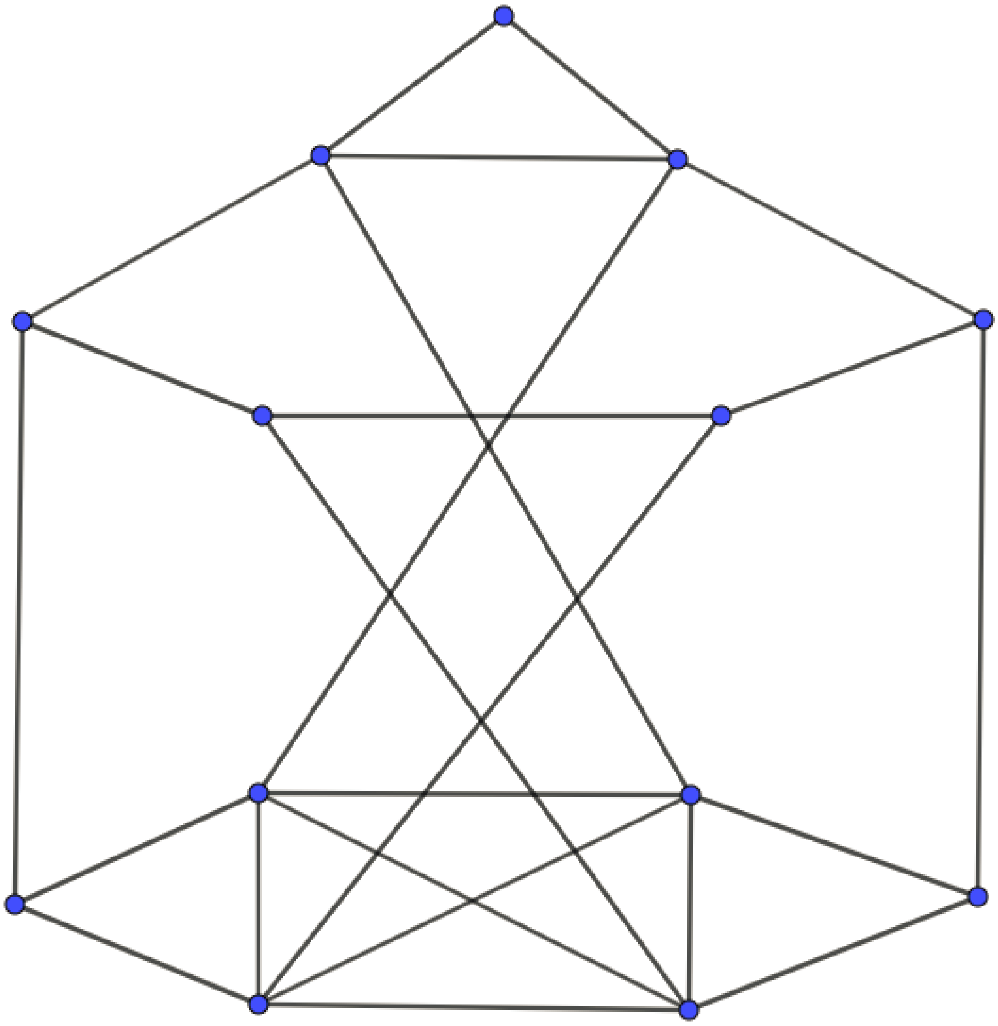
Which graphs produce irreducible copositive matrices?

Reinier de Zeeuw

June 2018



Contents

1 Introduction

Copositive matrices are of interest for various problems in operations research. The definition of a copositive matrix is given below.

Definition 1.1 (Copositive matrix)**.** *A symmetric matrix* $A \in S^n$ *(with* S^n *the set op symmetric matrices with n rows and columns) is copositive if* $x^T Ax \geq 0$ *for all nonnegative vectors* $x \in R^n_+$*.*

Definition 1.2. COP^n *is defined to be the set of copositive matrices.*

The concept of copositivity was first introduced by Prof. Dr. Theodore S. Motzkin in the field of algebra in the 1950s [5]. Due to the wide range of applications of copositive optimisation, copositivity is getting more and more interest over the last twenty years. The applications themselves will not be widely discussed here, but they spread from fields like Markov decision theory, game theory, model friction and queuing [6]. In this research the influence of copositive matrices on the stable set problem will be discussed.
The problem of checking whether a matrix is copositive however is NP-hard. This means the computation time needed to check whether a matrix is copositive will increase exponentially as the size of the matrix increases, unless P = NP. It will not be possible to check big matrices on copositivity in reasonable time. The goal of this research is to research problematic instances of copositive matrices. Because checking a matrix on copositivity is NP-hard, when optimising over COP^n, the copositive matrix is usually replaces by tractable inner approximation. It is useful to research how good such an approximation is, by researching whether difficult instances are contained in it. One of those difficult instances are called irreducible matrices. In this project graph theory will be used to generate such irreducible matrices considering properties of simple graphs. The definition of an irreducible matrix is given below.

Definition 1.3 (Irreducible matrix)**.** *A matrix* $A \in COP^n \backslash \{O\}$*, is irreducible if it is not the sum of a copositive matrix and a matrix in* $(PSD^n + N^n) \backslash \{O\}$*.*

Here PSD^n represents the set of positive semi-definite matrices, N^n the set of symmetric matrices that are entrywise nonnegative and O the matrix with all zeros.

This research will continue preliminary work by Peter J.C. Dickinson [1] and the goal of it is to find graphs that produce irreducible matrices. First all graphs of a small amount of vertices that produce irreducible matrices will be researched, then ways to combine graphs producing irreducible matrices to make bigger graphs producing irreducible matrices will be elaborated. Furthermore, the amount of graphs producing irreducible matrices with v(G) $\leq$ 13 will be discussed.

2 Literature Review

As mentioned before this report is continuing preliminary work by Peter J.C. Dickinson[1]. The main result of this, theorem 3.2 which will be key in this research. In this assignment, first some definitions were given, to be able to explain the transformation used between graphs and matrices. A summery of the theory and results will be discussed in section 3.

Graph theory with applications

The definitions and notations concerning graph theory defined in this book [3] were used in this research. For a list of notations used see page 257. Also the theory discussed in chapter 7 of this book was used in this research and in the assignments were some ideas for possible usable theorems and graph operations.

Nauty

Nauty [4] is the worlds fastest isomorphism checking program available. The program can be used to create a file of all graphs, with just one isomorphism of each, with certain conditions, for example connectivity or 2-connectivity. At the start of this research this program was used to create a file of all connected graphs with vertex number up to 9.

'On α-critical graphs and their construction'

The PhD research by Benjamin Small [2] is a closely related research as this one, researching α-critical graphs, instead of cop-irreducible graphs. In his research, Benjamin Small used the 'Smith' computer chain of the Washington state to create a file of all α-critical graphs up to 13 vertices. This file was used in this research to find all cop-irreducible graphs up to 13 vertices, as α-critical is one of the conditions for a graph to be cop-irreducible.
Furthermore, Benjamin Small defined some graph operations to create bigger α-critical graphs out of smaller ones. These operations were looked at, but unfortunately under his conditions, they did not turn out to preserve the α-covered condition of cop-irreducible graphs. Buckling, for example, seemed to hold for small graphs, but while buckling bigger cop-irreducible graphs, sometimes the result was not α-covered. Unfortunately, no conditions were found for when this would hold.

