Master Thesis

Modelling the FRTB's Default Risk Charge with a factor copula setup

Author:
T.C. Wisse

Supervisors:
Dr. B. Roorda (UT)
Dr. R.A.M.G. Joosten (UT)
S. Mellet (KPMG)

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Abstract

In this research we develop a factor copula model to calculate the capital charges for default risk (DRC) under the regulations on the Fundamental Review of the Trading Book by the Basel Committee on Banking Supervision. The main model requirement for the DRC is the use of two systematic risk factors. To determine which copula to use in the model, we calibrate three copula approaches (Gaussian, Student-t and Clayton) to historical default data. Calibration of the copulas is done through the setup according to Vašíček's Large Homogeneous Pool. All three copula calibrations indicate a good fit to historical default data in a one systematic factor setup. The Gaussian and Student-t copula allow for direct use of two multiple factors for default modelling, where for the Clayton copula a nested copula approach needed. To calculate the DRC, we construct a long S&P 500 portfolio and calculate the charge through the standardised approach (SA) and our internal models approach (IMA). The DRC model is constructed through regression analysis of standardised company returns against the Country and Sector returns. For model robustness, cluster analysis is performed through machine learning (regression tree). The DRC model is built using a Gaussian factor copula, and later enriched with the Student-t copula for more tail dependence. The Gaussian setup IMA charge of the highly diversified long portfolio is closely related to the SA charge. For multidirectional portfolios the DRC SA calculation proves to be very conservative compared to the developed model.

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Research Design

1.1 Introduction

The great global financial crisis of 2008 stressed the full spectrum of the financial system, all around the world. Interwoven financial products, complex models and especially inter-dependencies in the system have been identified as sources of the crisis the world experienced. The general view of the public was that such a crisis should never occur again, so the system had to change. As a response, more effort than ever before has been focused on proper financial risk management. After the 2008 financial crisis, the Basel Committee on Banking Supervision (BCBS) overhauled large parts of the regulations for financial markets. The biggest change in decades for market risk is planned: the Fundamental Review of the Trading Book.

Fundamental Review of the Trading Book
The BCBS published the proposed regulation in January 2016 as: 'Minimum capital requirements for market risk'. By the financial institutions, this regulation is more commonly known as the Fundamental Review of the Trading Book (FRTB). A schematic overview of the capital calculation under FRTB is depicted in Figure 1.1. Financial institutions can apply a standard approach or develop an own model to calculate the FRTB capital charge.

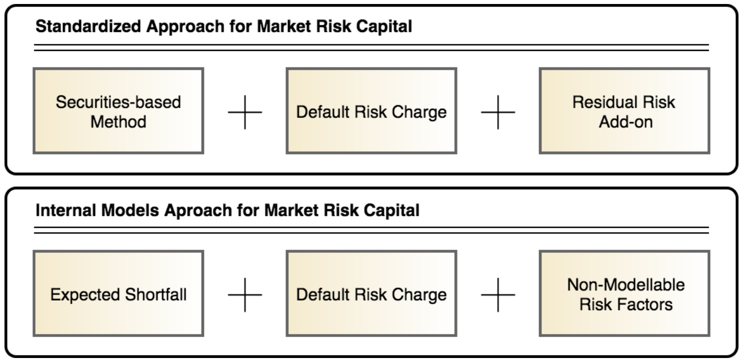


Figure 1.1: Calculation of capital charges under FRTB.

The main goal of the FRTB is to put appropriate capital charges on risks in the trading book. Previous regulation gave the opportunity to gain regulatory arbitrage by shifting credit related products from the banking book to the trading book and vice versa. The FRTB requires a different treatment of credit, and a sharper defined boundary between the trading and banking books. The FRTB regulation has not only consequences for capital calculation, but as well effects the granularity of reporting. Previously, reporting took place at company-level, under FRTB it has to be done at trading-desk level. The BCBS also sets new regulations on the calculation of capital resulting from the risk of default: the Default Risk Charge (DRC).

1.2 Research introduction

The thesis focuses on the Default Risk Charge. The DRC intends to capture the risk of an issuer of equity or bond to default. One of the key challenges for banks is the requirement to model defaults using two systematic factors, instead of the single systematic factor used before.

Historically the Asymptotic Single Factor (ASRF) model described by Gordy (2003) has been used for determining capital charges for credit risk. This single factor model has been built on the foundations of the work by Merton (1974) and Vašíček (1987), and by modifying the model it can also be applied to default risk in trading portfolios. The use of factor models is a popular tool to model correlations in large portfolios. Research has been done by Pykhtin (2004) and Schönbucher (2002) on modelling with multiple systematic factors, using a Merton-type model.

The work of Vašíček (2002) led to the widespread use of the Large Homogeneous Pool (LHP) model, which at first was used to model defaults in loan portfolios. In the LHP model lies the implicit assumption that defaults are correlated through a bivariate Gaussian copula. The use of the Gaussian copula method became conventional after the publication of the work of Li (2000) on the correlation of defaults. This led to a widespread application of the Gaussian copula in the world of finance. However, after the 2008 global financial crisis the bivariate Gaussian copula received heavy criticism. The main critique was on the lack of tail dependence implied by the Gaussian copula. Other copulas like the Student-t copula and the Clayton copula exist, which are known to imply fatter tails. The BCBS, 2016a does give financial institutions the freedom of developing their own default models, as long as they are compliant to the BCBS' requirements for internal models on market risk. This in combination with the global introduction of FRTB regulation brings momentum to (re-)develop default models.

1.3 Research objective, questions and model

From the introduction of the research the following **objective** arises:

Develop a model to calculate the capital charge for default risk, using copulas for default dependence, compliant to the BCBS regulation on the Fundamental Review of the Trading Book.

To reach this objective, we define the following main research question:

Main RQ: How to develop a model to calculate the FRTB's capital charge for default risk, using a factor copula model with two systematic factors?

To answer the main research question and reach the research objective, we define the following sub-questions:

RQ 1: How does the BCBS' FRTB regulation change the capital charge calculation for default risk?

1. How did market risk capital requirements change over time?
2. What are the FRTB requirements for default risk modelling?

RQ 2: How can we model (correlated) defaults?

1. How can we model correlation between companies?
2. What is the relationship between probability of default and loss given default?
3. How does the choice of a copula affect the default correlation?

RQ 3: Why did default models not suffice in the 2008 financial crisis?

1. What went wrong with default models during the 2008 financial crisis?
2. How does the FRTB regulation address shortcomings in the existing Basel 2.5 regulation?

RQ 4: How can we use the Large Homogeneous Pool model for default modelling under FRTB regulation?

1. How can we calibrate copulas according to the LHP?
2. How do different copulas fit to historically observed default rates?
3. How can we use the calibrated copula models for a FRTB compliant model?

RQ 5: How do FRTB's capital charges on default risk relate?

1. How does capitalization take place under the standardised approach for default risk?
2. What is the difference between the capital charges from the standardised approach with the capital charges from an internal model?

Research model

To give a proper overview of the thesis project, we visualise the working process in a phase model. This is depicted in Figure 1.2.

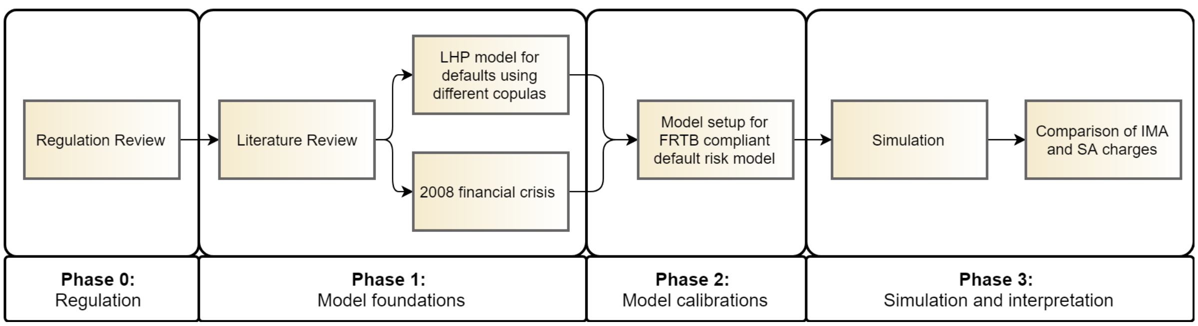


Figure 1.2: Research model.

We introduce every phase briefly. In **Phase 0** we review the FRTB regulation. **Phase 1** provides the model's foundations, we show how we can use theoretic concepts and match these to the LHP approach of Vašíček. We also link theory to events in the financial crisis of 2008. Afterwards in **Phase 2**, we use the proceedings of Phase 1 to build a model for default risk compliant to the FRTB regulation. **Phase 3** involves simulation with the programmed model, and gives a comparison and interpretation of the differences between the model built and the standard default risk calculation .

1.4 Methodology

We supply the reader with some guidance on how this thesis is structured.

Chapter 1 contains the research introduction and defines the structure of the rest of the thesis. In **Chapter 2** (**RQ 1**) we introduce the Fundamental Review of the Trading Book regulation. We do this by describing the historical development of the Basel regulations and by specifying what the FRTB is mainly about. We end Chapter 2 by specifying the Default Risk Charge and its modelling requirements. **Chapter 3** (**RQ 2**) supplies the theoretical structure to the research. We outline fundamental models in default modelling. We elaborate on copulas, and demonstrate the application to default modelling in a factor copula setup. After the theory, **Chapter 4** (**RQ 3**) covers the 2008 global financial crisis and describes criticism on default modelling with copulas. By describing this, we address formerly experienced problems in default modelling which the FRTB regulation tries to overcome in the future. In **Chapter 5** (**RQ 4**) we build a model for default risk according to the Large Homogeneous Pool model by Vašíček. We calibrate the model for the Gaussian, Student-t and the Clayton copulas. In-depth analysis of the calibrations are provided, and we show similarities between the copulas. This chapter functions as the foundation structure towards the FRTB compliant model. **Chapter 6** (**RQ 5**) starts with the theoretical setup of the proposed factor model. The setup is underpinned by a calibration process, after which we describe the methods applied in the DRC model. The factors are retrieved through regression analysis on a large dataset. We ensure model robustness by the application of a machine learning technique (regression tree) for clustering the companies. Afterwards we are able to compare the capital charge of the proposed model with the standard approach charge. **Chapter 7** is the final part of the thesis, where we conclude and reflect on the answer to the main research question. We end with some directions for further research.

2

Fundamental Review of the Trading Book

In this chapter, we introduce the BCBS and the Basel Accords, whereafter we specify the Fundamental Review of the Trading Book in greater detail. This chapter answers **Research Question 1**.

2.1 Introduction to the Basel Accords

The Basel Committee on Banking Supervision (BCBS) sets guidelines for worldwide regulation on the conduct of banking. The BCBS was established in 1974 in the aftermath of serious disturbances in international currency and banking markets (Goodhart, 2011). The main reason for the foundation of the BCBS arose from globalisation of financial intermediation (Goodhart, 2011). The G10 decided to establish the BCBS to improve financial stability by enhancements to the quality of banking supervision worldwide. Since inception, the BCBS expanded from 10 to 45 institutions from 28 jurisdictions (BCBS, 2018).

From 1974 till now, the BCBS worked on its aim to close gaps in international supervisory coverage such that (i) no banking establishment escapes supervision; and (ii) supervision is adequate and consistent across member jurisdictions (BCBS, 2018). The regulation on banking supervision is presented in the Basel Accords. We introduce the historical milestones in the accords briefly.

Basel I
In 1988 the BCBS published a set of minimum capital requirements for banks. This became known as Basel I, with the primary focus on credit risks and aimed to ensure that measurement practices of different countries converged. Risk-weighting of assets was done on all categories of credit risk, expressing the risk involved in certain asset categories. The Accord required banks to keep capital to at least a level of 8% of risk-weighted assets (RWA).

Basel II
In 2004, a new capital adequacy framework was published to replace the Basel I Accord. This because the 1988 Accord had been criticised as being too simple (BCBS, 2018). Basel II brought a key conceptual change, by the introduction of a three-pillar concept. The three pillars were introduced to achieve a more holistic approach to risk management (McNeil et al., 2005), the pillars are:

- Minimum Capital Requirements
- Supervisory Review Process
- Market Discipline

The framework was designed to better reflect the underlying risks to which banks are exposed. A focus was put on the disclosure requirements, which gives other market participants more information about the capital adequacy of financial institutions.

Even before the 2008 financial crisis, the need for improvements on Basel II became apparent (BCBS, 2018). The main limitation of the Basel II regulation were inadequate liquidity buffers and too much leverage in the banking sector. The collapse of Lehman Brother in September 2008 brought this all to light (Akkizidis and Kalyvas, 2018). The high leverage in combination with poor governance and perverse incentive structures led to the need for a revised regulatory framework.

Basel III
The members of the BCBS agreed on the introduction of Basel III in 2010. Key elements in Basel III are a minimum leverage ratio and liquidity ratios. These elements where based on experience with the 2008 global financial crisis. The 2008 crisis witnessed showed the need for liquidity, both on the short term and in a longer period of financial distress.

2.2 Introduction to FRTB's Default Risk Charge

In this section we describe the FRTB DRC regulation. We do this by shortly introducing the goals of the FRTB regulation in general whereafter we describe the development of capital charges for market risk and stating the DRC model requirements.

The FRTB in general aims to minimise regulatory arbitrage, improve both the standardised approach and internal modal approach, introduce a more granular framework, and increase transparency. Appendix A.1 explains the general focus of the FRTB regulation in greater detail.

2.2.1 Capital charges for market risk

To introduce capital charges for market risk, we concisely describe development of market default risk regulation over the last decade. In 2005, the Basel Committee became concerned about the distinction between the trading book and the banking book. The BCBS noticed that similar positions in both books resulted in lower capital charges in the trading book. This gave the opportunity to profit from a regulatory arbitrage strategy. Next to this, in 2005 the regulatory framework assumed that trading book positions were liquid over a 10-day horizon. The 2008 crisis disproved this assumption (BCBS, 2013), this led to the Incremental Risk Charge (IRC) in Basel 2.5 (Laurent and Gregory, 2005).

Basel 2.5 on market risk
In the Basel 2.5 regulation, the IRC was formulated as follows (BCBS, 2009): *"The IRC represents an estimate of the default and migration risks of unsecuritised credit products over a one-year capital horizon at a 99.9 percent confidence level, taking into account the liquidity horizons of individual positions or sets of positions."* The introduction of IRC intended to prevent regulatory arbitrage resulting from the fact that banks kept credit-dependent instruments in the trading book.

The IRC calculation took into account the liquidity horizons applicable to individual positions. Here, a constant level of risk assumption over a one-year horizon was taken. This implied that we assumed that a 3 month B-rated bond was 'rolled-over' with a 3-month rated bond for the capital calculation. The IRC model also captured recovery risk, and assumed that average recoveries were lower when default rates are higher.

The IRC's main drawback was that the variability of the VaR was big because also credit migrations were taken into account (BCBS, 2013). Another deficiency from the IRC was that it did not allow diversification effects between certain credit-related risks and other risks. These drawbacks have been addressed in the FRTB DRC regulation which we will present hereafter.

2.2.2 Default Risk Charge model requirements

The DRC *"captures default risk of credit and equity trading book exposures with no diversification effects allowed with other market risks"* (BCBS, 2016a). As stated, banks could both use the standardised approach as an internal models approach. Compared to the IRC, the main change of the DRC is that credit migrations are not taken into account anymore. This ensures that the variability of the VaR is reduced under the new default risk measure.

The standardised Default Risk Charge is calibrated to the credit risk treatment in the banking book (BCBS, 2016b). This reduces the potential discrepancies in capital requirements for similar risk exposures in the banking and trading books. The calculation of the standardised approach (SA) Default Risk Charge is included in Appendix A.7.

Below we list and explain some of the main elements and requirements for an internal models approach for the DRC. The DRC requirements are derived from (BCBS, 2016a, art. 186).[1]

- Default Risk: Default risk is defined as the risk of direct loss due to an obligor's defaults as well as the potential for indirect losses that may arise from a default event. The default risk must be measured for each obligor. PDs implied from market prices are not acceptable, and PDs are subject to a floor of 0.03%.
- LGD: The Loss Given Default (LGD) must be based on an amount of historical data that is sufficient to derive robust, accurate estimates. On top of this, LGD rates should be dependent on the realization of the systematic factors in the PD model.
- Model: Default risk must be measured using a Value-at-Risk (VaR) model. The VaR (99.9%) calculation should be done weekly and based on a one-year time horizon. The model must consist of two types of systematic risk factors.
- Correlations: Correlations must be calibrated on data covering a period of 10 years that includes a period of stress. The correlations must be measured over a liquidity horizon of one year. Default correlations must be based on either credit spreads or on listed equity prices.
- Validation: Validation of a DRC model necessarily must rely more heavily on indirect methods including but not limited to stress tests, sensitivity analyses and scenario analysis.
- Calculation: The calculation of the actual Default Risk Charge is subject to persistence in case of a recent extreme observation. The actual DRC is *"the greater of: (1) the average of the Default Risk Charge model measures over the previous 12 weeks; or (2) the most recent Default Risk Charge model measure."*

[1] Only the main and relevant modelling requirements of the regulation are listed here

These formulated model requirements are implemented in the FRTB compliant DRC model in Chapter 6.

Default simulation model

The BCBS provides a description on the use of a default simulation model. It states that banks must use a default simulation model with two types of systematic risk factors (BCBS, 2017).

The BCBS supplies the following Merton-type example model:

An obligor i defaults when asset return X_i falls below a specified threshold. Systematic risk can be described via M regional factors $Y_j^{region}(j = 1, ..., M)$ and N industry factors $Y_j^{industry}(j = 1, ..., N)$. For each obligor i, different region factor loadings $\beta_{i,j}^{region}$ and industry factor loadings $\beta_{i,j}^{industry}$ need to be chosen.

The asset return of obligor i can be represented as:

$$X_i = \sum_{j=1}^{M} \beta_{i,j}^{region} Y_j^{region} + \sum_{j=1}^{N} \beta_{i,j}^{industry} Y_j^{industry} + \gamma_i \epsilon_i \tag{2.1}$$

Here ϵ_i is the idiosyncratic risk factor and γ_i is the corresponding factor loading.

2.3 Chapter Conclusion

In this chapter we investigated **Research Questions 1**: *"How does the BCBS' FRTB regulation change the capital charge calculation for default risk?"*.

The FRTB represents the largest overhaul of calculating market risk capital charges since the inception of the Basel regulations. A key change is the requirement of using two systematic factors to calculate the Default Risk Charge. Comparing the DRC with the previous IRC, the additions are that credit migrations are not taken into account anymore. In relation, the DRC is a simpler model than the previous IRC model leading to less variability. The model requirements described for default risk modelling will be applied in Chapter 6.

3

Literature Framework

In this chapter we formulate the theoretical backbone of the thesis. This enables us to answer **Research Question 2**. We first provide a short introduction to some key concepts in default modelling. Afterwards we provide the fundamental default models of Merton (1974) and Vašíček (1987). The final part of this chapter is on copulas.

3.1 Default Theory

A default represents the situation that an obligor is unable to make a required payment on its outstanding financial obligations. A greater probability of default (PD) of a given obligor results in a bigger risk involved for the lender/investor (and therefore a higher compensation is required). The BCBS defines default risk as *"the risk of direct loss due to an obligor's default as well as the potential for indirect losses that may arise from a default event"* (BCBS, 2016a). We use this as definition for default risk.

Default correlation
From history we know that defaults for different companies do not occur independently. The tendency of two companies to default at about the same time is known as default correlation (Hull and White, 2004). In fierce economic conditions the correlation tends to increase, leading to a higher financial risk for the owner of a portfolio (Hull and White, 2004). Multiple correlation measures

exist, the most commonly known one is Pearson correlation coefficient which measures the linear correlation. However, also rank correlation coefficients like Kendall tau and Spearman rho are popular in use. Rank correlation measures assess the ordinal (rank) relationship between two observed quantiles of random variables. The Pearson correlation coefficient of two random variables X & Y:

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} \tag{3.1}$$

Default models
Default correlation models have been introduced to model the correlation between portfolio constituents efficiently. A popular means of modelling correlated defaults is through factor models. Factor models are a practical way of modelling correlated default events and portfolio loss distributions (Bluhm et al., 2010). The main feature of such models is that default events, conditionally on market/industry factors, are independent (Laurent and Gregory, 2005). A simple one-factor model is shown below:

$$Y_i = \rho_i F + \sqrt{1 - \rho_i^2} Z_i \tag{3.2}$$

In Equation 3.2 F is the systematic factor, and Z_i is an idiosyncratic factor for company i. F and Z_i's are independent and $N(0,1)$ distributed. Multi-factor models are represented analogous to Equation 3.2. In Appendix A.2 an overview is given of common setups of multi-factor models.

