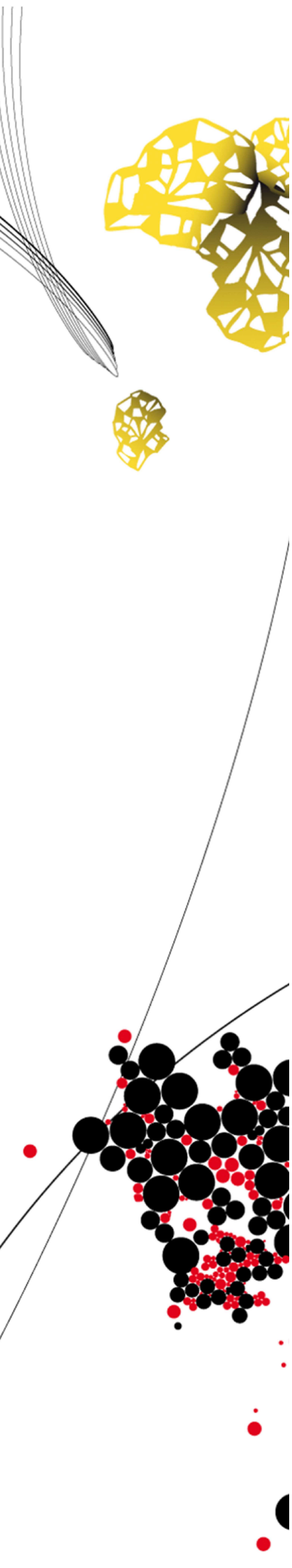

UNIVERSITY OF TWENTE.

Faculty of Behavioural, Management and Social sciences

Computing threat points in two-player ETP-ESP games

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The first lecture I attended when I started studying was covering statistics and probability. However, being a freshman I entered the wrong lecture hall and ended up at a lecture about economics in health sciences. After I recognized, quite late, that I was sitting at the wrong lecture, I found my way to the correct lecture but had to stand for the remaining hour. I did not know a lot about probability back then, but I did know that the probability of having two of these major mistakes was quite rare. The beginning of the Bachelor was mediocre, but the contrast with the ending of the Master is big.

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Abstract

This thesis focuses on the development of algorithms for computing the threat point in Frequency-Dependent-games, with the Endogenous Transition Probabilities (ETP) Endogenous Stage Payoffs (ESP) game as the most complex type of game. We limit ourselves to two-state, two-player ergodic stochastic games with or without a FD payoff. Also incorporated into an algorithm are so-called Endogenous Transition Probabilities, i.e., transition probabilities that depend on the history of the play. Several algorithms have been developed, one of them (SciPy algorithm) only works on non-FD Type I games while the other (Relative Value Iteration algorithm) is limited to use on non-FD Type II games. The Jointly-Convergent Pure-Strategy algorithm works on the broad spectrum of stochastic games described in this thesis, i.e., (non)-FD Type I, Type II and Type III games. We test the algorithms in terms of speed and accuracy and reflect on them including a disquisition of the risks surrounding some of the algorithms. We find that the SciPy algorithm is superior to the Jointly-Convergent Pure-Strategy algorithm in terms of accuracy and speed when computing the threat point in non-FD Type I games. The Relative Value Iteration algorithm works better in terms of accuracy when compared to the Jointly-Convergent Pure-Strategy algorithm but lags in terms of speed. The Jointly-Convergent Pure-Strategy algorithm fits all types of games while finding reasonably accurate solutions in a reasonable time.

Contents

List of acronyms

AIT	Action Independent Transition
CAIT	Current Action Independent Transition
CSP	Constant Stage Payoffs
CTP	Constant Transition Probabilities
ESP	Endogenous Stage Payoffs
ETP	Endogenous Transition Probabilities
FD	Frequency-Dependent
JCPS	Jointly-Convergent Pure-Strategy
MDP	Markov Decision Process
NaN	Not a Number
RVI	Relative Value Iteration

Chapter 1

Research Design

We start by elucidating the design of our research. We use a well-known method by Verschuren, Doorewaard, and Mellion (2010). Beginning by stating the research objective and the surrounding context. In the second section we derive a framework based on the research objective and context. In order to realize this framework we state research questions (third section) used as a guideline throughout our endeavours. Finally, we conclude by prescribing the structure of this thesis and its aim.

1.1 Research objective and context

Modern game theory is relatively new as a subject in science. Everyone with a little background knowledge in the subject will relate game theory to persons like John Nash Jr. and John von Neumann. The most well-known result in game theory is the Nash equilibrium, but science has evolved and game theory has moved on. Neymann introduced a new class in game theory called stochastic games. These stochastic games have been at interest of game theorists all around the globe, but also the subject of new discoveries. One of these discoveries was done in 2003, when Joosten, Brenner, and Witt (2003) published research in the field of Frequency-Dependent games.[1] Frequency-Dependent games are stochastic games in which the stage payoffs depend on the history of the play. The major theorem in this paper involves so-called 'threats'. These play an important role in game-theoretical analysis in which they determine the set of possible equilibria. Joosten (2009) stated: "Unfortunately, there exists no general theory on (finding) threat points in FD-games, yet." During the years, more research has been done in the domain of FD-games, e.g. (Mahohoma, 2014), (Joosten, 2015) and (Joosten & Samuel, 2018). Even though the first and last examples focus on computations within FD-games, none of these are able to state an exact threat point.

The introduction of Endogenous Transition Probabilities by Joosten and Meijboom (2010) added a novel dimension to the framework of FD-fishery games making analysis

[1] See Section 2.5 for a background on FD-games.

of such games more complex. This clearly leaves room for research on the topic of computations of threat points in FD-games. One should see our research as the next step into developing a general theory in finding threat points. We focus on the algorithm building aspect of finding these threat points. Our research objective is to calculate the threat point in ETP-ESP games by developing an algorithm.

1.2 Research framework

In order to complete our research objective we make use of a research framework. In this research framework we describe a few important pillars on which it will rely. The framework can be seen as a stepwise guide being followed in order to complete the research objective. Our research framework is built on the following five pillars.

1. Gather and summarize relevant literature in the game-theoretical domain with a focus on FD-games.
2. Search for and prepare programming tools necessary for the development of the algorithms with a focus on speed.
3. Develop the algorithms.
4. Test the algorithms in terms of speed, accuracy and scalability.
5. Reflect and conclude on the research performed, propose possibilities for future research.

The intended chronological element of our research framework should be clear. We start with a literature study, continue with a search for programming tools which are necessary for the next step: developing the algorithms. At last we test the algorithms and conclude the research in order to give recommendations for later research.

1.3 Research questions

The research framework constructed in the last section serves as a basis for the development of research questions. The research questions should result in knowledge necessary for achieving the research objective (Verschuren et al., 2010). Each core-question can be answered with the help of so-called sub-questions. In this research we only ask one core-question which we define as the main research question. The main research question is also a reflection of the research objective in question form. We state the main research question as:

How do we compute threat points efficiently in ETP-ESP games?

We answer this main research question with help of five sub-questions. These are:

1. What has been written in the literature about game theory and Frequency-Dependent games?
2. Which programming tools are available for game-theoretical computations?
3. Which algorithms can find a threat point and how do they work?
4. How do the algorithms perform in terms of speed and accuracy when scaled to a larger game?
5. What can we conclude and recommend based on this research?

As one can see, these sub-questions have a direct link to the research framework. Thus, the answering of the sub-questions should result in completing the steps described in the research framework which in its part should result in fulfillment of the research objective.

1.4 Thesis design and aim

This chapter describes the research design. In the second chapter we elaborate on all relevant literature surrounding the topic, we start with basic game theory in a gradual fashion ending up at Frequency-Dependent games. Chapter 3 focuses on developing the algorithm, starting with an explanation of the relevant programming tools and later in the chapter the development, testing and practical implications. We conclude and propose recommendations for future research in the fourth chapter.

Our aim of this thesis is to offer the reader a comprehensive summary of game theory in order to cope with a potential lack of knowledge in this domain. The more experienced reader could skip Chapter 2 and could continue at Chapter 3. Both experienced and inexperienced readers should be able to follow the main purpose of this thesis, developing an algorithm for calculation of threat points in ETP-ESP games.

Chapter 2

Game Theory Literature

In the previous chapter we laid a foundation for our research, but in this chapter we start with the necessary background knowledge in the domain of game theory. We start at the origin of game theory and progressively work towards FD-games. In between we discuss important aspects like the Nash equilibrium, repeated games and stochastic games in general. At the end of this chapter a non-expert reader should have the necessary background knowledge in order to follow the steps taken in the development of the algorithm in Chapter 3.

2.1 The Origin of Game Theory

A lot of scientific literature starts with cliché introductions on the complexity of the world we live and breath in. Although it may be a cliché, it is true to a large extent. In order to create structure in the chaos called life, scientists have developed theories and models based on empirical evidence to explain the complexity of life. In social sciences the focus is on explaining behaviour of individuals in a social context. Game theory is a mathematical discipline which forms the (descriptive) basis of analysis of rational decision makers in a strategic environment, in which their decisions have a direct or indirect effect on their individual payoff. According to Hyksova (2004) the first game-theoretical analysis was done by Charles Waldegrave in 1713 as documented in Pierre Rémond de Montmort's "Essay d'analyse sur les jeux de hazard", it comprises a strategic solution to the French card game le Her (Bellhouse, 2007).

We limit ourselves in the remainder to the domain of non-cooperative games, we start with an n players non-cooperative one-shot game in which $n = 2$. Players play a game consisting of states, denoted by the state space S, a nonempty and finite set. In a one-shot game players are limited to a game with only 1 state. Player 1 can choose an action from a nonempty and finite action set I_s when in state $s \in S$. Player 2 can choose an action from a nonempty and finite action set J_s when in state $s \in S$ (Flesch, 1998).

