The quality of equilibria in generalized market sharing games

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Abstract
We analyze the quality of several equilibria for generalized market sharing games. Generalized market sharing games model n selfish players selecting subsets of a finite set of items, where the payoff of an item is divided among all players choosing that item. Market sharing games are a special case of this, where the available subsets are restricted by budget constraints. Market sharing games were studied by Goemans et al. [6], who showed that the price of anarchy is at most 2 by showing that those games belong to the class of games known as valid utility games, as defined by Vetta [24].
We show that generalized market sharing games are valid utility games as well, yet sharpen the bounds on the price of anarchy further to show that it is exactly equal to $2 - \frac{1}{n}$. We do this using a smoothness argument, allowing this price of anarchy to extend to different equilibria. These tight bounds also hold for special cases such as symmetric or singleton. Furthermore we study the sequential version of the game and define a class of games called shared misery games. For this class of games we set up a framework which can be used to provide upper bounds on the sequential price of anarchy. Among other results, this shows that any subgame perfect equilibrium in a symmetric singleton congestion game is also a Nash equilibrium in the simultaneous variant of the game. For symmetric singleton generalized market sharing games, this means that the sequential price of anarchy is also $2 - \frac{1}{n}$. Lastly, we show that for an egalitarian social function the sequential price of anarchy differs significantly. Here the sequential price of anarchy is n, yet the symmetric singleton variant has a sequential price of anarchy of 1.

1 Introduction

Game theory is the study of mathematical models of cooperation or conflict between rational decision makers. These rational decision makers are called players, and their decisions are called their strategies. Game theory is a relatively new branch of mathematics, it was not formally developed before 1944, when John von Neuman published his book "Theory of Games and Economic Behavior" [19]. Since then game theory has grown considerably and is now used in many fields, including economics, psychology and computer science.

Goemans et al. [6] studied market sharing games. A market sharing game consists of players and resources. Each player chooses a subset of these resources. The players receive payoff from all the resources they choose, but have to share the payoff if multiple players choose the same resource. Each resource has a fixed payoff available and this payoff is divided equally among all players choosing the resource. To determine which subset of resources a player can choose, all players have a budget and all resources have a cost that a player needs to pay for using that resource. The sum over the costs of all resources chosen by a player may not exceed that player's budget. Each player wants to maximize his own payoff. The sum over the payoff of all players is called the social value. The social value is equal to the total payoff of the resources chosen.

There is only a finite number of resources available. Competition over these resources can make a system inefficient, since every player only tries to maximize his own payoff, not the social value. A state in which no player can improve his payoff unilaterally is called a Nash equilibrium. A Nash equilibrium can have significantly less social value than the optimal solution. The inefficiency a Nash equilibrium might cause is measured with the price of anarchy, which is the worst case ratio of the social value of an optimal solution over the social value of a Nash equilibrium.

Recently, it has been brought up that sequential games might provide better quality equilibria. In a sequential game players pick their strategies after each other, instead of simultaneously. A solution of a sequential game where all players take the strategies of all other players into account and pick to maximize their own payoff is called a subgame perfect equilibrium. The inefficiency a subgame perfect equilibria might cause is measured with the sequential price of anarchy, which is the worst case ratio of the social value of an optimal solution over the social value of a subgame perfect equilibrium equilibrium.

In this work we investigate a more general setting of market sharing games where the budget and cost constraints are replaced with a strategy set for each player, which contains all possible sets a player can choose. This allows us to model more situations. These games are called generalized market sharing games. For these games we analyze the quality of a Nash equilibria and subgame perfect equilibria. We measure the efficiency of these equilibria with the price of anarchy or sequential price of anarchy. Furthermore we investigate special cases such as symmetric games, where all players have the same strategy sets, and singleton games, where all players only choose one resource.

2 Related work and contribution

Generalized market sharing games are a subclass of congestion games. A congestion game consists of players, resources and payoff functions. Each player chooses a subset of the resources and pays costs or receives payoff based only upon which resources he has chosen. Each resource has its own payoff function, and the payoff a resource gives to a player depends only on the number of players choosing that resource. Congestion games were introduced by Rosenthal in 1973 [22] and he proved they all have at least one pure Nash equilibrium.

We use the price of anarchy and the sequential price of anarchy to measure the potential inefficiency of Nash equilibria and subgame perfect equilibria respectively. The price of anarchy was introduced by Koutsoupias et al. [13] in 1999 as a way of measuring the cost to society due to players acting selfishly . The sequential price of anarchy was introduced in 2012 by Paes Leme, Syrgkanis and Tardos [14]. They claimed that for many games “the subgame perfect equilibrium of their sequential version is a much more natural prediction, ruling out unreasonable equilibria, and leading to much better quality solutions." To substantiate these claims, they gave bounds on the sequential price of anarchy for four classes of games, which were an improvement over the price of anarchy. Bilò et al [2] showed that these claims were a bit too optimistic, and that their results only hold for generic games, which are are games where no ties occur. Following these papers there have been results showing that the sequential price of anarchy is lower than price of anarchy in some cases and higher in other cases.

The price of anarchy has been studied for many types of games. A first bound on the price of anarchy in generalized market sharing games comes from showing that they are valid-utility systems. Valid utility systems were introduced by Vetta in 2002 [24]. A utility system is a game with a submodular social function, where the payoff a player receives is at least as great as the loss of social value if the player does not participate, while the rest of the players do not change their strategies. If the sum over the payoff of the players is smaller or equal to the social value, then it is a valid utility system. Vetta showed that for any valid-utility system the price of anarchy is at most 2.

The sequential price of anarchy has only been studied for a handful of games. De Jong and Uetz investigated the sequential price of anarchy in linear congestion games and showed that the sequential price of anarchy is smaller than the price of anarchy for 2 and 3 players, but might grow larger for more players [9]. Furthermore they showed that the sequential price of anarchy is 4/3 for the symmetric singleton linear congestion games. An extensive overview of the results and literature on the sequential price of anarchy can be found in another work by de Jong [11].

The maximum coverage problem is a well studied problem in computer science and operations research. Generalized market sharing games can be seen as the maximum coverage problem in a game theoretic setting. The maximum coverage problem consists of resources and sets of these resources. The goal is to cover as many resources as possible using at most n sets. The difference with generalized market sharing games is that there is only one player, and any collection of n sets is possible. In generalized market sharing games each player has his own sets to choose from and plays only for his own interest.

Market sharing games are another adaptation of the maximum coverage problem. In market sharing games players have a budget to buy a subset of items. All items have fixed costs and values, and the value of an item is evenly distributed among all players that chose this item. The Market sharing games were analyzed by Goemans et al. in 2006 [6], and they proved that the price of anarchy is bounded from above by 2 by showing that market sharing games are valid-utility systems, and gave a lower bound example to show that this is tight when the amount of players is taken to infinity.

There also exist models that do not distribute the value of an item completely. Gairing investigated covering games [5], which are games where players can choose a subset of resources. All resources have a fixed value, but this value does not need to be divided equally or even completely among all the players choosing the resource. The only restrictions on the payoff functions are that the payoff function must e non-increasing payoff in the amount of players choosing a resource, and the total payoff provided to all players by one resource cannot extend the value of that resource. Gairing investigated which payoff functions would provide the best price of anarchy, and determined a symmetric payoff function which improved the price of anarchy from 2 to $\frac{e}{e-1} \approx 1.58$.

Expanding further, Marden and Wierman investigated welfare distribution games [16], where the total value of a resource would depend on the set of players choosing it. They examined different ways to divide the value of a resource, similar to what Gairing did with covering games. Here the payoff each player can get from each item depends on which specific players would also choose that item. The game which are closest to generalized market sharing games as studied in this research, are anonymous equally shared welfare distribution games. The only difference is that items might increase in value when chosen by multiple players.

In a recent paper Paccagnan and Marden also showed results for set covering games [20, 21], in which they extended the work of Gairing on covering games. Here resources still have a fixed value and payoff must be divided equally among the players choosing the resource, but they let go of the restrictions that the payoff function must be non-increasing and that the total value divided from one resource must be less than the value of that resource. The social value is still equal to the sum over the values of the resources chosen. Paccagnan and Marden compare different payoff functions, including the Shapley value payoff functions, dividing the total value of the resource among the players choosing the resource, and the new payoff functions found by Gairing [5]. They show that even without the restrictions used in covering games the payoff function found by Gairing still provides the lowest price of anarchy of $\frac{e}{e-1} \approx 1.58$. Furthermore they use simulations to show that this new payoff function provides better Nash equilibria in practice.

An important difference between market sharing games and covering games is the difference in strategies that are possible. Market sharing games use player budgets and resource costs, while other covering games provide a collection of sets to choose from. These lead to different possible strategy sets. Another approach was presented by set packing games, which were introduced by de Jong and Uetz in 2017 [11]. Set packing games do not take into account budget and item cost, but provide each player with a specific set of possible strategies, that has to be *downwards closed*. This means if a certain set is possible, any subset of that set is also a feasible strategy. Furthermore, set packing games also assume every item can be chosen only once. De Jong and Uetz investigated that the quality of equilibria of set packing games of both the simultaneous and sequential version of the game and proved that the sequential price of anarchy is only $\frac{e}{e-1} \approx 1.58$, while the price of anarchy is 2.

These games based on the maximum coverage problem can be used to model competing players, but they can also be used to approximate maximum coverage problems, which are hard to solve in a centralized manner. Goemans et al. [6] used market sharing games to model caching in a wireless networks, but they can also be used for vehicle-target assignment problems [15] or sensor allocation problems [1]. In these problems a finite number of vehicles or sensors have to be allocated to certain locations. However, these are difficult problems to solve and it might take too much time to find the optimal solution. An approximation can be made by having every vehicle or sensor act as a selfish player. All of these players try to optimize their own payoff. The social value of the market sharing game then represents the value of the solution in the original maximum coverage problem. The optimal solution is still hard to find, but can be approximated with a Nash equilibrium.

In this work we also consider bottleneck games, which are games where each player chooses a subset of resources and a player's payoff is determined solely by the resource which provides him the least payoff. Linear bottleneck congestion games were studied before by de Keijzer, Schäfer and Telelis [12], who investigated the price of anarchy and strong price of anarchy, and showed that the strong price of anarchy is 2 for symmetric games, but cannot be bounded by a constant in general.

Utilitarian social function			
Game	**PoA**	**SPoA**	**Section**
General	$2-\frac{1}{n}$	$\leq n$	5
2 player	$\frac{3}{2}$	$\frac{3}{2}$	6
3 player	$\frac{5}{3}$	$\leq 1.816...$	6
Singleton symmetric	$2-\frac{1}{n}$	$2-\frac{1}{n}$	7,8

Egalitarian social function			
Game	**Pure PoA**	**SPoA**	**Section**
General	n	n	9
Symmetric	$2-\frac{1}{n}$	$\geq 2-\frac{1}{n}$	9
Singleton	$> c$	$> c$	10
Singleton symmetric	1	1	7,9

Figure 1: Main results for generalized market sharing games

Our contribution

First we define generalized market sharing games in Section 4 and show that they are a generalization of market sharing games. Furthermore we show there always exists a pure Nash equilibria by proving that generalized market sharing games are potential games and show that the price of anarchy is at most 2 by proving that generalized market sharing games are valid utility systems.

Next, we look into the price of anarchy in Section 5. We provide direct proofs for a matching upper and lower bound of $2-\frac{1}{n}$ on the price of anarchy. We also give a direct proof for mixed strategies to show that the price of anarchy stays the same. We extend these results to other equilibria by showing that the robust price of anarchy is $2-\frac{1}{n}$ as well.

In Section 6 we investigate the sequential version of the game and study how sequential play influences the equilibria by investigating the sequential price of anarchy. In many cases the sequential price of anarchy is more difficult to bound than the simultaneous version. Here we attempt a variety of proof ideas. Our main result for this section follows from a linear program, showing that the sequential price of anarchy is higher than the price of anarchy for three players, and giving a strong indication that this might be the case for any number of players larger or equal to three.

One of the main results of our research is described in Section 7. Here we define a class of games called shared misery games. For this class of games we set up a framework which can be used to provide upper bounds on the sequential price of anarchy for any shared misery game. For symmetric games we can specify the results even further. We show that any subgame perfect equilibrium in a symmetric singleton congestion game is also a Nash equilibrium in the simultaneous variant of the game. Furthermore we show that these subgame perfect equilibria are optimal for an egalitarian social function.

We show that singleton generalized market sharing games are shared misery games in Section 8 and apply the results found in Section 7. This leads to a price of anarchy of $2-\frac{1}{n}$ for singleton generalized market sharing games and a sequential price of anarchy of $2-\frac{1}{n}$ for symmetric singleton generalized market sharing games. Furthermore we proof that the sequential price of anarchy of $2-\frac{1}{n}$ holds for generic singleton generalized market sharing games as well.

The results for shared misery games also have implications for generalized market sharing games with an egalitarian social function. This objective function does not sum over the payoff gained by the players, but only takes the player with the least amount of payoff into account. In Section 9 we look into these games and show that the price of anarchy and sequential price of anarchy are 1 for the singleton symmetric variant, by using the results for shared misery games.

Lastly, we look into bottleneck games in Section 10 and show that the price of anarchy and sequential price of anarchy cannot be bounded by a constant, even in the singleton variant.

A summary of bounds proven in this research can be found in Figure 1.

3 Preliminaries

3.1 Games

In this section we introduce concepts and notation that we use throughout this work.

Definition 3.1 Strategic-form game.
A game in *strategic-form*, also known as *normal-form* or *matrix form*, is a tuple $(N, S_1, \ldots, S_n, C_1, \ldots, C_n)$ that consists of a finite set N of players $1, \ldots, n$, and for each player i, a finite set S_i of pure strategies and a payoff function $C_i : S_1 \times \cdots \times S_n \to \mathbb{R}$.

In a strategic form game each player i chooses a *strategy* σ_i from a finite strategy set $\mathbf{S}_i$. We examine two kinds of strategies, *pure* strategies and *mixed* strategies. A pure strategy of player i is any action player i can make and a mixed strategy of player i is a probability distribution on the set of pure strategies of player i. The set of all pure strategies for player i is called the *strategy set* $\mathbf{S}_i$ of player i.

Pure strategies are a subset of mixed strategies, since every pure strategy can be written as a probability distribution where one strategy is chosen with with probability 1. In each section, we make clear whether we investigate only pure strategies or allow both kinds of strategies.

The outcome of a game is determined by the strategies of all the players. The vector with the strategy of each player is called a *strategy profile*.

Definition 3.2 Strategy Profile.
A *strategy profile* is an ordered set $\sigma = (\sigma_1, \ldots, \sigma_n)$ of strategies $\sigma_i \in \mathbf{S}_i$.

Given a strategy profile σ, define $\sigma_{-i} := (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ as the strategies of all players except i. Now that we have the strategies defined, we formalize the types of games. There are many sort of games, but we consider mainly two variants, namely *simultaneous* games and *sequential* games.

Definition 3.3 Simultaneous game.
A *simultaneous game* is a game where all players choose their strategy at the same time, without knowledge of the strategies of other players.

A classical example of a simultaneous game is rock-paper-scissors, where both players decide what to do at the same time, without knowing what the other player is going to play. We assume *perfect information*, which means that each player is aware of the strategies of the other players, which in the case of rock-paper-scissors could be rock, paper and scissor all with $\frac{1}{3}$ chance. While the players are aware of the strategy set in advance, they do not know which strategy is played before they choose their strategy. Unless specified otherwise, we assume a game is simultaneous.

By imposing an order on the players, we obtain a sequential game. For clarity in the coming definition, we denote a pure strategy from the simultaneous game as an action.

Definition 3.4 Sequential game.
A *sequential game* is a game where all players choose their strategy in some order, say $1, \ldots, n$. All players have knowledge of the actions chosen by all players before them.

Consider the sequential version of rock-paper-scissors, where the players take turns choosing their actions. No matter what action the first player chooses, the second player can choose his action accordingly such that he always wins. In a sequential game a strategy more complex than in a simultaneous game, as it must specify one action for all possible combinations of actions of preceding players.

3.2 Quality of equilibria

All players get payoff depending on their strategy and the strategies of the other players. In strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, player i plays strategy σ_i and the other players play σ_{-i}. The payoff player i receives is denoted as $w_i(\sigma_i, \sigma_{-i})$. To judge the quality of a strategy profile, we look at the *social value* it provides. The social value is determined by the *social function*.

Definition 3.5 Social function.
The *social function* is a function that maps the strategy profile to $\mathbb{R}$. The two variants we consider are:

- *Utilitarian* ($U(\sigma)$): The sum of payoff over all players, i.e. $U(\sigma) = \sum_{i=1}^{n} w_i(\sigma_i, \sigma_{-i})$
- *Egalitarian* ($C(\sigma)$): The minimum payoff over all players, i.e. $C(\sigma) = \min_{i \in (1,\ldots,n)} w_i(\sigma_i, \sigma_{-i})$

For cost minimization games, the egalitarian social function is equal to the maximum cost over all players instead. Unless specified otherwise, we use a utilitarian social function. An optimal solution for a game is a solution which gives the highest social value. This is the *social optimum*.

Definition 3.6 Social optimum.
In any game, a *social optimum* is a strategy profile σ^{OPT} that maximizes the social welfare function.

However, the players are not interested in the social value, they are only interested in their own payoff. We assum that they make decisions to maximize their own payoff. Therefore we investigate equilibria where no player can improve his payoff. For a simultaneous game, that is a *Nash equilibrium* [18].

Definition 3.7 Nash equilibrium.
For any game a *Nash equilibrium* $\sigma^{NE} = (\sigma_1^{NE}, \ldots, \sigma_n^{NE})$ is a strategy profile where no player can improve his payoff unilaterally by choosing a different strategy, i.e. for any player i and any other strategy $\sigma'_i \in \mathbf{S}_i$:

$$w_i(\sigma_i^{NE}, \sigma_{-i}^{NE}) \geq w_i(\sigma'_i, \sigma_{-i}^{NE}) \tag{1}$$

We call (1) the Nash property. A Nash equilibrium consisting of only pure strategies is called a pure Nash equilibrium.

Any Nash equilibrium in a simultaneous game is also a Nash equilibrium in the sequential version of that game, but that does not mean that no player can improve his payoff. Sometimes a player can play a different action causing the players after him to play different actions as well, causing him to end up with more payoff. In a sequential game an equilibria where no player can improve his payoff by switching strategies is a *subgame perfect equilibrium*.

Definition 3.8 Subgame perfect equilibrium.
A *subgame perfect equilibrium* (SPE) is a strategy profile $\sigma^{SPE} = (\sigma_1^{SPE}, \ldots, \sigma_n^{SPE})$ in a sequential game, that induces a Nash equilibrium in every subgame of the original game.

All games considered here are games with perfect information. This means that in a subgame perfect equilibrium every player knows the actions of all players before them and knows what actions all players after them will choose, for each action they can play. Each player selects utility-maximizing actions at each level of the game tree. Therefore all subgame perfect equilibria can be determined with backwards induction.

Nash equilibria and subgame perfect equilibria are the two major equilibria we look at, but there are many kinds of equilibria possible. We also show results for two other kinds of equilibria, *correlated equilibria* and *coarse correlated equilibria*.

Definition 3.9 Correlated equilibrium
Given is a probability distribution **p** over all possible outcomes. In this equilibrium a central authority selects a strategy profile according to the probability distribution **p** and recommends each player the corresponding strategy privately. When the players are recommended their strategy they are unaware of the strategies recommended to the other players, but are aware of probability distribution **p**. This probability distribution **p** is a *correlated equilibrium* if no player i can improve his expected payoff by switching from the recommended strategy σ_i to another strategy σ'_i, i.e.

$$\sum_{\sigma} p(\sigma|\sigma_i)\, w_i(\sigma_i, \sigma_{-i}) \geq \sum_{\sigma} p(\sigma|\sigma_i)\, w_i(\sigma'_i, \sigma_{-i}), \qquad \forall i, \sigma_i, \tag{2}$$

where $p(\sigma|\sigma_i)$ where is the conditional probability for σ assigned by **p**, when σ_i is recommended to player i.

Definition 3.10 Coarse correlated equilibrium
A *coarse correlated equilibrium* is a probability distribution **p** over all possible outcomes. Similar to correlated equilibria, a central authority selects a strategy profile according to the probability distribution **p**. However, players need to decide whether they will follow the recommended strategy before they know what the strategy is. Therefore, in a coarse correlated equilibrium it only needs to hold that no player i can improve his expected payoff by choosing another strategy σ'_i, without even knowing what the recommended strategy would be, i.e.

$$\sum_{\sigma} p(\sigma)\, w_i(\sigma_i, \sigma_{-i}) \geq \sum_{\sigma} p(\sigma)\, w_i(\sigma'_i, \sigma_{-i}), \qquad \forall i, \sigma_i, \tag{3}$$

where $p(\sigma)$ where is the probability for σ assigned by **p**.