An early famous default correlation model is the structural model by Merton (1974). In a structural model it is assumed that assets of companies follow correlated stochastic processes (Hull and White, 2004). Another default correlation model type is a reduced form model, where it is assumed that default intensities of different companies follow correlated stochastic processes. The the best known reduced form model is described in Duffie and Singleton (1999). We describe Merton (1974) model in Section 3.2.1. Nowadays, the most common way to model correlation is by the use of factor copula models. The main reason for this is that both reduced form models and structural models are computationally very time consuming for certain financial instruments (Hull and White, 2004). We elaborate on copula models in Section 3.3.

PD/LGD relationship
In finance, many studies have been performed to find metrics on calculating the probability of default (PD) of a given obligor. Compared to the research on PD rates, relatively little research has been done in the direction of the relationship between the probability of default and the loss given default (LGD). Frye and Jacobs (2012) describe that *"continuous LGD is more subtle than binary default and LGD data are fewer in number and lower in quality"*. As for default probabilities, modelling is often done through structural form and reduced form models (Altman et al., 2004).

On the relationship it is commonly accepted that greater default rates go hand in hand with greater loss rates (Altman et al., 2004). The literature poses the relationship works as follows: fierce economic conditions lead to a lower value of the collateral assets, which in turn results in lower recovery rates (Frye, 2000). From historical data it could be observed that in fierce economic conditions PD rates rise, implying a strengthened negative effect on portfolio results in an economic downturn. Altman and Kuehne (2012) use data on the average recovery rates in the time interval 1982-2011 to get the following linear relation:[1]

$$\begin{aligned} &\text{Recovery Rate} = 1 \text{ - Loss Given Default} \\ &\text{Recovery Rate} = -2.3137 * \text{Default Rate} + 0.5 \end{aligned} \tag{3.3}$$

This result by Altman and Kuehne (2012) should be interpreted as an indication, and not as a perfect relationship.[2] However, it could be used as a handle for modelling the PD/LGD relationship.

3.2 Default Models

In the management of large portfolios, the main risk involved is the occurrence of (dis-proportionally) large joint defaults of the portfolios obligors. An appropriate default model is able to capture the dependence between these different obligors (Frey and McNeil, 2001). The default models of Merton (1974) and Vašíček (1987) form the theoretical backbone of many risk models in the financial industry. Therefore we also use these as the foundation structure for our DRC model of Chapter 6. Both models are introduced shortly.

3.2.1 Merton Model

We introduce the Merton (1974) model, based on the notations supplied by Gray et al. (2007). Mertons model is applied by financial institutions to understand an obligor's capabilities of meeting its financial obligations in the future. In the Merton model, the total value of assets follows a geometric Brownian motion. We represent this with the stochastic differential equation denoted in Equation 3.4. Here μ_A represents the mean rate of return on the asset, σ_A the asset's volatility.

$$dA_t = \mu_A A_t dt + \sigma_A A_t W_t \tag{3.4}$$

The total value of assets of a firm is equal to the market value of the claims on the assets. Firms are assumed to be funded by equity (E) and debt (D). In the Merton model it is assumed that debts consists of a single outstanding bond with face value K to be paid at maturity (T). A firm defaults when $A_T < D$, intermediate defaults while $t < T$ are assumed to be impossible.

[1] This is the result from linear regression. Altman and Kuehne (2012) also show multivariate regressions, these are omitted for the sake of conciseness.

[2] The relationship explains 55% of the variance observed (R^2).

Merton's model could be interpreted as the value of the firm's equity as a (European) call option on the value of the company's assets, with strike price equal to the debt repayment.

$$E_t = max(A_t - D, 0) \tag{3.5}$$

Now we apply the Black-Scholes-Merton formula (for a European call option) to calculate the value of equity today (at t = 0):

$$E_0 = A_0 N(d_1) - De^{-rT} N(d_2) \tag{3.6}$$

$$d_1 = \frac{\ln(A_0/D) + (r + \sigma^2)/2}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T} \tag{3.7}$$

The 'risk-adjusted' default probability is $N(-d_2)$. To calculate this we need to have the values for A_0 and σ_A, which are not observable. By applying Itô's lemma we set up the following equation:[3]

$$\sigma_E E_0 = \frac{\partial E}{\partial A} \sigma_A A_0 \tag{3.8}$$

We can find fitting data for A_0 and σ_A by setting up a numerical solver to the equation. From this we can calculate the 'risk-adjusted' default probability.

3.2.2 Vasicek model

With the underpinnings of the Merton (1974) model, Vašíček (1987) developed a method for generating loss distributions for large portfolios (Pykhtin, 2004). The Vašíček model assumes that the asset value is given by both a systematic and an idiosyncratic factor. In the model, an obligor i defaults if a random variable Y_i falls below a certain threshold. The assets value of the obligor is given by the following equation:[4]

$$Y_i = \rho_i F + \sqrt{1 - \rho_i^2} Z_i \tag{3.9}$$

Here F represents the systematic factor, and Z_i the idiosyncratic factor, ρ_i represents the exposure to the market factor for obligor i. F and Z_i's are independent and $N(0, 1)$ distributed. The threshold condition for a default is:

$$\text{default if } Y_i < c$$

Here c is determined on the PD value for the obligor's credit rating class ($c = PD_i$). Since Y_i is also $N(0, 1)$ distributed, we can read the default equation

[3] Presenting the proof of Itô's lemma is beyond the scope of this thesis, for an accessible presentation of the proof we refer the reader to Hull (2015).

[4] Equation 3.9 could also be expressed as $Y_i = \sqrt{\rho_i} F + \sqrt{1 - \rho_i} Z_i$. In the remaining of this thesis we choose for the specification in Eq. 3.9 since ρ_i denotes the correlation to the systematic factor, which gives Eq. 3.9 a more intuitive representation.

that a default will happen in $PD\%$ of the cases.

The conditional probability of default ($DR(F)$), given the realization of the systematic risk factor can be written as (Bluhm et al., 2010):

$$\begin{aligned} DR(F) &= Pr[Y_i < \Phi^{-1}(PD_i)|F] \\ &= Pr[\rho_i F + \sqrt{1-\rho_i^2} Z_i < \Phi^{-1}(PD_i)|F] \\ &= Pr[Z_i < \frac{\Phi^{-1}(PD_i) - \rho_i F}{\sqrt{1-\rho_i^2}}|F] \\ &= \Phi\left(\frac{\Phi^{-1}(PD_i) - \rho_i F}{\sqrt{1-\rho_i^2}}\right) \end{aligned} \tag{3.10}$$

Application of the Merton model by Vašíček on large loan portfolios
Schönbucher (2002): *"In an influential paper, Vašíček (1987, 1997) showed that in a simplified multi-obligor version of the Merton (1974) credit risk model, the distribution of the losses of a large loan portfolio can be described by the inverse Gaussian distribution function"*. This makes that the Vašíček model is known as a one-factor default-mode Merton-type model (Pykhtin, 2004). In the setup by Vašíček (1987) the fraction L of defaults in the portfolio is less than a given level q is given by the formula:

$$P[L \leq q] = \Phi\left(\frac{1}{\rho}\left(\sqrt{1-\rho^2}\Phi^{-1}(q) - \Phi^{-1}(PD)\right)\right) \tag{3.11}$$

Here PD is the default probability of the individual obligors, ρ is the asset value correlation between any two obligors. This model is also known as the Asymptotic Risk Factor (ASRF). The most renown example of the application is the credit risk capital charge in the Basel II accord (Rosen and Saunders, 2010). For the ASRF model we should keep in mind that it is build on two important assumptions (Aas, 2005):

- First, in the ASRF model, it is assumed that the portfolio is infinitely fine-grained.
- Second, there is a single, common systematic risk factor that drives all the dependence across losses in the portfolio.

In Chapter 5 we use the Vašíček model to calibrate historical datasets in the Large Homogeneous Pool model. In Chapter 6 we extend the method to a FRTB compliant model for calculating the Default Risk Charge.

3.3 Copulas

In this section we supply the core information on copulas, needed for the scope of this thesis. We refer to Nelsen (2007) for a complete overview of the copula spectrum. After the copula fundamentals we specify the copulas we apply in Chapter 5 and 6 in greater detail. We finish with a visual comparison. The reader should be aware that we describe copulas here in a technical way, in Chapter 4 we give a more practical insight in the application of copulas in finance.

3.3.1 Copula fundamentals

A mathematical description of the concept of copulas is given by the formal definition of a copula.

Definition: Copula: *"A d-dimensional copula is a distribution function on* $[0,1]^d$ *with standard uniform marginal distributions"* (McNeil et al., 2005).

For a copula, three axiomas must hold (McNeil et al., 2005):

1. $C(u_1, \ldots, u_d)$ is increasing in each component u_i.
2. $C(1, \ldots, 1, u_1, 1, \ldots, 1) = u_i$ for all $i \in \{1, \ldots, d\}, u_i \in [0,1]$.
3. For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0,1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0, \tag{3.12}$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \ldots, d\}$

In 1959 Sklar used the word 'copula' for this mathematical concept, which he deemed to be the most appropriate name for *"functions that could be defined on the unit n-cube linking n-dimensional distributions to their one-dimensional margins"* (Sklar, 1996). In Latin, copula stands for 'link' or 'tie', so the name points out that it couples elements. Sklar (1959) developed a theorem which describes the functions that join together one-dimensional distribution functions to form multivariate distribution functions (Nelsen, 2007). One is free to decide what kind of marginals one couples, so the marginals could have any distribution. Therefore, copulas are convenient since they facilitate a bottom-up approach for multivariate model building (McNeil et al., 2005). For intuition we provide Sklar's theorem based on Rüschendorf (2013) and McNeil et al. (2005).

Sklar's Theorem: Let F be a n-dimensional distribution function with marginals $F_i, ..., F_n$. Then there exists a copula C : $[0,1]^d \to [0,1]$, i.e. a mapping of the unit hypercube into the unit interval. Mathematically:

$$\begin{aligned} F(x_1, ..., x_n) &= P(U_1 \leq F_1(x_1), ..., U_1 \leq F_n(x_n) \\ &= C(F_1(x_1), ..., F_n(x_n)) \end{aligned}$$

If the marginal distributions of $F_1, ..., F_n$ are continuous, then the copula C is unique. The converse is also true, this could be written as:

$$C(u_1, u_2, ..., u_N) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_N^{-1}(u_N)) \tag{3.13}$$

The implication of Sklar's Theorem is that one can work with a copula function C, in addition to the marginal functions $F_1, ..., F_n$ instead of with a multivariate function F. This separates the choice of marginals from the choice of dependence structure (O'Kane, 2011).

Independence copula
When defaults are independent, the independence copula could be used to model the multivariate distribution. This is the most straight forward copula, also known as the product copula.

$$C(u_1, u_2, ..., u_n) = u_1 u_2 ... u_n = \prod_{i=1}^{N} u_i \tag{3.14}$$

Fréchet-Hoeffding bounds
The Fréchet-Hoeffding bounds specify the mathematical bounds for any copula. When we have a bivariate copula, the bounds are:

$$\max(u + v - 1, 0) \leq C(u_1, u_2) \leq \min\{u, v\} \tag{3.15}$$

Tail dependence
The coefficient of tail dependence explains the relation between extreme values of bivariate distributions. Embrechts et al. (2001) describe that tail dependence between two continuous random variables is a copula property. Each copula therefore has its own tail dependence structure. In the DRC model we develop in Chapter 6, the 99.9% percentile is of interest. This stresses the need for a correct tail dependence for unidirectional portfolios. The coefficients for upper (λ_u) and lower (λ_l) tail dependence are:

$$\lambda_u(X, Y) = \lim_{\alpha \to 1} P(Y \leq F_Y^{-1}(\alpha) | X \leq F_X^{-1}(\alpha)) \tag{3.16}$$

$$\lambda_l(X, Y) = \lim_{\alpha \to 0} P(Y \leq F_Y^{-1}(\alpha) | X \leq F_X^{-1}(\alpha)) \tag{3.17}$$

For the specific copulas we investigate, we provide the tail dependence coefficients in the next section after its general specification.

3.3.2 Families of copulas

Numerous copulas could be found in the literature. Every copula has its own properties, and several families of copulas exist. The choice of the copula governs the nature of the default dependence (Hull and White, 2004). We present two families of copulas: elliptical copulas and Archimedean copulas. The Gaussian and the Student-t copula are examples of elliptical copulas. Examples of Archimedean copulas are the Clayton and Gumbel copula.

Elliptical copulas

Elliptical copulas are defined as copulas corresponding to elliptical distributions (Embrechts et al., 2001). The copulas can be derived from certain families of multivariate distributions using Sklar's Theorem (Yan et al., 2007). Elliptical copulas are popular to use because of the relatively easy implementation compared to other copulas. They are also known as implicit copulas.

Gaussian copula

If $X_1, ...X_n$ have a multivariate normal distribution with covariance matrix $\sum$ and mean zero, we have the Gaussian copula (Embrechts et al., 2001).

$$C(x_1, ..., x_n) = \Phi_{\sum}(\Phi^{-1}_{\sigma^2_{1,1}}(x_1), ..., \Phi^{-1}_{\sigma^2_{n,n}}(x_n)) \tag{3.18}$$

where $\Phi_{\sigma^2}(x)$ is the univariate cumulative normal distribution function with variance σ^2 and mean zero, and $\Phi_{\sum}$ the multivariate cumulative normal distribution function with covariance matrix $\sum$.

In the bivariate case, a Gaussian copula could be represented as a joint distribution of two random variables X and Y.

$$H(x, y) = \Phi_\rho(x, y) = C_\rho(\Phi(x), \Phi(y)) \tag{3.19}$$

Here Φ denotes standard univariate normal, Φ_ρ denotes standard bivariate normal distribution function, with correlation parameter ρ. The copula function can be written in analytic form as follows

$$\begin{aligned} C_\rho(a, b) &= \Phi_\rho(\Phi^{-1}(a), \Phi^{-1}(b)) \\ &= \int_{-\infty}^{\Phi^{-1}(a)} \int_{-\infty}^{\Phi^{-1}(b)} \frac{1}{2\pi\sqrt{(1-\rho)}} \exp\left\{\frac{-(x^2 + y^2 - 2\rho xy)}{2(1-\rho^2)}\right\} dxdy \end{aligned} \tag{3.20}$$

For the Gaussian copula, the coefficients for lower (λ_l) and upper (λ_u) tail dependence are (Aas, 2004):

$$\lambda_l(X, Y) = \lambda_u(X, Y) = 2 \lim_{x \to -\infty} \Phi\left(x \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) = 0 \tag{3.21}$$

Equation 3.21 implies that no matter how big the correlation factor ρ is, there is no tail dependence with a Gaussian copula.

One-factor Gaussian copula model:
A common representation of the Gaussian copula is the one-factor Gaussian copula (OFGC) model. The OFGC model originates from the work of Vašíček (1987), which we described in Section 3.1. We can represent Vašíček model in a one-factor Gaussian copula model mathematically as follows (derived from (O'Kane, 2011)):

$$\begin{aligned} Pr(Y_i < c_i, Y_j < c_j) &= Pr(Y_i < \Phi^{-1}(PD_i), Y_j < \Phi^{-1}(PD_j)) \\ &= \Phi_\rho(\Phi^{-1}(PD_i), \Phi^{-1}(PD_j)) \qquad (3.22) \\ &= C_\rho^{\mathrm{GC}}(PD_i, PD_j) \qquad (3.23) \end{aligned}$$

Here Y_i is the asset return, the default condition is fulfilled when Y_i falls below the specified default threshold (c_i which is set as $\Phi^{-1}(PD_i)$). Here PD_i represents the probability of default within a one-year time frame. From the bivariate distribution in Equation 3.22 we recognise a Gaussian copula model shown in Equation 3.19.

The one-factor Gaussian copula model offers analytic tractability by the assumption that the underlying portfolio of assets is large and homogeneous. Therefore, this approach by Vašíček (2002) is referred to as the Large Homogeneous Pool (LHP) Model. We give a detailed representation of the LHP in Chapter 5.

Student-t copula
The Student-t copula allows for joint fat tails and an increased probability of joint extreme events compared with the Gaussian copula (Aas, 2004). In the bivariate case, the Student-t copula is written as follows (Embrechts et al., 2001).

$$C_{\rho,\nu}(a,b) = \int_{-\infty}^{t_\nu^{-1}(a)} \int_{-\infty}^{t_\nu^{-1}(b)} \frac{1}{2\pi\sqrt{(1-\rho)}} \left\{1 + \frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right\}^{-(\nu+2)/2} dsdt \qquad (3.24)$$

Here ρ and ν are the parameters of the copula, t_ν^{-1} is the inverse of the univariate Student-t distributions with ν degrees of freedom. The Student-t dependence structure introduces an additional parameter compared with the Gaussian copula, namely the degrees of freedom ν. The degrees of freedom parameter allows for more dependence of joint extreme events (Aas, 2004). Increasing the value of ν decreases the tendency to exhibit extreme co-movements. The Student-t copula nests the Gaussian copula as a limiting case, when we increase ν to infinity the Student-t copula converges to the Gaussian copula (Kole et al., 2007).

For the Student-t copula, the coefficients for lower (λ_l) and upper (λ_u) tail dependence are (Aas, 2004):

$$\lambda_l(X,Y) = \lambda_u(X,Y) = 2t_{\nu+1}\left(- \sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) = 0 \tag{3.25}$$

Equation 3.25 shows that the higher the correlation parameter ρ and the lower the degrees of freedom ν, the heavier the tail dependence.

Archimedean copulas

Archimedean copulas are also known as explicit copulas. Where elliptical copulas could be modelled using multivariate distribution functions using Sklar's Theorem, this is not possible for Archimedean copulas. Archimedean copulas admit to an explicit closed from expressions (Embrechts et al., 2001). One can construct an Archimedean copulas through generator function ψ (Nelsen, 2007).

$$C(x_1, ..., x_n) = \psi^{-1}\{\psi(x_1), ...\psi(x_n)\} \tag{3.26}$$

The generator function uniquely determines an Archimedean copula. Different Archimedean copulas offer different dependence structures, which could be focused for example on (left or right) tail dependence. One of the attractive characteristics of Archimedean copulas is that they are easy to relate to dependence measures like the Kendall tau. Another advantage of the use of Archimedean copulas is the fact that there is one parameter governing the dependence structure, where in elliptical multivariate copulas many parameters are applicable.

Clayton copula

The Clayton copula is an asymmetric copula, exhibiting greater tail dependence in the lower tail than in the upper tail (Aas, 2004). The lower-tail dependence is the reason why we pick the Clayton copula for investigation, since default correlation is about the lower tail correlation. The Clayton copula is an Archimedean copula, so an explicit (closed-form) equation exists for the copula (Nelsen, 2007).

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}} \tag{3.27}$$

Here the generator function is $\psi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, where $0 < \theta < \infty$. θ is the parameter determining the dependence structure. We have perfect dependence if $\theta \to \infty$, and independence if $\theta \to 0$ (Aas, 2004).

The Clayton copula has upper tail dependence $\lambda_u = 0$. The coefficient for the lower tail dependence is:

$$\lambda_l(X,Y) = 2^{1/\theta} \tag{3.28}$$

3.3.3 Visualizing copula families

For intuition, we supply visualizations of copulas on the [0,1] square. We visualise the bivariate Gaussian copula, Student-t copula, and the Clayton copula in Figure 3.1. In this figure the marginal distributions are similar, so only the effect of the copula is visualised. Appendix A.3 shows the simulation algorithms for the different copulas depicted.

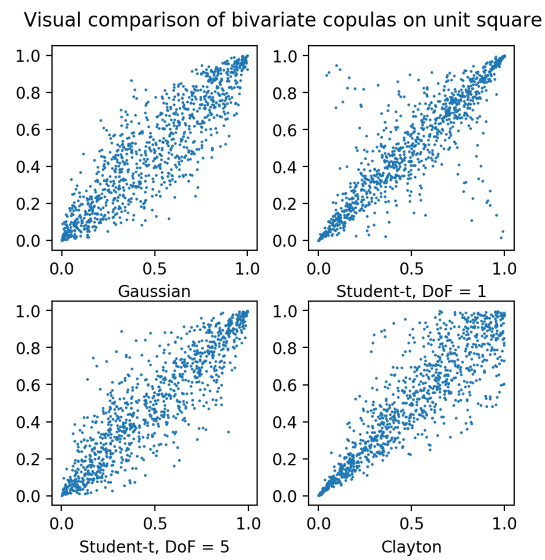


Figure 3.1: Comparison of several bivariate copulas.

The copulas depicted in Figure 3.1 share their rank correlation parameter.[5] By this we are able to visualise the effect of the copula on the dependence structure. We see that the different copulas result in different dependence, this is explicitly visible in the tails.

[5] Figure 3.1 shows a random sampling from the bivariate Gaussian copula with parameter $\rho = 0.9$, from the bivariate Student-t copula with parameters $\nu = 1$ and $\nu = 5$, $\rho = 0.9$ and from the bivariate Clayton copula with parameter $\theta = 4.9654$. The parameter for the Clayton copula could be found via backwards by the analytic expression to calculate the Kendall tau.