Players can choose an action with probability 1, but they can also randomize by using a mixed action. For player 1 we denote this mixed action by x_s in state $s \in S$, which is a probability distribution on I_s. Player 2 can choose a mixed action y_s in state $s \in S$, which is a probability distribution on J_s. In a one-shot game with two actions per player, $x \in \Delta^1 = \{x \in \mathbb{R} | x_i \geq 0$ for $i = 1, 2$ and $\sum_{i=1}^{2} x_i = 1\}$ and $y \in \Delta^1 = \{y \in \mathbb{R} | y_i \geq 0$ for $i = 1, 2$ and $\sum_{i=1}^{2} y_i = 1\}$. The combination of the (mixed) actions from both players result in payoffs $r^1_{s^1}(i^1_{s^1}, j^1_{s^1})$ for player 1 and $r^2_{s^1}(i^1_{s^1}, j^1_{s^1})$ for player 2 (Flesch, 1998).

Now that we have a mathematical basis we look back at the year 1921. Émile Borel published a few notes from 1921 until 1927 containing the first formalizations of pure and mixed strategies, and even a first attempt at a minimax solution (Borel, 1927). But the real breakthrough in game theory occured when Von Neumann and Morgenstern in 1944 published their work. However this breakthrough was based on earlier work by Von Neumann in 1928, this work already stated the minimax theorem, a revolution in game theory:

Theorem Von Neumann (1928): "For a two-person zero-sum[1] matrix game (denoted by M) in which player 1 chooses a mixed strategy denoted by p and player 2 a mixed strategy denoted by q. There always exists a pair of mixed strategies (p^*, q^*) such that."

$$M(p^*, q^*) = \max_p \min_q M(p, q) = \min_q \max_p M(p, q).$$

[1] A zero-sum game is a game in which the payoff of one player is the negative payoff of the other player and visa versa

It is best to demonstrate this with an example, consider matrix game zero-sum M which contains the payoffs for player 1 as[2]:

$$\begin{bmatrix} 7 & 3 \\ 2 & 2 \end{bmatrix}$$

Figure 2.1: Matrix Game M.

Player 1 is the row player using strategy p and player 2 is the column player using strategy q. First we look at $\max_p \min_q M(p, q)$. Player 1 wants to maximize his own payoff and therefore prefers playing the first row over the second one. Given that Player 1 uses his first action (row), Player 2, who wants to minimize the payoff of player 1, will play the right column over the left one because 3 is smaller than 7. The result of $\max_p \min_q M(p, q) = 3$. Player 2 who wants to minimize his opponents expected payoffs uses his second action (column) because this action weakly dominates his first action, meaning that no matter what Player 1 does, Action 2 always has a lower payoff to 1 than the alternative[3]. The result of minimax is $\min_q \max_p M(p, q) = 3$.

[2]We only show the payoffs for player 1 as a positive payoff, the payoffs for player 2 are the negative payoff of player 1 because it is a zero-sum game.

[3]In general such strategies need not be pure.

2.2 The Nash Equilibrium

Where von Neumann can be seen as one of the founding fathers of modern game theory, John Nash Jr. (from here on: Nash) can be seen as an absolute rockstar in the world of game theory. In 1951 Nash published his PhD. thesis. A major implication of this thesis was also published in a note called “Equilibrium Points in N-Person Games" (Nash, 1950). Nash won a Nobel Prize in Economics together with John Harsanyi and Reinhard Selten in 1994. He received this prize for the discovery of the solution concept for non-cooperative general-sum games. This discovery was later honoured by calling it the Nash equilibrium. Nash stated the following.

Theorem Nash: “Every finite normal form game has at least one equilibrium point in mixed strategies (Hyksova, 2004)."

Again let’s demonstrate this theorem with the help of two examples. In the first one we demonstrate that in a non zero-sum game there exist two Nash equilibria that are accessible by using a pure strategy for both players. Again player 1 is the row player with payoff matrix A and player 2 is the column player with payoff matrix B.

$$A = \begin{bmatrix} 9 & 5 \\ 9 & 3 \end{bmatrix}$$

Payoff Matrix A for Player 1.

$$B = \begin{bmatrix} 1 & 5 \\ 8 & 5 \end{bmatrix}$$

Payoff Matrix B for Player 2.

We combine these matrices to end up with the bimatrix game:

$$\begin{bmatrix} 9,1 & 5,5 \\ 9,8 & 3,5 \end{bmatrix}$$

Figure 2.2: Bimatrix Game $[A, B]$, first (second) entry is the payoff to player $1(2)$.

In order to check whether a pure Nash equilibrium exists, we set the strategy of one player on a fixed pure strategy and determine the best response of the other player. If a best response of player 1 is located in the same matrix location as the best response of player 2 then we can state that the corresponding actions for both players are a pure Nash equilibrium. Because they are best responses for both players, no player has the incentive to deviate from the Nash equilibrium. Applying this to bimatrix game $[A, B]$ gives us a new matrix in which we show the best responses of a player with a checkmark.

$$\begin{bmatrix} \checkmark 9, 1 & \checkmark 5, 5 \checkmark \\ \checkmark 9, 8 \checkmark & 3, 5 \end{bmatrix}$$

Figure 2.3: Bimatrix Game with pure best responses $[A, B]$.

As can be seen in Figure 2.3 the bimatrix game $[A, B]$ has two pure Nash equilibria, one in which player 1 plays a pure strategy of playing the bottom row and player 2 the left column. This pure Nash equilbrium is the Pareto optimal result[4] in this game. The other equilibrium is when player 1 plays the upper row with a pure strategy and player 2 the right column with a pure strategy. This equilibrium is a Pareto inferior result in comparison to the other pure Nash equilibrium.

A pure Nash equilibrium does not have to exist in a bimatrix game, which we show. Again we construct a bimatrix game in which player 1 is the row player with payoffs in matrix A and player 2 is the column player with payoffs in matrix B, resulting in the combined bimatrix game $[A, B]$:

$$\begin{bmatrix} 3, 8 & 7, 5 \\ 8, 1 & 6, 2 \end{bmatrix}$$

Figure 2.4: Bimatrix Game $[A, B]$.

Again we look for the pure best responses, resulting in the following bimatrix:

$$\begin{bmatrix} 3, 8 \checkmark & \checkmark 7, 5 \\ \checkmark 8, 1 & 6, 2 \checkmark \end{bmatrix}$$

Figure 2.5: Bimatrix Game $[A, B]$.

So Figure 2.5 shows that the bimatrix game $[A, B]$ does not have a pure Nash equilibrium. However, Nash showed that every bimatrix game in normal form has a mixed Nash equilibrium. For example, player 1 chooses to play the upper row with probability p and the lower row with probability $(1 - p)$. Player 2 on the other side chooses the left column with probability q and the right column with probability $(1 - q)$.

The solution for the mixed equilibrium is that player 1 plays strategy $(\frac{1}{4}, \frac{3}{4})$ and player 2 strategy $(\frac{1}{6}, \frac{5}{6})$. Player 1 receives a payoff of $6\frac{1}{3}$ and his opponent $2\frac{3}{4}$.

[4] See for an explanation on Pareto efficiency Page 10.

One of the greatest misconceptions surrounding the Nash equilibrium is that it is the optimal outcome for both players in a game. The Nash equilibrium does not have to be the highest possible payoff for an individual player nor for the two players. However, it is the payoff for which each individual player has no rational incentive to unilaterally deviate from. This is demonstrated by the use of the so-called prisoner's dilemma. In this dilemma two 'future prisoners' have the option to cooperate, i.e., not betraying the other player by not telling about the crime, in order to avoid going to prison. The other option is to defect, i.e., selling out the other player by telling. If both players choose to defect, then both go to jail for nine years. If one player defects while the other cooperates, then the player who cooperates is a free man while the other ends up in jail for ten years, if both cooperate they will only serve a one year sentence. This can be shown with the help of bimatrix game D representing the prisoner's dilemma.

$$\begin{array}{c} \\ cooperate \\ defect \end{array} \begin{array}{c} \begin{array}{cc} cooperate & defect \end{array} \\ \begin{bmatrix} -1,-1 & -10,0\checkmark \\ \checkmark 0,-10 & \checkmark -9,-9\checkmark \end{bmatrix} \end{array}$$

Figure 2.6: Prisoner's dilemma D.

As can be seen in Figure 2.6, the Nash equilibrium is pure, and implies that both players will defect. Therefore both end up in prison for nine year while a better outcome would be to cooperate by both. But because each individual player has an incentive to cheat (if the other player cooperates) and to defect (the result is not ending up in jail if the counter party does cooperate), the result is that both players end up with a Pareto inferior result. Pareto inferiority is linked to the concept of Pareto efficiency, in a situation in which Pareto efficiency has been achieved, no improvement in the allocation of the payoffs of the players can be made without sacrificing the payoff of another player. A solution is Pareto inferior if there is a possible improvement in payoffs for a player without reducing the payoff of another player. In the case of the prisoner's dilemma, the players can improve their payoff bilaterally by cooperating and obtain a Pareto optimal result, i.e., no additional Pareto improvements can be made.

2.3 Repeated Games

In the previous section we briefly discussed so-called one-shot zero-sum games and general-sum bimatrix games, i.e., games in which the game is only played once, resulting in a single payoff. A repeated game has similarities to the one-shot game. It is a one-shot game but now played for a finite or infinite number of times. The standard assumption is that players are aware of all the (possible) actions and resulting payoffs of the game until t. We call this complete information (Peters, 2015).

