertex and $u \in V_N$. Then*

$$D^* = V_N.$$

Proof. According to Theorem 18 $V_N \subset D^*$. So we only need to prove $D^* \subset V_N$ or that if $v \in D^*$ then $v \in V_N$. We will prove the converse statement.

$$v \notin V_N \Rightarrow v \notin D^*.$$

Let y be the index of the observed vertex and suppose $v \notin V_N$. Then by definition of V_N there is a path P from v to y. We will prove $v \notin D^*$ via induction on the length m of P.

We start with $m = 0$. In that case $v = y$ and it is clear that $v \notin D^*$.

Nowo suppose that if there is path P of length m from vertex v to y, then $v \notin D^*$.

Let v be a vertex such that $(v, v_2, v_3, \ldots, v_m, y)$ is the shortest path from v to y. The length of this path is $m + 1$. Clearly there is a path $(v_2, v_3, \ldots, v_m, y)$ of length m from v_2 to y and by the induction hypothesis we have $v_2 \notin D^*$. In that case v has a neighbour v_2 that is not in D^* so $N^{out}(v) \not\subseteq D^*$. Then by Theorem 20 $v \notin D^*$.

So via induction we have proven that if there is path from v to y ($v \notin V_N$) then $v \notin D^*$. So $D^* \subset V_N$ and $D^* = V_N$. $\square$

Because u is an element of $V_N = D^*$, $B = e_u \in \mathscr{V}$. Therefore according to Theorem 8, we can chose F, the feedback matrix to solve the DDP, to be zero.

Now that we have shown what happens if $u \in V_N$, see Theoren 23, we assume in the remaining of this paragraph that $u \notin V_N$.

Lemma 24. *Let $G = (V, E)$ be a graph satisfying Assumption 10 with $d_v^{in} \leq 1$ for every $v \in V$ and $u \notin V_N$. Then there is a unique path from u to y.*

Proof. Suppose there are two different paths P and Q, with $d(P) \geq d(Q)$. Then there is at least one vertex $v \in P$ such that $v \notin Q$. Now let $P' = (v, v_1, \ldots, y)$ be the path from v to y and let v_m be the first element on P' that is also on Q. Because Q and P both end in y this element exists. By definition $v_{m-1} \notin Q$. Then there exists an edge $(q_{m-1}, v_m) \in Q$ and a different edge $(v_{m-1}, v_m) \in P$ which means that v_m has two different incoming edges which is in contradiction with our assumption that the maximum in-degree in G was 1. So by contradiction we have proven that the path from u to y is unique. $\square$

Theorem 25. *Let $G = (V, E)$ be a graph satisfying Assumption 10 with $d_v^{in} \leq 1$ for every $v \in V$, $u \notin V_N$ and P the unique path from the controlled vertex u to the observed vertex y, then*

$$P = O^* \tag{5.1}$$

Proof. First we renumber the vertices such that $P = \{u, u-1, \dots, 2, 1, 0 = y\}$. So for $i \in \{0, 1, 2, \dots, u-1, u\}$ we have $(Ax)_i = a_{i+1}x_{i+1}$ and $a_{i+1} > 0$. Please note that only in this proof the first row of A has index 0.

Via induction we prove that $O^\mu = \{0, 1, \dots, \mu\}$ for $0 \leq \mu \leq u$.

For $\mu = 0$ we have

$$\begin{aligned} &V^0 = \mathscr{K} = \operatorname{Span}\{e_i \mid i \neq y\} \\ &\Rightarrow D^0 = V \setminus \{y\} \\ &\Rightarrow O^0 = \{y\} = \{0\}. \end{aligned} \tag{5.2}$$

and the statement holds for $\mu = 0$. Now suppose the statement holds for some $\mu = \mu_0 \leq u - 1$.

$$\begin{aligned} &O^{\mu_0} = \{0, 1, \dots, \mu_0\} \\ &\Rightarrow D^{\mu_0} = V\{0, 1, \dots, \mu_0\} \\ &\Rightarrow V^{\mu_0} = \operatorname{Span}\{e_i \mid i \notin \{0, 1, \dots, \mu_0\}\}. \end{aligned} \tag{5.3}$$

Now $\mu_0 \leq u - 1$ implies $i \leq \mu_0 < u$ which means $\mathscr{B} \subset \mathscr{V}^{\mu_0}$ and so $\mathscr{B} + \mathscr{V}^{\mu_0} = \mathscr{V}^{\mu_0}$. Thus

$$V^{\mu_0+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu_0}). \tag{5.4}$$

By (2.16) we know $\mathscr{K}$ and by (2.10) we know $A^{-1}(V^{\mu_0}) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^{\mu_0}\}$. Now by (5.3)

$$\begin{aligned} &Ax \in \mathscr{V}^{\mu_0} \Leftrightarrow (Ax)_i = 0 \text{ for } i \in \{0, 1, \dots, \mu_0\} \\ &\Leftrightarrow a_{i+1}x_{i+1} = 0 \text{ for } i \in \{0, 1, \dots, \mu_0\} \\ &\Leftrightarrow x_{i+1} = 0 \text{ for } i \in \{0, 1, \dots, \mu_0\}, \end{aligned}$$

where we have used $a_{i+1} > 0$ and so

$$\begin{aligned} &x \in A^{-1}(\mathscr{V}^{\mu_0}) \Leftrightarrow x_{\mu_0+1} = \dots = x_2 = x_1 = 0 \\ &\Rightarrow x \in \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu_0}) \Leftrightarrow x_{\mu_0+1} = \dots = x_2 = x_1 = x_0 = 0 \end{aligned}$$

Now by (5.4) we have

$$\begin{aligned} &\mathscr{V}^{\mu_0+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu_0}) = \operatorname{Span}\{e_i \mid i \notin \{0, 1, 2, \dots, \mu_0 + 1\}\} \\ &\Rightarrow D^{\mu_0+1} = V \setminus \{0, 1, 2, \dots, \mu_0 + 1\} \\ &\Rightarrow O^{\mu_0+1} = \{0, 1, 2, \dots, \mu_0 + 1\}. \end{aligned}$$

So via induction we have proven that the statement is true for $0 \leq \mu \leq u$. In particular $O^u = \{0, 1, 2, \dots, u\} = P$. So now we have to prove $O^u = O^*$. Using Theorem 22 and O^μ we have

$$\begin{aligned} &D^u + u = (V \setminus \{0, 1, 2, \dots, u\}) + u = V \setminus \{0, 1, 2, \dots, u - 1\} = D^{u-1} \\ &\Rightarrow D^u = D^* \\ &\Rightarrow O^u = O^*. \end{aligned}$$

So we have $O^* = O^u = P$ completing the proof. □

In this graph we also have a unique feedback matrix F on $\mathscr{V}^*$.

Theorem 26. *Let $G = (V, E)$ be a graph satisfying Assumption 10 with $d_v^{in} \leq 1$ for every $v \in V$, $u \notin V_N$ and P the unique path from the controlled vertex u to the observed vertex y. Then if u has an incoming edge $(u+1, u)$, $F = (-e_{u+1})^T$ is a feedback matrix unique on $\mathscr{V}^*$ such that $\mathscr{V}^*$ is (A, B)-invariant.*

Proof. Let $\{e_i \mid i \in D^*\}$ be a basis for $\mathscr{V}^*$. Now if $Ae_i + BFe_i \in \mathscr{V}^*$ for all $i \in D^*$ then F is the feedback matrix unique on $\mathscr{V}^*$ such that $\mathscr{V}^*$ is (A, B)-invariant.

Take $O^* = P = \{u, u-1, \ldots, 2, 1, 0 = y\}$. Because for every $i \in V$ we have $d_i^{in} \leq 1$ also $|N^{in}(i)| \leq 1$. Then for every $i \in P$, $i \neq u$ we have

$$N^{in}(i) = \{i+1\} \in O^*.$$

That means u is the only vertex in O^* with an out-neighbor in D^* (namely $u+1$) and thus $u+1$ is the only vertex in D^* with an in-neighbor in O^*.

Now by Lemma 19, Ae_i is a column vector where the v-th entry is non-zero if and only if v is an out-neighbour of i. So for every $i \in D^* \setminus \{u+1\}$ we have

$$\begin{aligned} &Ae_i \in \operatorname{Span}\left\{e_v \mid v \in N^{out}(i)\right\} \subset \operatorname{Span}\{e_j \mid j \in D^*\} \\ &\Rightarrow Ae_i \in \mathscr{V}^*. \end{aligned}$$

Choosing $F = (-e_{u+1})^T$ we have $Fe_i = 0$ for $i \in D^* \setminus \{u+1\}$ which implies that $Ae_i + BFe_i = Ae_i + B \cdot 0 = Ae_i \in \mathscr{V}^*$.

For $i = u+1$ we know that $u+1$ has one neighbour u in O^* and an unknown number of neighbours in D^*. So for $a_i \in \mathbb{R}$ we have

$$Ae_{u+1} = e_u + \sum_{i \in D^*} a_i e_i.$$

Now choosing $F = (-e_{u+1})^T$ we have $F \cdot e_{u+1} = -1$ and by Assumption 10 we have $B = e_u$, which we get

$$\begin{aligned} Ae_{u+1} + BFe_{u+1} &= \sum_{i \in D^*} a_i e_i + e_u + e_u \cdot -1 \\ &= \sum_{i \in D^*} a_i e_i \in \operatorname{Span}\{e_i \mid i \in D^*\} = \mathscr{V}^* \end{aligned}$$

which means that $Ae_{u+1} + BFe_{u+1} \in \mathscr{V}^*$ and we have proven that $F = (-e_{u+1})^T$ is a feedback matrix unique on $\mathscr{V}^*$ such that $\mathscr{V}^*$ is (A, B)-invariant. $\square$

In this paragraph we have proven that for graphs where each vertex has an in-degree equal or less than 1 we can find D^*. This means that every disturbance acting on one or multiple vertices in D^* can be decoupled using the feedback matrix we constructed in Theorem 26. Since $\operatorname{Span}\{e_i \mid i \in \mathscr{V}^*\} = V^*$ we know that these are also the only disturbances that can be decoupled. If a disturbance acts on a vertex outside of D^*, then this disturbance will not be decoupled and hence observed by y.

5.2 General graphs

In this paragraph we look at an arbitrary graph satisfying Assumption 10. In this case it is possible, that $\mathscr{V}^*$ is not the span of standard unit vectors. Therefore we cannot use Theorem 18 and Theorem 20 in Section 4. However, in Appendix A we have proven a similar theorems using D_*.

Theorem 27. *Let graph $G = (V, E)$ satisfy Assumption 10. Then for $v \in V$ we have*

(a) $v \in V_N \Rightarrow e_v \in V^*$.

(b) $d(v, y) < d(u, y) \Rightarrow e_v \notin \mathscr{V}^*$.

(c) $d(v, y) = d(u, y) \Rightarrow e_v \notin \mathscr{V}^*$.

(d) If for every $v \in V$ satisfying $d(v, y) = d(u, y) + 1$ there holds $e_v \in \mathscr{V}^*$ then $d(v, y) > d(u, y)$ implies that $e_v \in \mathscr{V}^*$.

Proof. (a) By the proof of Theorem 33 we have for $v \in V_N$. $e_v \in \mathscr{N} \subset \mathscr{V}^*$.

(b) First we prove via induction for $0 \le \mu < d(u, y)$ that

$$O_\mu = \{v \in V \mid 0 \le d(v, y) \le \mu\}. \tag{5.5}$$

$$\mathscr{B} \subset \mathscr{V}^\mu. \tag{5.6}$$

For $\mu = 0$ we have

$$\mathscr{V}^0 = \mathscr{K} = \text{Span}\,\{e_i | i \neq y\}.$$

In this case V^0 is the greatest span of unit vectors in V^0 and thus

$$\Rightarrow D_0 = V \setminus \{y\}$$
$$\Rightarrow O_0 = \{y\}.$$

Since $\{v \in V \mid d(v, y) = 0\} = \{y\}$, (5.5) holds for $\mu = 0$. Furthermore, $\mathscr{B} = \text{Span}\,\{e_u\} \subset \text{Span}\,\{e_i | i \neq y\} = \mathscr{V}^0$ so (5.6) holds as well for $\mu = 0$.

Now assuming (5.5) and (5.6) hold for μ, we can prove that (5.5) and (5.6) also hold for $\mu + 1$.

Let v be an arbitrary vertex with $d(v, y) = \mu + 1$. That means there is a path $P = (v, v_1, \dots, y)$ of length $\mu + 1$. Then there is path $(v_1, \dots, y)$ of length μ from v_1 to y which means that $d(v_1, y) \le \mu$ so by the induction hypothesis $v_1 \in O_\mu$ and subsequently $v_1 \notin D_\mu$.

Now v has the out-neighbour $v_1 \notin D_\mu$ so $N^{out}(v) \not\subset D_\mu$. Since $\mathscr{B} \subset \mathscr{V}^\mu$ we can use Theorem 34 and then $N^{out}(v) \not\subset D_\mu$ implies $v \notin D_{\mu+1}$ so $v \in O_{\mu+1}$. v was an arbitrary vertex and hence (5.6) holds for $\mu + 1$.

Moreover, $\{e_i \mid d(i, v) > \mu + 1\} \subset \{e_i \mid i \notin O_\mu\} = \{e_i \mid i \in D_\mu\} \subset \mathscr{V}^{\mu+1}$. By assumption $d(u, v) > d(v, y) = \mu + 1$ so $e_u \in \mathscr{V}^{\mu+1}$. Hence $\mathscr{B} = \text{Span}\{e_u\} \subset \mathscr{V}^{\mu+1}$ and thus (5.6) also holds for $\mu + 1$.