For the Gaussian and Student-t copula, the formula for the Kendall tau is:

$$\tau = \frac{2}{\pi} \arcsin \rho$$

For the Clayton copula, the formula for the Kendall tau is:

$$\tau = \frac{\theta}{\theta + 2} \text{ rewriting gives } \theta = \frac{2\tau}{1 - \tau}$$

3.4 Chapter Conclusion

In this chapter we investigated **Research Questions 2**: *"How can we model (correlated) defaults?"*.

Essential in default modelling is the specification of default correlation between the different obligors in the portfolio. An effective way to do is by using a copula function which specifies the dependence structure of the defaults. Specifying a different copula with given marginal distributions leads to different multivariate distribution. The Gaussian copula does not exhibit tail dependence, where the Student-t and Clayton copulas feature tail dependence. The theoretical foundations of Merton (1974) and Vašíček (1987) enable us to model defaults through a factor copula setup. A factor setup is a popular means to model correlation, since it puts a significant lower demand on computation time. The proceedings of Altman and Kuehne (2012) on the relationship of the probability of default and the loss given default give us the ability to model the loss size given the probability of default. These findings will be applied in the DRC model in Chapter 6.

4

The 2008 global financial crisis link with copulas

Now a decade ago, the financial system was rocked to its foundations. The great financial crisis observed in the period 2007-2009 was called the largest shock to the global economy since the Great Depression of the 1930s. In this chapter we mainly investigate the role of copulas within the financial crisis. The description of the 2008 global financial crisis and the role of copulas in this enables us to answer **Research Question 3**.

One of the main events in the financial crisis was the unexpectedly high default rate on debt. High default rates led to shock in the financial system through collateralised debt obligations (CDOs). The financial system was intertwined up to a high degree, leading to a domino effect after the first defaults. Over the first decade of the 21st century, CDO markets had grown at a magnificent pace. To illustrate the growth, we present the CDO issuance over the period 1995-2017 in Figure 4.1.

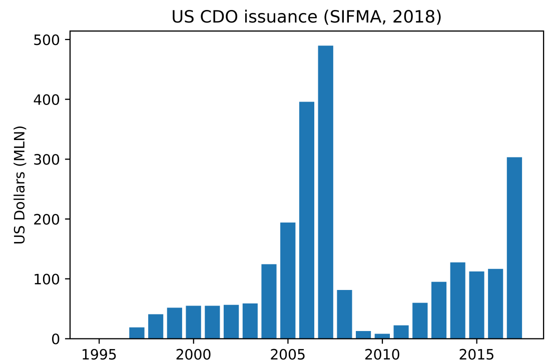


Figure 4.1: CDO issuance over the period 1995-2017, data from: SIFMA (2018).

Route towards the crisis

After the global financial crisis the questions how things could turn out so bad was asked many times. Salmon (2012) states: "*Investors like risk, as long as they can price it*". During the '90s global financial markets expanded and trillions of dollars were waiting to be loaned to borrowers. The hard part was putting a number on the default correlation between all the loans made. Salmon describes that the one who solved this, "*would earn the eternal gratitude of Wall Street and quite possibly the attention of the Nobel committee as well*". The need for a proper model was there, and after Li (2000) popularised the Gaussian copula model, the CDO market expanded quickly (see Figure 4.1).

MacKenzie (2011) describes one of the causes of the 2008 financial crisis as described hereafter.[1] The CDO market expansion led to an increase of banking book size on the side of the financial institutions. Asset-backed security collateralised debt obligations (ABS CDOs) enabled the banks ability of buying risk exposures at a particular credit rating. Rating agencies (e.g. Standard & Poors, Moody's, Fitch) rated the asset backed securities with high ratings since the idiosyncratic risk was diversified for the reason of being collateralised. Afterwards, pooling of ABSs took place in the bank. At the banks the assumption again was that this 'diversified' the idiosyncratic ABS risk away. This second 'diversification' practice led to a low risk assumption, where in fact no extra risk was diversified. MacKenzie (2011) describes this as a *"free lunch, eaten twice"*.

[1] The explanation of the causes on the global financial crisis by MacKenzie (2011) is only one out of many. The author chose to represent mainly this perspective, since the research by MacKenzie has been performed over a long period and even started before the crisis took place. However, the reader should be aware that this is not the 'complete picture'.

4.1 Copula adoption

As we introduced in Chapter 3.2.2, Vašíček (1987) developed a widely-known model. The Vašíček model is able to model the loss distribution for large homogeneous portfolios, given a single systematic risk factor. This model, known as the one-factor Gaussian copula (OFGC) model, became an industry standard for credit risk modelling from around 2000. The model was very convenient since when one underlying factor represents the state of the economy, the defaults by different companies could be treated as independent events (MacKenzie and Spears, 2014b).

The work of Vašíček on the Large Homogeneous Pool model was circulating through the banking industry, however it was never officially published (before the Journal of Risk published it in 2002). David Li: "*I was aware of Vašíček's work, I found that was one of the most beautiful math I had ever seen in practice.*" (MacKenzie and Spears, 2014b). The main problem of the Vašíček model according to Li was that it was a one period model. Li (2000) proposed a model which specifies the joint survival time distribution between marginal distributions. Li (2000) enabled the application of copula functions to CDO tranche pricing, something which was not done before. With the copula function, Li was able to create a link between the marginal default distributions developing a joint default distribution for the CDO portfolios. By this, Li popularised the use of the Gaussian copula calibrated on market prices. In the years which followed, the work of Li would be applied in finance all around the world (Salmon, 2012).

We concisely list some of the essential reasons for applying copula functions to the world of finance:

- Copulas are popular because of their simplicity. When the marginals are known, they can be plugged into the copula function. This is a very convenient modelling practice.
- Many dependence structures could be modelled, since a wide range of different copulas exists. A crucial point is however that one should accurately capture the dependence features of the data (Zimmer, 2012).
- The Gaussian copula was widely applied, which made it a good predictor of price movements (MacKenzie and Spears, 2014a).
- The Gaussian copula model became a widely applied model, this resulted in that also non-modellers (f.e. accountants) started favouring the model (MacKenzie and Spears, 2014a).

Before the 2008 financial crisis took place, several researchers and practitioners already questioned the use of the Gaussian copula for CDO pricing (MacKenzie and Spears, 2014b). MacKenzie and Spears (2014a) interviewed 29 quants on the Gaussian copula model, prior and after the 2008 financial crisis (8 were interviewed prior to the crisis). MacKenzie and Spears (2014b) reports in the pre-crisis interviews that quants were aware of the shortfalls of the Gaussian copula. The interviewees even described the method as unsatisfactory, and even not worth the term 'model' but an interpolation. The Gaussian copula model the situation was that there is a price derived from consensus: "*Since everyone kind of uses the same model, .., everyone kind of agrees on the same price.*" (MacKenzie and Spears, 2014b)

4.2 2008 financial crisis

In 2007, the financial world started to show cracks. When sub-prime lenders US started to default, it did not take long before housing bubble burst in 2008. In September 2008, Lehman Brothers collapsed. This event is seen as the defining event of the financial crisis, however it only was the start. The financial system was interwoven to a high degree, leading to a doom scenario for many financial institutions (FCIC, 2011). The recent developments and product innovations in the market of credit derivatives gave large exposures to portfolios of financial companies. Defaults rates rose to levels which were thought impossible with the Gaussian copula, which resulted in the fact that institutions faced risks which were much bigger than previously thought. For more conclusions on the causes of the financial crisis we refer to the document by the Financial Crisis Inquiry Commission in the US (FCIC, 2011).

In March 2009, Salmon wrote an article in technology magazine Wired, specified on the application of the Gaussian copula model by David Li. The article headlined "*Recipe for Disaster: The Formula That Killed Wall Street*" (Wired, 2009). Salmon blames Li for applying the Gaussian copula formula to CDO pricing, concluding that Li's instrument forced the global financial system to its knees. Salmon (2012) describes the advances of Li in the field of correlation modelling, what he did with a *"simple and elegant mathematical formula"*.

The main criticism from Salmon was on the specification of correlation structure. Li's model based itself on CDS data to calculate correlations. CDS contracts had been in existence only for less than a decade, a decade in which house prices had soared. Salmon (2012) argues that this was a fatal flaw in the model of Li, since adverse price movement of house prices led to a different correlation number.

In the literature, other authors are less outspoken about role of Li. MacKenzie and Spears (2014b) question themselves whether the Gaussian copula was the 'formula that killed Wallstreet'. One of the first observations MacKenzie and Spears (2014b) denote is that the amount of research into the origins of the Gaussian copula is very scarce, especially when compared to the amount of research on how to apply the Gaussian copula for CDO default modelling. Zimmer (2012) describes the major drawback of the Gaussian copula as follows: "*corre-*

lated events are asymptotically independent such that extreme events appear to be unrelated". Zimmer (2012) argues that this drawback "*might be innocuous in normal times, but not during extreme events such as the housing crisis*". The main conclusion after investigation on housing price data is that the Gaussian copula was unable to accommodate the tail dependence observed in the housing crisis.

4.3 Aftermath conclusions

Now, with the event of the great financial crisis a decade ago, it is time to reflect. Salmon (2012) blames Li because his model assumed that correlation was a constant rather than a stochastic process. Another fatal flaw in the copula model of Li (2000) was the low tail dependence in the standard Gaussian copula application. Next to this, the fact that different CDO tranches had different implied correlations was counter-intuitive (and even impossible), since the CDO underlyings were the same.

MacKenzie and Spears (2014a) conclude that David Li cannot be blamed, and neither can the Gaussian copula in its essence be blamed. The Gaussian copula model is known for its shortfalls, the main thing that went wrong with this was the organizational process around it. Since all market participants (from investment banks to rating agencies and regulators) were using a similar model, it became the market standard for determining CDO prices. As long as the market stayed away from extreme situations, everything would go right. But when extreme events occurred, like the 2008 crisis, things would turn out very bad. On top of this, David Li came up with the explicit use of fully fledged copula functions, but when the crisis hit in 2008 rating agencies had only moved partially into the direction of Li's model (MacKenzie and Spears, 2014a).

We propose a proper overall conclusion where both Li, Salmon and Mackenzie & Spears would agree on: The work of Li on the Gaussian copula was a 'recipe for disaster' when users would not understand the essence and limitations of the model. Which got clear so far, is that the application of the static Gaussian copula model was at least partially blamed for the crisis. As a consequence, after the crisis richer correlation for credit risk approaches were introduced as dynamic copula models (Albanese et al., 2013).

It should be noted that the work of Li (2000) did not address the CDOs marginal default distributions, but just how to link given marginal default distributions (MacKenzie and Spears, 2014b). After the 2008 financial crisis it became clear that banks and rating agencies overly assumed diversification benefits from ABS CDOs (*"a free lunch, eaten twice"*), leading to low expected joint default probabilities. This turned out different, and resulted in higher default rates than expected with the assumed marginals and Gaussian copula.

4.4 The future with copulas: DRC

Chapter 2 introduced the FRTB regulation and in specific the model requirements for calculating the Default Risk Charge. In Chapter 3 we described how we could apply factor copula models to model default risk. However, after reading the critiques by e.g. Salmon (2012) on the (Gaussian) copula practices, the reader could pose himself the question why we would continue with the application of the Gaussian copula. We could write lengthy articles about this, but we want to keep the reasons clear and concise. We itemise several arguments for modelling the FRTB's Default Risk Charge with a factor copula set up below:

- We know that the Gaussian copula approach lacks tail dependence. However, DRC - IMA regulation prescribes that calibration of the parameters in the model should be covering a period of >10 years, including a period of stress. This implies that parameters are fitted to historical data including also bad economic times. On top of this, a market trading portfolio is almost never unidirectional. Because of this, the tails of the loss distribution are not per say matching the tails in the Gaussian copula.
- The critique of Salmon (2012) is on the Gaussian copula application on securitised financial products (CDOs). Within the FRTB regulation these products are directly subject to the standardised approach of the BCBS. Therefore, the modelling issues of capital charges for CDOs in an internal models approach are not applicable.
- The FRTB regulation prescribes backtesting methods for internal models. First, internal models are granted on desk level after proofs of correctness. Second, DRC-IMA models are subject to backtesting and P&L attribution procedures. This ensures that the regulator oversees whether modelling practices are done appropriately.
- Quick adaptation and persistence is present in the DRC risk measure. DRC-IMA requires that the calculation for IMA DRC is *"the greater of: (1) the average of the Default Risk Charge model measures over the previous 12 weeks; or (2) the most recent Default Risk Charge model measure."*

4.5 Chapter Conclusion

In this chapter we investigated **Research Question 3**: "*Why did default models not suffice in the 2008 financial crisis?*".

In the years up to 2008, innovative financial products like CDOs quickly became popular. The proposed model by Li (2000) to price CDOs with copulas popularised the use of the Gaussian copula. In the year's following, the full market spectrum from traders to regulators started using Li's approach to calibrate correlation from market prices. For this reason, the method was quickly seen as the 'real' CDO price. However, the 2008 financial crisis showed a fatal flaw of the Gaussian copula: the lack of tail dependence. Another flaw was that rating agencies and banks assumed too much diversification in and between CDOs than applicable. The widespread misuse of the Gaussian copula for CDO pricing led to unexpectedly high default rates in CDO portfolios. Within the FRTB regulation, several measures have been taken to prevent similar situations in the future. One of these measures is that CDOs are subject to the standardised approach by the BCBS.

5

Modelling Default Risk with LHP

In this chapter we describe our default risk model when we use the assumptions of the Large Homogeneous Pool (LHP) model by Vašíček. This chapter answers **Research Question 4**. The LHP by Vašíček is based on the one-factor Gaussian copula, we modify this into a Student-t and a Clayton copula model. We end this chapter by calibrating the different copulas on historical data, whereafter we compare the differences between the copulas used for calibration.

5.1 LHP model

In Section 3.2.2 we described the model of Vašíček (1987). From here we continue using the results of Vašíček (1987) to show the derivation of the Large Homogeneous Pool (LHP) approximation. The LHP approximation uses a one-factor Gaussian copula to represent the default correlation structure. The portfolio contains an infinite number of entities, which all have the same characteristics (e.g. PD, LGD, notional amount).

Below we show the derivation of the closed form result of Vašíček (1987). We use a slightly simplified notation compared to Section 3.2.2, changing PD_i and ρ_i in PD and ρ. Equation 5.1 is the Vašíček LHP approximation (Bluhm et al., 2010).

$$\begin{aligned}
G(x) = P(DR(F) \leq x) &= P\left(\Phi\left(\frac{\Phi^{-1}(PD) - \rho F}{\sqrt{1-\rho^2}}\right) \leq x\right) \\
&= P\left(\frac{\Phi^{-1}(PD) - \rho F}{\sqrt{1-\rho^2}} \leq \Phi^{-1}(x)\right) \\
&= P\left(-F \leq \frac{\sqrt{1-\rho^2}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\rho}\right) \\
&= P\left(F \geq \frac{\Phi^{-1}(PD) - \sqrt{1-\rho^2}\Phi^{-1}(x)}{\rho}\right) \\
&= \Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\rho}\right)
\end{aligned} \tag{5.1}$$

Here, $G(x)$ represents the cumulative default distribution (CDF). After the CDF we want to derive the probability density function (PDF) for the default distribution. We do this by taking the derivative of the CDF $G(x)$, this results in the PDF $g(x)$ (Bluhm et al., 2010).

$$\begin{aligned}
g(x) &= \frac{\partial G(x)}{\partial x} \\
&= \sqrt{\frac{1-\rho^2}{\rho^2}} \exp\left(-\frac{1}{2\rho^2}\Big((1-2\rho^2)(\Phi^{-1}(x))^2 - \right. \\
&\qquad \left. 2\sqrt{1-\rho^2}\Phi^{-1}(x)\Phi^{-1}(PD) + (\Phi^{-1}(x))^2\Big)^2\right) \\
&= \sqrt{\frac{1-\rho^2}{\rho^2}} \exp\left\{\frac{1}{2}\big(\Phi^{-1}(x)\big)^2 - \frac{1}{2\rho^2}\left(\sqrt{1-\rho^2}\Phi^{-1}(x) - \Phi^{-1}(PD)\right)^2\right\}
\end{aligned} \tag{5.2}$$

LHP model with other dependence structures

The LHP approach shown is built on the Gaussian one-factor copula. As described in Section 3.3, the Gaussian copula particularly lacks tail dependence. Therefore we present several other methods to incorporate tail dependence in the LHP approach. First, we can modify the LHP approach towards a Student-t LHP approach. Schloegl and O'Kane (2005) derived a closed form solution with the Student-t copula. Second, the LHP approach can be changed to a model with Archimedean copula functions. Schönbucher (2002) worked out multiple Archimedean copula functions to estimate the default distributions. We work out a closed form solution for the LHP model with the Clayton copula.

LHP with Student-t copula

A common approach to induce more tail observations than resulting from the normal distribution would be to work with a Student-t distribution. The same holds for copulas. The Gaussian copula has no tail dependence, where the Student-t copula does (when $\nu \neq \infty$). Because of its capability of tail modelling, the Student-t copula is a widely applied copula in financial modelling (O'Kane, 2011).

Student-t copula approach by Schloegl and O'Kane (2005)
Schloegl and O'Kane (2005) extended the LHP approximation of Vašíček (1987). Where Vašíček assumed the asset returns are normally distributed, Schloegl and O'Kane (2005) assume that asset returns follow a multivariate Student-t distribution. The asset return Y_i of asset i is:

$$Y_i = \sqrt{\frac{\nu}{Q}}\left(\rho_i F + \sqrt{1-\rho_i^2} Z_i\right) \tag{5.3}$$

Here F, and all Z_i's are independent, and N(0,1) distributed. Q is a $\chi^2(\nu)$ independent random variable with ν degrees of freedom. The default condition is: $Y_i < D$. This could also be written as $\sqrt{1-\rho^2} Z_i \leq D\sqrt{\frac{Q}{\nu}} - \rho F$. The portfolio model could be interpreted as a mixing model. The mixing variable in this case is $\eta := D\sqrt{\frac{Q}{\nu}} - \rho F$. Now the conditional default probability can be written as

$$Pr[Y_i \leq D|\eta] = \Phi\left(\frac{\eta}{\sqrt{1-\rho^2}}\right) \tag{5.4}$$

Schloegl and O'Kane (2005) use the conditional default probability function to develop a cumulative distribution function for defaults. This method is however computationally intensive and results in model complexity. Therefore we propose a Monte Carlo algorithm to model the cumulative default distribution for the Student-t copula approach for the LHP model.

Student-t copula approach using Monte Carlo
The Monte Carlo is set up by simulating the asset return Y_i as formulated in Equation 5.3. We write the default condition as: $Y_i < t_v^{-1}(PD_i)$. Using the formula for asset return Y_i, we can write the conditional default probability (DR(F,Q)) as:

$$
\begin{aligned}
DR(F,Q) &= Pr[Y_i < t_\nu^{-1}(PD_i)] \\
&= Pr\left[\sqrt{\frac{\nu}{Q}}\left(\rho_i F + \sqrt{1-\rho_i^2} Z_i\right) < t_\nu^{-1}(PD_i)\right] \\
&= Pr\left[Z_i < \frac{\sqrt{\frac{Q}{\nu}} t_\nu^{-1}(PD_i) - \rho_i F}{\sqrt{1-\rho_i^2}}\right] \\
&= \Phi\left(\frac{\sqrt{\frac{Q}{\nu}} t_\nu^{-1}(PD_i) - \rho_i F}{\sqrt{1-\rho_i^2}}\right) \qquad (5.5)
\end{aligned}
$$

By simulating the default rate (DR) 100.000 times, we obtain the distribution of default rates and we can present this in a CDF function. We come back to this in Section 5.2.2. Similar to what we showed for the Gaussian copula, we could also write the CDF and the PDF function for the Student-t copula. The CDF function is:

$$G(x) = \Phi\left(\frac{\sqrt{1-\rho_i^2}\Phi^{-1}(x) - \sqrt{\frac{Q}{\nu}} t_\nu^{-1}(PD)}{\rho_i}\right) \qquad (5.6)$$

The PDF function is (Bluhm et al., 2010):

$$g(x) = \sqrt{\frac{1-\rho^2}{\rho^2}} \exp\left\{\frac{1}{2}\left(\Phi^{-1}(x)\right)^2 - \frac{1}{2\rho^2}\left(\sqrt{1-\rho^2}\Phi^{-1}(x) - \sqrt{\frac{Q}{\nu}} t_\nu^{-1}(PD)\right)^2\right\}$$

LHP with Clayton copula

The Clayton copula was introduced in Clayton (1978). The Clayton copula has dependence in the lower tail, which implies that extreme movements only cluster in one direction (Schönbucher, 2002). So, the Clayton copula allows for occurrence of extreme downside events, this results in improved statistical performance compared to elliptical copulas (Low et al., 2013).