3 Preliminary results

In this section a summary of the preliminary results discussed in [1] will be discussed, along with some additional definitions and lemmas, first starting with the important definitions needed to understand the transformation between matrices and graphs.
In the transformation from graphs to matrices, only simple graphs are used. Also, two graphs that are isomorphic to each other have the same properties and the produce the similar matrices (only some rows and columns are swapped). So only in this project only one isomorphism of a graph is looked into. In the rest of this project, therefore when considering graphs, just one isomorphism of simple graphs are considered.
To be able to explain the transformation from a graph to a matrix, first the term stability number needs to be introduced.

Definition 3.1. *A stable set of a graph G is a subset S of the vertices of G so that no pair of vertices in S are connected by an edge. A stable set is called a maximum stable set if there is no stable set with strictly greater cardinality.*

Definition 3.2 (Stability number)**.** *The stability number of a graph G denoted by α(G) is the cardinality of the maximum stable sets of G.*

Definition 3.3 (α-critical)**.** *A graph G is α-critical if α(G - {e})> α(G) for each edge e $\in$ E(G).*

Remark 3.1. *If G is α-critical, for any edge e $\in$ E(G), α(G - {e})= α(G) + 1.*

Proof. See [2] page 4 proposition 2.1. □

Furthermore, for a graph of n vertices let A_G be the adjacency matrix of G, $E \in S^n$ the matrix with n rows and columns with all ones and $I \in S^n$ the identity matrix with n rows and columns. By corollary 2.4 from [7] $\lambda(I + A_G)$ - E $\in COP^n$ if and only if $\lambda \geq \alpha_G$ and for $\lambda > \alpha_G$ $\lambda(I + A_G)$ - E $\in COP^n$ is not irreducible. So $Z_G = \alpha_G(I + A_G)$ - E $\in COP^n$ is a copositive matrix that might be irreducible depending on the graph G. To make writing this report easier, we will define the following:

Definition 3.4. *A graph G is called an cop-irreducible graph if its corresponding matrix Z_G is irreducible.*

Definition 3.5 (α-covered)**.** *A graph G is α-covered if and only if for any two vertices (i, j) which are not connected by an edge, there is a maximum stable set of G which contains both i and j.*

Theorem 3.2. [1] *Given a simple graph G with stability number α(G) $\geq$ 2, G is cop-irreducible if and only if the following three conditions hold:*

1. *G is connected;*
2. *G is α-critical;*
3. *G is α-covered;*

Lemma 3.1. *A simple graph G with stability number α(G) = 1, can not be cop-irreducible.*

Proof Since a graph with stability number α(G) = 1 must be a complete graph, its adjacency matrix A_G plus the identity matrix is the matrix with all ones. Filling this in in the equation for

[1] Source and introduction for this theorem: personal communication with Peter J.C.Dickinson

Z_G, gives the zero matrix, which by definition of an irreducible matrix is excluded.

This theorem can be used to find all cop-irreducible graphs by checking all possible graphs one these three conditions. An example of cop-irreducible graphs for example is every odd cycle.

4 Methods

At the start of the project the program Nauty [4] was used to get a data set of all connected graphs. This data set automatically rules out any graph that does not satisfy condition one of theorem 3.2. This was loaded in the program Wolfram Mathematica, for Nauty produces a graph set in .g6 format, which Mathematica can load. To check the graphs on the second en third condition of theorem 3.2 an script that is pretty much a brute force algorithm was used. Since brute forcing took little enough computation time, no smart algorithm was necessary.
However the file with all graphs of 10 vertices is already to big for Mathematica to load. However in his research on α-critical graphs [2], Benjamin Small created a file with all α-critical graphs up to 13 vertices. For this he also used the program Nauty, but he had access to the 'Smith' computer cluster of the Washington State University, to be able to handle the amount of graphs there exist. Replacing checking for the second condition of theorem 3.2 with simply checking connectives of the graphs in the script, and running the script on the results of Benjamin Small, made it possible to produce all cop-irreducible graphs with v(G)$\leq$ 13. The computation time of brute forcing went up pretty quickly as the graphs got bigger to slightly uncomfortable but still manageable. To reduce the computation time first all graphs that contain a duplicated vertex (which will be introduced in section 6.1) were ruled out, because these are cop-irreducible if and only if the graph, with one of the duplicated vertices removed, is cop-irreducible. The computation time of checking on a duplicated vertex is namely far smaller then checking α-coveredness and it was expected that a significant part of the cop-irreducible graphs contained a duplicated vertex. Every cop-irreducible graph with n - 1 vertices can namely be used to create a couple of different cop-irreducible graphs of n vertices. This extend to which this reduced the computation time was in the end far less then expected, as will be explained in the results in section 7.
After having obtained all cop-irreducible graphs patterns and properties of these graphs were researched, and ways of creating bigger graphs were researched.

5 Theory

Before getting into the theory of graph operations, in this section first some properties concerning cop-irreducible graphs will be discussed.

Proposition 5.1. *If G is α-critical, for any edge $e = uv \in E(G)$, any maximum stable set of G - $\{e\}$ must contain u and v and have cardinality $\alpha(G) + 1$.*

Proof. Suppose there exists a maximum stable set S of G - $\{e\}$ that does not contain u or v. S has cardinality α(G) + 1 , since by remark 3.1 α(G - $\{e\}$) = α(G) + 1. Since S contains at most one vertex of u and v S is also a stable set of G. So there exists a stable set of G containing α(G) + 1, so α(G) $\geq$ α(G) + 1, which is a contradiction. Thus all maximum stable sets S of G - $\{e\}$ contain both u and v.
Since by remark 3.1, α(G - $\{e\}$) = α(G) + 1, so any maximum stable set of G - $\{e\}$ has cardinality α(G) + 1.

□

Theorem 5.1. *An cop-irreducible graph G has no cutvertices. (see assignment 7.1.2 on page 103 of [3])*

Proof. [2]
Proof by contradiction: suppose there exist a vertex v $\in$ V(G) that is a cutvertex.
Since v is a cutvertex, G - v could be divided into two disjoint subgraphs H and J that are not connected with an edge. Since G is connected by theorem 3.2 there must be an edge uv for a vertex u in H and vw for a vertex w in J. By definition of H and J, u and w are not connected by an edge. Since G is α-critical G - $\{uv\}$ there must be a maximum stable set of G - $\{uv\}$ with cardinality |S| = α(G) + 1.
For the same reason an stable set T must exist in G - $\{vw\}$; with |T| = α_g + 1.
Call the set of vertices of in S $\cap$ H, S_H, in S $\cap$ J S_J, in T $\cap$ H T_H and in T $\cap$ J T_J. S consists of S_H, S_J and v, so $|S_H| + |S_J|$ = |S| - 1 = α(G). Similarly it can be shown that $|T_H| + |T_J|$ = α(G), so $|S_H| + |S_J| + |T_H| + |T_J|$ = 2α(G). This gives the following equality: $|S_H| + |S_J| + |T_H| + |T_J| + |v|$ = 2α(G) + 1.