We have now proven that if $d(v, y) < d(u, y)$, then $v \in O_{d(v,y)}$ and since $O_\mu \le O_{\mu+1}$ we have $O_{d(v,y)} \subset O_*$. By the definition of D_* and O_* we have $v \in O_*$ and thus $v \notin D_*$. Because

$\operatorname{Span}\{e_i \mid i \in D_*\}$ is the greatest span of unit vectors contained in V^* this means that $e_v \notin \mathscr{V}^*$ completing the proof.

(c) In part (b) we saw that $O_\mu = \{v \in V \mid 0 \leq d(v, y) \leq \mu\}$ and $B \subset \mathscr{V}^\mu$ for $\mu < d(u, v)$.

Let $d(u, v) = m$, then $m - 1 < d(u, v)$ and

$$O_{m-1} = \{v \in V \mid 0 \leq d(v, y) \leq m - 1\} \tag{5.7}$$

$$\mathscr{B} \subset \mathscr{V}^{m-1}. \tag{5.8}$$

Now let v be an arbitrary vertex with $d(v, y) = m$ and let $P = (v, v_1, \ldots, y)$ be a path of length m from v to y. Then there is a path $(v_1, \ldots, y)$ of length $m - 1$ from v_1 to y. This means that $d(v_1, y) \leq m - 1 < m = d(u, y)$ and by (5.7) $v_1 \in O_{m-1}$ and thus $v_1 \notin D_{m-1}$. Since $v_1 \in N^{out}(v)$, we have $N^{out}(v) \not\subset D^{m-1}$. By (5.8) we can use Theorem 34 so

$$\begin{aligned} &v \notin D_m \\ &\Rightarrow v \notin D_*. \end{aligned}$$

Again because $\operatorname{Span}\{e_i \mid i \in D_*\}$ is the greatest span of unit vectors contained in V^* this means that $e_v \notin \mathscr{V}^*$.

(d) We prove that for every $v \in V$ satisfying $d(v, y) > d(u, y)$, $e_v \in \mathscr{V}^\mu$ for $\mu = 0, 1, 2, \ldots$.

For $\mu = 0$ we have $v \neq y$ and thus $e_v \in \mathscr{K} = \mathscr{V}^0$.

Now suppose that $e_v \in \mathscr{V}^{\mu_0}$ for every $v \in V$ satisfying $d(v, y) > d(u, y)$. We prove that $e_v \in \mathscr{V}^{\mu_0+1}$.

By assumption we have that if $d(v, y) = d(u, y) + 1$ then $v \in \mathscr{V}^*$ and thus $v \in \mathscr{V}^{\mu_0+1}$.

So all that is left to prove is that $e_v \in \mathscr{V}^{m_0+1}$ for every $v \in V$ satisfying $d(v, y) > d(u, y) + 1$. We know that

$$\mathscr{V}^{\mu_0+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu_0} + \mathscr{B}).$$

When $d(v, y) > d(u, y) + 1$, we have $v \neq y$ and thus $e_v \in \mathscr{K}$. So we only need to prove that $e_v \in A^{-1}(\mathscr{V}^{\mu_0} + \mathscr{B})$.

$A^{-1}(\mathscr{V}^{\mu_0} + \mathscr{B}) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^{\mu_0} + \mathscr{B}\}$. By Lemma 19 Ae_i is a vector where the j-th entry is non-zero if and only if j is an out-neighbour of i. Now for v with $d(v, y) > d(u, y) + 1$, we have for every out-neighbour w of v that $d(w, y) > d(u, y)$ and by induction hypothesis $e_w \in \mathscr{V}^{\mu_0}$ for every $w \in V$ satisfying $d(w, y) > d(u, y)$. Then

$$Ae_v \in \operatorname{Span}\left\{e_w \mid w \in N^{out}(v)\right\} \subset \operatorname{Span}\{e_j \mid d(j, y) > d(u, y)\} \subset \mathscr{V}^{\mu_0}.$$

So we have $Ae_v \in \mathscr{V}^{\mu_0} + \mathscr{B}$ and thus $e_v \in A^{-1}(\mathscr{V}^{\mu_0} + \mathscr{B})$. Combining this with $e_v \in \mathscr{K}$, we have proven that $e_v \in \mathscr{V}^{\mu_0+1}$.

So for every $v \in V$ satisfying $d(v, y) > d(u, y)$ we have that $e_v \in \mathscr{V}^\mu$ for $\mu = 0, 1, \ldots$, which means $e_v \in \mathscr{V}^*$. $\square$

In part (d) we assumed that $e_v \in \mathscr{V}^*$ for every $v \in V$ satisfying $d(v, y) = d(u, y) + 1$. We suppose that this holds in all graphs satisfying Assumption 10, but it remains to be proven. However, if we can prove that this is true for an arbitrary graph then we can find the largest span of standard unit vectors ($\mathscr{E}_* = \text{Span}\,\{e_i \mid i \in D_*\}$) contained in $\mathscr{V}^*$. That means that every disturbance acting on one or multiple vertices in D_* can be decoupled. It is also important to note that since $\mathscr{E}_* \subset \mathscr{V}^*$, it is possible that a disturbance acting on some other vertices that are not elements of D_* can also be decoupled.

6 Conclusion

Our research question was: *In a network where information travels with constant speed, which is observed at a single vertex and which can be controlled at another vertex: in what subpart of the network can disturbances acting on one or multiple vertices be counteracted by the controller?*

This problem is known as the disturbance decoupling problem (DDP) in systems theory. Hence we combined theorems in literature [1, Chapter 4] regarding the disturbance decoupling problem with graph theory by providing a method to rewrite graphs as discrete time systems. This way we were able to solve the problem for a few basic graphs. However these computations were complex and lenghty and the results were only relevant for a very small group of graphs. However we could solve the problem for a larger group of graphs, by using the observations we made during these computations.

To achieve this result, we created two groups of graphs, thereby creating two sub problems from the research question. (1) how to solve the disturbance decoupling problem in a network where vertices cannot have multiple incoming edges are not allowed and (2) how to solve the disturbance decoupling problem in networks where vertices are allowed to have multiple incoming edges.

We have found a solution to the first subproblem, for graphs where vertices cannot have multiple incoming edges. For these graphs we showed how to construct a subset of vertices D^* such that a disturbance can be decoupled if and only if that disturbance acts on one or multiple vertices of this subset.

For the second problem we only managed to find a partial solution. For these graphs, we can create another subset D_* such that if a disturbance acts on one or multiple vertices of this subset then it can be decoupled. This means that it is possible that there are more types of disturbances that can also be decoupled, which we did not find. Moreover, we can only create D_* if an extra assumption is fulfilled. We do have a strong suspicion that this assumption holds for every graph but that remains to be proven. So in order to find every type of disturbance that can be decoupled in an arbitrary graph further research is required.

We can also continue the research in other areas. On the one hand we could develop an algorithm to solve the problem for very large graphs using a computer. On the other hand we could use the knowledge we gained by solving the disturbance decoupling problem where the control is fixed on one vertex to figure out how to place the controller in the network such that the maximum amount of disturbances can be decoupled.

7 Overview of all symbols and notations

To help the reader we have compiled a list of the notations and symbols that are used.

- $\mathscr{X} :=$ state space

- $A : \mathscr{X} \to \mathscr{X}$
- $B : \mathbb{R} \to \mathscr{X}$
- $C : \mathscr{X} \to \mathbb{R}$
- $S : \mathbb{R} \to \mathscr{X}$
- $F : \mathscr{X} \to \mathbb{R}$
- A^T := the transpose of matrix A
- $\mathscr{S} := \mathrm{im}\,(S)$
- $\mathscr{B} := \mathrm{im}\,(B)$
- $\mathscr{K} := \ker\,(C)$
- $\dim \mathscr{V}$:= the dimension of subspace $\mathscr{V}$
- V := set of all the nodes in graph $G = (V, E)$
- E := set of all the edges in graph $G = (V, E)$
- $|V|$:= the cardinality of set V
- $n := |G(V)|$
- $x(k)$:= state x on time k
- x_i := the i-th element of state x
- $\mathscr{N} \subset \mathscr{X}$:= the unobservable subspace
- V_N := the subset of vertices that have no path to the observer y
- $d(v, w)$:= the length of the path between v and w
- $\mathfrak{J}\,(A, B; \mathscr{K})$:= the class of (A, B)-invariant subspaces contained in $\mathscr{K}$
- $\langle A + BF|\mathscr{S}\rangle := \mathscr{S} + (A + BF)\mathscr{S} + \ldots + (A + BF)^{n-1}\mathscr{S}$
- $A^{-1}(\mathscr{V}) := \{x \in \mathscr{X} | Ax \in \mathscr{V}\}$
- e_i := the standard unit vector
- d_i^{out} := the out-degree of vertex i
- d_i^{in} := the in-degree of vertex i
- $N^{out}(i)$:= the subset of out-neighbors of vertex i
- $N^{in}(i)$:= the subset of in-neighbors of vertex i
- D^μ := the set of vertices such that $\{e_i | i \in D^\mu\}$ forms a basis for $\mathscr{V}^\mu$
- O^μ := the complement of D^μ in V
- D_μ := the set of vertices such that $Span\{e_i | i \in D^\mu\}$ is the largest span of unit vectors in $\mathscr{V}^\mu$
- O_μ := the complement of D_μ in V

References

[1] Wonham, W.M. (1985). *Linear Multivariable Control: A Geometric Approach* (3rd ed.). New York, NY: Springer-Verlag.

[2] Meinsma, G. (2015). *Inleiding wiskundige systeemtheorie.* Enschede: Universiteit Twente.

[3] Fagnani, F. & Frasca, P. (2014). *Introduction to Averaging Dynamics over Networks* (1st ed.). Springer International Publishing AG.

Appendices

A Proofs

Theorem 28. *Let $\mathscr{V}$ be a subspace. Then*

$$\mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{cl} \Leftrightarrow \mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{ol}$$

Proof. Suppose $\mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{cl}$ that means there exists a linear mapping $F : \mathscr{X} \to \mathbb{R}$ such that that for every $x(0) \in \mathscr{V}$ and $k = 0, 1, 2, \ldots$

$$\begin{aligned}(A + BF)^k x(0) &\in \mathscr{V} \\ C(A + BF)^k x(0) &= 0\end{aligned}$$

Let $x(0)$ be an arbitrary element of $\mathscr{V}$. For $k = 0$ we know

$$\begin{aligned}x(0) &\in \mathscr{V} \\ y(0) = Cx(0) = C(A + BF)^0 x(0) &= 0.\end{aligned}$$

That means that $x(0) \in \mathscr{V}$ and $y(0) = 0$.

For $k = 1$ choose $u(0) = Fx(0)$.

$$\begin{aligned}x(1) &= Ax(0) + Bu(0) \\ &= Ax(0) + BFx(0) \\ &= (A + BF)x(0) \in \mathscr{V} \\ y(1) &= Cx(1) \\ &= C(A + BF)x(1) = 0\end{aligned}$$

That means that there exists a $u(0)$ such that $x(1) \in \mathscr{V}$ and $y(1) = 0$.

Now suppose for k that there is a u such that $x(k) \in \mathscr{V}$ and $y(k) = 0$. Then for $k + 1$ choose $u(k) = Fx(k)$.

$$\begin{aligned}x(k + 1) &= Ax(k) + Bu(k) \\ &= Ax(k) + BFx(k) \\ &= (A + BF)x(k) \in \mathscr{V} \\ y(k + 1) &= Cx(k + 1) \\ &= C(A + BF)x(k) = 0\end{aligned}$$

Since $x(0)$ was chosen at random we know that for every $x(0)$ there is a $u(k)$ for $k = 0, 1, 2, ...$ such that $x(k) \in \mathscr{V}$ and $y(k) = 0$. So $\mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{ol}$.

Conversely, suppose $\mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{ol}$. That means that for every $x(0) \in \mathscr{V}$ there is a $u(k)$ such that $x(k) \in \mathscr{V}$ and $y(k) = 0$. Now let $\{\nu_1, \nu_2, \ldots, \nu_\mu\}$ be a basis for $\mathscr{V}$. Then for every ν_i, for $i = 0, 1, 2, \ldots, \mu$ there is a control u_i such that

$$A\nu_i + Bu_i \in \mathscr{V} \tag{A.1}$$

$$y_i = C\nu_i = 0 \tag{A.2}$$

Define $F_0 : \mathscr{X} \to \mathbb{R}$ by $F_0\nu_i = u_i$ and let F be some linear extension of F_0 to $\mathscr{X}$. We prove via induction that $(A+BF)^k\nu_i \in \mathscr{V}$ and $C(A+BF)^k\nu_i = 0$. For $k = 0$ we have

$$(A+BF)^0\nu_i = \nu_i \in \mathscr{V}$$
$$C(A+BF)^0\nu_i = C\nu_i = 0$$

And suppose for $k = s$ we have $(A+BF)^s\nu_i \in \mathscr{V}$ and $C(A+BF)^s\nu_i = 0$.