The CDF formula for Clayton copula one-factor model is (Schönbucher, 2002):

$$F(q) = 1 - G\left(-\frac{\ln q}{\phi(p)}\right) \qquad (5.7)$$

Here q represents the quantile at which we review the cumulative distribution, $\phi()$ is the generator function for the Clayton copula, and p is the default probability of any individual obligor.

From the CDF, the PDF formula for Clayton copula can be derived. This yields:

$$f(q) = \frac{1}{q\phi(p)} g\left(-\frac{\ln q}{\phi(p)}\right) \qquad (5.8)$$

Here $g(x)$ represents the Gamma distribution with parameter α. Here $\alpha = 1/\theta$, and θ is the parameter of the Clayton copula, ($\theta > 0$). The more the θ moves towards 0, the lower the dependence implied (Burtschell et al., 2009).

$$g(x) = \frac{\alpha^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\alpha x) \tag{5.9}$$

Since the Clayton copula is an Archimedean copula, we need a generator function. The generator function for the Clayton copula is (Schönbucher, 2002):

$$\phi(p) = \frac{p^{-\theta} - 1}{\theta} \tag{5.10}$$

5.2 Calibration of the LHP

In this section we calibrate the LHP model. We start with the traditional LHP approximation, the Gaussian one-factor copula. Afterwards we do the same for the Student-t and the Clayton copula. We calibrate the models using the historical default rates per rating class. This is done by moment matching procedures. We use historical data on default percentage per rating class by S&P for the period 1981 till 2016 (Standard & Poor's Financial Services, 2017).

Please note the following things when reading the calibration of the LHP models hereafter. The visualization of the moment matching is based on default data on S&P rating category B.[1] Rating categories are defined according to 'S&P Domestic Long Term Issuer Credit Rating'. Appendix A.5 presents the complete data table on default rates per rating category for 1981-2016 (Table A.1) and the descriptive statistics (Table A.2).

5.2.1 Calibrating LHP - Gaussian

To calibrate the Gaussian one-factor copula, we apply a moment matching procedure. The matching of moments is performed on the variance of the default rate. We first define the mathematical setup, before we show the calibration in practice.

Mathematical calibration method

Here we show our calibration method. The proof of this moment matching method is given by Gordy (2000), which is provided in Appendix A.4.

[1] The procedure can also be applied to rating classes AA, A, BBB, BB and CCC/C. This yields similar results. Since rating class AAA did not experience any defaults in its history, the moment matching procedure here yields a flat curve.

Definition of variables:

$$
\begin{aligned}
N &= \text{Number of obligors} \\
\sigma^2 &= \text{Variance of default} \\
PD &= \text{Probability of default, for a rating class (avg. PD over a time frame)} \\
X_i &= \text{N(0,1) asset return of obligor i} \\
\rho &= \text{the correlation coefficient} \\
\sigma(X_i X_j) &= \text{the covariance between obligor } X_i \text{ and } X_j \\
\Phi &= \text{Gaussian univariate distribution} \\
\Phi_\rho &= \text{Bivariate Gaussian cumulative distribution value with parameter } \rho
\end{aligned}
$$

We can write the variance of the mean default rate as the variance of a sum of correlated variables:

$$
\begin{aligned}
Var(DR) &= \frac{\sigma^2}{N} + \frac{N-1}{N}\rho\sigma^2 \\
&= \frac{\sigma^2}{N} + \frac{N-1}{N}\frac{\sigma(X_i X_j)}{\sigma^2}\sigma^2 \\
&= \frac{\sigma^2}{N} + \frac{N-1}{N}\sigma(X_i X_j)
\end{aligned}
\tag{5.11}
$$

We know the following about the variance and the covariance:

$$
\begin{aligned}
\sigma^2 &= PD(1-PD) \\
\sigma(X_i X_j) &= \Phi_\rho(\Phi^{-1}(PD), \Phi^{-1}(PD)) - PD^2
\end{aligned}
$$

When $N \to \infty$, Equation 5.11 changes since the first term drops. In this case we get:

$$
\begin{aligned}
Var(DR) &= \sigma(X_i X_j) \\
Var(DR) &= \Phi_\rho(\Phi^{-1}(PD), \Phi^{-1}(PD)) - PD^2
\end{aligned}
\tag{5.12}
$$

For our calibration to the historical data we assume $N \to \infty$, so we express Var(DR) as in Equation 5.12. Recall that we can calculate the first term in Equation 5.12 according to the Gaussian copula function.

$$
\begin{aligned}
C_\rho(a,b) &= \Phi_\rho(\Phi^{-1}(a), \Phi^{-1}(b)) \\
&= \int_{-\infty}^{\Phi^{-1}(a)} \int_{-\infty}^{\Phi^{-1}(b)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right\} \mathrm{dx\ dy}
\end{aligned}
\tag{5.13}
$$

There are no analytic ways to compute the value of Equation 5.13, therefore we need another method to find the value. For this we program a function in our Python model which gets the arguments $\Phi^{-1}(a)$, $\Phi^{-1}(b)$ and ρ (Genz and Bretz, 2009).

Practical calibration method

We continue with the practical calibration method. Previously we showed that we match the variance of the observed default rate (from historical data) with the variance rate resulting from Equation 5.12. To do the moment matching, we need inputs to our model. The inputs needed are unconditional PDs and the correlation factor ρ.

We assume that unconditional PDs are published by S&P, this assumption is common when an IRB approach is not applicable (Rutkowski and Tarca, 2015).[2] To find for which ρ the variance resulting from the data equals the variance resulting from Equation 5.12 we use Newton's method, which is a root-finding algorithm.

To get an insight in the defaults, we first plot the observed defaults. We plot a step function for observed defaults in Figure 5.1. We give an example how to read the graph: Figure 5.1 indicates that in 80% of the observed historical years, the default rate was equal to or lower than 8%.

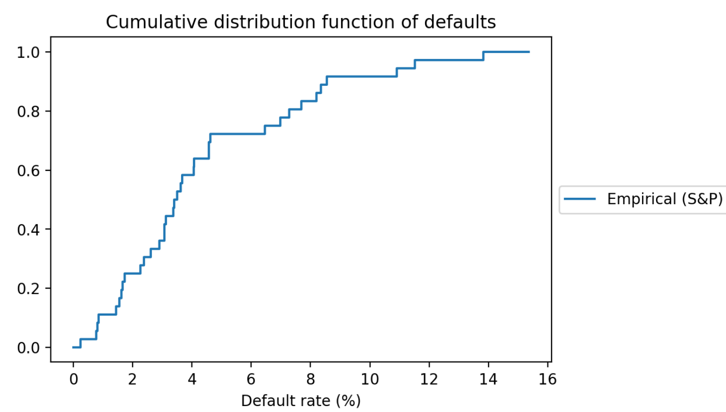


Figure 5.1: Observed defaults for B rating.

Using the information from Figure 5.1 we calibrate a Gaussian copula to the observed data with our Python model. The one-factor Gaussian copula is the LHP model by Vašíček, for which we showed the derivation in Section 5.1.

$$P(DR(F) \leq x) = \Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\rho}\right) \tag{5.14}$$

Here PD is the average probability of default on the data by S&P. DR is the default rate, which was also plotted on the horizontal axis in Figure 5.1. We

[2] "*Where an institution has approved PD estimates as part of the internal ratings-based (IRB) approach, this data must be used*" (BCBS, 2016a). Since this is not the case for our hypothetical situation, this does not apply and we get the PDs from market data.

plot the formula for the default rates on the interval [0,16], the result obtained is presented in Figure 5.2.

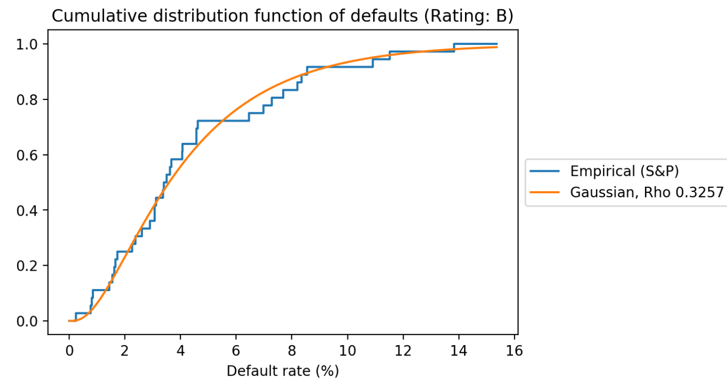


Figure 5.2: Observed defaults for B rating and fitted copula function.

From Figure 5.2 we see that the Gaussian copula fits the empirical cumulative default distribution from the data well at first sight. We continue with calibrating the other copulas, in Section 5.4 we comment on the differences.

5.2.2 Calibrating LHP - Student-t

For the Gaussian and Clayton calibrations, we have analytic formulas for calibration. However as we described in Section 5.1 directly fitting to an analytic equation is not possible for the Student-t copula. The reason for this is that the analytic equations do not exist. Therefore we propose a simulation algorithm. From Section 5.1 we know that we can calculate the conditional default rate with Equation 5.15.

$$DR(F,Q) = \Phi\left(\frac{\sqrt{\frac{Q}{\nu}}t_{\nu}^{-1}(PD_i) - \rho_i F}{\sqrt{1-\rho_i^2}}\right) \tag{5.15}$$

By simulating the default rate many times, we can build a cumulative density function for the distribution of defaults. In every simulation run, F gets a random (normal) realisation and Q gets a random (chi-square) realisation. After 100.000 simulations, we construct the cumulative distribution function from the gathered data.

The challenge with the Student-t copula calibration is that we should find two parameters, the ρ and ν. We choose for an approach where we assume ν, and calibrate ρ accordingly. The degrees of freedom observed are: ν = [5, 10, 20, 100]. The optimal value is determined by a minimizing optimization over the sum of squares between the historical distribution and the Student-t(ν, ρ) fitting. We do this for all values for ρ on [0,1]. The result is shown in Figure 5.3.

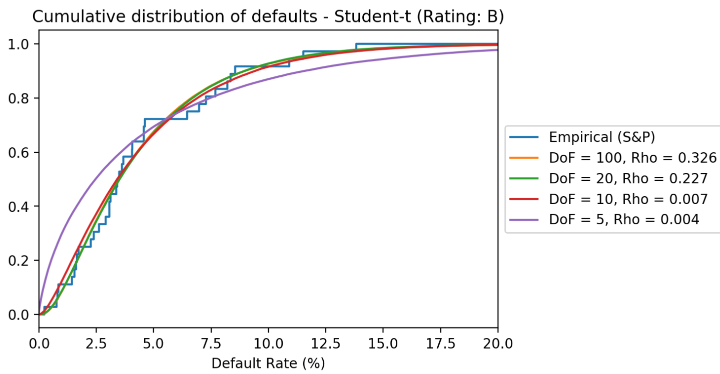


Figure 5.3: Calibration of Student-t copula to observed default rates.

As one can see from Figure 5.3 the calibration for a high degrees of freedom is almost analogous to the Gaussian/Clayton copula calibrations we saw before. For a low degrees of freedom the fit of the CDF is found by a very low ρ parameter, providing a bad fit to historical data. When we compare the sum of squares from low ν with high ν, the sum decreases when we choose a higher degrees of freedom ν.

To explain the effect of the Student-t copula compared to the Gaussian copula, we show the effect of having the same ρ parameter various degrees of freedom. This implies a less good fit to the historical default rates, but ensures fatter tails. The effect is shown in Figure 5.4, the application is relevant for the DRC model in Chapter 6.

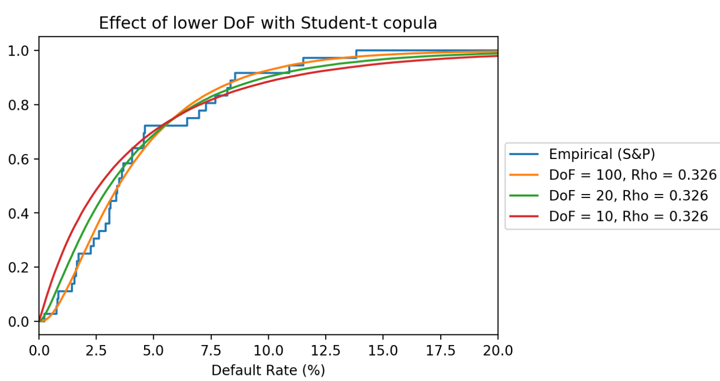


Figure 5.4: Effect of default rates with lower ν for Student-t copula.

5.2.3 Calibrating LHP - Clayton

The calibration method for the Clayton copula in the LHP model is roughly the same as for the Gaussian model. The procedure here also involves moment matching, described in Section 5.2.1. Like for the one-factor Gaussian copula approach to the LHP model, we use a Newton method for the calibration process. Here we calibrate the Clayton copula parameter θ. The Clayton copula was described in Section 3.3.2. Equation 5.12 changes to Equation 5.16 for the Clayton copula.

$$Var(DR) = \phi^{-1}(2\phi(PD)) - PD^2 \tag{5.16}$$

Here $\phi(PD)$ is the generator for parameter PD. The generator function is $\phi(p) = p^{-\theta} - 1$, as defined in Schönbucher (2002). The inverse generator function is in this case $\phi^{-1}(s) = (1+s)^{-1/\theta}$. Using these results, we can calculate $Var(DR)$ from Equation 5.16 as:

$$\begin{aligned} Var(DR) &= \phi^{-1}(2\phi(PD)) - PD^2 \\ &= \phi^{-1}(2(PD^{-\theta} - 1) - PD^2 \\ &= (2PD^{-\theta} - 1)^{-1/\theta} - PD^2 \end{aligned} \tag{5.17}$$

We use Equation 5.17 in our root finding algorithm. As for the one-factor Gaussian copula, we match the observed variance from the data, with the variance according to Equation 5.17 for a certain value of θ. From this method we find the optimised fitting parameter value.

To map the cumulative distribution of defaults, we use the formula below:

$$F(q) = 1 - G\left(-\frac{\ln q}{\phi(p)} \right) \tag{5.18}$$

The function $G()$ is an cumulative distribution function for the gamma distribution. This is the regularised gamma function, which we can calculate after calculating the lower incomplete gamma function. This is implemented in the code in Python. The results of the calibration for the Clayton LHP approach in Figure 5.5.

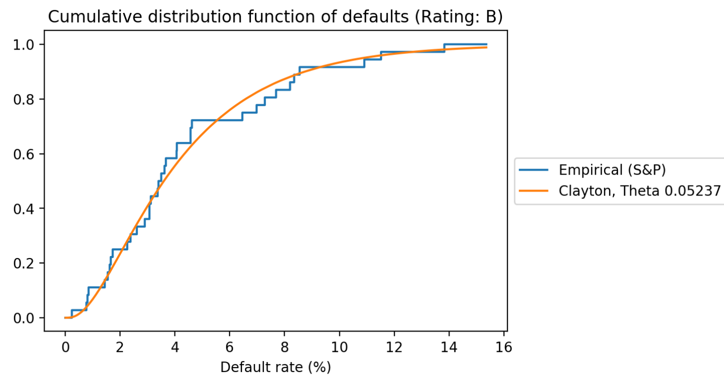


Figure 5.5: Observed defaults for B rating and fitted Clayton copula function.

In Figure 5.5 observe that the results look very similar to the Gaussian LHP approach. We come back to this in Section 5.4.

5.3 Empirical Evaluation

In this section we provide an empirical evaluation of the calibrations previously shown. We first provide confidence intervals for the data, afterwards we calibrate the copulas on the tails.

Building confidence intervals

We build confidence intervals around the data, since the data set contains only yearly observation points over 36 years. We build a confidence interval around the mean PD, and around variance of the PD. In Appendix A.6 we provide the methods for developing the confidence intervals in detail. Confidence intervals are shown for the Gaussian copula. Doing this for the Student-t or Clayton copula yields similar results, so we limit ourselves to one copula.

We build the confidence interval using formulas from statistics, next to this we compute confidence intervals with bootstrapping. The confidence interval for the mean PD is [4.25%,4.62%]. The confidence interval for the variance of the PD is [0.00052, 0.0016].

To show the combination of both the interval of the mean PD and the interval of the PD variance, we plot the possible outcomes. This gives the outer ranges for the confidence interval and is shown in Figure 5.6.

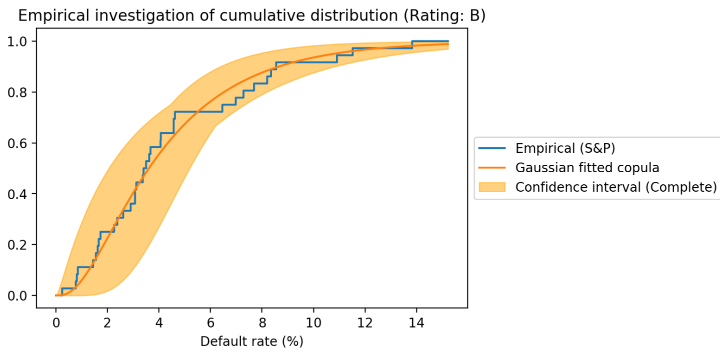


Figure 5.6: Fitted Gaussian copulas for 95% interval for the PD and PD variance.

From Figure 5.6 we can observe that the outer ranges of the copula calibrations lie around the observed historical data. This is a valuable result, since it visualises the uncertainty there implicitly is in the calibration.

Tail calibration

So far, we are calibrating the LHP models with the different copulas over the complete default distribution. However, in Chapter 4 we read that in 2008 the models were unable to model tail correlation correctly. For this reason, we perform a calibration on the tail in specific. From the historical default observations, we should determine where the tail starts. We calibrate the model again with the tail start value at 9% default rate and at 10% default rate.

The calibration is performed with the same approach as denoted in Section 5.2. Again, we determine the best fitting copula as the copula with the lowest result from the numerical optimization (least squares). We test the following setups: Gaussian, Clayton, Student-t(ν = 5), Student-t(ν = 10), Student-t(ν = 20), Student-t(ν = 50), Student-t(ν = 100). Figure 5.7 depicts the results. The shaded areas represent the area for the tail calibration.

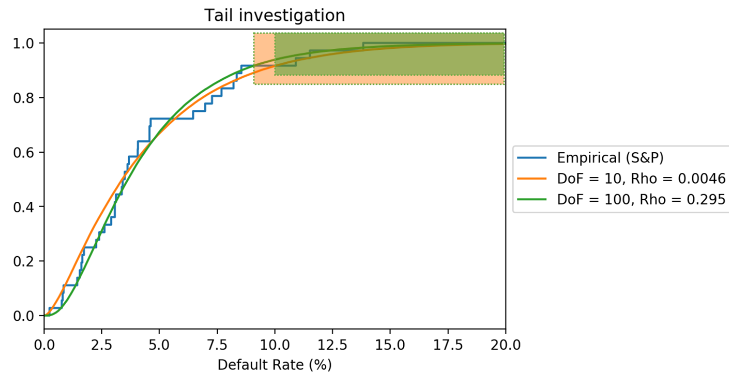


Figure 5.7: Calibration of the copulas on the tails

We see that the Student-t($\nu = 10, \rho = 0.0046$) copula fits the tail best when we determine the tail to starts at $DR = 9\%$. However, when we determine the tail to start at $DR = 10\%$, the Student-t($\nu = 100, \rho = 0.295$) copula fits best. So, from Figure 5.7 we conclude that the determination of the start of the tail influences the copula which fits best. This result indicates that one should be aware of the copula function applied when calibration should be done on the tails of the default distribution. For example this is relevant when we work with unidirectional portfolios/strategies from (e.g.) hedge funds.

Lastly, it should be noted that a trading book portfolio is often multidirectional. This is an important difference between credit exposures in a banking book with similar exposures in the trading book. Since the trading books are not unidirectional, the 99.9% tail from the Gaussian copula does not necessarily correspond to the 99.9% tail of the loss distribution. We elaborate on this in the sensitivity analyses in Section 6.4.

5.4 Comparison and inference

In this section we compare the three different calibrations of the LHP models from Section 5.2. Afterwards we explain on this how we can infer the LHP model to a FRTB compliant model.

5.4.1 Visualizing the similarities

First, we visually observed that the Gaussian copula provides a good fit to the historical data plotted. After working out the Clayton and the Student-t copulas, we saw very similar calibration results on the CDF plots. By performing a least squares method for all copulas, we observe a lower sum of squares for the Gaussian copula than for the Clayton and Student-t copulas investigated.[3]

[3] For a Student-t copula with very much degrees of freedom, the Gaussian case is reached. So in fact the Student-t copula also provides the optimal fit.

To investigate the differences between the calibration, we plot the probability density functions according to the formulas denoted in Section 5.1. We plot the probability density functions for the defaults with the different copulas in Figure 5.8. In the background we plot the histogram of historical defaults.