Since S is an stable set u is the only vertex that is an neighbour of v in G in S. u is in H so there are no neighbours of v in S_J. The same way it can be concluded that there are no neighbours of v in T_H. This means S_J + T_H + v is an stable set of G, so has cardinality smaller or equal to α(G). Also, clearly S_H + T_J is an stable set in G, because by definition of H and J there are no edges between these subgraphs. This also means the cardinality of S_H+T_J is smaller or equal to α(G). Adding these two together gives the following inequality: $|S_H| + |S_J| + |T_H| + |T_J| + |v| \leq$ 2α(G). This is in contradiction the equality $|S_H| + |S_J| + |T_H| + |T_J| + |v|$ = 2α(G) + 1 stated earlier. So G has no cutvertices.

□

Corollary 5.1. *An cop-irreducible graph G has no vertices of degree 1.*

Proof. Proof by contradiction: suppose there exists a vertex v $\in$ V(G) with degree 1. Then v has only one neighbour w. If w is connected to another vertex u then v, w would be a cutvertex, for removing w would disconnect u and v. If w is not connected to another vertex then v, G is either

[2] I am aware that there also exists a proof in [2], but I made my own proof at the start of my bachelor-project, so for completeness I included my own proof

the complete graph with 2 vertices, which has stability number α(G) = 1 or G would not be a connected graph. Both of these are a contradiction, for by theorem 3.2 an cop-irreducible graph has α(G)$\geq$ 2 and has to be connected.

□

Proposition 5.2. *If G is cop-irreducible, for any $v \in$ V(G), there exists a maximum stable set S which contains v.*

Proof. Assume, there exists a vertex v $\in$ V(G) for which there is no maximum stable set S which contains v. Because by theorem 3.2 G is α-covered there is no vertex in V(G) - $\{v\}$ that is not adjacent to v.
Since G is cop-irreducible α(G)$\geq$ 2 by lemma 3.1. Since G is cop-irreducible it is also α-critical, for some vertex w by proposition 5.1 there must exist a maximum stable set T of G - $\{vw\}$ that contains both v and w with cardinality α(G)+1 $\geq$ 2+1 = 3. However, v is connected to all vertices except w in G - $\{vw\}$, so T can contain no other vertices then v and w, which is a contradiction. So there must exists a maximum stable set S which contains v.

□

6 Graph operations

In this section, some graph operations, that can be used to create bigger cop-irreducible graphs out of smaller ones, will be discussed. A couple other bigger graph operations were researched, but did not turn out to preserve cop-irreducibility, like graph composition ([3] page 108, assignment 7.2.4).

6.1 Vertex duplication

Vertex duplication is the smallest operation used in this research. It takes a graph G and a vertex v ∈ V(G) and duplicates it with a vertex v'. It then creates an edge between v' and all vertices in the neighbourhood of v N(v), and with v. The formal definition of vertex duplication is given below. Vertex duplication is an operation that creates a graph with one additional vertex and the same stability number as the original graph. In figure 1 an example of vertex duplication as G the cycle of 5 vertices is given. The edges added by the vertex duplication are given by dashed lines.

Definition 6.1 (Vertex duplication)**.** *Duplicating vertex v in graph G produces graph H with vertexset V(G) + $\{v'\}$ and edgeset E(G) + $\{\Sigma_{u\in N(v)} uv'\} + \{v'v\}$.*

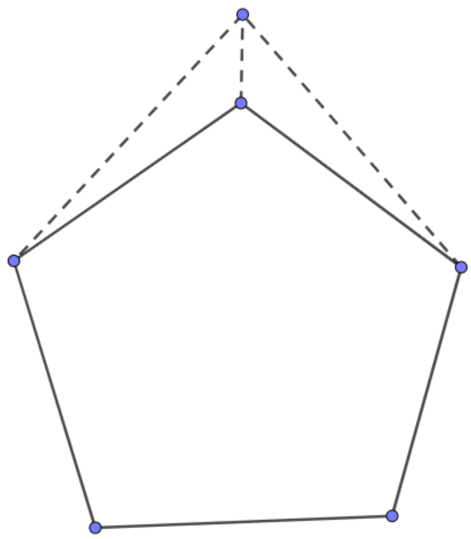


Figure 1: Vertex duplication with the C5

Theorem 6.1. *Given a graph G and a vertex $v \in V(G)$, the graph H with $V(H) = V(G) + \{v'\}$ and $E(H) = E(G) + \{\Sigma_{u\in N(v)} uv'\} + \{v'v\}$., is cop-irreducible if and only if G is cop-irreducible.*

Proof. Since G is connected by condition 1 of theorem 3.2, and v is connected with at least one edge to G, H is also connected, and satisfies condition 1.1.

From corollary 3.6 from [2] follows that H is α-critical if and only if G is α-critical, and thus satisfies condition 2 of theorem 3.2.

Since G satisfies condition 3 of theorem 3.2, any pair of vertices in H that does not include vertex u satisfies condition 3 of theorem 3.2. Vertex v is connected with vertex u in H so, the pair u and v satisfies condition 1.3. All other vertices x in H that are not connected to u, are also not connected to v, so for every vertex x there is a maximum stable set S including x and v. Since u is only connected with v and all vertices that are connected with v, S - $\{v + u\}$ is a stable set in H, and since this set has the same cardinality, this is also a maximum stable set. So H satisfies condition 3 of theorem 3.2, so H is an cop-irreducible graph. □

6.2 Line stretching

In order to understand line stretching first a line needs to be defined.

Definition 6.2 (line)**.** *A line is a set of vertices of a graph G $v_1, v_2, v_3..., v_n$, with $n \geq 3$, $v_i v_{i+1} \in E(G)\ \forall\ i \leq n-1$, $v_1 v_n \notin E(G)$ and degree$(v_i) = 2\ \forall\ i$ with $1 < i < n$.*

Line stretching is an operation on a line within a graph, that adds more vertices with degree 2 in the middle of the line to make it longer. The formal definition of line stretching is given below. As will be proven later, line stretching with an even amount of vertices preserves cop-irreducibility. Line stretching can be used to create cop-irreducible graphs out of a graph with 2m extra vertices and a stability number of m higher.

Definition 6.3 (Line stretching)**.** *Stretching a line $v_1, v_2, v_3..., v_n$, with $n \geq 3$ by 2m vertices for some $m \in \mathbb{N}$, in a graph G produces a graph H with vertexset $V(H) = V(G) + u_1 + u_2... + u_{2m}$ and edgeset $E(H) = E(G)$ - $\{v_{n-1}v_n + v_{n-1}u_1 + \Sigma_{j \leq 2m-1} u_j u_{j+1} + u_{2m}v_n\}$.*

Remark 6.2. *Note that stretching a line with and odd amount vertices can never preserve cop-irreducibility, for the stretching any line in an odd cycle with an odd amount of vertices produces an even cycle, which is not cop-irreducible.*

In figure 2 an example of line stretching is given. In this figure, the original line is coloured red, the added vertices are coloured yellow, the added edges are represented by dashed lines and the removed edge $v_{n-1}v_n$ is represented by a dot line. Another easy example would be stretching any odd cycle to a bigger odd cycle.