Repeated games have an advantage over one-shot games in the sense that they can reach rewards[5] which would normally be unattainable (Mertens, 1990). As we stated earlier, repeated games can be played for a finite or infinite number of times. In the first case this means that the game is played repeatedly for a finite number of times T. The other option is to play the game forever. In game theory the infinitely repeated type of game is studied more frequently than the finitely repeated game. A reason for this is that it is usually unknown how long T is for a game. Also infinitely repeated games could be more interesting because finitely repeated games can in specific cases be rewritten as a one-shot game and lack interesting features (Vrieze, 2003b). However, both options result in cumulative payoffs, there are two common methods for the analysis of these cumulative payoffs.

The first common method is the discounted reward criterion, with this criterion we analyze the game by assuming that any future payoff is valued less than the same payoff in the present. Therefore we discount future payoffs with a factor δ ($0 < \delta < 1$). We define the discounted criterion as follows. Player k receives a discounted reward (γ^k) in a two player game in which player 1 plays strategy π and player 2 plays strategy σ. The expected payoff of player k at time t under (π, σ) is defined by $R_t^k(\pi, \sigma)$. The discounted reward for player k is:

$$\gamma^k(\pi, \sigma) = \sum_{t=0}^{T} \delta^t R_t^k(\pi, \sigma)$$

Another way to analyze repeated games is by stating that the player values the present equally to any other period in the future. Therefore the player is indifferent with respect to receiving a payoff now or at a future stage. In this case we are looking at an infinite game with the limiting average reward criterion (Sorin, 2003). For player k this is:

$$\gamma^k(\pi, \sigma) = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} R_t^k(\pi, \sigma)$$

In this thesis we focus on the limiting average criterion. For repeated games the use of this limiting average reward criterion could result in the same rewards as just analyzing a single-period (bi)matrix game.

[5]Rewards are a result of cumulative payoffs based on a strategy pair, we introduce rewards in this same page.

2.4 Stochastic Games

We move on to stochastic games. Stochastic games were introducted by Shapley (1953). Stochastic games are much more complex in comparison to repeated games. The only thing in common is that they both are played for a(n) (in)finite number of periods T ($T \geq 2$). The difference lies within the state that game is in. In a repeated game there is only a single state which is played for period T ($T \geq 2$). In a stochastic game the state can change, based on the transition probabilities of the state in which the game currently is. Such transition probabilities represent the probability of moving to the current state or going to another state. Collectively states can be seen as multiple games that can be played but it depends on the transition probabilities whether they are played or not. Characteristics of one-shot games, repeated games and stochastic games are stated in Table 2.1.

Game type	**One-shot game**	**Repeated game**	**Stochastic game**
T	1	≥ 2	≥ 2
States	1	1	≥ 2

Table 2.1: Characteristics of different types of games.

In order to formalize the stochastic game in general we build further on the framework of a non-cooperative game stated in Section 2.1. Again we have n-players $(n \geq 2)$, but this time there are a multiple states denoted in the state space S. Player i has k number of actions depending on the state being in. A strategy is stating exactly to a player which action to play in each state at each point in time given any history of the play. We usually denote a strategy with π for player 1 and σ for player 2. The probability of playing a state depends on the transition probabilities, denoted by matrix p. In p the rows (r) represent the current states while the columns (c) are the possible future states. An entry in p with row r and column c is the probability of ending up in future state c from current state r. The rows in p should always sum up to 1 $(\sum_{c=0}^{n} p_{r,c} = 1)$. The combination of states S and pair of strategies (π, σ) result together with transition probabilities p in a reward set for players denoted by R.

In order to make it conceivably we show a simplified example, taken from Samuel's Master Thesis (2017). The example concerns a two player, two state stochastic game. The payoffs of both players in state 1 and state 2 are given by θ_{S_1} and θ_{S_2}.

$$\theta_{S_1} = \begin{bmatrix} 16,16 & 14,28 \\ 28,14 & 24,24 \end{bmatrix} \qquad \theta_{S_2} = \begin{bmatrix} 4.0,4.0 & 3.5,7.0 \\ 7.0,3.5 & 6.0,6.0 \end{bmatrix}$$

Ending up in a certain state depends on transition probabilities p, which for both states are given as:

$$p^{S_1} = \begin{bmatrix} 0.8,0.2 & 0.7,0.3 \\ 0.7,0.3 & 0.6,0.4 \end{bmatrix} \qquad p^{S_2} = \begin{bmatrix} 0.5,0.5 & 0.4,0.6 \\ 0.4,0.6 & 0.15,0.85 \end{bmatrix}$$

This example game is a 'commons'-type game (Hardin, 1968). In a 'commons' game there is a renewable common-pool resource, players share this common-pool and use it for their individual gain which may result in depletion of the resource. A type of game modelled as a 'commons'-type game is the fishery wars, analyzed by Levhari and Mirman (1980) and Joosten (2007). In their fishery wars game the players catch fish from the ocean, but overfishing can result in a depletion of the fish stock. In the example game the depletion of the fish takes place in state S_2, while state S_1 represents a higher amount of fish in the common-pool. Overfishing not only results in a higher probability of ending up in state S_2, but it also affects the transition probabilities. When overfishing, the transition probabilities change, the transition probability of ending up in state S_2 increases while the probability of ending up in state S_1 decreases.

However, the big question is now, how do we analyze these games? Shapley (1953) proposed using the discounted criterion and proved that there is a value[6] for the infinite zero-sum game and also showed that minimax and maximin solutions exist. Gillette (1957) introduced the limiting average criterion and showed that a value exists for infinite zero-sum games with perfect information. The absolutely compelling proof came from Mertens and Neyman (1981). They showed that a value exists for all zero-sum stochastic games and Neyman (2003a) demonstrated that all stochastic games have a minimax and maximin solution.

[6]The value of a game is the reward of the game when $M(p^*, q^*) = \max_p \min_q M(p,q) = \min_q \max_p M(p,q)$

Types of strategies and Folk Theorem

Finding an equilibrium, a value, a minimax or a maximin solution depends on the type(s) of strategy being chosen by both players. In stochastic games several types of strategies are possible, which we now describe.

Behavioral strategies are strategies in which a player takes into account the history of play leading up to a certain state. Therefore behavioral strategies are history dependent and require a player to remember the history of play in order to determine the optimal strategy.

Markov strategies are strategies in which the player does not take the entire history of play into account, but decides on the basis of the current state the play is in and current period how to play.

Stationary strategies are strategies in which the decision of the player does not depend on the history of play, nor on the current time, it only depends on the current state being in. Therefore a player always plays the same (mixed) action in each state.

Behavioral strategies are the most complex strategy type described here because they require a memory for the player. For infinite stochastic games this requires a large memory, and therefore a lot of storage capacity. However behavioral strategies are not always necessary. In this thesis we limit ourselves to irreducible stochastic games under the limiting average criterion. Vrieze (2003a), showed that for irreducible stochastic games stationary strategies are optimal. For reducible stochastic games, the zero-sum game 'Big Match' by Blackwell and Ferguson (1968) shows that not all limiting average stochastic games have a value that corresponds to a stationary strategy.

Another important result in literature is the Folk Theorem, stating that subgame perfect Nash equilibria can be reached in repeated games. When players are patient and far-sighted (as is the case with the limiting average criterion) then, as long as the payoffs are feasible and strictly individually rational (Levin, 2002), any outcome satisfying the conditions can be obtained by using a Nash equilibrium. The difference with playing a mixed action in a one-shot game is that in practice a mixed action will be converted to a pure action, one cannot actually randomize, but only randomize the probability that a certain action is chosen. Therefore, the Folk Theorem and repeated games define necessary conditions in order to actually obtain rewards. The Folk Theorem therefore forms a basis for further analysis of equilibria in stochastic games.

2.5 Frequency-Dependent Games

This brings us to the class of Frequency-Dependent games, introduced in 2003 by Brenner and Witt. Frequency-Dependent games are infinitely repeated games in which the stage payoffs depend on the frequencies of play up until that stage and including the current period in time. The first analysis was done by Brenner & Witt (2003), but Joosten et al. (2003) continued by deepening on feasible results and equilibria.

This was done by introducing the concept of jointly-convergent pure-strategy pairs. An important part of this is the relative frequency vector denoted by x^t. The relative frequency vector x^t states the relative frequency of play ending up at a certain payoff cumulatively at stage t. As an example we look at a simple $2x2$ bimatrix game. The relative frequency vector at stage t for this game looks as follows:

$$x^t = \begin{bmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{bmatrix}$$

Suppose we play this bimatrix game for a period $t = 100$, the strategy pair (π, σ) results in the game ending up in x_1 for 75 times, in x_2 20 times and x_4 5 times. The result is the relative frequency vector x^{100} with:

$$x^{100} = \begin{bmatrix} 0.75 & 0.20 \\ 0.00 & 0.05 \end{bmatrix}$$

For which:

$$\sum_{i=1}^{i=4} x_i = 1$$

The relative frequency vector x^t therefore changes based on frequency of the history of play and as stated earlier plays a large role in the concept of jointly-convergent pure-strategy pairs. A pure strategy is a strategy in which at any stage t in any state a player chooses a certain action with probability 1. A strategy pair (π, σ) is jointly-convergent for strategy of player 1 π, strategy of player 2 σ and relative frequency vector x if and only if (Joosten et al., 2003):

$$\limsup_{t \to \infty} Pr_{\pi,\sigma}[|x_i^t - x_i| > \epsilon] = 0$$

Jointly-convergent pure-strategy pairs are strategy pairs for which the relative frequency vector converges to a vector consisting of fixed numbers with probability 1 when t goes to infinity (Billingsley, 2008). Based on this it was possible to compute the set of feasible jointly-convergent pure-strategy rewards. The main result however of jointly-convergent pure-strategy pairs is the following:

Theorem: "Each pair of individually-rational jointly-convergent pure-strategy rewards can be supported by an equilibrium. Moreover, each pair of jointly-convergent pure-strategy rewards giving each player strictly more than the threat-point reward, can be supported by a subgame-perfect equilibrium. (Joosten et al., 2003)"

Key in this theorem is the so-called threat point reward. The threat point $v = (v^1, v^2)$ is defined for two players as the point in which $v^1 = \min_\sigma \max_\pi \gamma^1(\pi, \sigma)$ and $v^2 = \min_\pi \max_\sigma \gamma^2(\pi, \sigma)$. So both players can threaten each other by minimizing the payoff of the other player while the player under threat tries to maximize its own payoff. The threat point is the absolute minimum players can reach as a payoff equilibrium and therefore each individual player has an individual rational payoff if he can at least receive the threat point reward (Joosten et al., 2003). But the major result by Joosten et al. (2003) is the following theorem:

Theorem: "Each pair of rewards in the convex hull of all individually-rational jointly-convergent pure-strategy rewards can be supported by an equilibrium. Moreover, each pair of rewards in the convex hull of all jointly-convergent pure strategy rewards giving each player strictly more than the threat-point reward, can be supported by a subgame-perfect equilibrium (Joosten et al., 2003)."