Every correlated equilibrium is also a coarse correlated equilibrium. Nash equilibria have been used to predict behavior of players in many different applications. However, the social value of a Nash equilibrium may be much lower than the social value of the optimal solution. The *price of anarchy* (PoA) [13] is a way to measure the inefficiency of a system due to players behaving selfishly.

Definition 3.11 Price of anarchy.
For any game I, the *price of anarchy* is the ratio between the social value of the optimal solution σ^{OPT} and the worst Nash equilibrium σ^{NE}, i.e.

$$PoA(I) = \max_{\sigma^{NE} \in \sigma^{NE}(I)} \frac{w(\sigma^{OPT})}{w(\sigma^{NE})} \tag{4}$$

where $\sigma^{NE}(I)$ denotes the set of all Nash equilibria in the game I. The price of anarchy of any class of games C is defined as:

$$PoA(C) = \sup_{I \in C} PoA(I) \tag{5}$$

For cost minimization games the price of anarchy we take the maximum over $w(\sigma^{NE})/w(\sigma^{OPT})$ instead. We write PoA if the game or class of games is clear from the context.

To measure the inefficiency of subgame perfect equilibria in sequential games, there is a similar ratio called the *sequential price of anarchy*.

Definition 3.12 Sequential price of anarchy.
For any sequential game I, the *sequential price of anarchy* (SPoA) is the ratio of the value of an optimal solution σ^{OPT} over the social value of a worst subgame perfect equilibrium σ^{SPE}, i.e.

$$SPoA(I) = \max_{\sigma^{SPE} \in \sigma^{SPE}(I)} \frac{w(\sigma^{OPT})}{w(\sigma^{SPE})} \tag{6}$$

where $\sigma^{SPE}(I)$ denotes the set of all subgame perfect equilibria in the game I. The sequential price of anarchy of any class of games C is defined as:

$$SPoA(C) = \sup_{I \in C} SPoA(I) \tag{7}$$

For cost minimization games we again take the maximum over $w(\sigma^{SPE})/w(\sigma^{OPT})$ instead. We write SPoA if the game or class of games is clear from the context.

Definition 3.13 Price of stability.
For any game I, the *price of stability* is the ratio between the social value of the optimal solution σ^{OPT} and the best Nash equilibrium σ^{NE}, i.e.

$$PoA(I) = \min_{\sigma^{NE} \in \sigma^{NE}(I)} \frac{w(\sigma^{OPT})}{w(\sigma^{NE})} \tag{8}$$

where $\sigma^{NE}(I)$ denotes the set of all Nash equilibria in the game I. The price of stability of any class of games C is defined as:

$$PoA(C) = \sup_{I \in C} PoA(I) \tag{9}$$

3.3 Classes of games

One well known class of games is *congestion games*, introduced by Rosenthal in 1973 [22].

Definition 3.14 Congestion games
A *congestion game* is a tuple $(N, R, (\mathbf{S}_i)_{i\in N}, (p_j)_{j\in R})$ where $N = (1,\ldots,n)$ denotes the set of players, $R = \{1,\ldots,m\}$ the set of resources, $\mathbf{S}_i$ the strategy space of player i, and p_j the payoff or cost function of resource j. Each player chooses a subset of resources $\sigma_i \in \mathbf{S}_i$ and receives payoff from all the resources according to their payoff functions. The payoff function of a resource is only dependent on the number of players choosing that resource.

Definition 3.15 Symmetric games
A *symmetric* game is a game where all players have the same strategy space, i.e for all players $i, j \in N$, it holds that $\mathbf{S}_i = \mathbf{S}_j$.

Definition 3.16 Generic games
A *generic* game is a game where no two different strategies could give a player the same cost or payoff, i.e. for any two different strategies σ_i, σ'_i, it holds that $w_i(\sigma_i, \sigma_{-i}) \neq w_i(\sigma'_i, \sigma_{-i})\ \forall\ \sigma_i, \sigma'_i, \sigma_{-i}$.

Definition 3.17 Singleton congestion games
A *singleton* congestion game is a game where all strategies only consist of a single resource, i.e. $|\sigma_i| = 1$ for all $i, \sigma_i \in \mathbf{S}_i$.

Definition 3.18 Anonymous games
A *anonymous* game is a game where the payoff of a player is not dependent on his identity. i.e. for any two players i and j and any strategies σ_k, σ_{-k} it holds that $w_i(\sigma_k, \sigma_{-k}) = w_j(\sigma_k, \sigma_{-k})$.

In this work we mainly investigate generalized market sharing games, but study a few other games as well. We define all of these games in their respective sections, but also note them here for easy reference.

Definition 3.19 Market sharing games
A *market sharing game* is a tuple $(N, R, (b_i)_{i\in N}, (c_j)_{j\in R}, (q_j)_{j\in R})$ where $N = (1,\ldots,n)$ denotes the set of players, $R = \{1,\ldots,m\}$ the set of resources, b_i the budget of player i, c_j the cost of resource j and q_j the value of resource j. Lastly there is a bipartite graph $G = (N \cup J, E)$ in which an edge between player i and market j means that player i is capable of servicing market j. A player i can choose a subset σ_i of markets if player i can service all these markets and the total cost of these markets is within the budget of player i, i.e. $\sum_{j\in\sigma_i} C_j \leq B_i$. The payoff of a market is shared between all players who service that market.

In market sharing games, the strategies of players are determined by budgets of players and costs of resources. This might be restrictive in some applications, and therefore we consider the following generalization to arbitrary strategy sets:

Definition 3.20 Generalized market sharing game.
A *generalized market sharing game* is a tuple $(N, J, (\mathbf{S}_i)_{i\in N}, (w_j)_{j\in J})$ where $N = (1, \ldots, n)$ denotes the set of players, $J = \{1, \ldots, m\}$ the set of items, $\mathbf{S}_i$ the finite set of pure strategies of player i, and w_j the value of item j. Each player chooses a subset of items and the value of an item is shared between all player choosing that item.

Definition 3.21 Valid utility systems
There are three conditions for a game to be a valid utility system:

- The social function is submodular and nondecreasing
- The payoff a player receives is at least as great as the loss of social payoff if the player does not participate, while the rest of the players do not change their actions
- The sum of the payoff for the players must be less or equal to the social function

Definition 3.22 Set packing games
There is a set N of n players, and a finite ground set J of items. Each item $j \in J$ has a value w_j. For $S \subseteq J$, we let $w(S) := \sum_{j\in S} w_j$. Each player i has a strategy set $\mathbf{S}_i \subseteq 2^J$ which is downward closed, i.e., if $\sigma_i \in \mathbf{S}_i$, then $T_i \in \mathbf{S}_i$ for all $T_i \subseteq \sigma_i$. For a strategy profile $\sigma = w_i(\sigma_i, \sigma_{-i})$ the payoff for player i is defined as :

$$w_i(\sigma_i, \sigma_{-i}) = \begin{cases} w_i(\sigma_i) & \text{if } \sigma_i \cap \sigma_k = \emptyset \text{ for all } k \neq i \\ -\infty & \text{otherwise} \end{cases} \tag{10}$$

Definition 3.23 covering games.
There is a set N of n players, and a finite ground set J of items. Each item $j \in J$ has a value w_j. For $S \subseteq J$, we let $w(S) := \sum_{j\in S} w_j$. Each player i has a strategy set $\mathbf{S}_i \subseteq 2^J$. The payoff of an item is shared between all players who chose that item.

4 Introducing generalized market sharing games

In this section we formally define generalized market sharing games and show some of their properties. First we show the difference to market sharing games as defined by Goemans et al. [6] and fix some more notation. Next we show that every generalized market sharing game has a pure Nash equilibrium, by proving that every generalized market sharing game is a potential game. This is done by providing the potential function. Furthermore we prove that generalized market sharing games are valid utility systems, which yields that the price of anarchy is at most 2.

4.1 Generalized market sharing games

Market sharing games were investigated by Goemans et al. in 2006 [6], who used them to model non-cooperative caching in wireless networks. Market sharing games are defined as follows:

Definition 4.1. Market sharing games
There is a set N of n players and a set J of m markets. Each market j has a payoff w_j, and a cost c_i. Each player has a budget b_i, which they can use to service markets. Lastly there is a bipartite graph $G = (N \cup J, E)$ in which an edge between player i and market j means that player i is capable of servicing market j. A player i can choose a subset σ_i of markets if player i can service all these markets and the total cost of these markets is within the budget of player i, i.e. $\sum_{j \in \sigma_i} C_j \leq B_i$. The payoff of a market is shared between all players who service that market.

Goemans et al. [6] studied the efficiency of market sharing games if all players try to maximize their individual gain by looking into the price of anarchy. They proved an upper bound on the price of anarchy of 2, and a lower bound example that shows the bound is tight as n goes to infinity.

In market sharing games, the strategies of players are determined by budgets of players and costs of resources. That might be restrictive in some applications. For instance, the price to service a certain market may differ per player. Or some combination of markets is easier to service for logistical reasons. There might even be political reasons, which causes a player to have to choose between two markets, by not allowing the player to service them both. The following example provides a game which cannot be modeled with budget constraints.

Example 4.2. Consider a game with 2 players and 3 items. Each item has value 1. Player one can pick either items 1 and 2 both, or only item 3. Player two can pick either items 2 and 3 both, or only item 1.

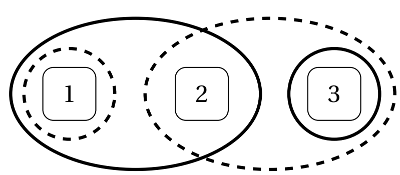


Figure 2: Graphic representation of the game in Example 4.2. Squares represent items. Player one can pick one of the solid circles, player two can pick one of the dashed circles.

Example 4.2 is a game that cannot be modeled as a market sharing game. Since both players have access to all items, it would only be possible to turn this into a market sharing game if the strategy set of one of the players was equal to or a subset of the strategy set of the other player. This is not the case here, since they both have a unique strategy available that the other player does not have. Therefore it is impossible to turn this into a market sharing game.

In order to allow games like Example 4.2 to be modeled, we introduce generalized market sharing games.

Definition 4.3 Generalized Market sharing games
There is a set N of n players, and a finite ground set J of items. Each item $j \in J$ has a value w_j. For $S \subseteq J$, we let $w(S) := \sum_{j \in S} w_j$. Each player i has a strategy set $\mathbf{S}_i \subseteq 2^J$ which is downward closed, i.e., if $\sigma_i \in \mathbf{S}_i$, then $T_i \in \mathbf{S}_i$ for all $T_i \subseteq \sigma_i$. The payoff of an item is shared between all players who chose that item.

Generalized market sharing games are similar to market sharing games. Both are congestion games with players and resources where the payoff of an item is shared evenly among all players who chose that item. The difference lies in the strategy sets that are allowed. Instead of the budget and cost per resource, generalized market sharing games only have the restriction that the strategy set must be downwards closed. This allows situations such as Example 4.2 to be modeled.

It is interesting to note that the strategy sets of market sharing games are also always downwards closed, all costs are positive so a subset of a strategy always fall within the budget as well. So every market sharing game can be written as a generalized market sharing game. The reverse does not hold, as was shown with Example 4.1. Therefore we can say that generalized market sharing games are a generalization of market sharing games, and this means that any upper bound found on the price of anarchy and its variants also applies to the corresponding market sharing game.

4.2 Potential Game

Before we say anything about the quality of Nash equilibria, we first prove that generalized market sharing games always have a pure Nash equilibrium. We do this by giving a potential function, which is a function that strictly increases whenever a player changes his strategy to improve his own payoff. A potential function is called an exact potential function if it increases with the same amount as the players payoff. A game where a potential function or exact potential function can be defined are called potential games or exact potential games respectively. We show that generalized market sharing games are exact potential games by giving an exact potential function. An exact potential function is formally as follows:

Definition 4.4 Exact potential function
A function $\phi : \sigma \to \mathbb{R}$ is called an exact potential function if:

$$\phi(\sigma_i, \sigma_{-i}) - \phi(\sigma'_i, \sigma_{-i}) = w_i(\sigma_i, \sigma_{-i}) - w_i(\sigma'_i, \sigma_{-i}) \qquad \forall \sigma_i, \sigma'_i \in \mathbf{S}_i, \forall \sigma_{-i} \in \mathbf{S}_{-i}. \tag{11}$$

Theorem 4.5 *Generalized market sharing games are exact potential games.*

Proof. An exact potential function for generalized market sharing games can be constructed in the following manner: Let $n_j(\sigma)$ be the number of players choosing item j in strategy profile σ, i.e. $n_j(\sigma) = \left|\left\{k | j \in \sigma_k, k \in \{1 \ldots n\}\right\}\right|$. Define for given strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$:

$$\phi_j(\sigma) = w_j + \frac{1}{2} w_j + \frac{1}{3} w_j + \cdots + \frac{1}{n_j(\sigma)} w_j \tag{12}$$

$$\phi(\sigma) = \sum_{j \in J} \phi_j(\sigma) \tag{13}$$

Here $\phi(\sigma)$ is an exact potential function since every time a player increases his payoff, the value of the potential function increases with the same amount. If one player switches his strategy from σ_i to σ'_i, the potential function changes as follows:

$$\phi(\sigma'_i, \sigma_{-i}) - \phi(\sigma_i, \sigma_{-i}) = \sum_{j \in \sigma'_i \setminus \sigma_i} \frac{1}{n_j(\sigma) + 1} w_j - \sum_{j \in \sigma_i \setminus \sigma'_i} \frac{1}{n_j(\sigma)} w_j = w(\sigma'_i, \sigma_{-i}) - w_i(\sigma_i, \sigma_{-i}) \tag{14}$$

So the change in the potential function equals the change in payoff for player i. ■

Corollary 4.6 *Generalized market sharing games always have a pure Nash equilibrium.*

This follows from Monderer and Shapley [17], who showed that every potential game has at least one pure Nash equilibrium. This is because there are a finite amount of items and strategies, therefore the potential function attains its maximum for some σ. In this maximum no player can improve its payoff, otherwise the function could improve as well. This means that the strategy which gives this maximum is a pure Nash equilibrium. Since every generalized market sharing game has a potential function, every generalized market sharing game must have a pure Nash equilibrium.

4.3 Valid utility system

To find a first bound on the price of anarchy, we show that generalized market sharing games are valid utility systems. Valid utility systems are a class of games introduced by Vetta in 2002 [24], for which he proved an upper bound on the price of anarchy of 2. By showing generalized market sharing games are valid utility systems, this upper bound also holds for generalized market sharing games. The proof we give is similar to the proof used by Goemans et al. for market sharing games [6].

Valid utility systems require the social function to be a set function. Before we proof that generalized market sharing games are valid utility systems, we have to change the notation of the social function to a set function. We also change the notation of the strategy profile σ to a single set. We do this by defining a corresponding set function and a corresponding strategy profile, which are only different in notation.

A strategy profile $\sigma = (\sigma_1, \ldots \sigma_n)$ describes what strategy all players choose and determines which items are chosen by which players. The social function $U_u(\sigma)$ determines the social value that the strategy profile σ provides and since we use a utilitarian social function, it is defined as the sum over the payoff of all the players. This is equal to the sum over the value of all the items that were chosen at least once, i.e. $U_u(\sigma) = \sum_{i=1}^{n} w_i(\sigma_i, \sigma_{-i}) = \sum_{j\in\sigma} w_j$.

For any strategy profile σ, we define a corresponding set $H_\sigma = \{(k,i) | 1 \le k \le n, i \in \sigma_k\}$ called the *pair set* for σ. It has an element for each item choice of each player. For the social function U_u we define a corresponding set function U_u^s. The corresponding set function is identical to the original social function $U_u(\sigma)$, with the exception that it uses the pair set H_σ as input instead of the strategy set σ. This means that for any strategy set σ with a corresponding pair set H_σ, the social function and the corresponding set function give the same result, i.e. $U_u(\sigma) = U_u^s(H_\sigma)$. Now we can replace $U_u(\sigma)$ with U_u^s and σ with H_σ and still have the same game with all the same outcomes. Now that we have a set function, we can define what a submodular function is.

Definition 4.7 Submodular function
Given a universe H, a function of the form $f : 2^H \to \mathbb{R}$ is called a set function, where 2^H denotes the power set of H. A set function f is submodular if $f(\emptyset) = 0$ and for any two sets $A, B \subseteq H, f(A) + f(B) \ge f(A \cap B) + f(A \cup B)$. Furthermore, f is a nondecreasing function, if for any two sets X, Y with $X \subseteq Y \subseteq H$, it holds that $f(X) \le f(Y)$.

This definition of a submodular function is not always easy to use. Therefore we give an equivalent definition of a submodular function: A set function f is submodular if for any subsets A, B, such that $A \subseteq B$ and any element $i \notin B$, it holds that:

$$f(A \cup i) - f(A) \ge f(B \cup i) - f(B). \tag{15}$$

Now that we have the corresponding set function, the pair set and the definition of submodular, we can prove that generalized marker sharing games are valid utility systems.

Definition 4.8 Valid utility systems
There are three conditions for a game to be a valid utility system:

- The social function is submodular and nondecreasing
- The payoff a player receives is at least as great as the loss of social payoff if the player does not participate, while the rest of the players do not change their actions
- The sum of the payoff for the players must be less or equal to the social function

Theorem 4.9 *Generalized market sharing games are valid utility systems.*

Proof. We prove this by showing the social function of generalized market sharing games satisfies all 3 conditions. For the utilitarian social function $U_u(\sigma)$, we have $U_u^s(\sigma)$ as the corresponding set function.

- First we show that the social function is submodular and nondecreasing. For that we need two strategy profiles $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\sigma' = (\sigma'_1, \ldots, \sigma'_n)$ with $H_\sigma \subseteq H_{\sigma'}$. Each element of the pair set H_σ represents a player choosing a certain item, so $H_\sigma \subseteq H_{\sigma'}$ means that in σ' every player still chooses all the items they chose in σ, but they might choose some additional items, i.e. $\sigma_i \subseteq \sigma'_i \forall i$.
  For any two strategy profiles σ, σ' with $H_\sigma \subseteq H_{\sigma'}$, it holds that:

  $$U^s(H_{\sigma'}) = \sum_{j \in \sigma'} w_j = \sum_{j \in \sigma} w_j + \sum_{j \in (\sigma' \cap \overline{\sigma})} w_j \geq \sum_{j \in \sigma} w_j = U^s(H_\sigma),$$

  since all items have nonnegative values. Thus the social function is nondecreasing.
  To show that the social function is submodular, we use the equivalent definition in (15). Take again two strategy profiles σ, σ' with $H_\sigma \subseteq H_{\sigma'}$. Let (i,k) be an element with $(i,k) \notin H_{\sigma'}$. Adding (i,k) to $H_{\sigma'}$ means item k gets added to the strategy of player i. If item k was not chosen by any other player before, then the social function increases with value w_k. If it was chosen before, the value of the social function stays the same:

  $$U^s(H_{\sigma'} \cup (i,k)) - U^s(H_{\sigma'}) = \begin{cases} w_k & \text{if } k \notin \sigma' \\ 0 & \text{else} \end{cases}$$

  The same holds for σ. Since $H_\sigma \subseteq H_{\sigma'}$, it holds that if the item was not chosen in σ', then it was not be chosen in σ either. This means that if the social value increases with w_k for σ', it must increase with w_k for σ as well. This gives:

  $$U^s(H_{\sigma'} \cup (i,k)) - U^s(H_{\sigma'}) \geq U^s(H_\sigma \cup (i,k)) - U^s(H_\sigma),$$

  so the social function U^s_u is submodular.

- Let player i be the player which does not participate. The loss of social payoff when player i does not participate is equal to the value of all the items that were only chosen by player i. The value that player i gained from the items also chosen by other players is not lost, since it will now be divided among those other players. Since player i gets the full value of all the items only he chooses, his payoff must be at least equal to the value of all those items:

  $$U(\sigma) - U(\sigma_{-i}) = \sum_{j \in \sigma} w_j - \sum_{j \in \sigma_{-i}} w_j = \sum_{j \in (\sigma_i \cap \overline{\sigma_{-i}})} w_j \leq w_i(\sigma_i, \sigma_{-i})$$

  This means the payoff of player i is equal or higher than the loss in social payoff when he does not participate, so the second condition holds.