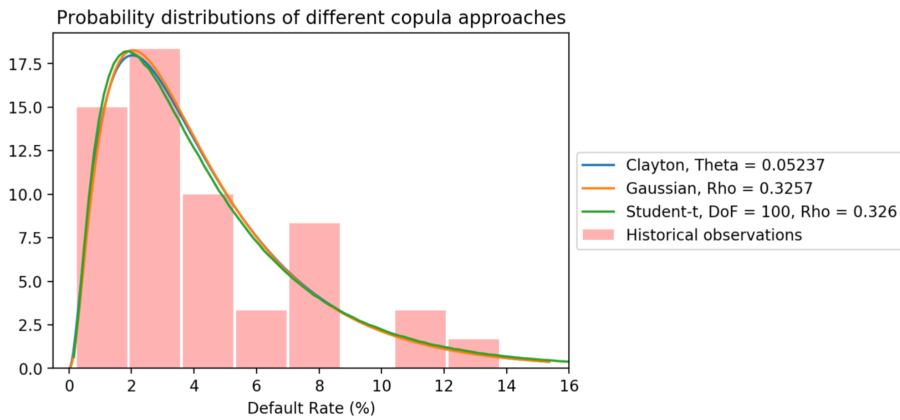


Figure 5.8: PDF visualization of the different copula approaches.

Figure 5.8 shows us again that there are minimal differences in the calibrations of the copula approaches to the data. The Student-t result converges to the Gaussian result if we choose a larger amount of degrees of freedom, and run the Monte Carlo algorithm with a larger amount of simulations. So, the difference between the Gaussian copula and the Student-t copula could be solely explained by the calibration method, which is in the one case with an analytic equation and Monte Carlo in the other.

5.4.2 Explaining the similarities

As we observe in Figure 5.8, the peak of the Clayton copula is only slightly different from the Gaussian copula. The similarities observed between the copulas are consistent with the results by Schloegl and O'Kane (2005). Schloegl and O'Kane (2005) investigated the differences in the LHP when other copula functions would be applied for credit risk modelling. The results of Schloegl and O'Kane are on the comparison of the Value at Risk risk measure, resulting from different copulas (Gaussian, Student-t, Clayton and Gumbel) in a LHP setup. The most striking result is the comparability of the results of the VaR measure of the Gaussian, Student-t and Clayton copula. This is only the case when constant bivariate default probabilities are assumed.

Schönbucher (2002) shows that in case of a Clayton copula, the difference to the Gaussian copula assumption is minor when constant bivariate default probabilities are assumed. Burtschell et al. (2009) explain the 'strikingly similar' results of the Gaussian copula and the Clayton copula from the fact that the conditional default probability is a key input. Since the both one-factor models base itself on the same (constant) input, the results obtained are also similar.

5.4.3 Inference of LHP model to FRTB compliant model

In the preceding sections we described how the LHP model could be applied to fit historical default rates under different copula regimes. However, in our research we try to model the default risk in a situation compliant to the FRTB regulation. Therefore we describe here how we can infer the one-factor models to a FRTB compliant model.

The Gaussian, Student-t and Clayton copulas allow to be defined through a one-factor copula model. For the Gaussian and the Student-t copula it is rather easily possible to transform it to a multi-factor setup. However, a straightforward transformation of the Clayton copula to a multi-factor setup is not found in the literature. As we read in Chapter 2, FRTB regulation prescribes that the DRC model is based on two systematic risk factors. We want to model these systematic factors through a factor model, therefore we leave the Clayton copula approach for now. In Chapter 7's Further Research, we comment on applying the Clayton copula (or other Archimedean copulas) in a nested copula approach.

5.5 Chapter Conclusion

In this chapter we investigated **Research Question 4**: "*How can we use the Large Homogeneous Pool model for default modelling under FRTB regulation?*".

We can calibrate the LHP model for the Gaussian, Student-t and Clayton copulas by moment matching procedures. The calibrations give the insight that the Gaussian, Student-t and Clayton copulas all provide a very comparable fit to historical default data. This can be explained with literature on bivariate default modelling (f.e. Schönbucher (2002) and Burtschell et al. (2009)).

From the calibration of the LHP model we conclude that the Gaussian copula fits the data best when calibrated to the complete default distribution. This changes when we calibrate to the tails of the distribution. In this case the Student-t copula allows for a better fit. One should be aware of the limitations of the calibrations with the Gaussian copula. Since on the average it could fit the shape of the historical data very well, however in the tails of the distribution the copula functions less well. However, trading portfolios are (almost) never unidirectional, which implies that average fit of the copula is more relevant than the tails of the copula.

In the following chapter, we append the Gaussian and Student-t copulas to a multi-factor setup as the FRTB regulation prescribes. Yet, there is no straightforward method to use the Clayton copula with multiple systematic factors. The Gaussian copula acts as the backbone of the DRC model, where the Student-t copula induces more tail dependence in the model.

6

Modelling the DRC compliant to FRTB

In this chapter we describe our approach to model the Default Risk Charge according to our own developed internal model. The modelling is done compliant to the DRC model requirements from Chapter 2.2.2. After this chapter we are be able to answer **Research Question 5**. To build our own internal models approach for calculating the DRC we perform several steps. We start with describing the model setup, both theoretical and practical. Also, we describe the methods applied and calibrations performed. Afterwards we are able to compare the DRC from our model with the standardised approach DRC.

6.1 Model setup

In this section we describe the theoretical model setup to model the Default Risk Charge. We do this by describing the model inputs, the methods and the calibrations which are performed. The model is compliant to the DRC regulations presented in Section 2.2.2.

The model setup is inspired by the work of Wilkens and Predescu (2015). The factor model by Wilkens and Predescu (2015) is provided in Appendix A.2, the main difference is the use of the R^2 parameters. We first specify the IMA DRC model with a Gaussian factor copula structure in Section 6.1.1, we modify the model to a Student-t factor copula structure in Section 6.4.

6.1.1 Theoretical model

BCBS (2016a) describes that the model for default risk should be a VaR model, where *"banks must use a default simulation model with two types of systematic factors"*. As proposed in Chapter 5.1 this can be done through a factor model.

Factor model:
A factor model with two systematic factors is shown in Equation 6.1.

$$Y_i = \rho_{C,i} F_C + \rho_{S,i} F_S + \sqrt{1 - \rho_{C,i}^2 - \rho_{S,i}^2 - 2\rho_{C,i}\rho_{S,i} cov(\rho_{C,i}, \rho_{S,i})} Z_i \tag{6.1}$$

As shown for a one-factor approach in Chapter 3.2.2, the two-factor model of Equation 6.1 could be written as a conditional probability of default equation[1]:

$$DR(F_C, F_S) = \Phi\left(\frac{\Phi^{-1}(PD_i) - (\rho_{C,i} F_C + \rho_{S,i} F_S)}{\sqrt{1 - \rho_{C,i}^2 - \rho_{S,i}^2 - 2\rho_{C,i}\rho_{S,i} cov(\rho_{C,i}, \rho_{S,i})}} \right) \tag{6.2}$$

In Equation 6.1 $\rho_{C,i}$ and $\rho_{S,i}$ are the factor loadings to the country factor (F_C) and the sector factor (F_S) respectively. F_C should be interpreted as the current realisation of the country's macroeconomic situation, F_S is defined accordingly. Z_i represents the company idiosyncratic factor. The covariance factor is the covariance between F_C and F_S. Y_i is the asset return of company i. The factors F_C, F_S and Z_i are standard normally distributed, by its mathematical properties this ensures that the asset return, Y_i, is also standard normally distributed.

The factor loadings are gotten through multi-factor regression analysis. To make the model robust, we apply a machine learning technique to cluster the data. After the clusters have been made, we use bootstrapping to provide descriptive statistics of the different clusters and to ensure that the clusters are statistically different from each other.

[1] Note that Equation 6.2 incorporates that the two systematic factors are not independent from each other.

Simulation:
After the model setup, we run a simulation. In the simulation, an obligors defaults when the default condition is fulfilled. The default condition is:

$$Y_i < N^{-1}(PD_i) \tag{6.3}$$

Here N^{-1} represents the inverse of the cumulative standard normal distribution function. The default probability (PD) is dependent on the rating class of obligor i.

PD-LGD model
The regulation prescribes that the LGD should be dependent on the systematic factors (BCBS, 2016a). We propose a factor setup for this PD-LGD relationship in which we also incorporate the linear function defined by Altman and Kuehne (2012) in Chapter 3.1. Altman and Kuehne (2012) defined the relationship between the default rate and the recovery rate of obligor i as follows:

$$RR_{\text{Altman},i} = -2.3137 * [\text{Default Rate}]_i + 0.5 \tag{6.4}$$

In Equation 6.4, the $[\text{Default Rate}]_i$ is the percentile point of $N^{-1}(\mu_i, \sigma_i)$ where the percentile is the realization of the systematic factor. Here μ_i and σ_i represent the mean and standard deviation of the yearly default rates of obligor i's rating class over the period 1981-2016 (Standard & Poor's Financial Services, 2017).

In our simulation model, the recovery rate (RR) for obligor i is defined according to the following equations:

$$\begin{aligned} U_i &= \gamma U_{\text{Altman},i} + \sqrt{1-\gamma^2}U_{\text{idio,i}} \\ &\text{with} \\ U_{\text{Altman},i} &= N(0,1) \text{ value of } RR_{\text{Altman},i} \\ U_{\text{idio},i} &= \text{random draw N(0,1)} \end{aligned} \tag{6.5}$$

The recovery rate of obligor i (RR_i) is the $N^{-1}(0,1)$ value of U_i. The proposed model ensures the dependence with the systematic factor through the $[\text{Default Rate}]_i$ by factor loading γ. We set $\gamma = 0.7$.[2]

[2] In practice γ is derived from internal historical data by the bank. Since we don't have data availability on this we set $\gamma = 0.7$ to give a +/- 50% loading to both the recovery rate by Altman as to the idiosyncratic recovery rate.

6.1.2 Data

We use various data for our model from different sources:

- **Global equity data**: We use the S&P Global 1200 Index as a gauge for the global equity returns. We have availability of the daily returns data in the period 2008-2018 (S&P Indices, 2018).
- **Country equity data**: We use US S&P 500 index returns (Yahoo Finance, 2018). We have availability of a long history of data, but use the daily returns data in the period 2008-2018 .
- **Sector equity data**: For every sector, we use the sector indices on 11 sectors according to the Global Industry Classification Standard (GICS). We have availability of the daily returns data in the period 2008-2018 of the S&P Global Indices (S&P Indices, 2018).[3]
- **Security data**: We use individual security returns of all S&P 500 constituents. We have availability of a long history of data, but use the daily returns data in the period 2008-2018 from the Compustat Capital IQ database (Standard & Poors, 2018).
- **Company information**: We use descriptive company information as rating, industry, net income, total assets, and so on. We have availability of a long history of data, but use the most recent set of descriptive company data of every S&P 500 constituent (2017) from the Compustat Capital IQ database ('Fundamentals Annual') (Standard & Poors, 2018).
- **Default data**: We use data on corporate defaults over the period 1981-2016 with the defaults per rating category. Appendix A.5 provides an overview on the default data from Standard & Poor's Financial Services (2017).

6.1.3 Model calibrations

In this subsection we explain the model calibrations performed. We both explain how we calibrate in the model using data from a period of stress and we investigate how big the optimal return window should be.

Finding the period of stress

According to the regulation we should calibrate on a dataset of 10 years that includes a period of stress (BCBS, 2016a, 186 b). Since the 2008 financial crisis is within the dataset we assume the dataset includes a period of stress. We calculate the pairwise correlations on all S&P 500 pairs (124.750 pairs) to find the period with the highest pairwise correlations. Since the regulations

[3] We use data on the S&P indices since there is free data availability for a period of 10 years, this is not the case for the MSCI World Indices.

does not prescribe the type of correlation, we use the (standard) Pearson correlation. This method is also faster than rank order correlation methods as Kendall/Spearman. The correlation of a pair is calculated on a one year time interval. The results are visualised Figure 6.1, where we plot the 5%, 50% and 95% quantiles of the pairwise correlations.

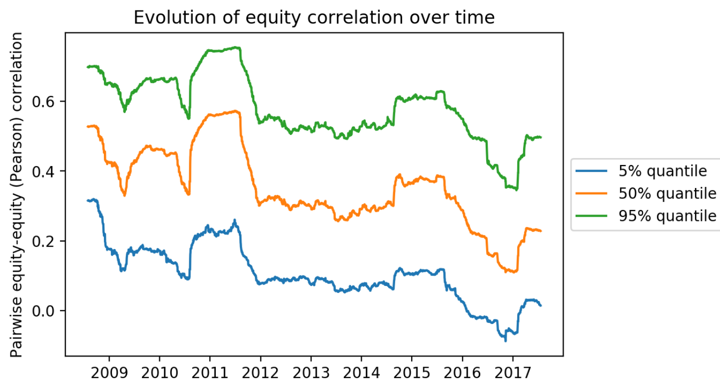


Figure 6.1: Moving pairwise correlations (Pearson) on the 2008-2018 interval.

In Figure 6.1 we clearly observe the period of stress. The correlations were highest in 2011.

Finding the optimal return window

A next step is to determine which time return window should be used. It could for example be the case that the one-day security returns do not strongly follow the one-day index returns, where the one-week or one-month security returns are highly related. The reason for this is that market response times are delayed (by any reason). Therefore we investigate the results of regression over different time intervals, from 1 day to 40 days (8 business weeks).

The greater the R^2 factor, the bigger the part of the returns could be explained. The quantiles of the R^2 factors for different return observation periods (on the data 2008-2018) are plotted in Figure 6.2. We plot this for the full 10 year time frame, since this gives a complete picture of the most appropriate return observation period. A similar analysis, but than for the 2011 year of stress, leads to the same conclusion.

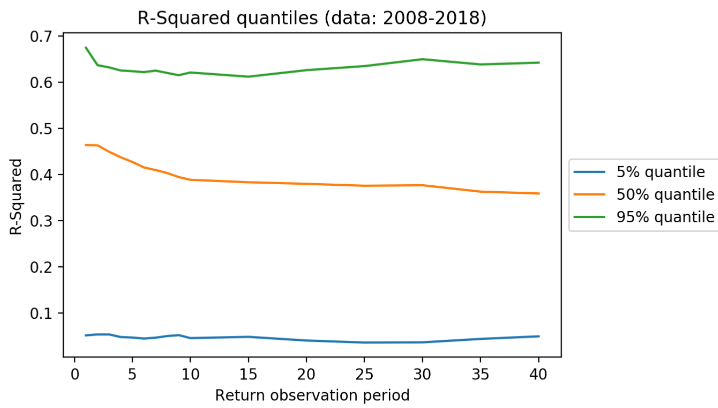


Figure 6.2: R^2 value of different return observations periods (data: 2008-2018).

As we see from Figure 6.2, the one-day return observation period yield the highest R^2 values. We do not need to look at longer return periods. This is the proof that markets are responding quick, and the daily sector/country index changes explain daily changes of a security best. Another interesting fact observable from Figure 6.2 is that from a 10-day observation period, the R^2 values keep roughly constant.

6.2 Model methods

In this section, we describe the key elements applied in the analysis. These are multi-factor regression and clustering analysis.

6.2.1 Regression

For the regression we first standardise the returns to $N(0,1)$. This is a quantile-quantile mapping of the return data on a $N(0,1)$ distribution. This implies that the 1% worst return is mapped to the 1% quantile of the standard normal distribution. This is done on the full regression interval for the sector and country indices, as for the individual securities.

Afterwards we perform a regression of the standardised security returns against the standardised sector and country returns. This is done on the 10 year regulatory prescribed period: 2008 - 2018. The independent variables are country and sector returns. The dependent variable is the security return.

For every S&P 500 constituent, the regression results in a country factor loading (ρ_C) and the sector factor loading (ρ_S). The covariance factor is determined as the covariance between F_C and F_S for the data. Also the R^2 value is gotten from the regression, explaining how much of the variance has been explained.

6.2.2 Cluster analysis

After the regression, we have a country factor loading and a sector factor loading for every S&P 500 constituent. Figure 6.3 shows a visual plot of the results.

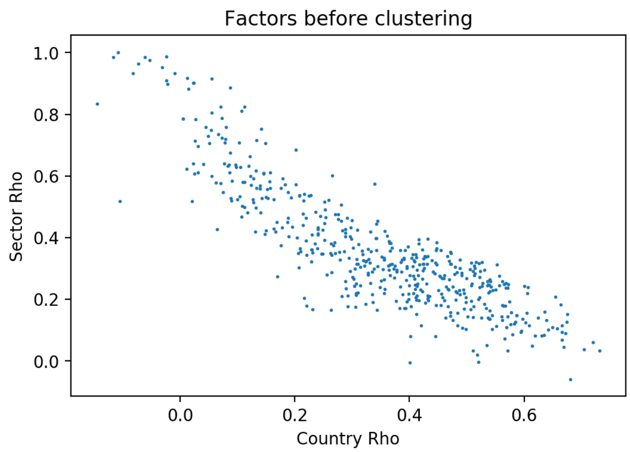


Figure 6.3: Visualization of the factor values for all S&P 500 constituents.

Figure 6.3 indicates that many of the values are closely tied together. For robustness of the model, cluster analysis should be performed. Clustering makes it possible that new S&P 500 constituents could be classified, without re-running the model. In practice this is a big advantage, since rebuilding the model is (time) costly. There are many machine learning techniques to perform clustering. Since the response variables are known for the cluster analysis, a supervised learning procedure suffices. We choose for a regression tree method because it's rule based and intuitive to follow, by this we prevent the model becoming a 'black box'. A regression tree is a tree-based method where the target variable takes on a continuous value (James et al., 2013). In this case the regression trees target variable is the factor (either country or sector factor).

Black box models

After the 2008 financial crisis and events like the 2010 and 2015 flash crashes, there is a continuous call to prevent 'black box modelling'. In the FRTB regulation for calculating the Default Risk Charge (as described in Chapter 2), the BCBS describes the model requirements very broadly. However, since all internal models by financial institutions are subject to review and acceptance by the BCBS, 'black box'-like models would probably not be accepted for the internal models approach. Therefore we address why the proposed technique of applying regression trees is not a black box model.

A regression tree is a method which allows to be interpreted by human logic. The inner components/rules are open for inspection, and since it is a supervised learning algorithm we can easily track and check performance. A regression tree is visualised trough a dendrogram, which makes it very easy readable and in fact a 'white box'. To prevent over-fitting in the method, we prune the tree to its smallest but most effective size. This yields for more intuition, since the tree depth is reduced.

Obviously, also different clustering methods are available. However, one should note that when choosing for an unsupervised learning method the technique is not intuitive anymore. Currently, random forests are popular in application, since they provide reasonable results even with small data sets as input (Hastie et al., 2001). But, the lack of interpretability results easily in problems as overfitting causing big errors. This makes a random forest clustering method a black box.

Regression tree building

The tree building takes place through splitting rules, the splitting is based on descriptive company data supplied into the algorithm. After the tree is build, (post-)pruning is done. This is a method to simplify the tree, while yielding the same or a better prediction accuracy. Namely, a smaller tree with less splits might lead to lower variance (James et al., 2013). This is known as the bias-variance trade-off. To prune the tree, we apply cost complexity pruning through an algorithm supplied by James et al. (2013).[4] Appendix A.8 shows the process of the clustering step-wise.

We supply the tree building algorithm with 42 explanatory variables, see Table 6.1. The explanatory variables include sector classifications, credit ratings, stock market capitalisation and so on. The source of the data is the Compustat - Capital IQ database (Standard & Poors, 2018). Selection of the explanatory variables took place based on data completeness for all S&P 500 constituents. Time dependent variables are based on companies' 2017 year report values.

[4] Since Python does not contain this method, we use R for the tree building and the cost complexity pruning

Explanatory Variables (1-21)	**Explanatory Variables (22-42)**
Sector	Inventories - Total
Current Assets - Total	Current Liabilities - Total
Accounts Payable - Trade	Liabilities - Total
Acquisitions	Net Income (Loss)
Assets - Total	Order Backlog
Capital Expenditures	Revenue - Total
Common/Ordinary Equity - Total	Sales/Turnover (Net)
Common Equity - Liquidation Value	Stockholders Equity - Parent
Cash	Receivables (Net)
Cash and Short-Term Investments	Common Shares Traded - Annual - Calendar
Comprehensive Income - Total	Market Value - Total - Fiscal
Cost of Goods Sold	GIC Groups
Long-Term Debt - Total	GIC Industries
Dividends - Total	GIC Sectors
Earnings Before Interest and Taxes	International, Domestic, Both Indicator
Employees	Standard Industry Classification Code
Property, Plant, and Equipment - Buildings at Cost	S&P Industry Sector Code
Goodwill	S&P Economic Sector Code
Gross Profit (Loss)	S&P Quality Ranking - Current
Invested Capital - Total	State/Province
Intangible Assets - Total	S&P Domestic Long Term Issuer Credit Rating

Table 6.1: Explanatory Variables for the regression tree.