The threat point is key in finding equilibria supported by jointly-convergent pure-strategy rewards. Threat points can be calculated analytically (e.g.,(Joosten & Meijboom, 2018))[7], but analytical methods may be too cumbersome, and as the complexity of a game increases the need for a fast algorithmic solution to compute the threat point increases equally. Joosten and Samuel (2018) have explored the domain of stochastic games and tried to order them. Games are divided into three major categories. Type I, Type II and Type III games. Type I games are games in which the play is repeated and transition probabilities per state are fixed, Type II games are stochastic games with constant transition probabilities and Type III games are stochastic games in which the transition probabilities are endogenous, which means that they depend on the history of play.

[7]The first working paper by Joosten and Meijboom was released in 2010, but published for the first time in a book in 2018.

Within these three categories there is also a distinction in whether the payoffs are frequency-dependent. If they are frequency-dependent we call them Endogenous Stage Payoffs (ESP) games. Without the frequency-dependent stage payoffs they are called Constant Stage Payoffs (CSP) games. The same goes for the transition probabilities, with frequency-dependent transition probabilities they are called Endogenous Transition Probabilities (ETP) games and Constant Transition Probabilities (CTP) games. Combining these in a table has been done by Joosten and Samuel (2018) and gives the following:

	CTP		**ETP**
	Type I (p_0)	**Type II** (x, p_0)	**Type III** $(x, p(x))$
CSP (θ_0)	AIT games	Stochastic games	CAIT-games ETP-games
ESP $(\theta(x))$	FD games AIT FD games	Stochastic FD games	CAIT-ESP games ETP-ESP games

Table 2.2: Ordering of different types of stochastic games (Joosten & Samuel, 2018).

Stochastic Frequency-Dependent (FD) games were first named this way by Mahohoma (2014), Joosten and Samuel (2018) call this term “a tautology in the broad interpretation." Mahohoma (2014) analyzed stochastic FD games while Joosten and Meijboom (2018) analyzed ETP-CSP games. Joosten and Samuel (2018) were the first to calculate feasible rewards in ETP-ESP games. In general, the use of the term Frequency-Dependent has been expanding over the years.

The first FD games introduced by Brenner and Witt (2003) and Joosten et al. (2003) were additive. Later, Joosten (2007) introduced an adaptation on the Small Fish Wars which contained FD games that are multiplicative and nonlinear. Joosten (2009) introduced the first FD games which are jointly frequency dependent. In 2010, the first FD games with Endogenous Transition Probabilities were developed by Joosten and Meijboom. The last and most complex development in the domain of FD games was tackled by Joosten and Samuel (2018) in the domain of ETP-ESP games.

Last but not least we clarify AIT and CAIT games. AIT stands for Action Independent Transition (AIT) (probability) games and CAIT games are Current Action Independent Transition (CAIT) (probability) games. AIT-games are games in which the play is repeated and for which transition probabilities are fixed and equal in each state, i.e., independent from the action pair chosen. The main focus of this thesis is to develop a threat point algorithm which is fast and suits all of these types of stochastic games. Therefore we start at the simpler Type I games, then continue with Type II games and finally end up with Type III games. As the complexity of the games increases, the complexity of the computations also rises.

Chapter 3

Building the algorithm

In this chapter we build the algorithm from the ground up, we start by introducing earlier work which partly forms a basis for the algorithm created. Then we clarify the relation between Markov Chains, Markov Decision Processes and Stochastic Games. We introduce necessary limitations on stochastic games in order for our algorithm to work as intended. We also introduce the programming tools used to develop the algorithm. Once these things are clear, we start by describing the algorithm(s) for Type I games, then continue with Type II games and end with Type III games. All code corresponding to the algorithms can be found in Appendix A.

3.1 Earlier Work

Before we start with the building blocks of the algorithm, we look at earlier work done in the area of computations in stochastic games. In literature, research has been conducted on algorithms for stochastic games. Raghavan and Filar (1991) have conducted a survey in which they limited to algorithms for zero-sum and non-zero-sum stochastic games with complete information and stationary strategies. A lot of research has been done, but mostly regarding the discounted reward criterion (Breton, Filar, Haurle, & Schultz, 1986, e.g.). Vrieze (1981) has developed a linear programming approach to an undiscounted stochastic game based on earlier algorithms by Filar and Raghavan (1979). However it is unclear whether these approaches work on the broad range of stochastic games and if they do, are they able to find the so-called threat point in every type of game?

In 2014 Mahohoma tried to partially overcome this in his Master thesis. He analyzed so-called stochastic FD-games (Type II, CTP-ESP). Part of his thesis is a simulation based approach in MATLAB in order to determine equilibria, feasible rewards but also threat points. However, one of the problems in the approach taken by Mahohoma lies in the simulation approach and the resulting complexity of the code. As an example we

look at the determination of threat points. Mahohoma simulates the stochastic game for a fixed period t, number of times n. Therefore a total of n simulation results is combined and gives as an average an approximation of the threat point. Not only does this algorithm lack the ability to give an exact result, the algorithm is also of a high computational complexity $O(n^4)$. This does not mean that the approach of Mahohoma is useless, it could be useful in games in which non-stationary strategies are the only way of reaching the threat point. However, when looking at stochastic games in the broader sense, in games in which stationary strategies always cover the threat point, we think that this could be done in an exact and more optimized matter.

The most recent work however has been done by Joosten and Samuel (2018). Samuel (2017) did a first analysis in her thesis. There she started the work which is also partly used for the calculation of threat points in this thesis. Samuel defines three algorithms in MATLAB used for computing feasible rewards in Type I, Type II and Type III games, with or without FD payoff. The basis for the computations of the set of feasible rewards are jointly-convergent pure-strategy rewards in which the algorithm is limited to games with communicating states.[1] This work has been formalized and analyzed more rigorously by Joosten and Samuel (2018). The algorithm works well and the threat point is part of the set of feasible rewards found. However, the algorithm does not explicitly find this threat point. Another potential problem with the algorithm is that the computational running time for a large set of rewards possibly is unnecessarily long.

3.2 From Markov Chain to Stochastic Game

In literature there is a clear link between Markov chains on one side, and stochastic games on the other end of the spectrum. Markov chains were introduced by and named after the Russian mathematician Andrey Markov (1971). Markov chains are stochastic processes in which the process is memoryless. The Markov property states that it does not matter what the history is before the present, the only thing relevant for the future is the present. We denote a random variable in a stochastic process at time t by X_t, the current value of the variable is denoted by x. Mathematically, memorylessness has the following effect on a stochastic process.

$$Pr(X_{t+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = Pr(X_{t+1} = x | X_t = x_t) \tag{3.1}$$

[1] See next paragraph on communicating states

So Equation 3.1 formalizes that the probability of ending up in another state $(X_{t+1} = x)$ only depends on the current state $(X_t = x_t)$ and not on the history of the states visited. Markov chains are therefore memoryless and time homogeneous (Häggström, 2002).

An important aspect of Markov chains used in this thesis is that we take a Markov chain as irreducible. We refer to a definition by Olle Häggström (2002):

Definition Irreducible Markov Chains: *"A Markov chain $(X_0, X_1, ...)$ with state space $S = \{s_1, ..., s_k\}$ and transition matrix P is said to be **irreducible** if for all $s_i, s_j \in S$ we have that $s_i \Leftrightarrow s_j$. Otherwise the chain is said to be reducible (Häggström, 2002)."*

Irreducible Markov chains are Markov chains in which all states communicate, which is denoted by $s_i \Leftrightarrow s_j$. So from any state all other states are reachable in an irreducible Markov chain. This can be seen with the help of an example. We analyze the weather and in our model there are two states in which the weather could be: it rains or it is sunny. The probability that it is now sunny and it stays sunny is 0.75, the same goes for the probability that it rains and it will keep on raining tomorrow. The probability that it will get sunny tomorrow while it rained today is 0.25, the probability that it will rain tomorrow while it is sunny today is also 0.25. This is visualized with Figure 3.1.

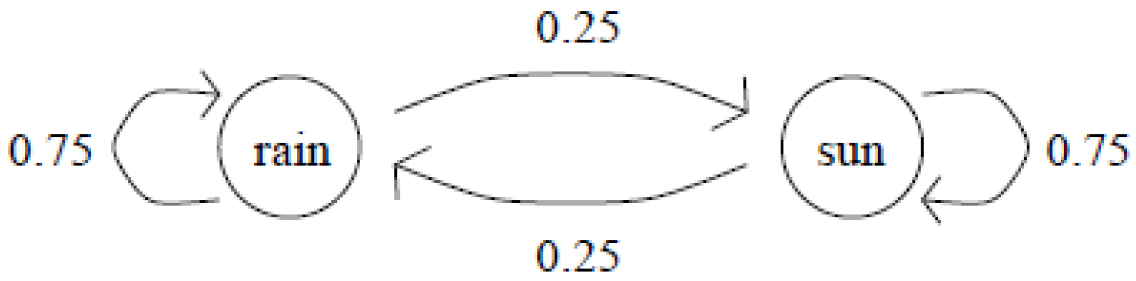


Figure 3.1: Markov chain: Weather example (Häggström, 2002).