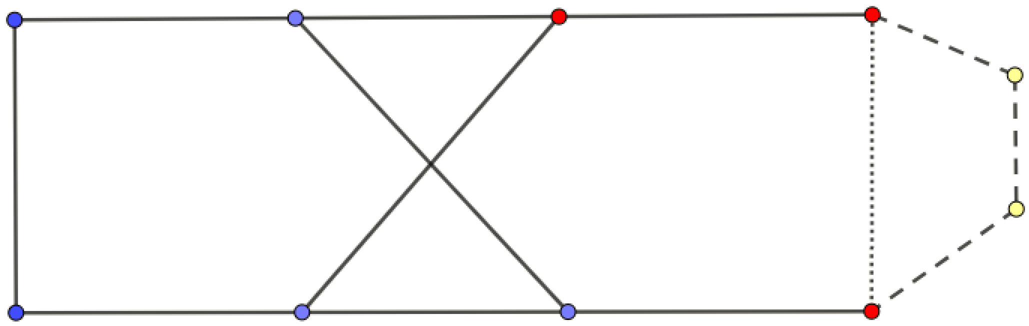


Figure 2: Line stretching with the C5

Lemma 6.1. *The graph H constructed by stretching a line $v_1, v_2..., v_n$, with $n \geq 3$ in a graph G by 2 vertices (so m = 1), has stability number $\alpha(H) = \alpha\ (G) + 1$.*

Proof. Since v_{n-1} and v_n are connected in G any maximum stable set S can contain at most one of them. Therefor if S does not contain v_{n-1}, adding u_2 to S gives a stable set of cardinality α(G) + 1, and if S does not contain v_n adding u_1 to S gives a stable set of cardinality α(G) + 1, so

$\alpha(H) \geq \alpha$(G) + 1.
All remains to prove that there can not exists a stable set of H with stability number α(G) + 2. Proof by contradiction so assume there exists a stable set T of H with stability number α(G) + 2.
Case 1: v_{n-1} and v_n both $\in T$ T can not contain u_1 or u_2, since both of these vertices are either connected to either v_{n-1} or v_n, so T - $\{v_n\}$ is a stable set of G with cardinality α(G) + 1, which is a contradiction.
Case 2: at most one vertex of $\{v_{n-1}, v_n\} \in T$ Since u_1 and u_2 are connected, T can contain only one vertex w of $\{u_1, u_2\}$. Then T - $\{w\}$ would be a stable set of G with cardinality α(G) + 1, which is again a contradiction.
So α(H) = α(G) + 1. □

Theorem 6.3. *The graph H constructed by stretching a line $v_1, v_2 ..., v_n$, with $n \geq 3$ in a graph G by 2 vertices, is cop-irreducible if and only if G is cop-irreducible.*

Proof. It is easy to see stretching a line does not change connectivity in a graph, so H is connected if and only if G is connected.

Suppose G is cop-irreducible. First prove that H is α-critical, for any e $\in$ E(H) α(H - $\{e\}$) > α(H).
Case 1: e $\in$ E(G)$\setminus v_{n-1}v_n$
Since G is α-critical there exists a maximum stable set S of G - $\{e\}$ with cardinality α(G) + 1 (proposition 5.1. This set can not contain both v_{n-1} and v_n, because these are connected in G. If S does not contain v_{n-1}, adding u_2 to S gives a stable set of cardinality α(G) + 1 + 1, and if S does not contain v_n adding u_1 to S gives a stable set of cardinality α(G) + 1 + 1 > α(G) + 1 = α(H) (lemma 6.1). So removing e increases the stability number of H.
Case 2: e = $v_{n-1}u_1$ or e = u_2v_n
When removing edge e $v_{n-1}u_1$ or u_2v_n, there exists a maximum stable set S of $G - \{v_{n-1}v_n\}$ containing both v_{n-1} and v_n with cardinality α(G) + 1. Therefor if S does not contain v_{n-1}, adding u_2 to S gives a stable set of cardinality α(G) + 1 + 1, and if S does not contain v_n adding u_1 to S gives a stable set of cardinality α(G) + 1 + 1 > α (G) + 1 = α(H). So removing e increases the stability number of H.
Case 3: e = u_1u_2 Since G is alpha-critical there exists a maximum stable set S of G - $\{v_{n-2}v_{n-1}\}$ of cardinality α(G) + 1 consisting v_{n-1} (proposition 5.1). Since v_{n-1} and v_n are connected in G - $\{v_{n-2}v_{n-1}\}$, S can not contain v_n. So the set S - $\{v_{n-1}\}$ is a stable set of G pf cardinality α(G) + 1 - 1 = α(G) that does not contain v_n or v_{n-1}. Since u_1 and u_2 are only connected to v_{n-1} and v_n respectively in H - $\{e\}$, the set S - $\{v_{n-1} + u_1 + u_2\}$ is a stable set of H - $\{e\}$, with cardinality α(G) + 2 > α (G) + 1 = α(H) (lemma 6.1). So removing e increases the stability number of H.
So H is α-critical if G is cop-irreducible.
Now prove H is α-covered, so for any combination of vertices $w_1, w_2 \in$ V(H) that are not connected by an edge there exists a maximum stable set of H containing both of them.
Case 1: $w_1, w_2 \in$ V(G), $w_1w_2 \notin$ E(G)
Since G is α-covered, there exists a maximum stable set S of G containing w_1 and w_2, which does not contain both v_{n-1} and v_n because they are connected in G. If S does not contain v_{n-1}, adding u_2 to S gives a stable set of cardinality α(G) + 1, and if S does not contain v_n adding u_1 to S gives a stable set of cardinality α(G) + 1 = α(H).
Case 2: $w_1 = v_{n-1}$, $w_2 = v_n$
Since G is α-critical, there exists a maximum stable set of G - $\{e\}$ with cardinality α(G) + 1 (proposition 5.1 consisting w_1 and w_2 . S is also a stable set of H, and since α(G) + 1 = α(H) (lemma 6.1), it is a maximum stable set.
Case 3: $w_1 \in$ V(G), $w_2 \in \{u_1, u_2\}$
w_2 is connected to one vertex y $\in \{v_{n-2}, v_{n-1}\}$ and not connected to one vertex x $\in \{v_{n-2}, v_{n-1}\}$.
If w_1 is not connected to x, there exists a maximum stable set S in G consisting w_1 and x, since x is connected to w_1. Adding w_2 to S gives a stable set of cardinality α(G) + 1.
If w_1 is connected to x, then there exists a maximum stable set S of G - $\{w_1x\}$ with cardinality α(G) + 1 containing x and w_1. S can not contain y, since x and y are connected, so w_2 is not

connected to a vertex in S. Removing x and adding w_2 from S, produces a stable set containing w_1 and w_2 of H, with cardinality α(G) + 1 - 1 + 1 = α(H).
Since there is an edge between all other pairs of vertices in H, H is α-covered.