Clustering results

In Appendix A.8 the regression trees are visualised. We show them both before pruning, and after pruning took place. Also we show why we prune the trees to a certain size, based on the relative error with a certain tree size. The three most important explanatory variables for explaining both the sector factor and the country factor are (listed in order):

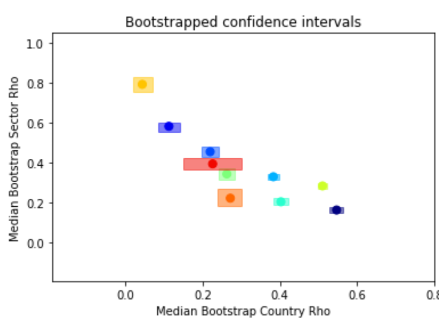
1. Standard Industry Classification Code
2. Global Industry Classification Industries
3. Global Industry Classification Groups

With the trees built, shown in Appendix A.8, we can plot the results from the cluster analysis. Figure 6.4 visualises the resulting clusters.

Figure 6.4: Visualization of the clustered factor values for all S&P 500 constituents.

Figure 6.4 indicates with small coloured dots in which cluster it belongs, the big dots show the values assigned to the clusters[5].

In Figure 6.4 we see 10 clusters. To check if all clusters are significantly different from each other, we perform a bootstrap method on the medians (2-dimensional) by re-sampling the dots. Figure 6.5 (left) depicts the initial confidence intervals around the median values of the clusters. We observe that the red cluster ('SC 4, CC 3', which contains only 2 companies) has an overlap with other confidence intervals. To make the model more robust, we update the classification of these two companies. The cluster assigned after reclassification 'SC 4, CC4'. The final clusters are depicted in Figure 6.4 (right), here we clearly see that all the clusters are distinctive from each other.



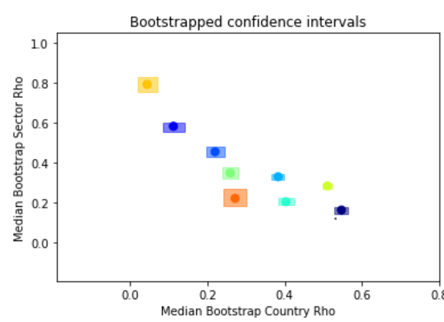


Figure 6.5: Bootstrapped confidence of the mean and updated clusters.

[5] The meaning of 'SC 3, CC 7' is that the sector cluster (SC) has label 3 (label for the leaf in the tree) and country cluster (CC) has label 7, respectively.

6.3 Simulation and results

In Section 6.1 we defined a factor model for calculating the asset return of company i. We use Equation 6.6 to simulate asset returns for all S&P 500 constituents.

$$Y_i = \rho_{C,i}F_C + \rho_{S,i}F_S + \sqrt{1 - \rho_{C,i}^2 - \rho_{S,i}^2 - 2\rho_{C,i}, \rho_{S,i}cov(\rho_{C,i}, \rho_{S,i})}Z_i \tag{6.6}$$

We assume that we have a S&P 500 long portfolio with equal weights. If the default condition is hit, the obligor (the S&P 500 constituent) defaults on its obligations. Thereafter the LGD is calculated for the defaults in the portfolio. By performing a large set of simulation runs we are able to calculate the Default Risk Charge afterwards.

Calculation of the Default Risk Charge

From the simulation results we find the DRC scenario, being the 99.9% percentile of defaults.[6] We run the simulation with a growing amount of simulation runs. The reason for this is that we want to find out for which amount the result converges. The results are shown in Figure 6.6.

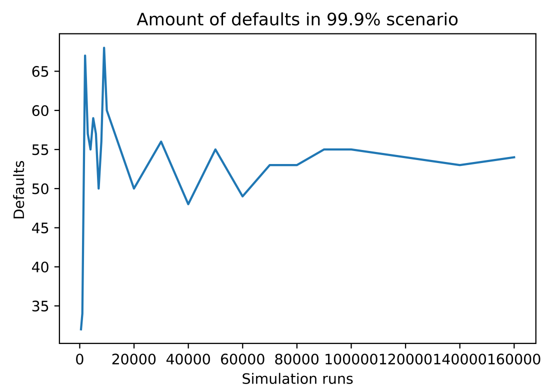


Figure 6.6: Convergence of DRC.

We run this simulation 100.000 times, since Figure 6.6 the DRC result is reasonably stable after 10 years. Figure 6.7 shows the histogram of observed defaults per simulation run. In this figure, the y-axis is limited to 100 observations, since the first histogram bins contain many observations.

[6] The 99.9% percentile is seen as the amount of defaults occurred in the 99.9% worst simulation. This ensures that the result always is an integer, which is common logic in case of defaults.

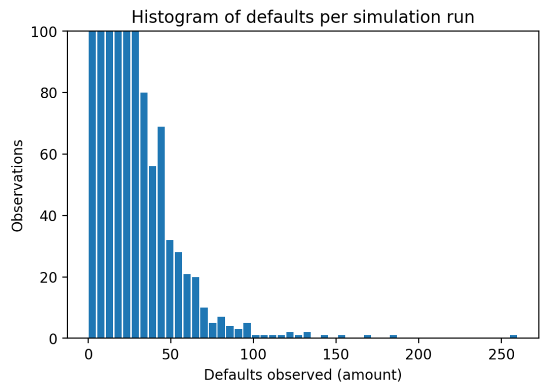


Figure 6.7: Histogram of counted defaults (100.000 simulations).

Hereafter, we find the LGD of the defaulted portfolio constituents, which is based on the systematic factors. From this we calculate the Default Risk Charge as a percentage of our portfolio in the 99.9% worst case scenario.[7] Please bear in mind that these results are only for a S&P 500 Long portfolio.[8]
Default Risk Charge (IMA): 6.66%

We compare this results when we would apply the standardised approach to the Default Risk Charge. The SA calculation of default risk is split in three different classes. The class comparable with the developed DRC-IMA model we develop is the calculation of DRC for Non-Securitisations. The standardised approach calculation for the DRC is given in Appendix A.7.
Default Risk Charge (SA): 6.46%

These results indicate that the Default Risk Charge resulting from the internal model developed are very similar to the charge resulting from the standardised approach. For a long S&P 500 this makes sense, since the DRC standardised approach is calibrated with a large dataset according to the credit risk treatment in the banking book (BCBS, 2016a). The S&P 500 represents a large part of the US economy, and in a unidirectional (long) portfolio it is plausible that the DRC IMA charge is close to the DRC SA charge.

[7] We calculate the DRC VaR 99.9% using Extreme Value Theory (EVT). Appendix A.9 presents in detail how EVT works.

[8] We do not specify the type of financial product or the seniority of the products, since this complicates the comparability of the both approaches.

6.4 Model extensions and sensitivity analysis

In this section we elaborate on some model extensions and we perform sensitivity analyses. The specific objective of this is to see what the main influencing factors for the model outcomes are, and to induce more variability into the model. These extensions and sensitivity analyses are useful since the Gaussian model setup has it's limitations.

6.4.1 Student-t copula

We analyse the impact of changing the copula in the DRC model from Gaussian to Student-t. We assess the effect of the Student-t copula with several degrees of freedom. For this we choose 5, 10 and 20 degrees of freedom. We use the factor model for the Student-t copula as presented in Chapter 5.1.

As described in Chapter 5, the Gaussian copula provides a very good fit to the data. However, as noted before, problems could arise in case we have a unidirectional exposure. Since a trading book portfolio is (almost) never unidirectional, we can fairly argue that the Default Risk Charge (at Value at Risk 99.9%) can be derived with the Gaussian copula setup. The extensions shown below is mainly to show the effect of the Student-t copula's feature to ensure fatter tails.

In Figure 6.8 we present the results of the simulation (100.000 runs) for the DRC model with 5, 10 and 20 degrees of freedom (DoF). As we expect, with lower degrees of freedom very high default rates could be observed. If we would take the degrees of freedom higher, the model would converge to the Gaussian model.

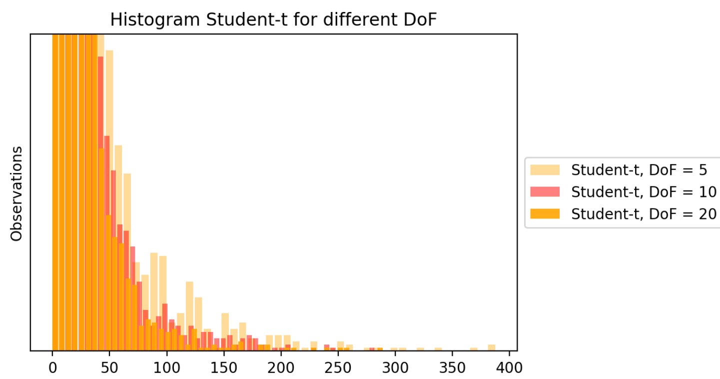


Figure 6.8: Defaults of Student-t copula for different degrees of freedom.

There are plenty of more options for model changes by adapting to a different copula structure. For example with a nested copula, we are able to use

Archimedean copulas in a multi-factor setup. Since this goes beyond the scope of this thesis, we address some of these opportunities in the further research in Chapter 7.

6.4.2 Parameter sensitivity

We assess the developed model for its sensitivity to certain parameters. First, we investigate the effect of a different clustering with regards to the cluster approach described in Section 6.2.2. Second, we inspect the sensitivity of the outcome to the covariance between ρ_C and ρ_S.

Sensitivity of the clustering
To inspect the effect of a different clustering approach, we change the complexity parameter in the tree pruning method. James et al. (2013) describe that CP should be chosen based on visual inspection of the relative error based on the tree size (see Appendix A.8). Previously in Section 6.2.2, we chose CP 0.035 for the country factor, and CP 0.045 for the sector factor. To inspect the sensitivity we choose different values for CP. For the sake of simplicity, we set sector/country CP to the same value. We explore the effect of CP 0.01, 0.02 and 0.10.

Sensitivity of the covariance
Examining the sensitivity to the covariance is relevant since we imply a period of higher and lower stress in the financial market. We both investigate the effect of a 10% higher covariance and the effect of a 10% lower covariance factor. These situations could be compared with a period of 'more stress' and 'less stress' in the financial markets.

The parameter sensitivity effects are shown in Table 6.2 and assessed in the next subsection.

6.4.3 Risk measures

We investigate the risk measures resulting from the different model extensions and sensitivity measures stated in the previous paragraphs. For estimating the VaR and ES measure, we apply Extreme Value Theory from which we know that the tails of a loss distribution converge to a generalised Pareto distribution. A theoretical description of Extreme Value Theory is described in Appendix A.9. We use 100.000 simulations for the different setups investigated. From this we are able to calculate the Value at Risk measure as the Expected Shortfall measure at 99.9%. The results are visualised in Table 6.2.

The regulation on the Default Risk Charge prescribes the use of the VaR 99.9%, however we also show the Expected Shortfall at 99.9% result because this gives us a good indication of the shape of the loss distribution's tail. The ES and VaR measures allow us to compare the effect of the sensitivity and the extensions.

Model	VaR 99.9%	ES 99.9%	Default Rate VaR 99.9%	Default Rate ES 99.9%
DRC - Standardised approach	6,46%	-	-	-
Gaussian model	6,66%	9,70%	12,0%	17,5%
Student-t model (DoF = 5)	14,77%	22,42%	26,6%	40,4%
Student-t model (DoF = 10)	11,53%	16,81%	20,8%	30,3%
Student-t model (DoF = 20)	8,83%	13,94%	15,9%	25,1%
Clustering CP = 0.01	6,45%	10,11%	11,6%	18,2%
Clustering CP = 0.02	6,29%	8,99%	11,3%	16,2%
Clustering CP = 0.10	7,51%	10,87%	13,5%	19,6%
Covariance +10%	6,61%	10,04%	11,9%	18,1%
Covariance -10%	6,41%	9,23%	11,5%	16,6%

Table 6.2: Risk measures outcomes of the different models.

Interpretation of the results
From Table 6.2 we observe the effects of the Student-t copula on the Gaussian model developed, but also the effects of the sensitivity analyses performed. The Student-t copula induces, as we expected from what we saw in Chapter 5, more tail risks into the model. This leads to a significantly higher DRC.

The sensitivity of changing the parameters in the cluster analysis (and so, leading to different clusters) is less big. For CP 0.01 or 0.02 the effects are minimal. With lower CP, tree size grows. From the ES we can observe here that the variance increases when compared with the original model when we reduce the CP. This makes sense according to the the theory described in Section 6.2.2 and Appendix A.8. When we set CP = 0.10 tree size reduces. As we see from Table 6.2 this leads to higher risk measures. Setting CP = 0.10 is therefore not optimal.

Changing the covariance between ρ_C and ρ_S leads to minimal changes. The effects are that with higher covariance, the ES measure increases. The opposite holds for when the covariance is lower.

6.4.4 Extra portfolios

So far we only worked out the model with a unidirectional (long) portfolio composition, representing a diversified portfolio. From the results we saw that the IMA charge is closely related to the SA charge.

However, many more portfolios could be designed. We investigate three other portfolio compositions:

- High quality long portfolio: Containing S&P 500 constituents with credit rating AAA, AA or A.
- Low quality long portfolio: Containing S&P 500 constituents with credit rating BBB, BB, B or CCC.
- Multidirectional portfolio: Containing 50% long and 50% short exposures.

Table 6.3 shows the Default Risk Charge for the three specified portfolio compositions. We both show the standardised approach charge as the internal models approach charge.

Portfolio	**SA Charge**	**IMA charge**
High quality long portfolio	2,8%	2,9%
Low quality long portfolio	8,4%	8,2%
Multidirectional portfolio	3,2%	0,9%

Table 6.3: Default Risk Charge for different portfolio compositions

From Table 6.3 we see that the two unidirectional (long) portfolios result in a similar charge for both the SA and IMA. However, we see that the multidirectional portfolio IMA leads to a default risk charge 3.5 times lower than the SA charge. This can be explained by the fact that the SA model (see Appendix A.7) is very conservative in its 'hedge benefit ratio' for multidirectional portfolios. This shows that an internal DRC model can lead to significantly lower risk charges than the standard DRC model.

6.5 Chapter Conclusion

In this chapter we proposed a default model to investigate **Research Question 5**: "*How do FRTB's capital charges on default risk relate?*".

Under the standardised approach, the calculation of default risk is split in three different classes. The class comparable with the DRC-IMA model we develop is the calculation of DRC for Non-Securitisations (Appendix A.7). The offsetting rules are strictly defined in the standardised approach.

For a long portfolio, containing all S&P 500 constituents, the DRC from the IMA and the SA model are closely related. This is reasonable for a highly diversified portfolio, since the SA has been calibrated along the treatment of credit risk in the banking book. For multidirectional portfolios, the IMA results in a lower default risk capital charge compared to the SA.

7

Conclusion, Discussion and Further Results

In the final chapter we conclude on the results and provide an answer to the main research question. Also we pose discussion points and we suggest directions for further research.

7.1 Conclusion

In Chapter 1 we defined the main research question as follows:

Main RQ: How to develop a model to calculate the FRTB's capital charge for default risk, using a factor copula model with two systematic factors?

The approach set to answer the main research question was by answering five sub-questions. From Chapter 2 to Chapter 6 we provided answers to these sub-questions. Now we provide an answer to the main research question.

The BCBS regulation on the Fundamental Review of the Trading book stipulates the use of two systematic factors for determining the Default Risk Charge through an internal models approach. To determine which copula we would apply in a factor setup, we calibrated three different copulas (Gaussian, Student-t, Clayton) to historical default data with the Large Homogeneous Pool model by Vašíček (1987). From this we concluded that the Gaussian, Clayton and Student-t copula can fit the historical observations of defaults very well. We drop the Clayton copula at this point, since this Archimedean copula can't be set to a two-factor model except when we use a nested copula approach (see Further Research).

After the 2008 financial crisis, the Gaussian copula method received a lot of criticism. The downsides of the Gaussian copula learned from the 2008 financial crisis do not fully apply for the Default Risk Charge. The main reason for this is that CDOs are subject to the DRC SA by regulation. However, we should be aware for the limitations of the Gaussian copulas with the default distribution of unidirectional portfolios.

We developed a Gaussian copula factor model to calculate the Default Risk Charge compliant to the FRTB regulation. The systematic factors determined are a country factor and a sector factor. We used historical data on the S&P 500 constituents for regression against the country and sector performance. By a machine learning clustering approach we were able to cluster the companies to achieve model robustness. The calibrated internal DRC model leads to a Default Risk Charge highly similar to the standardised Default Risk Charge in a unidirectional portfolio. By investigating the model for multidirectional portfolios, we observed that Default Risk Charges are lower through our internal models than the standardised approach. The main reason for this is the conservative 'hedge benefit ratio' set by the BCBS. Finally, we worked out some model enrichment as ensuring more tail dependence by changing the Gaussian copula to a Student-t copula.

7.2 Discussion and Further Research

In this section, we reflect on some points of the thesis. We combine this with some directions for further research.

Factor model
In this research we work with a factor model in a straightforward setup. However, many different model configurations are possible (as shown in Appendix A.2). It would be interesting to compare the results from the factor model in our situation, for example with the model applied by Wilkens and Predescu (2015). Next, in the DRC model there are different systematic factors we could have applied for modelling the capital charge. The 'sector' and 'country' factor are the two systematic factors the BCBS suggests, however the regulation allows for different factors. Laurent et al. (2016) propose to calibrate according to a latent variable factor model, with uncorrelated factors (f.e. in a principal component analysis). Here, the factors are not observable. This allows for a more detailed specification of correlation structure, however the model's readability is reduced.

Copulas
After Chapter 5 we left the Clayton copula since it did not allow for a straightforward application as a two-factor copula. However, it is possible to model the DRC with many different copula structures. We can develop a nested copula, which consists of multiple (Archimedean) copulas to enable a more complex dependence structure (Otani and Imai, 2013). It should be noted that nested copula structures make a model less intuitive since any dependence relationship can be configured. This makes them less fundamental than the dependence structures shown.

On top of this, for the copulas we applied there is room for specifying the marginals differently than done. As denoted in Chapter 3 specifying the marginals correctly is of major importance, since with incorrect marginals and whatever copula does not lead to a sensible result.

Data and simulation
In Chapter 6 the main IMA DRC model developed is based on a S&P 500 long portfolio. In practice, market portfolios of banks are often not unidirectional, because of hedging strategies and exposure offsets taken by financial institutions. The IMA and SA model outcomes are highly comparable in the case of a unidirectional portfolio, but we showed that offset gains could be reached with the IMA model in multidirectional portfolios. Further research could and should be performed in the direction of the model outcomes for different trading portfolios. The model now allows to calculate the DRC for portfolios containing S&P 500 companies only. In further research this could be extended by a wider scope of companies, different seniority of exposures towards one obligor, and by analysing companies from different countries and continents.

An important point of discussion is the method of determining the Loss Given Default in the model. In Section 3.1 we described the method of Altman et al. (2004) to determine the PD-LGD relationship. Since this was not the major focus of the research, the relationship specified is elementary. Financial institutions are required by the BCBS to base model assumptions (with regards to the PD-LGD relationship) on historical data. This leaves room for improvement in the defined model.

Another direction for further research is found in Chapter 6. In Figure 6.2 we observed that regression of the one-day observation period data resulted in the highest R^2 values. The BCBS does not prescribe which observation periods should be used, however it would be interesting to investigate the cause and effect of different observation periods in the model.

Cluster analysis
Finally, for the cluster analysis in Section 6.2.2 we apply regression trees as clustering technique. As described there (and in Appendix A.8), there are plenty

of other (un-)supervised learning techniques for cluster analysis. Nowadays, the use of random forest algorithms is growing. James et al. (2013) describe the advantages of random forests, which are mainly that they are low in variance and risk of overfitting is reduced. However, random forests work more like a 'black box' model, since it is hard and not intuitive to interpret the predictions. After all, the random forests and other machine learning clustering techniques could improve the DRC model, so investigation of this is recommended for further research.

Next to this, for the clustering we used an explanatory dataset, containing data on the constituents in 2017. Since explanatory variables like 'Rating grade' and 'Market capitalisation' evolve over time, it would be a welcome addition for the cluster analysis to use time series data for all the explanatory variables.

Since the FRTB regulation is in place from January 2022, financial institutions have multiple years ahead in which the DRC models could be developed and configured.

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A

Appendices

A.1 Appendix 1: FRTB

In this appendix we present key elements of the Fundamental Review of the Trading Book (FRTB) in general.[1] We explain three of the main elements of the FRTB are introduced and their implications are set out. The topics are stronger defined trading and banking book boundary, the revised standard and internal model approach and desk eligibility.

Trading book/Banking book Boundary
The BCBS believes the regulatory boundary between the trading book and the banking book was a major weakness. The previous definition introduced arbitrage when transferring trades between the banking and trading book. The FRTB aims to minimise regulatory arbitrage by introducing stronger boundaries between the banking and trading book. Supervisory powers include a required approval requirement when a bank deviates from the presumptive list of trading book instruments and ability to override allocation of a bank if considered in appropriate.

