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MASTER THESIS

BIFURCATIONS IN NEURAL FIELD MODELS WITH TRANSMISSION DELAYS AND DIFFUSION

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Abstract

Neural Field Equations model the large scale behaviour of large groups of neurons. In this context gap junctions, electrical connections between neurons, are modelled by adding diffusion to the neural field. In this work we study the role of diffusion next to the usual connectivity with transmission delay. We extend known sun-star calculus results for delay equations to be able to include diffusion. Consequently, we are able to compute the spectral properties and normal form coefficients on the center manifold for Hopf and Pitchfork-Hopf bifurcations. By examining a numerical example, we find that the addition of diffusion suppresses spatial modes, while leaving temporal modes unaffected.

Chapter 1

Introduction

The field of computational neuroscience uses mathematical models to further our understanding of the complex nature of the nervous system. Hodgkin and Huxley (1952) created a model which describes how action potentials are propagated across neurons, for which they received the 1963 Nobel Prize in Physiology or Medicine. Neurons propagate these action potentials across axons to a synapse, where the action potential is passed along to the dendrites of some other neuron. Later networks of these individually modelled cells with their interactions were constructed, e.g. (van Drongelen *et al.*, 2004). However the large number of parameters and variables make it hard to study these networks analytically and costly to simulate numerically.

In response to these problems, neural field models were developed by Wilson and Cowan (1972) and Amari (1977) among others. These models replace the large network of neurons by a continuous spatial approximation and replace the individual spikes by a time-averaged spiking rate. Neural field models are usually formulated as non-linear integro-differential equations. One major addition to these models is the inclusion of delays, e.g. (Coombes and Laing, 2009). These delays have there origin in the finite propagation speed of action potentials along axons, synaptic processing and dendritic integration (Campbell, 2007). Local bifurcation theory for these models was developed in van Gils *et al.* (2013) and was expanded in Dijkstra *et al.* (2015).

One possible modification to these models is the addition of diffusion, which models the interactions through gap-junctions between neurons (Gibson *et al.*, 1999). The implications of adding diffusion on the analytic setting and bifurcation theory are an open question. Some preliminary work has been done by Bellingacci (2017) on the numerics and by (Diekmann *et al.*, 2014) on the analysis in an unpublished manuscript. This thesis will expand on these ideas and the work already done in van Gils *et al.* (2013) and Dijkstra *et al.* (2015).

In section (2) we extend the analytical work of (Diekmann *et al.*, 2014) on delay equations and proof a result on the essential spectrum of the linearised problem. In section (3) we develop the sun-star calculus with respect to the diffusion operator and state some its spectral properties. In section (4) we show how the spectrum and resolvent can be computed explicitly for specific choices for our connectivity and delay functions. In (5) we show how the normal form coefficients can be computed explicitly for the Andronov-Hopf and Pitchfork-Hopf bifurcations. In (6) we investigate the effect of diffusion on neural field equations by evaluating the normal form coefficients for a particular example and confirm our results by a numerical simulation of the discretised problem.

1.1 Neural Field Model

We analyse a neural field model of N connected populations with delays and diffusion. For the spatial domain we take $\Omega = [-1, 1]$ and for the time domain we take $\mathbb{R}^+$. As we introduce diffusion, it is necessary to introduce boundary conditions. We choose Neumann boundary conditions, i.e. the spatial derivative is zero at the boundary. As this equation is a delay equation, we need an initial condition which is given for some history $[-h, 0]$. We want to find solutions $\mathbf{u} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^N$ of the following system of partial differential delay equations (PDDE) for $i \in \{1, \cdots, N\}$

$$\begin{cases} \frac{\partial u_i}{\partial t}(t,x) = d_i \frac{\partial^2 u_i}{\partial x^2}(t,x) - \alpha_i u_i(t,x) & \\ \qquad + \sum_{j=1}^{N} \int_\Omega J_{i,j}(x,x') S_j(u_j(t-\tau_{i,j}(x,x'),x'))\mathrm{d}x' & \text{for } x \in \Omega,\ t \in \mathbb{R}^+ \\ \frac{\partial u_i}{\partial x}(t,x) = 0 & \text{for } x \in \partial\Omega,\ t \in \mathbb{R}^+ \\ u(t,x) = \phi_i(t,x) & \text{for } x \in \Omega,\ t \in [-h,0] \end{cases} \tag{PDDE}$$

We assume for $i,j \in \{1,\cdots,N\}$ that $d_i, \alpha_i > 0$ and that $h := \sup_{x,x' \in \Omega, i,j \in \{1,\cdots,N\}} \tau_{i,j}(x,x') < \infty$, $J_{i,j} \in C(\Omega \times \Omega)$ and $S_j \in C^\infty(\mathbb{R})$.

We define the following Banach Spaces $Y := C(\Omega; \mathbb{R}^N)$ and $X := C([-h,0];Y)$ with their corresponding supremum-norms. We introduce the notation $\mathbf{u}_t(\theta) := \mathbf{u}(t+\theta)$[1] for $\theta \in [-h,0]$ so we have that $\mathbf{u}_t \in X$.

We can write (PDDE) as an abstract delay differential equation (ADDE)

$$\begin{cases} \dot{\mathbf{u}}(t) = B\mathbf{u}(t) + G(\mathbf{u}_t) \\ \mathbf{u}_0 = \phi \in X \end{cases} \tag{ADDE}$$

Where the linear operator $B : D(B) \to Y$ is defined as

$$(B\mathbf{u})_i := d_i \Delta u_i - \alpha_i \tag{1.1}$$

Here Δ is the second derivative operator

$$\Delta u_i(x) := \frac{\partial^2 u_i}{\partial x^2}(x) \tag{1.2}$$

We take the domain of B as the twice continuously differentiable functions with Neumann boundary conditions

$$D(B) := D(\Delta) := \{\mathbf{y} \in Y | \mathbf{y} \in C^2(\Omega, \mathbb{R}^N), \mathbf{y}'(\partial\Omega) = 0\} \tag{1.3}$$

The non-linear operator $G : X \to Y$ for $\phi \in X$ is defined as

$$(G(\phi))_i(x) := \sum_{j=1}^{N} \int_\Omega J_{i,j}(x,x') S_j(\phi_j(-\tau_{i,j}(x,x'),x'))\mathrm{d}x'$$

This operator has the following properties.

Lemma 1.1.1. *(van Gils et al., 2013, Lemma 3, Proposition 11) G is compact, globally Lipschitz continuous and k times Fréchet differentiable for any $k \in \mathbb{N}$. Furthermore the kth Fréchet derivative of G at $\psi \in X$, $D^k G(\psi) : X^k \to Y$ is compact and given by*

$$(D^k G(\psi)(\phi^1, \cdots, \phi^k))_i(x) = \sum_{j=1}^{N} \int_\Omega J_{i,j}(x,x') S_j^{(k)}(\psi_j(-\tau_{i,j}(x,x'),x') \prod_{m=1}^{k} (\phi_j^m(-\tau_{i,j}(x,x'),x'))\mathrm{d}x'$$

[1] As a notational convention, all bold variables correspond to vectors $\mathbf{u} = (u_1, \cdots, u_N)^T$, where the length of the vector is clear from the context.

Chapter 2

Analytic Setting

In this chapter we will analyze the (ADDE) in a general setting where $X = C([-h, 0]; Y)$ and Y is some Banach space over $\mathbb{R}$ or $\mathbb{C}$. Let S be some strongly continuous semigroup on Y with its generator B and let $G : X \to Y$ be a non-linear, globally Lipschitz continuous operator.

$$\begin{cases} \dot{\mathbf{u}}(t) = B\mathbf{u}(t) + G(\mathbf{u}_t) \\ \mathbf{u}_0 = \phi \in X \end{cases} \tag{ADDE}$$

On X we consider the strongly continuous semigroup T_0 defined by

$$(T_0(t)\phi)(\theta) := \begin{cases} \phi(t+\theta) & t+\theta \in [-h, 0] \\ S(t+\theta)\phi(0) & t+\theta > 0 \end{cases} \tag{2.1}$$

Here $\phi \in X, t \geq 0$ and $\theta \in [-h, 0]$. This semi-group is also called the shift-semigroup and is related to the problem for $G = 0$:

$$\begin{cases} \dot{\mathbf{v}}(t) = B\mathbf{v}(t) & \text{for } t > 0 \\ \mathbf{v}(t) = \phi(t) & \text{for } t \in [-h, 0] \end{cases} \tag{2.2}$$

The solution of problem (2.2) is then given by $\mathbf{v}(t) := (T_0(t)\phi)(0)$.

The infinitesimal generator of A of a semigroup T is defined as the limit $A\phi = \lim_{t \downarrow 0} \frac{1}{t}(T(t)\phi - \phi)$, with its domain $D(A)$ the set in X where this limit exists. It is well known that the generator of translation is differentiation, see for instance Engel and Nagel (1999, Theorem VI.6.1). The generator A_0 of the shift-semigroup T_0 is given by

$$A_0\phi = \dot{\phi}, \qquad D(A_0) = \{\phi \in C^1([-h, 0]; Y) | \phi(0) \in D(B) \text{ and } \dot{\phi}(0) = B\phi(0)\} \tag{2.3}$$

We will interpret the (ADDE) as problem (2.2) with some non-linear perturbation G and use the variation-of-constants formula to obtain results about the perturbed problem. However, this results in technical complications as the rule for extending a function beyond its original domain, i.e. $\dot{\phi}(0) = B\phi(0)$, is incorporated in $D(A_0)$. So a perturbation leads to a change in the domain of A_0. This is the source of a lot of confusion in delay equations. We can avoid these complications by embedding X into a larger space, where this rule for extension is not incorporated in the domain of A_0. We construct this larger space, $X^{\odot *}$, using sun-star calculus. First we restrict X^*, the dual space of X, to the space on which T_0^*, the dual semigroup of T_0, is strongly continuous. This restricted space $X^\odot$ can be found by taking the closure of $D(A^*)$ with respect to the norm on X^*. By taking the dual of this space, we end up at $X^{\odot *}$, which is the like a second dual space with some restrictions based on T_0. It is convenient to present the relationship of the various spaces schematically in the following 'duality' diagram, see figure (2.1).

2.1 Sun-star calculus

We will now construct the sun-star calculus with respect to T_0. We can represent X^*, the dual space of X, as $NBV([0, h]; Y^*)$. This follows from a generalization of the Riesz Representation Theorem for Y-valued functions proven by Gowurin (1936). We use the results from Diekmann *et al.* (2014) to find a representation of $X^\odot$ and $T_0^\odot$.

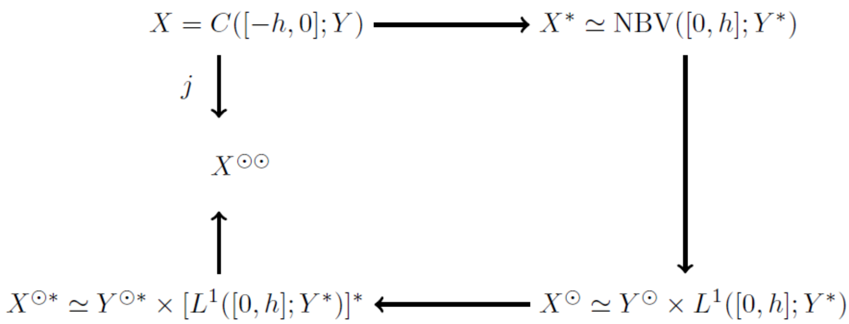

Figure 2.1: A schematic representation of the various spaces in sun-star calculus.

Theorem 2.1.1. *(Diekmann et al., 2014, Theorem 2.2) The sun-dual of X with respect to T_0, can be represented as $X^{\odot} = Y^{\odot} \times L^1([0,h];Y^*)$, where $L^1([0,h];Y^*)$ is the Banach space of Bochner integrable Y^* valued functions. The duality pairing between $\phi \in X$ and $\phi^{\odot} = (y^{\odot}, g) \in X^{\odot}$ is given by*

$$\langle \phi^{\odot}, \phi \rangle := \langle y^{\odot}, \phi(0) \rangle + \int_0^h \langle g(\theta), \phi(-\theta) \rangle \mathrm{d}\theta \tag{2.4}$$

Moreover, for the action of $T_0^{\odot}$ on $X^{\odot}$ we have

$$T_0^{\odot}(t)(y^{\odot}, g) := (S^{\odot}(t)y^{\odot} + \int_0^{\min\{t,h\}} S^*(t-\theta)g(\theta)\mathrm{d}\theta, T_1(t)g) \tag{2.5}$$

where the integral is a weak Lebesque integral with values in $Y^{\odot}$ and $T_1(t)$ is defined as translation to the left by zero for $g \in L^1([0,h];Y^*)$*

$$(T_1(t)g)(\theta) := \begin{cases} g(t+\theta) & t+\theta \in [0,h] \\ 0 & t+\theta > h \end{cases} \tag{2.6}$$

Using this theorem we are able to find $A_0^{\odot}$ and its domain $D(A_0^{\odot})$

Theorem 2.1.2. *For the sun-dual of A_0 on $X^{\odot}$ we have that*

$$D(A_0^{\odot}) = \{(y^{\odot}, g) | y^{\odot} \in D(B^{\odot}), g \in AC([0,h);Y^*), g(0) \in Y^{\odot}, g(\theta) = 0 \text{ for } \theta \geq h\} \tag{2.7}$$

and $A_0^{\odot}(y^{\odot}, g) = (B^{\odot}y^{\odot} + g(0), \dot{g})$, with $\dot{g}$ some function in $L^1([0,h];Y^{\odot})$ such that

$$g(\theta) = g(0) + \int_0^{\theta} \dot{g}(s)\mathrm{d}s \tag{2.8}$$

for $\theta \in [0,h)$

Proof. As $T_0^{\odot}$ is a strongly continuous semigroup on $X^{\odot}$, by definition

$$D(A_0^{\odot}) = \{\phi^{\odot} \in X^{\odot} | \lim_{t \downarrow 0} \left\| \frac{1}{t}(T_0^{\odot}(t)\phi^{\odot} - \phi^{\odot}) - \psi^{\odot} \right\|_{X^{\odot}} = 0 \text{ for some } \psi^{\odot} \in X^{\odot}\}, \quad A_0^{\odot}\phi^{\odot} = \psi^{\odot} \tag{2.9}$$

Let $(y^{\odot}, g) \in D(A_0^{\odot})$ and $A_0^{\odot}(y^{\odot}, g) = (z^{\odot} + g(0), f)$. Then we have that

$$\begin{aligned} \lim_{t \downarrow 0} \left\| \frac{1}{t}(T_0^{\odot}(t)(y^{\odot}, g) - (y^{\odot}, g)) - (z^{\odot} + g(0), f) \right\|_{X^{\odot}} &\leq \lim_{t \downarrow 0} \left\| \frac{1}{t}(S^{\odot}(t)y^{\odot} - y^{\odot}) - z^{\odot} \right\|_{Y^{\odot}} \\ &+ \lim_{t \downarrow 0} \left\| \frac{1}{t} \int_0^t S^*(t-\theta)g(\theta)\mathrm{d}\theta - g(0) \right\|_{Y^{\odot}} \\ &+ \lim_{t \downarrow 0} \left\| \frac{1}{t}(T_1(t)g - g) - f \right\|_{L_1} \end{aligned}$$

The first limit exists if and only if $y^{\odot} \in D(B^{\odot})$ and with $z^{\odot} = B^{\odot}y^{\odot}$. The last limit exists if and only if $g \in AC([0,h);Y^*)$ and $g(\theta) = 0$ for $\theta \geq h$ with $f = \dot{g}$ by Diekmann *et al.* (2012, theorem A.2.3).

The second limit requires some attention. The norm on $Y^{\odot}$ is defined as

$$||y^{\odot}||_{Y^{\odot}} = \sup\{|\langle y^{\odot}, y \rangle| \; |y \in Y, ||y||_Y \leq 1\} \tag{2.10}$$

By the definition of the weak* integral we have for $y \in Y, ||y||_Y \leq 1$ that

$$\begin{aligned}\left|\left\langle \frac{1}{t}\int_0^t S^*(t-\theta)g(\theta)\mathrm{d}\theta - g(0), y\right\rangle\right| &= \left|\frac{1}{t}\int_0^t \langle S^*(t-\theta)g(\theta), y\rangle\,\mathrm{d}\theta - \langle g(0), y\rangle\right| \\ &= \left|\frac{1}{t}\int_0^t \langle g(\theta), S(t-\theta)y\rangle - \langle g(0), y\rangle\mathrm{d}\theta\right| \\ &= \left|\frac{1}{t}\int_0^t \langle g(\theta), S(t-\theta)y - y\rangle + \langle g(\theta)-g(0), y\rangle\mathrm{d}\theta\right| \\ &\leq \frac{1}{t}\int_0^t ||g(\theta)||_{Y^*}||S(t-\theta)-I||||y||_Y + ||g(\theta)-g(0)||_{Y^*}||y||_Y\mathrm{d}\theta\end{aligned}$$

Hence we have that we can bound the limit by

$$\lim_{t\downarrow 0}\left\|\frac{1}{t}\int_0^t S^*(t-\theta)g(\theta)\mathrm{d}\theta - g(0)\right\|_{Y^\odot} \leq \lim_{t\downarrow 0}\frac{1}{t}\int_0^t ||g(\theta)||_{Y^*}||S(t-\theta)-I|| + ||g(\theta)-g(0)||_{Y^*}\mathrm{d}\theta$$

As $||g||_{Y^*}$ is continuous on $[0,h)$, the first term vanishes due to the strong continuity of S. As the limit $\lim_{t\downarrow 0}||g(\theta)-g(0)||_{Y^*} = 0$, the second term vanishes. □

If we take the dual of $X^\odot$, we see that $X^{\odot *}$ can be represented as $Y^{\odot *} \times (L^1([-h,0];Y^*))^*$. If Y is a reflexive separable Banach space then we can identify $X^{\odot *}$ as $Y^{\odot *} \times L^\infty([-h,0];Y)$, (Zeidler *et al.*, 1994, Problem 23.12d). However, in general this is not the case, still we can show that $L^\infty([-h,0];Y)$ is a subspace of $(L^1([-h,0];Y^*))^*$ with an explicit pairing.

Theorem 2.1.3. *$Y^{\odot *}\times L^\infty([-h,0];Y) \subset X^{\odot *}$ with the duality pairing for $(y^{\odot *}, f) \in Y^{\odot *}\times L^\infty([-h,0];Y)$ and $(y^\odot, g) \in Y^\odot \times L^1([0,h];Y^*)$*

$$\langle (y^{\odot *}, f), (y^\odot, g)\rangle = \langle y^{\odot *}, y^\odot\rangle + \int_0^h \langle g(\theta), f(-\theta)\rangle\mathrm{d}\theta \tag{2.11}$$

Proof. It is sufficient to proof that $\langle (y^{\odot *}, f), \cdot\rangle$ as above is a bounded linear functional on $X^\odot$. We have that $\langle y^{\odot *}, \cdot\rangle$ is a bounded linear functional on $Y^\odot$ and for all $g \in L^1([0,h];Y^*)$. Furthermore

$$\left|\int_0^h \langle g(\theta), f(-\theta)\rangle\mathrm{d}\theta\right| \leq \int_0^h ||f(-\theta)||_Y||g(\theta)||_{Y^*}\mathrm{d}\theta$$

As $f \in L^\infty([-h,0];Y)$ we have that $||f(-\theta)||_Y$ is bounded on $[0,h]$ and as $g \in L^1([0,h];Y^*)$ we get that $\langle g(\theta), f(-\theta)\rangle$ is integrable on $[0,h]$. Hence $\langle (y^{\odot *}, f), \cdot\rangle$ is a bounded linear functional on $X^\odot$. □

The canonical embedding $j: X \to X^{\odot *}$ for which $\langle j(\phi), \phi^\odot\rangle = \langle \phi^\odot, \phi\rangle$ is given by $j(\phi) = (j_Y\phi(0), \phi)$. We are now able to proof a fundamental lemma, which touches on the main benefit of using sun-star calculus. We find that on the larger space $X^{\odot *}$ the rule for extension $\dot\phi(0) = B\phi(0)$ is no longer embedded in the domain of $D(A_0^{\odot *})$, but is instead included in the action of $A^{\odot *}$.

Lemma 2.1.4. *(Fundamental Lemma of the Sun-Star Calculus) If $\phi \in \{\phi \in C^1([-h,0];Y) | \phi(0) \in D(B)\}$ then $j\phi \in D(A_0^{\odot *})$ and $A_0^{\odot *}j\phi = (j_Y B\phi(0), \dot\phi)$*

Proof. Let $\phi \in \{\phi \in C^1([-h,0];Y) | \phi(0) \in D(B)\}$ and let $(y^\odot, g) \in D(A_0^\odot)$. As $(j_Y B\phi(0), \dot\phi) \in Y^{\odot *} \times L^\infty([-h,0];Y)$, theorem (2.1.3) applies. We use the integration by parts formula for Bochner-integrals[1] to find that

$$\begin{aligned}\langle (j_Y B\phi(0), \dot\phi), (y^\odot, g)\rangle &= \langle j_Y B\phi(0), y^\odot\rangle + \int_0^h \langle g(\theta), \dot\phi(-\theta)\rangle\mathrm{d}\theta \\ &= \langle y^\odot, B\phi(0)\rangle - \langle g(\theta), \phi(-\theta)\rangle|_0^h + \int_0^h \langle \dot g(\theta), \phi(-\theta)\rangle\mathrm{d}\theta \\ &= \langle B^\odot y^\odot + g(0), \phi(0)\rangle + \int_0^h \langle \dot g(\theta), \phi(-\theta)\rangle\mathrm{d}\theta \\ &= \langle A_0^\odot(y^\odot, g), \phi\rangle = \langle j\phi, A_0^\odot(y^\odot, g)\rangle\end{aligned}$$

Hence $j\phi \in D(A_0^{\odot *})$ and $A_0^{\odot *}j\phi = (j_Y B\phi(0), \dot\phi)$ □

[1] Zeidler *et al.* (1994) has a proof a this formula for $\phi \in W^{1,p}([-h,0];Y)$ for $1 < p < \infty$ but not for $p = 1$ and $p = \infty$, but I presume an equivalent statement should exist, although I do not have a reference to the literature for it.