Figure 3.1 is an example of an irreducible Markov chain. It is always possible to get a sunny day after a rainy day and visa versa. One could add a third weather option 'snow' for example. This adjusted Markov chain will be irreducible if it would be able to get a snowy day after a rainy day, a sunny day and after a snowy day. It should be possible to reach a sunny day and a rainy day after a snowy day. Another important property of Markov chains is so-called aperiodicity. If a Markov chain is aperiodic, there is no fixed period in which a certain state is always visited. Also, if a state is aperiodic, then all states are aperiodic and therefore the whole chain is aperiodic. More formally this leads to the following definition:

Definition Aperiodic Markov Chains: *"A Markov chain is said to be **aperiodic** if all its states are aperiodic. Otherwise the chain is said to be periodic (Häggström, 2002)."*

Figure 3.1 is an aperiodic Markov chain. It is not the case that after a certain fixed number of rainy days that there always will be a sunny day. Aperiodicity and irreducibility are two properties which form the basis of an interesting insight into Markov chains. When an aperiodic and irreducible Markov chain is run for a long time, it is unclear in which state the Markov chain is at a certain period in time. However, running the Markov chain for an infinite period of time will result in the Markov chain settling in a stationary distribution. This stationary distribution describes the probability of visiting a certain state when time goes to infinity. Therefore with great precision we know the frequency of being in a certain state when the Markov chain is run for an infinite period of time. We therefore present an important resulting theorem which forms an important part of this research:

Theorem: *"For any irreducible and aperiodic Markov chain, there exists at least one stationary distribution (Häggström, 2002)."*

But how are Markov chains linked to stochastic games? Neymann described that "Markov chains and Markov decision processes are special cases of stochastic games (Neyman, 2003a)." Markov chains are necessary to model the dynamics of a system. They state the transition probabilities of a stochastic game. In between the Markov chain and the stochastic game is the Markov Decision Process (MDP). The MDP is a reduction of the stochastic game in which there is only one player. Therefore the player is able to control the play and hence the corresponding payoffs on his own. Filar and Vrieze (1997) describe the stochastic game in terms of a competitive MDP.

In the case of one player under the limiting average criterion they describe the MDP as follows. The player starts in initial state s while playing stationary strategy f, the reward at time t is defined by R_t. The value of this irreducible limiting average MDP is defined as (Filar & Vrieze, 1997):

$$v_\alpha(f) := \lim_{T\to\infty} \left[\left(\frac{1}{T+1}\right) \sum_{t=0}^{T} \mathbb{E}_{sf}[R_t] \right]$$

The individual rational player always wants to maximize his own payoff. So the problem is an optimal control problem in which the player wants to:

$$\max v_\alpha(f)$$

This problem can also be seen as an optimal control problem in which the player tries to control the process in such a matter that his own value is maximized (Blackwell, 1962). In literature these MDP models are not only used for stochastic games but for a wide range of applications. They are used for decision-theoretic planning, learning robot control and ofcourse stochastic games. MDPs are the standard for learning sequential decision making (Otterlo, 2009). Algorithms in order to find optimal values for an MDP are divided into two categories. Model-free and model-based algorithms. The first category is also known as reinforcement learning and generates approximations while the second one is exact and uses dynamic programming as a basis (Blackwell, 1962), (Otterlo, 2009).

Because we are dealing with stochastic games in which we assume perfect information we have all information available in order to calculate an exact result. We shall therefore only look at model-based algorithms. These algorithms work on optimizing value functions by either iterating over the value function (value iteration) or by changing the so-called policy (policy iteration). The policy of an MDP can be seen as a fixed pure strategy which always is taken when in a certain state. At the heart of these algorithms is the Bellman equation. The Bellman equation defines the relation between the value function and the recursive process in order to determine the result of the value function (Otterlo, 2009). The equation is stated as (Otterlo, 2009):

$$V^{\pi}(s) = \mathbb{E}_{\pi}\{r_t + V^{\pi}(s_{t+1})|s_t = s\}$$

$$= \sum_{s'} T(s, \pi(s), s')\big(R(s, a, s') + V^{\pi}(s')\big)$$

In which policy is represented by π, the transitions by T, current state by s, rewards by R and current reward by r. This Bellman equation is important when we come to an algorithm for Type II games. But for now it is most important to acknowledge that stochastic games can be seen as competitive MDPs in which Markov chains are responsible for the transition dynamics between the states.

3.3 Limitations game usage

In literature on stochastic games several classes of games have been distinguished. Flesch (1998) gives an overview of the different types of stochastic games. He defines eight types of special classes and shows important results for these games in both the zero-sum case and general-sum case. Without going into much detail on the characteristics of these special classes we only focus on important results. The zero-sum cases of these special classes show mostly that there are optimal stationary strategies.

For the general-sum case there exist ϵ-equilibria[2], but optimal stationary equilbria are only known to exist in a select number of cases (Flesch, 1998).

We limit ourselves in the creation of the algorithm to types of games for which we know that the algorithm works. Earlier we stated Markov chain properties which guarantee a stationary distribution of the chain, i.e., irreducibility and aperiodicity. Therefore we focus on games in which the Markov chain governing the transition matrix is ergodic.[3] Games with absorbing states or more than one ergodic set are currently left out of scope. This however, does not mean that the algorithm constructed does not cope with any of the special classes described by Flesch (1998). We think that this leaves room for future research. For now we focus on the above stated characteristics and type of games displayed in Figure 2.2. We limit ourselves to two-player, two-state games.

3.4 Programming Tools

Creating an algorithm has to be done with the help of programming languages and tools. Earlier we stated that work has been done by Mahohoma (2014) and Samuel (2017) in MATLAB. We choose to deviate from MATLAB and take a different approach. First, MATLAB is a commercial programming language used mostly in the educational domain, but because of the commercial licensing necessary to run, it limits the applicability of the algorithm without a license. We are however in favor of free open-source solutions. Secondly, because MATLAB is a commercial product we cannot always get a grasp on what is happening under the hood of the program. In case of algorithmic optimization this can be problematic. So a choice has been made to not continue using MATLAB but to transfer to Python. We describe Python and the other programming tools used briefly to give the reader an impression of their possibilities.

3.4.1 Python

We first introduce the main programming language used. Python was developed in the early 90's by Guido van Rossum. The goal of Python is to provide easy and understandable syntax, which should result in highly readable code. Python does this by using indentation as a major part of the programming syntax. In comparison to other programming languages, Python has a dynamic type system, i.e., that Python does not require the programmer to define the type of variables used. This also improves readability a lot. On the other side, Python is an interpreted language and also a rather slow one. In comparison to C++ or Java, Python lacks raw speed. However, Python

[2] ϵ-equilibria are equilibria that approximately satisfy the condition associated with the Nash equilibrium. The incentive for a player to deviate from this equilibrium is ϵ small.

[3] Ergodic Markov chain are always irreducible and aperiodic

is a flexible language for which additional packages can be used to cope with this. A lot of additional and highly optimized packages are developed for Python and can be used.

3.4.2 NumPy

One of these packages is NumPy and was introduced in 2005 into Python. NumPy is written in C and offers Python users multi-dimensional array and matrix tools. It is seen as the go to solution for numerical computations with high-level[4] mathematical functions. Because NumPy has been written and compiled in C the operations are comparable to MATLAB in terms of speed. NumPy is also open-source and development is ongoing, so it is well-suited for the building of the algorithm in this thesis.

3.4.3 SciPy

Another comparable package is SciPy, SciPy is based on NumPy and uses a lot of attributes from NumPy in order to provide scientific computing functions. SciPy contains for example linear algebra functions and optimization functions using linear programming.

3.4.4 MDP Toolbox

The last tool used is the Python MDP Toolbox. This toolbox contains multiple algorithms (model-free and model-based) which can be used in order to solve an MDP. Most algorithms are written with the help of NumPy and therefore speed should not be an issue.

3.5 Type I games

We start building the algorithm by looking at Type I games. They are simple in essence because the play in these games is repeated. This makes Type I games less hard to analyze in comparison to Type II, let alone Type III games. But first we should state how we enter a game into the algorithm(s). We use NumPy in order to declare matrices. In Type I games, only two things are relevant when entering the game into an algorithm. Most important are the reward matrices for each player, the reward matrices display the payoffs that a player can achieve when playing pure strategies. Next is the frequency-dependent function, the frequency-dependent function declares what the relation is

[4] High-level in this sense means that the programming language is of a higher abstraction in comparison to low-level programming languages or machine language.

between a potential change in payoff and the frequency of play of certain strategy combinations. Because both can be defined in several ways, we adapt to the model used by Samuel (2017). Samuel (2017) uses a game in which the Nash equilibrium is also Pareto superior. However, she states that when playing this equilibrium this will eventually have an impact on the renewable common-pool resource and will therefore influence the payoff in the future (Samuel, 2017).

The combined payoff matrix used in order to test the algorithm is as follows:

$$\theta_{S_1} = \begin{bmatrix} 16, 16 & 14, 28 \\ 28, 14 & 24, 24 \end{bmatrix}$$

Samuel also created a linearly decreasing function as the frequency-dependent function. This was specifically created for the game stated above, so we also use this function in order to test the algorithm within this thesis.[5]

The frequency-dependent function for the Type I game is determined by:

$$FD_I^t = 1 - \frac{1}{4}(x_2^t + 2x_3^t) - \frac{2x_4^t}{3}$$

Hence, if the payoffs are frequency-dependent, they become:

$$\theta_{S_1}^t = FD_I^t \cdot \begin{bmatrix} 16, 16 & 14, 28 \\ 28, 14 & 24, 24 \end{bmatrix}$$

The upper-left corner of the payoff matrix (with payoff 16 to both players) can be seen as the responsible result of the game for which the depletion of the common-pool resource is non-existent. The upper-right corner results in a slight depletion, while the bottom-left corner is double the depletion compared to the upper-right corner. The worst outcome for the common-pool resource would be the bottom-right corner, it would result in the largest depletion effect on the renewable common-pool resource. Therefore the players also have to take into account the long-term effect on the common-pool resource.