Standard approach and internal model
<u>Standardised Approach</u>: The FRTB standard approach provides a more 'risk sensitive' framework compared to previous regulation. It comprises of a Default Risk Charge, risk sensitivity risk charge and residual add-on. The default risk charge is intended to capture jump-to-default-risk. The sensitivities risk charge adds up charges from delta risk, vega risk and curvature risk. The residual risk add-on is to be calculated separately for all instruments bearing residual risk, residual risks could for example be gap risk and correlation risk.

<u>Internal Models Approach</u>: IMA allows banks to model their risks on an individual and more appropriate basis and would generally lead to lower capital charges for complex portfolios. It comprises of an Expected Shortfall, a Default Risk Charge and a stressed capital add-on. The expected shortfall, with consolations for varying liquidity horizons, depending on the financial product. The DRC capitalises potential losses from immediate defaults of credit and equity issuers. The stressed capital add-on is the aggregate regulatory capital measure for non-modellable risk factors (NMRFs).

Desk eligibility
FRTB brings approval for IMA down from entity level to desk level. If a desk is deemed to be in-scope for IMA, model approval is required. In that case an assessment of the trading desk takes place and is rated against quantitative criteria. The approval is based on multiple elements, which are P&L attribution, backtesting and a model-independent risk assessment tool.

[1] This appendix is developed with information of KPMG's internal training on FRTB.

A.2 Appendix 2: Factor Models

Different representations of factor models exist. We introduce and review the most common factor models. For readability, we describe them through a two-factor setup.

Two-factor model Hull and White (2004)):

$$U_i = \rho_{i1} F_1 + \rho_{i2} F_2 + \sqrt{1 - \rho_{i1}^2 - \rho_{i2}^2} Z_i \tag{A.1}$$

U_i shows the performance of company i. This depends on two systematic factors (F_1 and F_2) and one idiosyncratic factor (Z_i). $F_{1,2}$ and Z_i are distributed according to a certain distribution, different choices gives different copulas. When $F_{1,2}$ and Z_i are N(0,1) distributed and independent, we assume that the U_i have a multivariate normal distribution.

A default is triggered when $U_i < c$, where c is a company dependent default threshold.

Two-factor model Pykhtin (2004)
We have the following model for defaults. We define $V_{i,j}$ as a creditworthiness index for company i in rating class j, when $V_{i,j} < 0$ a default is triggered.

$$V_{i,j} = -\lambda x_i + r_i \tag{A.2}$$

Here $-\lambda$ is a vector containing the $N^{-1}(PD_j)$ values for every rating class j. x_j is a indicator vector that indicates the rating class. r_i is the asset return for company i. So, in this case a default is triggered if $r_i < N^{-1}(PD_j)$. We define the asset return as:

$$r_i = \rho_i Y_i + \sqrt{1 - \rho_i^2} \epsilon_i \tag{A.3}$$

Here Y_i is a composite factor consisting of two systematic risk factors F_1 and F_2.

$$Y_i = w_{1,i} F_1 + w_{2,i} F_2 \tag{A.4}$$

Where $w_{1,i}^2 + w_{2,i}^2 = 1$. To ensure that Y_i has unit variance (given the fact that F_1 and F_2 are uncorrelated). For a guarantee of this condition, each factor is divided by $\sqrt{w_{1,i}^2 + w_{2,i}^2}$.

Two-factor model Schönbucher (2001)
We find a slight different representation of a two-factor model compared to Pykhtin (2004). In this representation the asset value of a firm is driven by a composite systematic factor Y composed of J (in this case J = 2) driving factors. Every driving factor influences the value of the firm n's assets with a weight β_n^j.

$$V_n = \sum_{j=1}^{2} \beta_n^j Y_j + \epsilon_n \tag{A.5}$$

Firm n defaults when it's firm value falls below a barrier: $V_n \leq K_n$. The factors and errors are independent and distributed:

- $Y \sim N(0, \Omega_Y)$
- $\epsilon_n \sim N(0, \omega_n^2)$

Many other similar two-factors models described are defined in the literature. Examples could be found in Crouhy et al. (2000); Lütkebohmert (2008); Rosen and Saunders (2009); Skoglund and Chen (2015). Of the two-factor representations most models are similar to the model of Pykhtin (2004). An advantage of the Pykhtin (2004) model is the possibility to apply the Asymptotic Risk Factor (ASRF) model by Gordy (2003), since this model involves only one fitting parameter.

Two-factor model Wilkens and Predescu (2015)
Wilkens and Predescu (2015) also built a multiple factor model, in specific to model the Default Risk Charge from the FRTB regulation. They do this using three factors: global, a company's country and a company's sector. The factor loadings are gotten through regression analysis.

Contrary to other models, the model involves the R^2 values. Giving a higher weight to the systematic factors in case the factors from regression are well in describing the company returns.

Simulating the country returns takes place through the equations hereafter:

$$\hat{R}_{C(j)} = \sqrt{R^2_{C(j)}} Z_G + \sqrt{1 - R^2_{C(j)}} Z_{C(j)} \tag{A.6}$$

And for the industry returns:

$$\hat{R}_{I(j)} = \sqrt{R^2_{I(j)}} Z_G + \sqrt{1 - R^2_{I(j)}} Z_{I(j)} \tag{A.7}$$

For the returns of single corporates, the following factor representation is applicable:

$$R_{i,t} = \gamma_G R_{G,t} + \sum_{j=1}^{N_C} \gamma_{C(j)} \epsilon_{C(j),t} + \sum_{j=1}^{N_I} \gamma_{I(j)} \epsilon_{I(j),t} + \epsilon_{i,t} \tag{A.8}$$

Here γ_G is the sensitivity of the company i to the global factor. $\epsilon_{C(j)}$ and $\epsilon_{I(j)}$ are the residuals from the country and industry regressions. $\epsilon_{i,t}$ is the residual for the corporate.

Simulation of individual returns
The simulation from the individual returns is as following:

$$\hat{R}_i = \beta_i \sqrt{\frac{R_i^2}{\Psi_i}} \left(\gamma_G \sigma_G Z_G + \gamma_{C(i)} \sigma_{C(i)} Z_{C(i)} + \gamma_{I(i)} \sigma_{I(i)} Z_{I(i)} \right) + \sqrt{1 - R_i^2} \epsilon_i \tag{A.9}$$

Here β_i represents the coefficient for the systematic factor in the regression of obligor i's return on the aggregated systematic return. $\Psi_i = \gamma_G^2\sigma_G^2 + \gamma_{C(i)}^2\sigma_{C(i)}^2 + \gamma_{I(i)}^2\sigma_{I(i)}^2$, so Ψ_i functions as a normalization coefficient.

Default model
A possible default is simulated in the model of Wilkens and Predescu (2015) according to:

$$V_i = \Lambda_i + \hat{R}_i \tag{A.10}$$

When $V_i < 0$, a default of obligor i is observed. $\Lambda_i = -N^{-1}(PD)$ where the PD is derived from the rating class of the (sovereign or corporate) obligor. N^{-1} represent the inverse of the standard normal distribution.

A.3 Appendix 3: Copula simulation algorithms

These simulation algorithms are presented slightly modified representations compared to algorithms directly observed in the literature. The algorithms are based on Embrechts et al. (2001).

Algorithm 1: Bivariate Gaussian copula

1. Draw a random variate $\mathbf{X}$ ($\mathbf{X} \sim (X_1, X_2)'$) from bivariate normal distribution, with mean (0,0) and correlation matrix R. $R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
2. Set $u_1 = \Phi(X_1)$ and $u_2 = \Phi(X_2)$ for all n variates. Here Φ is the standard normal cumulative distribution.
3. Repeat step 1 to 3 n times to get n draws of the bivariate Gaussian copula.
4. Scatter plot of n draws, which represent all u_1 and u_2 combinations. This gives $C_R^{Gaussian}$ the Gaussian copula with correlation matrix R.

Algorithm 2: Bivariate Student-t copula

1. Draw a random variate $\mathbf{X}$ ($\mathbf{X} \sim (X_1, X_2)'$) from bivariate normal distribution, with mean (0,0) and correlation matrix R. $R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
2. Draw a random variate s, independent from X, from the χ_v^2 distribution with v degrees of freedom.
3. Set $\mathbf{Y} = \sqrt{\frac{v}{s}}\mathbf{X}$
4. Set $u_1 = t_v(Y_1)$ and $u_2 = t_v(Y_2$ for all n variates. Here t_v is the Student-t cumulative distribution, with v degrees of freedom.
5. Scatter plot of n draws, which represent all u_1 and u_2 combinations. This gives $C_{R,v)}^{Student-t}$, which is the Student-t copula with v degrees of freedom and correlation matrix R.

Algorithm 3: Bivariate Clayton copula

1. Draw two random variates X_1 and X_2 from the uniform distribution ($U(0,1)$).
2. Set $u_1 = X_1$ and set $u_2 = (u_1^{-\theta}(X_2^{-\theta/(1+\theta)} - 1) + 1)^{-1/\theta}$.
3. Scatter plot of n draws, which represent all u_1 and u_2 combinations. This gives $C_\theta^{Clayton}$, which is the Clayton copula with parameter θ.

A.4 Appendix 4: Calibration method - Moment matching

Proof of calibration (Gordy, 2000):

$$Var(DR) = \phi_{\rho^2}(\phi^{-1}(PD), \phi^{-1}(PD)) - PD^2 \tag{A.11}$$

We assume that X_1 and X_2 are two latent variables for obligor i and j, which are both ψ rated. We assume that there is one systemic risk factor, to which both obligors have the same weight. This implies.

$$X_i = \rho F + \sqrt{1-\rho^2} Z_i \tag{A.12}$$

Conditional on F, defaults events for the two obligors are independent.

$$\begin{aligned} Pr(X_1 < c_\psi, X_2 < c_\psi | F) &= Pr(X_1 < c_\psi | F) Pr(X_2 < c_\psi | F) \\ &= \Phi\left(\frac{c_\psi - \rho F}{\sqrt{1-\rho^2}}\right)^2 = PD^2 \end{aligned}$$

Where c_ψ represents the default threshold. Therefore, we can write Var(DR) as:

$$\begin{aligned} Var(DR) &= E[(DR)^2] - E[DR]^2 \\ &= E[Pr(X_1 < c_\psi \& X_2 < c_\psi | F)] - PD^2 \end{aligned} \tag{A.13}$$

X_1 and X_2 are N(0,1) and have correlation ρ^2. We can write the $E[Pr(X_1 < c_\psi \text{ \& } X_2 < c_\psi | F)]$ as $\Phi(c_\psi, c_\psi, \rho^2)$. This gives the result as required.

$$Var(DR) = \phi_{\rho^2}(\phi^{-1}(PD), \phi^{-1}(PD)) - PD^2 \tag{A.14}$$

A.5 Appendix 5: S&P data

This appendix includes data by S&P on the global annual corporate default rates. Data is taken from Standard & Poor's Financial Services (2017).

Corporate Annual Default Rates by Rating Category (%)

	AAA	*AA*	*A*	*BBB*	*BB*	*B*	*CCC/C*
1981	0.00	0.00	0.00	0.00	0.00	2.27	0.00
1982	0.00	0.00	0.21	0.34	4.22	3.13	21.43
1983	0.00	0.00	0.00	0.32	1.16	4.58	6.67
1984	0.00	0.00	0.00	0.66	1.14	3.41	25.00
1985	0.00	0.00	0.00	0.00	1.48	6.47	15.38
1986	0.00	0.00	0.18	0.33	1.31	8.36	23.08
1987	0.00	0.00	0.00	0.00	0.38	3.08	12.28
1988	0.00	0.00	0.00	0.00	1.05	3.63	20.37
1989	0.00	0.00	0.18	0.60	0.72	3.38	33.33
1990	0.00	0.00	0.00	0.58	3.57	8.56	31.25
1991	0.00	0.00	0.00	0.55	1.69	13.84	33.87
1992	0.00	0.00	0.00	0.00	0.00	6.99	30.19
1993	0.00	0.00	0.00	0.00	0.70	2.62	13.33
1994	0.00	0.00	0.14	0.00	0.28	3.08	16.67
1995	0.00	0.00	0.00	0.17	0.99	4.58	28.00
1996	0.00	0.00	0.00	0.00	0.45	2.91	8.00
1997	0.00	0.00	0.00	0.25	0.19	3.51	12.00
1998	0.00	0.00	0.00	0.41	0.82	4.63	42.86
1999	0.00	0.17	0.18	0.20	0.95	7.29	33.33
2000	0.00	0.00	0.27	0.37	1.16	7.70	35.96
2001	0.00	0.00	0.27	0.34	2.96	11.53	45.45
2002	0.00	0.00	0.00	1.01	2.89	8.21	44.44
2003	0.00	0.00	0.00	0.23	0.58	4.07	32.73
2004	0.00	0.00	0.08	0.00	0.44	1.45	16.18
2005	0.00	0.00	0.00	0.07	0.31	1.74	9.09
2006	0.00	0.00	0.00	0.00	0.30	0.82	13.33
2007	0.00	0.00	0.00	0.00	0.20	0.25	15.24
2008	0.00	0.38	0.39	0.49	0.81	4.08	27.27
2009	0.00	0.00	0.22	0.55	0.75	10.92	49.46
2010	0.00	0.00	0.00	0.00	0.58	0.86	22.62
2011	0.00	0.00	0.00	0.07	0.00	1.67	16.30
2012	0.00	0.00	0.00	0.00	0.30	1.56	27.52
2013	0.00	0.00	0.00	0.00	0.10	1.63	24.34
2014	0.00	0.00	0.00	0.00	0.00	0.78	17.13
2015	0.00	0.00	0.00	0.00	0.16	2.39	25.88
2016	0.00	0.00	0.00	0.00	0.47	3.68	32.67

Table A.1: Global Corporate Annual Default Rates by Rating Category (%).

Descriptive Statistics On One-Year Global Default Rates

	AAA	*AA*	*A*	*BBB*	*BB*	*B*	*CCC/C*
Minimum	0.00	0.00	0.00	0.00	0.00	0.25	0.00
Maximum	0.00	0.38	0.39	1.01	4.22	13.84	49.46
Weighted long-term average	0.00	0.02	0.06	0.18	0.72	3.76	26.78
Median	0.00	0.00	0.00	0.07	0.64	3.46	23.71
Standard deviation	0.00	0.07	0.10	0.26	1.01	3.29	11.76
2008 default rates	0.00	0.38	0.39	0.49	0.81	4.08	27.27

Table A.2: Descriptive Statistics On One-Year Global Default Rates.

One-Year Global Corporate Default Rates By Rating Modifier

	AAA	*AA+*	*AA*	*AA-*	*A+*	*A*	*A-*	*BBB+*	*BBB*	*BBB-*	*BB+*	*BB*	*BB-*	*B+*	*B*	*B-*	*CCC/C*
1981	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.28	0.00	0.00
1982	0.00	0.00	0.00	0.00	0.00	0.33	0.00	0.00	0.68	0.00	0.00	2.86	7.04	2.22	2.33	7.41	21.43
1983	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.33	2.17	0.00	1.59	1.23	9.80	4.76	6.67
1984	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.40	0.00	0.00	1.64	1.49	2.15	3.51	7.69	25.00
1985	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.64	1.49	1.33	2.61	13.11	8.00	15.38
1986	0.00	0.00	0.00	0.00	0.00	0.00	0.76	0.00	0.78	0.00	1.82	1.18	1.12	4.68	12.16	16.67	23.08
1987	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.83	1.31	5.95	6.82	12.28
1988	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	2.34	1.99	4.50	9.80	20.37
1989	0.00	0.00	0.00	0.00	0.00	0.00	0.58	0.90	0.78	0.00	0.00	0.00	2.00	0.43	7.80	4.88	33.33
1990	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.76	0.00	1.10	2.78	3.09	4.50	4.89	12.26	22.58	31.25
1991	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.83	0.74	0.00	3.70	1.14	1.05	8.72	16.25	32.43	33.87
1992	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.72	14.93	20.83	30.19
1993	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.94	0.00	1.30	5.88	4.17	13.33
1994	0.00	0.00	0.00	0.00	0.46	0.00	0.00	0.00	0.00	0.00	0.00	0.86	0.00	1.84	6.58	3.13	16.67
1995	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.64	0.00	1.56	1.12	2.77	8.00	7.50	28.00
1996	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.65	0.56	2.37	3.74	3.85	8.00
1997	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.37	0.35	0.00	0.00	0.00	0.41	0.72	5.30	14.58	12.00
1998	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.27	1.06	0.67	1.06	0.72	2.60	7.56	9.46	42.86
1999	0.00	0.00	0.00	0.36	0.00	0.24	0.27	0.00	0.28	0.31	0.55	1.34	0.91	4.22	10.45	15.60	33.33
2000	0.00	0.00	0.00	0.00	0.00	0.24	0.57	0.00	0.26	0.89	0.00	0.82	2.06	5.83	10.04	11.61	35.96
2001	0.00	0.00	0.00	0.00	0.58	0.25	0.00	0.24	0.49	0.28	0.52	1.22	5.60	5.84	17.24	22.46	45.45
2002	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.10	0.88	1.07	1.59	1.78	4.81	3.27	10.23	19.85	44.44
2003	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.20	0.54	0.51	0.98	0.28	1.72	5.34	9.52	32.73
2004	0.00	0.00	0.00	0.00	0.00	0.24	0.00	0.00	0.00	0.00	0.00	0.67	0.53	0.46	2.35	2.84	16.18
2005	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.17	0.00	0.38	0.00	0.51	0.79	2.64	2.96	9.09
2006	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.50	0.55	0.82	1.57	13.33
2007	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.31	0.24	0.19	0.00	0.90	15.24
2008	0.00	0.00	0.44	0.41	0.32	0.21	0.60	0.19	0.61	0.73	1.22	0.66	0.68	3.14	3.45	7.59	27.27
2009	0.00	0.00	0.00	0.00	0.30	0.40	0.00	0.42	0.19	1.14	0.00	1.05	0.98	5.96	10.82	17.99	49.46
2010	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.85	0.36	0.56	0.00	0.75	2.13	22.62
2011	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.42	1.28	4.47	16.30
2012	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.79	0.60	1.44	3.50	27.52
2013	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.26	0.77	0.83	4.66	24.34
2014	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.19	0.33	2.74	17.13
2015	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.26	0.22	1.75	2.02	4.25	25.88
2016	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.23	0.00	1.10	0.91	2.29	10.53	32.67
Average	0.00	0.00	0.01	0.02	0.05	0.05	0.08	0.13	0.22	0.26	0.53	0.75	1.28	2.20	6.26	9.16	23.96
Minimum	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Maximum	0.00	0.00	0.44	0.41	0.58	0.40	0.76	1.10	1.40	1.33	3.70	3.09	7.04	8.72	17.24	32.43	49.46

Table A.3: One-Year Global Corporate Default Rates By Rating Modifier.

A.6 Appendix 6: Empirical evaluation of the Gaussian LHP copula

In this appendix we provide an empirical evaluation to the LHP (Gaussian). We both make a confidence around the mean PD and around the variance of the PD. This is a valuable analysis, since we should keep in mind that the dataset contained observations over 36 years. With the help of statistical methods we can obtain confidence intervals, and show the results of calibration over the full confidence interval.

Confidence intervals for the mean and the variance of the PD
CI around mean PD: Figure 5.2 gives the fitted solution according to the fact that the PD is as observed from the data points by S&P. However, the dataset contains only 36 observations. Therefore we use statistical methods to obtain a confidence interval for the probability of default.

The confidence interval could be set up according to the formula:

$$\left[\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right] \tag{A.15}$$

Since the σ^2 is estimated and not known, we use $t_{n-1,\alpha/2}$ instead of $z_{\alpha/2}$. We can also set up a confidence interval using a bootstrapping technique. The bootstrap method is programmed in the model. To be sure that we represent a correct confidence interval given the observed data, we test both methods. The bootstrapping (10.000 runs) result gives a wider interval (formula-interval: [4.25%,4.62%], bootstrap-interval: [3.43%,5.53%]). The formula interval is significantly smaller, this could be explained by the amount of observations(36). The effect is that the value of $t_{n-1,\alpha/2}$ goes to $z_{\alpha/2}$, because of the sample size, but from the resampling in the bootstrap method we see that this is not appropriate. We go on with the bootstrapping method only, since this method will be more powerful in explaining observed default rates for different rating classes.

The results obtained of a 95% confidence interval are shown in Figure A.1. The interval for the probability of default is [3.43%,5.53%].

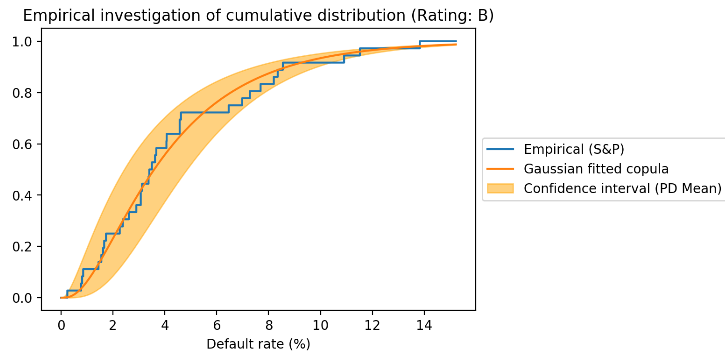

Figure A.1: Fitted Gaussian copulas for 95% interval.