Since $(A+BF)^s\nu_i \in \mathscr{V}$ we have $(A+BF)^s\nu_i = \sum_{i\in\{0,1,\dots,\mu\}} a_i\nu_i$ with $a_i \in \mathbb{R}$. Then for $k = s+1$:

$$\begin{aligned}
(A+BF)^{s+1}\nu_i &= (A+BF)(A+BF)^s\nu_i \\
&= (A+BF)\sum_{i\in\{0,1,\dots,\mu\}} a_i\nu_i \\
&= \sum_{i\in\{0,1,\dots,\mu\}} a_i(A+BF)\nu_i \\
&= \sum_{i\in\{0,1,\dots,\mu\}} a_i(A\nu_i + BF\nu_i) \\
&= \sum_{i\in\{0,1,\dots,\mu\}} a_i(A\nu_i + Bu_i)
\end{aligned}$$

We know by (A.1) that $A\nu_i + Bu_i \in \mathscr{V}$. Since subspace $\mathscr{V}$ is closed under addition and scalar multiplication this means $\sum_{i\in\{0,1,\dots,\mu\}} a_i(A\nu_i + Bu_i) \in \mathscr{V}$ and so $(A+BF)^{s+1}\nu_i \in \mathscr{V}$. Then also $(A+BF)^{s+1}\nu_i = \sum_{i\in\{0,1,\dots,\mu\}} b_i\nu_i$ with $b_i \in \mathbb{R}$. Combining this with (A.2) we have

$$\begin{aligned}
C(A+BF)^{s+1}\nu_i &= C\sum_{i\in\{0,1,\dots,\mu\}} b_i\nu_i \\
&= \sum_{i\in\{0,1,\dots,\mu\}} b_iC\nu_i \\
&= \sum_{i\in\{0,1,\dots,\mu\}} b_i \cdot 0 = 0
\end{aligned}$$

So we have proven for $k = 0, 1, 2, \dots$ that for every ν_i with $i \in \{0, 1, \dots, \mu\}$

$$(A+BF)^k\nu_i \in \mathscr{V}$$
$$C(A+BF)^k\nu_i = 0$$

And since this holds for a basis of $\mathscr{V}$ we have constructed a linear mapping $F : \mathscr{X} \to \mathbb{R}$ such that for every $x(0) \in \mathscr{V}$ for $k = 0, 1, 2, \dots$

$$(A+BF)^k x(0) \in \mathscr{V}$$
$$C(A+BF)^k x(0) = 0$$

which means that $\mathscr{V} \in \mathfrak{J}\left(A, B; \mathscr{K}\right)_{cl}$. □

In the next couple of lemmas and theorems we will prove that the supremal element of $\mathfrak{J}\left(A, B; \mathscr{K}\right)$ exists and is unique. To make the proofs more readable we will write $\mathfrak{J} = \mathfrak{J}\left(A, B; \mathscr{K}\right)$ and $\sup\mathfrak{J} = \sup\left(\mathfrak{J}\left(A, B; \mathscr{K}\right)\right)$.

Lemma 29. *Let* $\mathscr{V} \subset \mathscr{K}$ *and* $\mathscr{B} = \text{im}\,(B)$*. Then*

$$\mathscr{V} \in \mathfrak{J} \Leftrightarrow A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$$

Proof. Suppose $\mathscr{V} \in \mathfrak{J}$ and let $\nu \in \mathscr{V}$. By definition $\mathscr{V}$ is (A, B)-invariant and so $(A+BF)\nu = \omega$ for some $\omega \in \mathscr{V}$. Thus

$$A\nu = \omega - BF\nu \Rightarrow A\nu \in \mathscr{V} + \mathscr{B}.$$

Since ν is arbitrary this means that $A\nu \in \mathscr{V} + \mathscr{B}$ for every $\nu \in \mathscr{V}$ and thus $A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$.

Conversely, let $\mathscr{V} \subset \mathscr{K}$ be a subspace such that $A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$ and let $\{\nu_1, \nu_2, \ldots, \nu_\mu\}$ be a basis for $\mathscr{V}$. Then for every ν_i, for $i = 0, 1, 2, \ldots, \mu$ there exist a $\omega_i \in \mathscr{V}$ and $u_i \in \mathbb{R}$ such that

$$A\nu_i = \omega_i - Bu_i$$

Now define $F_0 : \mathscr{X} \to \mathbb{R}$ by $F_0\nu_i = u_i$ and let F be some linear extension of F_0 to $\mathscr{X}$. Then

$$\begin{aligned} A\nu_i &= \omega_i - BF\nu_i \\ A\nu_i + BF\nu_i &= \omega_i \\ (A + BF)\nu_i &= \omega_i \end{aligned}$$

And since this holds for a basis of $\mathscr{V}$ we know $(A + BF)\mathscr{V} \subset \mathscr{V} \subset \mathscr{K}$. So $\mathscr{V} \in \mathfrak{J}$. □

Lemma 30. $\mathfrak{J}$ *is closed under the operation of subspace addition.*

Proof. Let $\mathscr{V}_1, \mathscr{V}_2 \in \mathfrak{J}$. Then by Lemma 29

$$\begin{aligned} &A\mathscr{V}_1 \subset \mathscr{V}_1 + \mathscr{B}, \text{ and} \\ &A\mathscr{V}_2 \subset \mathscr{V}_2 + \mathscr{B}. \end{aligned}$$

Thus $A(\mathscr{V}_1 + \mathscr{V}_2) = A(\mathscr{V}_1) + A(\mathscr{V}_2) \subset \mathscr{V}_1 + \mathscr{V}_2 + \mathscr{B}$. Now by Lemma 29 we know $\mathscr{V}_1 + \mathscr{V}_2 \in \mathfrak{J}$ and so $\mathfrak{J}$ is closed under the operation of subspace addition. □

Theorem 31. $\sup \mathfrak{J}$ *exists.*

Proof. First we will prove that the supremal element of $\mathfrak{J}$ exists. We know that $\mathfrak{J}$ is a class of subspaces of $\mathscr{K}$ and since $\mathscr{K}$ is finite-dimensional there must be an element $\mathscr{V}^*$ of greatest dimension. Now let $\mathscr{V}$ be an arbitrary element of $\mathfrak{J}$. We have $\mathscr{V} + \mathscr{V}^* \in \mathfrak{J}$, because by Lemma 30 $\mathfrak{J}$ is closed under the operation of subspace addition. Now let $\dim(\mathscr{V})$ be the dimension of subspace $\mathscr{V}$, then

$$\dim(\mathscr{V}^*) \geq \dim(\mathscr{V} + \mathscr{V}^*) \geq \dim(\mathscr{V}*).$$

That means that $d(\mathscr{V}^*) = d(\mathscr{V} + \mathscr{V}^*)$ and thus $\mathscr{V}^* = \mathscr{V}^* + \mathscr{V}$, which means that $\mathscr{V} \subset \mathscr{V}^*$. Since $\mathscr{V}$ was arbitrary this holds for every subspace $\mathscr{V} \in \mathfrak{J}$ and so $\mathscr{V}^* = \sup \mathfrak{J}$ exists.

Now we will show that $\mathscr{V}^*$ is unique. Suppose there is another supremal element $\mathscr{V}$. By definition of the supremal element $\mathscr{V} \in \mathfrak{J}$ so $\mathscr{V} \subset \mathscr{V}^*$. Also $\mathscr{V}^* \in \mathfrak{J}$ so $\mathscr{V}^* \subset \mathscr{V}$. Which means that $\mathscr{V} = \mathscr{V}^*$ and $\mathscr{V}^*$ is unique. □

Lemma 32. *Let $G = (V,E)$ be a graph satisfying Assumption 10 and matrix A defined as in (2.15). Then for all $v, w \in V$ and $k = 1, 2, \dots$ if there is no path from v to w then $(A^k)_{wv} = 0$.*

Proof. The statement is proven by induction on k. By the definition of matrix A in (2.15) this is true for $k = 1$.

Now we suppose the statement is true for k and we will prove it is also true for $k+1$. We know that if the edge (v, i) exists there is no path from i to w (otherwise there would be a path from v to w).

$$\begin{aligned}
(A^{k+1})_{wv} &= \sum_{i \in V} (A^k)_{wi} A_{iv} \\
&= \sum_{i \in V | (v,i) \in E} (A^k)_{wi} A_{iv} + \sum_{i \in V | (v,i) \notin E} (A^k)_{wi} A_{iv} \\
&= \sum_{i \in V | (v,i) \in E} 0 \cdot A_{iv} + \sum_{i \in V | (v,i) \notin E} (A^k)_{wi} \cdot 0 \\
&= 0 + 0 = 0.
\end{aligned}$$

So by induction the statement is true for all $k = 1, 2, \dots$ □

Theorem 33. *Let $G = (V, E)$ be a graph satisfying Assumption 10 where the observer has index y, then*

$$V_N \subset D_*.$$

Proof. By Lemma 32 in Appendix A we know $(A_{yi})^k = 0$ for every $i \in V_N$ for $k = 0, 1, 2, \dots$. Then for $i \in V_N$

$$\forall k = 0, 1, 2, \dots \;\; CA^k e_i = C \begin{bmatrix} (A^k)_{1i} \\ (A^k)_{2i} \\ \vdots \\ (A^k)_{yi} \\ \vdots \\ (A^k)_{ni} \end{bmatrix} = (A^k)_{yi} = 0$$

and thus $e_i \in \mathscr{N}$ for every $i \in V_N$.

Because a subspace is closed under addition and scalar multiplication that means by Theorem 16

$$\operatorname{Span}\{e_i \mid i \in V_N\} \subset \mathscr{N} \subset \mathscr{V}^*$$

Now since $\mathscr{E}_*$ is the largest span of unit vectors such that $\mathscr{E}_* \subset \mathscr{V}^*$ we have

$$\begin{aligned}
&\operatorname{Span}\{e_i \mid i \in V_N\} \subset \mathscr{E}_* = \operatorname{Span}\{e_i \mid i \in D_*\} \\
&\Rightarrow \operatorname{Span}\{e_i \mid i \in V_N\} \subset \operatorname{Span}\{e_i \mid i \in D_*\} \\
&\Rightarrow V_N \subset D_*
\end{aligned}$$

. □

Theorem 34. *Let $G = (V, E)$ be a graph satisfying Assumption 10 and let vertex $w \in V$, not the observed vertex. Now if $\mathscr{V}^\mu$ is the subspace after μ iterations of the $\mathscr{V}^*$-algorithm and $\mathscr{B} \subset \mathscr{V}^\mu$, then*

$$N^{out}(w) \subset D_\mu \Leftrightarrow w \in D_{\mu+1}$$

Proof. Suppose $N^{out}(w) \subset D_\mu$ then by the definition of D_μ

$$\text{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \subset \text{Span}\left\{e_i \mid i \in D_\mu\right\} \subset \mathscr{V}^\mu.$$

By Lemma 19 Ae_w is a column vector where the i-th entry is non-zero if and only if i is an out-neighbour of w. That means

$$\begin{aligned} &Ae_w \in \text{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \\ &\Rightarrow Ae_w \in \mathscr{V}^\mu \end{aligned} \tag{A.3}$$

Since w is not the observed vertex we have $e_w \in \mathscr{K} \subset \mathscr{X}$ and by (A.3) we get

$$\begin{aligned} &e_w \in A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &\Rightarrow e_w \in \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &\Rightarrow e_w \in \mathscr{V}^{\mu+1} \end{aligned}$$

And since $\mathscr{E}_{\mu+1}$ is the largest largest span of unit vectors in $\mathscr{V}^\mu$ we have

$$\begin{aligned} &e_w \in \mathscr{E}_{\mu+1} = \text{Span}\left\{e_i \mid i \in D_{\mu+1}\right\} \\ &\Rightarrow w \in D^{\mu+1} \end{aligned}$$

Conversely, suppose $w \in D_{\mu+1}$. Then by definition of $D_{\mu+1}$ and Theorem 7

$$\begin{aligned} &e_w \in \mathscr{E}_{\mu+1} \subset \mathscr{V}^{\mu+1} \\ &\Rightarrow e_w \in \mathscr{V}^{\mu+1} \\ &\Rightarrow e_w \in \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu+1} + \mathscr{B}) \\ &\Rightarrow e_w \in A^{-1}(\mathscr{V}^\mu) \\ &\Rightarrow Ae_w \in \mathscr{V}^\mu \end{aligned}$$

Now Ae_w is a column-vector where the i-th entry is non-zero if and only if i is an out-neighbour of w. That means that

$$\begin{aligned} &Ae_w = \sum_{i \in N^{out}(w)} a_i e_i \in \mathscr{V}^\mu \quad \text{with } a_i \in \mathbb{R} \setminus \{0\} \\ &\Rightarrow e_i \in \mathscr{V}^\mu \ \forall i \in N^{out}(w) \\ &\Rightarrow \text{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \subset \mathscr{V}^\mu \end{aligned}$$

And again since $\mathscr{E}_{\mu+1}$ is the largest largest span of unit vectors in $\mathscr{V}^\mu$ we have

$$\begin{aligned} &\text{Span}\left\{e_i \mid i \in N^{out}(w)\right\} \subset \text{Span}\left\{e_j \mid j \in D_\mu\right\} \\ &\Rightarrow N^{out}(w) \subset D_\mu \end{aligned}$$

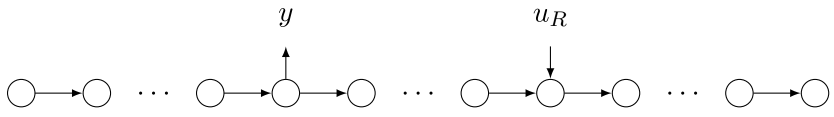
□

B Examples of Disturbance Decoupling in basic graphs

In this section we will show a precise computation of $\mathscr{V}^*$ for the graphs, described in Section 3. For simplicity we will show the graphs in each step of the algorithm with D^μ and O^μ by coloring the vertices in D^μ white and the vertices in O^μ black. Also x_i is the i-th element of vector x.

B.1 Computation of $\mathscr{V}^*$ in a line graph with the controller on the right of the observer

We compute $\mathscr{V}^*$ for a line graph with the controller to the right of the observer and the matrices in (B.1), (B.2) and (B.3). We will assume that $n \geq 4$ so that $1, y, u_R$ and n are distinct vertices.