- Since the social function is defined as the sum of the payoff for the players, the third condition follows immediately.

■

Corollary 4.10 *$PoA \leq 2$ for generalized market sharing games.*

It follows that the price of anarchy is at most 2, since Vetta [24] proved this must hold for all valid utility systems. By providing this bound through valid utility systems we have kept the proof very general. This way this proof can be used for many different variants of generalized market sharing games. For instance, the proof does not even contain the way the value of an item is shared, so as long as the total value of an item is shared somehow among the players choosing it, the price of anarchy does not exceed 2. In the next section we improve this upper bound for generalized market sharing games.

5 Exact price of anarchy for generalized market sharing games

In the last section we showed that the price of anarchy of generalized market sharing games is at most 2. In this section we provide direct proofs for the price of anarchy for pure and mixed strategies and show that the price of anarchy is exactly equal to $2-\frac{1}{n}$ by providing tight bounds. Furthermore we generalize these results by proving that the robust price of is exactly equal to $2-\frac{1}{n}$, which means that these results even extend to correlated equilibria and coarse correlated equilibria. Lastly we discuss what these results mean for market sharing games and set packing games.

5.1 Exact price of anarchy for pure and mixed strategies

In this subsection we show that the price of anarchy of generalized market sharing games is $2-\frac{1}{n}$ by directly proving a tight upper bound and lower bound. We start off with pure strategies and afterwards generalize the proof to show it still holds for mixed strategies.

5.1.1 Exact price of anarchy for pure strategies

We begin by giving the upper bound on the price of anarchy for arbitrary generalized market sharing games with pure strategies.

Lemma 5.1 PoA $\leq 2-\frac{1}{n}$ *for generalized market sharing games with pure strategies.*

Proof. Take any instance with Nash equilibrium σ^{NE} and optimum σ^{OPT}, and let σ_i^{NE}, and σ_i^{OPT}, for $i = 1\dots n$, be the items selected by player i in the Nash equilibrium and the optimum, respectively. Let $w(\sigma) = \sum_{i=1}^{n} w_i(\sigma_i, \sigma_{-i})$. Furthermore let S^{NE} be all items chosen in the Nash equilibrium and S^{OPT} all items chosen in the optimum. For any $S \subseteq J$, let $\overline{S} = J \setminus S$ be the complement of S in J. Lastly let $w(S) = \sum_{j\in S} w_j$.

In a Nash equilibrium no player can improve his payoff by switching to another strategy. The strategy player i chose in the optimal solution, σ_i^{OPT}, is still available to him, but because it is a Nash equilibrium, it cannot give player i more payoff. So we have for all players i that $w_i(\sigma_i^{NE}, \sigma_{-i}) \geq w_i(\sigma_i^{OPT}, \sigma_{-i})$. Furthermore, since we consider pure strategies, we can split the set of items into three relevant categories: items chosen in the optimal strategy, items chosen in the Nash equilibrium and items chosen in both. We disregard items chosen in neither, since they are not relevant for the proof.

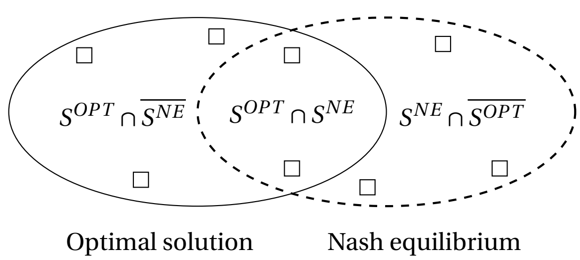


Figure 3: Splitting relevant items into three categories. The solid set of items represents the optimal solution, the dashed set of items represents the Nash equilibrium S

The strategy of any player consist of the items they choose. We split these strategies into different parts in the same way. The optimal strategy of player i consists of the items chosen by player i that are only chosen in the optimal strategy, $\sigma_i^{OPT} \cap \overline{S^{NE}}$, and the items that are chosen both in the optimal solution and the Nash equilibrium,

$\sigma_i^{opt} \cap S^{NE}$. This gives us the following lower bound on the social value of Nash equilibrium σ^{NE}:

$$
\begin{aligned}
w(\sigma^{NE}) &= \sum_{i=1}^{n} w_i(\sigma_i^{NE}, \sigma_{-i}^{NE}) \qquad (16)\\
&\geq \sum_{i=1}^{n} w_i(\sigma_i^{OPT}, \sigma_{-i}^{NE})\\
&= \sum_{i=1}^{n} \left(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{NE}) + w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}) \right)
\end{aligned}
$$

Since all items in $\sigma_i^{OPT} \cap \overline{S^{NE}}$ are not chosen by any players in the Nash equilibrium, player i must receive full value if he chooses those items and the rest of the players plays their Nash strategies. Therefore it must hold that $w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{NE}) \geq w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{OPT})$. Additionally, the payoff of player i can be made at most n times as bad by the strategy of other players. This would only happen if every other player chooses all the same items as player i. It follows that $w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}) \geq \frac{1}{n} w_i(\sigma_i^{OPT} \cap S, \sigma_{-i}^{OPT})$. Using this we can continue the proof:

$$
\begin{aligned}
w(\sigma^{NE}) &\geq \sum_{i=1}^{n} \left(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{NE}) + w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}) \right)\\
&\geq \sum_{i=1}^{n} \left(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{OPT}) + w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}) \right)\\
&\geq \sum_{i=1}^{n} \left(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{OPT}) + \frac{1}{n} w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{OPT}) \right)\\
&= w(\sigma^{OPT} \cap \overline{S^{NE}}) + \frac{1}{n} w(\sigma^{OPT} \cap S^{NE}). \qquad (17)
\end{aligned}
$$

Here the last step follows from $w(\sigma^{OPT}) = \sum_{i=1}^{n} w_i(\sigma_i^{OPT}, \sigma_{-i}^{OPT})$. Because all strategies are pure, all items are either chosen or not with certainty. This gives

$$
w(\sigma^{NE}) = w(S^{NE}) \geq w(S^{OPT} \cap S^{NE}) = w(\sigma^{OPT} \cap S^{NE}), \qquad (18)
$$

since $(S^{OPT} \cap S^{NE}) \subseteq S^{NE}$. We can use (17) and (18) to finish the proof:

$$
\begin{aligned}
\left(2 - \frac{1}{n}\right) w(\sigma^{NE}) &= \left(1 - \frac{1}{n}\right) w(\sigma^{NE}) + w(\sigma^{NE}) \qquad (19)\\
&\geq \left(1 - \frac{1}{n}\right) w(\sigma^{OPT} \cap S^{NE}) + w(\sigma^{NE})\\
&\geq \left(1 - \frac{1}{n}\right) w(\sigma^{OPT} \cap S^{NE}) + w(\sigma^{OPT} \cap \overline{S^{NE}}) + \frac{1}{n} w(\sigma^{OPT} \cap S^{NE})\\
&= w(\sigma^{OPT} \cap S^{NE}) + w(\sigma^{OPT} \cap \overline{S^{NE}})\\
&= w(\sigma^{OPT})
\end{aligned}
$$

This gives $2 - \frac{1}{n} \geq \frac{w(\sigma^{OPT})}{w(\sigma^{NE})}$. Since this holds for any Nash equilibrium, it gives PoA $\leq 2 - \frac{1}{n}$.

■

Now that we have an upper bound, we give a matching lower bound. We provide a lower bound by giving an example.

Lemma 5.2 PoA $\geq 2 - \frac{1}{n}$ *for generalized market sharing games with pure strategies.*

Proof. The lower bound follows from the following example:

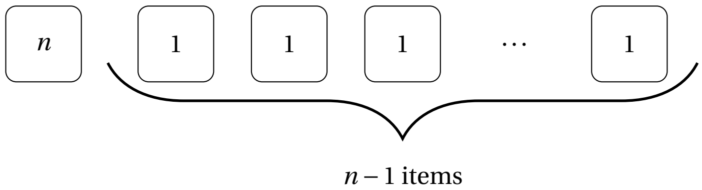


Figure 4: Graphic representation of the game in Example 5.3

Example 5.3 Consider a game with n players and n items. There are $n-1$ items with value 1, and one item with value n. Any player can choose any one item.

Proof of the lower bound. The optimal solution of Example 5.3 consists of 1 player taking the item worth n and the other $n-1$ players all choosing an item worth 1, giving a total value of $2n-1$. However, a Nash equilibrium occurs when every player chooses the item of value n, which gives each player individually the value 1. No player can improve his gained value by switching to an item of value 1. The total value gained is n.

This means the price of anarchy in this example is $\frac{2n-1}{n} = 2 - \frac{1}{n}$, so we conclude that PoA $\geq 2 - \frac{1}{n}$ for generalized market sharing games. ■

We now have a matching upper and lower bound. We summarize as follows.

Theorem 5.4 PoA $= 2 - \frac{1}{n}$ *for generalized market sharing games with pure strategies.*

Another interesting observation about Example 5.3 is when the value of the first item is increased a little from n to $n+\epsilon$, for small $\epsilon > 0$, the Nash equilibrium is unique. This means that the price of stability is bounded by $2 - \frac{1}{n} = \text{PoA} \geq \text{PoS} \geq 2 - \frac{1+\epsilon}{n}$. This gives PoS $= 2 - \frac{1}{n}$ by taking the limit.

Corollary 5.5 PoS $= 2 - \frac{1}{n}$ *for generalized market sharing games with pure strategies.*

5.1.2 Exact price of anarchy for mixed strategies

Now we look at how allowing mixed strategies influence the price of anarchy. Mixed strategies can sometimes create worse Nash equilibria than pure strategies. Observe the following example:

Example 5.6 Consider a game with n players and n items. All items have value 1 and any player can choose any one item.

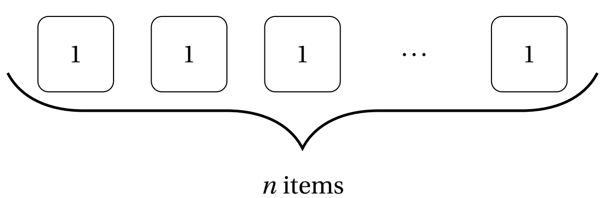


Figure 5: Graphic representation of the game in Example 5.6

Since every player can only select one item, the optimal solution is each player choosing one item and the social payoff is n. This is also the only Nash equilibrium if only pure strategies are allowed. However, when mixed strategies are allowed, worse Nash equilibria are possible. For instance, a Nash equilibrium occurs if every player chooses every item with probability $\frac{1}{n}$. This would give an expected social payoff of $n(1-\frac{1}{e}) \approx 0.632n$. So in this specific case the price of anarchy would go from 1 to $\frac{e}{e-1} \approx 1.58$.

To see if mixed strategies can make the price of anarchy worse for generalized market sharing games in general, we give a direct proof for an upper bound. But first we state the following lemma, to make the upcoming proof easier.

Lemma 5.7 *Generalized market sharing games always have an optimal solution consisting of only pure strategies.*

Proof. Assume this is not true. Then there exists a game without a pure optimal solution. Let $\sigma^{OPT} = (\sigma_1^{OPT}, \ldots, \sigma_n^{OPT})$ be the optimal solution of that game with the least players playing a mixed strategy. Since there is no pure optimal solution, there exists at least one player i playing a mixed strategy in σ^{OPT}, namely strategy σ_i^{OPT}. This mixed strategy is a combination of k different pure strategies $A_i^1, \ldots, A_i^k$, with $k \geq 2$. If any of these pure strategies is chosen by player i instead of σ_i^{OPT}, the social value would decrease, otherwise σ^{OPT} is not the optimal solution with the least players playing a mixed strategy. So for all A_i^ℓ holds:

$$U_u(A_i^\ell \cap \sigma_{-i}^{OPT}) < U_u(\sigma_i^{OPT} \cap \sigma_{-i}^{OPT}) = U_u(\sigma^{OPT})$$

Let $p_1, \ldots, p_k$ be the probabilities of each pure strategy being chosen. Because of the linearity of the expectation we get that

$$U_u(\sigma_i^{OPT} \cap \sigma_{-i}^{OPT}) = \sum_{l=1}^{k} \left(p_l \cdot U_u(A_i^\ell \cap \sigma_{-i}^{OPT})\right) < \sum_{l=1}^{k} \left(p_l \cdot U_u(\sigma_i^{OPT} \cap \sigma_{-i}^{OPT})\right) = U_u(\sigma_i^{OPT} \cap \sigma_{-i}^{OPT}).$$

This is a contradiction, so therefore there must always be an optimal solution consisting of only pure strategies. ■

Now we look at the bounds on the price of anarchy again.

Lemma 5.8 PoA $\leq 2 - \frac{1}{n}$ *for generalized market sharing games with mixed strategies.*

Proof. The first part of the proof is similar to the proof of Lemma 5.1. Once again, take any instance with Nash equilibrium σ^{NE} and optimum σ^{OPT}.

$$\begin{aligned}
U_u(\sigma^{NE}) = E(w_i(\sigma^{NE})) &= \sum_{i=1}^{n} E(w_i(\sigma_i^{NE}, \sigma_{-i}^{NE})) \\
&\geq \sum_{i=1}^{n} E(w_i(\sigma_i^{OPT}, \sigma_{-i}^{NE})) \\
&= \sum_{i=1}^{n} \left(E(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{NE})) + E(w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}))\right) \\
&\geq \sum_{i=1}^{n} \left(E(w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}, \sigma_{-i}^{OPT})) + E(w_i(\sigma_i^{OPT} \cap S, \sigma_{-i}^{NE}))\right) \\
&= E(w(\sigma^{OPT} \cap \overline{S^{NE}})) + \sum_{i=1}^{n} \left(E(w(\sigma_i^{OPT} \cap S, \sigma_{-i}^{NE}))\right)
\end{aligned} \tag{20}$$

Let p_j be the probability that item j is chosen at least once in Nash equilibrium σ^{NE}, so $E(U_u(\sigma^{NE})) = \sum_{j \in S^{NE}} p_j \cdot w_j$. Because of Lemma 5.7 we can assume that σ^{OPT} is a pure strategy, so $U_u(\sigma^{OPT}) = \sum_{j \in S^{OPT}} w_j$. Now we can better describe the payoff for a player i if he switches to a pure strategy. If player i chooses an item j while the rest of the players play their Nash strategies, then there is a probability of at least $1 - p_j$ that player i is the only player to choose item j. In this case player i receives the full value w_j from item j. There is also a probability of at most p_j that player i shares item j with one or more other players. Even if item j is shared with every other player, player i still gets at least value $\frac{w_j}{n}$ from item j. Using this, we can bound $E(w(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE}))$ by bounding the value that each item would provide for player i:

$$
\begin{aligned}
E(w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE})) &\geq \sum_{j \in (\sigma_i^{OPT} \cap S^{NE})} \left((1-p_j)w_j + \frac{p_j}{n} w_j\right) \\
\sum_{i=1}^{n} E(w_i(\sigma_i^{OPT} \cap S^{NE}, \sigma_{-i}^{NE})) &\geq \sum_{i=1}^{n} \sum_{j \in (\sigma_i^{OPT} \cap S^{NE})} \left((1-p_j)w_j + \frac{p_j}{n} w_j\right) \\
&\geq \sum_{j \in (S^{OPT} \cap S^{NE})} \left((1-p_j)w_j + \frac{p_j}{n} w_j\right)
\end{aligned} \tag{21}
$$

Where the last step holds since $\bigcup_{i=1}^{n}(\sigma_i^{OPT} \cap S^{NE}) = S^{OPT} \cap S^{NE}$. This can be further rewritten to:

$$
\begin{aligned}
\sum_{j \in (S^{OPT} \cap S^{NE})} \left((1-p_j)w_j + \frac{p_j}{n} w_j\right) &= \sum_{j \in (S^{OPT} \cap S^{NE})} \left(w_j - \left(1 - \frac{1}{n}\right) p_j w_j\right) \\
&= \left(\sum_{j \in (S^{NE} \cap S^{OPT})} w_j\right) - \left(1 - \frac{1}{n}\right) \sum_{j \in (S^{NE} \cap S^{OPT})} (p_j w_j) \\
&= w(\sigma^{OPT} \cap S^{NE}) - \left(1 - \frac{1}{n}\right) E(w(\sigma^{NE}))
\end{aligned} \tag{22}
$$

Combining these derivations gives:

$$
\begin{aligned}
E(w(\sigma^{NE})) &\geq w(\sigma^{OPT} \cap \overline{S^{NE}}) + E(w(\sigma^{OPT} \cap S^{NE})) - \left(1 - \frac{1}{n}\right) E(w(\sigma^{NE})) \\
\left(2 - \frac{1}{n}\right) E(w(\sigma^{NE})) &\geq w(\sigma^{OPT} \cap \overline{S^{NE}}) + w(\sigma^{OPT} \cap S^{NE}) \\
&= w(\sigma^{OPT})
\end{aligned} \tag{23}
$$

This gives $2 - \frac{1}{n} \geq \frac{w(\sigma^{OPT})}{E(w(\sigma^{NE}))} = \frac{U_u(\sigma^{OPT})}{U_u(\sigma^{NE})}$ and therefore PoA $\leq 2 - \frac{1}{n}$. ■

Lemma 5.9 PoA $\geq 2 - \frac{1}{n}$ *for generalized market sharing games with mixed strategies.*

Proof. Example 5.3 once again provides a price of anarchy of $2 - \frac{1}{n}$, even if mixed strategies are allowed. The Nash equilibrium and the optimal solution stay the same. See the proof of Lemma 5.2 for more on Example 5.3. ■

Lemma 5.10 PoA $= 2 - \frac{1}{n}$ *for generalized market sharing games with mixed strategies.*

The upper bound from Lemma 5.8 can also be extracted from the work of Gairing [5]. He provided an upper bound on the price of anarchy for covering games. Where generalized market sharing games distribute the value of a resource equally and completely among all players choosing a resource, covering games do not have a fixed payoff function. When equal sharing is chosen as the payoff function in the upper bound of Gairing, it provides an upper bound of $2 - \frac{1}{n}$.

5.2 The Robust Price of Anarchy in generalized market sharing games and set packing games

In the previous subsections we have shown that the price of anarchy for pure and mixed Nash equilibria is $2-\frac{1}{n}$. In this subsection we generalize these results by proving that the robust price of anarchy is $2-\frac{1}{n}$ as well. The robust price of anarchy was introduced in 2009 by Roughgarden [23]. He established the smoothness argument and showed that if an upper bound on price of anarchy for pure strategies can be shown though a smoothness argument, then in automatically extends to mixed Nash equilibria, correlated equilibria and coarse correlated equilibria. The lowest upper bound possible though the smoothness argument is called the robust price of anarchy.