We define $l : Y \to X^{\odot *}$ as in Diekmann et al. (2014, Lemma 3.13) by $l\mathbf{y} = (j_Y \mathbf{y}, 0)$. We can rewrite equation (ADDE) into (AIE), which is a variation of constants formula.

$$\mathbf{u}(t) = T_0(t)\phi + j^{-1}\int_0^t T_0^{\odot *}(t-s)lG(\mathbf{u})(s)\mathrm{d}s \tag{AIE}$$

It is possible find a unique solution of (AIE) using a Banach fixed point argument and the fact that S is a semigroup.

Theorem 2.1.5. *(Diekmann et al., 2014, Corollary 2.9) For every initial condition $\phi \in X$ there exists a unique solution $\mathbf{u} \in C(\mathbb{R}^+, Y)$ to equation (AIE).*

This solution of (AIE) implies a solution of (ADDE), however there are 2 solution concepts which are relevant.

Definition 2.1.6. A function $\mathbf{u} : [-h, \infty) \to Y$ is called a classical solution of (ADDE) if $\mathbf{u}$ is continuous on $[-h, \infty)$, continuously differentiable on $\mathbb{R}^+$ and $\mathbf{u}(t) \in D(B)$ for all $t \geq 0$ and $\mathbf{u}$ satisfied (ADDE).

A function $\mathbf{u} : [-h, \infty) \to Y$ is called a mild solution of (ADDE) if $\mathbf{u}$ is continuous, $\mathbf{u}_0 = \phi$ and satisfies

$$\mathbf{u}(t) = S(t)\phi(0) + \int_0^t S(t-s)G(\mathbf{u}_s)\mathrm{d}s \tag{2.12}$$

If $\mathbf{u}$ is a classical solution then it also is a mild solution. The converse holds when $\phi(0) \in D(B)$.

Theorem 2.1.7. *(Diekmann et al., 2014, Corollary 2.16) For every initial condition $\phi \in X$ there exists a unique mild solution of (ADDE). When $\phi(0) \in D(B)$ then this solution is a classical solution.*

2.2 Linearisation

We want to investigate the behaviour around a fixed point, which we take without loss of generality to be $\mathbf{u} \equiv 0$. Linearising equation (ADDE) around the trivial fixed point $\mathbf{u} \equiv 0$ results in the linear problem (LINP).

$$\begin{cases} \dot{\mathbf{u}}(t) = B\mathbf{u}(t) + DG(0)\mathbf{u}_t \\ \mathbf{u}_0 = \phi \in X \end{cases} \tag{LINP}$$

We can now define a new semigroup for (LINP), $T(t)\mathbf{u}_s = \mathbf{u}_{s+t}$, where $\mathbf{u}_t$ is the solution of (LINP). This semigroup is in fact strongly continuous.

Theorem 2.2.1. *(Diekmann et al., 2014, Theorem 3.5) $T(t)$ is the unique strongly continuous semigroup such that*

$$T(t)\phi = T_0(t)\phi + j^{-1}\int_0^t T_0^{\odot *}(t-s)lDG(0)T(s)\phi\mathrm{d}s \tag{2.13}$$

for all $\phi \in X$ and for all $t \geq 0$

This semigroup T inherits some properties from S.

Theorem 2.2.2. *(Engel and Nagel, 1999, Theorem 6.6 and 6.9) If $S(t)$ is immediately norm continuous then $T(t)$ is norm continuous for $t > h$. If $S(t)$ is immediately compact then $T(t)$ is eventually compact for $t > h$*

This semigroup T has a generator A for which is given by the following theorem.

Theorem 2.2.3. *(Diekmann et al., 2014, Theorem 3.12) For the generator A of the semigroup T we have*

$$\begin{aligned} D(A) &= \{\phi \in X | j\phi \in D(A^{\odot *}) \text{ and } A_0^{\odot *}j\phi + lDG(0)\phi \in j(X)\} \\ A &= j^{-1}(A_0^{\odot *}j + lDG(0)) \end{aligned} \tag{2.14}$$

We will now proof an equivalent lemma to lemma (2.1.4) for $A^{\odot *}$. For our purposes later it is sufficient to have a representation of $A^{\odot *}$ restricted on $j(X)$.

Lemma 2.2.4. *If* $\phi \in \{\phi \in C^1([-h,0];Y) | \phi(0) \in D(B)\}$ *then* $j\phi \in D(A^{\odot *})$ *and* $A^{\odot *}j\phi = (j_Y B\phi(0), \dot{\phi}) + (j_y DG(0)\phi, 0)$

Proof. By Diekmann *et al.* (2014, Corollary 3.11), it holds that $D(A^{\odot *}) \cap j(X) = D(A_0^{\odot *}) \cap j(X)$ and $A^{\odot *} = A_0^{\odot *} + lDG(0)j^{-1}$ on this space. Clearly, we have that $j\phi \in j(X)$. By lemma (2.1.4) we get that $j\phi \in D(A_0^{\odot *})$ and that

$$A^{\odot *}j\phi = A_0^{\odot *}j\phi + lDG(0)\phi = (j_Y B\phi(0), \dot{\phi}) + (j_y DG(0)\phi, 0) \qquad \square$$

For an operator A on X the resolvent set $\rho(A)$ is the set of all $z \in \mathbb{C}$ such that the operator $A - z$ has a bounded inverse. The resolvent operator $R(z, A) : X \to D(A)$ is then defined as $R(z, A) = (A - z)^{-1}$ for $z \in \rho(A)$. The spectrum of A, $\sigma(A) = \mathbb{C}/\rho(A)$, can be decomposed into the point spectrum $\sigma_p(A)$ and the essential spectrum $\sigma_{ess}(A)$. We use Weyl's definition of the essential spectrum, i.e. $\sigma_{ess}(A) := \{\lambda \in \mathbb{C} | A - \lambda I$ is not a Fredholm operator$\}$. Then $\sigma_P(A) = \sigma/\sigma_{ess}(A)$ is the discrete spectrum, i.e. isolated eigenvalues with a finite dimensional eigenspace.

Lemma 2.2.5. *For the following spectra we have that* $\sigma(A_0) = \sigma(A_0^*) = \sigma(A_0^{\odot}) = \sigma(A_0^{\odot *}) = \sigma(B)$. *Furthermore* $\sigma_{ess}(A_0) = \sigma_{ess}(B)$.

Proof. By Engel and Nagel (1999, Proposition IV.2.18), we have that $\sigma(A_0) = \sigma(A_0^*) = \sigma(A_0^{\odot}) = \sigma(A_0^{\odot *})$. We show that $\sigma(A_0) = \sigma(B)$ by proving the converse $\rho(A_0) = \rho(B)$.

If $z \in \rho(B)$ then we can find the resolvent of A_0 explicitly as for all $\phi \in X$ and $\theta \in [-h, 0]$

$$[R(z, A_0))\phi](\theta) = e^{z\theta}R(\lambda, B)\phi(0) + \int_{\theta}^{0} e^{z(\theta - s)}\phi(s)\mathrm{d}s \tag{2.15}$$

Hence $z \in \rho(A_0)$.

Suppose that $z \in \rho(A_0)$ en let $\mathbf{y} \in Y$. Then the constant function $\psi(\theta) = y$ for $\theta \in [-h, 0]$ is in X and hence $\phi = R(z, A_0)\psi \in D(A_0)$. This implies that $\mathbf{q} = \phi(0) \in D(B)$ and $(B - z)\mathbf{q} = \dot{\phi}(0) - z\phi(0) = ((A_0 - z)\phi)(0) = \psi(0) = \mathbf{y}$. We conclude that $q = R(z, B)y$ and $z \in \rho(B)$.

We can explicitly find the point spectrum $\sigma_p(A_0)$. For some $\lambda \in \sigma(A_0)$, we need to find a $\phi \in D(A_0)$ such that $\dot{\phi} = \lambda\phi$. Clearly this is the case if and only if $\phi(\theta) = qe^{\lambda\theta}$ for $\theta \in [-h, 0]$, with $q \in D(B)$ and $Bq = B\phi(0) = \dot{\phi}(0) = \lambda q$. Therefore $\lambda \in \sigma_p(A_0)$ if and only if $\lambda \in \sigma_p(B)$ and the corresponding eigenspaces have the same dimension. This implies that $\sigma_{ess}(A_0) = \sigma_{ess}(B)$. $\square$

If $DG(0)$ is compact then we can make inferences on the spectrum of A from the spectrum of A_0.

Theorem 2.2.6. *If* $DG(0)$ *is compact then* $\sigma_{ess}(A) = \sigma_{ess}(B)$. [2]

Proof. Due to Diekmann *et al.* (2014, Corollary 3.8) We have that $A^* = A_0^* + (lDG(0))^*$ with $D(A^*) = D(A_0^*)$. As l is a bounded embedding, we have that $lDG(0)$ is compact. Due to Schauder's theorem, (Kato, 2013, Theorem III.4.10), we have that $(lDG(0))^*$ is compact if and only if $lDG(0)$ is compact. Hence A^* is a compact perturbation of A_0^*. Then due to the stability theorem of Kato (2013, Theorem IV.5.35), we get that $\sigma_{ess}(A^*) = \sigma_{ess}(A_0^*)$. As a consequence of Kato (2013, Theorem IV.5.14), we have that $\sigma_{ess}(A) = \sigma_{ess}(A^*) = \sigma_{ess}(A_0^*) = \sigma_{ess}(A_0) = \sigma_{ess}(B)$ $\square$

[2] Note that for $B = -\alpha I$ and dim$(Y) = \infty$, we get that $\sigma_{ess}(A) = \sigma_{ess}(B) = \{-\alpha\}$ as the corresponding eigenspace has infinite dimension.

Chapter 3

Properties of the diffusion operator

Before we dive into explicit computations of the spectrum of A, we first do some analysis when B is a diffusion operator on $Y = C(\Omega; \mathbb{R}^N)$, with $\Omega = [-1, 1]$. We can interpret the operator B as a diagonal matrix of operators $d_i\Delta - \alpha_i$ which act on a single component u_i. So all properties of these individual operators immediately generalize to B. So for this section without loss of generality we assume that $N = 1$, so $B = d\Delta - \alpha$.

3.1 Semigroup and resolvent

B is an unbounded, closed, linear operator acting on Y with its domain

$$D(B) = \{y \in Y | y \in C^2(\Omega), y'(\partial\Omega) = 0\} \tag{3.1}$$

Lemma 3.1.1. *(Engel and Nagel, 1999, Proposition VI.19). The operator $(B, D(B))$ generates a strongly continuous, positive and immediately compact semigroup $(S(t))_{t\geq 0}$.*

We can explicitly find an explicit representation of the semi-group $S(t)$ by finding the spectrum of B and employing a linear combination of the eigenvectors.

Lemma 3.1.2. *The point spectrum of B consists of simple eigenvalues: even eigenvalues $\lambda_n^{even} = -dn^2\pi^2 - \alpha$ with eigenvectors $\cos(n\pi x)$ and odd eigenvalues $\lambda_n^{odd} = -d(n+\frac{1}{2})^2\pi^2 - \alpha$ with eigenvectors $\sin((n+\frac{1}{2})\pi x)$ for all $n \in \mathbb{N}_0$. The semigroup S can be explicitly written as a convolution with a Green's function G.*

$$S(t)\phi(x) = \int_\Omega \phi(x')G(t, x, x')\,\mathrm{d}x'$$
$$G(t, x, x') := \sum_{n=0}^{\infty} \left((1 + \delta_n)^{-1} \cos(n\pi x)\cos(n\pi x')e^{(-dn^2\pi^2-\alpha)t} + \sin((n + \frac{1}{2})\pi x)\sin((n + \frac{1}{2})\pi x')e^{(-d(n+\frac{1}{2})^2\pi^2-\alpha)t} \right) \tag{3.2}$$

Here δ_n is the Kronecker delta function, 1 for $n = 0$ and vanishing elsewhere.

Proof. We find the point spectrum by finding $q \in D(B)$ which solves the following equation for some $\lambda \in \mathbb{C}$

$$(B - \lambda)q(x) = dq''(x) - (\alpha + \lambda)q(x) = 0$$

We find even solutions $q(x) = \cos(n\pi x)$ with $\lambda_n^{even} = -dn^2\pi^2 - \alpha$ and odd solutions $q(x) = \sin((n+\frac{1}{2})\pi x)$ with $\lambda_n^{odd} = -d(n + \frac{1}{2})^2\pi^2 - \alpha$ for all $n \in \mathbb{N}_0$. Next we can construct $S(t)$ as a linear combination of $\cos(n\pi x)e^{t\lambda_n^{even}}$ and $\sin((n + \frac{1}{2})\pi x)e^{t\lambda_n^{odd}}$.

$$S(t)\phi(x) = \sum_{n=0}^{\infty} \left(a_n \cos(n\pi x)e^{(-dn^2\pi^2-\alpha)t} + b_n \sin((n + \frac{1}{2})\pi x)e^{(-d(n+\frac{1}{2})^2\pi^2-\alpha)t} \right) \tag{3.3}$$

Here the coefficients are given by

$$\begin{aligned} a_n &:= \frac{1}{1+\delta_n}\int_\Omega \phi(x')\cos(n\pi x')\mathrm{d}x' \\ b_n &:= \int_\Omega \phi(x')\sin((n+\frac{1}{2})\pi x')\mathrm{d}x' \end{aligned} \tag{3.4}$$

We can rewrite this using a Green function G

$$\begin{aligned} S(t)\phi(x) &= \int_\Omega \phi(x')G(t,x,x')\,\mathrm{d}x' \\ G(t,x,x') &:= \sum_{n=0}^{\infty}\Big((1+\delta_n)^{-1}\cos(n\pi x)\cos(n\pi x')e^{(-dn^2\pi^2-\alpha)t} + \sin((n+\frac{1}{2})\pi x)\sin((n+\frac{1}{2})\pi x')e^{(-d(n+\frac{1}{2})^2\pi^2-\alpha)t}\Big) \end{aligned}$$

The terms of sum are bounded by $e^{-dn^2\pi^2 t}$ for every $t>0$. As the sum $\sum_{n=0}^{\infty} e^{-dn^2\pi^2 t}$ converges, we have that for every $t>0$, G converges uniformly in x and x'. $\square$

We can also explicitly find an expression for the resolvent of B by again utilizing the completeness of its eigenvectors.

Lemma 3.1.3. *The spectrum of B contains only eigenvalues; $\sigma(B)=\sigma_p(B)$. The resolvent $R(\lambda,B): Y\to D(B)$ for $\lambda\in\rho(B)$ is given by*

$$\begin{aligned} R(\lambda,B)y(x) &= \int_\Omega y(x')G^\lambda(x,x')\mathrm{d}x' \\ G^\lambda(x,x') &:= \sum_{n=0}^{\infty}(1+\delta_n)^{-1}(-dn^2\pi^2-\alpha-\lambda)^{-1}\cos(n\pi x)\cos(n\pi x') \\ &\quad + (-d(n+\frac{1}{2})^2\pi^2-\alpha-\lambda)^{-1}\sin((n+\frac{1}{2}\pi x)\sin((n+\frac{1}{2})\pi x') \end{aligned} \tag{3.5}$$

Proof. We need find a $q\in D(B)$ which solves $(B-\lambda)q=y$. First we assume that we can decompose q as

$$q(x)=\sum_{n=0}^{\infty}\Big(a_n\cos(n\pi x)+b_n\sin((n+\frac{1}{2})\pi x)\Big) \tag{3.6}$$

Next we substitute this into the equation $Bq=y$.

$$y(x)=\sum_{n=0}^{\infty}\Big(a_n(\lambda_n^{even}-\lambda)\cos(n\pi x)+b_n(\lambda_n^{odd}-\lambda)\sin((n+\frac{1}{2})\pi x)\Big) \tag{3.7}$$

We find that the coefficients are given by

$$\begin{aligned} a_n &:= \frac{1}{\lambda_n^{even}-\lambda}\frac{1}{1+\delta_n}\int_\Omega y(x')\cos(n\pi x')\mathrm{d}x' \\ b_n &:= \frac{1}{\lambda_n^{odd}-\lambda}\int_\Omega y(x')\sin((n+\frac{1}{2})\pi x')\mathrm{d}x \end{aligned} \tag{3.8}$$

We can again rewrite this using a Green's function $G^\lambda(x,x')$

$$\begin{aligned} R(\lambda,B)y(x) &= \int_\Omega y(x')G^\lambda(x,x')\mathrm{d}x' \\ G^\lambda(x,x') &:= \sum_{n=0}^{\infty}(1+\delta_n)^{-1}(-dn^2\pi^2-\alpha-\lambda)^{-1}\cos(n\pi x)\cos(n\pi x') \\ &\quad + (-d(n+\frac{1}{2})^2\pi^2-\alpha-\lambda)^{-1}\sin((n+\frac{1}{2}\pi x)\sin((n+\frac{1}{2})\pi x') \end{aligned}$$

The terms of sum are bounded by $(dn^2\pi^2+\alpha+\lambda)^{-1}$ for $\lambda\notin\sigma_p(B)$. As the sum $\sum_{n=0}^{\infty}(dn^2\pi^2+\alpha+\lambda)^{-1}$ converges, we have that for every $\lambda\in\sigma_p(B)$, G^λ converges uniformly in x and x'. Therefore the resolvent $R(\lambda,B)$ exists and is bounded for every $\lambda\notin\sigma_p(B)$, hence $\sigma(B)=\sigma_p(B)$. $\square$

3.2 Sun-star calculus

We will now develop the sun-star calculus when B is a diffusion operator. Without loss of generality, we can take $d = 1$ and $\alpha = 0$ for this section, hence $B = \Delta$. This is due to the linearity of the sun-star calculus, $B^* = d\Delta^* - \alpha$ with $D(B^*) = D(\Delta^*)$ and hence $B^\odot = d\Delta^\odot - \alpha$ with $D(B^\odot) = D(\Delta^\odot)$, etc.

As a consequence of the Riesz representation theorem, Y^* can be represented as $NBV(\Omega)$. A function $y \in NBV(\Omega)$ is said to be of normalized bounded variation if $y(-1) = 0$, y is continuous from the right on the open interval $(-1,1)$ and y is of bounded variaton. The corresponding norm of $NBV(\Omega)$ is the Total Variation norm

$$||y^*||_{Y^*} := \sup_P \sum_{i=1}^{n_P} |y^*(x_{i+1}) - y^*(x_i)| \tag{3.9}$$

Here P is any partition of Ω. The duality pairing is given by the Riemann-Stieltjes integral:

$$\langle y^*, y\rangle := \int_{-1}^{1} y \, \mathrm{d}y^* \tag{3.10}$$

We will now try to find a representation for B^*.

Theorem 3.2.1. *The dual space Y^* can be represented as $NBV(\Omega)$. Further, $y^* \in D(B^*)$ if and only if for $x \in (-1,1]$*

$$y^*(x) = c_1 + \int_{-1}^{x} \left(c_2 + \int_{-1}^{s} z^*(x')\mathrm{d}x' \right) \mathrm{d}s \tag{3.11}$$

Where $c_1, c_2 \in \mathbb{R}$ and $z^* \in NBV(\Omega)$ with $z^*(1) = 0$. For such y^* we have that $B^*y^* = z^*$

Proof. Let $y^* \in D(B^*)$, $y \in D(B)$ and $z^* = B^*y^*$, then $\langle y^*, By\rangle = \langle z^*, y\rangle$. Let

$$w^*(s) = c_2 + \int_{-1}^{s} z^*(x')\mathrm{d}x' \tag{3.12}$$

for some $c_2 \in \mathbb{R}$. As $y \in C^2(\Omega)$ and $y'(\pm 1) = 0$ we get that using partial integration

$$\begin{aligned}
\int_{-1}^{1} y''(x)\mathrm{d}y^*(x) &= \langle y^*, By\rangle = \langle z^*, y\rangle \\
&= \int_{-1}^{1} y \mathrm{d}z^* \\
&= z(x)y(x)|_{-1}^{1} - \int_{-1}^{1} y'(x)z^*(x)\mathrm{d}x \\
&= z(1)y(1) + \int_{-1}^{1} y''(x)w(x)\mathrm{d}x
\end{aligned}$$

If we take y as a constant function we immediately see that $z(1) = 0$ is a necessary condition. For any $-1 < x' < x < 1$ we can take a sequence of $y_n \in D(B)$ such that $y_n''(s)$ converges monotone to the characteristic function on the interval $[x', x]$. Then by the Lebesque monotone convergence theorem we get that

$$y^*(x) - y^*(x') = \int_{x'}^{x} \mathrm{d}y^*(s) = \int_{x'}^{x} w^*(s)\mathrm{d}s$$

Letting $x' \downarrow -1$ we get that

$$y^*(x) = y^*(-1+) + \int_{-1}^{x} w^*(s)\mathrm{d}s$$

So we can write this y^* as

$$y^*(x) = c_1 + \int_{-1}^{x} \left(c_2 + \int_{-1}^{s} z^*(x')\mathrm{d}x' \right) \mathrm{d}s \tag{3.13}$$

Conversely let y^* have the form in equation (3.11) with $z(1) = 0$. Then for all $y \in D(B)$ we have that using Diekmann *et al.* (2012, theorem I.1.4 and I.1.5)

$$\begin{aligned}\langle y^*, By\rangle &= \int_{-1}^{1} y''(x)\mathrm{d}y^* \\ &= \int_{-1}^{1} y''(x)w^*(x)\mathrm{d}x \\ &= -\int_{-1}^{1} y'(x)z^*(x)\mathrm{d}x \\ &= \int_{-1}^{1} y\mathrm{d}z^* = \langle z^*, y\rangle\end{aligned}$$

Hence we can conclude that $y^* \in D(B^*)$ and $B^*y^* = z^*$. □

Now we are in a position to find the sun-dual of Y with respect to B, $Y^{\odot}$, which is the closure of $D(B^*)$ with respect to the Total Variation Norm.