$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \tag{B.1}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{B.2}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{y} & 0 & \cdots & 0 & \underbrace{0}_{u_R} & 0 & \cdots & 0 \end{bmatrix} \tag{B.3}$$

Preliminaries

We also need to find the subspaces $\mathscr{K}$, $\mathscr{B}$ and define the operation Ax. Since (2.16) and (2.17) tell us how to find $\mathscr{K}$ and $\mathscr{B}$ we will compute these first.

$$\mathscr{K} = \text{Span}\{e_i \mid i \neq y\}$$
$$\mathscr{B} = \text{Span}\{e_{u_R}\}.$$

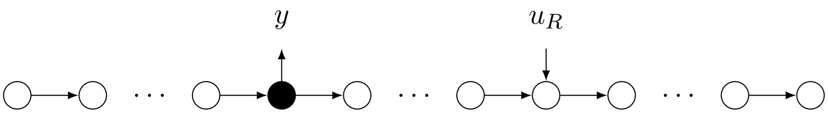
By B.1 we see that $(Ax)_1 = 0$ and $(Ax)_i = x_{i-1}$ for $i \in \{2, 3, \ldots, n\}$.

Step 1: Computation of $\mathscr{V}^0$
We start by computing $\mathscr{V}^0$ the first element in the sequence $\mathscr{V}^\mu$.

$$\mathscr{V}^0 = \mathscr{K} = \mathrm{Span}\,\{e_i \mid i \neq y\}\,.$$

By definition of D^μ this means:

$$D^0 = V \setminus \{y\}.$$



Step 2: Computation of $\mathscr{V}^\mu$ for $0 \leq \mu \leq y-1$
We want to show that for $0 \leq \mu \leq y-1$

$$\mathscr{V}^\mu = \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, y-\mu\}\}. \tag{B.4}$$

We have already shown that (B.4) holds for $\mu = 0$.

Now suppose (B.4) holds for μ ($\mu \leq y-2$). We will show that (B.4) also holds for $\mu+1$. By Theorem 7 we know that

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^\mu).$$

Because u_R is on the right of y, $u_R \notin \{y, y-1, \ldots, y-\mu\}$. This means that $\mathscr{B} = \mathrm{Span}\,\{e_{u_R}\} \subset \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, y-\mu\}\} = \mathscr{V}^\mu$ and so

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu).$$

Since $\mathscr{K}$ is known we are only interested in $A^{-1}(\mathscr{V}^\mu)$.

$$A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}\,.$$

Now $Ax \in \mathscr{V}^\mu$ if and only if $(Ax)_i = 0$ for $i = y, y-1, \ldots, y-\mu$. By our definition of the operation Ax this means that $Ax \in \mathscr{V}^\mu$ if and only if $x_{i-1} = (Ax)_i = 0$ for $i = y, \ldots, y-\mu$ and if $i \geq 2$ (which is true if $\mu \leq y-2$). Then

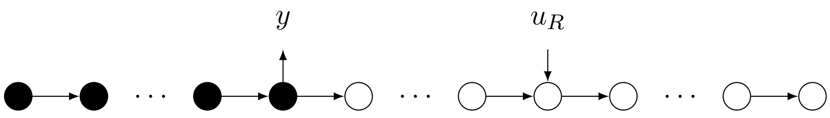
$$\begin{aligned}
&x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots = x_{y-\mu-1} = 0 \\
&\Rightarrow A^{-1}(\mathscr{V}^\mu) = \mathrm{Span}\,\{e_i \mid i \notin \{y-1, y-2, \ldots, y-\mu-1\}\} \\
&\Rightarrow \mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu) = \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, y-2, \ldots, y-\mu-1\}\}.
\end{aligned}$$

So by induction we showed that (B.4) is true for $0 \leq \mu \leq y-1$. Thus, especially

$$\begin{aligned}
\mathscr{V}^{y-1} &= \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, y-y-1\}\} \\
&= \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, 1\}\}.
\end{aligned}$$

Using the definition of D^{μ} for $\mu = y - 1$ we have

$$D^{y-1} = V \setminus \{y, y-1, \ldots, 1\}.$$



Step 3: Termination of the algorithm while computing $\mathscr{V}^{y}$.
We will now show that the algorithm terminates if we compute $\mathscr{V}^{y}$. That means that we will show that $\mathscr{V}^{y} = \mathscr{V}^{y-1}$, and hence $\mathscr{V}^{*}$.

$$\mathscr{V}^{y} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-1} + \mathscr{B}).$$

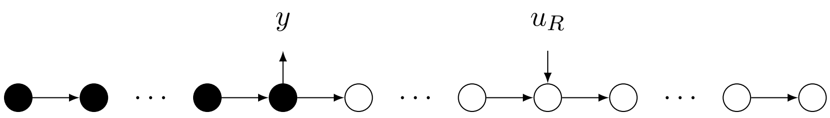
Again $u_R \notin \{1, 2, \ldots, y\}$ so $\mathscr{B} \subset \mathscr{V}^{y-1}$ and thus

$$\mathscr{V}^{y} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-1})$$

$\mathscr{K}$ is still know so we will compute $A^{-1}(\mathscr{V}^{y-1}) = \left\{x \in \mathscr{X} \mid Ax \in \mathscr{V}^{y-1}\right\}$. And $Ax \in \mathscr{V}^{y-1}$ if and only if $(Ax)_i = 0$ for $i = \{1, 2, \ldots, y\}$. Because $(Ax)_1 = 0$ for every $x \in \mathscr{X}$ that means $Ax \in \mathscr{V}^{y-1}$ if and only if $x_{i-1} = (Ax)_i = 0$ for $i = \{2, 3, \ldots, y-2\}$ or

$$\begin{aligned}
&x \in A^{-1}(\mathscr{V}^{y-1}) \Leftrightarrow x_1 = x_2 = \cdots = x_{y-1} = 0 \\
&\Rightarrow A^{-1}(\mathscr{V}^{y-1}) \operatorname{Span}\{e_i \mid i \notin \{1, 2, \ldots, y-1\}\} \\
&\Rightarrow \mathscr{V}^{y} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-1}) = \operatorname{Span}\{e_i \mid i \notin \{1, 2, \ldots, y\}\} = \mathscr{V}^{y-1}
\end{aligned}$$

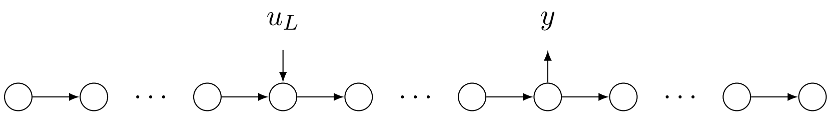
Because $\mathscr{V}^{y-1} = \mathscr{V}^{y}$ the algorithm terminates and we have found $\mathscr{V}^{*} = \mathscr{V}^{y-1}$.



Observations made during the computation
While computing $\mathscr{V}^{*}$ we noticed the following things in the structure of the graph. In the first step of the algorithm only vertex y belongs to O^{0} and then in each consecutive step the vertex to the left of the vertex that was added to $O^{\mu-1}$ will be added to O^{μ}. Furthermore, only the vertices that cannot reach vertex y are contained in D^{*}.

B.2 Computation of $\mathscr{V}^{*}$ in a line graph with the controller on the left of the observer

We compute $\mathscr{V}^{*}$ for a line graph with the controller to the right of the observer and the matrices in (B.5), (B.6) and (B.7). We will again assume that $n \geq 4$ so that $1, y, u_R$ and n are distinct vertices.



$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \tag{B.5}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{B.6}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{0}_{u_L} & 0 & \cdots & 0 & \underbrace{1}_{y} & 0 & \cdots & 0 \end{bmatrix}. \tag{B.7}$$

Preliminaries
As before we start of by defining the subspaces $\mathscr{K}$, $\mathscr{B}$ and the operation Ax. Because matrix A and C are equal to the matrices A and C in section B.1, $\mathscr{K}$ and Ax are the same as well.

$$\begin{aligned} \mathscr{K} &= \text{Span}\,\{e_i \mid i \neq y\} \\ (Ax)_1 &= 0 \\ (Ax)_i &= x_{i-1} \text{ for } i \in \{2, 3, \ldots, n\}. \end{aligned}$$

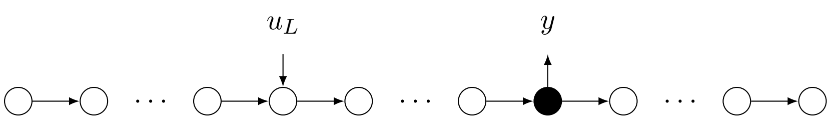
In this case however control u_L is on the left of the observer y so B and thus $\mathscr{B}$ are slightly different. The u_L-th entry of B is the only non-zero entry and so

$$\mathscr{B} = \text{Span}\,\{e_{u_L}\}$$

Step 1: Computation of $\mathscr{V}^0$
Since $\mathscr{K}$ is equal to the $\mathscr{K}$ in section B.1. $\mathscr{V}^0$, the first element in the sequence $\mathscr{V}^\mu$, will not change and so

$$\mathscr{V}^0 = \mathscr{K} = \text{Span}\,\{e_i \mid i \neq y\}$$

Thus $D^0 = V \setminus \{y\}$.



Computation of $\mathscr{V}^\mu$ for $0 \leq \mu \leq y - u_L$
For $0 \leq \mu \leq y - u_L$ we want to prove that

$$\mathscr{V}^\mu = \operatorname{Span}\{e_i \mid i \neq \{y, y-1, \dots, y-\mu\}\}\,. \tag{B.8}$$

(B.8) holds for $\mu = 0$.

Now we assume that (B.8) holds for μ ($\mu \leq y - u_L - 1$), and prove that (B.8) holds for $\mu + 1$ as well. Using Theorem 7 we know

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^{\mu+1})$$

Because $\mu \leq y - u_L - 1$, the index $y - \mu \geq y - y + u_L + 1 = u_L + 1 > u_L$. So $\mathscr{B} = \operatorname{Span}\{e_{u_L}\} \subset \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, y-\mu\}\} = \mathscr{V}^\mu$ and thus

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu).$$

Since we already know $\mathscr{K}$ we want to find $A^{-1}(\mathscr{V}^\mu)$ again.

$$A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}\,.$$

$Ax \in \mathscr{V}^\mu$ if and only if $(Ax)_i = 0$ for $i \in \{y, y-1, \dots, y-\mu\}$ or $Ax \in \mathscr{V}^\mu$ if and only if $x_{i-1} = (Ax)_i = 0$ for $i \in \{y, y, \dots, y-\mu\}$. Then

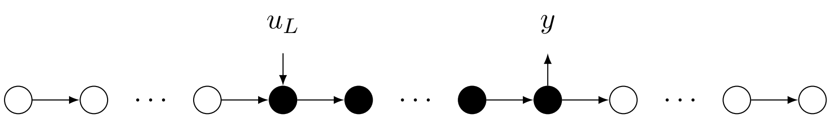
$$\begin{aligned}
&x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots = x_{y-\mu-1} = 0\\
&\Rightarrow (A^{-1}\mathscr{V}^\mu) = \operatorname{Span}\{e_i \mid i \notin \{y-1, y-2, \dots, y-\mu-1\}\}\\
&\Rightarrow \mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu) = \operatorname{Span}\{e_i \mid i \notin \{y, y-1, y-2, \dots, y-\mu-1\}\}
\end{aligned}$$

So we have proven by induction that (B.8) holds for $0 \leq \mu \leq y - u_L$. In particular

$$\begin{aligned}
\mathscr{V}^{y-u_L} &= \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, y-(y-u_L)\}\}\\
&= \operatorname{Span}\{e_i \mid i \notin \{u_L, u_L+1, \dots, y\}\}.
\end{aligned}$$

Again by the definition of D^μ.

$$D^{y-u_L} = V \setminus \{u_L, u_L+1, \dots, y\}.$$



Step 3: Termination of the algorithm while computing $\mathscr{V}^{y-u_L+1}$.
We continue by computing $\mathscr{V}^{y-u_L+1}$;

$$\mathscr{V}^{y-u_L+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^{y-u_L}).$$

In this case we find

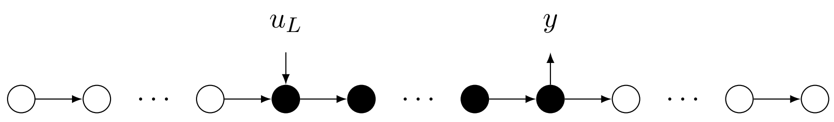
$$\begin{aligned}\mathscr{B} + \mathscr{V}^{y-u_L} &= \operatorname{Span}\{e_{u_L}\} + \operatorname{Span}\{e_i \mid i \notin \{u_L, u_{L+1}, \dots, y-1, y\}\} \\ &= \operatorname{Span}\{e_i \mid i \notin \{u_{L+1}, \dots, y-1, y\}\} = \mathscr{V}^{y-u_L-1}\end{aligned}$$

Thus $\mathscr{V}^{y-u_L+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-u_L-1}) = \mathscr{V}^{y-u_L}$ by (2.8) in Theorem 7.

Because $\mathscr{V}^{y-u_L} = \mathscr{V}^{y-u_L+1}$ we know

$$\mathscr{V}^{y-u_L} = \mathscr{V}^*$$

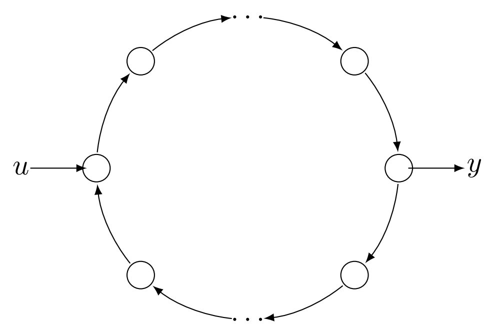
and the algorithm terminates.