Now suppose H is cop-irreducible. First prove G is α-critical, so for any e $\in$ E(G) α(G - $\{e\}$) > α(G).
Case 1: $e \in E(G) \backslash v_{n-1}v_n$
Since H is cop-irreducible and thus α-critical, there exists a maximum stable set S of cardinality α(H) + 1 of H - $\{e\}$. If S contains both v_{n-1} and v_n, then S can not contain u_1 or u_2 so S - $\{v_n\}$ is a stable set of G - $\{e\}$ with cardinality α(H) > α(G).
If S contains at most one vertex out of $\{v_{n-1}, v_n\}$, then, because u_1 and u_2 are connected in H so only one of them can be in S, S $\cap$ $\{u_1, u_2\}$ is a stable set of G - $\{e\}$ with cardinality $\geq$ α(H) + 1 - 1 > α(G).
case 2: $e = v_{n-1}v_n$ Since H is cop-irreducible, it is *alpha*-covered, and thus there is a maximum stable set S of H containing v_{n-1} and v_n. Since u_1 and u_2 are connected to either v_{n-1} or v_n, S can not contain them and thus only contains vertices of G, so S is also a stable set of G - $\{e\}$ with cardinality α(H) > α(G). So G is is α-critical if H is cop-irreducible.

Now prove that G is α-covered, so for any combination of vertices $w_1, w_2 \in$ V(G) that are not connected by an edge there exists a maximum stable set of H containing both of them.
Any combination of vertices $w_1, w_2 \in$ V(G) that is not connected by an edge G is not connected by an edge in H, so since H is α-covered, there exists a maximum stable set S of H containing both w_1, w_2. Since in H u_1 and u_2 are connected, S can contain at most one of them. So S $\cap$ $\{u_1, u_2\}$ is a stable set of G, containing both w_1, w_2 of cardinality $\geq$ α(H) - 1 = α(G), so by definition of stability number S must have cardinality equal to α(G). So for any combination of vertices w_1, w_2 $\in$ V(G) that are not connected by an edge there exists a maximum stable set of H containing both of them.
Thus G α-covered if H is cop-irreducible.

Thus H is cop-irreducible if and only if H is cop-irreducible.

□

Corollary 6.1. *The graph H constructed by stretching a line* $v_1, v_2..., v_n$ *in a graph G by 2m vertices with* $m \in \mathbb{N}$*, is cop-irreducible if and only if G is cop-irreducible and has stability number* α*(H)* = α*(G) + m.*

Proof. Proof by induction:
Induction basis: by theorem 6.3 stretching $v_1, v_2..., v_n$ in G by 2 vertices produces a graph H that is cop-irreducible if and only if G is cop-irreducible and has stability number α(H) = α(G) + 1 by lemma 6.1.
Induction hypothesis: suppose the graph I constructed by stretching $v_1, v_2..., v_n$ in G by 2p vertices $u_1, u_2, ..., u_{2p}$ is cop-irreducible if and only if G is cop-irreducible and has stability number α(I) = α(G) + p.
Induction step: from the definition of line stretching follows that $v_1, v_2...v_n, u_1, u_2...u_{2p}$ is still a line, thus by theorem 6.3 stretching it with 2 vertices produces a graph J that is cop-irreducible if and only if I is cop irreducible and by lemma 6.1 has α(J) = α(G) + p + 1.
Thus the graph H constructed by stretching a line $v_1, v_2..., v_n$ in a graph G by 2m vertices, is cop-irreducible if and only if G is irreducible and has stability number α(G) + m. □

6.3 Line duplication

Line duplication is again an operation on a line within a graph, this time duplicating it. It takes a line, duplicates all vertices of it and connects all vertices that were connected in the original line in the duplicated line. Then it connects the start vertex in the duplicated line and connects it to all vertices to which the original start-vertex was connected; same with the end-vertex. Then in order to preserve cop-irreducibility either the start-vertex or the end-vertex of the duplicated line is connected to the end- or start-vertex respectively of the original line. As will be proven duplicating a line of odd length preserves cop-irreducibility and can be used to create bigger cop-irreducible graphs. Below the formal definition of line duplication.

Definition 6.4 (Line duplication)**.** *Duplicating a line $v_1, v_2, ...v_n$, with $n = 2k+1$ for some $k \geq 1 \in \mathbb{N}$ in a graph G produces a graph H with vertexset $V(G) + \{v'_1, v'_2..., v'_n\}$ and edgeset $E(G) + \Sigma_{u\in\{N(v_1)-v_2\}} uv'_1 + \Sigma_{w\in\{N(v_n)-v_2\}} wv'_n$ + either the edge $v_1v'_n$ or the edge $v_nv'_1$.*

Remark 6.4. *The graphs created when adding edge $v_1v'_n$ or the edge $v_nv'_1$ are isomorphic to each other, so only one option needs to be discussed.*

In figure 3 the example of line duplication with the cycle of five vertices and a line of three vertices is given. In this figure, the original line is coloured red, the duplication of the line is coloured yellow and the edges added in the line duplication are given by dashed lines.

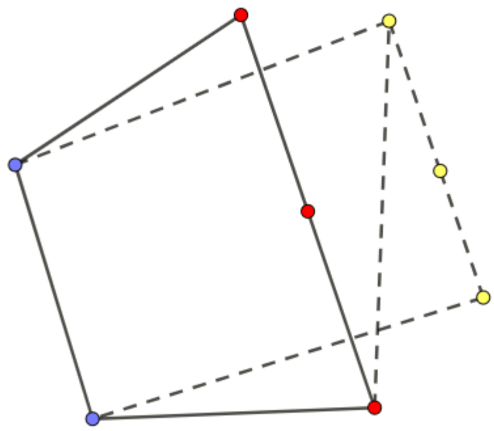


Figure 3: Line-duplication with the C5

Lemma 6.2. *Suppose H is the graph constructed by taking the line duplication of a graph G and the line $\{v_1, v_2, v_3\}$. Then the stability number of H α(H) is equal to α(G) + 1.*

Proof. Suppose there exists a stable set S of H with cardinality |S| = α(G) + 2. If S contains less then 2 vertices from $\{v'_1, v'_2, v'_3\}$ then S $\cap$ $\{v'_1, v'_2, v'_3\}$ would be a stable set in G of cardinality $>$ α(G), which is a contradiction, so S contains at least 2 vertices from $\{v'_1, v'_2, v'_3\}$. Since v'_2 is connected to both v'_1 and v'_3, S must contain v'_1 and v'_3. Since either v'_1 is connected to v_3 or v'_3 to v_1, and v_2 is connected to both v_1 and v_3 S can contain only one vertex w. Thus the set T = S - $\{w - v_3 - v'_3 + v_1 + v_3\}$ is a stable set in G with cardinality α(G) + 1 $>$ α(G), which is again a contradiction. Thus there can not exists a stable set with Suppose S is a maximum stable set of H, and |S| = α(G) + c for some $c \in \mathbb{N}$. If S contains less then c vertices out of the set $\{v'_1, v'_2...v'_{2k+1}\}$, then S $\cap$ $\{v'_1, v'_2...v'_{2k+1}\}$ is a stable set of G, with cardinality cardinality |S| = α(G) + 2, so α(H) $\leq$ α(G) + 1.

Adding v_2 to any maximum stable set R in G gives a stable set of H with cardinality |R| = α(G)

+ 1, so α(H) $\geq$ α(G) + 1, thus α(H) = α(G) + 1.