Theorem 3.2.2. *The sun dual $Y^{\odot}$ can be represented as $\mathbb{R} \times L^1(\Omega)$.* [1] *For the sun dual of B we have that*

$$D(B^{\odot}) := \{(c, w^{\odot}) \in \mathbb{R} \times L^1(\Omega) | c \in \mathbb{R}, (w^{\odot})' \in AC[-1,1], (w^{\odot})'(1) = 0\} \tag{3.14}$$

and $B^{\odot}(c, w^{\odot}) := ((w^{\odot})'(-1), (w^{\odot})'')$, where $(w^{\odot})''$ is some L^1 function such that

$$(w^{\odot})'(x) = (w^{\odot})'(-1) + \int_{-1}^{x} (w^{\odot})''(s)\mathrm{d}s \tag{3.15}$$

Proof. Let $y^* \in D(B^*)$. Again using the notation that for $x, s \in (-1, 1]$,

$$\begin{aligned}y^*(x) &= c_1 + \int_{-1}^{x} w^*(s)\mathrm{d}s \\ w^*(s) &= c_2 + \int_{-1}^{s} z^*(x')\mathrm{d}x'\end{aligned}$$

for some $c_1, c_2 \in \mathbb{R}$ and $z^* \in NBV(\Omega)$ with $z^*(1) = 0$, we can rewrite the Total Variation norm as:

$$||y^*||_{Y^*} = |c_1| + ||w^*||_{L^1} \tag{3.16}$$

For the space

$$W := \left\{ c + \int_{-1}^{s} z^*(x')\mathrm{d}x' \middle| c \in \mathbb{R}, z^* \in NBV(\Omega), z^*(1) = 0 \right\} \tag{3.17}$$

we have that $\{w^* \in C^2 | (w^*)'(-1) = 0\} \subset W \subset L^1$. As this first space of C^2 functions is dense in L^1, we have that W is dense in L^1. Hence, we can represent $Y^{\odot}$ as the space

$$\left\{ y^{\odot} \in NBV(\Omega) \middle| y^{\odot}(x) = c + \int_{-1}^{x} w^{\odot}(s)\mathrm{d}s \text{ where } c \in \mathbb{R}, w^{\odot} \in L^1(\Omega) \text{ for } x \in (-1, 1] \right\} \tag{3.18}$$

which are the absolutely continuous functions on $(-1, 1]$ with a jump from 0 to c at $x = -1$. We can equivalently express $Y^{\odot}$ as $\mathbb{R} \times L^1(\Omega)$ where $y^{\odot} = (c, w^{\odot})$ with $c \in \mathbb{R}$ and $w^{\odot} \in L^1(\Omega)$ equipped with the norm

$$||y^{\odot}||_{Y^{\odot}} := |c_1| + ||w^{\odot}||_{L^1} \tag{3.19}$$

The domain of $B^{\odot}$ is defined as $D(B^{\odot}) = \{y^{\odot} \in D(B^*) | B^*y^{\odot} \in Y^{\odot}\}$. Using equation (3.11) we have $B^*y^* = z^*$. If $z^* \in Y^{\odot}$ then z^* must be absolutely continuous on $(-1, 1]$. So for $y^{\odot} = (c, w^{\odot})$ we find that $(w^{\odot})' = z^*$ is absolutely continuous on $(-1, 1]$. Thus we can write that $B^{\odot}(c, w^{\odot}) = ((w^{\odot})'(-1), (w^{\odot})'')$, where $(w^{\odot})''$ is an L^1 function such that

$$(w^{\odot})'(x) = (w^{\odot})'(-1) + \int_{-1}^{x} (w^{\odot})''(s)\mathrm{d}s$$

The boundary condition $z(1) = 0$ is transformed into $(w^{\odot})'(1) = 0$ □

Now we can take the dual again and end up at the sun-star dual $Y^{\odot *}$.

[1] The sun-dual $Y^{\odot}$ is almost the same as in Diekmann *et al.* (2012, Theorem II.5.2), where it is taken with respect to the first derivative with the condition $\dot{y}(0) = 0$. However in that case there was an extra condition in $Y^{\odot}$ that functions $g \in L^1$ could be extended be zero for $\theta \geq h$. In our case with diffusion we have a fixed domain on which the diffusion takes place, so this condition is not present.

Theorem 3.2.3. *The sun-star dual $Y^{\odot*}$ can be represented as $\mathbb{R} \times L^\infty(\Omega)$. For the sun-star dual of B we have that*

$$D(B^{\odot*}) = \{(\gamma, w^{\odot*}) | (w^{\odot*})' \text{ is Lipschitz continous, } w^{\odot*}(-1) = \gamma, (w^{\odot*})'(\pm 1) = 0\} \tag{3.20}$$

and $B^{\odot*}(\gamma, w^{\odot*}) := (0, (w^{\odot*})'')$, where $(w^{\odot*})''$ is an $L^\infty(\Omega)$ function such that

$$(w^{\odot*})'(x) = \int_{-1}^{x} (w^{\odot*})''(s)\mathrm{d}s \tag{3.21}$$

By Diekmann *et al.* (2012, lemma I.5.4) the sun-star dual $Y^{\odot*}$ can be represented as $\mathbb{R} \times L^\infty(\Omega)$ with the duality pairing between $Y^{\odot*}$ and $Y^{\odot}$ being given by

$$\langle (\gamma, w^{\odot*}), (c, w^{\odot}) \rangle := \gamma c + \int_{-1}^{1} w^{\odot*}(x) w^{\odot}(x)\mathrm{d}x \tag{3.22}$$

Let $(\gamma, w^{\odot*}) \in D(B^{\odot*})$ and $B^{\odot*}(\gamma, w^{\odot*}) = (\beta, z^{\odot*})$. Let

$$v^{\odot*}(x) := v^{\odot*}(-1) + \int_{-1}^{x} z^{\odot*}(s)\mathrm{d}s \tag{3.23}$$

which is a Lipschitz continuous function as $z^{\odot*} \in L^\infty(\Omega)$. Then for all $(c, w^{\odot}) \in D(B^{\odot})$ we get that

$$\begin{aligned}
\gamma(w^{\odot})'(-1) + \int_\Omega w^{\odot*}(x)(w^{\odot})''(x)\mathrm{d}x &= \langle (\gamma, w^{\odot*}), B^{\odot}(c, w^{\odot}) \rangle \\
&= \langle (\beta, z^{\odot*}), (c, w^{\odot}) \rangle \\
&= \beta c + \int_\Omega z^{\odot*}(x) w^{\odot}(x)\mathrm{d}x \\
&= \beta c + v^{\odot*}(x) w^{\odot}(x)|_{-1,1} - \int_\Omega v^{\odot*}(x)(w^{\odot})'(x)\mathrm{d}x \\
&= \beta c + v^{\odot*}(x) w^{\odot}(x)|_{-1,1} + \gamma(w^{\odot})'(-1) \\
&\qquad + \int_\Omega \left(\gamma + \int_{-1}^{x} v^{\odot*}(s)\mathrm{d}s\right)(w^{\odot})''(x)\mathrm{d}x
\end{aligned}$$

Here we used that $(w^{\odot})' \in AC[-1, 1]$ and $(w^{\odot})'(1) = 0$. As c and $w^{\odot}(\pm 1)$ are arbitrary we see that necessarily $\beta = 0, v^{\odot*}(\pm 1) = 0$. Furthermore,

$$w^{\odot*}(x) = \gamma + \int_{-1}^{x} v^{\odot*}(s)\mathrm{d}s \tag{3.24}$$

which implies that $(w^{\odot*})' = v^{\odot*}$ and $w^{\odot*}(-1) = \gamma$. □

Finally we characterize the sun-sun dual $Y^{\odot\odot}$ which is the closure of $D(B^{\odot*})$ with respect to the $Y^{\odot*}$-norm, which is a supremum norm.

Theorem 3.2.4. *The sun-sun dual $Y^{\odot\odot}$ can be represented as $\{(\gamma, w^{\odot\odot}) | w^{\odot\odot} \in C(\Omega), w^{\odot\odot}(-1) = \gamma\}$. The canonical embedding $j_Y : Y \to Y^{\odot*}$ is given by $j_Y(y) = (y(-1), y)$ and $j_Y(Y) = Y^{\odot\odot}$, i.e. Y is sun-reflexive.*

Proof. Let $y^{\odot*} = (\gamma, w^{\odot*}) \in Y^{\odot*}$. As the supremum norm does not preserve derivatives, i.e. the C^2 functions are dense in C^0 with respect to the supremum norm, we have that only the continuity and the condition $w^{\odot*}(-1) = \gamma$ remain. The canonical embedding between Y and $Y^{\odot*}$ is an isomorphism between Y and $Y^{\odot\odot}$, hence Y is sun-reflexive. □

Chapter 4

Spectral Properties of the Linearized Problem

We will now compute the spectrum of A, (2.14) for specific choices for the functions $\tau, \mathbf{S}$ and J. For $i, j \in \{1, \cdots, N\}$, the delay is chosen to be an intrinsic delay $\tau^0_{i,j} \in \mathbb{R}$ plus delay with a finite propagation speed $\nu_{i,j} \in \mathbb{R}$

$$\tau_{i,j}(x, x') := \tau^0_{i,j} + \frac{|x - x'|}{\nu_{i,j}} \tag{4.1}$$

Where $\tau^0_{i,j}, \nu_{i,j} > 0$. The connectivity is chosen to be a single exponential

$$J_{i,j}(x, x') := \eta_{i,j} e^{-\mu_{i,j}|x-x'|} \tag{4.2}$$

Where $\eta_{i,j}, \mu_{i,j} \in \mathbb{R}$, with $\mu_{i,j} > 0$. The firing rate function S_j is chosen to be an odd sigmoid with steepness parameter $\gamma_j \in \mathbb{R}$, with $\gamma_j > 0$.

$$S_j(u) := \frac{1}{1 + e^{-\gamma_j u}} - \frac{1}{2} \tag{4.3}$$

As $S(0) = 0$ we get that the (ADDE) has a fixed point $\mathbf{u} \equiv 0$. We define $\theta_j = S'_j(0) = \dfrac{\gamma}{4}$.

To find the spectrum of A we introduce a family of operators $K^z : Y \to Y$ parametrized by $z \in \mathbb{C}$. They are defined such that for $\mathbf{q} \in Y$ we have that $K^z\mathbf{q} = DG(0)\phi$, with $\phi(\theta) = qe^{\lambda\theta}$.

For each $z \in \mathbb{C}$, K^z is a matrix of Hilbert-Schmidt integral operators $K^z_{i,j}$

$$K^z_{i,j} y(x) := c_{i,j}(z) \int_{-1}^{1} e^{-k_{i,j}(z)|x-x'|} y(x') \mathrm{d}x' \tag{4.4}$$

for all $i, j \in \{1, \cdots, N\}$ and $y \in Y$ and $x \in \Omega$.

$$c_{i,j}(z) := \theta_j \eta_{i,j} e^{-\tau^0_{i,j} z}, \quad k_{i,j}(z) := \mu_{i,j} + \frac{z}{\nu_{i,j}} \tag{4.5}$$

We have that K^z is a compact operator for all $z \in \mathbb{C}$. We define also the family of operators $H^z : X \to X$ and $W^z : X \to Y$ as

$$\begin{aligned} (H^z\phi)(\theta) &:= \int_{\theta}^{0} e^{z(\theta-s)} \phi(\theta) \mathrm{d}s \\ W^z\phi &:= \phi(0) + DG(0)H^z\phi \end{aligned} \tag{4.6}$$

for $\theta \in [-h, 0]$ and $\phi \in X$. Now we formulate the main theorem of this section which allows us to compute the spectrum.

Theorem 4.0.1. *(Engel and Nagel, 1999, Proposition VI.6.7) For every $z \in \mathbb{C}$, $\phi \in \mathcal{R}(z - A)$ if and only if*

$$(B - z + K^z)q = W^z\phi$$

has a solution $q \in Y$. Moreover $z \in \rho(A)$ if and only if q is also unique. In that case the resolvent is given by

$$(R(z, A)\psi)(\theta) = e^{z\theta} R(z, B + K^z) W^z\phi + (H^z\psi)(\theta)$$

where $\theta \in [-h, 0]$, $\psi \in X$. Furthermore, W^z is surjective for every $z \in \mathbb{C}$, so $z \in \sigma(A)$ if and only if $z \in \sigma(B+K^z)$. Finally $\psi \in D(A)$ is an eigenvector corresponding to $\lambda \in \sigma_p(A)$ if and only if $\psi(\theta) = e^{\lambda\theta}\mathbf{q}$, where $q \in D(B)$ satifies $(B - \lambda + K^\lambda)\mathbf{q} = 0$.

Due to theorem (3.2.4), we have that $\sigma_{ess} = \sigma_{ess}(B) = \emptyset$. So now it remains to compute the point spectrum, i.e. the eigenvalues of A. Due to the theorem above this is equivalent to finding non-trivial solutions $\mathbf{q} \in D(B)$ for some $z \in \mathbb{C}$ of the following differo-integral equation.

$$(B - z + K^z)\mathbf{q} = 0 \tag{IE}$$

4.1 Eigenvalues

We want to find $z \in \mathbb{C}$ such that there is a non-trivial solution $q \in D(B)$ of the spectral equation (IE).

Lemma 4.1.1. *All solutions of* (IE) *are* $C^\infty(\Omega)$.

Proof. As $\mathbf{q} \in C^2(\Omega)$ and the range of K^z is contained in $C^3(\Omega)$ we have that $\Delta\mathbf{q} \in C^2(\Omega)$, which means that $\mathbf{q} \in C^4(\Omega)$. By induction, we conclude that $\mathbf{q} \in C^\infty(\Omega)$. □

We will now solve (IE) by transforming it into a differential equation. We define $L^z_{i,j} : C^\infty(\Omega) \to C^\infty(\Omega)$ for $i, j \in \{1, \cdots, N\}$ as

$$L^z_{i,j} := k^2_{i,j}(z) - \partial^2_x \tag{4.7}$$

Hence for $i, j \in \{1, \cdots, N\}$:

$$L^z_{i,j} K^z_{i,j} q_j = 2c_{i,j}(z)k_{i,j}(z)q_j \tag{4.8}$$

By applying $L^z_{i,j}$ for $j \in \{1, \cdots, N\}$ to each row $i \in \{1, \cdots, N\}$ of (IE), it follows that all solutions $\mathbf{q}$ must satisfy the linear differential equation $M^z\mathbf{q} = 0$, where the linear differential operator M^z is defined by introducing $M^z_{i,j}$ for $i, j \in \{1, \cdots, N\}$ as

$$M^z_{i,j} := (B_{i,i} - z)\delta_{i,j} \prod_{p=1}^{N} L_{i,p}(z) + 2c_{i,j}(z)k_{i,j}(z) \prod_{\substack{p=1 \\ p\neq j}}^{N} L_{i,p}(z) \tag{DE}$$

We try a solution of the form $\mathbf{q}(x) = e^{\rho x}\mathbf{q}^0$. Then $M^z e^{\rho x}\mathbf{q}^0 = e^{\rho x}P^z(\rho)\mathbf{q}^0 = 0$, where $P^z(\rho)$ is a complex valued matrix, $P^z(\rho) \in \mathbb{C}^{N\times N}$, which is defined for $i, j \in \{1, \cdots, N\}$ as:

$$P^z_{i,j}(\rho) := (d_i\rho^2 - \alpha_i - z)\delta_{i,j} \prod_{p=1}^{N}(k_{i,p}(z)^2 - \rho^2) + 2c_{i,j}(z)k_{i,j}(z) \prod_{\substack{p=1 \\ p\neq j}}^{N}(k_{i,p}(z)^2 - \rho^2) \tag{4.9}$$

Where $\delta_{i,j}$ is the Kronecker delta, i.e 1 where $i = j$ and vanishing elsewhere. The characteristic equation $P^z(\rho)\mathbf{q}^0 = 0$ has a non-trivial solution if and only if $\det(P^z(\rho)) = 0$. The individual $P^z_{i,j}(\rho)$ are even polynomials in ρ and have a degree of $2(N + 1)$ for $i = j$ or a degree of $2(N - 1)$ for $i \neq j$. This implies that $\det(P^z(\rho))$ is an even polynomial in ρ and the highest order term in this polynomial has degree $2N(N + 1)$.

To simplify the analysis we assume the generic case where $\det(P^z(\rho))$ has distinct $2N(N + 1)$ zeros, $\pm\rho_1, \cdots, \pm\rho_{N(N+1)}$. We find a $\mathbf{q}^{0,m}(z) \in \mathbb{C}^N$ by taking a vector from the nullspace of $P^z(\pm\rho_m(z))$. As the null-space is one-dimensional, $\mathbf{q}^{0,m}(z)$ is unique up to scalar multiplication. This means that if we repeat the calculations below with different choices of $\mathbf{q}^{0,m}(z)$ we get the same results.

The general solution of (DE) is then given by

$$\mathbf{q}^z(x) := \sum_{m=1}^{N(N+1)} \left[a_m \cosh(\rho_m(z)x) + b_m \sinh(\rho_m(z)x)\right] \mathbf{q}^{0,m}(z) \tag{4.10}$$

Now that we have a solution $\mathbf{q}^z$ for (DE) we want to see for which $z \in \mathbb{C}$ this is also a solution of (IE). Before we plug $\mathbf{q}^z$ into (IE) we first state some preliminary results.

Lemma 4.1.2. *If the characteristic polynomial* $\det(P^z(\rho))$ *has* $2N(N + 1)$ *distinct roots, then* $\rho_m \neq 0$ *for* $m \in \{1, \cdots, N(N + 1)\}$ *and* $k_{i,j}(z) \neq 0$ *for all* $i, j \in \{1, \cdots, N\}$.

Proof. Suppose that $\det(P^z(\rho))$ has a root $\rho = 0$. We have that if ρ is a root then $-\rho$ is also a root. So $\det(P^z(\rho))$ has a double root at zero. Hence we have proven the first part of the lemma by contradiction.

Suppose without loss of generality that $k_{1,1}(z) = 0$ then the first line in the characteristic equation $P^z(\rho)\mathbf{q}^0 = 0$ becomes:

$$(D_1\rho^2 - \alpha_1 - z)\rho^2 q^0_1 \prod_{p=2}^{N}(k_{1,p}(z)^2 - \rho^2) + 2\rho^2 \sum_{j=1}^{N} c_{1,j}(z)k_{1,j}(z)q^0_j \prod_{p=2}^{N}(k_{1,p}(z)^2 - \rho^2) = 0$$

So the equation $P^z(\rho)\mathbf{q}^0 = 0$ has a solution for $\rho = 0$ with $\mathbf{q}^0 = \mathbf{e}_1$, the first unit vector. Hence we have proven the lemma by contradiction. $\square$

We define the set $\mathcal{S}$ as

$$\mathcal{S} := \{z \in \mathbb{C} | \exists i, j \in \{1, \cdots N\}, m \in \{1, \cdots, N(N+1)\} \text{ such that } k_{i,j}(z) = \pm\rho_m(z)\} \tag{4.11}$$

For $z \notin \mathcal{S}$ we have that $\det(P^z(k_{i,j})) \neq 0$ for $i, j \in \{1, \cdots, N\}$ then we can rewrite the characteristic equation $P^z(\pm\rho_m(z))\mathbf{q}^{0,m} = 0$ for $i \in \{1, \cdots, N\}$

$$\sum_{j=1}^{N}\left[(d_i\rho_m^2 - \alpha_i - z)\delta_{i,j} + \frac{2c_{i,j}(z)k_{i,j}(z)}{k_{i,j}(z)^2 - \rho_m^2(z)}\right] q_j^{0,m}(z) \prod_{p=1}^{N}(k_{i,p}(z)^2 - \rho_m^2(z)) = 0$$

We can divide out the product to conclude that for $i \in \{1, \cdots, N\}$

$$(d_i\rho_m^2(z) - \alpha_i - z)q_i^{0,m}(z) + \sum_{j=1}^{N}\frac{2c_{i,j}(z)k_{i,j}(z)}{k_{i,j}(z)^2 - \rho_m^2(z)}q_j^{0,m}(z) = 0 \tag{4.12}$$

Next we find the expressions for $K_{i,j}^z\cosh(\rho_m(z)x)$ and $K_{i,j}^z\sinh(\rho_m(z)x)$. To compute these integrals we need to split the interval $[-1, 1]$ into the intervals $[-1, x]$ and $[x, 1]$. On these intervals $e^{-k|x-x'|}$ is an C^1 function in x' so we can compute the following anti-derivatives for these smooth branches.

$$\begin{aligned}
\int^{x'} e^{-k|x-s|}\cosh(\rho s)\mathrm{d}s &= \begin{cases} e^{-k|x-x'|}\dfrac{(\ k\cosh(\rho x') - \rho\sinh(\rho x'))}{k^2-\rho^2} + const. & -1 \leq x' < x \leq 1 \\ e^{-k|x-x'|}\dfrac{(-k\cosh(\rho x') - \rho\sinh(\rho x'))}{k^2-\rho^2} + const. & -1 \leq x < x' \leq 1 \end{cases} \\
\int^{x'} e^{-k|x-s|}\sinh(\rho s)\mathrm{d}s &= \begin{cases} e^{-k|x-x'|}\dfrac{(\ k\sinh(\rho x') - \rho\cosh(\rho x'))}{k^2-\rho^2} + const. & -1 \leq x' < x \leq 1 \\ e^{-k|x-x'|}\dfrac{(-k\sinh(\rho x') - \rho\cosh(\rho x'))}{k^2-\rho^2} + const. & -1 \leq x < x' \leq 1 \end{cases}
\end{aligned} \tag{4.13}$$