Observations made during the computation
While using Theorem 7 to find $\mathscr{V}^*$ we made the following observations in the graph. In the first step of the algorithm only the observed vertex is contained in O^0 and then in each consecutive step one vertex to the left of the the original vertex y will be added to O^μ until the controlled vertex u_L is added to O^μ and the algorithm terminates. Again vertices that can't reach the observer are a part of D^*.

B.3 Computation of $\mathscr{V}^*$ in a circle graph

In this paragraph we will compute $\mathscr{V}^*$ for the directed circle graph of Figure 5. We will use the matrices as constructed in (B.9), (B.10) and (B.11), see Section 3.2. Similar to the previous sections we will assume that $n \geq 3$ such that $1, u$ and n are distinct vertices.



$$
A = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & \ddots & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 1 & 0
\end{bmatrix} \tag{B.9}
$$

$$
B = \begin{bmatrix}
0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0
\end{bmatrix} \tag{B.10}
$$

$$
C = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{0}_{u} & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{B.11}
$$

Preliminaries
To use Theorem 7 we need to find subspaces $\mathscr{K}$, $\mathscr{B}$ again.

$$
\begin{aligned}
\mathscr{K} &= \operatorname{Span}\{e_i \mid i \neq n\} \\
\mathscr{B} &= \operatorname{Span}\{e_u\}
\end{aligned}
$$

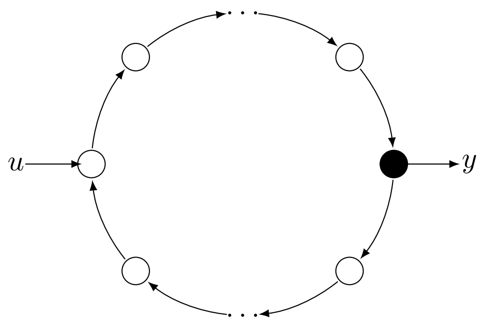
We also know $(Ax)_1 = x_n$ and for $i \in \{2, 3, \ldots, n\}$ we have $(Ax)_i = x_{i-1}$, by (B.9).

Computation of $\mathscr{V}^0$
We start again with the computation of $\mathscr{V}^0$.

$$
\mathscr{V}^0 = \mathscr{K} = \operatorname{Span}\{e_i \mid i \neq n\}
$$

In that case

$$
D^0 = V \setminus n
$$



Computation of $\mathscr{V}^\mu$ for $0 \leq \mu \leq n-u$
For $0 \leq \mu \leq n-u$ we want to prove that

$$\mathscr{V}^\mu = \text{Span}\{e_i \mid i \neq \{n, n-1, \ldots, n-\mu\}\} \tag{B.12}$$

Note that this equation is similar to equation (B.8) if we replace n by y.

(B.12) is true for $\mu = 0$ as we have seen above.

We prove that if (B.12) holds for μ ($\mu \leq n-u-1$), then it also holds for $\mu+1$. By Theorem 7 we have

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^{\mu+1}).$$

Because $\mu \leq n-u-1$, the index $n-\mu \geq n-n+u+1 = u+1 > u$. Thus $\mathscr{B} = \text{Span}\{e_u\} \subset \text{Span}\{e_i \mid i \notin \{n, n-1, \ldots, n-\mu\}\} = \mathscr{V}^\mu$ and so

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu).$$

Since we already know $\mathscr{K}$ we only compute $A^{-1}(\mathscr{V}^\mu)$,

$$A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}.$$

$Ax \in \mathscr{V}^\mu$ if and only if $(Ax)_i = 0$ for $i \in \{n, n-1, \ldots, n-\mu\}$ or $Ax \in \mathscr{V}^\mu$ if and only if $x_{i-1} = (Ax)_i = 0$ for $i \in \{n, n-1, \ldots, n-\mu\}$. Then

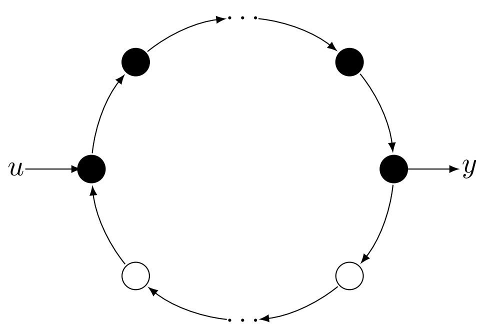
$$\begin{aligned}
&x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{n-1} = x_{n-2} = \cdots = x_{n-\mu-1} = 0\\
&\Rightarrow (A^{-1}\mathscr{V}^\mu) = \text{Span}\{e_i \mid i \notin \{n-1, n-2, \ldots, n-\mu-1\}\}\\
&\Rightarrow \mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu) = \text{Span}\{e_i \mid i \notin \{n, n-1, n-2, \ldots, n-\mu-1\}\}.
\end{aligned}$$

By induction we have proven that (B.12) holds for $0 \leq \mu \leq n-u$. In particular,

$$\begin{aligned}
\mathscr{V}^{n-u} &= \text{Span}\{e_i \mid i \notin \{n, n-1, \ldots, n-(n-u)\}\}\\
&= \text{Span}\{e_i \mid i \notin \{u, u+1, \ldots, n\}\}.
\end{aligned}$$

Again by the definition of D^μ.

$$D^{n-u} = V \setminus \{u, u+1, \ldots, n\}.$$



Termination of the algorithm while computing $\mathscr{V}^{n-u+1}$
To find $\mathscr{V}^{n-u+1}$ we again use

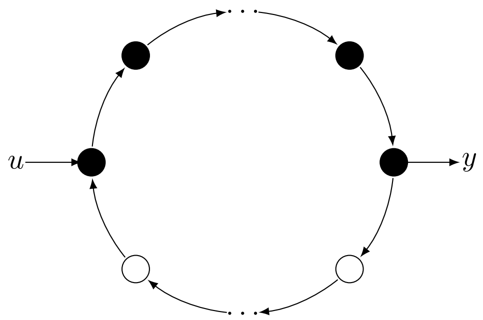
$$\mathscr{V}^{n-u+1} = \mathscr{K} \cap A^{-1}(\mathscr{B} + \mathscr{V}^{n-u})$$

The subspace $\mathscr{B} + \mathscr{V}^{n-u}$ can be calculated.

$$\begin{aligned}\mathscr{B} + \mathscr{V}^{n-u} &= \operatorname{Span}\{e_u\} + \operatorname{Span}\{e_i \mid i \notin \{u, u+1, \ldots, n-1, n\}\} \\ &= \operatorname{Span}\{e_i \mid i \notin \{u+1, \ldots, n-1, n\}\} \\ &= \mathscr{V}^{n-u-1}.\end{aligned}$$

Thus $\mathscr{V}^{n-u+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{n-u-1}) = \mathscr{V}^{n-u}$, and thus the algorithm terminates and we have

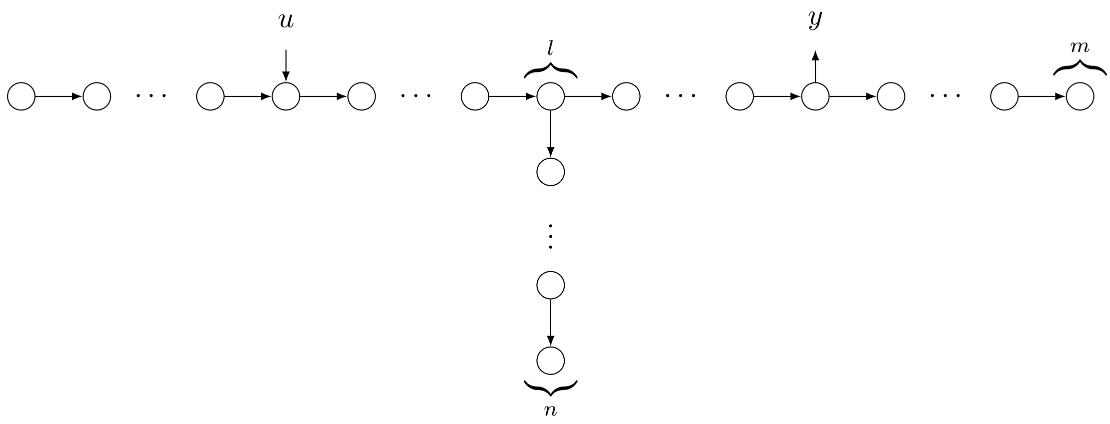
$$\mathscr{V}^* = \mathscr{V}^{n-u}$$



Observations made during the computation
During the compution of $\mathscr{V}^*$ in the circle graph we made the following observations. In both cases the observed vertex is the only element of O^0 and for each consecutive O^μ the vertex with the highest index that is not in $O^{\mu-1}$ will be added to O^μ until the construction of O^{n-u} when the controlled vertex u is added.

B.4 Computation of $\mathscr{V}^*$ in a graph where one vertex has two outgoing edges

Now we will compute $\mathscr{V}^*$ for a graph where one vertex l has two outgoing edges, see also Figure 7. In order to use Theorem 7 we will use the matrices in (B.13), (B.14) and (B.15). To help the reader, we marked the vertices l, m and n and we assume that these vertices, including y and u are distinct, so $n \geq 5$.



$$A = \begin{bmatrix}
0 & 0 & 0 & & & & & \cdots & & & & 0 & 0 \\
1 & 0 & 0 & & & & & \cdots & & & & 0 & 0 \\
0 & 1 & 0 & & & & & \cdots & & & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & & & & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & & & & \cdots & & & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & & & \cdots & & & 0 & 0 \\
 & & & & 0 & 1 & 0 & & & & & & \\
\vdots & \vdots & & & & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\
 & & & & & & 0 & 1 & 0 & & & & \\
0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & & & & & & & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & & & & & \cdots & & & & 0 & 1 & 0
\end{bmatrix} \tag{B.13}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{B.14}$$

$$C = \begin{bmatrix} 0 & \cdots & \underbrace{0}_{l} & \cdots & 0 & 1 & 0 & \cdots & \underbrace{0}_{m} & \cdots & 0 \end{bmatrix} \tag{B.15}$$

Preliminaries
As usual we first define $\mathscr{K}$, $\mathscr{B}$.

$$\mathscr{K} = \operatorname{Span}\{e_i \mid i \neq y\}$$
$$\mathscr{B} = \operatorname{Span}\{e_u\}.$$

By (B.13) we have for the operation Ax

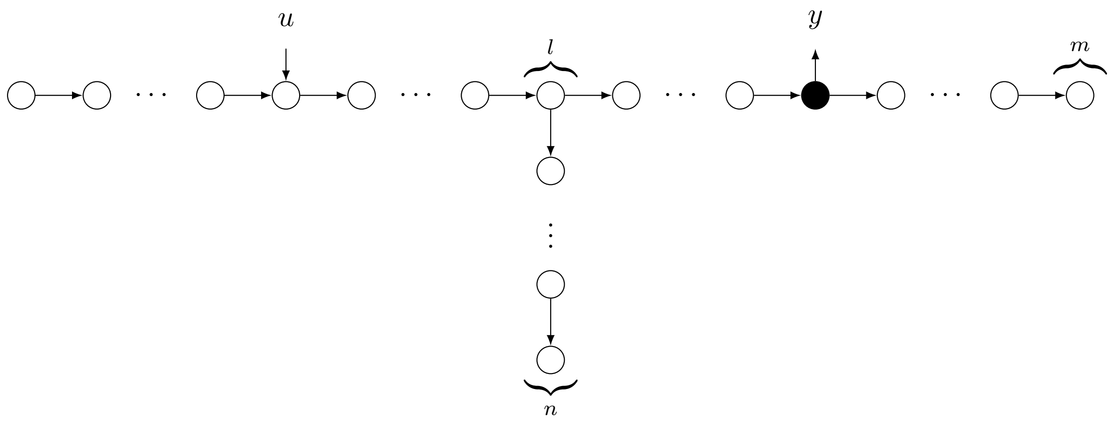
$$\begin{cases} (Ax)_i = 0 & \text{if } i = 1 \\ (Ax)_i = \frac{1}{2}x_l & \text{if } i = l+1 \text{ or } i = m+1 \\ (Ax)_i = x_{i-1} & \text{elsewhere} \end{cases}$$

Step 1: Computation of $\mathscr{V}^0$

$$\mathscr{V}^0 = \mathscr{K} = \text{Span}\{e_i \mid i \neq y\}$$

With $\mathscr{V}^0$ we can compute D^0 by using the definition.

$$D^0 = V \setminus \{y\}$$



Step 2: Computation of $\mathscr{V}^\mu$ for $0 \leq \mu \leq y - l - 2$
By induction on μ we will show that $\mathscr{V}^\mu$ for $0 \leq \mu \leq y - l - 2$ we have

$$\mathscr{V}^\mu = \text{Span}\{e_i \mid i \neq \{y, y-1, \ldots, y-\mu\}\} \tag{B.16}$$

Equation (B.16) is again very similar to (B.4) and (B.12).