□

Theorem 6.5. *The graph H constructed by duplicating the line v_1, v_2, v_3 of an graph G, is cop-irreducible if if G is cop-irreducible.*

Proof. Since adding a line is mutually and there is at least one edge between the newly duplicated line and G, H is connected if G is connected.

Now prove that H is α-critical, so removing any e in H increases the stability number of H. *Case 1: e $\in$ E(G)* Since G is cop-irreducible it is α-critical, so there exists a maximum stable set S of G - $\{e\}$ with cardinality $\alpha(G) + 1$. Adding v_2 to S gives a stable set of H - $\{e\}$ with cardinality $\alpha(G) + 2 > \alpha$(H).
Case 2: e $\in \{v_1'v_2', v_2'v_3'\}$ Call the vertex v_1' or v_3' that is incident to e w'. Since G is cop-irreducible, it is α-covered and thus by proposition 5.2 there exists a maximum stable set S in G containing w (with w = v_1 if w' = v_1' and w = v_3 if w' = v_3'). Then the set S + $\{w' + v_2\}$ is a stable set of H with cardinality $\alpha(G) + 2 > \alpha$(H).

Now prove that H is α-covered, so for any combination of vertices (u, w) in H not connected by an edge there exists a maximum stable set of H that contains both u and w.
Case 1:, u,w $\in$ V(G) For any combination of vertices u, w in H with u, w $\in$V(G), because G is cop-irreducible, by theorem 3.2 there exists a maximum stable set S of G containing u, w. Adding v_2 to S, produces a stable set of H, containing u and w with cardinality α(G) + 1 = α(H). Thus S is a maximum stable set of H containing u and w.
Case 2: u $\in$ V(G), w = v_2 Since G is cop-irreducible and u $\in$ V(G), by proposition 5.2, there exists a maximum stable set S of G, containing u. So the set S + $\{w\}$ is a stable set of H, containing u and w with cardinality α(G) + 1 = α(H). Thus S is a maximum stable set of H containing u and w.
Case 3: u = v_1', w = v_3' Since G is α-covered, there exists a maximum stable set S of G containing v_1 and v_3. Replacing v_1 and v_3 by u and w and adding v_2 to S gives a stable set of H containing u and w with cardinality α(G) + 1 = α(H). Thus S is a maximum stable set of H containing u and w.
Case 4: u $\in$ V(G), w $\in \{v_1', v_3'\}$: Name x to be v_1 if w = v_1' and v_3 if w = v_3. Since u and w are not connected and x is the original version of w, u and x can not be connected, and thus there must exist a maximum stable set S in G that contains u and x. Adding w to S gives a stable set of H containing u and w with cardinality α(G) + 1 = α(H). Thus S is a maximum stable set of H containing u and w.
Thus H is *alpha*-covered, and so by theorem 3.2 H is cop irreducible.

□

Corollary 6.2. *The graph H constructed by duplicating the line $v_1, v_2, ...v_n$, with n $\geq$ 3 and uneven in a graph G, is cop-irreducible if G is cop-irreducible.*

Proof. If n = 3 rename G as I, and if n > 3 G can be created by stretching a line L with 3 vertices in some graph I. By corollary 6.1 I is cop-irreducible if and only if G is cop-irreducible (in the case that n = 3, I = G so trivially I is also cop-irreducible if G is cop-irreducible). The graph J created by duplicating L in I is cop-irreducible by theorem 6.5.
Then stretching both L and the duplicate of L in I with n - 3 vertices produces a graph isomorphic to H, and this graph is cop-irreducible by corollary 6.1. Thus the graph H constructed by duplicating the line $v_1, v_2, ...v_n$, with n $\geq$ 3 and uneven in a graph G, is cop-irreducible if G is cop-irreducible.

□

7 Results

By checking all α-critical graphs with vertex number ≤ 13 on cop-irreducibility, first ruling out graphs with duplicated vertices, a graph file was created with all cop-irreducible graphs without duplicated vertices with vertex number ≤ 13. The cop-irreducible graphs without duplicated vertices of vertex number ≤ 9 are drawn below.