Using these anti-derivatives, we can evaluate the integrals $K_{i,j}^z\cosh(\rho_m(z)x)$ and $K_{i,j}^z\sinh(\rho_m(z)x)$. For clarity we omit the dependence on z in the remainder of this section.

$$\begin{aligned}
K_{i,j}\cosh(\rho_m x) &= \frac{2c_{i,j}k_{i,j}\cosh(\rho_m x) - 2c_{i,j}e^{-k_{i,j}}\cosh(k_{i,j}x)(k_{i,j}\cosh(\rho_m) + \rho_m\sinh(\rho_m))}{k_{i,j}^2 - \rho_m^2} \\
K_{i,j}\sinh(\rho_m x) &= \frac{2c_{i,j}k_{i,j}\sinh(\rho_m x) - 2c_{i,j}e^{-k_{i,j}}\sinh(k_{i,j}x)(\rho_m\cosh(\rho_m) + k_{i,j}\sinh(\rho_m))}{k_{i,j}^2 - \rho_m^2}
\end{aligned}$$

We will now substitute $\mathbf{q}^z$ into (IE). For $i \in \{1, \cdots, N\}$ it should be that

$$\begin{aligned}
&\sum_{m=1}^{N(N+1)}[a_m\cosh(\rho_m x) + b_m\sinh(\rho_m x)]\left[(d_i\rho_m^2(z) - \alpha_i - z)q_i^{0,m}(z) + \sum_{j=1}^{N}\frac{2c_{i,j}k_{i,j}}{k_{i,j}^2 - \rho_m^2}q_j^{0,m}\right] + \\
&\sum_{j=1}^{N}c_{i,j}e^{-k_{i,j}}\left[\cosh(k_{i,j}x)\sum_{m=1}^{N(N+1)}a_m\frac{k_{i,j}\cosh(\rho_m) + \rho_m\sinh(\rho_m)}{k_{i,j}^2 - \rho_m^2}q_j^{0,m}\right. \\
&\qquad\left. + \sinh(k_{i,j}x)\sum_{m=1}^{N(N+1)}b_m\frac{\rho_m\cosh(\rho_m) + k_{i,j}\sinh(\rho_m)}{k_{i,j}^2 - \rho_m^2}q_j^{0,m}\right] = 0
\end{aligned} \tag{4.14}$$

Due to the characteristic equation (4.12) the first line in equation (4.14) vanishes. The second line vanishes if and only if the following conditions hold

$$\begin{aligned}
\sum_{m=1}^{N(N+1)} S_{i+(j-1)N,m}^{z,even}a_m &= 0 \\
\sum_{m=1}^{N(N+1)} S_{i+(j-1)N,m}^{z,odd}b_m &= 0
\end{aligned} \tag{4.15}$$

Here we defined matrices $S^{z,even}$ and $S^{z,odd}$ as

$$\begin{aligned}
S_{i+(j-1)N,m}^{z,even} &:= \frac{k_{i,j}\cosh(\rho_m) + \rho_m\sinh(\rho_m)}{k_{i,j}^2 - \rho_m^2}q_j^{0,m} \\
S_{i+(j-1)N,m}^{z,odd} &:= \frac{\rho_m\cosh(\rho_m) + k_{i,j}\sinh(\rho_m)}{k_{i,j}^2 - \rho_m^2}q_j^{0,m}
\end{aligned} \tag{4.16}$$

We also need to take the boundary conditions into account as $\mathbf{q^z} \in D(B)$

$$(q_i^z)'(\pm 1) = \sum_{m=1}^{N(N+1)} [b_m \rho_m \cosh(\rho_m) \pm a_m \rho_m \sinh(\rho_m)]\, q_i^{0,m}) = 0 \tag{4.17}$$

To satisfy the boundary conditions, we augment the matrices $S^{z,even}$ and $S^{z,odd}$ as follows:

$$\begin{aligned} S^{z,even}_{N^2+i,m} &:= \rho_m \sinh(\rho_m) q_i^{0,m} \\ S^{z,odd}_{N^2+i,m} &:= \rho_m \cosh(\rho_m) q_i^{0,m} \end{aligned} \tag{4.18}$$

Now we have square matrices $S^{z,even}$ and $S^{z,odd}$ of size $N(N+1)$. So we have a non-trivial solution $\mathbf{q}^z \in D(B)$ of (4.14) and thus also (IE) if and only if $\det(S^{z,even}) = 0$ or $\det(S^{z,odd}) = 0$

Theorem 4.1.3. *Suppose* $\det(P^\lambda(\rho))$ *has* $2N(N+1)$ *distinct roots and* $\lambda \notin \mathcal{S}$ *for some* $\lambda \in \mathbb{C}$ *then we have that* $\lambda \in \sigma_p(A)$ *if and only if* $\det(S^{\lambda,even})\det(S^{\lambda,odd}) = 0$. *The eigenvalue* λ *is called even if* $\det(S^{\lambda,even}) = 0$ *and odd if* $\det(S^{\lambda,odd}) = 0$.

The corresponding eigenvector $\psi^\lambda \in X$ *for even eigenvalues is given by*

$$\psi^\lambda(\theta)(x) := e^{\lambda\theta} \sum_{m=1}^{N(N+1)} a_m \cosh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda) \tag{4.19}$$

Where $\mathbf{a}$ *is a vector in the nullspace of* $S^{\lambda,even}$. *For every* $\theta \in [-h, 0]$, ψ^λ *is an even function in* x.

The corresponding eigenvector $\psi^\lambda \in X$ *for odd eigenvalues is given by*

$$\psi^\lambda(\theta)(x) := e^{\lambda\theta} \sum_{m=1}^{N(N+1)} b_m \sinh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda) \tag{4.20}$$

Where $\mathbf{b}$ *is a vector in the nullspace of* $S^{\lambda,odd}$. *For every* $\theta \in [-h, 0]$, ψ^λ *is an odd function in* x

Proof. By theorem (4.0.1) we have that $\lambda \in \sigma_P(A)$ if and only if there a non-trivial solution $q \in D(B)$ of (IE). Suppose there exists such solution q for a certain $\lambda \notin \mathcal{S}$ for which $\det(P^\lambda(\rho))$ has $N(N+1)$ distinct roots. Then by theorem (4.1.1) $q \in C^\infty$ so it is also a solution of (DE). This limits q to the form of equation (4.10). Finally by the computations above we have shown that is a necessary condition that $\det(S^{\lambda,even})\det(S^{\lambda,odd}) = 0$ for a non-trivial q to solve (IE) and satisfy the boundary conditions $q \in D(B)$. Conversely if we have that $\det(S^{\lambda,even})\det(S^{\lambda,odd}) = 0$ then by the above calculations we can construct a solution $q \in D(B)$ of the form of (4.10) which solves (IE). □

4.2 Resolvent

We want to find a solution of the resolvent problem, when $z \in \rho(A)$:

$$(A - z)\phi = \psi \tag{4.21}$$

Where $\phi \in D(A)$ and $\psi \in X$. Due to theorem (4.0.1), we can reduce this resolvent problem for A to a resolvent problem for $B + K^z$.

$$(B - z + K^z)\mathbf{q} = \mathbf{y} \tag{RE}$$

Here we have that $\mathbf{q} \in D(B)$ and $\mathbf{y} \in Y$.

Lemma 4.2.1. *(Fredholm Alternative) For any given* $\mathbf{y} \in Y$ *and* $z \in \rho(A)$ *there exists a unique* $\mathbf{q} \in D(B)$ *which solves* (RE).

The resolvent operator $R(z, B + K^z) : Y \to D(B)$ is defined as $R(z, B + K^z)\mathbf{y} = \mathbf{q}$, with $\mathbf{q}$ the unique solution to (RE). We construct a solution $\mathbf{q} = \mathbf{q}^z$ to (RE) using a variation-of-constants Ansatz and substituting it into (RE). But first we introduce the matrix $\hat{Q}(z) \in \mathbb{C}^{N(N+1)\times N(N+1)}$

$$\hat{Q}_{i+(j-1)N,m} := \begin{cases} \dfrac{1}{k_{i,j}^2 - \rho_m^2} q_j^{0,m} & \text{for } j \in \{1, \cdots, N\} \\ q_i^{0,m} & \text{for } j = N+1 \end{cases} \tag{4.22}$$

We define our exception set

$$\mathcal{L} := \sigma(B) \cup \mathcal{S} \cup \{z \in \mathbb{C} | \det(P^z(\rho)) \text{ has less than } 2N(N+1) \text{ distinct zeros}\} \cup \{z \in \mathbb{C} | \det(\hat{Q}(z)) = 0\} \tag{4.23}$$

With $\mathcal{S}$ as in (4.11).

Theorem 4.2.2. *For $z \in \rho(A)$ with $z \notin \mathcal{L}$ the unique solution $q := q^z \in D(B)$ of (RE) is given by*

$$\mathbf{q}^z(x) := R(z, B)\mathbf{y}(x) + \sum_{m=1}^{N(N+1)} [a_m(x) \cosh(\rho_m(z)x) + b_m(x) \sinh(\rho_m(z)x)]\, \mathbf{q}^{0,m}(z))$$

Where $R(z, B)$ is the resolvent operator of B as in (3.5) and $\mathbf{a}(x)$ and $\mathbf{b}(x)$ as in (4.38)

Proof. For this proof we suppress the dependencies on z. Our variation-of-constants Ansatz $\mathbf{q}^z$ needs to satisfy 3 conditions. It must solve (RE), $(B - z + K^z)\mathbf{q}^z = \mathbf{y}$, it must satisfy the boundary conditions $(\mathbf{q}^z)'(\pm 1) = 0$ and the regularity condition $\mathbf{q}^z \in C^2(\Omega)$. As $R(z, B)$ maps into $D(B)$, it suffices that $\mathbf{a}(x), \mathbf{b}(x) \in C^2(\Omega)$

To aid in the calculation of $(B - z + K^z)\mathbf{q}^z$, we first compute some integrals up front. We can integrate by parts by splitting the interval $[-1, 1]$ into $[-1, x)$ and $(x, 1]$ and using the anti-derivatives in (4.13) to end up at

$$\begin{aligned}
K^z_{i,j} a_m(x) \cosh(\rho_m x) &= a_m(x) \cosh(\rho_m x) \frac{2c_{i,j} k_{i,j}}{k^2_{i,j} - \rho^2_m} \\
&+ c_{i,j} e^{-k_{i,j}(1+x)} a_m(-1) S^{z,even}_{i+(j-1)N,m} + c_{i,j} e^{-k_{i,j}(1-x)} a_m(1) S^{z,even}_{i+(j-1)N,m} \\
&+ c_{i,j} \int_{-1}^{1} \frac{a'_m(x)}{k^2_{i,j} - \rho^2_m} e^{-k_{i,j}|x-x'|} (\operatorname{sgn}(x - x') k_{i,j} \cosh(\rho_m x') - \rho_m \sinh(\rho_m x'))\, dx' \\
K^z_{i,j} b_m(x) \sinh(\rho_m x) &= b_m(x) \sinh(\rho_m x) \frac{2c_{i,j} k_{i,j}}{k^2_{i,j} - \rho^2_m} \\
&- c_{i,j} e^{-k_{i,j}(1+x)} b_m(-1) S^{z,odd}_{i+(j-1)N,m} + c_{i,j} e^{-k_{i,j}(1-x)} b_m(1) S^{z,odd}_{i+(j-1)N,m} \\
&+ c_{i,j} \int_{-1}^{1} \frac{b'_m(x)}{k^2_{i,j} - \rho^2_m} e^{-k_{i,j}|x-x'|} (\operatorname{sgn}(x - x') k_{i,j} \sinh(\rho_m x') - \rho_m \cosh(\rho_m x'))\, dx'
\end{aligned} \tag{4.24}$$

Now we substitute $\mathbf{q}^z$ into (RE) and collect the terms. Using the above calculations and the fact that $(B - z)R(z, B)\mathbf{y} = \mathbf{y}$, we get for $i \in \{1, \cdots, N\}$

$$\begin{aligned}
&\sum_{m=1}^{N(N+1)} d_i \left[(a''_m(x) + 2b'_m(x)\rho_m) \cosh(\rho_m x) + (b''_m(x) + 2a'_m(x)\rho_m) \sinh(\rho_m x) \right] q^{0,m}_i \\
&+ \sum_{m=1}^{N(N+1)} [a_m(x) \cosh(\rho_m x) + b_m(x) \sinh(\rho_m x)] \left[(d_i \rho^2_m(z) - \alpha_i - z) q^{0,m}_i(z) + \sum_{j=1}^{N} \frac{2c_{i,j} k_{i,j}}{k^2_{i,j} - \rho^2_m} q^{0,m}_j \right] \\
&+ \sum_{j=1}^{N} c_{i,j} e^{-k_{i,j}(1+x)} \left[\sum_{m=1}^{N(N+1)} a_m(-1) S^{z,even}_{i+(j-1)N,m} - \sum_{m=1}^{N(N+1)} b_m(-1) S^{z,odd}_{i+(j-1)N,m} \right] \\
&+ \sum_{j=1}^{N} c_{i,j} e^{-k_{i,j}(1-x)} \left[\sum_{m=1}^{N(N+1)} a_m(1) S^{z,even}_{i+(j-1)N,m} + \sum_{m=1}^{N(N+1)} b_m(1) S^{z,odd}_{i+(j-1)N,m} \right] \\
&+ \sum_{j=1}^{N} c_{i,j} \int_{-1}^{1} e^{-k_{i,j}|x-x'|} \Big[R(z, B_{i,i}) y_i(x) \\
&\qquad + \sum_{m=1}^{N(N+1)} \frac{a'_m(x)}{k^2_{i,j} - \rho^2_m} q^{0,m}_j (\operatorname{sgn}(x - x') k_{i,j} \cosh(\rho_m x') - \rho_m \sinh(\rho_m x')) \\
&\qquad + \sum_{m=1}^{N(N+1)} \frac{b'_m(x)}{k^2_{i,j} - \rho^2_m} q^{0,m}_j (\operatorname{sgn}(x - x') k_{i,j} \sinh(\rho_m x') - \rho_m \cosh(\rho_m x')) \Big]\, dx' = 0
\end{aligned} \tag{4.25}$$

The second line vanishes naturally due to characteristic equation in (4.12). We seek functions $\mathbf{a}(x), \mathbf{b}(x)$ such that all the functions inside large square brackets in (4.25) vanish.

As $R(z,B)$ maps into $D(B)$, the boundary conditions on $\mathbf{q}^z$, $(\mathbf{q}^z)'(\pm 1) = \mathbf{0}$, reduce for $i \in \{1, \cdots, N\}$ to

$$\sum_{m=1}^{N(N+1)} \left[(a_m'(\pm 1) + b_m(\pm 1)\rho_m)\cosh(\rho_m) \pm (b_m'(\pm 1) + \rho_m a_m(\pm 1))\sinh(\rho_m) \right] q_i^{0,m} = 0 \tag{4.26}$$

We can split the above into 3 equations (4.27). Then (4.26) vanishes, when all three equations vanish in (4.27).

$$\begin{aligned} \sum_{m=1}^{N(N+1)} \left[a_m'(\pm 1)\cosh(\rho_m) \pm b_m'(\pm 1)\sinh(\rho_m) \right] q_i^{0,m} = 0 \\ \sum_{m=1}^{N(N+1)} \left[b_m(1)\rho_m \cosh(\rho_m) + \rho_m a_m(1)\sinh(\rho_m) \right] q_i^{0,m} = 0 \\ \sum_{m=1}^{N(N+1)} \left[b_m(-1)\rho_m \cosh(\rho_m) - \rho_m a_m(-1)\sinh(\rho_m) \right] q_i^{0,m} = 0 \end{aligned} \tag{4.27}$$

Note that the last 2 lines of (4.27) are equivalent to

$$\begin{aligned} \sum_{m=1}^{N(N+1)} a_m(-1) S^{z,even}_{N^2+i,m} - \sum_{m=1}^{N(N+1)} b_m(-1) S^{z,odd}_{N^2+i,m} = 0 \\ \sum_{m=1}^{N(N+1)} a_m(1) S^{z,even}_{N^2+i,m} + \sum_{m=1}^{N(N+1)} b_m(1) S^{z,odd}_{N^2+i,m} = 0 \end{aligned} \tag{4.28}$$

If we combine the equations (4.28) with the third and fourth line in (4.25) this is equivalent to the following equations (4.29)

$$\begin{aligned} S^{z,even}\mathbf{a}(-1) - S^{z,odd}\mathbf{b}(-1) = \mathbf{0} \\ S^{z,even}\mathbf{a}(1) + S^{z,odd}\mathbf{b}(1) = \mathbf{0} \end{aligned} \tag{4.29}$$

We can split the first line in (4.25) into two equations (4.30). When these vanish, then so does the first line of (4.25).

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{m=1}^{N(N+1)} \left[a_m'(x)\cosh(\rho_m x) + b_m'(x)\sinh(\rho_m x) \right] q_i^{0,m} = 0 \\ \sum_{m=1}^{N(N+1)} \left[b_m'(x)\rho_m \cosh(\rho_m x) + a_m'(x)\rho_m \sinh(\rho_m x) \right] q_i^{0,m} = 0 \end{aligned} \tag{4.30}$$

We see that in the first line of (4.30) the sum should be constant. Using the first line of (4.27) we see that this constant is zero

$$\sum_{m=1}^{N(N+1)} \left[a_m'(x)\cosh(\rho_m x) + b_m'(x)\sinh(\rho_m x) \right] q_i^{0,m} = 0 \tag{4.31}$$

The last line of (4.25) vanishes when the following conditions hold for $i \in \{1, \cdots, N\}$

$$\begin{aligned} \sum_{m=1}^{N(N+1)} \left[\frac{a_m'(x)}{k_{i,j}^2 - \rho_m^2} k_{i,j}\cosh(\rho_m x') + \frac{b_m'(x)}{k_{i,j}^2 - \rho_m^2} k_{i,j}\sinh(\rho_m x') \right] q_j^{0,m} = 0 \\ \sum_{m=1}^{N(N+1)} \left[\frac{a_m'(x)}{k_{i,j}^2 - \rho_m^2} \rho_m \sinh(\rho_m x') + \frac{b_m'(x)}{k_{i,j}^2 - \rho_m^2} \rho_m \cosh(\rho_m x') \right] q_j^{0,m} = R(z, B_{i,i}) y_i(x) \end{aligned} \tag{4.32}$$

Equations (4.32), (4.31) and the second line of (4.30) form a system of differential equations with boundary conditions (4.29). We can rewrite these equations by introducing some matrices.

We define the diagonal matrices $\hat{A}(x)$, $\hat{B}(x) \in C(\Omega, \mathbb{C}^{N(N+1)\times N(N+1)})$, the square matrices $\hat{K}$, $\hat{M}$,

$\hat{Q} \in \mathbb{C}^{N(N+1)\times N(N+1)}$ and the vector $\mathbf{r}(x) \in C(\Omega, \mathbb{C}^{N(N+1)})$ as follows

$$\begin{aligned}
\hat{A}_{m,m}(x) &= \cosh(\rho_m x) \\
\hat{B}_{m,m}(x) &= \sinh(\rho_m x) \\
\hat{K}_{i+(j-1)N,m} &= \rho_m \hat{Q}_{i+(j-1)N,m} \\
\hat{M}_{i+(j-1)N,m} &= k_{i,j} \hat{Q}_{i+(j-1)N,m} \\
\hat{Q}_{i+(j-1)N,m} &= \begin{cases} \dfrac{1}{k_{i,j}^2 - \rho_m^2} q_j^{0,m} & \text{for } j \in \{1,\cdots,N\} \\ q_i^{0,m} & \text{for } j = N+1 \end{cases} \\
r_{i+(j-1)N}(x) &= \begin{cases} R(z, B_{i,i}) y_i(x) & \text{for } j \in \{1,\cdots,N\} \\ 0 & \text{for } j = N+1 \end{cases}
\end{aligned} \tag{4.33}$$

Here $m \in \{1, \cdots, N(N+1)\}$, $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, N+1\}$ and we define $k_{i,N+1} := 1$.

We seek functions $\mathbf{a}$ and $\mathbf{b}$ which solve the system of differential equations

$$\begin{aligned}
\hat{M}(\hat{A}(x)\mathbf{a}'(x) + \hat{B}(x)\mathbf{b}'(x)) &= \mathbf{0} \\
\hat{K}(\hat{B}(x)\mathbf{a}'(x) + \hat{A}(x)\mathbf{b}'(x)) &= \mathbf{r}(x)\mathbf{v}
\end{aligned} \tag{4.34}$$

with boundary conditions

$$\begin{aligned}
S^{z,even}\mathbf{a}(-1) - S^{z,odd}\mathbf{b}(-1) &= \mathbf{0} \\
S^{z,even}\mathbf{a}(1) + S^{z,odd}\mathbf{b}(1) &= \mathbf{0}
\end{aligned} \tag{4.35}$$

For $z \in \rho(A)$ we have that $S^{z,odd}$ and $S^{z,even}$ are invertible. We can write the determinant of $\hat{K}$ and $\hat{M}$ in terms of the determinant of $\hat{Q}$, $|\hat{K}| = |\hat{Q}| \prod_{i,j=1}^{N} k_{i,j}$ and $|\hat{M}| = |\hat{Q}| \prod_{m=1}^{N(N+1)} \rho_m$. Due to lemma (4.1.2), $k_{i,j} \neq 0$ and $\rho_m \neq 0$. Hence $\hat{M}$ and $\hat{K}$ are invertible if and only if $\hat{Q}$ is invertible. We assumed the invertibility of $\hat{Q}$ by taking $z \notin \mathcal{L}$.