From Figure A.1 we see that for the B-rating the interval shows that the empirical observed defaults fall almost always within the 95% ranges.

<u>CI around variance of PD:</u> However, we should take into account that there is more uncertainty involved. We obtained the sample variance given the data, but also here a confidence interval is applicable. Therefore we also set up a confidence interval for the variance of the probability of default.

We can set up the confidence interval with:

$$\left[\frac{(n-1)S^2}{\chi_{1-\alpha/2}}, \frac{(n-1)S^2}{\chi_{\alpha/2}} \right] \tag{A.16}$$

Again, we also set up a bootstrapping method to find the confidence interval. The comparison of the results from the bootstrapping method are again slightly different than the results from the formula estimation(formula-interval: [0,00073, 0,0019], bootstrap-interval: [0.00052, 0.0016]). We continue with the bootstrap results for the variance of the PD, with the same reasoning as before.

We plot the confidence interval for the variance of the probability of default, Figure A.2 shows this. For graphing reasons, we show only the result from the mean PD (4.43%) from the historical data. The interval for the variances of the probability of default is [0.00052, 0.0016]

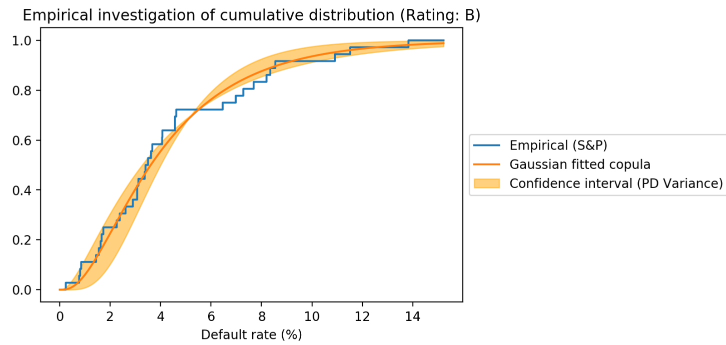

Figure A.2: Fitted Gaussian copulas for 95% interval for the PD variance.

From Figure A.2 we see that the variance has a significant input on the resulting fitted Gaussian copula. To show the combination of both the PD-interval and the Var PD-interval, we plot everything together. This gives the outer ranges for the confidence interval and is done in Figure A.3.

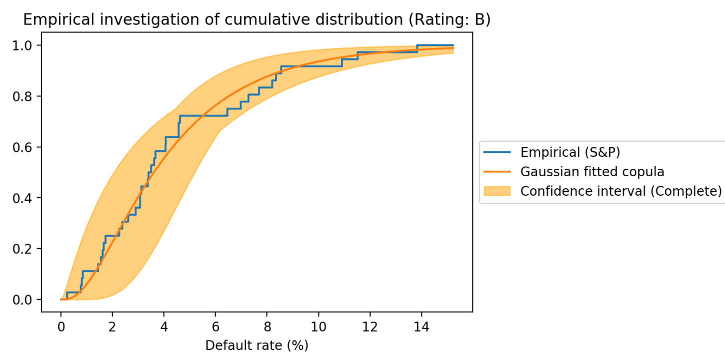

Figure A.3: Fitted Gaussian copulas for 95% interval for the PD and PD variance.

From Figure A.3 we can observe that the outer ranges of the copula fittings around the observed historical data. This is a valuable result for later, when we do the copula fitting on the tail in specific.

A.7 Appendix 7: Default Risk Charge - SA

Here we outline the Default Risk Charge calculation for the Standardised Approach (SA). This is based on Basel Committee on Banking Supervision (2016b).

The goal of the DRC is to capture the jump-to-default (JTD) risk of:

- Non-securitisations. F.e. bonds, equity, derivatives (options) and CDS.
- Securitisations (non-correlation trading portfolio).
- Securitisations (correlation trading portfolio).

Capturing the default risk and calculating the capital charge goes with the following steps:

1. Compute JTD risk of each instrument separately. This gives the gross-JTD risk.
2. Apply offsetting rules. This gives the net JTD risk positions for every obligor. F.e. if we have both a long and a short bond on the same obligor.
3. Allocation to buckets. Within the buckets, net short exposures are discounted by a hedge benefit ratio.
4. Afterwards weighting takes place by prescribed default risk weights. With this we arrive at the capital charge for default risk.

DRC for Non-Securitisations

Here we show the steps we need to take for calculating the Default Risk Charge for non-securitisations. This is based on BCBS, 2016a articles 139 to 156.

Gross JTD risk position.
The gross JTD risk is computed exposure by exposure.

$$JTD(long) = \max(LGD * notional + P\&L, 0)$$
$$JTD(short) = \min(LGD * notional + P\&L, 0)$$

Here notional is the bond-equivalent notional (or face value) of the position, and P&L is equal to $market\ value - notional$. $LGD = 1 - RR$. This implies that we can also write the gross JTD risk positions as:

$$JTD(long) = max(\text{Market Value} - RR * notional, 0)$$
$$JTD(short) = min(\text{Market Value} - RR * notional, 0)$$

As we can derive, this simplifies to $JTD(long) = \max(\text{Market Value}, 0)$ when RR = 0%. Long and short refers to the credit exposure, in a long exposure results from an instrument for which the default of the underlying obligor results

Credit quality category	Default risk weight
AAA	1%
AA	2%
A	3%
BBB	6%
BB	15%
B	30%
CCC	50%
Unrated	15%
Defaulted	100%

Table A.4: Default risk weights assigned to net JTD by credit quality categories

in a loss for the holder of the instrument.

For the loss given default (LGD) values for certain portfolio elements, the following values apply: Equity instruments: 100%, non-senior debt instruments 100%, senior debt instruments 75%, covered bonds 25%.

Net JTD risk positions
The gross JTD amounts are scaled for exposures with a maturity which is less than the one year capital horizon. These exposures are scaled by the fraction of the year, with a floor of 0.25 (3 months). Cash equity positions are assigned a maturity of either > 1 year or 3 months [2]. The gross JTD amounts of long and short exposures to the same obligors may be offset when the seniority of the short exposure is equal or lower to the long exposure.

Bucketing, weighting and capital charge
The buckets are: corporates, sovereigns and local government/municipalities. For every bucket, a hedge benefit ratio is computed. The hedge benefit ratio (Weighted to Short ratio, WtS) is calculated as follows:

$$WtS = \frac{\sum \text{net } JTD_{\text{long}}}{\sum \text{net } JTD_{\text{long}} + \sum |\text{net } JTD_{\text{short}}|} \tag{A.17}$$

The risk weights (RW) are assigned to the net JTD positions, according to the credit quality category of the counterparty. The risk weights are shown in Table A.4. The overall capital charge for each bucket can now be calculated with:

$$DRC_b = max\left[\left(\sum_{i \in \text{long}} RW_i \text{net } JTD_i\right) - WtS\left(\sum_{i \in \text{short}} RW_i |\text{net } JTD_i|\right); 0\right] \tag{A.18}$$

The total capital charge is the sum of the bucket-level capital charges from Equation A.18. The three buckets are: corporates, sovereigns, and local governments/municipalities.

[2] $PD_{3 \text{ months}} = 1 - (1 - PD_{\text{annual}})^{1/4}$

A.8 Appendix 8: Regression tree building

We introduced the clustering procedure in Section 6.2.2. In this appendix we show in detail how the regression trees are developed to determine the clusters. The trees are built using R. We use a method involving cross-validating the estimated factors. Afterwards we apply a pruning method to prevent over fitting the model yielding a too complex model not improving the model outcomes. As said in Section 6.2.2 algorithms and methods in this Appendix are taken from James et al. (2013).

First, we show how the clustering for the sector-factor takes place, afterwards we show the same for the country-factor.

Clustering the sector factors
Figure A.4 shows the regression tree built on 42 explanatory variables. Afterwards, Figure A.5 shows the appropriate size of the tree. In the cost-complexity algorithm which we apply for the pruning, we should set the complexity parameter (CP). We derive this from the regression tree. We investigate Figure A.5 to choose CP such that the model does not over fit the outcomes. We set CP = 0.045[3]. This value is chosen by applying the 'elbow-rule' on the relative error observed (James et al., 2013)[4]. From Figure A.5 we choose the size of the tree to be 4 (leave nodes). The reason for this is that the relative error does not improve from here anymore. James et al. (2013) describe that this visual inspection is 'inherently ad hoc', however there is no general accepted objective way of determining the tree size. The resulting tree is depicted in Figure A.6.

[3] In Chapter 6 we later apply sensitivity analysis where we check for the effect of changing this factor

[4] Please note that we plot the relative error, and not the absolute error in R. When we would plot the 'xerror' we would observe the minimum variance at the 'elbow' point from the relative error plot.

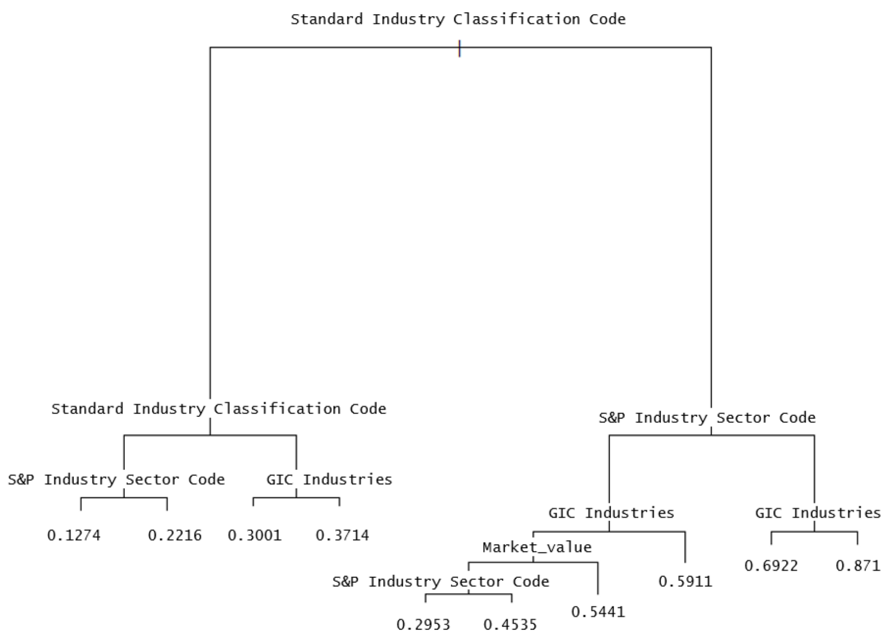

Figure A.4: Regression tree before pruning (Sector).

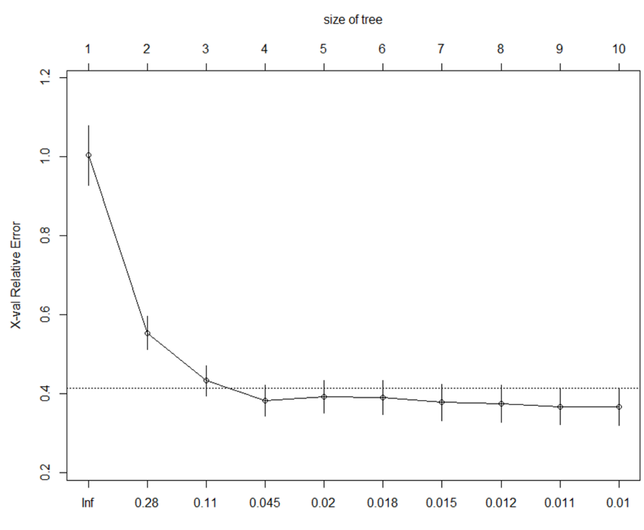

Figure A.5: Regression tree optimal size (Sector).

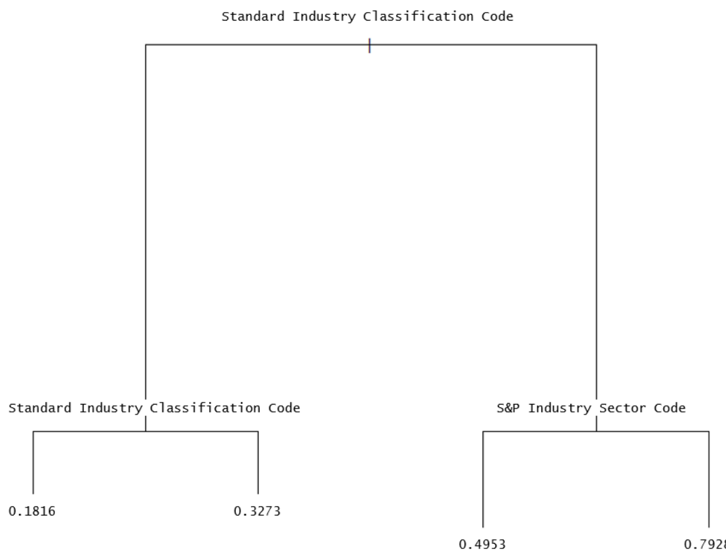

Figure A.6: Regression tree after pruning (Sector).

Clustering the country factors

Figure A.7 shows the regression tree built on 42 explanatory variables. Afterwards, Figure A.8 shows the appropriate size the tree should have. In our cost-complexity algorithm where we prune with, we should set the complexity parameter (cp). We derive this from the regression tree. We investigate Figure A.8 to choose cp such that the model does not over fit the outcomes. We set cp = 0.035. Again the value is chosen by applying the 'elbow-rule' on the relative error observed (James et al., 2013). From Figure A.8 we now observe that the size of the tree will be 4 (leave nodes) when we set cp = 0.035. The resulting tree is depicted in Figure A.9.

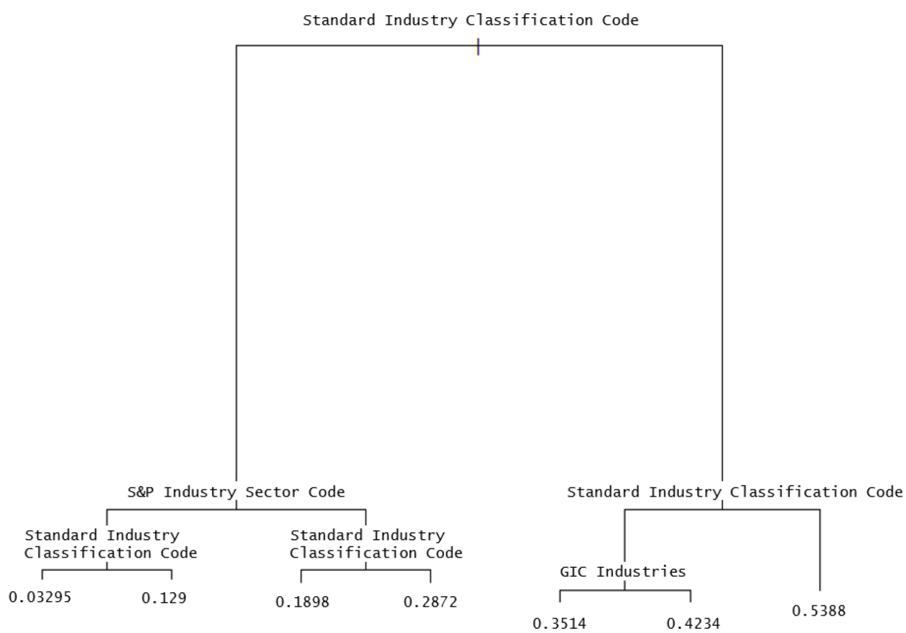

Figure A.7: Regression tree before pruning (Country).

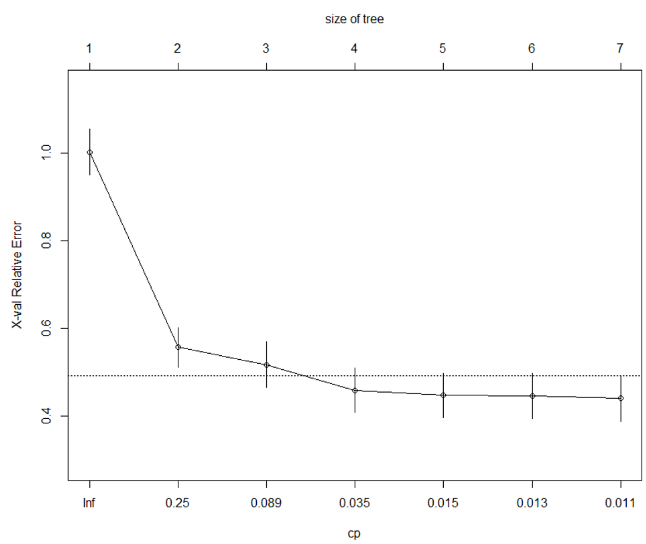

Figure A.8: Regression tree optimal size (Country).

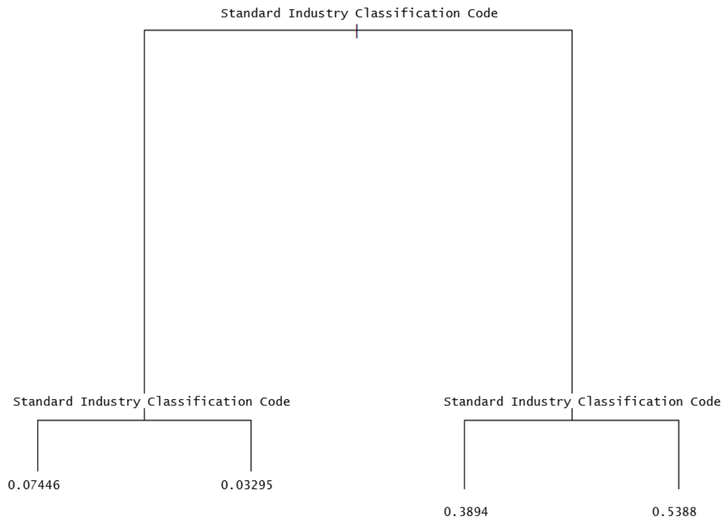

Figure A.9: Regression tree after pruning (Country).

A.9 Appendix 9: Extreme Value Theory

Extreme value theory is the science of estimating the tails of a distribution. For this appendix we use the presentation and notation by Hull (2015). According to Gnedenko (1943), the tails of a wide range of probability distributions converge to the generalised Pareto distribution:

$$G_{\xi,\beta}(y) = 1 - \left[1 + \xi \frac{y}{\beta}\right]^{-\frac{1}{\xi}} \tag{A.19}$$

where ξ is the shape parameter that determines the heaviness of the tail and β determines the scale. Both parameters can be estimated by the maximum likelihood methods.

We should choose u such that it is sufficiently high to ensure that we are investigating the tail. In practice a method which works well is to choose u as the 95% percentile (Hull, 2015).

The simulation observations where the threshold u is exceeded are ranked. Now the likelihood function can be maximised via it's logarithm function. The logarithm of the likelihood function is:

$$\sum_{i=1}^{n_u} \ln\left[\frac{1}{\beta}\big(1 + \frac{\xi(v_i - u)}{\beta}\big)^{\frac{-1}{1-\xi}}\right] \tag{A.20}$$

The values for ξ and β are found by a numerical optimization (maximization) of Equation A.20.

Calculating the risk measures with Extreme Value Theory
The Value at Risk (VaR) and Expected Shortfall (ES) are calculated using the following formulas, also taken from Hull (2015):

$$\text{VaR} = u + \frac{\beta}{\xi}\left\{\left[\frac{n}{n_u}(1-q)\right]^{-\xi} - 1\right\} \tag{A.21}$$

and

$$\text{ES} = \frac{VaR + \beta - \xi u}{1 - \xi} \tag{A.22}$$

Here n is the amount of simulation runs, and n_u is the amount of simulation runs where the threshold of u is exceeded.

A.10 Appendix 10: List of Abbreviations

Abbreviation	Meaning
ABS	Asset Backed Security
ASRF	Asymptotic Single Risk Factor
BCBS	Basel Committee for Banking Supervision
CDO	Collateralised Debt Obligation
CDS	Credit Default Swap
CI	Confidence interval
CP	Complexity parameter
DR	Default rate
DRC	Default Risk Charge
ES	Expected Shortfall
FCIC	Financial Crisis Inquiry Commission
FRTB	Fundamental Review of the Trading Book
IDRC	Incremental Default Risk Charge
IMA	Internal Models Approach
IRC	Incremental Risk Charge
JTD	Jump-to-default
LGD	Loss Given Default
LHP	Large Homogeneous Pool
OFGC	One-Factor Gaussian Copula
PD	Probability of Default
RR	Recovery rate
RW	Risk weight
S&P	Standard & Poors
SA	Standard Approach
VaR	Value at Risk

Table A.5: List of Abbreviations