But now we focus on the building of the algorithm. Because there are different ways to skin a cat, there are also different ways to build an algorithm. We have tested two approaches and will start with the approach which uses a SciPy optimizer. The second approach is an adaptation of the algorithm by Joosten and Samuel (2018).

[5]Other games were tested by the author, but for the sake of simplicity we keep it at just one game within the thesis

3.5.1 SciPy optimizer

SciPy as stated earlier is home to scientific computing packages. One of these is the optimization package. The optimization package contains several mathematical optimizers which are used for finding local or global optima. Mostly these optimizers rely on methods which calculate derivatives in order to find the optimum of a function. However a problem with these methods is that they can get stuck in local optima which could potentially not be the global optimum (Knowles, Watson, & Corne, 2001). Using these optimizers therefore has an inherent risk, which is that they provide possible sub-optimal solutions.

For Type I games with no frequency-dependent function this problem is redundant. Type I games as stated earlier are simple games which are repeated for a(n) (in)finite number of periods. Therefore optimizing over the payoffs is like optimizing a linear function when searching for a minimum. We start by using the optimizer for non-FD Type I games.

Because we are looking at the threat point for two players we start by restating what the threat point is. The threat point v is the point in which player 1 plays strategy π and player 2 plays strategy σ for which:

$$v = (v^1, v^2) \qquad \text{with:}$$

$$v^1 = \min_{\sigma^1} \max_{\pi^1} \gamma^1(\pi^1, \sigma^1)$$

$$v^2 = \min_{\pi^2} \max_{\sigma^2} \gamma^2(\pi^2, \sigma^2)$$

So $v^{1\,(2)}$ is the amount which player $1(2)$ can guarantee himself if player $2(1)$ tries to minimize the payoff of player $1(2)$.

In order to find the threat point we start in Python by initializing the strategy of the player who is threatening the other player with an empty strategy (an array containing only zeros). Then we have to set a condition for which the stationary strategy must hold. This condition declares that the probability assigned to all possible actions within the strategy must sum to 1. For example, if we have a game with one-state and two possible actions for player 2, if σ_1, σ_2 are the representation of player 2 choosing respectively action 1, action 2 then

$$\sum_{i=1}^{2} \sigma_i = 1$$

So again, the player has to choose a stationary strategy in which there is a summed probability of 1 over all possible actions. Additionally we have to declare to the algorithm that the bounds for assigning a probability to an action lie between 0 and 1:

$$0 \leq \sigma_i \leq 1$$

We also declare a threat function, this is the function that needs to be optimized. In this case we first multiply the rewards with the stationary strategy chosen by the threatening player, i.e., for v_1 we multiply γ^1 with σ^1. Therefore function v_1 has the form:

$$v^1 = \max_{\pi^1}(\gamma^1 \cdot \sigma^1)$$

And for v^2:

$$v^2 = \max_{\sigma^2}(\gamma^2 \cdot \pi^2)$$

This threat function is then applied within the optimizer, together with the sum constraint and the boundary conditions. The optimizer adjusts the strategy for the player under threat in order to receive the maximum reward possible. The optimizer is run based on sequential quadratic programming and finds a solution based on the given input. Summarizing the algorithm works as follows:

Algorithm 1 Threat point algorithm Type I non-FD game with SciPy optimizer.

Input: Reward matrix Player 1, Player 2

Output: Threat point $v = (v^1, v^2)$

1: Declare that strategies must sum to 1
2: Declare that each action has a probability bound between 0 and 1
3:
4: Initialize strategy pair σ^1
5: Declare threat function v^1
6: Run SLSQP optimizer with constraints and bounds for v^1
7: Print the result of v^1
8:
9: Initialize strategy pair π^2
10: Declare threat function v^2
11: Run SLSQP optimizer with constraints and bounds for v^2
12: Print the result of v^2
13:
14: Return threat point $v = (v^1, v^2)$

Running this algorithm for the example game stated above finds the threat point in 0.005 seconds[6]. The algorithm returns as a threat point $v = (24, 24)$, which is the

[6]Running time is dependent not only on the algorithm used, but also on the CPU and RAM within the computer

correct threat point for this game. We have confirmed this by computing the maximin value of the game, which was $23.\overline{9}$. We were not able yet to incorporate the frequency-dependent function into the SciPy algorithm. We have looked into multiple methods but have not found an efficient and satisfactory solution.

3.5.2 Jointly-Convergent Pure-Strategy Algorithm

As stated earlier, the previous algorithm is one way to find the threat point. We have developed another algorithm based on earlier work by Joosten and Samuel (2018). However, we keep in mind that the implementation of Samuel (2017) could be optimized further. This algorithm also contains necessary building blocks for the algorithms of Type II and Type III games. Therefore we gradually build the algorithm and state necessary functions for all algorithms (Type I, II and III) in this section.

We start with the first element. Again we want to fix the strategy for the player who is threatening the other player. In this algorithm however, we want to draw random strategies from a β-distribution. We draw them from the β-distribution because the β-distribution has nice properties. One of these is that when set accordingly ($\alpha = \beta = 0.5$) the β-distribution tends to draw more values at the edges of the distribution (at 0 and 1). In these 'less likely' values there is a higher chance that there are more interesting things to occur (Samuel, 2017).

Function wise we built the following function:

Algorithm 2 Function: Draw Random Strategy.

Input: Total points to generate, number of total actions for player who threatens
Output: Random strategy matrix

1: Draw *points* number of strategies with a length of *number of total actions* from a β-distribution with $(\alpha = \beta = 0.5)$
2: Normalize the drawn strategies, such that each individual strategy sums to one
3: Return the random strategy matrix

Samuel used the β-distribution to draw frequency vectors, but she implemented this with an unoptimized for-loop making computing time much longer than necessary. In this version we vectorize the code as much as possible in order to improve computing speed. Vectorization delivers a direct improvement in computing speed by reducing complexity of code.

Now that we have a function to draw random strategies for the player who threatens the other we also have to build a function which is a best reply. Hordijk, Vrieze, and Wanrooij (1983) state that the optimal response to a mixed stationary strategy is always a pure stationary strategy. Because the best reply is always a pure stationary

strategy, we only consider pure stationary strategies for the player who is being threatened. Therefore we have to convert the strategy of the player who is threatening into a frequency vector based on the pure stationary strategy reply of the player under threat.

Based on this we built a function which sorts strategies into frequency vectors. For each individual player we built a separate function, but this is due to storage differences, in essence they are comparable. The function contains:

Algorithm 3 Function: Create Frequency Vector.

Input: Total points generated, number of total actions for both players, random strategy matrix

Output: Frequency pairs based on pure best replies from player under threat

1: Initialize frequency vector with dimensions: (total number of points $\cdot$ total number of actions Player 1 , total number of actions Player $1 \cdot$ total number of actions Player 2)
2: **for** i in range *total number of actions player under threat* **do**
3: **for** j in range *total number of actions threatening player* **do**
4: **if** v^1 is searched **then** frequency vector[total number of points $\cdot$ $(i-1)$:total number of points i,(number of actions player $2 \cdot i)+j$] = random strategy matrix[:,j]
5: **end if**
6: **if** v^2 is searched **then** frequency vector[total number of points $\cdot$ $(i-1)$:total number of points i,(number of actions player $1 \cdot j)+i$] = random strategy matrix[:,j]
7: **end if**
8: **end for**
9: **end for**
10: Return frequency vector

The result is a frequency vector in which only the pure stationary best replies of the player under threat contain a non-zero frequency of play. As an example, if a player under threat has two pure stationary strategies (as an example: left and right) as a reply, then if there are 200 random strategies drawn for the threatening player, the result is 400 frequency pairs in the frequency vector. The first 200 represent the first (left) pure stationary strategy of the player under threat, the second 200 represent the other (right) pure stationary strategy. We visualize this with an example, suppose the player threatening plays a mixed stationary strategy in which up is played with probability 0.8 and down with probability 0.2. Algorithm 3 then sorts this strategy in two best replies (left and right) for the player under threat. Resulting in the following frequency matrix, shown in Figure 3.2:

$$\begin{bmatrix} 0.8 & 0 \\ 0.2 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0.8 \\ 0 & 0.2 \end{bmatrix} \tag{3.2}$$

Figure 3.2: Best reply frequency matrices.

There is a trade-off taking place in this frequency vector function. By creating this function it was inevitable to use a for-loop. Because we want to create a function which can scale with larger games with more actions, we had to use a for-loop. However, this for-loop was done with a NumPy for-loop, an optimized way of running a for-loop in large matrices. Performance gains can be made if the number of actions for a game is always fixed at the same number, in those cases a hard-coded function without for-loops will result in a significant gain in computational performance.

At last we need to build a function which stores all the corresponding drawn random mixed strategies with pure best responses into a logical matrix which makes extracting the threat point easy. Therefore we built a function called payoff sort. This function sorts the payoffs which are computed from the generated random mixed strategies based on there relationship. So the function stores all pure best replies to a certain drawn strategy in the same row. Function wise this has resulted in the following:

Algorithm 4 Function: Payoff sort.