5.2.1 The Robust Price of Anarchy in generalized market sharing games

A payoff-maximization game is (λ,μ)-smooth [23] if for every two strategy profiles σ and σ':

$$\sum_{i=1}^{n} w_i(\sigma'_i,\sigma_{-i}) \geq \lambda \cdot w(\sigma') - \mu \cdot w(\sigma). \tag{24}$$

If a game is (λ,μ)-smooth (with $\lambda > 0$ and $\mu > -1$), it easily follows that for Nash equilibrium σ^{NE} and optimal solution σ^{OPT} :

$$w(\sigma^{NE}) \geq \sum_{i=1}^{n} w_i(\sigma_i^{OPT},\sigma_{-i}^{NE}) \geq \lambda \cdot w(\sigma^{OPT}) - \mu \cdot w(\sigma^{NE}) \tag{25}$$

$$(1+\mu)\cdot w(\sigma^{NE}) \geq \lambda \cdot w(\sigma^{OPT})$$

$$\frac{1+\mu}{\lambda} \geq \frac{w(\sigma^{OPT})}{w(\sigma^{NE})}$$

Therefore $\frac{1+\mu}{\lambda}$ is an upper bound for the price of anarchy. The robust price of anarchy [23] is defined as the best (i.e. least) upper bound on the POA that is provable via a smoothness argument.

$$\text{Robust PoA } = \inf\left\{\frac{1+\mu}{\lambda} : (\lambda,\mu) \text{ s.t. the game is } (\lambda,\mu)\text{-smooth}\right\}. \tag{26}$$

Lemma 5.11 *Generalized market sharing games are* $(1,\frac{n-1}{n})$*-smooth*

Proof. The proof that generalized market sharing games are smooth uses the same argument as Lemma 5.1. Let σ and σ' be any two pure strategy profiles. To show that generalized market sharing games are $(1,\frac{n-1}{n})$-smooth, we need to show the following:

$$\sum_{i=1}^{n} w_i(\sigma'_i,\sigma_{-i}) \geq 1 \cdot w(\sigma') - \frac{n-1}{n} \cdot w(\sigma). \tag{27}$$

Let S be the set of items chosen in strategy σ. We start out with dividing $w_i(\sigma'_i,\sigma_{-i})$ into the items that are in S and the items in $\overline{S}$.

$$\sum_{i=1}^{n} w_i(\sigma'_i,\sigma_{-i}) = \sum_{i=1}^{n}\Big(w_i(\sigma'_i\cap\overline{S},\sigma_{-i}) + w_i(\sigma'_i\cap S,\sigma_{-i})\Big) \tag{28}$$

Here we can use that $w_i(\sigma'_i\cap\overline{S},\sigma_{-i}) \geq w_i(\sigma'_i\cap\overline{S},\sigma'_{-i})$, since σ_{-i} does not choose an item from $\overline{S}$. This means player i gets full value from the items in $(\sigma'_i\cap\overline{S})$, so this value can only go down if other players change their strategy. We can also use $w_i(\sigma'_i\cap S,\sigma_{-i}) \geq \frac{1}{n} w_i(\sigma'_i\cap S,\sigma'_{-i})$, since the value a player receives from an item can be made at most n times as small. Finally, since we only look at pure strategies, the following holds:

$$\sum_{i=1}^{n} w_i(\sigma'_i\cap S,\sigma'_{-i}) = w(\sigma'\cap S) \leq w(\sigma'). \tag{29}$$

Using this we can complete the proof:

$$\begin{aligned}\sum_{i=1}^{n} w_i(\sigma'_i, \sigma_{-i}) &= \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma_{-i}) + w_i(\sigma'_i \cap S, \sigma_{-i}) \Big) \\ &\geq \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma'_{-i}) + w_i(\sigma'_i \cap S, \sigma_{-i}) \Big) \\ &\geq \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma'_{-i}) + \frac{1}{n} w_i(\sigma'_i \cap S, \sigma'_{-i}) \Big) \\ &= w(\sigma' \cap \overline{S}) + \frac{1}{n} w(\sigma' \cap S) \\ &= w(\sigma' \cap \overline{S}) + w(\sigma' \cap S) - \frac{n-1}{n} w(\sigma' \cap S) \\ &= w(\sigma') - \frac{n-1}{n} w(\sigma' \cap S) \\ &= w(\sigma') - \frac{n-1}{n} w(\sigma).\end{aligned} \tag{30}$$

This is the equation we wanted, so generalized market sharing games are $(1, \frac{n-1}{n})$-smooth. ∎

Since generalized market sharing games are $(1, \frac{n-1}{n})$-smooth, this yields another proof of the upper bound on the price of anarchy of $2 - \frac{1}{n}$. We have already shown a matching lower bound, so it is impossible to get a lower smoothness result. Therefore this smoothness result is equal to the robust PoA.

Theorem 5.12 *The robust PoA is* $2 - \frac{1}{n}$ *for generalized market sharing games*

The advantage of the robust price of anarchy is that it does not only hold for pure Nash equilibria, but it also extends to other equilibria. Its upper bound also extends to the price of anarchy of mixed Nash equilibria, correlated equilibria and coarse correlated equilibria [23]. See Definition 3.9 and Definition 3.10 for more on correlated equilibria and coarse correlated equilibria.

5.2.2 The Robust Price of Anarchy in set packing games

The arguments needed for the proof of Lemma 5.1 are the same arguments needed to conclude that the robust price of anarchy is $2 - \frac{1}{n}$ for generalized market sharing games. However, this is not always the case. De Jong and uetz studied set packing games, which are games closely related to generalized market sharing games. The only difference is that in set packing games each item can only be chosen once. De Jong and Uetz proofed that set packing games have a price of anarchy of 2 [11]. The proof for the upper bound is similar to the proof of Lemma 5.1, and on first glance seems to use the smoothness argument as well. In this subsection we show that this is not the case.

Definition 5.13 Set packing games
There is a set N of n players, and a finite ground set J of items. Each item $j \in J$ has a value w_j. For $S \subseteq J$, we let $w(S) := \sum_{j \in S} w_j$. Each player i has a strategy set $\mathbf{S}_i \subseteq 2^J$ which is downward closed, i.e., if $\sigma_i \in \mathbf{S}_i$, then $T_i \in \mathbf{S}_i$ for all $T_i \subseteq \sigma_i$. For a strategy profile $\sigma = w_i(\sigma_i, \sigma_{-i})$ the payoff for player i is defined as :

$$w_i(\sigma_i, \sigma_{-i}) = \begin{cases} w_i(\sigma_i) & \text{if } \sigma_i \cap \sigma_k = \emptyset \text{ for all } k \neq i \\ -\infty & \text{otherwise} \end{cases} \tag{31}$$

The proof by the Jong and Uetz that the price of anarchy of set packing games is at most 2 is given using only sets of items. To clarify why it does not use the smoothness argument, we first change the notation of the proof to strategies:

$$
\begin{aligned}
2w(\sigma^{NE}) &\geq w(\sigma^{NE}) + w(\sigma^{OPT} \cap S^{NE}) \\
&= \sum_{i=1}^{n} w_i(\sigma_i^{NE}) + w(\sigma^{OPT} \cap S^{NE}) \\
&\geq \sum_{i=1}^{n} w_i(\sigma_i^{OPT} \cap \overline{S^{NE}}) + w(\sigma^{OPT} \cap S^{NE}) \\
&= w(\sigma^{OPT})
\end{aligned}
$$

It follows here that the PoA ≤ 2. When we extract the part that looks like the smoothness argument, we get:

$$\sum_{i=1}^{n} w_i(\sigma_i^{OPT} \cap \overline{S}, \sigma_{-i}^{S}) \geq w(\sigma^{OPT}) - w(\sigma_S). \tag{32}$$

This actually holds for any two strategies, since we did not use the Nash property for this part of the proof. However, the smoothness argument states that the following must hold for any two strategies σ and σ':

$$\sum_{i=1}^{n} w_i(\sigma_i', \sigma_{-i}) \geq \lambda \cdot w(\sigma') - \mu \cdot w(\sigma) \tag{33}$$

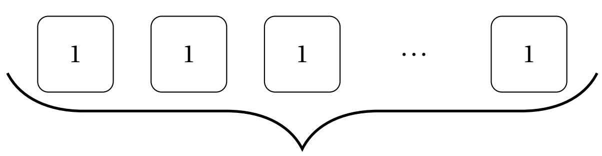
This second statement does not generally follow from the first statement. It only shows that the second statement holds when the strategy σ' has no items of σ, i.e. $\sigma \cap \sigma' = \emptyset$. Obviously this is not always the case, so it does not follow that the game is smooth.

In fact, it can be shown that the robust price of anarchy for set packing games is ∞. Since the robust price of anarchy provides an upper bound for mixed strategy games as well, it must be equal or larger than the price of anarchy for mixed strategies.

Theorem 5.14 PoA $= \infty$ *for set packing games , assuming players are allowed to play mixed strategies.*

Proof. To show that PoA $= \infty$, we only need to show that PoA $\geq \infty$. This is done by providing a lower bound example

Example 5.15 Consider a game with 2 players and c items. Each item has value 1. Both players have access to all items, but the first player can pick only one item, while the second player can pick as may items as he wants.



c items

Proof of the lower bound. In the optimal solution the first player picks one item, and the second player picks the rest of the items. Therefore the optimal value is c. Now consider the Nash equilibrium where the first player picks each item with probability $\frac{1}{c}$. Now the second player won't pick anything, since picking any item gives him a $\frac{1}{c}$ chance of a payoff of $-\infty$. Therefore he is better off picking nothing. The first player cannot improve either, his payoff is 1 and that is his maximum payoff possible. The total value of this Nash equilibrium is 1. This gives a price of anarchy of c. Since c can be taken arbitrarily large, this provides a lower bound on the price of anarchy of ∞. ■

Modification of payoff function for set packing games
The gap between the price of anarchy and the robust price of anarchy for set packing games is quite large. This

follows from the way the payoff function is defined (31). It is interesting to investigate what happens if the payoff for item sharing was defined in a different way. For instance, what if it was defined as follows:

$$w_i(\sigma_i, \sigma_{-i}) = \sum_{j \in \sigma_i} c_j w_j, \qquad \text{where } c_j = \begin{cases} 1 & \text{if } w_j \notin \sigma_k \text{ for all } k \neq i \\ -\epsilon & \text{otherwise} \end{cases} \tag{34}$$

This means whenever there are multiple players choosing the same item j, the players now receive $-\epsilon w_j$ for that item. This still means that any pure Nash equilibrium cannot have multiple players choosing the same item. The players receive more payoff by dropping any contested items from their strategy. This is always possible, since the sets are downward closed.

The new payoff function has interesting consequences. We can show that the game is $(1, 1+\epsilon)$ smooth, since the following holds:

$$\sum_{i=1}^{n} w_i(\sigma'_i, \sigma_{-i}) \geq w(\sigma') - (1+\epsilon) w(\sigma) \tag{35}$$

Theorem 5.16 *Set packing games with a payoff function as in* (34) *are* $(1, 1+\epsilon)$ *smooth.*

Proof. Let σ and σ' be any two pure strategies, and let S be the set of items chosen in strategy σ.

$$\begin{aligned} \sum_{i=1}^{n} w_i(\sigma'_i, \sigma_{-i}) &= \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma_{-i}) + w_i(\sigma'_i \cap S, \sigma_{-i}) \Big) \\ &\geq \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma'_{-i}) + w_i(\sigma'_i \cap S, \sigma_{-i}) \Big) \\ &\geq \sum_{i=1}^{n} \Big(w_i(\sigma'_i \cap \overline{S}, \sigma'_{-i}) - \epsilon w_i(\sigma'_i \cap S, \sigma'_{-i}) \Big) \\ &= w(\sigma' \cap \overline{S}) - \epsilon w(\sigma' \cap S) \\ &= w(\sigma' \cap \overline{S}) + w(\sigma' \cap S) - (1+\epsilon) w(\sigma' \cap S) \\ &= w(\sigma') - (1+\epsilon) w(\sigma' \cap S) \\ &\geq w(\sigma') - (1+\epsilon) w(S) \\ &= w(\sigma') - (1+\epsilon) w(\sigma) \end{aligned} \tag{36}$$

Taking the limit of ϵ to zero makes the robust price of Anarchy equal to 2, since the robust price of anarchy is defined as the infimum in (26). This means that the price of anarchy for mixed strategies is suddenly bounded from above by two. Since the lower bound example for pure strategies still holds, this means that the price of anarchy for mixed strategies must also be equal to 2. ■

Marginal contribution

The same result can be found by using a slightly different model then set packing games. In 2013 Marden and Wierman [16] studied distributed welfare games with marginal contribution. Here the social value would still be equal to the value of all items chosen at least once, but the value of an item j, $w_j(\sigma)$, would depend on the set of players choosing it. Furthermore, the payoff players would receive from an item j would be equal to extra social value they created by choosing that item, i.e.

$$w_i(\sigma_i, \sigma_{-i}) = \sum_{j \in \sigma_i} w_j(\sigma) - w_j(\sigma_{-i}) \tag{37}$$

Notice that this could cause the sum of the value gained by all the players to be different than the social value. If all the item costs are kept constant regardless of the set of people choosing it, then it would be very similar to set packing games. If two players would choose the same item, they would both receive nothing. However, the difference is that the the value of the item does count towards the social function. Marden and Wierman showed that with this property the game is a valid utility game, which provides an upper bound of 2 on the price of anarchy.

6 The Sequential Price of Anarchy for generalized market sharing games

The sequential price of anarchy was introduced by Paes Leme, Syrgkanis and Tardos [14]. They advocated that letting players choose their strategies sequentially would sometimes allow to improve the quality of equilibria. However, this did not always turn out to be the case. There exist results showing that for some games the equilibria are better in the sequential version, and for some games they are worse. In this section we investigate sequential generalized market sharing games, and see hows sequentially choosing strategies influences the strategies of players and the quality of equilibria. We try different proof techniques to bound the sequential price of anarchy.

When players are allowed to choose sequentially, it is important that the strategy sets are *downwards closed*. This means for any strategy σ_i of player i, every strategy σ'_i with $\sigma'_i \subset \sigma_i$ is also feasible. For the simultaneous version this does not matter as much, since players never play a strategy σ'_i in a Nash equilibrium, since that strategy is dominated by σ_i. However, in a subgame perfect equilibrium it can happen that a player improves his payoff by choosing less items. In our examples we describe strategy sets by the maximal subsets that can be chosen.

We first introduce a concept to describe a connection between the simultaneous and sequential representation of a game:

Definition 6.1 Nash-stable
We call a subgame perfect equilibrium Nash-stable if the actions taken by the players in the subgame perfect equilibrium induce a Nash equilibrium in the simultaneous version of the game.

We start out with some general bounds on the sequential price of anarchy:

Theorem 6.2 SPoA $\geq 2 - \frac{1}{n}$ *for generalized market sharing games.*

Proof. In the preceding section we used Example 5.3 as a lower bound example for the price of anarchy. When we impose an order $1, \dots, n$ on the players, we find that the subgame perfect equilibrium has the same actions as the Nash equilibrium in the simultaneous version of the game. It still involves every player choosing the item of value n, while the optimal solution has value $2n-1$. This means SPoA $\geq 2 - \frac{1}{n}$. ■

Theorem 6.3 SPoA $\leq n$ *for generalized market sharing games.*

Proof. When a player chooses a strategy, the worst outcome for that player would be if every other player chooses the same items. He would only receive $\frac{1}{n}$ of the value of those items. But since this is the worst case scenario, this means every player gets at least $\frac{1}{n}$ of the value of the items he chooses. Since the items of the optimal solution are always available, it holds for all players that they must be able to get at least $\frac{1}{n}$ of the value they receive in the optimal solution:

$$w(\sigma_i^{SPE}, \sigma_{-i}^{SPE}) \geq \frac{1}{n} w(\sigma_i^{OPT}, \sigma_i^{OPT})$$
$$\frac{w(\sigma^{OPT}, \sigma^{OPT})}{w(\sigma^{SPE}, \sigma^{SPE})} \leq n.$$

So the sequential price of anarchy must be smaller or equal to n. ■

Together Theorem 6.2 and Theorem 6.3 provide $2 - \frac{1}{n} \leq$ SPoA $\leq n$. These are still fairly large gaps, so in the coming sections we try to improve these bounds. Most proofs on the sequential price of anarchy use properties very specific to the game being investigated, and those proofs cannot be used in other games. In this subsection we consider a few properties of sequential generalized market sharing games and see if they could be used to bound the SPoA.

One such property was used by de Jong and Uetz, who investigated the sequential price of anarchy of set packing games [11]. They showed that in set packing games each subgame perfect equilibrium was Nash-stable. However, this is not the case for generalized market sharing games. See the following example:

Example 6.4 Consider a game with 2 players and 4 items. The items have values 6,8,4 and 7. Player 1 can choose the items with value 6 and 8 or the items with value 8 and 4. Player 2 can pick the items with value 8 and 4 or the item with value 7. Both players can also pick any subset of the strategies available.

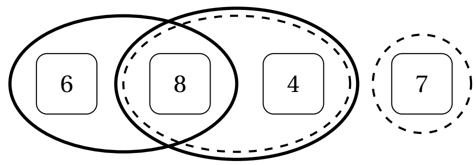


Figure 6: Example 6.4. Solid lines represent strategies for player 1, dashed lines represent strategies for player 2.

If the items are chosen simultaneously, then there is only one Nash equilibrium. In this Nash equilibrium player 1 chooses the items with values 6 and 8, and player 2 chooses the items with values 8 and 4. This gives player 1 a payoff of 10 and player 2 a payoff of 8, for a total social value of 18.

If the items are chosen sequentially, then player 1 should pick the items with value 8 and 4, which leads to player 2 choosing the item with value 7. This gives player 1 a payoff of 12 and player 2 a payoff of 7. This subgame perfect equilibrium is not Nash-stable, since player 1 could switch to choosing {6,8} and increase his payoff to 14, as long as player 2 does not change his strategy.

In fact, even in symmetric games a subgame perfect equilibrium is not necessarily Nash-stable. The following example is a symmetric sequential generalized market sharing game, where the subgame perfect equilibrium is not Nash-stable.

Example 6.5 Consider a game with 2 players and 3 items. The items have values 6,8 and 12. Both players have the same strategies available. They can pick either the items worth 6 and 8, or the items worth 8 and 12. They can also choose any subset of those strategies.

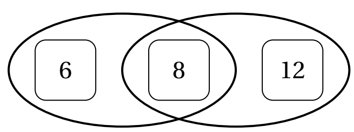


Figure 7: Example 6.5. The circles represent available strategies. both players the same strategies available.

In the game of Example 6.5 a subgame perfect equilibrium occurs when player 1 picks only the item of value 12, and player 2 then picks the two items of value 8 and 6, giving him a total value of 14. This is not a Nash equilibrium, because player 1 can get a better payoff by taking the item of 8 together with the item of 12 if the choice of player 2 stays the same. There is two interesting things to note from this subgame perfect equilibrium. The first is that this is a symmetric sequential game, where the second player receives more payoff than the first player. The second is that this subgame perfect equilibrium does follow from a tie, after player 1 has chosen the item of value 12, player 2 can get a payoff of 14 by choosing the items of value 6 and 8, or by choosing the items of 8 and 12. However, if player 2 would choose the items with value 8 and 12 in such a situation, then player 1 would have never chosen just item of value 12 and the subgame perfect equilibrium changes.

Generic games are games where ties do not occur and in certain cases improve the sequential price of anarchy [7]. However, even in generic games subgame perfect equilibria are not necessarily Nash-stable. Example 6.4 is a generic game that demonstrates this.

What does stand out in the previous examples is that the sequential price of anarchy is better that the price of anarchy. In Example 6.2 the subgame perfect equilibrium gives a social value of 19, while the Nash equilibrium in the simultaneous variant has a social value of 18. So in this example making the game sequential improves the social value. This is not always the case, as is demonstrated in the next example:

Example 6.6 Consider a game with 2 players and 4 items. The items have values 12,8,17 and 18. Player 1 can choose the items with value 12 and 8 or the item with value 17. Player 2 can pick the items with value 8 and 17 or the item with value 18.

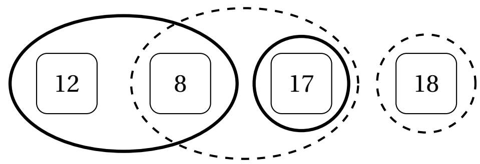


Figure 8: Example 6.6. Solid lines represent strategies for player 1, dashed lines represent strategies for player 2.

In Example 6.6 the optimum strategy is for player one to choose the items with values 12 and 8 and for player two to pick the item with value 18, which gives a social value of 38. However, the only Nash equilibrium is {12,8} and {8,17}, which gives a value of 37. Finally, the subgame perfect equilibrium gives strategy {17} for player 1 and {18} for player 2. This gives a total value of 35. So making a game sequential can also make it worse.

While Example 6.6 provides a subgame perfect equilibrium with less social value than the Nash equilibrium in the simultaneous variant, player 1 still improves his payoff. But even player 1 does not necessarily improve his payoff in the subgame perfect equilibrium, as compared to the worst Nash equilibrium. See the following example:

Example 6.7 Consider a game with 2 players and 3 items. The items have values 3,4 and 2. Player 1 can choose the item with value 3 or the item with value 4. Player 2 can pick the item with value 4 or the item with value 2.

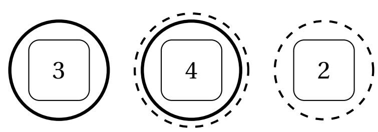


Figure 9: Example 6.7. Player 1 can choose between all sets of solid circles, player 2 can choose any set with dashed circles.

In Example 6.7 the worst Nash equilibrium is for player 1 to choose the item of value 4 and player two to pick the item with value 2. However, if player 1 picks the item of value 4 in the sequential version of the game, then player 2 might pick that item as well, since it gives him the same payoff of 2. In this case it is a subgame perfect equilibrium when player 1 is picks the item of value 3, and player 2 picks the item of value 4. While this is also a Nash equilibrium in the simultaneous game, it has a worse payoff for player 1 then the worst Nash equilibrium of the game.