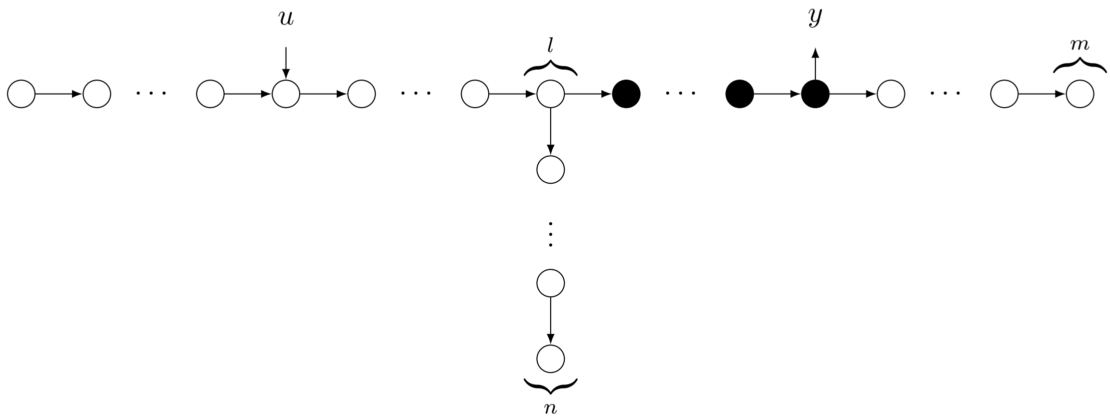
In the first step we showed that (B.16) holds for $\mu = 0$. Now assuming that (B.16) holds for μ we will prove that it also holds for $\mu+1$. We know $\mathscr{B} = \text{Span}\{e_u\} \subset \text{Span}\{e_i \mid i \neq y, y-1, \ldots, y-\mu\} = \mathscr{V}^\mu$ so

$$\begin{aligned} \mathscr{V}^{\mu+1} &= \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &= \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu). \end{aligned}$$

Now for $0 \leq \mu \leq y - l - 3$ we have $(Ax)_i = x_{i-1}$ and using (B.16) we can find $A^{-1}(\mathscr{V}^\mu)$.

$$\begin{aligned} &Ax \in \mathscr{V}^\mu \Leftrightarrow x_{i-1} = (Ax)_i = 0 \text{ for all } i \in \{y, y-1, \ldots, y-\mu\} \\ &\Rightarrow x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots = x_{y-\mu-1} = 0 \\ &\Rightarrow A^{-1}(\mathscr{V}^\mu) = \text{Span}\{e_i \mid i \notin \{y-1, y-2, \ldots, y-\mu-1\}\} \\ &\Rightarrow \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu) = \text{Span}\{e_i \mid i \neq y\} \cap \text{Span}\{e_i \mid i \notin \{y-1, y-2, \ldots, y-\mu-1\}\} \\ &\Rightarrow \mathscr{V}^\mu = \text{Span}\{e_i \mid i \notin \{y, y-1, y-2, \ldots, y-\mu-1\}\}. \end{aligned}$$

In particular $\mathscr{V}^{y-l-2} = \text{Span}\,\{e_i \mid i \notin \{y, y-1, y-2, \ldots, l+1\}\}$ and $D^{y-l-2} = V \backslash \{l+1, \ldots, y-1, y\}$.



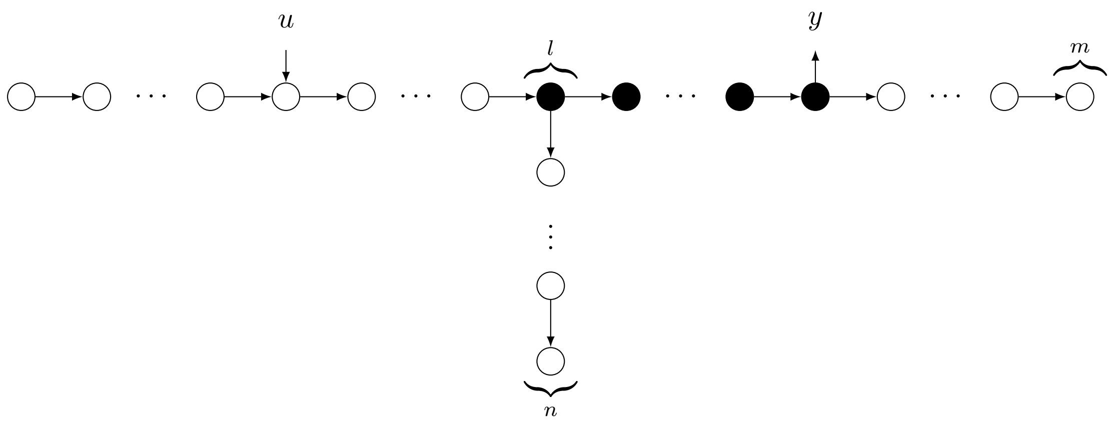
Step 3: Computation of $\mathscr{V}^{y-l-1}$
We continue by computing $\mathscr{V}^{y-l-1}$. Because $u < l+1$ we still have $\mathscr{B} \subset \mathscr{V}^{y-l-2}$ and so

$$\begin{aligned}\mathscr{V}^{y-l-1} &= \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-l-2} + \mathscr{B}) \\ &= \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-l-2}).\end{aligned}$$

We know $\mathscr{K}$ so we only want to find $A^{-1}(\mathscr{V}^{y-l-2})$. Remember $(Ax)_i = x_{i-1}$ for $i \in l+2, l+3, \ldots, y$, $(Ax)_{l+1} = \frac{1}{2}x_l$ and $\mathscr{V}^{y-l-2} = \text{Span}\,\{e_i \mid i \notin \{y, y-1, y-2, \ldots, l+1\}\}$ so

$$\begin{aligned}&Ax \in \mathscr{V}^{y-l-2} \Leftrightarrow (Ax)_y = (Ax)_{y-1} = \cdots = (Ax)_{l+2} = (Ax)_{l+1} = 0 \\ &\Rightarrow x \in A^{-1}(\mathscr{V}^{y-l-2}) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots x_{l+1} = \frac{1}{2}x_l = 0 \\ &\Rightarrow x \in A^{-1}(\mathscr{V}^{y-l-2}) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots x_{l+1} = x_l = 0 \\ &\Rightarrow A^{-1}(\mathscr{V}^{y-l-2}) = \text{Span}\,\{e_i \mid i \notin \{y-1, \ldots, l+1, l\}\} \\ &\Rightarrow \mathscr{V}^{y-l-1} = \mathscr{K} \cap A^{-1}\mathscr{V}^{y-l-2} = \text{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, l+1, l\}\}.\end{aligned}$$

Then by definition of D^μ we know $D^{y-l-1} = V \setminus \{l, \ldots, y-1, y\}$.



Step 4: Computation of $\mathscr{V}^\mu$ for $y - l \leq \mu \leq y - u$
Via induction we can prove that (B.16) also holds for $y - l \leq \mu \leq y - u$. In Step 3, we already

saw that (B.16) holds for $\mu = y - l$. Now we prove that if (B.16) holds for μ with $\mu \leq y - u - 1$ then it also holds for $\mu + 1$.

$\mu \leq y - u - 1$ so we get $i \geq y - \mu \geq y - (y - u - 1) = u + 1 > u$, thus $\mathscr{B} \subset \mathscr{V}^\mu$ and we have

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu).$$

We are familiar with $\mathscr{K}$ so we want to find $A^{-1}(\mathscr{V}^\mu)$.

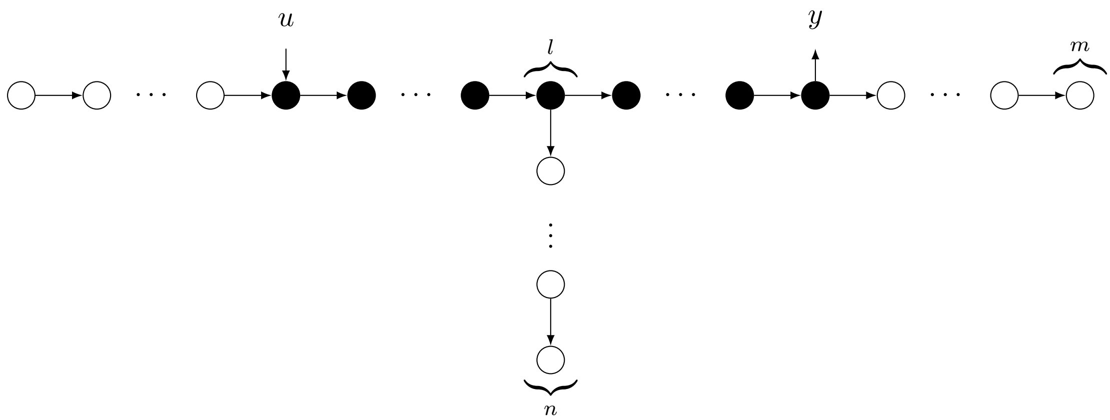
$$A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}.$$

By (B.16) we know $Ax \in \mathscr{V}^\mu$ if and only if $(Ax)_i = 0$ for $i \in \{y, y-1, \dots, y-\mu\}$ or $x \in A^{-1}(\mathscr{V}^\mu$ if and only if $x_{i-1} = (Ax)_i = 0$ for $i = \{y, y-1, \dots, y-\mu\} \setminus \{l+1\}$ and $x_l = 2(Ax)_{l+1} = 0$ so

$$\begin{aligned}
&x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots = x_l = \cdots = x_{y-\mu-1} = 0 \\
&\Rightarrow A^{-1}(\mathscr{V}^\mu) = \mathrm{Span}\,\{e_i \mid i \notin \{y-1, y-2, \dots, l, \dots, y-\mu-1\}\} \\
&\Rightarrow \mathscr{V}^{\mu+1} = A^{-1}(\mathscr{V}^\mu) \cap \mathscr{K} = \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, y-2, \dots, y-\mu-1\}\}
\end{aligned}$$

Now we have proven that (B.16) also holds for $y - l \leq \mu \leq y - u$ and in particular

$$\begin{aligned}
\mathscr{V}^{y-u} &= \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, y-2, \dots, y-(y-u)\}\} \\
&= \mathrm{Span}\,\{e_i \mid i \notin \{u, u+1, \dots, y\}\}, \\
D^{y-u} &= V \setminus \{u, u+1, \dots, y\}.
\end{aligned}$$



Step 5: Termination of the algorithm by computing $\mathscr{V}^{y-u+1}$
Now we show that $\mathscr{V}^{y-u+1} = \mathscr{V}^{y-u}$ and thus $\mathscr{V}^{y-u} = \mathscr{V}^*$. We still have

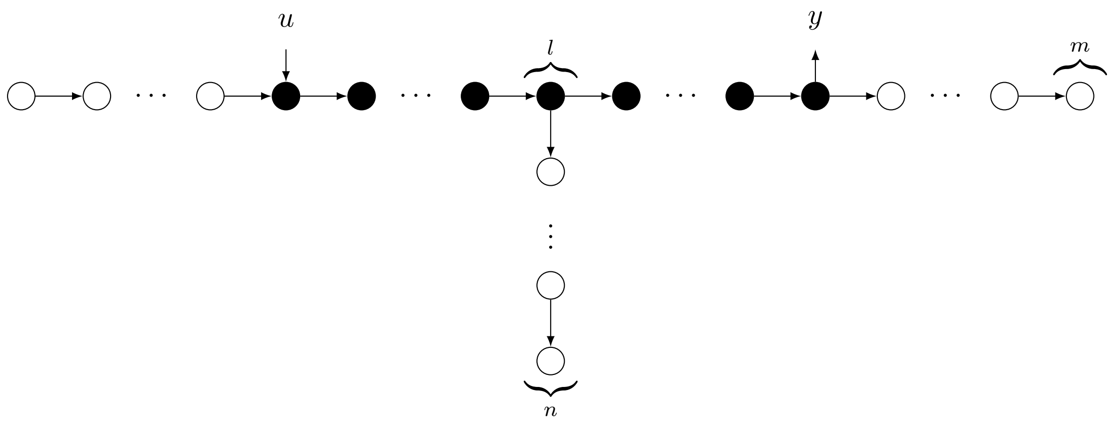
$$\mathscr{V}^{y-u+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-u} + \mathscr{B})$$

and we also know that $V^{y-u-1} = \mathrm{Span}\,\{e_i \mid i \notin \{u+1, \dots, y\}\}$. Furthermore

$$\begin{aligned}
\mathscr{V}^{y-u} + \mathscr{B} &= \mathrm{Span}\,\{e_i \mid i \notin u, u+1, \dots, y\} + \mathrm{Span}\,\{e_u\} \\
&= \mathrm{Span}\,\{e_i \mid i \notin u+1, \dots, y\} = \mathscr{V}^{y-u-1}.
\end{aligned}$$

That means that $\mathscr{V}^{y-u+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-u} + \mathscr{B}) = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-u-1} + \mathscr{B}) = \mathscr{V}^{y-u}$ and thus

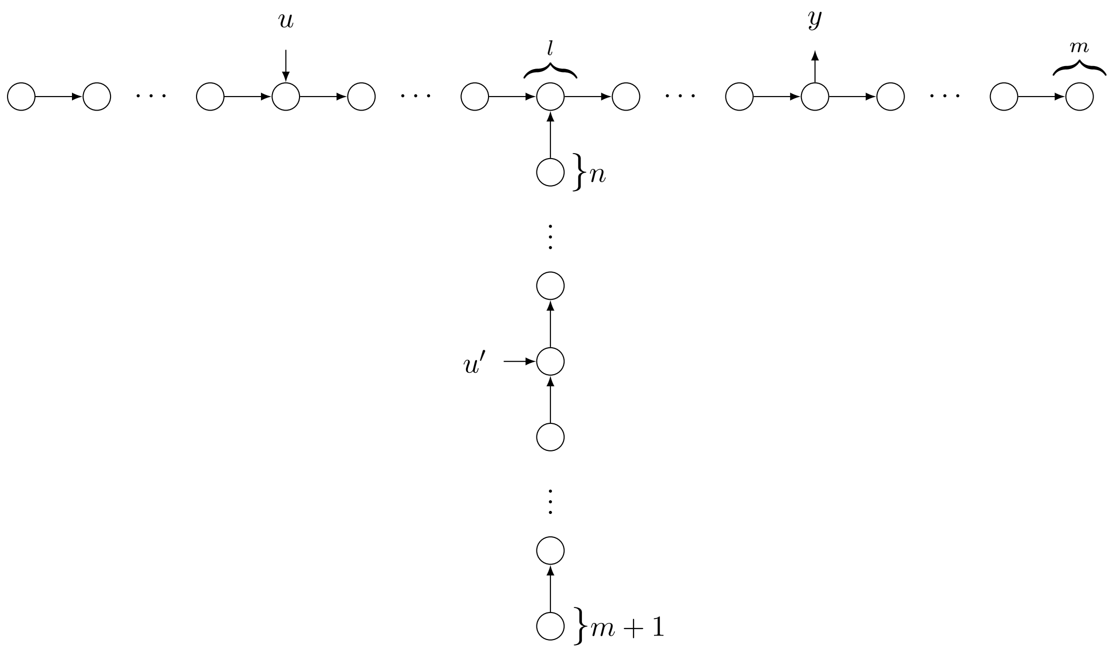
$$\begin{aligned} V^* &= \operatorname{Span}\{e_i \mid i \notin u, u+1, \ldots, y\} \\ D^* &= V \setminus \{u, u+1, \ldots, y\} \end{aligned}$$



Observations made during the computation.
First, we see that the result we get is very similar to the result for a linegraph with the observer to the right of the controller. Only the path from u to y is not in O^*. Secondly, only the vertex in D^μ that has an outgoing edge to O^μ is added to $O^{\mu+1}$, just like in the last three examples. Also like in the last three examples, all the vertices that cannot reach y are elements of D^*.