Now we multiply the first line of (4.34) by $\hat{A}(x)\hat{M}^{-1}$ and second line by $\hat{B}(x)\hat{K}^{-1}$

$$\begin{aligned}
\hat{A}(x)^2\mathbf{a}'(x) + \hat{A}(x)\hat{B}(x)\mathbf{b}'(x) &= \mathbf{0} \\
\hat{B}(x)^2\mathbf{a}'(x) + \hat{A}(x)\hat{B}(x)\mathbf{b}'(x) &= \hat{B}(x)\hat{K}^{-1}\mathbf{r}(x)
\end{aligned} \tag{4.36}$$

If we now subtract these equations and use the trigonometric identity $\hat{A}(x)^2 - \hat{B}(x)^2 = I$, we arrive at the following equation

$$\begin{aligned}
\mathbf{a}'(x) &= -\hat{B}(x)\hat{K}^{-1}\mathbf{r}(x) \\
\mathbf{b}'(x) &= \quad \hat{A}(x)\hat{K}^{-1}\mathbf{r}(x)
\end{aligned} \tag{4.37}$$

Here we get the second line by a similar procedure. We note that $\mathbf{r}(x) \in C^2(\Omega)$ and $A(x), B(x) \in C^\infty(\Omega)$, which implies that $\mathbf{a}(x), \mathbf{b}(x) \in C^3(\Omega)$. Hence we satisfy the regularity condition.

We can now find $\mathbf{a}(x)$ and $\mathbf{b}(x)$ by taking an anti-derivative plus some constants of integration, $\mathbf{a}^c$ and $\mathbf{b}^c$. To satisfy the boundary equations (4.35), we take an anti-derivative such that $\mathbf{a}(-1)+\mathbf{a}(1) = \mathbf{a}^c$ and $\mathbf{b}(-1) + \mathbf{b}(1) = \mathbf{b}^c$.

$$\begin{aligned}
\mathbf{a}(x) &= \mathbf{a}^c - \frac{1}{2}\left(\int_{-1}^{x} \hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' - \int_{x}^{1} \hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \\
\mathbf{b}(x) &= \mathbf{b}^c + \frac{1}{2}\left(\int_{-1}^{x} \hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' - \int_{x}^{1} \hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right)
\end{aligned} \tag{4.38}$$

By adding and subtracting the boundary equations (4.35) we find that the constants of integration equal

$$\begin{aligned}
\mathbf{a}^c &= -(S^{z,even})^{-1}S^{z,odd}\left(\int_{-1}^{1} \hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \\
\mathbf{b}^c &= \quad (S^{z,odd})^{-1}S^{z,even}\left(\int_{-1}^{1} \hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \qquad \square
\end{aligned} \tag{4.39}$$

Chapter 5

Normal Forms on the Center manifold

In this chapter we will investigate an Andronov-Hopf and a Pitchfork-Hopf-bifurcation and deduce formulas for the normal form coefficients on the center manifold.

When we have a pair of eigenvalues on the imaginary axis, then generically we have an Andronov-Hopf-bifurcation[1]. In this bifurcation a limit cycle appears when an equilibrium changes stability. The bifurcation is called supercritical if the limit cycle is stable and subcritical if it is unstable.

Before we discuss Pitchfork bifurcations, we first observe some symmetries of our problem. For our choices of J, τ and $\mathbf{S}$, the (ADDE) has two $\mathbb{Z}_2$ symmetries generated by the linear involutions $\kappa_1 : Y \to Y$ and $\kappa_2 : Y \to Y$ defined by

$$(\kappa_1 \mathbf{y})(x) = \mathbf{y}(-x), \qquad (\kappa_1 \mathbf{y})(x) = -\mathbf{y}(-x) \tag{5.1}$$

for all $\mathbf{y} \in Y$ and $x \in \Omega$. The fixed subspaces of κ_1 and κ_2 are composed of the even and odd functions on Ω respectively. Due to these symmetries, all zero eigenvalues correspond to Pitchfork bifurcations (Kuznetsov, 2013, Theorem 7.7). In this bifurcation two equilibria with the same stability type appear, when an equilibrium changes stability. The bifurcation is called supercritical if the additional equilibria are stable and subcritical if they are unstable.

We will investigate the case of the Andronov-Hopf bifurcation and the Pitchfork-Hopf bifurcation, when such a Pitchfork bifurcation coincides with an Andronov-Hopf-bifurcation in which we have a pair of purely imaginary eigenvalues. We will follow the reasoning in Dijkstra *et al.* (2015) and make adjustments where necessary.

5.1 The Critical Center Manifold for Andronov-Hopf

Suppose that $\sigma(A)$ contains a pair of simple purely imaginary eigenvalues $\lambda = \pm i\omega$ with $\omega > 0$ and no other eigenvalues on the imaginary axis. Let $\psi \in X$ be the corresponding eigenvector of A and $\psi^{\odot} \in X^{\odot}$ be the corresponding eigenvectors of A^* respectively,

$$A\psi = i\omega\psi, \qquad A^*\psi^{\odot} = i\omega\psi^{\odot} \tag{5.2}$$

We impose a 'bi-orthogonality' condition on these vectors by scaling them such that

$$\langle \psi^{\odot}, \psi \rangle = 1 \tag{5.3}$$

The center subspace X_0 is spanned by the basis $\Psi = \{\psi, \bar{\psi}\}$ of eigenvectors corresponding to the critical eigenvalues of A. If $\zeta \in X_0$ then $\zeta = z\psi + \bar{z}\bar{\psi}$ for some $z \in \mathbb{C}$.

Due to theorem (2.1.7) the (ADDE) and (AIE) formulations are equivalent. The (AIE) is a variation-of-constants formula in the state space X, *for which we assume we can construct a locally invariant critical center manifold* $\mathcal{W}^c_{loc} \subset X$. The critical center manifold has the formal expansion

$$\mathcal{H}(z, \bar{z}) = z\psi + \bar{z}\bar{\psi} + \sum_{j+k\geq 2} \frac{1}{j!k!} h_{jk} z^j \bar{z}^k \tag{5.4}$$

By weak* differentiation of (AIE) and exploiting the finite dimensionality of $\mathcal{W}^c_{loc}$, one can show that a solution $\mathbf{u}$ of (AIE) satisfies the abstract ODE

$$\dot{\mathbf{u}}_t = j^{-1}(A^{\odot *} j\mathbf{u}_t + R(\mathbf{u}_t)) \tag{5.5}$$

[1] Also known as just a Hopf-bifurcation

Where the non-linearity $R : X \to X^{\odot *}$ is given by

$$R(\phi) = l(G(\phi) - DG(0)(\phi)) = lD^2G(0)(\phi,\phi) + lD^3G(0)(\phi,\phi,\phi) + \mathcal{O}(|\phi|^4) \tag{5.6}$$

Recall that the Fréchet derivatives of G are given by (1.1.1). Let $\zeta(t)$ be the projection of $\mathbf{v}(t)$ onto the center subspace X_0. This ODE is smoothly equivalent to the Poincaré normal form

$$\dot{z} = i\omega z + g_{21} z|z|^2 + \mathcal{O}(|z,\bar{z}|^4) \tag{5.7}$$

Where $z, g_{21} \in \mathbb{C}$. By substituting $z = re^{i\theta}$ we get rewrite (5.7) as

$$\begin{cases} \dot{r} &= \ell_1 r^3 + \mathcal{O}(|r|^4) \\ \dot{\theta} &= \omega + \mathcal{O}(|r|^2) \end{cases} \tag{5.8}$$

Here ℓ_1 is the first Lyapunov coefficient determined by the formula

$$\ell_1 = \frac{1}{\omega}\mathrm{Re}(g_{21}) \tag{5.9}$$

It is well known, see for instance Kuznetsov (2013), that in generic unfoldings of (5.7) $\ell_1 < 0$ implies a supercritical bifurcation of a limit cycle, while ℓ_1 implies a subcritical bifurcation of a limit cycle.

5.2 Normal Form coefficients for Andronov-Hopf

The critical center manifold $\mathcal{W}^c_{loc}$ has the expansion (5.4) and due to the time-invariance of $\mathcal{W}^c_{loc}$ we have

$$\mathcal{H}(z(t),\bar{z}(t)) = \mathbf{u}(t) \tag{5.10}$$

If we differentiate both sides with respect to time and use (5.5) we arrive at the homological equation

$$A^{\odot *} j\mathcal{H}(z,\bar{z}) + R(\mathcal{H}(z,\bar{z})) = \mathcal{H}_z(z,\bar{z})\dot{z} + \mathcal{H}_{\bar{z}}(z,\bar{z})\dot{\bar{z}} \tag{5.11}$$

We can substitute in the expansion of the non-linearity (5.6), the normal form (5.7) and the expansion of the critical center manifold (5.4) into the homological equation (5.11) to derive the normal form coefficients. If we equate coefficients of the corresponding powers of z and $\bar{z}$ we get the following equations

$$\begin{aligned} A^{\odot *} jh_{20} &= -lD^2G(0)(\psi,\bar{\psi}) \\ (A^{\odot *} - 2i\omega)jh_{11} &= lD^2G(0)(\psi,\psi) \\ (A^{\odot *} - i\omega)jh_{21} &= lD^3G(0)(\psi,\psi,\bar{\psi}) + lD^2G(0)(\bar{\psi},h_{20}) + 2lD^2G(0)(\psi,h_{11}) - 2g_{21}j\psi \end{aligned} \tag{5.12}$$

These are all equations of the form

$$(A^{\odot *} - \lambda)\phi^{\odot *} = \psi^{\odot *}$$

Here $\lambda \in \mathbb{C}$ and $\psi^{\odot *} \in X^{\odot *}$ are given. When $\lambda \in \rho(A)$ then this has a unique solution. However if $\lambda \in \sigma(A)$ then a solution $\phi^{\odot *}$ doesn't necessarily exist for all $\psi^{\odot *}$. The following lemma, which is equivalent to van Gils *et al.* (2013, Lemma 33), provides a condition for solvability.

Lemma 5.2.1. *(Fredholm solvability) Let $\lambda \notin \sigma(B)$ and suppose that K^λ is bounded. Then $A^\odot - \lambda : D(A^\odot) \to X^\odot$ has closed range. In particular $(A^{\odot *} - \lambda)\psi^{\odot *} = \phi^{\odot *}$ is solvable for $\phi^{\odot *} \in D(A^{\odot *})$ given $\psi \in X^{\odot *}$ if and only if $\langle \psi^{\odot *}, \phi^{\odot} \rangle = 0$ for all $\phi^\odot \in \mathcal{N}(A^* - \lambda)$.*

Proof. Due to theorem (4.0.1), we have that $\phi \in \mathcal{R}(A - z)$ if and only if $W^z\phi \in \mathcal{R}(B - z + K^z)$. As for $z \in \rho(B)$, $\mathcal{R}(B - z)$ is closed. We have by Kato (2013, Theorem IV.1.1) that $\mathcal{R}(B - z + K^z)$ is closed. Now let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}(A - z)$ converging to some $\phi \in X$. Then the sequence $(W^z\phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{R}(B - z + K^z)$ as $\mathcal{R}(B - z + K^z)$ is closed, hence $\phi \in \mathcal{R}(A - z)$. This implies that $\mathcal{R}(A - z)$ is closed and $\mathcal{R}(A^* - \lambda)$ too. Then the rest of the proof immediately carries over from van Gils *et al.* (2013, Lemma 33). □

As $0, 2i\omega \in \rho(A)$ we can just use the resolvent, however $i\omega \in \sigma(A)$. The null-space $\mathcal{N}(A^* - \lambda)$ for λ is spanned by ψ. Hence we can solve for the normal form coefficient

$$\begin{aligned} jh_{20} &= -R(0, A^{\odot *})lD^2G(0)(\psi,\bar{\psi}) \\ jh_{11} &= R(2i\omega, A^{\odot *})lD^2G(0)(\psi,\psi) \\ g_{21} &= \frac{1}{2}\langle lD^3G(0)(\psi,\psi,\bar{\psi}) + lD^2G(0)(\bar{\psi},h_{20}) + 2lD^2G(0)(\psi,h_{11}), \psi^{\odot} \rangle \end{aligned} \tag{5.13}$$

Due to our choice of $\mathbf{S}$ as an odd function, we get that $D^2G(0) \equiv 0$. Hence $h_{20} = h_{11} = 0$ and

$$g_{21} = \frac{1}{2}\langle l\mathbf{y}_{21}, \psi^{\odot}\rangle \tag{5.14}$$

Where $\mathbf{y}_{21}$ is defined as

$$\mathbf{y}_{21} = D^3G(0)(\psi_2, \psi_2, \bar{\psi}_2) \tag{5.15}$$

We are not yet able to compute the normal form coefficient as we don't have an explicit representation of $\psi^{\odot}$.

Now suppose that $\lambda \in \sigma_p(A)$ is a simple eigenvalue with corresponding eigenvector $\psi \in D(A)$. Furthermore, let $\psi^{\odot} \in D(A^*)$ be the eigenvector of A^* corresponding to λ such that, without loss of generality, $\langle \psi^{\odot}, \psi\rangle = 1$. Let $P^{\odot}$ and $P^{\odot *}$ be the spectral projections on $X^{\odot}$ and $X^{\odot *}$, respectively. Then $P^{\odot *}\phi^{\odot *} = \nu j\psi$ for some $\nu \in \mathbb{C}$ and

$$\langle \phi^{\odot *}, \psi^{\odot}\rangle = \langle \phi^{\odot *}, P^{\odot}\psi^{\odot}\rangle = \langle P^{\odot *}\phi^{\odot *}, \psi^{\odot}\rangle = \nu\langle j\psi, \psi^{\odot}\rangle = \nu$$

Hence we seek to determine ν. From the Dunford integral representation follows that

$$P^{\odot *}\phi^{\odot *} = \frac{1}{2\pi i}\oint_{\partial C_\lambda} R(z, A^{\odot *})\phi^{\odot *}\mathrm{d}z = \nu j\psi \tag{5.16}$$

Where C_λ is a sufficiently small open disk centered at λ and ∂C_λ its boundary.

The element on the left in the pairing (5.14) is in the range of l, $Y^{\odot *} \times \{0\} \subset X^{\odot *}$. For $\phi^{\odot *} = l\mathbf{y}$, we can reduce $R(z, A^{\odot *})\phi^{\odot *}$ to $R(z, B + K^z)\mathbf{y}$ due to the following lemma, which is an adaptation of van Gils *et al.* (2013, lemma 36).

Lemma 5.2.2. *Suppose that $\lambda \in \rho(A)$ and $\lambda \notin \mathcal{L}$. For each $\mathbf{y} \in Y$ the function $\phi \in X$, defined as $\phi(\theta) = e^{\lambda\theta}R(\lambda, B + K^\lambda)\mathbf{y}$ for $\theta \in [-h, 0]$, is the unique solution in $\{\phi \in C^1([-h, 0]; Y) | \phi(0) \in D(B)\}$ of the system*

$$\begin{cases} (B - \lambda)\phi(0) + DG(0)\phi = \mathbf{y} \\ \dot{\phi} - \lambda\phi = 0 \end{cases} \tag{5.17}$$

Moreover, $\phi^{\odot *} = j\phi$ is the unique solution in $D(A^{\odot *})$ of $(A^{\odot *} - \lambda)\phi^{\odot *} = l\mathbf{y}$.

Proof. Since $\lambda \in \rho(A)$, by theorem (4.2.2) it follows that $R(\lambda, B + K^\lambda)$ exists. We start by showing that ϕ as defined above solves (5.17). Clearly $\phi \in C^1([-h, 0]; Y)$ and $\phi(0) = R(\lambda, B + K^\lambda)\mathbf{y} \in D(B)$. Moreover, ϕ satisfies the second equation in (5.17). Recall from the definition of K^z that for $\mathbf{q} \in Y$, $DG(0)\mathbf{q}e^{\lambda\theta} = K^\lambda\mathbf{q}$. Therefore,

$$(B - \lambda)\phi(0) + DG(0)\phi = (B - \lambda)R(\lambda, B + K^\lambda)\mathbf{y} + K^\lambda R(\lambda, B + K^\lambda)\mathbf{y} = \mathbf{y}$$

Lemma (2.2.4) implies that $j\phi \in D(A^{\odot *})$ and

$$(A^{\odot *} - \lambda)\phi^{\odot *} = (j_Y(B - \lambda)\phi(0), -\lambda\phi) + (j_Y DG(0)\phi, \dot{\phi}) = (j_Y\mathbf{y}, 0) = l\mathbf{y}$$

But $\sigma(A^{\odot *}) = \sigma(A)$, so $\phi^{\odot *} = j\phi$ is the unique solution of $(A^{\odot *} - \lambda)\phi^{\odot *} = l\mathbf{y}$. Consequently, ϕ itself is the unique solution in $\{\phi \in C^1([-h, 0]; Y) | \phi(0) \in D(B)\}$. ☐

Now we are able to state our final result for the calculation of ν.

Theorem 5.2.3. *Let $\lambda \in \sigma_p(A)$ be a simple eigenvalue such that there exists a sufficiently small closed disk C_λ such that $\mathcal{L} \cap C_\lambda = \emptyset$ and $C_\lambda \cap \sigma(A) = \{\lambda\}$.*

If λ is an even eigenvalue such that

$$\psi(0)(x) = \sum_{m=1}^{N(N+1)} a_m \cosh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda) \tag{5.18}$$

for all $x \in \Omega$, where $\mathbf{a}$ is a non-trivial solution of $S^{\lambda,even}\mathbf{a} = 0$. Then the formula $P^{\odot *}l\mathbf{y} = \nu j\psi$ is equivalent to

$$\frac{-\mathrm{adj}(S^{\lambda,even})}{\frac{d}{dz}(\det(S^{z,even}))|_{z=\lambda}} S^{z,odd}\int_{-1}^{1} \hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' = \nu\mathbf{a} \tag{5.19}$$

For all $y \in Y$, where $\mathrm{adj}(S^{\lambda,even})$ denotes the adjugate of $S^{\lambda,even}$ and using the definitions in (4.33).

If λ is an odd eigenvalue such that

$$\psi(0)(x) = \sum_{m=1}^{N(N+1)} b_m \sinh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda) \tag{5.20}$$

for all $x \in \Omega$, where $\mathbf{b}$ is a non-trivial solution of $S^{\lambda,odd}\mathbf{b} = 0$. Then the formula $P^{\odot*}l\mathbf{y} = \nu j\psi$ is equivalent to

$$\frac{\mathrm{adj}(S^{\lambda,odd})}{\frac{d}{dz}(\det(S^{z,odd})|_{z=\lambda}} S^{z,even} \int_{-1}^{1} \hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' = \nu\mathbf{b} \tag{5.21}$$

For all $\mathbf{y} \in Y$, where $\mathrm{adj}(S^{\lambda,odd})$ denotes the adjugate of $S^{\lambda,odd}$ and using the definitions in (4.33).

Proof. Suppose λ is an even eigenvalue. As $\mathcal{L} \cap \partial C_\lambda = \sigma(A) \cap \partial C_\lambda = \emptyset$, lemma (5.17) states that $P^{\odot*}l\mathbf{y} = \nu j\psi$ is equivalent to

$$\frac{1}{2\pi i}\oint_{\partial C_\lambda} R(z, B + K^z)\mathbf{y}\mathrm{d}z = \nu\psi(0) \tag{5.22}$$

As $\mathcal{L} \cap C_\lambda = \emptyset$ and $\sigma(A) \cap C_\lambda = \{\lambda\}$, we have that the $R(z, B + K^z)\mathbf{y}$ is given by theorem (4.2.2). We observe that all components of the resolvent are analytic for all $z \in C_\lambda$ expect for the constants of integration $\mathbf{a}^c(z)$. This analyticity simplifies (5.22) to

$$\frac{1}{2\pi i}\sum_{m=1}^{N(N+1)} \cosh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda)\oint_{\partial C_\lambda}\mathbf{a}^c(z)\mathrm{d}z = \nu\sum_{m=1}^{N(N+1)} a_m\cosh(\rho_m(\lambda)x)\mathbf{q}^{0,m}(\lambda)$$

for all $x \in \Omega$. We can substitute (4.39) and use the residue formula

$$\frac{1}{2\pi i}\oint_{\partial C_\lambda}(S^{z,even})^{-1}\mathrm{d}z = \mathrm{Res}\left(\frac{\mathrm{adj}(S^{z,even})}{\det(S^{z,even})}, \lambda\right) = \frac{\mathrm{adj}(S^{\lambda,even})}{\frac{d}{dz}(\det(S^{z,even}))|_{z=\lambda}}$$

This results in the formula

$$\frac{-\mathrm{adj}(S^{\lambda,even})}{\frac{d}{dz}(\det(S^{z,even}))|_{z=\lambda}} S^{z,odd}\int_{-1}^{1}\hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' = \nu\mathbf{a}$$

The reasoning for odd eigenvalues is similar. □

The condition that $\mathcal{L} \cap C_\lambda = \emptyset$ and $C_\lambda \cap \sigma(A) = \{\lambda\}$ is not very restrictive, as the eigenvalues in $\sigma(A)$ and $\sigma(B)$ are isolated and $\rho_m(z), \det(P^z(k_{i,j}(z)))$ and $\det(\hat{Q}(z))$ are smooth in z. Hence $\lambda \notin \mathcal{L}$ is a sufficient condition that such a C_λ exists.