6.1 Bounding the SPoA with Nash-difference

In a sequential game, players may make decisions that are sub-optimal at that moment, but better in the span of the whole game. Therefore not all decisions have to fulfill the Nash property. Examples in the previous subsection already showed that sequential games can lead to worse equilibria than simultaneous games. However, the difference in social value between the subgame perfect equilibria and the Nash equilibria in the simultaneous version of the game was not that high. If we can bound this difference, then we can use this in combination with the bounds on the price of anarchy to bound the sequential price of anarchy.

A first thing to notice is that the last player always picks as much value as he can, so he always fulfills the Nash property in the subgame perfect equilibrium. This can be used to prove upper bounds for 2 players. In this subsection we bound the ratio between the social value of the worst Nash equilibrium and the subgame perfect equilibrium for generic games and symmetric generic games for 2 players. We measure this with the Nash-difference.

Definition 6.8 Nash-difference
For any game I, the *Nash-difference* is the maximum ratio between the social value of a subgame perfect equilibrium σ^{SPE} and the worst social value of a Nash equilibrium σ^{NE} in the simultaneous version of the game, i.e.

$$\text{Nash-difference}(I) = \max_{\sigma^{APE} \in \sigma^{SPE}(I)} \frac{w(\sigma^{NE})}{w(\sigma^{SPE})} \tag{38}$$

where $\sigma^{SPE}(I)$ denotes the set of all subgame perfect equilibria in the game I. The Nash-difference of any class of games C is defined as:

$$\text{Nash-difference}(C) = \sup_{I \in C} \text{Nash-difference}(I) \tag{39}$$

We write Nash-difference if the game or class of games is clear from the context.

Generic generalized market sharing games with 2 players

Theorem 6.9 *The Nash-difference* $\leq \frac{3}{2}$ *for generic generalized market sharing games with 2 players*

Proof. Let $(\sigma_1^{SPE}, \sigma_2^{SPE})$ be the actions of a subgame perfect equilibrium, and $(\sigma_1^{NE}, \sigma_2^{NE})$ be the strategies of the worst Nash equilibrium in the simultaneous version of the game.

In the sequential game, player 1 has the option to play according to the worst Nash equilibrium, i.e. σ_1^{NE}. This will be followed up by σ_2^{NE}, there is no strategy that would give player 2 more, otherwise it would not be a Nash equilibrium. There is also no other strategy that would give player 2 equal payoff to σ_2^{NE} since the game is generic. So in the subgame perfect equilibrium, player 1 is guaranteed at least as much payoff as when he plays strategy σ_1^{NE}. Therefore:

$$w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) \geq w(\sigma_1^{NE}, \sigma_{-1}^{NE}) \tag{40}$$

Note that this does not need to hold when the game is not generic, see Example 6.7.
For the second player σ_2^{SPE} must be the strategy that gives him the most value when player 1 plays σ_1^{SPE}. Furthermore, since there are 2 players, any strategy can be made at most twice as bad by the strategy of the other player. Using this gives:

$$\begin{aligned} w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) &\geq w(\sigma_2^{NE}, \sigma_{-2}^{SPE}) \\ &\geq \frac{1}{2} w(\sigma_2^{NE}, \sigma_{-2}^{NE}) \end{aligned}$$

Adding (40) yields:

$$w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) + 2w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) \geq w(\sigma_1^{NE}, \sigma_{-1}^{NE}) + w(\sigma_2^{NE}, \sigma_{-2}^{NE}) = w(\sigma^{NE}) \tag{41}$$

While player 1 can take away at most half of the payoff of player 2, he cannot take away more value than his own payoff. This gives:

$$\begin{aligned} w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) &\geq w(\sigma_2^{NE}, \sigma_{-2}^{SPE}) \\ &\geq w(\sigma_2^{NE}, \sigma_{-2}^{NE}) - w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) \end{aligned}$$

Adding (40) yields:

$$2w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) + w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) \geq w(\sigma_1^{NE}, \sigma_{-1}^{NE}) + w(\sigma_2^{NE}, \sigma_{-2}^{NE}) = w(\sigma^{NE}) \tag{42}$$

Adding (41) and (42) together gives:

$$\begin{aligned} 3w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) + 3w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) &\geq 2w(\sigma^{NE}) \\ \frac{3}{2} w(\sigma^{SPE}) &\geq w(\sigma^{NE}) \end{aligned} \tag{43}$$

Which gives an upper bound of $\frac{3}{2}$ on the Nash-difference for generalized market sharing games with 2 players. ■

Next is a lower bound example where the subgame perfect equilibrium is significantly worse than the Nash equilibrium in the simultaneous variant.

Theorem 6.10 *The Nash-difference* $\geq \frac{4}{3}$ *for generic generalized market sharing games with 2 players*

Proof. Example 6.11 Consider a game with 2 players and 4 items. The items have values $(2-2\epsilon)$, 3ϵ, 2 and $1+4\epsilon$. Player 1 can choose the items with value $(2-2\epsilon)$ and 3ϵ, or the item with value 2. Player 2 can pick the items with value 2 and 3ϵ or the item with value $1+4\epsilon$.

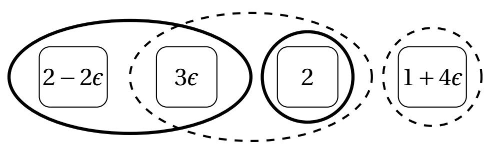


Figure 10: Example 6.8. Player 1 can choose between all sets of solid lines, player 2 can choose any set with dashed line. Social value Nash equilibrium = $4+\epsilon$, subgame perfect equilibrium = $3+4\epsilon$

In Example 6.8 the only Nash equilibrium is when player 1 chooses the set of the items with value $(2-2\epsilon)$ and 3ϵ, and player 2 chooses the items with value 2 and 3ϵ. This gives a social value of $4+\epsilon$. However, in the sequential variant player 1 can improve his payoff from $2-\frac{1}{2}\epsilon$ to 2 by switching his strategy to choosing the item of value 2. This will lead player 2 to choose the item with value $1+4\epsilon$. This gives a social value of $3+4\epsilon$ So in this example the social value of the Nash equilibrium is about $\frac{4}{3}$ times the social value of the subgame perfect equilibrium. ■

It is worth noting that the Nash equilibrium In Example 6.11 is also the optimal solution.

Symmetric generic generalized market sharing games with 2 players
In generic symmetric sequential games, we can assume without loss of generality that $w(\sigma_1^{NE}, \sigma_{-1}^{NE}) > w(\sigma_2^{NE}, \sigma_{-2}^{NE})$. If this does not hold, player 1 would simply choose σ_2^{NE}, and σ_1^{NE} would be the best response to that. So for any Nash equilibrium, the reverse order is also a Nash equilibrium.

Theorem 6.12 *The Nash-difference* $\leq \frac{4}{3}$ *for generic symmetric generalized market sharing games with 2 players*

Proof. Because the game is generic, we once again have

$$w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) \geq w(\sigma_1^{NE}, \sigma_{-1}^{NE}) \tag{44}$$

In the same way as for non-symmetric games we can deduce:

$$
\begin{aligned}
w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) &\geq w(\sigma_2^{NE}, \sigma_{-2}^{SPE}) \\
&\geq \frac{1}{2} w(\sigma_2^{NE}, \sigma_{-2}^{NE}) \\
w(\sigma_1^{SPE}, \sigma_{-1}^{SPE}) + w(\sigma_2^{SPE}, \sigma_{-2}^{SPE}) &\geq w(\sigma_1^{NE}, \sigma_{-1}^{NE}) + \frac{1}{2} w(\sigma_2^{NE}, \sigma_{-2}^{NE}) \\
w(\sigma^{SPE}) &\geq w(\sigma^{NE}) - \frac{1}{2} w(\sigma_2^{NE}, \sigma_{-2}^{NE})
\end{aligned}
$$

But since we now have $w(\sigma_1^{NE}, \sigma_{-1}^{NE}) \geq w(\sigma_2^{NE}, \sigma_{-2}^{NE})$, we get that $w(\sigma_2^{NE}, \sigma_{-2}^{NE}) \leq \frac{1}{2} w(\sigma^{NE})$. Using this gives:

$$
\begin{aligned}
w(\sigma^{SPE}) \geq w(\sigma^{NE}) - \frac{1}{2} w(\sigma_2^{NE}, \sigma_{-2}^{NE}) \geq w(\sigma^{NE}) - \frac{1}{4} w(\sigma^{NE}) = \frac{3}{4} w(\sigma^{NE}) \\
\frac{4}{3} w(\sigma^{SPE}) \geq w(\sigma^{NE}) \qquad (45)
\end{aligned}
$$

So the Nash-difference for generic symmetric generalized market sharing games with 2 players is at most $\frac{4}{3}$. ■

Theorem 6.13 *The Nash-difference* $\geq \frac{8}{7}$ *for generic symmetric generalized market sharing games with 2 players*

Proof. Example 6.14 gives a lower bound on the Nash-difference of $\frac{8}{7}$. See Figure 11.

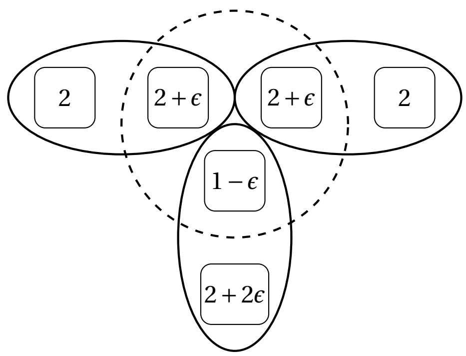


Figure 11: Example 6.14 Symmetric game, both players can pick any (sub)set between any circle, dashed or not. Value Nash equilibrium = $8 + 2\epsilon$, subgame perfect equilibrium = $7 + 3\epsilon$

In Example 6.14 a Nash equilibrium only occurs when player 1 and 2 both choose one of the sets with items of value 2 and $2 + \epsilon$, giving a social value of $8 + 2\epsilon$. However, in the sequential variant player 1 can improve his payoff from $4 + \epsilon$ to $4 + 2\epsilon$ by switching his strategy to choosing only both items of value $2 + \epsilon$. This leads player 2 to choose the two items with values $1 - \epsilon$ and $2 + 2\epsilon$}, giving a social value of $7 + 3\epsilon$. So in this example the Nash-difference is almost $\frac{8}{7}$. It can be made $\frac{8}{7}$ by taking the limit of ϵ to zero. ■

The PoA provides the maximum ratio between the optimal solution and the worst Nash equilibrium, and the Nash difference provides the maximum ratio between the worst Nash equilibrium and the worst subgame perfect equilibrium. Using the maximum Nash-difference of $\frac{4}{3}$ in combination with the price of anarchy of $\frac{3}{2}$ for 2 players gives a bound on the sequential price of anarchy of 2. This is no improvement over the bound from Theorem 6.3, which states SPOA $\leq n$. Since the Nash-difference method has not provided better bounds for the SPoA, even for very specific generalized market sharing games, we move on to different methods.

6.2 Using a linear program

The sequential price of anarchy of a class of games is equal to the highest sequential price of anarchy of any game in that class. In this subsection we build a linear program that can find the game with the highest SPoA for a fixed number of players. This means it even provides the sequential price of anarchy directly, instead of a upper or lower bound. This method was used by de Jong and Uetz [9], who used a linear program to find the worst case scenarios for affine congestion games with 2 or 3 players. This subsection is based upon their method.

In this section we show that we can bound the number of strategies and items necessary to find a game with the worst case sequential price of anarchy. As long as the number of items and strategies are reasonably small, it is possible to solve the linear program in reasonable time. However, when building a linear program for generalized market sharing games, it very quickly becomes too large to solve. This is caused by the downwards closed property. For example, if there exists an action which involves picking 10 items, then this means there are an additional $2^{10} - 1 = 1023$ other actions also possible, consisting of picking subsets of these 10 items. This will very quickly become too difficult to solve within reasonable time.

To avoid this we will be dropping the downwards closed property for now. This allows us to look only at the optimal and subgame perfect outcomes, which causes the LP to be small enough to solve. To make it clear when games do not have the downwards closed property, we use covering games, as introduced by Gairing [5].

Definition 6.15 Covering games.
There is a set N of n players, and a finite ground set J of items. Each item $j \in J$ has a value w_j. For $S \subseteq J$, we let $w(S) := \sum_{j \in S} w_j$. Each player i has a strategy set $\mathbf{S}_i \subseteq 2^J$. The payoff of an item is shared between all players who chose that item.

Gairing [5] studied the price of anarchy for covering games with different payoff functions. We only look into equally shared payoff functions. The important difference to generalized market sharing games is that the strategy sets of covering games are not downwards closed. The downwards closed property does not matter for Nash equilibria, since players always choose maximal subsets. Therefore covering games have the same price of anarchy as generalized market sharing games.

Note that every generalized market sharing game is a covering game, but not vice versa. This means that the SPoA of covering games is larger or equal to the SPoA of generalized market sharing games. So the SPoA of covering games provides an upper bound on the SPoA of generalized market sharing games.

To bound the number of actions we use adaptations of Lemma 3.10 and Lemma 3.11 used in [10] by de Jong. He used these lemmas to bound the number of actions for affine congestion games, here we adapt them to bound actions for covering games.

Recall that $\mathbf{S}_i$ is the strategy set of player i, so $|\mathbf{S}_i|$ is the number of different actions possible for player i.
Furthermore define the series:

$$z_1 = 2 \text{ and } z_i = 1 + \prod_{j<i} z_j \text{ for all } i \geq 2 \tag{46}$$

With this we can proof the following lemma:

Lemma 6.16 *For any covering game I, there exists a covering game I', such that $|\mathbf{S}_i| \leq z_i$, for all players $i \in N$, and such that $SPoA(I') = SPoA(I)$.*

Proof. Let σ^{OPT} denote an optimal solution and σ_{SPE} denote the subgame perfect equilibrium which maximizes the sequential price of anarchy. This lemma follows from the reasoning that if in a game an action is possible for a player, but it is not part the optimal solution σ^{OPT} or any state of the subgame perfect equilibrium σ_{SPE}, then the action can be removed from the game and the optimal solution and subgame perfect outcome stay the same. So for player 1 there only need to be two actions, the optimal one and the subgame perfect one. Player 2 has to have a subgame perfect action for both of these actions of player 1, and and optimal one. So player 2 needs a maximum of 3 different actions. Player 2 could have less actions if any of the subgame perfect actions or the optimal actions overlap, but he never needs more then 3. For any player it holds that he only need as much actions as all possible combinations of actions of the previous players, +1 for the optimal solution. This is exactly equal to z_i for any player i. Let I' be the game where all actions which are not part of either σ^{OPT} or σ_{SPE} are removed. I' then satisfies $|\mathbf{S}_i| \leq z_i$, for all players $i \in N$. Furthermore σ^{OPT} is still optimal and σ_{SPE} is still subgame perfect in I', since all removed actions were inferior or identical. Therefore $SPoA(I') = SPoA(I)$. ∎

With the number of actions now bounded, we can also bound the number of resources required. Let R be the set of resources, and thus $|R|$ the number of different resources.

Lemma 6.17 *For any covering game I, there exists a covering game I', such that $|\mathbf{S}_i| \leq z_i$, for all players $i \in N$, and for any pair of resources there exists an action that contains exactly one of these resources. Hence $|R| \leq 2^{\sum_{i \in N} |\mathbf{S}_i|}$ and such that $SPoA(I') = SPoA(I)$.*

Proof. By Lemma 6.10, we can restrict to games I with $|\mathbf{S}_i| \leq z_i$, for all players $i \in N$. If there exists any pair of items j, k in I such that every action either contains both j and k or contains neither, then those items can be replaced with a single item l. Let this single item have the value of j and k together, i.e. $w_l = w_j + w_k$. Then every action and every strategy profile still provides the same value to each player as in the original game I. Therefore the subgame perfect equilibrium and optimal solution are still the same, so $SPoA(I') = SPoA(I)$. This can be repeated for any two items, until there exists an action for any pair of resources that contains exactly one of these resources and hence $|R| \leq 2^{\sum_{i \in N} |\mathbf{S}_i|}$ ∎

6.2.1 Formulation of the linear program

Here we specify the linear program for finding the sequential price of anarchy for covering games with 3 players. Using the lemmas from the previous section, we bound the number of actions and resources required. Player 1 has 2 possible strategies, player 2 has 3 possible strategies and player 3 has 7 possible strategies. This means the LP uses a set of $(2^{2+3+7} - 1) = 4095$ different items. The -1 comes from the fact that the item which is not chosen in any action is not necessary for the model. This gives us the following sets:

Sets

	Index	
Actions	u, v, w	Set of all actions
Actions player 1	a, a'	Subset of Actions
Actions player 2	b, b'	subset of Actions
Actions player 3	c, c'	subset of Actions
Items	r	

For every item r, we use binary parameter $\delta_{r,u}$ to specify whether item r is in action u. Furthermore, we fix a subgame perfect equilibrium σ_{SPE} and use binary parameters z^1_a, z^2_{ab} and z^3_{abc} to specify if an action is prescribed by σ_{SPE}.

Parameters

$$\delta_{r,u} \quad \forall r, u \quad \begin{cases} 1 & \text{if } r \in u \\ 0 & \text{else} \end{cases}$$

$$z^1_a \quad \forall a \quad \begin{cases} 1 & \text{if } a \text{ is prescribed by } S_1^{SPE} \\ 0 & \text{else} \end{cases}$$

$$z^2_{ab} \quad \forall a, b \quad \begin{cases} 1 & \text{if } b \text{ is prescribed by } S_2^{SPE} \text{ in state } a \\ 0 & \text{else} \end{cases}$$

$$z^3_{abc} \quad \forall a, b, c \quad \begin{cases} 1 & \text{if } c \text{ is prescribed by } S_3^{SPE} \text{ in state } ab \\ 0 & \text{else} \end{cases}$$

Since we already have the number of items and actions, and determined which items are in which action, the only unknown left is the value of the items. This is denoted by the non-negative variable β_r. Using β_r, we can determine some other values we need for further constraints. We do this by creating a new variable which value is determined by an equality constraint. First one is the variable β_u, the value of all items in an action u, determined by constraint (47). Next is o_{uv}, the value of items which are in any two actions u and v, determined by constraint (48). o_{uvw} is the value of items which are in the intersection of any three actions u, v and w, is determined by constraint (49). Since we now have the value of any action and any intersection, we can also determine the profit any player gets in any strategy profile. C^i_{abc} is the value player i receives when players 1,2,3 choose actions a, b, c, determined by (50), (51) and (52). Finally, C^1_a and C^2_{ab} are the profits received by player 1 and 2 respectively, when their successors choose according to the subgame perfect equilibrium σ_{SPE}. Their values are fixed by constraints (53) and (54).

Variables

β_r	$\forall r$	value of item r
β_u	$\forall u$	value of all items in u
o_{uv}	$\forall u, v$	value of items in $u \cap v$
o_{uvw}	$\forall u, v, w$	value of items in $u \cap v \cap w$
C^i_{abc}	$\forall a, b, c, i$	Gained value of player i when players 1,2,3 choose actions a, b, c
C^1_a	$\forall a$	Profit of player 1 when he chooses action a and players 1 and 2 choose as prescribed by S^{SPE}
C^2_{ab}	$\forall a, b$	Profit of player 2 when players 1,2 choose actions a, b and 3 chooses as prescribed by S^{SPE}

Constraints

$$\beta_u = \sum_r \delta_{u,r}\beta_r \qquad \forall u \qquad (47)$$

$$o_{uv} = \sum_r \delta_{u,r}\delta_{v,r}\beta_r \qquad \forall u, v \qquad (48)$$

$$o_{uvw} = \sum_r \delta_{u,r}\delta_{v,r}\delta_{w,r}\beta_r \qquad \forall u, v, w \qquad (49)$$

$$C^1_{abc} = \beta_a - \frac{1}{2}o_{ab} - \frac{1}{2}o_{ac} + \frac{1}{3}o_{abc} \qquad \forall a, b, c \qquad (50)$$

$$C^2_{abc} = \beta_b - \frac{1}{2}o_{ab} - \frac{1}{2}o_{bc} + \frac{1}{3}o_{abc} \qquad \forall a, b, c \qquad (51)$$

$$C^3_{abc} = \beta_c - \frac{1}{2}o_{ac} - \frac{1}{2}o_{bc} + \frac{1}{3}o_{abc} \qquad \forall a, b, c \qquad (52)$$

$$C^1_a = C^1_{abc} \qquad \forall a, b | z^2_{a,b} = 1, c | z^3_{abc} = 1 \qquad (53)$$

$$C^2_{ab} = C^2_{abc} \qquad \forall a, b, c | z^3_{abc} = 1 \qquad (54)$$

$$C^3_{abc} \le C^3_{abc'} \qquad \forall a, b, c, c' | z^3_{abc'} = 1 \qquad (55)$$

$$C^2_{ab} \le C^2_{ab'} \qquad \forall a, b, b' | z^2_{a'} = 1 \qquad (56)$$

$$C^3_a \le C^3_{a'} \qquad \forall a, a' | z^1_{a'} = 1 \qquad (57)$$

$$\sum_{i \in \{1,2,3\}} C^i(1.2, 2.3, 3.7) = 1 \qquad (58)$$

$$\sum_{i \in \{1,2,3\}} C^i(1.1, 2.1, 3.1) \ge \sum_{i \in \{1,2,3\}} C^i(a, b, c) \qquad \forall a, b, c \qquad (59)$$

Constraints (55), (56) and (56) make sure that no player can gain more payoff by deviating from the subgame perfect equilibrium. We take 1.2, 2.3 and 3.7 as the actions in the subgame perfect equilibrium and 1.1, 2.1 and 3.1 as the actions in the optimal solution. Constraint (58) sets the value of the subgame perfect equilibrium to 1. Constraint (59) makes sure that (1.1, 2.1, 3.1) is the optimal solution.