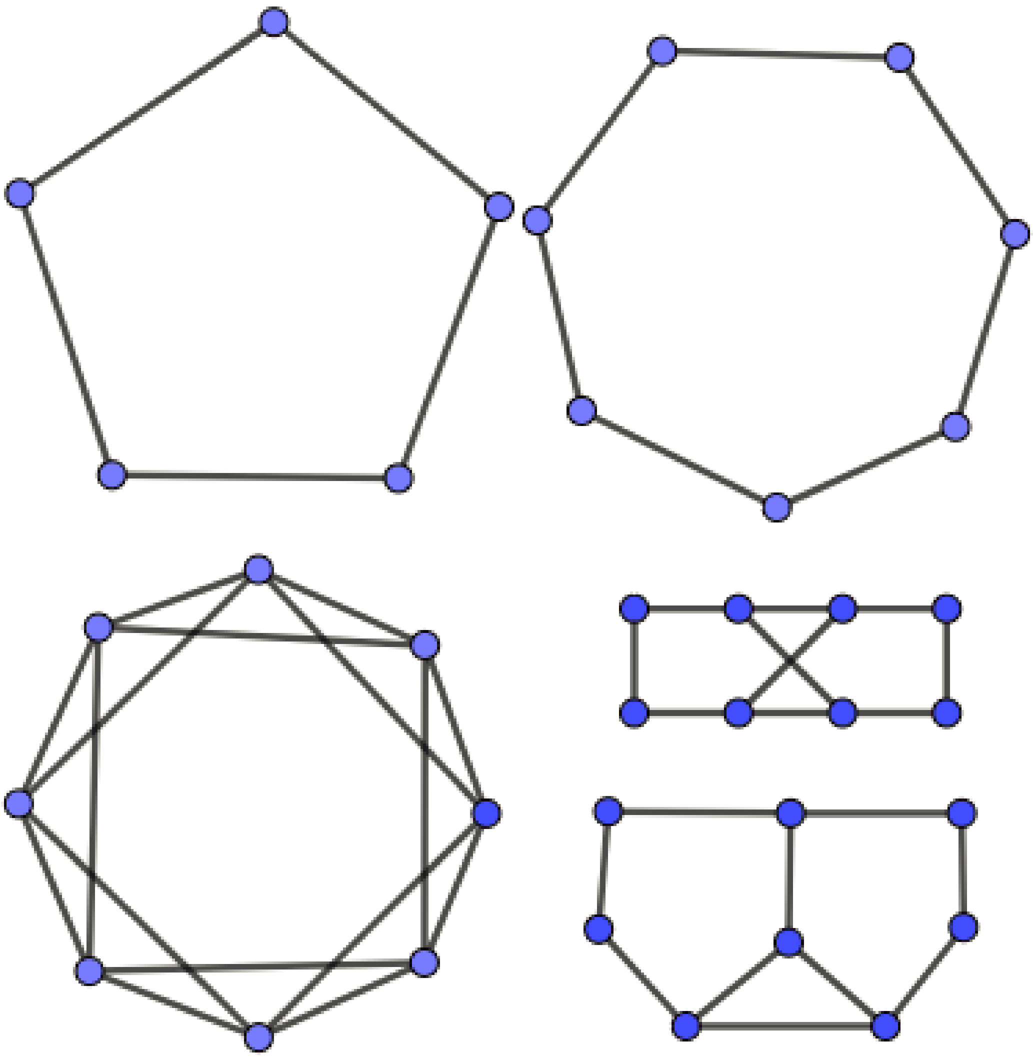


Figure 4: All cop-irreducible graphs without duplicated vertices with v(G) $<$ 9

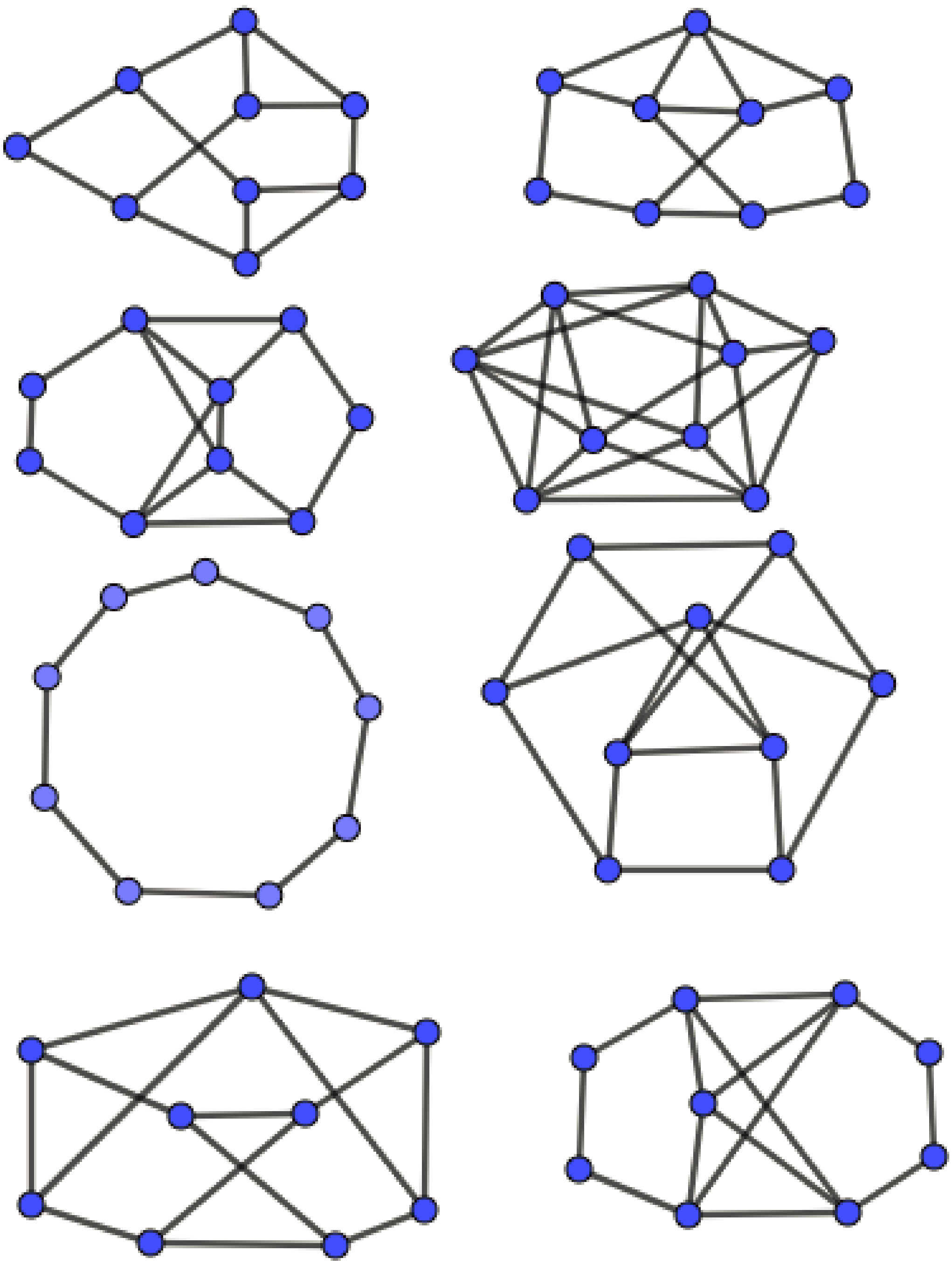


Figure 5: All cop-irreducible graphs without duplicated vertices with v(G) = 9

In table 1 the total number of cop-irreducible graphs per vertex-number and stability number is given.

As can be seen, these numbers grow extremely fast as the vertex-number of the graphs grows. In figure 6 the total number of graphs wihtout duplicated vertices with a certain vertex number

Table 1: Number of cop-irreducible graphs without duplicated vertices per α(G) and v(G)

	α(G)					
v(G)	2	3	4	5	6	Total
5	1	-	-	-	-	1
6	-	-	-	-	-	-
7	-	1	-	-	-	1
8	1	2	-	-	-	3
9	1	6	1	-	-	8
10	2	16	5	-	-	23
11	4	110	51	1	-	166
12	8	1021	497	10	-	1536
13	24	14340	10454	305	1	25124

is plotted against the vertex number. The number of graphs grows so fast the plot gets pretty unclear, but it seems to grow exponential. Therefor, in figure 7, the logarithm of the number of graphs is plotted.