Input: Total points generated, payoff matrix, total number of actions
Output: Sorted payoff matrix

```
Initialize empty sorted payoff matrix (number of points, number of actions)
for x in range total number of points do
    for i in range total number of actions do
        Sorted payoffs[x,i] = payoff matrix[points · i + x]
    end for
end for
Return sorted payoff matrix
```

Now we have all essential functions in order to construct the final algorithm which calculates the threat point for a Type I (non)-FD game. We construct the algorithm in such a way that it incorporates the frequency-dependent function for the payoffs. We combine the above stated functions in order to construct the following algorithm:

Algorithm 5 (Non)-FD Type I threat point algorithm.

Input: Type I Game, total number of points, activate FD function
Output: Threat point

If v^1 is calculated, then $X = 1$, $Y = 2$,
If v^2 is calculated, then $X = 2$, $Y = 1$

Turn reward matrix Player X into flattened reward vector
Threatening strategies Player Y = Draw Random Strategies
Best response Player X frequency vector = Create Frequency Vector
if Activate FD function = True **then**
    Activate and Calculate FD payoff Function result
end if
Payoffs Player X = Sum over all columns of: (Frequency Vector per row $\cdot$ flattened vector Player X)
if FD Function is active **then**
    Element wise multiplication of FD payoff function result with Payoffs Player X
end if
Sorted payoff matrix = Payoff sort
Pick the maximum value of each row of the sorted payoff matrix as best response of Player X
Pick the minimum value over all rows as the result of v^X

Return threat point $v = (v^1, v^2)$

Filling in a total number of points generated of 100 and the FD payoff function as not active results in a threat point of approximately $(24.009, 24.0006)$ in roughly 0.006 seconds. When activating the FD payoff function the algorithm finds a threat point with 100 points in roughly the same amount of time with as a result $(10.506, 8.0002)$.

3.5.3 Visualizing the results

In order to give an impression of the threat point within these games, we visualize both the threat point and the surrounding reachable payoffs for the players. We use the algorithm by Joosten and Samuel to calculate the set of rewards surrounding the threat point. We use the SciPy optimizer to calculate the threat point for the Type I non-FD game and the Jointly-Convergent Pure-Strategy (JCPS) algorithm for the Type I FD game.

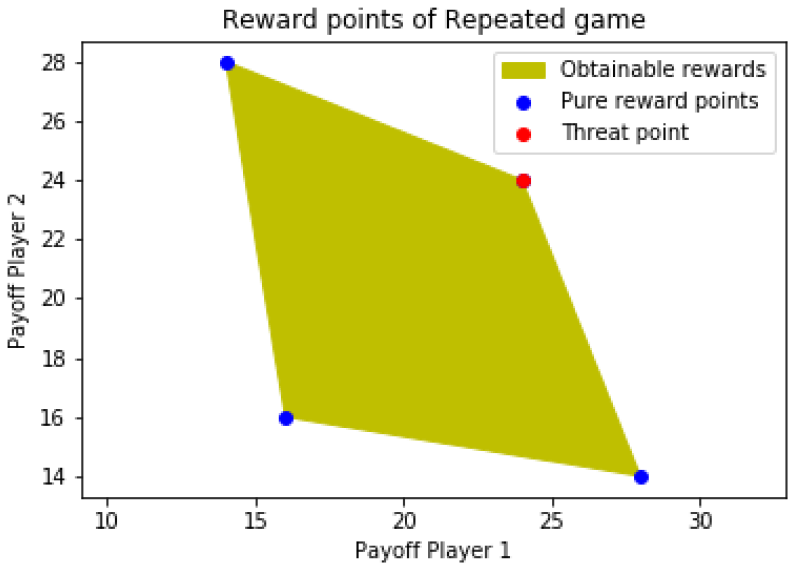


Figure 3.3: Payoffs including threat point of the Type I non-FD example game.

As can be seen in Figure 3.3 the threat point is also the Pareto optimal outcome for the game. The threat point is also the pure Nash equilibrium. Therefore the players have no incentive to deviate from the threat point and will always play the threat point in this case. As the example game is based on a common-pool resource game the threat point here is equal to depletion of the common-pool resource. However, this depletion only takes place when we incorporate the FD payoff function.

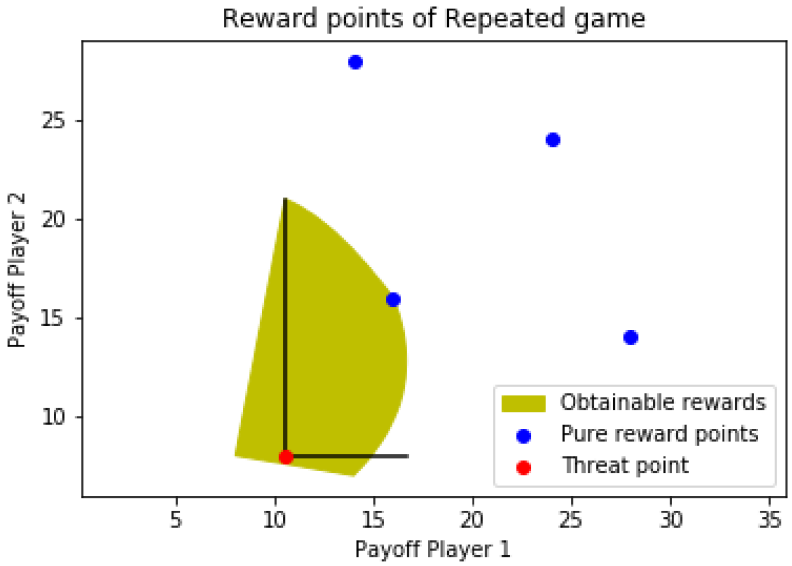


Figure 3.4: Payoffs including threat point of the Type I FD example game.

The effect of the depletion of the common-pool resource can clearly be seen in Figure 3.4. In this game in which the FD payoff function is activated the players still play the same payoff in case of the threat point, which is the bottom-right corner of the example game resulting in the depletion of the common-pool resource. The result of this depletion effect is that even the lowest possible pure payoff $(16, 16)$ in the non-FD situation is now a Pareto optimal solution. The players can threaten each other by depleting the common-pool resource and after wards can reach several payoffs denoted by the black lines north-east of the threat point.

3.5.4 Comparing the two algorithms

The question that still is unanswered is: which of the algorithms should we use for Type I games? For Type I FD-games it is quite easy to choose, because the SciPy algorithm does not incorporate FD payoff functions we cannot use this algorithm for Type I FD-games. However, for Type I non-FD-games we have a choice between the SciPy algorithm and the jointly-convergent pure-strategy algorithm. We test both algorithms in terms of running time and decimal accuracy. We construct three games to test:

1. The example game
2. A random game of size 4x4
3. A random game of size 10x10

For both algorithms these games are run, for the jointly-convergent pure-strategy algorithm we enter multiple amounts of points in order to increase accuracy. However, we must state that for the random games we simply do not know the exact threat point, we could compute this by hand, but because this takes a lot of time we trust that the algorithms should be a good approximation. We check this approximation by calculating the maximin result, subtracting the best approximation for the maximin result from the best approximation for the threat point should give us a reasonable indication of the algorithm accuracy. We start by testing both algorithms on the example game.

# points JCPS-Algorithm	**JCPS Algorithm Result**	**Run-time JCPS**	**SciPy Optimizer Result**	**Run-time SciPy**	**Difference JCPS - SciPy**
100	(24.009, 24.0006)	0.006 seconds	(24., 24.)	0.005 seconds	(0.009, 0.0006)
10,000	(24.00000002, 24.00000037)	0.29 seconds	(24., 24.)	0.005 seconds	(2 * 10^-8, 3.7 * 10^-7)
1,000,000	(24.000000000004057, 24.000000000000398)	26.34 seconds	(24., 24.)	0.005 seconds	(4.057 * 10^-12, 3.98 * 10^-13)

Table 3.1: Results of Example Game on algorithms.

As the results show in Table 3.1 the SciPy optimizer finds the exact threat point within 0.005 seconds while the jointly-convergent pure-strategy algorithm can find approximations with 100 points with an accuracy of two decimals. Increasing the number of points generated also increases accuracy a lot, to seven decimals, but also increasing the run-time of the algorithm to 0.29 seconds. For 1,000,000 points, the increase in accuracy is much lower compared to the increase time necessary to compute the algorithm. The SciPy algorithm is clearly the winner in this comparison in case of the example game.

The second game is a 4x4 bimatrix game randomly generated. The game is run by both algorithms and generates the following results:

# points JCPS-Algorithm	**JCPS Algorithm Result**	**Run-time JCPS**	**SciPy Optimizer Result**	**Run-time SciPy**	**Difference JCPS - SciPy**
100	(14.662, 12.547)	0.004 seconds	(13.736842, 12.49786)	0.006 seconds	(0.925158, 0.04914)
10,000	(13.759, 12.502)	0.25 seconds	(13.736842, 12.49786)	0.006 seconds	(0.022158, 0.00414)
1,000,000	(13.73994, 12.49384)	21.69 seconds	(13.736842, 12.49786)	0.006 seconds	(0.003098, -0.00402)

Table 3.2: Results of 4x4 Bimatrix Game on algorithms.

The best maximin result of this random 4x4 bimatrix game is $(13.7368419, 12.49205)$. The SciPy optimizer yields results which are respectively of six decimal accuracy and two decimal accuracy. As can be seen in Table 3.2 the JCPS algorithm is not very accurate at 100 points generated, but steadily increases the accuracy when the number of points generated increases. For 1,000,000 points generated it comes close to the best v^1 found by the SciPy optimizer, but it even finds a more accurate threat point for v^2. The downside to all of this is the computational time necessary for the JCPS algorithm. The SciPy optimizer is able to find an accurate result in a fraction of the time compared to the JCPS algorithm. However, the better result for v^2 at the JCPS algorithm suggests that the SciPy optimizer might have been stuck at a local optimum. This risk does not outweigh the benefits of the increase in computational time and accuracy when using the SciPy algorithm in this 4x4 random bimatrix situation.