We want to find the game with the maximum sequential price of anarchy. The sequential price of anarchy is defined as the ratio between the social value of the optimal solution and the worst subgame perfect equilibrium. Since the social value of the subgame perfect equilibrium is equal to 1, this ratio is equal to the value of the optimal solution. So the objective of the LP is to maximize the value of the optimal solution $(1.1, 2.1, 3.1)$.

6.2.2 Results for covering games with 2 players

Theorem 6.18 *SPoA* ≤ 1.5 *for generalized market sharing games with 2 players*

Proof. When the LP is implemented for 2 players, it finds a solution with only 2 items, one with value 1 and one with value $\frac{1}{2}$. In the optimal solution players pick different items, but in the subgame perfect equilibrium they both pick the item with value 1. This gives SPoA = 1.5 for covering games, which means SPoA ≤ 1.5 for generalized market sharing games. ■

Theorem 6.19 *SPoA* $= 1.5$ *for generalized market sharing games with 2 players*

The lower bound follows from Theorem 6.2. The game the LP found was actually equal to Example 5.3 for 2 players.

6.2.3 Results for covering games with 3 players

Theorem 6.20 *SPoA* $\leq 1.816155355139044\ldots$ *for generalized market sharing games with 3 players*

Proof. When the LP is used to find the worst case for covering games with 3 players, it provides a game with a sequential price of anarchy of 1.816155355139044. This means the sequential price of anarchy for covering games is 1.816155355139044..., which means it is a upper bound for generalized market sharing games. ■

The number 1.816155355139044 cannot be written shorter with a simple fraction. The number follows from a lot of items which in turn have long values, that are not simple fractions. To produce a clearer example, we changed the LP to an ILP with integer item values, and a subgame perfect equilibrium with a value of 1000. This forces items to have an integer value between 0 and 1000. The ILP found a solution with SPoA 1.815:

Example 6.21 Consider a game with 16 items and 12 actions. Player 1 has access to actions 1.1 and 1.2. Player 2 has access to actions 2.1, 2.2 and 2.3. Player 3 has access to actions 3.1 - 3.7. See Table 1 for the value of items and which items are in which action. See Figure 12 for the subgame perfect equilibrium.

In Example 6.21 the optimal solution is when the players choose strategies 1.1, 2.1 and 3.1. The total payoff for the optimal solution is 1815. However, a subgame perfect equilibrium occurs when players choose actions 1.2, 2.3 and 3.7. The payoff then is 1000. This means that the sequential price of anarchy = $\frac{1815}{1000} = 1.815$.

Figure 12 is the decision tree showing the different options for each player in Example 6.21. For clarity the figure does not display all actions of player 3, but only the optimal strategy and the sequential optimum of that branch. The expected payoff for each player is written above the edge.

Since the price of Anarchy is $2 - \frac{1}{n}$, this means that for 3 players the price of anarchy is $2 - \frac{1}{3} \approx 1.67$. So the sequential price of anarchy is actually worse here than the price of anarchy.

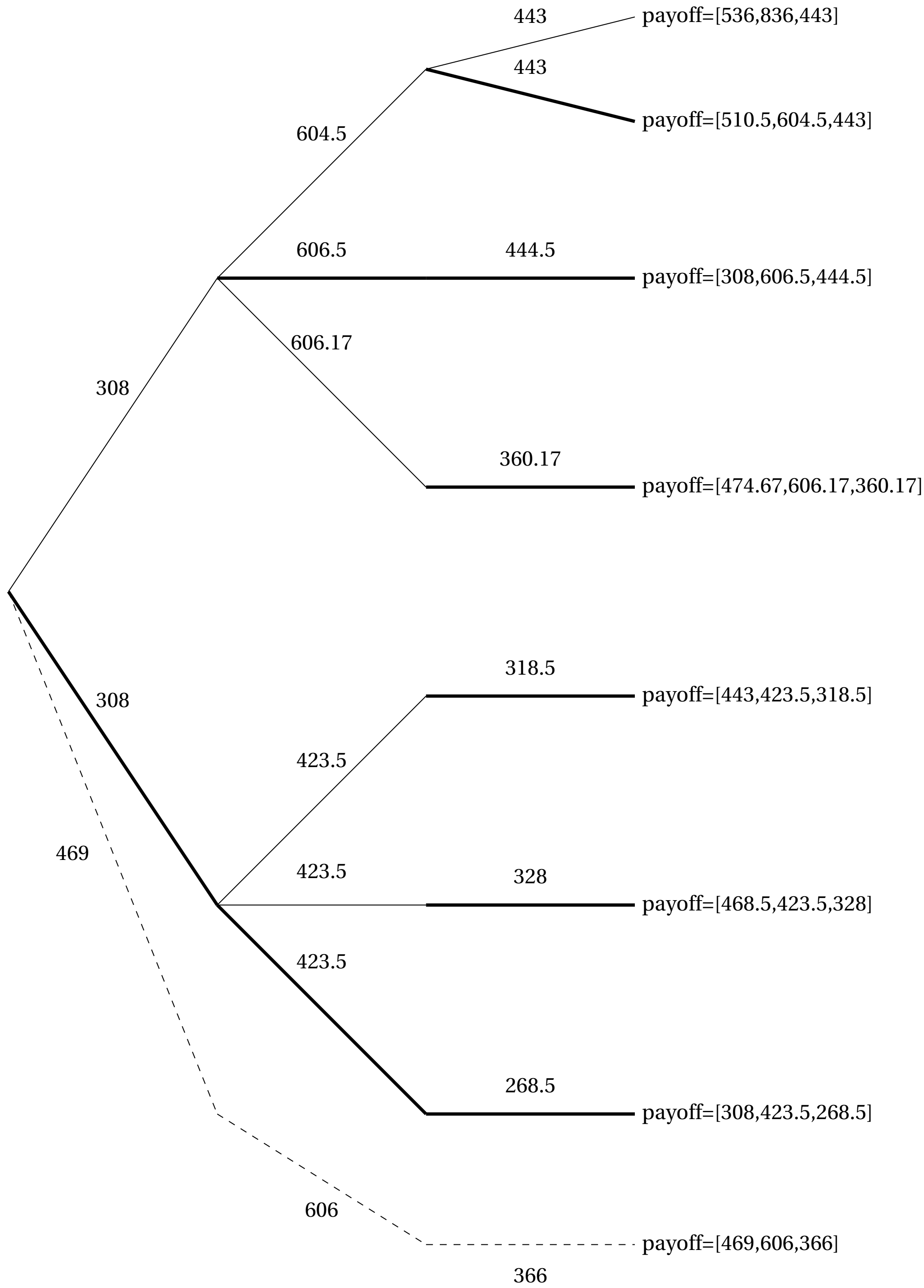


Figure 12: Decision tree for Example 6.21. The thick lines are subgame perfect outcomes. The dashed line is only available when all strategies are downwards closed.

Item	Value	1.1	1.2	2.1	2.2	2.3	3.1	3.2	3.3	3.4	3.5	3.6	3.7
1	80	x											
2	197			x									
3	220	x			x								
4	177						x						
5	141	x							x				
6	186		x			x	x	x	x				
7	65			x		x					x	x	
8	17					x	x				x	x	
9	12	x			x			x		x			x
10	2		x	x	x			x		x			x
11	63		x			x	x		x	x			x
12	14		x	x				x	x	x			x
13	44	x			x	x				x	x	x	x
14	39	x			x	x		x		x	x	x	x
15	447		x	x	x	x		x		x	x	x	x
16	111		x	x		x			x	x	x	x	x

Table 1: The values of all items and indication which sets they are a part of.

Adding downwards closed property to Example 6.21

The SPoA in Example 6.21 is not the same for downwards closed strategy sets. If this game was downwards closed there would be 126 actions for player 1 and 381 actions for player 2 to consider. For player 3 it would remain at the 5 unique actions, as the last player is always going to choose a maximum subset. The actions of taking a full set always dominate taking a subset for the last player.

When considering all these actions, it turns out that the subgame perfect equilibrium changes to player 1 picking a subset of action 1.1. Player one will now choose the item set $(1,3,5,13,14)$, so action 1.1 without item 9. This leads to player 2 picking action 2.3 and player 3 picking action 3.4. The payoff is then $[469,606,366]$, so in total 1441. This solution is displayed with the dashed line in the tree. As the optimal solution stays the same, this gives a sequential price of anarchy of $\frac{1815}{1441} \approx 1.26$. As we have already found a case where the SPoA is $2-\frac{1}{3} \approx 1.67$, it shows that this game is far from the worst case.

6.2.4 Results for covering games with 4 players

The LP can be expanded to 4 players. In that case player 4 would need 43 different actions. This would take too long to solve in reasonable time. However, a greedy approach can be taken by giving the players less actions. When looking at the result for 3 players, the third player only needed five unique actions. We give player 3 four different actions and player 4 five different actions. While this might not give the worst case scenario it does give a lower bound on the SPoA for covering games with 4 players. The previous LP used a binary parameter to determine in advance what action was the subgame perfect outcome in every case. This parameter is now turned into a binary variable, which changes the LP into an ILP.

Theorem 6.22 *The SPoA $\geq 2.0558498235\ldots$ for covering games with 4 players*

The solution found by the LP has an SPoA of 2.0558498235. It consists of 28 different items and took 6 hours and 44 minutes to solve. It is interesting to notice that the sequential price of anarchy for covering games is larger than 2 for four players, since the price of anarchy of covering games with equal sharing is $2-\frac{1}{n}$, so this can never exceed two. In fact, it might be larger than the price of anarchy any payoff function can provide. Covering games with different payoff functions have been studied [5, 16, 20, 21], but none of them found a payoff function that a price of anarchy larger than 2. As long as the players cannot receive negative payoff from an item and all value from an item is divided among the players, then the covering game is a valid utility game and the price of anarchy cannot be higher than 2.

7 Shared misery games

In this section we define a class of games called shared misery games, of which singleton generalized market sharing games are a special case. For this class of games we give a useful technical lemma, which we call the Payoff-Guarantee Lemma. While the lemma itself might seem straightforward, it allows to prove bounds on the sequential price of anarchy.

7.1 Payoff-Guarantee Lemma

In 2015 Hassin and Yovel [8] investigated the sequential price of anarchy of scheduling games on identical machines. They showed that if every player i can choose a machine so that job i's completion time is bound by U, regardless of the previous players, then each player can guarantee himself a job completion time of at most U. This was then used to prove tight bounds on the sequential price of anarchy.

The interesting aspect of their lemma was that it was able to turn the instantaneous availability of a completion time into a guaranteed maximum completion time. When investigating subgame perfect equilibria, it is notoriously hard to say something about the payoff of a player without observing all possible actions of all other players. However, using their lemma, Hassin and Yovel bounded the maximum cost of a player, regardless of the actions of the subsequent players.

In this subsection we generalize the idea of Hassin and Yovel. We define a class of games called shared misery games and provide a lemma that can be used for any game from this class. But before we can provide the lemma we need some definitions.

Definition 7.1 Instantaneous payoff.
Let there be a sequential game, where the payoff for all players is well defined for every subgame where only a subset of players have chosen a strategy, and the rest of the players chooses the empty set. The *instantaneous payoff* of a player i is the payoff player i would receive if the game would end after his choice, i.e. all subsequent players would choose the empty set as their strategy.
Let $\sigma_{<i} = (\sigma_1, \dots, \sigma_{i-1})$ for any player i. Then the instantaneous payoff for player i is defined as $w_i(\sigma_{<i}, \sigma_i)$. In this case all players $i+1, \dots, n$ do not have an action yet, so this is treated as if they all chose the empty set, i.e $w_i(\sigma_{<i}, \sigma_i) = w_i(\sigma_1, \dots, \sigma_i, \emptyset, \dots, \emptyset)$, where there are $n-i$ empty sets.

Instantaneous payoff can analogously be defined for cost minimization games. In that case it is called the instantaneous cost, and would be the cost a player would receive if the game would end after his choice. to avoid confusion we denote the payoff a player gets from a game as *final payoff* in this section.

The Payoff-Guarantee Lemma, that is to follow, holds for a specific class of games. We call this class of games *shared misery games*, and they are defined as follows:

Definition 7.2 Shared misery games.
Let there be a sequential (sub)game with non-increasing payoff functions in the number of players, where the instantaneous payoff of each player is well defined. In words, this game is a *shared misery game*, if a player 'later' in the game cannot decrease an 'earlier' player's instantaneous payoff to something lower than his own instantaneous payoff. W.l.o.g. let the order of the players be $1, 2, \dots, n$. Then for a shared misery game, if a player j decreases the instantaneous payoff of a player i with $i < j$, then the instantaneous payoff of player j cannot be higher then the new instantaneous payoff of player i, i.e.

$$\forall i < j, \text{ if } w_i(\sigma_{<j}) > w_i(\sigma_{<j}, \sigma_j), \text{ then } w_i(\sigma_{<j}, \sigma_j) \geq w_j(\sigma_{<j}, \sigma_j). \tag{60}$$

The game investigated by Hassin and Yovel, machine scheduling with a makespan objective, is a shared misery game. This holds because it is only possible for a player i to enlarge a job j's completion time if player i chooses the same machine as j. This would give player i the same completion time as player j, which means the instantaneous payoff of i is not larger than the instantaneous payoff of j. Furthermore any singleton congestion game is a shared misery game, because players only influence the final payoff of players with the same strategy, and they have the same payoff as these players. Singleton congestion games includes singleton generalized market sharing games, which we look into later. Other examples of shared misery games are bottleneck congestion games, which are games where the players choose a subset of resources with the goal to minimize the maximum cost of any of these resources.

Definition 7.3 $(l_1, \dots, l_n)$**-available.**
For a given vector $(l_1, \dots, l_n)$ with $l_1 \le \dots \le l_n$, we call a game $(l_1, \dots, l_n)$-available, if it holds that each player i can choose a strategy with an instantaneous payoff of at least l_i, regardless of the strategies of the previous players $1, \dots, i-1$, i.e.

$$\forall i, \sigma_{<i} \quad \exists \sigma_i, \text{ such that } w_i(\sigma_{<i}, \sigma_i) \ge l_i$$

Note that for any symmetric shared misery game, it always holds that the lower bounds $(l_1, \dots, l_n)$ are all the same value l, i.e. $l = l_1 = \dots = l_n$. This is because any strategy available to player n would have to be available any previous player as well, and the payoff functions are non-increasing. This gives $l_1 \ge \dots \ge l_n$, which combined with the condition that $l_1 \le \dots \le l_n$ means that $l_1 = \dots = l_n$. For any symmetric game we use l-available, to indicate $(l, \dots, l)$-available.

Lemma 7.4 (Payoff-Guarantee Lemma) *If a shared misery game is $(l_1, \dots, l_n)$-available, then it holds that every player i can guarantee himself a final payoff of at least l_i in any subgame perfect equilibrium.*

Proof. Let $l_1, \dots, l_n$ be as stated in the lemma. We show that every player i can guarantee himself a final payoff of at least l_i by backwards induction on player index. So we start with player n. Since there is a strategy available with instantaneous payoff at least l_n, player n can obviously guarantee himself l_n by choosing that strategy, since there is no player after him to affect his final payoff.

Now we need to show that the lemma holds for player i, when it holds for all players $i+1, \dots, n$. Given is that player i has a strategy σ_i available with instantaneous payoff at least l_i. That leaves us to show that if player i chooses strategy σ_i, he ends up with at least l_i final payoff. If none of the following players $i+1, \dots, n$ cause the final payoff of player i to decrease, then the claim follows immediately. So assume there are some players of $i+1, \dots, n$ that cause the final payoff of player i to decrease. By the induction hypothesis we have that all these players can guarantee themselves a final payoff of at least l_i. This means that in the subgame perfect equilibrium these players will end up with least l_i final payoff, otherwise it would be an irrational choice. Since the payoff functions are non-increasing, this means their instantaneous payoff has to be at least l_i as well. By the definition of shared misery games we have that the final payoff of player i cannot be decreased to below the instantaneous payoff of the subsequent players $i+1, \dots, n$, so the payoff of i remains at least l_i. Therefore player i can guarantee himself a final payoff of at least l_i by choosing σ_i. ■

To use the Payoff-Guarantee Lemma for cost minimization games, the lower bounds $l_1, \dots, l_n$ need to be replaced with upper bounds $u_1, \dots, u_n$ on costs. Furthermore, for cost minimization, a shared misery game means that no player can increase another player's cost above its own, and requires non-decreasing cost functions. When those changes are made, the Payoff-Guarantee Lemma is a generalization of the lemma for machine scheduling provided by Hassin and Yovel could therefore also be applied to their machine scheduling problem.

Even though the Payoff-Guarantee Lemma seems straightforward, it has some interesting implications. First of all, when it is used to show that every player i can guarantee himself a final payoff of l_i in the subgame perfect equilibrium, then that means that the social value of a subgame perfect equilibrium has to be larger than or equal to $\sum_{k=1}^{n} l_k$. This can be used to provide bounds on the sequential price of anarchy. This already is useful, but specifically for symmetric games, we get some interesting results.

7.2 Symmetric shared misery games

The Payoff-Guarantee Lemma provides some more specific results for symmetric shared misery games. As stated in Definition 7.3, when a symmetric shared misery games has $(l_1,\dots,l_n)$-availability, all the lower bounds have the same value l, i.e. $l = l_1 = \dots = l_n$. In the rest of this subsection we only use l as a value for the lower bound and use l-available to indicate $(l,\dots,l)$-available.

Because the payoff functions are non-increasing, whether a symmetric shared misery game is l-available only depends on whether l is available to the last player. This means that only looking at the worst case scenario for the last player can already give us bounds on the sequential price of anarchy.

Theorem 7.5 *For any symmetric shared misery game with l-availability, the social value of any pure Nash equilibrium or subgame perfect equilibrium is at least nl.*

Proof. It follows from the Payoff Guarantee Lemma that if the game has l-availability, then each of the players can guarantee himself a final payoff of at least l in the subgame perfect equilibrium and therefore the social value must be at least nl.
Now we only need to show that for a symmetric shared misery game the bound of nl extends to pure Nash equilibria. This is because the definition of l-availability requires that the last player has a strategy available with instantaneous payoff at least l, regardless of the strategies of the $n-1$ players before. Since the game is symmetric, all players have the same strategy sets available. This means that every player must have a strategy with at least instantaneous payoff l available, regardless of the strategies of the other $n-1$ players. This also holds in a pure Nash equilibrium, where it is given what the actions of all players are.
For example, if the strategies of the other $n-1$ players in a pure Nash equilibrium would cause a player to be unable to choose a strategy with final payoff at least l, then it would be possible to do these $n-1$ strategies in a subgame perfect equilibrium, causing player n to be unable to choose a strategy with instantaneous/final payoff of at least l. This would contradict the l-availability property. So that means all players in the Nash equilibrium must have final payoff at least l, so the social value must be at least nl. ■

The main result we get from the framework of shared misery games and the Payoff-Guarantee Lemma, is that we can bound the SPoA for symmetric singleton congestion games. In singleton congestion games all players get to choose one resource, and their final payoff is only dependent on the number of players choosing that resource. Furthermore all players choosing that resource get the same final payoff, and that final payoff is non-increasing, so singleton congestion games are shared misery games. These properties allows us to get concrete results about subgame perfect equilibria using the Payoff-Guarantee Lemma.