B.5 Computation of $\mathscr{V}^*$ in a graph where one vertex has two incoming edges

Finally we show the computation of $\mathscr{V}^*$ in a graph where one vertex l has two incoming edges, see the figure below. For clarity we marked the spots for the vertices m, $m+1$ and l. We also named one vertex u', this is the vertex such that $d(u,l) = d(u',l)$. We will assume that this vertex exist (so $d(m+1,l) > d(u,l)$) and we will assume that $n > 6$ such that $u, l, y, m, m+1$ and u' are distinct vertices. We also used the matrices in (B.17), (B.18) and (B.19).



$$A = \begin{bmatrix}
0 & & & & & & & & & & \cdots & & 0 & 0 \\
1 & 0 & & & & & & & & & \cdots & & 0 & 0 \\
0 & 1 & 0 & & & & & & & & \cdots & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & & & & & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & & & & & & \cdots & & 0 & 0 \\
0 & & \cdots & 0 & 1 & 0 & & & & & \cdots & & 0 & 1 \\
0 & & \cdots & & 0 & 1 & 0 & & & & \cdots & & 0 & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & & & & & \vdots & \vdots \\
0 & & \cdots & & & & 0 & 1 & 0 & & \cdots & & 0 & 0 \\
0 & & \cdots & & & & & 0 & 0 & 0 & \cdots & & 0 & 0 \\
0 & & \cdots & & & & & & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & & \cdots & & & & & & & & 0 & 1 & 0 & 0 \\
0 & & \cdots & & & & & & & & & 0 & 1 & 0
\end{bmatrix} \tag{B.17}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{B.18}$$

$$C = \begin{bmatrix} 0 & \cdots & \underbrace{0}_{l} & \cdots & 0 & 1 & 0 & \cdots & \underbrace{0}_{m} & 0 \end{bmatrix} \tag{B.19}$$

Preliminaries

In this example we have

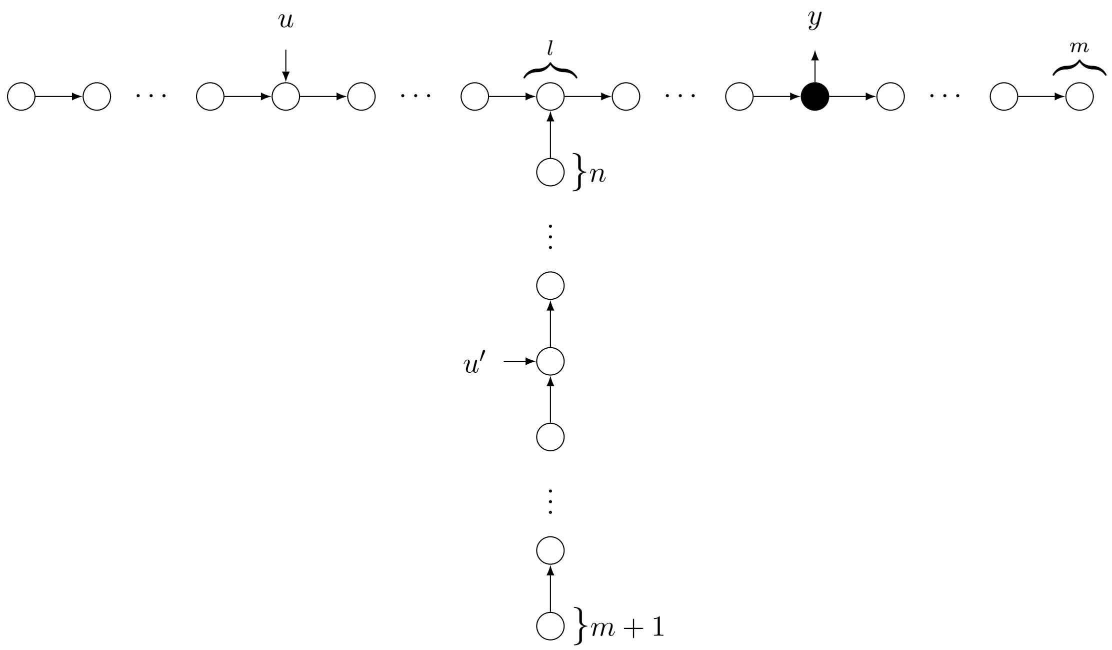
$$\mathscr{K} = \operatorname{Span}\{e_i \mid i \neq y\} \tag{B.20}$$

$$\mathscr{B} = \operatorname{Span}\{e_u\} \tag{B.21}$$

$$\begin{cases} (Ax)_i = 0 & \text{if } i \in \{1, m+1\} \\ (Ax)_i = x_{l-1} + x_n & \text{if } i = l \\ (Ax)_i = x_{i-1} & \text{elsewhere} \end{cases} \tag{B.22}$$

Step 1: Computation of $\mathscr{V}^0$
We start by calculating $\mathscr{V}^0$.

$$\begin{aligned} &\mathscr{V}^0 = \mathscr{K} = \operatorname{Span}\{e_i \mid i \neq y\} \\ &\Rightarrow D^0 = V \setminus \{y\} \end{aligned}$$



Step 2: Computation of $\mathscr{V}^\mu$ for $0 \leq \mu \leq y - l - 1$
Via induction on μ we prove that for $0 \leq \mu \leq y - l - 1$ we have

$$\mathscr{V}^\mu = \operatorname{Span}\{e_i \mid i \neq \{y, y-1, \ldots, y-\mu\}\} \tag{B.23}$$

Equation (B.23) is the same equation as (B.16), presumably because part of these graphs is similar.

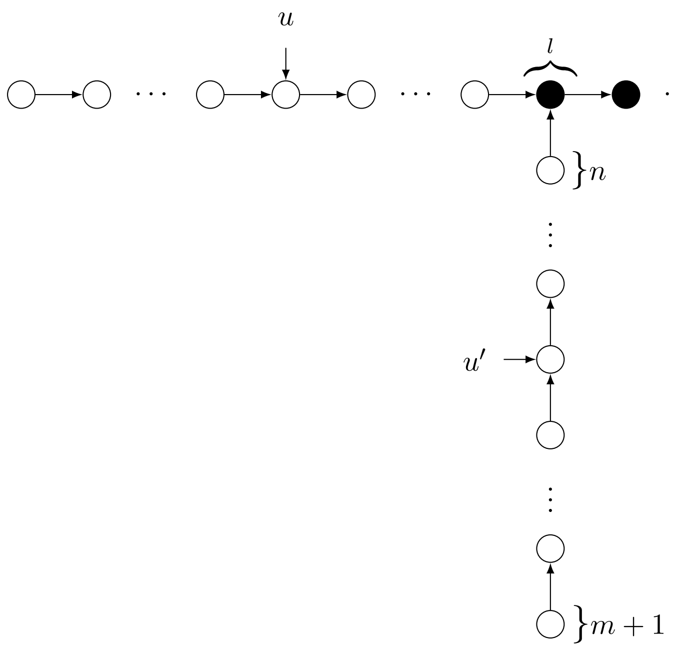
In the first step we proved that (B.23) holds for $\mu = 0$. Now we suppose that (B.23) holds for μ and we prove that (B.23) also holds for $\mu+1$. We know $\mathscr{B} = \operatorname{Span}\{e_u\} \subset \operatorname{Span}\{e_i \mid i \neq \{y, y-1, \ldots, y-\mu\}\} = \mathscr{V}^\mu$ so

$$\begin{aligned} \mathscr{V}^{\mu+1} &= \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu + \mathscr{B}) \\ &= \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu). \end{aligned}$$

Now for $0 \le \mu \le y-l-2$ we have $(Ax)_i = x_{i-1}$ and using (B.23) we can find $A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}$.

$$
\begin{aligned}
&Ax \in \mathscr{V}^\mu \Leftrightarrow x_{i-1} = (Ax)_i = 0 \text{ for all } i \in \{y, y-1, \ldots, y-\mu\} \\
&\Rightarrow x \in A^{-1}(\mathscr{V}^\mu) \Leftrightarrow x_{y-1} = x_{y-2} = \cdots = x_{y-\mu-1} = 0 \\
&\Rightarrow A^{-1}(\mathscr{V}^\mu) = \operatorname{Span}\{e_i \mid i \notin \{y-1, y-2, \ldots, y-\mu-1\}\} \\
&\Rightarrow K \cap A^{-1}(\mathscr{V}^\mu) = \operatorname{Span}\{e_i \mid i \neq y\} \cap \operatorname{Span}\{e_i \mid i \notin \{y-1, y-2, \ldots, y-\mu-1\}\} \\
&\Rightarrow \mathscr{V}^\mu = \operatorname{Span}\{e_i \mid i \notin \{y, y-1, y-2, \ldots, y-\mu-1\}\}.
\end{aligned}
$$

In particular $\mathscr{V}^{y-l-1} = \operatorname{Span}\{e_i \mid i \notin \{y, y-1, y-2, \ldots, l\}\}$ and thus $D^{y-l-1} = V \setminus \{l, \ldots, y-1, y\}$.



Step 3: Computation of $\mathscr{V}^{y-l}$
We continue by computing $\mathscr{V}^l$. Because $u < l$ we have $\mathscr{B} \subset \mathscr{V}^{y-l-1}$ and

$$\mathscr{V}^{y-l} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-l-1})$$

We know that $x \in \mathscr{K}$ if and only if $x_y = 0$ so we want to find $A^{-1}(\mathscr{V}^{y-l-1}) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^{y-l-1}\}$. We know

$$Ax \in \mathscr{V}^{y-l-1} \Leftrightarrow (Ax)_i = 0 \text{ for } i \in \{y, y-1, \ldots, l\}$$

and by (B.22) we have $x_{l-1} + x_n = (Ax)_l = 0$ and $x_{i-1} = (Ax)_i = 0$ for $i \in \{y, y-1, \ldots, l+1\}$

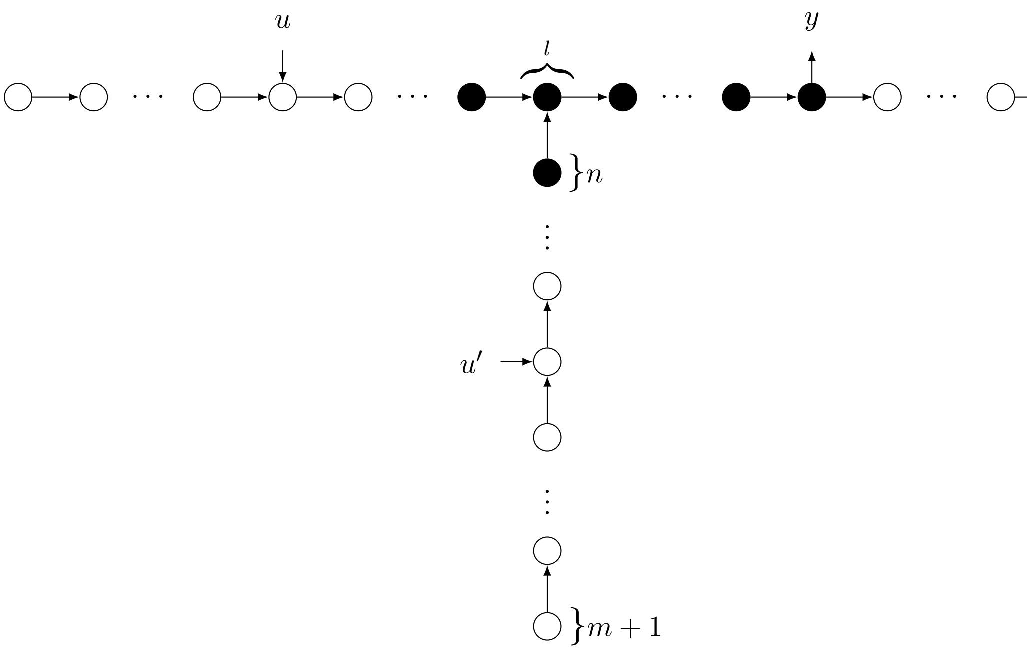
so

$$
\begin{aligned}
&x \in A^{-1}(\mathscr{V}^{y-l-1}) \Leftrightarrow \begin{cases} x_{l-1} + x_n = 0 \\ x_{y-1} = x_{y-2} = \cdots = x_{l+1} = 0 \end{cases} \\
&\Rightarrow x \in \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-l-1}) \Leftrightarrow \begin{cases} x_{l-1} = -x_n \text{ or } x_{l-1} = x_n = 0 \\ x_y = x_{y-1} = x_{y-2} = \cdots = x_{l+1} = 0 \end{cases} \\
&\Rightarrow \mathscr{V}^{y-l} = \mathscr{K} \cap A^{-1} = \mathrm{Span}\,\{e_i \mid i \notin \{y, y-1 \ldots, l+1, l, l-1, n\}\} + \mathrm{Span}\,\{e_{l-1} - e_n\}
\end{aligned}
$$

Now in this case $\mathscr{V}^{y-l}$ is not the span of unit vectors and the definition for D^* is not valid. Hence to visualize this step in our computation we use the definitions of D_{y-l} and O_{y-l}.

$$D_{y-l} = V \setminus \{y, y-1, \ldots, l, l-1, n\}$$

In the figure below we have given the vertices belonging to D_{y-l} a white color and the vertices be-



longing to O_{y-l} a black color.