5.3 The Critical Center Manifold for Pitchfork-Hopf

Suppose that $\sigma(A)$ contains a simple zero eigenvalue and a pair of simple purely imaginary eigenvalues.

$$\lambda_1 = 0, \qquad \lambda_2 = \pm i\omega$$

with $\omega > 0$ and no other eigenvalues lie on the imaginary axis. Let $\psi_1, \psi_2 \in X$ be the corresponding eigenvectors of A and $\psi_1^\odot, \psi_2^\odot \in X^\odot$ be the corresponding eigenvectors of A^* such that

$$A\psi_1 = 0, \qquad A\psi_2 = i\omega\psi_2, \qquad A^*\psi_1^\odot = 0, \qquad A^*\psi_2^\odot = i\omega\psi_2^\odot \tag{5.23}$$

We impose a 'bi-orthogonality' condition on these vectors by scaling them such that for $i, j \in \{1, 2\}$

$$\langle\psi_i^\odot, \psi_j\rangle = \delta_{i,j} \tag{5.24}$$

The center subspace X_0 is spanned by the basis $\Psi = \{\psi_1, \psi_2, \bar{\psi}_2\}$ of eigenvectors corresponding to the critical eigenvalues of A. If $\zeta \in X_0$ then $\zeta = s\psi_1 + z\psi_2 + \bar{z}\bar{\psi}_2$ for some $s \in \mathbb{R}$ and $z \in \mathbb{C}$.

Due to theorem (2.1.7) the (ADDE) and (AIE) formulations are equivalent. The (AIE) is a variation-of-constants formula in the state space X, for which we assume we can construct a locally invariant critical center manifold $\mathcal{W}^c_{loc} \subset X$. The critical center manifold has the formal expansion

$$\mathcal{H}(s, z, \bar{z}) = s\psi_1 + z\psi_2 + \bar{z}\bar{\psi}_2 + \sum_{i+j+k\geq 2}\frac{1}{i!j!k!}h_{ijk}s^i z^j \bar{z}^k \tag{5.25}$$

By weak* differentiation of (AIE) and exploiting the finite dimensionality of $\mathcal{W}^c_{loc}$, one can show that a solution $\mathbf{u}$ of (AIE) satisfies the abstract ODE

$$\dot{\mathbf{u}}(t) = j^{-1}(A^{\odot*}j\mathbf{u} + R(\mathbf{u}(t)) \tag{5.26}$$

Table 5.1: Different unfoldings of (AE) (Guckenheimer and Holmes, 1983, Table 7.5.2)

Case	Ia	Ib	II	III	IVa	IVb	V	VIa	VIb	VIIa	VIIb	VIII
d	+1	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1
b	+	+	+	-	-	-	+	+	+	-	-	-
c	+	+	-	+	-	-	+	-	-	+	+	-
$1 - bc$	+	-	(+)	(+)	+	-	(-)	+	-	+	-	(-)

Table 5.2: Classification of fixed points (AE) (Dijkstra *et al.*, 2015, Table 3)

Amplitude equation solution	Neural field solution
Trivial fixed point $(0, 0)$	Background state
Mode one fixed point $(\bar{s}, 0)$	Non-trivial stationary state
Mode two fixed point $(0, \bar{r})$	Oscillation around the background state
Mixed mode fixed point $(\bar{s}, \bar{r})$	Oscillation around a non-trivial state

Where the non-linearity $R : X \to X^{\odot *}$ is given by (5.6). Let $\zeta(t)$ be the projection of $\mathbf{v}(t)$ onto the center subspace X_0. Both involutions κ_1 and κ_2 act on X_0 as reflections

$$(s, z) \to (-s, z)$$
$$(s, z) \to (z, -s)$$

If X_0 is symmetric under these reflections, the coordinates of $\zeta(t)$ satisfy an ODE which is equivalent to (5.26) (Kuznetsov, 2013, Theorem 7.6). By Guckenheimer and Holmes (1983), this ODE is smoothly equivalent to the Poincaré normal form

$$\begin{cases} \dot{s} &= g_{300}s^3 + g_{111}s|z|^2 + \mathcal{O}(|s, z, \bar{z}|^5) \\ \dot{z} &= i\omega z + g_{210}zs^2 + g_{021}z|z|^2 + \mathcal{O}(|s, z, \bar{z}|^5) \end{cases} \tag{5.27}$$

Where $s, g_{300}, g_{111} \in \mathbb{R}$ and $z, g_{210}, g_{021} \in \mathbb{C}$.

5.4 The Canonical Pitchfork-Hopf Bifurcation

By substituting $z = re^{i\theta}$ we can write (5.27) in cylindrical coordinates

$$\begin{cases} \dot{s} &= p_{11}s^3 + p_{12}sr^2 + \mathcal{O}(|s, r|^5) \\ \dot{r} &= p_{21}rs^2 + p_{22}r^3 + \mathcal{O}(|s, r|^5) \\ \dot{\theta} &= \omega + \mathcal{O}(|s, r|^2) \end{cases} \tag{5.28}$$

where

$$p_{11} = g_{300}, \qquad p_{12} = g_{111}, \qquad p_{21} = \mathrm{Re}(g_{210}), \qquad p_{22} = \mathrm{Re}(g_{021}) \tag{5.29}$$

We assume that $p_{ij} \neq 0$ for all $\{i, j\} \in \{1, 2\}$ and $p_{11}p_{22} - p_{12}p_{21} \neq 0$. If we drop the higher order terms, we can decouple the equation for $\dot{\theta}$, which is just a rotation around the s-axis. Perturbing the equation for $\dot{s}$ with $\epsilon_1 s$ and the equation for $\dot{r}$ by $\epsilon_2 r$ leads to the amplitude equations

$$\begin{cases} \dot{s} &= s(\epsilon_1 + p_{11}s^2 + p_{12}r^2) \\ \dot{r} &= r(\epsilon_2 + p_{21}s^2 + p_{22}r^2) \end{cases} \tag{AE}$$

These amplitude equations are identical to the amplitude equations of the double Hopf bifurcation studied in Guckenheimer and Holmes (1983) and Kuznetsov (2013). The unfolding can be classified into twelve topologically different cases ,see table (5.1). In Dijkstra *et al.* (2015) the parametric portrait and corresponding phase portraits of unfolding Ib is given, see figure (5.1), with a classification of the different fixed points, see table (5.2).

5.5 Normal Form Coefficients

The critical center manifold $\mathcal{W}^c_{loc}$ has the expansion (5.25) and due to the time-invariance of $\mathcal{W}^c_{loc}$ we have

$$\mathcal{H}(s(t), z(t), \bar{z}(t)) = \mathbf{u}(t) \tag{5.30}$$

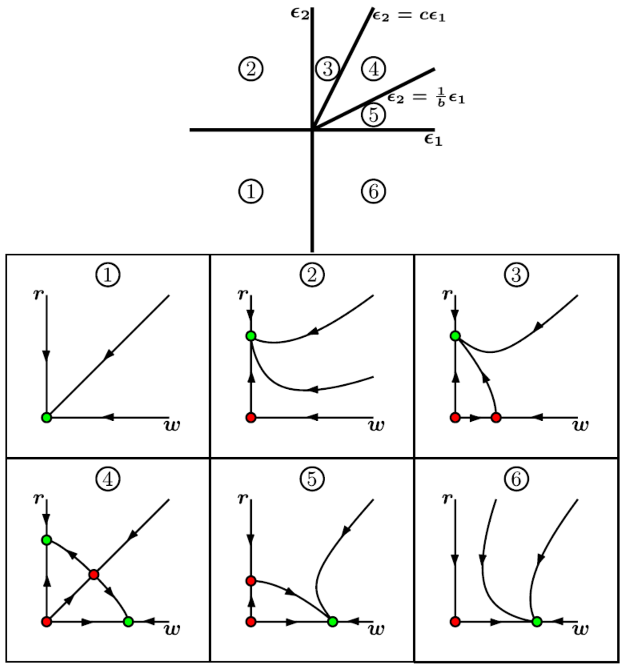

Figure 5.1: Parametric portrait and corresponding phase portraits of unfolding IB of (AE) (Dijkstra *et al.*, 2015, Figure 6)

If we differentiate both sides with respect to time and use (5.26) to get the homological equation

$$A^{\odot *} j\mathcal{H}(s, z, \bar{z}) + R(\mathcal{H}(s, z, \bar{z})) = j(\mathcal{H}_s(s, z, \bar{z})\dot{s} + \mathcal{H}_z(s, z, \bar{z})\dot{z} + \mathcal{H}_{\bar{z}}(s, z, \bar{z})\dot{\bar{z}}) \tag{5.31}$$

We can substitute in this expansion of the non-linearity (5.6), the normal form (5.27) and the expansion of the critical center manifold (5.25) into the homological equation (5.31) to derive the normal form coefficients. If we equate coefficients of the corresponding powers of s, z and $\bar{z}$, this leads to operator equations of the form

$$(A^{\odot *} - \lambda)\phi^{\odot *} = \psi^{\odot *}$$

Using the Fredholm solvability conditions, lemma (5.2.1), we can solve for the normal form coefficients, see Dijkstra *et al.* (2015, equation 72).

$$\begin{aligned} g_{300} &= \frac{1}{6}\langle l\mathbf{y}_{300}, \psi_1^{\odot}\rangle \\ g_{111} &= \langle l\mathbf{y}_{111}, \psi_1^{\odot}\rangle \\ g_{210} &= \frac{1}{2}\langle l\mathbf{y}_{210}, \psi_2^{\odot}\rangle \\ g_{021} &= \frac{1}{2}\langle l\mathbf{y}_{021}, \psi_2^{\odot}\rangle \end{aligned} \tag{5.32}$$

Where we defined

$$\begin{aligned} \mathbf{y}_{300} &:= D^3G(0)(\psi_1, \psi_1, \psi_1) \\ \mathbf{y}_{111} &:= D^3G(0)(\psi_1, \psi_2, \bar{\psi}_2) \\ \mathbf{y}_{210} &:= D^3G(0)(\psi_2, \psi_1, \psi_1) \\ \mathbf{y}_{021} &:= D^3G(0)(\psi_2, \psi_2, \bar{\psi}_2) \end{aligned} \tag{5.33}$$

We can compute the normal form coefficients using lemma (5.17) and (5.2.3).[2]

[2]Note that g_{021} corresponds to the Lyapunov coefficient of the Andronov-Hopf-bifurcation.

Chapter 6

Numerical example

In this chapter we will investigate a specific example and examine the bifurcations for this example. In order to limit the amount of free parameters and to keep the computation time manageable, we take a slight variant of (PDDE), where we have a single population u with a connectivity J which is a sum of exponentials

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d\frac{\partial^2 u}{\partial x^2}(t,x) - \alpha u(t,x) + \int_\Omega J(x,x')S(u(t-\tau(x,x'),x'))\mathrm{d}x' & \text{for } x \in \Omega,\ t \in \mathbb{R}^+ \\ \frac{\partial u}{\partial x}(t,x) = 0 & \text{for } x \in \partial\Omega,\ t \in \mathbb{R}^+ \\ u(t,x) = \phi(t,x) & \text{for } x \in \Omega,\ t \in [-h,0] \end{cases} \tag{PDDE}$$

This new (PDDE) can be understood a restriction of the original (PDDE) to the invariant linear space where $u_1 = u_2 = \cdots = u_N$. All the formulas change in a natural fashion, see appendix (A).

We will use a wizard-hat connectivity, the sum of a positive and a negative exponential, see also figure (6.1)

$$J(x,x') = \frac{25}{2}e^{-2|x-x'|} - 10e^{-|x-x'|} \tag{6.1}$$

This connectivity is used to model the interaction of a pair of excitatory and inhibitory populations of neurons. For τ and S we take the usual functions

$$\tau(x,x') = \tau^0 + |x-x'| \tag{6.2}$$

$$S(u) = \frac{1}{1+e^{-\gamma u}} - \frac{1}{2} \tag{6.3}$$

We take $\alpha = 1$ and $\tau^0 = \frac{3}{4}$. We take γ and d as bifurcation parameters.

6.1 Andronov-Hopf bifurcation

We will first investigate the effect of diffusion on an Andronov-Hopf-bifurcation. We use the formulas from Dijkstra *et al.* (2015) to compute the spectrum for the non-diffusive case.

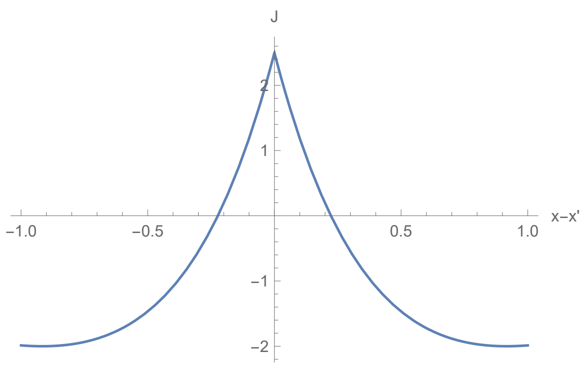

Figure 6.1: The wizard-hat connectivity of (6.1)

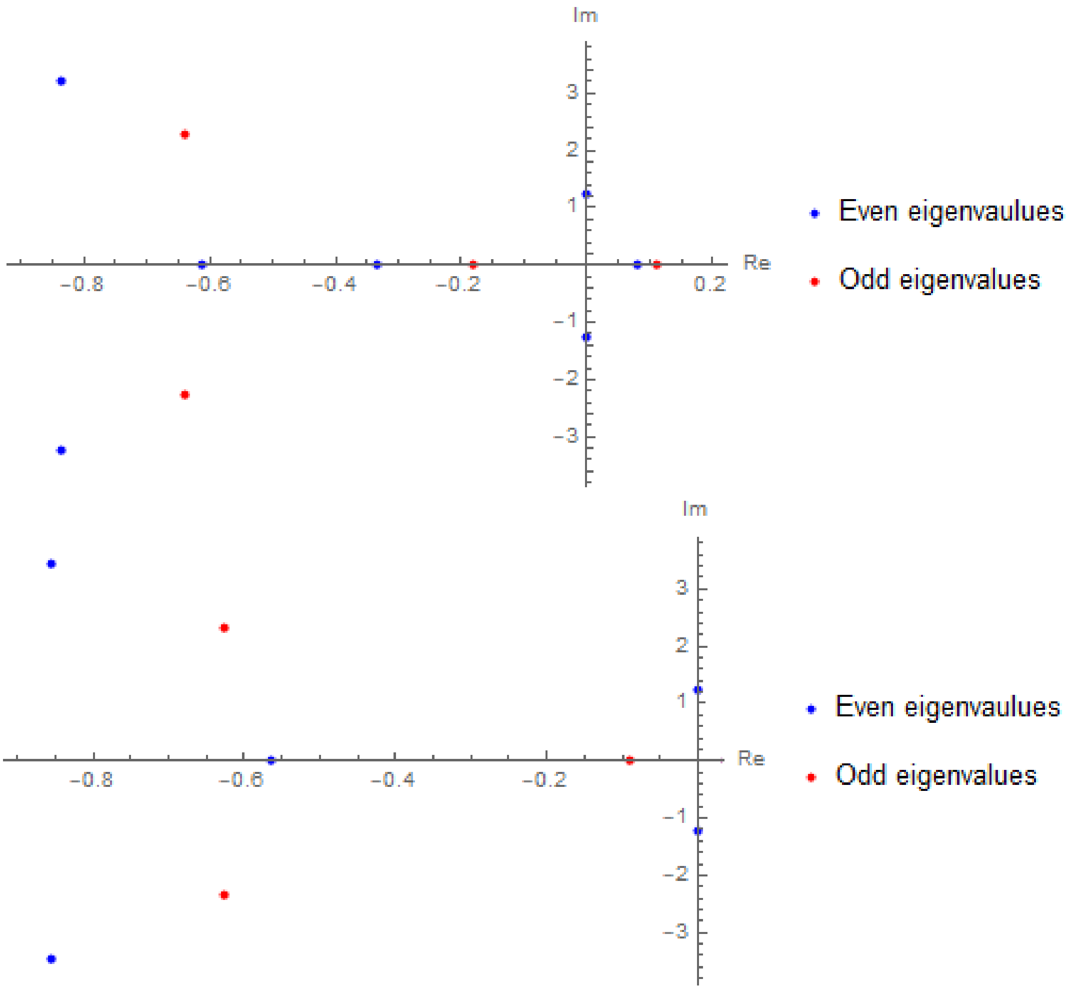

Figure 6.2: The eigenvalues at parameter values in (6.1) of the Andronov-Hopf bifurcation without and with diffusion respectively.

For $d = 0$ we have an Andronov-Hopf-bifurcation for $\gamma = 3.3482$ at $\lambda = 1.2403i$ with corresponding eigenvector

$$\psi(\theta)(x) = e^{1.2403i\theta}((-0.0178+0.0050i)\cosh((3.7185+3.2284i)x)+0.9998\cosh((0.2770-0.8878i)x) \quad (6.4)$$

The normal form coefficient $g_{21} = -2.2631 - 0.5641i$ and the Lyapunov coefficient $\ell_1 = -1.8247$ and hence the bifurcation is supercritical.

For $d = 0.2$ we have an Andronov-Hopf-bifurcation for $\gamma = 3.3094$ at $\lambda = 1.2379i$ with corresponding eigenvector

$$\psi(\theta)(x) = e^{1.2379i\theta}(0.9972\cosh((0.2535 - 0.8490i)x) + (-0.0727 - 0.0177i)\cosh((1.7315 + 3.2475i)x) + (0.0029 - 0.0060i)\cosh((3.90746 + 0.3586i)x) \quad (6.5)$$

The normal form coefficient $g_{21} = -2.30591 - 0.5170$ and the Lyapunov coefficient $\ell_1 = -1.8627$ and hence the bifurcation is supercritical.

As one might already have observed, the diffusion has little effect on the Andronov-Hopf bifurcation. The eigenvalues which are off the real axis are barely effected by the introduction of diffusion, while the eigenvalues on the real axis become more negative, see figure (6.2)[1]. This could be due to the fact the eigenvector corresponding to the eigenvalue on the imaginary axis has very little curvature, see figure (6.3). As diffusion penalizes curvature, its effect on this eigenvector would be small.

[1] Note that there is another positive $\lambda \in \mathbb{R}$, not shown in figure (6.2), which solves $\det(S^{\lambda,odd}) = 0$ and $\det(S^{\lambda,even}) = 0$, however this is a degenerate case as $P^{\lambda}(\rho)$ has a double root. Simulations of the linearised system with random initial conditions did not indicate the presence of an unstable mode, so we don't regard this point as an eigenvalue.

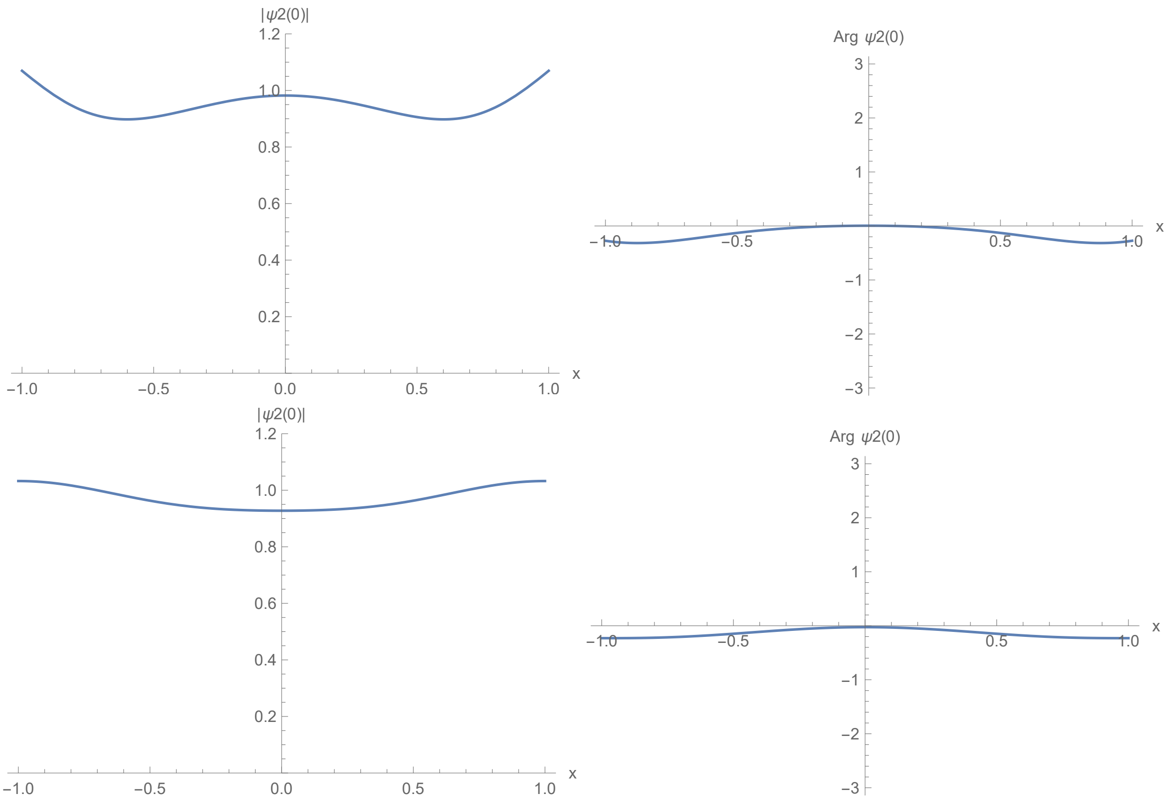

Figure 6.3: The eigenvectors at parameter values in (6.1) of the Andronov-Hopf bifurcation without and with diffusion respectively. Note that with diffusion the eigenvector has a derivative of zero at $x = 1$ and $x = -1$, while this is not the case without diffusion.