Theorem 7.6 *For symmetric singleton congestion games, every subgame perfect equilibrium is Nash-stable, and thus SPoA ≤ PoA.*

Proof. Assume this is not true. Then there exists a game with a subgame perfect equilibrium so that the resulting outcome is not a Nash equilibrium. That game must have one or more players who do not satisfy the Nash condition in the outcome of the subgame perfect equilibrium. This means there are strategies available for these players which would give a higher final payoff if the rest of the players play the same. Take the last player who's strategy is not Nash, let this player be player i. Because player i does not satisfy the Nash condition, there must exist a strategy σ_i, which gives player i a higher final payoff if he chooses σ_i while the rest of the players does not change their strategies. Say strategy σ_i is chosen k times in the subgame perfect equilibrium and let $w_{\sigma_i}(x)$ be the payoff function for strategy σ_i, where x is the number of players choosing strategy σ_i. In the the subgame perfect equilibrium player i does not play strategy σ_i, but since switching to σ_i would give him more, so the final payoff for player i in the subgame perfect equilibrium is $< w_{\sigma_i}(k+1)$. Since all players after i satisfy the Nash condition and the game is symmetric, their final payoff must be $\geq w_{\sigma_i}(k+1)$.

That means that at the moment player i chooses, there are still $N-i+1$ possibilities for a player to get a final payoff ≥ $w_{\sigma_i}(k+1)$, and all of these possibilities are available to all players since it is a symmetric game. It's impossible for a player to take away multiple of these possibilities, since every player only chooses one strategy, and the final payoff of a strategy is only dependent on the amount of players choosing that strategy. This means all remaining players have a strategy with instantaneous payoff $w_{\sigma_i}(k+1)$ available to them, when they have to choose. So we can apply the Payoff-Guarantee Lemma to this subgame and which guarantees all remaining players a final payoff of at least $w_{\sigma_i}(k+1)$. This means player i can guarantee himself a better final payoff by switching to strategy σ_i. So it was not a subgame perfect equilibrium, which is a contradiction. Therefore every subgame perfect equilibrium must also be a Nash equilibrium. It follows that $SPoA \leq PoA$. ■

This makes it considerably easier to determine bounds on the sequential price of anarchy. Note that Theorem 7.6 does not hold for non-anonymous games, i.e. when the final payoff also depends on the identity of the players, instead of just the number of players. Next is an example to show this.

Example 7.7 Let there be a machine scheduling problem with a makeshift objective function. There are 2 identical machines and 3 players, where the first two players have a job of length 1 and the third player has a job with length 2.

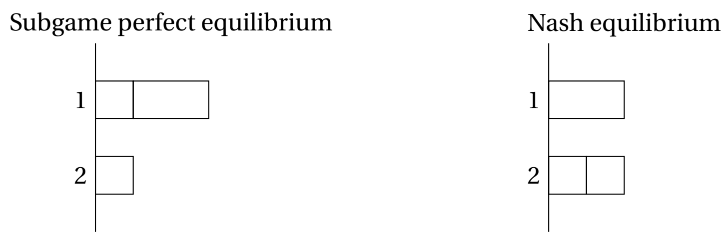


Figure 13: The subgame perfect equilibrium and the Nash equilibrium for the game in Example 7.7. Numbers represent machines, rectangles represent jobs.

In this game there exists a subgame perfect equilibrium where the first player picks the first machine, the second player picks the second machine and the last player picks the first machine again. In this case the cost for the first player is 3, the cost for the second player is 1 and the cost for the third player is 3 as well. This subgame perfect equilibrium is not Nash-stable, because in that case player 1 would switch to the second machine, and all players would have cost 2.

The Payoff-Guarantee Lemma also allows us to prove theorems for different social functions. Here we use the Payoff-Guarantee Lemma to prove that these subgame perfect equilibria always yield solutions with maximum minimum final payoff, i.e. solutions where the player with the lowest final payoff has as much final payoff as possible. This is optimal for an egalitarian social function.

The optimal solution in an egalitarian game is the solution that maximizes the minimum final payoff among all players. The price of anarchy in egalitarian games is the ratio between the worst minimum final payoff possible in a Nash equilibrium and the minimum final payoff in the optimal solution.

In cost minimization games the egalitarian social function is equal to the maximum cost instead of the minimum final payoff.

Theorem 7.8 *Given a symmetric shared misery game, where the final payoff of any strategy only depends on the number of players choosing the same strategy. Then any subgame perfect equilibrium of that game is optimal for an egalitarian social function, or in other words, the sequential price of anarchy for an egalitarian social function is 1.*

Proof. Let σ^E be the optimal egalitarian solution. Let σ^E exists of k unique strategies, of which each one may be chosen multiple times. Let l be the lowest final payoff in σ^E. To be able to use the previous lemma we need to show that each player i always has a strategy available with an instantaneous payoff of at least l, independent of the strategies of the previous players $1,\dots,i-1$. This follows from the fact that the instantaneous payoff of a strategy is only dependent on the number of players choosing that strategy. When player i must select his strategy, then for any strategy profile of players $1,\dots,i-1$, at least one of the k strategies in σ^E must have been chosen less times than in σ^E, since the amount of players is finite. Choosing this strategy must offer an instantaneous payoff of at least l, since it gives a final payoff of at least l in σ^E, where it is chosen less times and the payoff function is non-increasing. So now all conditions for the Payoff-Guarantee Lemma are met, so each player can guarantee himself a final payoff of l. Since this is the optimal egalitarian solution, this means that all subgame perfect equilibria are optimal egalitarian solutions. Therefore the sequential price of anarchy for an egalitarian social function is 1. ■

In the proof for Theorem 7.8 is used that the last player must have an instantaneous payoff of l available. This means Theorem 7.5 can also be used here, to show that the price of anarchy is 1 for pure Nash equilibria with an egalitarian social function.

8 Singleton generalized market sharing games

In this section we examine a specific variation of generalized market sharing games, named singleton generalized market sharing games. A singleton set is a set with exactly one element. Singleton generalized market sharing games are therefore games where the strategy sets consist of only strategies with one item. Every singleton generalized market sharing game can also be written as a singleton market sharing game.

8.1 Price of Anarchy for singleton generalized market sharing games

Singleton generalized market sharing games are a subset of generalized market sharing games, therefore the upper bound from Theorem 5.8 about generalized market sharing games still holds. The matching lower bound uses Example 5.3, which happens to be a symmetric singleton game. Therefore the price of anarchy of $2-\frac{1}{n}$ still holds for singleton generalized market sharing games.

Collorary 8.1 *$PoA = 2-\frac{1}{n}$ for singleton generalized market sharing games.*

Symmetric singleton generalized market sharing games are a variant of generalized market sharing games where every player can pick only one item and all items are available to everybody. Symmetric singleton generalized market sharing games are symmetric shared misery games, where the payoff of any strategy only depends on the number of players choosing the same strategy. This means that Theorem 7.6 can be applied, which gives $SPoA \leq PoA$. Since Example 5.3 still provides a lower bound for sequential symmetric singleton generalized market sharing games and the upper bound of $2-\frac{1}{n}$ on the price of anarchy still holds, we get:

Collorary 8.2 *SPoA $= 2-\frac{1}{n}$ for symmetric singleton generalized market sharing games.*

8.2 Relation to network congestion games

Symmetric singleton generalized market sharing games can be reduced to a symmetric network congestion game with parallel links. The reduction consists of making a directed edge for each item. The edge e for item i then has a cost of $-\frac{w_i}{n_e}$. Here n_e is the amount of players choosing edge e. Figure 14 is an example of such a reduction.

This reduction does cause every edge to have negative costs. This can be solved by adding the maximum item value to all edges, which makes all edges have non-negative costs.

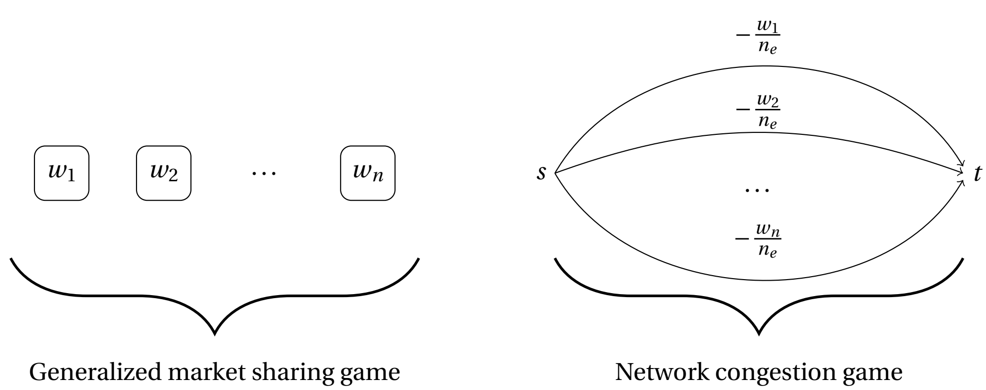


Figure 14: Reduction of symmetric singleton generalized market sharing game to network congestion game. Here n_e is the amount of players choosing edge e.

8.3 Non-symmetric singleton generalized market sharing games

When the game is not symmetric, Theorem 7.6 does not hold anymore, and thus subgame perfect equilibria are not necessarily Nash-stable. This was necessary for Corollary 8.2, where we bounded the SPoA for symmetric singleton games. The following example shows that subgame perfect equilibria can be not Nash-stable:

Example 8.3 There are 4 items, with values 1.5, 2, 2 and 2. There are also 3 different players, all capable choosing between exactly two different items. The first player picks between the item 1 and 2, so items with value 1.5 and 2. The second player picks between items 2 and 3, both value 2. Finally, the third player picks between item 3 and 4, also both value 2. See figure 15.

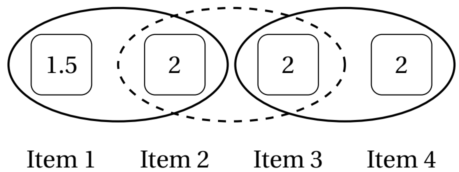


Figure 15: Example 8.3. A 3 player non-symmetric singleton generalized market sharing game.

Suggest now that the first player picks the second item, and that this is followed by the second player picking the third item and the third player picking the second item again. This means player 1 gets payoff 1, so he is better off choosing the first item, of value 1.5.

However, in this case player 2 chooses the fourth item and player 3 chooses the third item. This means that the second item is untouched. No player can make a different choice to improve his own payoff, so this is a subgame perfect equilibrium.

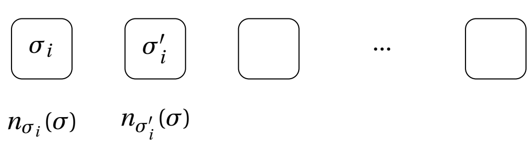
In this specific case player 1 does not abide to the Nash condition, but choosing it is part of a subgame perfect equilibrium. So subgame perfect equilibria are not necessarily Nash-stable.

8.4 Generic singleton generalized market sharing games

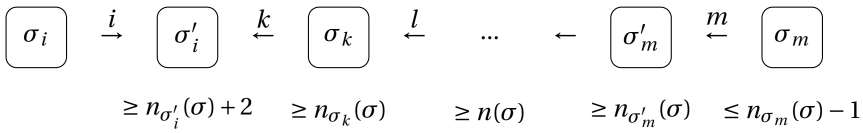
While Example 8.3 shows that in non-symmetric games a subgame perfect equilibrium is not necessarily Nash-stable, we can show it does hold for generic games.

Theorem 8.4: *A subgame perfect equilibrium of a generic singleton generalized market sharing game is always Nash-stable.*

Proof. Assume this is not true, then there exists a game where the subgame perfect equilibrium is not Nash-stable. So in the outcome of that subgame perfect equilibrium there must exist a player who does not satisfy the Nash condition. Let that player be player i. In the subgame perfect equilibrium player i chooses item σ_i, but item σ'_i would give higher payoff if the strategies of the other players stay the same, i.e. $w_i(\sigma_i, \sigma_{-i}) < w_i(\sigma'_i, \sigma_{-i})$. Let $n_j(\sigma)$ be the times the item j is chosen in the subgame perfect equilibrium σ. This is the situation:



Now take a look at the situation where player i chooses item σ'_i instead. Because this is not the choice player i makes in the subgame perfect equilibrium, player i cannot receive more payoff with this action. This means there must be at least one other player $k > i$ who chooses item σ'_i now, but not in the subgame perfect equilibrium, bringing the total of players choosing item σ'_i to at least $n_{\sigma'_i}(\sigma) + 2$.



The other player k who also chooses item σ' came from an item, which we call item σ_k. This player had the choice between item σ'_i and σ_k already in the subgame perfect equilibrium, but then chose σ_k. So in the subgame perfect equilibrium σ_k gave more payoff then σ'_i. In this case he chooses σ'_i instead, so here σ'_i must give higher expected payoff then σ_k. The payoff of σ'_i has only become lower due to player i choosing that as well, so this must mean that there must be at least one other player l choosing item σ_k, that did not choose this item in the subgame perfect equilibrium. So this player l forced player k to choose σ'_i by picking σ_k. For this player l then holds the same as for player k. He also could have done this in the subgame perfect equilibrium, since his new payoff was also available in the subgame perfect equilibrium. Therefore he must be forced away from his original item by at least one other player choosing that item.

This argument can be repeated, but since the amount of items and players is finite, the chain must eventually end in an item chosen less times in this new strategy profile than in the subgame perfect equilibrium. There must be a player m who went from an item σ_m to another item σ'_m, where σ'_m has at least as many players choosing that item as in the subgame perfect equilibrium, while σ_m has less players choosing it than in the subgame perfect equilibrium. This means that the payoff player m got from σ_m in the subgame perfect equilibrium is still available. But since player m does not choose it and the game is generic, he must receive more payoff now. Every other player in the chain has switched items because m did. But this means this option was also available to m in the subgame perfect equilibrium. If he had chosen σ'_m there, he would have caused the same chain of moving players, ending with player k choosing item σ'_i, and player m would have had a larger payoff. This means it was not a true subgame perfect equilibrium, which is a contradiction. ■

Using Theorem 8.4 we can find the SPoA for generic singleton generalized market sharing games.

Theorem 8.5 *SPoA* $= 2 - \frac{1}{n}$ *for generic singleton generalized market sharing games.*

Proof. It follows from Theorem 8.4 that SPoA ≤ PoA for generic singleton generalized market sharing games. It follows from Lemma 5.1 that PoA $\leq 2 - \frac{1}{n}$ for generic singleton generalized market sharing games. Example 5.3 generic singleton generalized market sharing game with SPoA $= 2 - \frac{1}{n}$. ■

9 Generalized market sharing games with an egalitairian social function

In this section we investigate generalized market sharing games with an egalitarian social function. In section 7 we already showed some results about shared misery games for an egalitarian social function. Since singleton generalized market sharing games are shared misery games, we have:

Collorary 9.1 SPoA= 1 *for symmetric singleton generalized market sharing games with an egalitarian social function.*

In this section we look into the egalitarian price of anarchy for different settings. In the following proofs we use $\sigma^{NE}_{\min}$ as the strategy of the player with the lowest payoff in the Nash equilibrium σ^{NE}. In the same way $\sigma^{OPT}_{\min}$ is used as the strategy of the player with the lowest payoff in the optimal strategy σ^{OPT}. The egalitarian price of anarchy is defined as $\frac{w(\sigma^{OPT}_{\min},\sigma^{OPT}_{-\min})}{w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min})}$.

9.1 Price of anarchy for generalized market sharing games with an egalitarian social function

Theorem 9.2 PoA $\leq n$ *and* SPoA $\leq n$ *for generalized market sharing games with an egalitarian social function.*

Proof. When a player chooses a strategy, the worst outcome for that player would be if every other player chooses the same items. He would only get $\frac{1}{n}$ of the value then. But it also can not get worse than that. So it holds for every player that he must be able to get at least $\frac{1}{n}$ of the value he receives in the optimal solution:

$$w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min}) \geq w(\sigma^{OPT}_{\min},\sigma^{NE}_{-\min}) \geq \frac{1}{n} w(\sigma^{OPT}_{\min},\sigma^{OPT}_{-\min})$$

$$\frac{w(\sigma^{OPT}_{\min},\sigma^{OPT}_{-\min})}{w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min})} \leq n.$$

So the price of anarchy must be smaller than n. The same holds for the sequential price of anarchy. ■

Theorem 9.3 PoA $\geq n$ *and* SPoA $\geq n$ *for generalized market sharing games with an egalitarian social function.*

Proof. The lower bound follows from the following example:

Example 9.4 Let there be a game with n players and n items. All items have value 1. the first player can only pick item 1. the rest of the players can pick any item, or a pair of items, namely item 1 and another item.

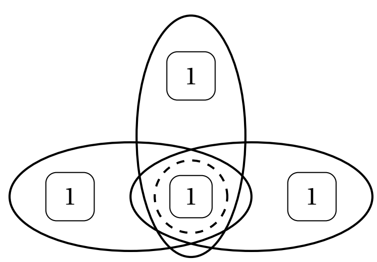


Figure 16: The game of Example 9.4 for 4 players. The dashed circle represent the available item for player 1. The other circles are the sets available to the other players.

Proof of the lower bound. In Example 9.4 the optimal solution is to let every player choose one item. Then every player gets value 1. However, the only Nash equilibrium is that player 1 picks item 1, and all other players pick the other items together with item 1. This means every item is chosen exactly once, except for item 1, which is chosen

n times. The payoff for player 1 is $\frac{1}{n}$, and the payoff for the rest of the players is $\frac{n+1}{1}$.

In the optimal solution the minimum payoff is 1. In the Nash equilibrium the minimum payoff is $\frac{1}{n}$. So for this game the price of anarchy is n. This still holds if this game is sequential and the player play in order $1,\ldots,n$. Therefore the price of anarchy and sequential price of anarchy are larger than or equal to n. ■

Corollary 9.5 PoA = *n and* SPoA = *n for generalized market sharing games with an egalitarian social function.*

Since the worst Nash equilibrium in the game of Example 9.4 is also the only Nash equilibrium in that game, it also follows that the price of stability is n for (sequential) generalized market sharing games with egalitarian objective function.

Corollary 9.6 PoS = *n for generalized market sharing games with an egalitarian social function.*

9.2 Price of anarchy for symmetric generalized market sharing games with an egalitarian social function and pure strategies

In this subsection we show that the price of anarchy is significantly lower for symmetric games with pure strategies. We use σ^{NE} as the strategy profile of the Nash equilibrium, and S^{NE} as the set of items chosen at least once in the Nash equilibrium σ^{NE}.