Step 4: Computation of $\mathscr{V}^{\mu}$ for $y - l \leq \mu \leq y - u - 1$
For $y - l \leq \mu \leq y - u + 1$ we will prove that

$$
\begin{aligned}
V^{\mu} = &\mathrm{Span}\,\{e_i \mid i \notin \{y, y-1, \ldots, y-\mu-1, n, n-1, \ldots, n+y-l-\mu\}\} \\
&+ \mathrm{Span}\,\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \ldots, l+\mu-y\}\}
\end{aligned} \tag{B.24}
$$

In Step 3 we have already shown that (B.24) holds for $\mu = y - l$. Now we show that if (B.24) holds for μ then it also holds for $\mu + 1$.

Since $i \geq y - \mu + 1 \geq y - (y - u - 2) - 1 = u + 1 > u$ we have $\mathscr{B} \subset \mathscr{V}^{\mu}$ and

$$\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{\mu})$$

Again we are familiar with $\mathscr{K}$ so we are only interested in $A^{-1}(\mathscr{V}^\mu) = \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^\mu\}$ and thus we want to know when $Ax \in \mathscr{V}^\mu$

$$Ax \in \mathscr{V}^\mu \Leftrightarrow \begin{cases} \text{(a)} \quad (Ax)_i = 0 \text{ for } i \in \{y, y-1, \dots, y-\mu-1, n, n-1, \dots, n+y-l-\mu\} \\ \text{(b)} \quad (Ax)_{l-1-j} = -(Ax)_{n-j} \text{ for } j \in \{0, 1, \dots, l+\mu-y\} \end{cases}$$

First we want to find x such that (a) holds. For $i \in \{y, y-1, \dots, l+1, l-1, \dots, y-\mu-1, n, n-1, \dots, n+y-l-\mu\}$ we have $x_{i-1} = 0$ and for $i = l$ we have $x_{l+1} = -x_n$ or $x_{l+1} = x_n = 0$.

Second we want to find x such that (b) holds. For $j \in \{0, 1, \dots, l+\mu-y\}$ we have $x_{l-2-j} = -x_{n-1-j}$. Summarizing we have

$$Ax \in \mathscr{V}^\mu \Leftrightarrow \begin{cases} x_{i-1} = 0 & \text{for } i \in \{y, y-1, \dots, y-l\} \\ x_{l-2-j} = -x_{n-1-j} \text{ or } x_{1-2-j} = x_{n-1-j} = 0 & \text{for } j \in \{-1, 0, 1, \dots, l+\mu-y\} \end{cases}$$

$$\Rightarrow x \in \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu) \Leftrightarrow \begin{cases} x_y = 0 \\ x_{y-1} = x_{y-2} = \cdots = x_{y-l} = 0 \\ (x_{l-1} = -x_n), (x_{l-2} = -x_{n-1}), \dots, (x_{y-\mu-2} = -x_{m+y-l-\mu-1}) \text{ or} \\ x_{l-1} = x_{l-2} = \cdots = x_{y-\mu-2} = x_n = x_{n-1} = x_{n-2} = \cdots = x_{n+y-l-\mu-1} = 0 \end{cases}$$

Since $\mathscr{V}^{\mu+1} = \mathscr{K} \cap A^{-1}(\mathscr{V}^\mu)$ this means

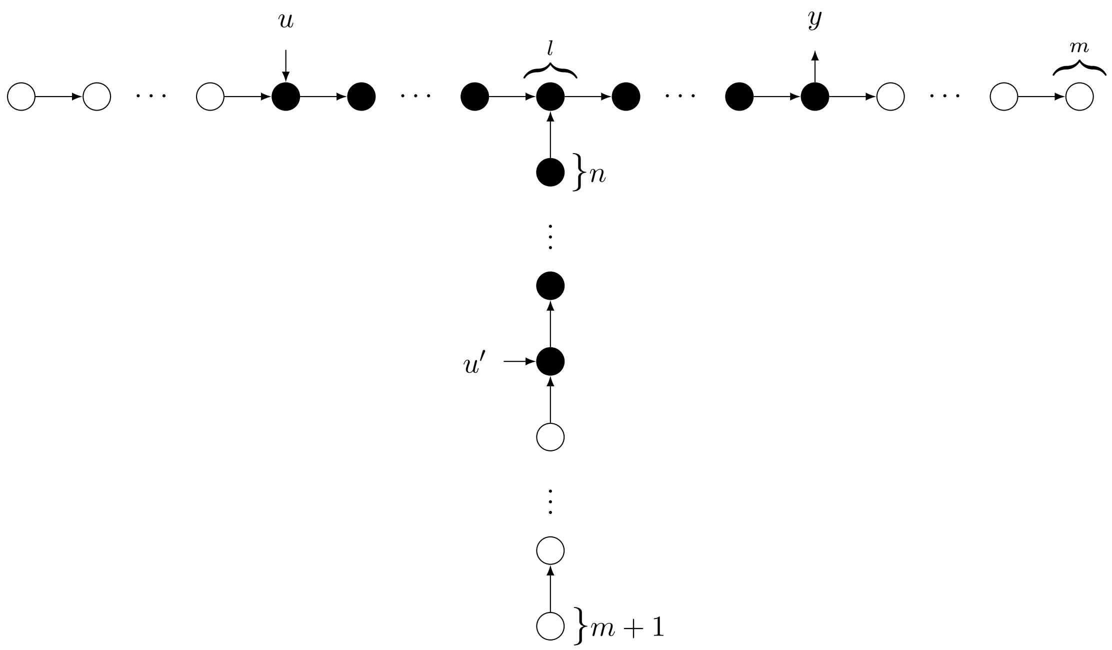
$$\begin{aligned} V^{\mu+1} =& \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, y-\mu-2, n, n-1, \dots, n+y-l-\mu-1\}\} \\ &+ \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l+\mu-y\}\} \end{aligned}$$

and we have proven that (B.24) holds for $\mu + 1$.

Now again $\mathscr{V}^\mu$ is not the span of unit vectors and we use of D_μ and O_μ. For $\mathscr{V}^{y-u-1}$ in particular

$$\begin{aligned} \mathscr{V}^{y-u-1} &= \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, y-(y-u-1)-1, n, n-1, \dots, n+y-l-(y-u-1)\}\} \\ &+ \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l+(y-u-1)-y\}\} \\ \mathscr{V}^{y-u-1} &= \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, u, n, n-1, \dots, n-l+u+1\}\} \\ &+ \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l-u-1\}\} \\ &\Rightarrow D_{y-u-1} = V \setminus \{y, y-1, \dots, u, n, n-1, \dots, n-l+u+1\} \end{aligned}$$

If we write $n-l+u+1 = u'$ and color every vertex in D_{y-u-1} white and every vertex in O_{y-u-1} black we get



Step 5: Termination of the algorithm by computing $\mathscr{V}^{y-u}$
Finally we show that the $\mathscr{V}^{y-u} = \mathscr{V}^*$, because $\mathscr{V}^{y-u} = \mathscr{V}^{y-u-1}$.

$$\mathscr{V}^{y-u} = \mathscr{K} \cap A^{-1}(\mathscr{V}^{y-u-1} + \mathscr{B})$$

$\mathscr{K}$ is still known so we are only interested in $A^{-1}(\mathscr{V}^{y-u-1} + \mathscr{B})$ and we know

$$\begin{aligned}
A^{-1}(\mathscr{V}^{y-u-1} + \mathscr{B}) &= \{x \in \mathscr{X} \mid Ax \in \mathscr{V}^{y-u-1} + \mathscr{B}\} \\
\mathscr{V}^{y-u-1} + \mathscr{B} &= \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, u, n, n-1, \dots, n-l+u+1\}\} \\
&\quad + \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l-u-1\}\} \\
&\quad + \operatorname{Span}\{e_u\} \\
&= \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, u+1, n, n-1, \dots, n-l+u+1\}\} \\
&\quad + \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l-u-1\}\}
\end{aligned}$$

Hence we are looking for $x \in \mathscr{X}$ such that $Ax \in \mathscr{V}^{y-u-1} + \mathscr{B}$ and we know

$$\begin{aligned}
Ax &\in \operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, u+1, n, n-1, \dots, n-l+u+1\}\} \\
&\Leftrightarrow (Ax)_i = 0 \text{ for } i \in \{y, y-1, \dots, u+1, n, n-1, \dots, n-l+u+1\} \\
Ax &\in \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l-u-1\}\} \\
&\Leftrightarrow (Ax)_{l-1-j} = -(Ax)_{n-j} \text{ or } (Ax)_{l-1-j} = (Ax)_{n-j} = 0 \text{ for } j \in \{0, 1, \dots, l-u-1\}
\end{aligned}$$

For $j = l-u-1$ we have $(Ax)_{n-l+u-1} = (Ax)_u$ where $(Ax)_u$ is free, so this condition can always be satisfied. Combining this with our knowledge about the operation Ax, see (B.22) we get

$$x \in A^{-1}(\mathscr{V}^{y-u-1} + \mathscr{B}) \Leftrightarrow \begin{cases} x_{i-1} = 0 & \text{for } i \in \{y, y-1, \dots, l+1\} \\ x_{l-1-j} = -x_{n-j} \text{ or } x_{l-1-j} = x_{n-j} = 0 & \text{for } j \in \{0, \dots, l-u-1\} \\ x \in \mathscr{K} & \Leftrightarrow x_y = 0 \end{cases}$$

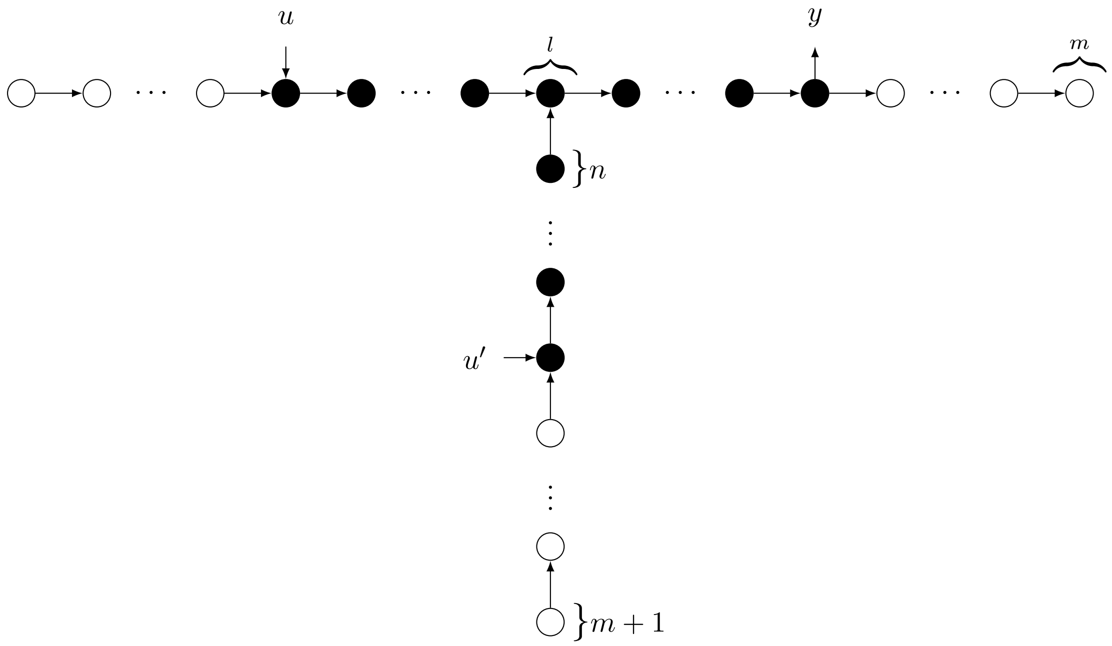
Then we have for $\mathscr{V}^{y-u}$

$$\begin{aligned}\mathscr{V}^{y-u} = &\operatorname{Span}\{e_i \mid i \notin \{y, y-1, \dots, u, n, n-1, \dots, n-l+u+1\}\} \\ &+ \operatorname{Span}\{e_{l-1-j} - e_{n-j} \mid j \in \{0, 1, \dots, l-u-1\}\} = \mathscr{V}^{y-u-1}\end{aligned}$$

Now we have proven that $\mathscr{V}^{y-u-1} = \mathscr{V}^*$. Again we use the definition for D_μ and get

$$D_{y-u-1} = V \setminus \{y, y-1, \dots, u, n, n-1, \dots, n-l+u+1 = u'\}$$

which we have visualized in the figure below by coloring the vertices in D_{y-u-1} white and the vertices in O_{y-u-1} black.



Observations made during the computation
While computing $\mathscr{V}^*$ for this graph we roughly did the same observations as in the last four examples. However we also observed that if there exists a vertex u' on the path from $m+1$ to l such that $d(u', l) = d(u, l)$ only the paths from u to y and u' to y were part of O_*.