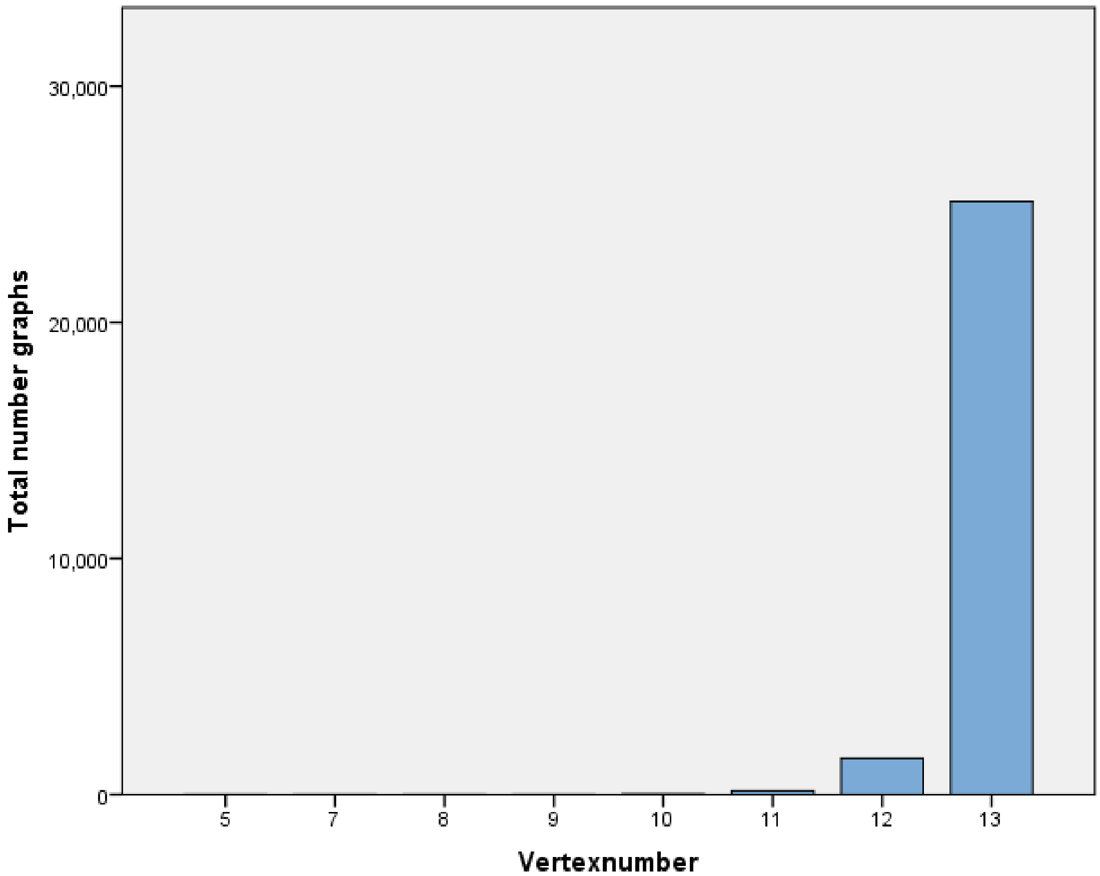


Figure 6: Number of graphs cop-irreducible graphs without duplicated vertices per vertex-number

In figure 7 also a fit line is drawn. It turns out that the number of cop-irreducible graphs without duplicated vertices per vertex-number could very well be approximated with a quadratic function. Thinking more about this, it actually makes perfect sense that the 2-log of the number of cop-irreducible graphs without duplicated vertices is a quadratic function. The total number of graphs counting every isomorphism once is namely $2^{n(n-1)}$, because there are n(n - 1) possible combinations of vertices that can be either connected by an edge or not. Therefor, the 2-log of the number of cop-irreducible graphs without duplicated vertices can definitely not grow faster then quadratic or else there would be a n so that the number of cop-irreducible graphs without duplicated vertices becomes bigger then the total amount of graphs. Just data up to a vertex-number of 13 is not enough proof, but figure 7 suggests that the 2 log of the number of cop-irreducible graphs without duplicated vertices grows grows quadratic.

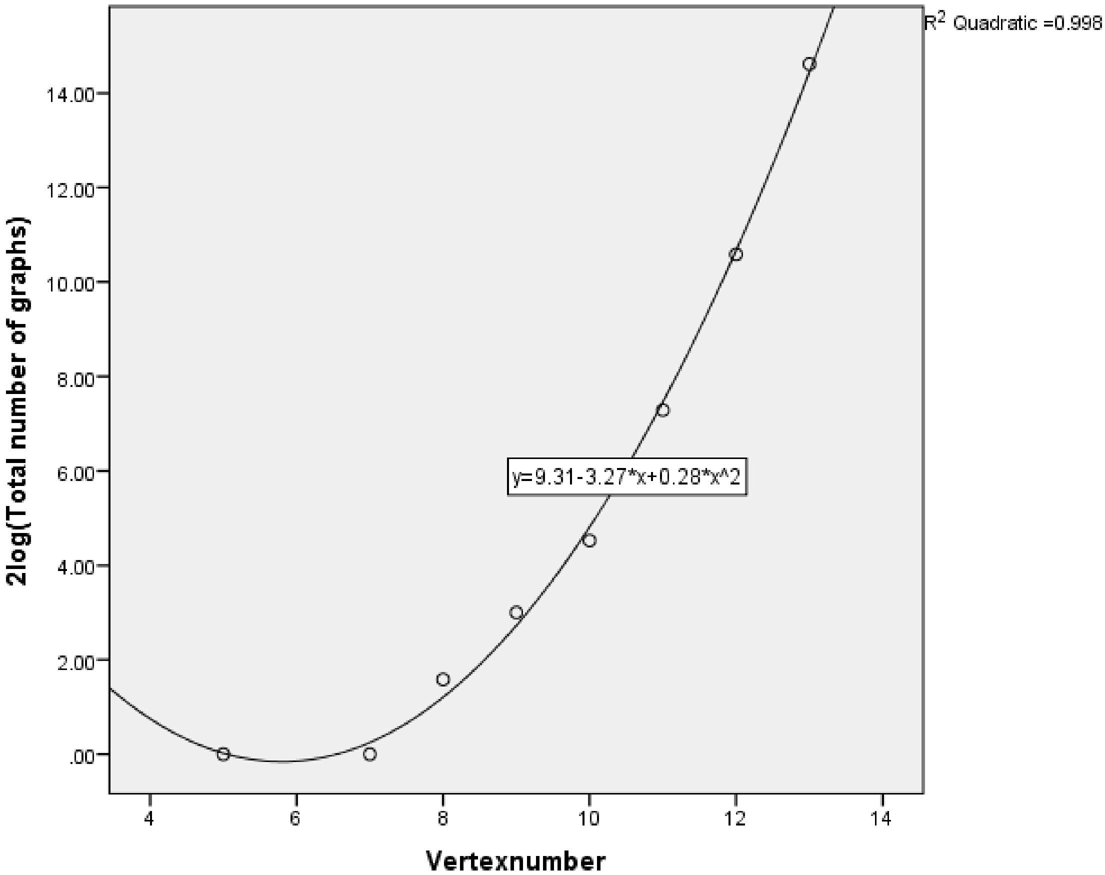


Figure 7: 2 Log of the total number of graphs cop-irreducible graphs duplicated vertices per vertex-number

8 Discussion

The graph set found in this research could be used to test the tractable inner approximation used in optimisation problems discussed in section 1. Also the graph operations defined in this research could be used to create bigger cop-irreducible graphs. For example starting with a cycle of five, duplicating a line, stretching the duplicated line, duplicating it again etc., would be a way to create big nontrivial cop-irreducible graphs.
With some more research it should be possible to find more direct ways to create big nontrivial cop-irreducible graphs. Also it would be valuable to research other difficult instances of copositive optimisation, like extreme copositive matrices (a special case of irreducible matrices which can not be written as the sum of two other copositive matrices.

References

[1] Peter J.C. Dickinson, *Proposed student assignment*,(2018)

[2] Benjamin L. Small, *On α-critical graphs and their construction*, PhD thesis,Washington State University, (2015)

[3] John Adrian Bondy and U. S. R. Murty. *Graph theory with applications*, TheMacmillan Press Ltd., 1976.

[4] B. D. McKay and A. Piperno, *Practical Graph Isomorphism, II*, *J. Symbolic Computation* **60**, 94-112, (2013)

[5] Peter J.C. Dickinson, *The Copositive Cone, the Completely Positive Cone and their Generalisations.* PhD thesis, University of Groningen, (2013)

[6] Immanuel M. Bomze, *Copositive optimisation — recent developments and applications*, ISOR, University of Vienna, (2012)

[7] Etienne de Klerk and Dmitrii V. Pasechnik., *Approximation of the stability number of a graph via copositive programming.*, SIAM Journal on Optimization, 12(4), page 875-892(2002)