Bifurcation	α	τ^0	η_1	η_2	μ_1	μ_2	d	γ	λ	ℓ_1
Andronov-Hopf 1	1	0.75	12.5	-10	2	1	0	3.3482	1.2403i	-1.8247
Andronov-Hopf 2	1	0.75	12.5	-10	2	1	0.2	3.3094	1.2379i	-1.8627

Table 6.1: Parameter values of the Andronov-Hopf bifurcation without and with diffusion respectively.

6.2 Discretisation

To obtain an approximate solution of (PDDE) we discretize the spatial domain Ω into an equidistant grid of n^x points, x_n, with a width of $\delta = \dfrac{2}{n^x - 1}$. We discretize the integral operator G using a Trapezoidal rule and the diffusion operator B using a central finite difference method and a reflection across the boundary for the boundary conditions. This results in a second order spatial discretisation. The discretization of the (PDDE) for $n \in \{1, \cdots, n^x\}$ and $t \in \mathbb{R}^+$ becomes a set of delay delay equations (DDE).

$$\begin{cases} \frac{\partial u}{\partial t}(t, x_n) & = \frac{d}{2\delta^2}(u(t, x_{n-1}) - 2u(t, x_n) + u(t, x_{n+1})) - \alpha u(t, x_n) \\ & +\delta \sum_{m=1}^{n^x} \int_{-1}^{1} \xi_m J(x_n, s_m) S(u(t - \tau(x_n, x_m), x_m)) \mathrm{d}x' \\ u(t, x_0) & = u(t, x_2) \\ u(t, x_{n^x+1}) & = u(t, x_{n^x-1}) \\ u(t, x_n) & = \phi(t, x_n) \end{cases} \tag{DDE}$$

Here ξ_m is defined as

$$\xi_m = \begin{cases} 1 & m \in \{2, \cdots, n^x - 1\} \\ \frac{1}{2} & m = 1 \vee m = n^x \end{cases} \tag{6.6}$$

Now we are left with a set of n^x ordinary delay equations which we solve with a dde-solver.

We will now do some simulations around the Andronov-Hopf-bifurcation with diffusion. We set $n^x = 50$ and take as initial conditions an odd function, an even function and a major component of the eigenvector (6.5).

$$\begin{aligned} \phi_1(\theta)(x) &= \frac{1}{5} \sin \frac{1}{2}\pi x \\ \phi_2(\theta)(x) &= \frac{1}{5} \cos \pi x \\ \phi_3(\theta)(x) &= \frac{1}{5}\mathrm{Re}(e^{1.2379i\theta} \cosh((0.2535 - 0.8490i)x)) \end{aligned} \tag{6.7}$$

For figure (6.4) we took $\gamma = 3$ and for figure (6.5) $\gamma = 4$.

For $\gamma = 3$ all initial conditions converge to the trivial steady-state. The odd function converges like a node to the equilibrium, while the even functions converges like a focus to the equilibrium. For $\gamma = 4$ oscillations emerge, which correspond to the stable limit cycle of the Andronov-Hopf bifurcation. However this limit cycle is not globally attractive as the odd initial condition converges to some non-trivial steady state.

6.3 Pitchfork-Hopf bifurcation

In figure (6.2), we notice that when the diffusion increases some eigenvalues cross the imaginary axis. In the bifurcation diagram with d and γ as bifurcation parameters, figure (6.6), we notice that there are indeed Pitchfork bifurcations of the even and odd type. There are two Pitchfork-Hopf bifurcations in this paramter range.

There is an κ_1 Pitchfork-Hopf bifurcation at $\gamma = 3.3301$ and $d = 0.0871$ with critical eigenvalues $\lambda_1 = 0$ odd and $\lambda_2 = 1.2385i$ even and corresponding eigenvectors

$$\begin{aligned} \psi_1(\theta)(x) &= 0.6015 \sin(1.3089x) + 0.7989 \sin(2.7469x) + 0.0045 \sinh(5.0738x) \\ \psi_2(\theta)(x) &= e^{1.2385i\theta}(0.998501 \cosh((0.2666 - 0.8695i)x) + (-0.0533 + 0.0123i) \cosh((2.1758 + 3.7681i)x) \\ &\quad + (-0.0001 - 0.0026i) \cosh((4.8897 + 0.5850i)x) \end{aligned} \tag{6.8}$$

The normal forms coefficients take the following values

$$\begin{pmatrix} g_{300} & g_{111} \\ g_{210} & g_{021} \end{pmatrix} = \begin{pmatrix} -0.9966 & -3.9596 \\ -2.7820 - 0.3043i & -2.2783 - 0.5364i \end{pmatrix} \tag{6.9}$$

Furthermore, $b = 1.738, c = 2.7914, 1 - bc = -3.8515$ and $p_{11}p_{22} = g_{300}\mathrm{Re}(g_{021}) = 2.2706$, which corresponds to case Ib of table (5.1)

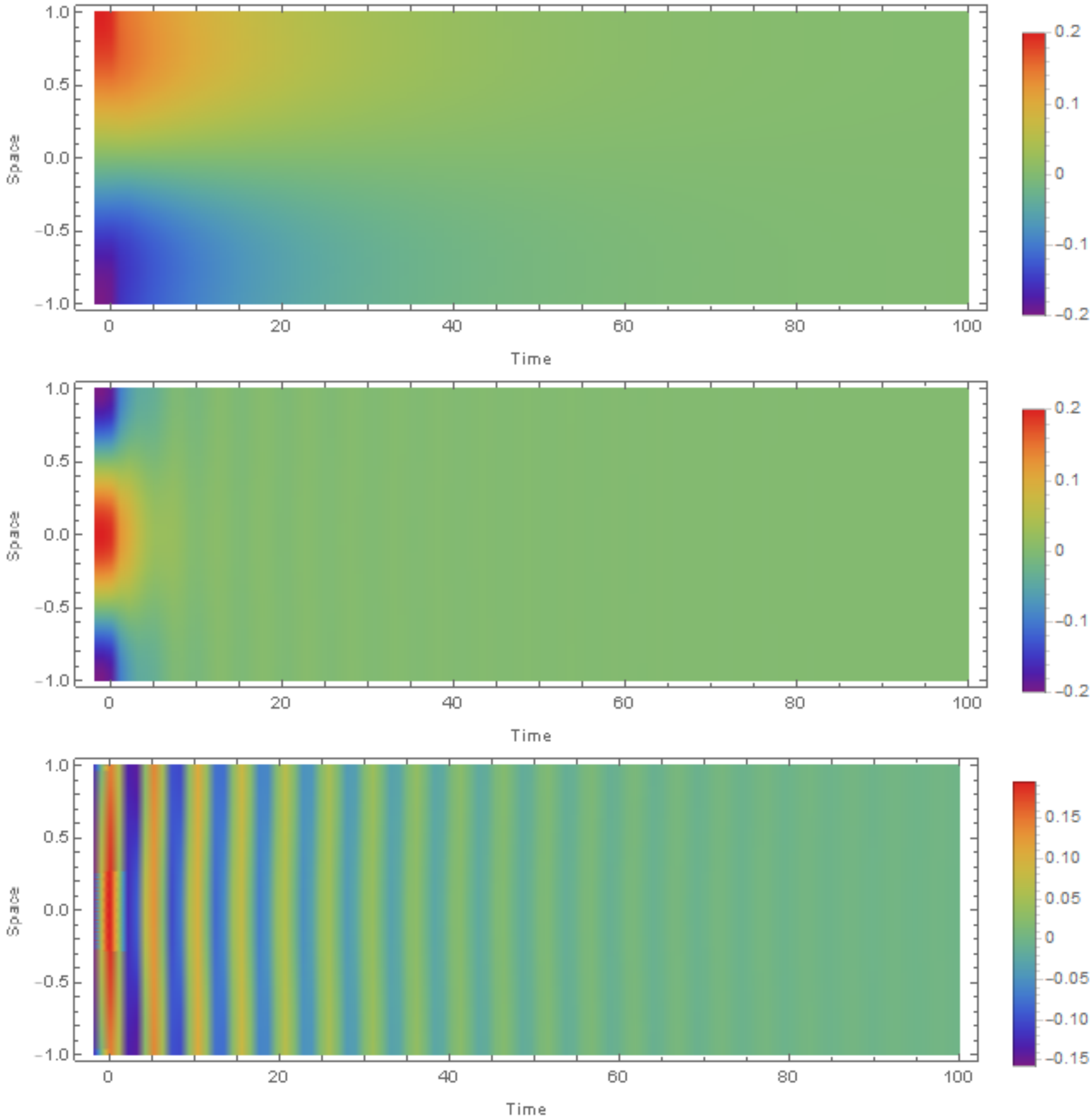

Figure 6.4: Simulation of (PDDE) with the initial conditions of (**??**) and $\gamma = 3$ close to the Andronov-Hopf-bifurcation with diffusion, see table (6.1)

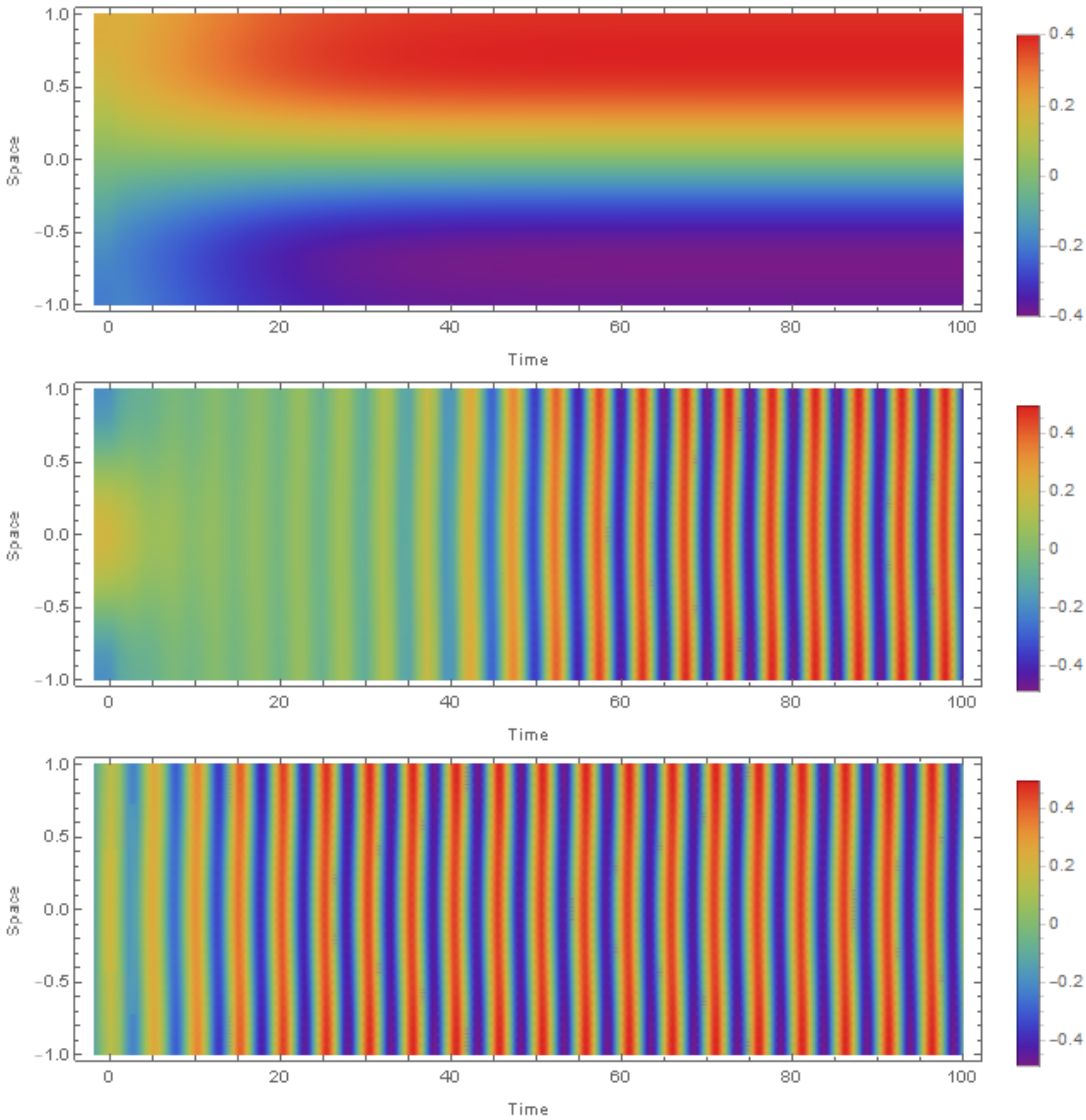

Figure 6.5: Simulation of (PDDE) with $\gamma = 4$ close to the Andronov-Hopf-bifurcation with diffusion, see table (6.1)

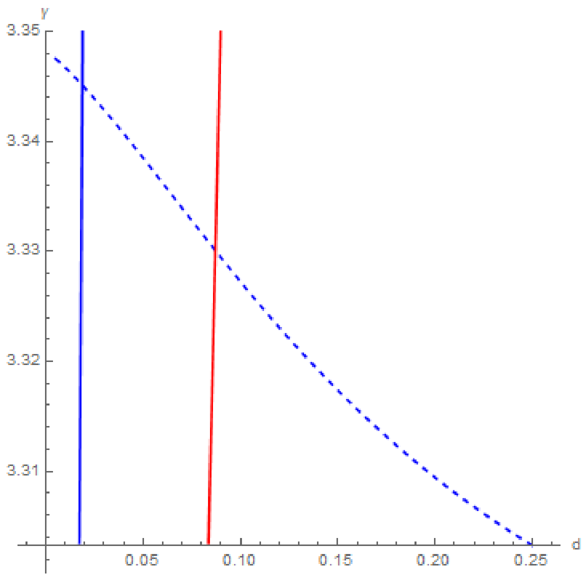

Figure 6.6: Bifurcation diagram of (PDDE) with bifurcation parameters γ and d

Bifurcation	α	τ^0	η_1	η_2	μ_1	μ_2	d	γ	λ_1	λ_2
κ_1 Pitchfork-Hopf	1	0.75	12.5	-10	2	1	0.0871	3.3301	0	1.2385i
κ_2 Pitchfork-Hopf	1	0.75	12.5	-10	2	1	0.0019	3.3345	0	1.2398i

Table 6.2: Parameter values of the Pitchfork-Hopf bifurcations.

There is an κ_2 Pitchfork-Hopf bifurcation at $\gamma = 3.3345$ and $d = 0.0019$ with critical eigenvalues $\lambda_1 = 0$ even and $\lambda_2 = 1.2398i$ even and corresponding eigenvectors

$$\begin{aligned}\psi_1(\theta)(x) &= 0.2345\cos(1.2636x) + 0.9721\cos(2.7469x) + 0.00006\cosh(8.5522x)\\ \psi_2(\theta)(x) &= e^{1.2398i\theta}(0.9997\cosh((0.8838-0.2748i)x) + (0.0217-0.0126i)\cosh((3.5769+3.8884i)x)\\ &\quad + (0.00004+0.0001i)\cosh((8.1405+2.8351i)x)\end{aligned} \tag{6.10}$$

The normal forms coefficients take the following values

$$\begin{pmatrix} g_{300} & g_{111} \\ g_{210} & g_{021} \end{pmatrix} = \begin{pmatrix} -0.8280 & -5.0333 \\ -1.5642 - 0.4584i & -2.2641 - 0.5591i \end{pmatrix} \tag{6.11}$$

Furthermore, $b = 2.2231, c = 1.8892, 1 - bc = -3.1999$ and $p_{11}p_{22} = g_{300}\mathrm{Re}(g_{021}) = 1.8746$, which corresponds to case Ib of table (5.1)

Both these cases gives the Ib-type Pitchfork-Hopf bifurcation which is the same case as found in Dijkstra *et al.* (2015). The direction of the Pitchfork bifurcation is such that the non-trivial equilibria vanish when the diffusion is increased. This makes sense since these equilibria have some spatial curvature which is penalized by the diffusion. This points to the general conclusion we take from this numerical example. The addition of diffusion suppresses spatial modes, but has no effect on the temporal modes.

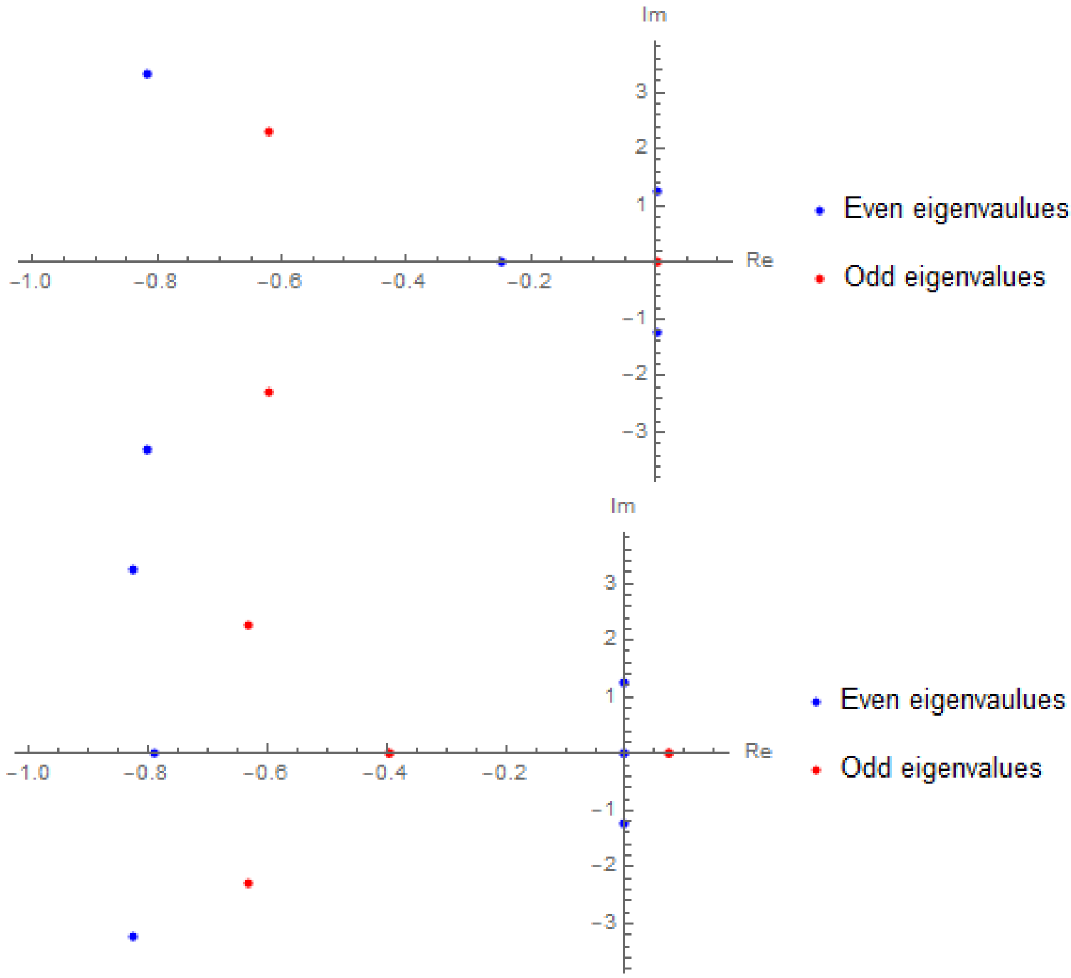

Figure 6.7: The eigenvalues at parameter values in (6.1) of the Andronov-Hopf bifurcation without and with diffusion respectively. Note that there is another positive $\lambda \in \mathbb{R}$, not shown here, which solves $\det(S^{\lambda,odd}) = 0$ and $\det(S^{\lambda,even}) = 0$, however this is a degenerate case as $P^{\lambda}(\rho)$ has a double root.