Theorem 9.7 *PoA* $\leq 2-\frac{1}{n}$ *for symmetric generalized market sharing games with pure strategies and an egalitarian social function.*

Proof. Let $\sigma^{NE}_{\min}$ be the strategy of the player with the least payoff in the Nash equilibrium σ^{NE}. Because it is a Nash equilibrium, this player cannot improve his payoff by switching to another strategy. Since the game is symmetric, all strategies from the optimal solution are available to player i. This gives the following:

$$\begin{aligned} w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min}) &\geq w_i(\sigma^{OPT}_i,\sigma^{NE}_{-\min}) \qquad \forall i \\ \Rightarrow n\cdot w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min}) &\geq \sum_{i=1}^{n} w_i(\sigma^{OPT}_i,\sigma^{NE}_{-\min}) \\ &= \sum_{i=1}^{n}\Big(w_i(\sigma^{OPT}_i\cap\overline{S^{NE}},\sigma^{NE}_{-\min}) + w_i(\sigma^{OPT}_i\cap S^{NE},\sigma^{NE}_{-\min})\Big) \end{aligned} \tag{61}$$

Since all items in $\sigma^{OPT}_i\cap\overline{S^{NE}}$ are not chosen by any player in the Nash equilibrium, player i must receive full value if he chooses those items and the rest of the players play their Nash strategies. So it must hold that $w_i(\sigma^{OPT}_i\cap\overline{S^{NE}},\sigma^{NE}_{-i})\geq w_i(\sigma^{OPT}_i\cap\overline{S^{NE}},\sigma^{OPT}_{-i})$. Therefore:

$$\begin{aligned} n\cdot w(\sigma^{NE}_{\min},\sigma^{NE}_{-\min}) &\geq \sum_{i=1}^{n}\Big(w_i(\sigma^{OPT}_i\cap\overline{S^{NE}},\sigma^{NE}_{-\min}) + w_i(\sigma^{OPT}_i\cap S^{NE},\sigma^{NE}_{-\min})\Big) \\ &\geq \sum_{i=1}^{n}\Big(w_i(\sigma^{OPT}_i\cap\overline{S^{NE}},\sigma^{OPT}_{-i}) + w_i(\sigma^{OPT}_i\cap S^{NE},\sigma^{NE}_{-\min})\Big) \\ &= w(\sigma^{OPT}\cap\overline{S^{NE}}) + \sum_{i=1}^{n}\big(w_i(\sigma^{OPT}_i\cap S,\sigma^{NE}_{-\min})\big) \end{aligned} \tag{62}$$

next we need to look at the payoff from the items in $(S^{NE}\cap S^{OPT})$ individually. All items in $(S^{NE}\cap S^{OPT})$ are chosen by at least one player in σ^{OPT}. Let $n_j(\sigma^{NE}_{-\min})$ be the number of times item j is chosen in $\sigma^{NE}_{-\min}$. When summed over all strategies of σ^{OPT}, we get at least payoff $\frac{1}{n_j(\sigma_{-\min})+1}w_j$ of each item j in $(S^{NE}\cap S^{OPT})$.

$$\sum_{i=1}^{n}\left(w_i(\sigma_i^{OPT}\cap S^{NE},\sigma_{-\min}^{NE})\right) \geq \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{1}{n_j(\sigma_{-\min})+1}w_j$$

$$\Rightarrow n\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) \geq w(\sigma^{OPT}\cap\overline{S^{NE}}) + \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{1}{n_j(\sigma_{-\min})+1}w_j \tag{63}$$

Before we can use this, we need to look into the value gained when switching to a strategy of another player in the Nash equilibrium. Since it is a Nash equilibrium, we have that the player with the least payoff cannot improve his payoff by switching to a strategy used by of any of the other $n-1$ players in the Nash equilibrium.

$$w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) \geq w_i(\sigma_i^{NE},\sigma_{-\min}^{NE}) \qquad \forall i$$

$$\Rightarrow (n-1)\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) \geq \sum_{\sigma_i^{NE}\in(\sigma^{NE}\setminus\sigma_{\min}^{NE})} w_i(\sigma_i^{NE},\sigma_{-\min}^{NE}) \tag{64}$$

Once again we look at the value gained per item. To do this we replace the payoff of a strategy by the sum over the payoffs of all the items in that strategy.

$$\begin{aligned}
(n-1)\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) &\geq \sum_{\sigma_i^{NE}\in\sigma_{-\min}^{NE}} w_i(\sigma_i^{NE},\sigma_{-\min}^{NE})\\
&= \sum_{\sigma_i^{NE}\in\sigma_{-\min}^{NE}}\sum_{j\in\sigma_i^{NE}}\frac{1}{n_j(\sigma_{-\min})+1}w_j\\
&= \sum_{j\in(S^{NE})}\sum_{(\sigma_i^{NE}\in\sigma_{-\min}^{NE}|j\in\sigma_i^{NE})}\frac{1}{n_j(\sigma_{-\min})+1}w_j\\
&= \sum_{j\in S^{NE}} n_j(\sigma_{-\min})\frac{1}{n_j(\sigma_{-\min})+1}w_j\\
&\geq \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{n_j(\sigma_{-\min})}{n_j(\sigma_{-\min})+1}w_j
\end{aligned} \tag{65}$$

Now we can use this to finish the proof:

$$\begin{aligned}
(2n-1)\,w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) &= n\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) + (n-1)\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE})\\
&\geq n\cdot w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) + \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{n_j(\sigma_{-\min})}{n_j(\sigma_{-\min})+1}w_j\\
&\geq w(\sigma^{OPT}\cap\overline{S^{NE}}) + \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{1}{n_j(\sigma_{-\min})+1}w_j + \sum_{j\in(S^{NE}\cap S^{OPT})}\frac{n_j(\sigma_{-\min})}{n_j(\sigma_{-\min})+1}w_j\\
&= w(\sigma^{OPT}\cap\overline{S^{NE}}) + \sum_{j\in(S^{NE}\cap S^{OPT})} w_j\\
&= w(\sigma^{OPT}\cap\overline{S^{NE}}) + w(\sigma^{OPT}\cap S^{NE})\\
&= w(\sigma^{OPT})\\
&\geq n\cdot w(\sigma_{\min}^{OPT},\sigma_{-\min}^{OPT})\\
\Rightarrow \left(2-\frac{1}{n}\right) w(\sigma_{\min}^{NE},\sigma_{-\min}^{NE}) &\geq w(\sigma_{\min}^{OPT},\sigma_{-\min}^{OPT})
\end{aligned} \tag{66}$$

■

Next is a matching lower bound example.

Theorem 9.8 *PoA* $\geq 2 - \frac{1}{n}$ *for symmetric generalized market sharing games with pure strategies and an egalitarian social function.*

Proof. The lower bound follows from the following example:

Example 9.9 Consider a symmetric game with n players and $2n$ items. There are n items with value n and n items with value $n-1$. Each player can either choose all items with value n, or one item with value n and one item with value $n-1$.

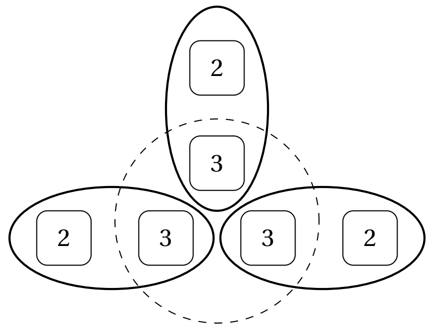


Figure 17: The game of Example 9.9 for 3 players. The circles are the sets available to the players. Every player can choose from all sets.

Proof of the lower bound. The optimal solution of the game in Example 9.9 is for all players to pick a set with one item of value n and one item of value $n-1$. This way all players gain $2n-1$. The worst Nash equilibrium is when all players pick the set with n items of value n. Since they now have to share all the payoff, each player has a payoff of n. So for an egalitarian social function the price of anarchy is $\frac{2n-1}{n} = 2 - \frac{1}{n}$. ■

Theorem 9.10 *PoA* $= 2 - \frac{1}{n}$ *for symmetric generalized market sharing games with pure strategies and an egalitarian social function.*

The worst Nash equilibrium in Example 9.9 also becomes the only Nash equilibrium when all items of value n are replaced with value $n+\epsilon$. This means the PoS $= 2 - \frac{1}{n}$ as well.

Corollary 9.11 PoS $= 2 - \frac{1}{n}$ *for symmetric generalized market sharing games with pure strategies and an egalitarian social function.*

So now we have that the worst-case egalitarian price of anarchy for pure Nash equilibria is $2 - \frac{1}{n}$ for an egalitarian social function. In 2016 Bilò [3] studied the robust price of anarchy, which was introduced by Roughgarden [23] for utilitarian social functions. Bilò investigated how far this could be extended to egalitarian social functions and he proved the following:

For a variety of utilitarian and egalitarian social functions, the worst-case price of anarchy of pure Nash equilibria coincides with that of coarse correlated equilibria in the class generalized weighted congestion games with non-negative latency functions.

Since symmetric generalized market sharing games are congestion games that fit the requirement he states in his paper, it follows from his work that the egalitarian price of anarchy of $2 - \frac{1}{n}$ for pure Nash equilibria extends to coarse correlated equilibria.

An overview of the bounds shown in this section can be found in Table 2:

Game	Egalitarian pure PoA	Egalitarian SPoA
Generalized market sharing games	n	n
Symmetric generalized market sharing games	$2-\frac{1}{n}$	$2-\frac{1}{n} \leq SPoA \leq n$
Singleton symmetric generalized market sharing games	1	1

Table 2: Bounds for generalized market sharing games with a egalitarian social function

10 Bottleneck congestion games

In this section we look into the bottleneck variant of generalized market sharing games. Bottleneck games are games where each player chooses a set of resources, but not with the goal of gaining as much value as possible, but of having a set of resources where the minimum value gained from a resource is as high as possible. Bottleneck congestion games are defined as follows:

Definition 10.1 Bottleneck congestion games
A *bottleneck congestion game* is a tuple $(N, R, (\mathbf{S}_i)_{i\in N}, (p_j)_{j\in R})$ where $N = (1,\ldots,n)$ denotes the set of players, $R = \{1,\ldots,m\}$ the set of resources, $\mathbf{S}_i$ the strategy space of player i, and p_j cost function of resource j. Each player chooses a subset of resources $\sigma_i \in \mathbf{S}_i$ and all resources have a cost, that is determined by their cost function. Each player has a weight x_i, which influences the cost function. The cost a player receives is equal to the payoff of the item chosen which provides the most costs, i.e. $w_i(\sigma_i, \sigma_{-i}) = \max_{j\in\sigma_i} p_j(y_j)$, where y_j is the total weight of the players choosing resource j.

We only investigate bottleneck games with an egalitarian social function. Bottleneck congestion games were studied by Busch and Magdon-Ismail [4] on network congestion games for an egalitarian social function. They show that the PoS = 1, which also holds for bottleneck generalized market sharing games.

Linear bottleneck congestion games are games where the cost function of each resource is linear. Linear bottleneck congestion games with egalitarian social function have been studied by de Keijzer et al. [12]. They provided bounds on the price of anarchy of $n \leq PoA \leq 2n-1$. Before specifying the results for generalized market sharing games , it's worth noting their lower bound on the PoA can be improved.

Theorem 10.2 PoA $\geq n+1$ *for Bottleneck congestion games*

Proof. Consider the following example:

Example 10.3 Consider a game for 2 players. Player 1 has weight 1 and player 2 has weight $\frac{1}{2}(\sqrt{5}-1) \approx 0.618$. There are three items, with payoff functions $p_1(y_1) = \frac{1}{2}\left(\sqrt{5}+1\right)y_1$, $p_2(y_2) = y_2$ and $p_3(y_3) = \left(\sqrt{5}+2\right)y_3$. Player 1 can choose only item 2 or both item 1 and item 2. Player 2 can choose item 1 or both item 2 and item 3.

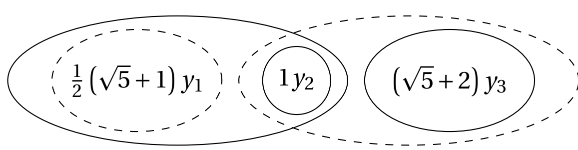
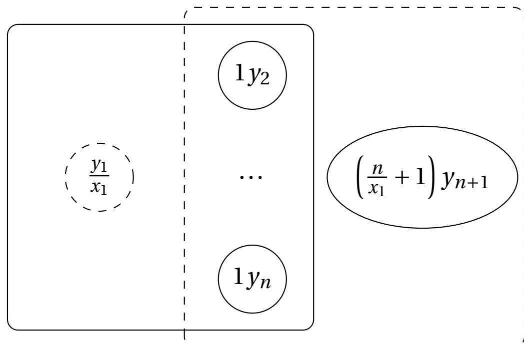


Figure 18: The game in Example 10.3. The circles are the sets available to the players. Player 1 can pick solid circles and player 2 can pick from the dashed circles. Player 1 has weight 1 and player 2 has weight $\frac{1}{2}(\sqrt{5}-1) \approx 0.618$.

In Example 10.3 the optimal solution is for player 1 to choose item 2 with cost function y_2, and for player 2 to choose item 1 with cost function $\frac{1}{2}\left(\sqrt{5}+1\right)y_1$. This gives both cost 1. However, the worst equilibrium is for player 1 to pick item 1 and 2, and player 2 picks items 2 and 3. This gives player 1 a cost of $\frac{1}{2}\left(\sqrt{5}+1\right)$, and player 2 a cost of $\frac{1}{2}(3+\sqrt(5)) \approx 2.618$, larger then the 2 provided by their lower bound.

This gives a price of anarchy of 2.618 for 2 players. This example can be generalized to n players, in the following way. Let there n players. Player 1 has weight x_1 and players $2,\dots,n$ have weight 1. Player 1 can choose only item 1 or all items $2,\dots,n+1$. Player i with $i \in (2,\dots,n)$ can choose item i or all items $2,\dots,n$. The cost function of item 1 is $\frac{y_1}{x_1}$. The cost function of item i with $i \in (2,\dots,n)$ is y_i. The cost function of item $n+1$ is $\left(\frac{n}{x_1}+1\right)y_{n+1}$.



We choose x_1 such that $\frac{n-1}{x_1} = n-1+x_1$, which gives $x_1 = \frac{1}{2}\left(\sqrt{(n+1)^2-4}-(n-1)\right)$. In this game the optimal solution is for every player i to choose item i, which gives each player cost 1. If each player goes for the other option they have available, player 1 gets cost $n+x_1$ while all other players get cost $n-1+x_1$. This gives a price of anarchy of $n+x_1 = n+\frac{1}{2}\left(\sqrt{(n+1)^2-4}-(n-1)\right)$, which asymptotically goes to $n+1$ as n goes to ∞. ■

Bottleneck generalized market sharing games

Now we move on to the bottleneck variant of generalized market sharing games. Bottleneck generalized market sharing games are defined as follows:

Definition 10.4 Bottleneck generalized market sharing games
A *bottleneck generalized market sharing game* is a tuple $(N, J, (\mathbf{S}_i)_{i\in N}, (w_j)_{j\in J})$ where $N = (1,\dots,n)$ denotes the set of players, $J = \{1,\dots,m\}$ the set of items, $\mathbf{S}_i$ the finite set of pure strategies of player i, and w_j the value of item j. Each player chooses a subset of items and the value of an item is shared between all player choosing that item. The payoff a player receives is equal to the payoff of the item chosen which provides the least value, i.e. $w_i(\sigma_i,\sigma_{-i}) = \min_{j\in\sigma_i}\frac{w_j}{n_j(\sigma)}$, where $n_j(\sigma)$ denotes the number of players choosing item j in strategy profile σ.

Theorem 10.5 PoA = *n for bottleneck generalized market sharing games.*

Proof. In bottleneck generalized market sharing games, we have identical players. This makes sure the $PoA \le n$, since the arguments used in Theorem 9.2 still hold: Any strategy σ_i^{OPT} chosen in the optimal solution can be made at maximum n times as bad, i.e. $w(\sigma_i^{OPT},\sigma_{-i}^{OPT}) \ge \frac{1}{n}w(\sigma_i^{OPT},\sigma_{-i}^{NE})$. This means $PoA \le n$. For the lower bound we use the following example:

Example 10.6 There are n players and n items, all value 1. Each player can pick any one item, or they can pick all items.

In this example all players can either pick 1 item, or they can pick all items. In the optimal solution everybody chooses only 1 item, giving each player a minimum payoff of 1. If everybody picks every item then a Nash equilibrium occurs, where all players have a minimal payoff of $\frac{1}{n}$, and nobody can improve that. ■

Example 10.3 shows that a Nash equilibrium can be n times worse than the optimal solution. But the subgame perfect equilibrium would avoid this bad Nash equilibrium and find the optimal solution. It is possible that im-

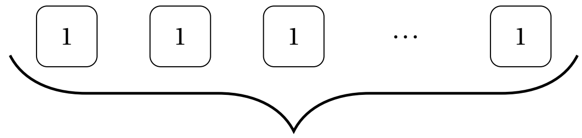


Figure 19: Example 10.3

posing a order on the players provides better results. With the arguments used in Theorem 9.2 we can show that the sequential price of anarchy is bounded from above by n. But the SPoA cannot be bounded from above by any constant:

Theorem 10.7 *The sequential price of anarchy of Bottleneck generalized market sharing games cannot be bounded from above by any constant.*

Proof. The proof uses the following example:

Example 10.8 There are 7 players and 8 items. The first 7 items have value 1 and the last item has value $\frac{1}{4}$. The items are divided into four groups, see figure 19. Players 1,2 and 3 can pick any set from group 1 or 2. Players 4,5 and 6 can pick any set from group 2 or 3. Player 4 can pick any set from group 3 or 4.

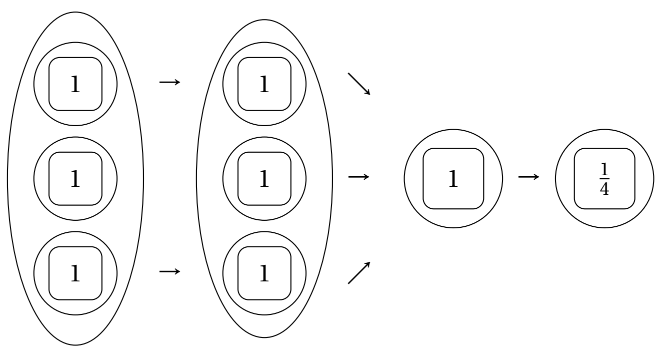


Figure 20: Game for 7 players. The circles are the sets available to the players. Players 1,2 and 3 can pick any set from group 1 or 2. Players 4,5 and 6 can pick any set from group 2 or 3. Player 4 can pick any set from group 3 or 4.

In the optimal solution players 1,2 and 3 all choose different items from group 1, players 4,5 and 6 all choose different items from group 2 and player 7 picks the item in group 3, giving each player a payoff of 1.

However, there is a subgame perfect equilibrium which gives player 7 a payoff of $\frac{1}{4}$. Here player 1 picks all items in group 1. This is followed up with players 2 and 3 picking all items in group 2. For players 4-6 there is no option to get more then $\frac{1}{3}$ from each item in their strategy, so they all pick the item from group 3. Finally, player 7 picks the item in group 4, giving him the minimum payoff of $\frac{1}{4}$.

Its possible to get any constant as SPoA using this method, by adding more groups and more players to each group. ■

The subgame perfect equilibrium in Example 10.8 is a Nash stable subgame perfect equilibrium. This subgame perfect equilibrium requires at least $1 + \frac{1}{2}SPoA(SPoA-1)$ players, which gives $SPoA \leq \frac{1}{2} + \sqrt{2n - \frac{3}{2}}$ for any game as in Example 10.8. Table 3 denotes how many players are needed to achieve a certain SPoA by using this method. For any given SPoA, the amount of players needed is at most $\sum_{i=1}^{SPoA} \prod_{k=1}^{i} (SPoA - k)$.

Example 10.8 can also be played with only singleton strategies. In that case it would use exactly $\sum_{i=1}^{SPoA} \prod_{k=1}^{i} (SPoA-$

SPoA	n	$\frac{1}{2} + \sqrt{2n - \frac{3}{2}}$
1	1	≈ 1.2
2	2	≈ 2.08
3	4	≈ 3.049
4	7	≈ 4.035
5	13	≈ 5.449

Table 3: Table containing SPoA using the method of Example 10.8

$k)$ players to archive a certain SPoA. When playing a singleton game, there is no difference between bottleneck games and non-bottleneck games, since all players try to optimize the value of their one item.

Corollary 10.9 *The sequential price of anarchy of singleton generalized market sharing games with a egalitarian social function cannot be bounded from above by any constant.*

11 Conclusion

In this research we investigated the quality of equilibria in many different kinds of resource allocation games and specifically generalized market sharing games. Our interest in generalized market sharing games originated from set packing games, where each item can be only chosen once. In generalized market sharing games the items can be chosen multiple times, which leads to sharing of the profit of items. We showed that this change has some counter intuitive results. First we proved that the price of anarchy of generalized market sharing games is $2 - \frac{1}{n}$, which is an improvement over the price of anarchy of set packing games of 2. Contrary to set packing games however, imposing a order on the players did not improve the quality of equilibria, but actually worsened it. It is interesting to see that such a change on the payoff functions can increase equilibria in the simultaneous variant, while decreasing quality of equilibria in the sequential variant.

We addressed several variations of generalized market sharing games and shown that the SPoA and PoA can vary greatly. Large improvements can be made by restricting the strategy sets. Specifically for an egalitarian social function we proved that the PoA and SPoA range from n for unrestricted strategy sets to 1 for singleton symmetric games.

In most versions of the game the quality of equilibria got worse in the sequential variant, or the SPoA was very difficult to bound. We applied a number of different methods in attempts to find bounds on the SPoA. Known methods did not always provide the wanted results, so we attempted new methods for proving upper bounds. The most fruitful method found during our research was for the class of games called shared misery games. This lead to our main result , which is that for symmetric singleton congestion games, every subgame perfect equilibrium is Nash-stable, and thus SPoA $\leq$ PoA (Theorem 7.6).

The framework used for this proof, specifically the guaranteed payoff Lemma, might prove useful for bounding the sequential price of anarchy in many other games. For instance, one interesting research topic for future work would be to investigate if more results can be found for anonymous symmetric shared misery games. If it can be shown that Theorem 7.6 can be extended to anonymous symmetric shared misery games, than this would prove SPoA $\leq$ PoA for a variety of games, including generalized market sharing games with an egalitarian social function and bottleneck generalized market sharing games.

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