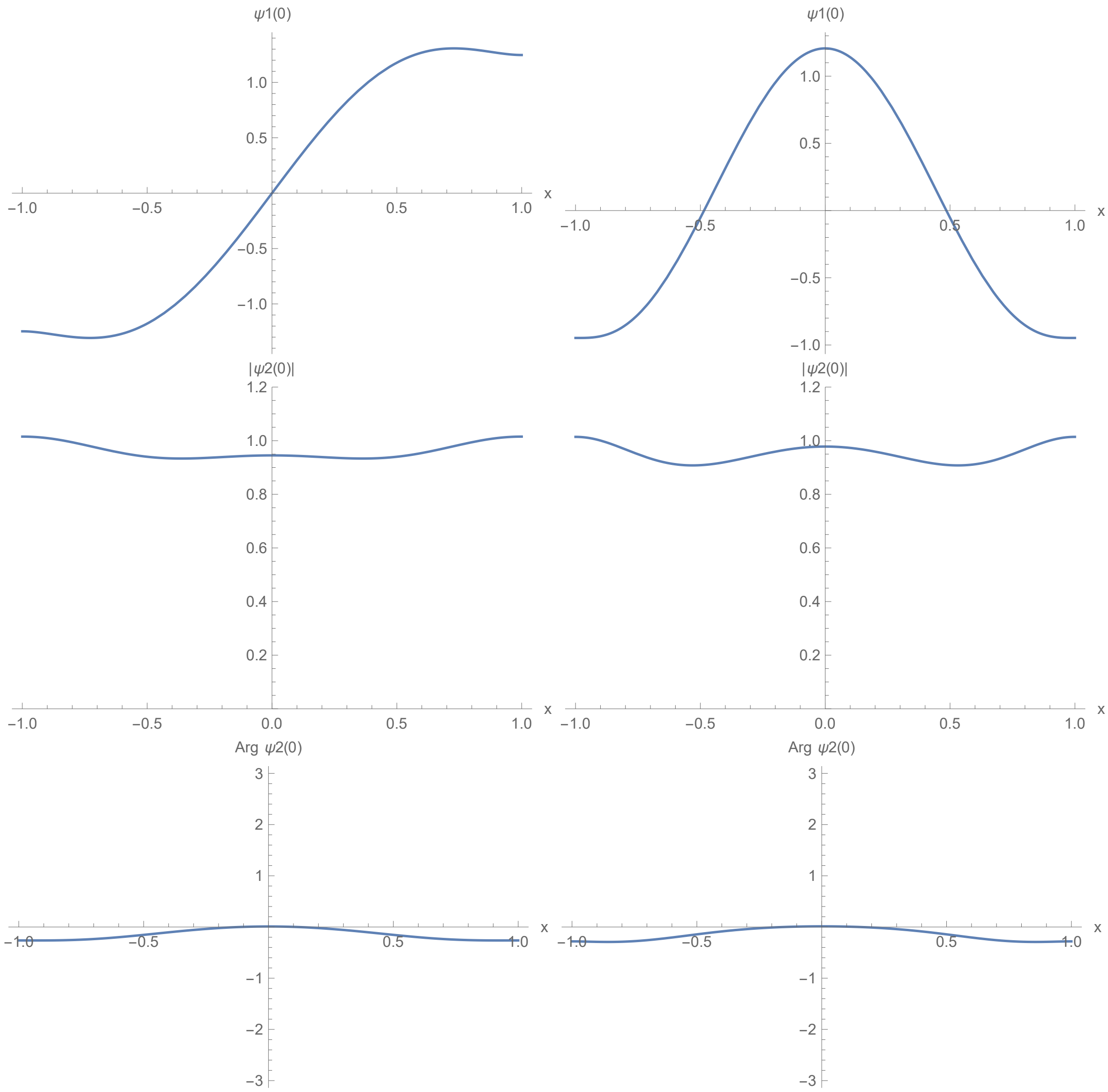

Figure 6.8: The eigenvectors at parameter values in (6.2), the left panel corresponds to the κ_1 Pitchfork-Hopf and the right panel corresponds to the κ_2 Pitchfork-Hopf

Chapter 7

Discussion

We have shown how a neural field model with delays and diffusion fits into the general sun-star framework for delay equations and proved a result on the essential spectrum. Furthermore we found an explicit characterisation of the point spectrum, resolvent and normal form coefficients for specific choices for the connectivity J, delay τ and firing rate function S. We examined Andronov-Hopf and Pitchfork bifurcations by evaluating normal form coefficients and confirming these results by simulating the discretised problem (DDE). We found that the addition of diffusion suppresses spatial modes, while having no effect on the temporal modes. In the context of neural fields this implies that the addition of diffusion synchronizes the neural field.

In the computation of the normal forms we used an odd firing rate function S as it was mathematically convenient. However, there is no biological reason why S should have this symmetry. For a more general form of S, equations (5.13) still hold, but require a more involved computation.

We chose Neumann boundary conditions which model a closed system. This effectively introduces a reflection across the boundary for the diffusion. An alternative is periodic boundary conditions, which effectively wraps the effect of the diffusion around the domain. A different option is using Dirichlet boundary conditions, which models an outside contribution at the boundary.

In the numerics, we used single population with a wizard-hat connectivity (6.1) originating from Amari (1977) instead of a full two population model like Wilson and Cowan (1972). This was mainly done to reduce the number of free parameters and computational costs. We hypothesize that our numerical results, i.e. that diffusion suppresses spatial modes, carries over to the full two population model, but it remains to be confirmed.

In chapter (5) we assumed the existence of the center manifold. In Diekmann *et al.* (2014) some progress has been made on the existence of stable and unstable manifolds. However there is not yet a proof of the existence of the center manifold.

One other possible inclusion tot the neural field model (ADDE) besides diffusion are second order synapses. This models the fact that synapses are not immediately at full strength when an action potential arrives. This would produce a model of the following form.

$$\left(1+\frac{1}{\beta}\frac{\partial u_i}{\partial t}(t,x)\right)\left(1+\frac{1}{\alpha}\frac{\partial u_i}{\partial t}(t,x)\right) = d_i\frac{\partial^2 u_i}{\partial x^2}(t,x)+\sum_{j=1}^{N}\int_{\Omega} J_{i,j}(x,x')S_j(u_j(t-\tau_{i,j}(x,x'),x'))\mathrm{d}x' \quad (7.1)$$

We obtain the orginial (PDDE) if $\beta \to \infty$. This model might produce more oscillations due to the second order temporal derivative.

Lastly, we considered the model on the one dimensional domain Ω. A more physiological appropriate domain would be planar of spherical domain. Visser *et al.* (2017) investigated the role of spherical topology on pattern formation and bifurcations. It is possible to extend the approach in this work to explore how diffusion affects the spherical harmonics and corresponding standing waves.

Appendix A

Reduction to a single population

In this appendix we elaborate on the reduction of (PDDE) to a single population u with a connectivity J, which is the sum N exponentials. We state the equivalent theorems and formulas which were used in the computations in (6) for completeness and reproducibility.

The partial differential delay equation (PDE) for a single population u with a connectivity J, which is the sum N exponentials, is given by

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = d\frac{\partial^2 u}{\partial x^2}(t,x) - \alpha u(t,x) + \int_\Omega J(x,x')S(u(t-\tau(x,x'),x'))\mathrm{d}x' \quad \text{for } x\in\Omega,\ t\in\mathbb{R}^+ \\ \frac{\partial u}{\partial x}(t,x) & = 0 \quad \text{for } x\in\partial\Omega,\ t\in\mathbb{R}^+ \\ u(t,x) & = \phi(t,x) \qquad \text{for } x\in\Omega,\ t\in[-h,0] \end{cases} \tag{PDE}$$

We can think of this (PDE) as a restriction of the original (PDDE), to the invariant linear space where $u_1 = u_2 = \cdots = u_N$. We define the following Banach Spaces $Y := C(\Omega;\mathbb{R})$ and $X := C([-h,0];Y)$ with their corresponding supremum-norms. We can formulate the (ADDE)

$$\begin{cases} \dot{u}(t) = Bu(t) + G(u_t) \\ \mathbf{u}_0 = \phi \in X \end{cases} \tag{ADDE}$$

Where the linear operator $B : D(B) \to Y$ is defined as

$$(Bu)(x) := d\frac{\partial^2 u}{\partial x^2}(x) - \alpha u(x) \tag{A.1}$$

With $d, \alpha > 0$. We take the domain of B as the twice continuously differentiable functions with Neumann boundary conditions: $D(B) = \{\mathbf{y} \in Y | \mathbf{y} \in C^2(\Omega), \mathbf{y}'(\partial\Omega) = 0\}$ The non-linear operator $G : X \to Y$ for $\phi \in X$ is defined as

$$(G(\phi))(x) := \int_\Omega J(x,x')S(\phi(-\tau(x,x'),x'))\mathrm{d}x' \tag{A.2}$$

The following lemma carries over.

Lemma A.0.1. *(van Gils et al., 2013, Lemma 3, Proposition 11) G is compact and globally Lipschitz continuous and k times Fréchet differentiable for any $k \in \mathbb{N}$. Furthermore the kth Fréchet derivative of G at $\psi \in X$, $D^kG(\psi) : X^k \to Y$ is compact and given by*

$$D^kG(\psi)(\phi^1,\cdots,\phi^k)(x) = \int_{-1}^{1} J(x,x')S^{(k)}(\psi(-\tau(x,x'),x')\prod_{m=1}^{k}(\phi^m(-\tau(x,x'),x'))\mathrm{d}x' \tag{A.3}$$

Due to the general results, chapter 2 and chapter 3 immediately carry over.

A.1 Spectral Properties

For the spectral properties we make the following choices for the functions τ, S and J.

$$\begin{aligned} \tau &:= \tau^0 + |x - x'| \\ J(x,x') &:= \sum_{j=1}^{N} J_j(x,x') \\ J_j(x,x') &:= \eta_j e^{-\mu_j|x-x'|} \\ S(u) &:= \frac{1}{1+e^{\gamma u}} - \frac{1}{2} \end{aligned} \tag{A.4}$$

Where $\tau^0, \mu_j, \gamma > 0$ and $\eta_j \in \mathbb{R}$ for $j \in \{1, \cdots N\}$. The operator K^z is given by $K^z = \sum_{j=1}^{N} K_j^z$ with

$$\begin{aligned} K_j^z y(x) &:= c_j \int_{-1}^{1} e^{-k_j(z)|x-x'|} y(x') \mathrm{d}x' \\ c_j(z) &:= \frac{\gamma}{4} \alpha \eta_j e^{-\tau^0 z} \\ k_j(z) &:= \mu_j + z \end{aligned} \tag{A.5}$$

Theorem (4.0.1) is still applicable, so we have that $z \in \sigma(A)$ if and only if $z \in \sigma(B + K^z)$. To find eigenvalues, we want to find $q \in D(B)$ such that

$$(B - z + K^z)q = 0 \tag{IE}$$

We define L_j^z for $j \in \{1, \cdots, N\}$.

$$L_j^z := k_j^2(z) - \partial_x^2$$

Similarly we find that

$$L_j^z K_j^z q = 2c_j(z)k_j(z)q$$

We apply L_j successively to (IE) for $j \in \{1, \cdots, N\}$ and end up with a linear differential equation $M^z q = 0$ where M^z is defined as

$$M^z := (B - z) \prod_{p=1}^{N} L_p(z) + 2 \sum_{j=1}^{N} c_j(z)k_j(z) \prod_{\substack{p=1 \\ p \neq j}}^{N} L_p(z) \tag{DE}$$

We try a solution of the form $q = e^{\rho x}$, which yields the characteristic polynomial $P^z(\rho)$

$$P^z(\rho) := (d\rho^2 - \alpha - z) \prod_{p=1}^{N} (k_p(z)^2 - \rho^2) + 2 \sum_{j=1}^{N} c_j(z)k_j(z) \prod_{\substack{p=1 \\ p \neq j}}^{N} (k_p(z)^2 - \rho^2) \tag{A.6}$$

Note that this is similar to (4.9), $P^z(\rho)\mathbf{q}^0$ with $q_i^0 = 1$ for $i \in \{1, \cdots N\}$. The polynomial $P^z(\rho)$ has at most $2(N+1)$ roots. When $P^z(\rho)$ has exactly $2(N+1)$ roots, the general solution of (DE) is given by

$$q^z(x) := \sum_{m=1}^{N+1} [a_m \cosh(\rho_m(z)x) + b_m \sinh(\rho_m(z)x)]) \tag{A.7}$$

We can formulate an equivalent lemma to lemma (4.1.2)

Lemma A.1.1. *If the characteristic polynomial $P^z(\rho)$ has $2(N+1)$ distinct roots then $\rho_m \neq 0$ for $m \in \{1, \cdots, N+1\}$ and $k_j(z) \neq 0$ for all $j \in \{1, \cdots, N\}$.*

In this setting we have an explicit characterisation for the set

$$\mathcal{S} := \{z \in \mathbb{C} | \exists j \in \{1, \cdots N\}, m \in \{1, \cdots, N+1\} \text{ such that } k_j(z) = \pm \rho_m(z)\} \tag{A.8}$$

Lemma A.1.2. *If characteristic polynomial P^z has $2(N+1)$ distinct roots then*

$$\mathcal{S} = \{z \in \mathbb{C} | \exists j, p \in \{1, \cdots N\}, j \neq p \text{ such that } k_j^2(z) = k_p^2(z)\}$$

Proof. We have that $z \in \mathcal{S}$ if and only if $P^z(k_j(z))) \neq 0$ for all $j \in \{1, \cdots, N\}$.

$$P^z(k_j(z)) = 2c_j(z)k_j(z)\prod_{\substack{p=1\\p\neq j}}^{N}(k_p^2(z) - k_j^2(z))$$

Hence $P^z(k_j(z)) \neq 0$ if and only if $k_j^2(z) = k_p^2(z)$ for some $p \in \{1, \cdots N\}, j \neq p$ □

We will now substitute $\mathbf{q}^z$ into (IE), which yields the equations $S^{z,even}\mathbf{a} = S^{z,odd}\mathbf{b} = 0$ after some rewriting, where the $N+1$ by $N+1$ matrices $S^{z,even}$ and $S^{z,odd}$ are given by

$$\begin{aligned} S^{z,even}_{j,m} &:= \begin{cases} \dfrac{k_j(z)\cosh(\rho_m(z)) + \rho_m(z)\sinh(\rho_m(z))}{k_j^2(z) - \rho_m^2(z)} & j \in \{1, \cdots, N\} \\ \rho_m(z)\sinh(\rho_m(z)) & j = N+1 \end{cases} \\ S^{z,odd}_{j,m} &:= \begin{cases} \dfrac{\rho_m(z)\cosh(\rho_m(z)) + k_j(z)\sinh(\rho_m(z))}{k_j^2(z) - \rho_m^2(z)} & j \in \{1, \cdots, N\} \\ \rho_m(z)\cosh(\rho_m(z)) & j = N+1 \end{cases} \end{aligned} \tag{A.9}$$

For the eigenvalues, we can formulate an equivalent theorem to theorem (4.1.3).

Theorem A.1.3. *Suppose $P^\lambda(\rho)$ has $2(N+1)$ distinct roots and $\lambda \notin \mathcal{S}$ for some $\lambda \in \mathbb{C}$ then we have that $\lambda \in \sigma_p(A)$ if and only if $\det(S^{\lambda,even})\det(S^{\lambda,odd}) = 0$. The eigenvalue λ is called even if $\det(S^{\lambda,even}) = 0$ and odd if $\det(S^{\lambda,odd}) = 0$.*

The corresponding eigenvector $\psi^\lambda \in X$ for even eigenvalues is given by

$$\psi^\lambda(\theta)(x) := e^{\lambda\theta}\sum_{m=1}^{N+1} a_m \cosh(\rho_m(\lambda)x) \tag{A.10}$$

Where $\mathbf{a}$ is a vector in the nullspace of $S^{\lambda,even}$. For every $\theta \in [-h, 0]$, ψ^λ is an even function in x.

The corresponding eigenvector $\psi^\lambda \in X$ for odd eigenvalues is given by

$$\psi^\lambda(\theta)(x) := e^{\lambda\theta}\sum_{m=1}^{N+1} b_m \sinh(\rho_m(\lambda)x) \tag{A.11}$$

Where $\mathbf{b}$ is a vector in the nullspace of $S^{\lambda,odd}$. For every $\theta \in [-h, 0]$, ψ^λ is an odd function in x

Also for the resolvent, we can formulate an equivalent theorem to theorem (4.2.2).

Theorem A.1.4. *For $z \in \rho(A)$ with $z \notin \mathcal{L}$ the unique solution $q := q^z \in D(B)$ of (RE) is given by*

$$q^z(x) := R(z, B)y(x) + \sum_{m=1}^{N+1}[a_m(x)\cosh(\rho_m(z)x) + b_m(x)\sinh(\rho_m(z)x)]$$

Where $R(z, B)$ is the resolvent operator of B given in equation (3.5) and $\mathbf{a}(x)$ and $\mathbf{b}(x)$ as

$$\begin{aligned} \mathbf{a}(x) &:= \mathbf{a}^c - \frac{1}{2}\left(\int_{-1}^{x}\hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' - \int_{x}^{1}\hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \\ \mathbf{b}(x) &:= \mathbf{b}^c + \frac{1}{2}\left(\int_{-1}^{x}\hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' - \int_{x}^{1}\hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \end{aligned} \tag{A.12}$$

Where $\mathbf{a}^c$ and $\mathbf{b}^c$ are defined as

$$\begin{aligned} \mathbf{a}^c &:= -(S^{z,even})^{-1}S^{z,odd}\left(\int_{-1}^{1}\hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \\ \mathbf{b}^c &:= \ (S^{z,odd})^{-1}S^{z,even}\left(\int_{-1}^{1}\hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x'\right) \end{aligned} \tag{A.13}$$

And the $N+1$ by $N+1$ matrices $\hat{A}, \hat{B}, \hat{K}$ and $\hat{Q}$ and the vector of length $N+1$, $\mathbf{r}$, are defined as

$$\begin{aligned}
\hat{A}_{m,m}(x) &:= \cosh(\rho_m(z)x) \\
\hat{B}_{m,m}(x) &:= \sinh(\rho_m(z)x) \\
\hat{K}_{j,m} &:= \rho_m(z)\hat{Q}_{j,m} \\
\hat{Q}_{j,m} &:= \begin{cases} \dfrac{1}{k_j^2(z) - \rho_m^2(z)} & \text{for } j \in \{1, \cdots, N\} \\ 1 & \text{for } j = N+1 \end{cases} \\
r_j(x) &:= \begin{cases} R(z,B)y(x) & \text{for } j \in \{1, \cdots, N\} \\ 0 & \text{for } j = N+1 \end{cases}
\end{aligned} \tag{A.14}$$

In this setting the exception set $\mathcal{L}$ (4.23) reduces to

$$\mathcal{L} = \sigma(B) \cup \mathcal{S} \cup \{z \in \mathbb{C} | P^z(\rho) \text{ has less than } 2N(N+1) \text{ distinct zeros}\} \tag{A.15}$$

due to the following lemma

Lemma A.1.5. *If $z \notin \mathcal{S}$ and $P^z(\rho)$ has $2N(N+1)$ distinct zeros then $\hat{Q}$ is invertible.*

Proof. By substituting $n_j = k_j^2$ and $p_m = \rho_m^2$, we can write $\hat{Q}$ as

$$\hat{Q}_{j,m} = \begin{cases} \dfrac{1}{n_j - p_m} & \text{for } j \in \{1, \cdots, N\} \\ 1 & \text{for } j = N+1 \end{cases}$$

As $z \notin \mathcal{S}$ and $P^z(\rho)$ has $2N(N+1)$ distinct zeros and using lemma A.1.2, we have that $n_i \neq n_j \neq p_m \neq p_l$ for $i, j \in \{1, \cdots, N\}, l, m \in \{1, \cdots, N+1\}, i \neq j, l \neq m$. We subtract the last column from the other columns; this does not change the determinant. We get the following matrix $\tilde{Q}$:

$$\hat{Q}_{j,m} = \begin{cases} \dfrac{1}{n_j - p_m} \dfrac{p_m - p_{N+1}}{n_j - p_{N+1}} & \text{for } i, m \in \{1, \cdots, N\} \\ \dfrac{1}{n_j - p_{N+1}} & \text{for } j \in \{1, \cdots, N\}, m = N+1 \\ 0 & \text{for } j = N+1, m \in \{1, \cdots, N\} \\ 1 & \text{for } j = m = N+1 \end{cases}$$

Now row j of matrix $\tilde{Q}$ contains the factor $\dfrac{1}{n_j - p_{N+1}}$ and column m contains the factor $p_m - p_{N+1}$ for $j, m \in \{1, \cdots, N\}$. Hence we can rewrite the determinant of $\hat{Q}$ as:

$$\|\hat{Q}\| = \|\tilde{Q}\| = \|\bar{Q}\| \prod_{i=1}^{N} \frac{p_i - p_{N+1}}{n_i - p_{N+1}}$$

Here $\bar{Q}$ is defined as:

$$\bar{Q}_{j,m} = \frac{1}{n_j - p_m} \quad \text{for } i, m \in \{1, \cdots, N\}$$

We observe that $\bar{Q}$ is a Cauchy matrix as $n_i \neq n_j \neq p_m \neq p_l$ for $i, j \in \{1, \cdots, N\}, l, m \in \{1, \cdots, N+1\}, i \neq j, l \neq m$ and hence invertible. Furthermore the product $\prod_{i=1}^{N} \dfrac{p_i - p_{N+1}}{n_i - p_{N+1}}$ is non-zero, so we conclude that $\hat{Q}$ is invertible. □

In chapter (5) the formulas of the normal forms and lemmas (5.2.1) and (5.17) are still applicable. Also for the computation of the normal form computation, we can formulate an equivalent theorem to theorem (5.2.3)

Theorem A.1.6. *Let $\lambda \in \sigma_p(A)$ be a simple eigenvalue such that there exists a sufficiently small closed disk C_λ such that $\mathcal{L} \cap C_\lambda = \emptyset$ and $C_\lambda \cap \sigma(A) = \{\lambda\}$.*

If λ is an even eigenvalue such that

$$\psi(0)(x) = \sum_{m=1}^{N+1} a_m \cosh(\rho_m(\lambda)x) \tag{A.16}$$

for all $x \in \Omega$, where $\mathbf{a}$ is a non-trivial solution of $S^{\lambda,even}\mathbf{a} = 0$. Then the formula $P^{\odot *}ly = \nu j\psi$ is equivalent to

$$\frac{-\mathrm{adj}(S^{\lambda,even})}{\frac{d}{dz}(\det(S^{z,even}))|_{z=\lambda}} S^{z,odd} \int_{-1}^{1} \hat{A}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' = \nu\mathbf{a} \tag{A.17}$$

For all $y \in Y$, where $\mathrm{adj}(S^{\lambda,even})$ denotes the adjugate of $S^{\lambda,even}$ and using the definitions in (A.14).

If λ is an odd eigenvalue such that

$$\psi(0)(x) = \sum_{m=1}^{N+1} b_m \sinh(\rho_m(\lambda)x) \tag{A.18}$$

for all $x \in \Omega$, where $\mathbf{b}$ is a non-trivial solution of $S^{\lambda,odd}\mathbf{b} = 0$. Then the formula $P^{\odot *}ly = \nu j\psi$ is equivalent to

$$\frac{\mathrm{adj}(S^{\lambda,odd})}{\frac{d}{dz}(\det(S^{z,odd})|_{z=\lambda}} S^{z,even} \int_{-1}^{1} \hat{B}(x')\hat{K}^{-1}\mathbf{r}(x')\mathrm{d}x' = \nu\mathbf{b} \tag{A.19}$$

For all $y \in Y$, where $\mathrm{adj}(S^{\lambda,odd})$ denotes the adjugate of $S^{\lambda,odd}$ and using the definitions in (A.14).

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