Last up is a random bimatrix game of the size 10x10. We again run the algorithms and obtain the following results:

# points JCPS-Algorithm	**JCPS Algorithm Result**	**Run-time JCPS**	**SciPy Optimizer Result**	**Run-time SciPy**	**Difference JCPS - SciPy**
100	(14.4978, 13.5343)	0.014 seconds	(12.28409, 12.12467)	0.006 seconds	(2.21371, 1.40963)
10,000	(13.3521, 12.6217)	0.75 seconds	(12.28409, 12.12467)	0.006 seconds	(1.06801, 0.49703)
1,000,000	(12.7718, 12.3507)	139.6 seconds	(12.28409, 12.12467)	0.006 seconds	(0.48771, 0.22603)

Table 3.3: Results of 10x10 Bimatrix Game on algorithms.

This last run summarized in Table 3.3 clearly shows that the JCPS algorithm fails to generate an accurate threat point. Even when ran for a large amount of points the algorithm is even not close to the threat point computed by the SciPy optimizer. One could improve this within the algorithm by looking at a best approximate, and then search locally for an improvement again. This could be repeated an infinite number

of times in order to find a good approximation. However, for now we deem this out of scope in this research. We also ran the maximin algorithm with a best approximation of $(12.16356, 12.05364)$. The SciPy optimizer is close to the maximin approximation, but a one decimal accuracy is still a bridge too far.

When only looking for the threat point, the SciPy optimizer is clearly the algorithm to use. There still is a risk that the SciPy optimizer may get stuck in a local optimum. But we have confirmed that the SciPy optimizer is able to get closer to the threat point in the 10x10 bimatrix game and the 4x4 bimatrix game, for the example game the optimizer is able to find the exact threat point. However, this does not mean that the JCPS algorithm is completely redundant. The JCPS algorithm has an advantage because it also computes a part of the set of rewards of the game. If these results are required, then one should favor the JCPS algorithm over the SciPy optimizer. The other case in which the JCPS algorithm should be used is when the repeated game has a FD payoff function incorporated. In all other cases, the SciPy optimizer is much faster and provides more accurate results.

3.6 Type II games

Next up are the Type II games. Again we propose two algorithms, one relies on an algorithm for solving a Markov Decision Process. The other one is again the JCPS algorithm which we adapt to incorporate stochastic Type II games now. We limit ourselves to Type II games with at maximum two states. Our algorithms can be adapted to incorporate more states, but this makes the code more complicated than is necessary for now.

As an example we again refer to the game used by Samuel (2017). But now we add another state and end up with the following reward matrices:

$$\theta_{S_1} = \begin{bmatrix} 16,16 & 14,28 \\ 28,14 & 24,24 \end{bmatrix} \qquad \theta_{S_2} = \begin{bmatrix} 4.0,4.0 & 3.5,7.0 \\ 7.0,3.5 & 6.0,6.0 \end{bmatrix}$$

And because we are now having a Type II game, we also have to deal with the following transition probabilities:

$$p^{S_1} = \begin{bmatrix} 0.8,0.2 & 0.7,0.3 \\ 0.7,0.3 & 0.6,0.4 \end{bmatrix} \qquad p^{S_2} = \begin{bmatrix} 0.5,0.5 & 0.4,0.6 \\ 0.4,0.6 & 0.15,0.85 \end{bmatrix}$$

Last but not least, we also update the FD payoff function for the ETP-ESP games. Because we now deal with two states the FD payoff function is transformed into:

$$FD^t_{II,III} = 1 - \frac{1}{4}(x^t_2 + x^t_3) - \frac{x^t_4}{3} - \frac{1}{2}(x^t_6 + x^t_7) - \frac{2x^t_8}{3}$$

This is an adaptation from the earlier FD payoff function now incorporating two states. The second state is the state in which the common-pool resource is affected the most. So again, the bottom-right corner is resulting in the largest deterioration of the common-pool resource. In general, when the state of the game is the second (or the low) one, this will have a larger influence on depletion than when the game is in the first (or the high) state.

3.6.1 Relative Value Iteration Algorithm

The first algorithm is based on the Relative Value Iteration algorithm, which, on its turn is an adaptation of the Value Iteration algorithm for an MDP. The algorithm was introduced by Bellman (1957). In this algorithm the value function is computed until it converges to an optimal value V^*. Based on the optimal value the optimal policy is retrieved by a backup operator. The Relative Value Iteration algorithm is different because it looks at the relative value, i.e., the limiting average reward criterion. If convergence of the value is guaranteed then the algorithm will find a ϵ-optimal value, i.e., the value found by the algorithm is the optimal value for the MDP.

Within Python we use the MDP package. But before we can apply the algorithm we should make it suitable for a stochastic game in general. As explained in Subsection 3.2, stochastic games are comparable to an MPD, except for that in an MDP there is only one player. In order to solve the stochastic game as an MDP we therefore have to fix the strategy of one of the players. Because we are looking for the threat point, we fix the strategy of the player who is threatening the other player. However, this already has an immediate effect on the payoffs and transition probabilities that the player under threat faces.

Therefore we build two functions in order to prepare the MDP for the player under threat. The first function calculates the adjusted transition probabilities, the second one adjusts the reward matrices. We start with the algorithm for the transition probabilities.

Algorithm 6 Calculate transition matrix MDP.

Input: Fixed strategy threatening player

Output: Adjusted transition probabilities player under threat

1: Transition probability matrix · fixed strategy threatening player
2: Normalize the adjusted transition probabilities per row
3: Return adjusted transition probability matrix

And we also construct a function in order to transform the reward matrix for the player under threat.

Algorithm 7 Calculate reward matrix MDP.

Input: Fixed strategy threatening player

Output: Adjusted reward matrix player under threat

1: Reward matrix Game 1 · Fixed strategy threatening player Game 1
2: Reward matrix Game 2 · Fixed strategy threatening player Game 2
3: Append reward matrices into one matrix
4: Return adjusted reward matrix player under threat

Now that we have these two functions, we build a function which initiates the MDP with the Relative Value Iteration algorithm for the player under threat.

Algorithm 8 Markov Decision Process solver.

Input: Fixed strategy threatening player

Output: Optimal value V^*, optimal policy

1: Calculate adjusted transition matrix
2: Calculate adjusted reward matrix
3: Initiate Relative Value Iteration algorithm with ϵ accuracy
4: Run the Relative Value Iteration algorithm
5: Return optimal value V^* and optimal policy

The biggest setback when using the MDP algorithm is that it can only solve the MDP for one fixed strategy of the threatening player at a time. Therefore we are forced to generate new strategies in a for-loop. The MDP calculates the maximum value of the player under threat, we store the result of the MDP if it is the lowest result encountered by the threatening player. Running for an large amount of fixed strategies should result in the threat point v. Therefore we apply the following algorithm which should find the threat point.

Algorithm 9 Threat point MDP.

Input: Number of points to generate
Output: Threat point $v = (v^1, v^2)$

1: **for** i in number of points to generate **do**
2: Generate random fixed strategy Player $1/2$
3: Normalize the fixed strategy for Player $1/2$ per state
4: Run the Markov Decision Process solver
5: If the result of the solver is lower than earlier found results, then store the new result and strategy
6: **end for**
7: Display the found result and prepare for local search
8: Repeat 1 to 6 while searching in local region of found result
9: Return threat point $v = (v^1, v^2)$

A remark has to be made on Algorithm 9. We apply a global search for a best approximation in the first loop, later we apply a local search to find an even better approximation of the threat point. The local search works by altering the fixed strategy for the threatening player by small margins, therefore we are able to find more precise results without computing unnecessary values which are very likely to give worse results. For the example game we are able to find a threat point of $v = (13.75003, 13.75002)$, in around 32 seconds by generating $10,000$ points.

The MDP could theoretically find an exact threat point, but for this one has to generate a lot of points in order to increase accuracy. Therefore finding the threat point with this algorithm should be a balance between accuracy and computational time. The major problem is that the MDP currently does not have way to incorporate a FD payoff function into the equation. Therefore the MDP algorithm is useful for stochastic games, but for stochastic FD-games we need another solution.

3.6.2 Jointly-Convergent Pure-Strategy Algorithm

We adapt the jointly-convergent pure-strategy algorithm as it has been useful in Type I games to incorporate the FD payoff function. We adjust the Type I Algorithm 5 so that it incorporates stochastic games. Therefore we alter the Functions 2, 3 & 4. These functions now scale with two states as an input. However, the most important challenge is to deal with the uncertainty regarding the transition probabilities.

Joosten and Samuel (2018) cope with these difficulties by introducing the so-called balance equation. But first we have to refer to the restrictions we have on the type of games for which the algorithm works. We look at games in which the transition

probabilities in Markov chain form are irreducible and aperiodic[7]. If they are both, then they are guaranteed to converge to a stationary distribution. This stationary distribution can be found with the balance equation. If x represents the frequency vector of play and p represents the transition probabilities, then the balance equation is given for a two-player two state stochastic game with two actions per state is:

$$\sum_{i=1}^{4} x_i(1-p_i) = \sum_{i=5}^{8} x_i p_i \tag{3.3}$$

In which must hold that:

$$\sum_{i=1}^{4} x_i(1-p_i) \neq 0 \tag{3.4}$$

$$\sum_{i=5}^{8} x_i p_i \neq 0 \tag{3.5}$$

Equations 3.4 & 3.5 state that Equation 3.3 cannot end up with absorbing states. We limit ourselves to transition probabilities which are in Markov chain form irreducible and aperiodic, so therefore absorbing states should not occur. In order to incorporate Equation 3.3 into the algorithm we build a function for it.

In this function we calculate the stationary distribution of the Markov chain. We do this in a similar way as Samuel (2017), by introducing intermediate vector y and variable Q, such that:

$$y_i^{S_1} = \frac{x_i^{S_1}}{\sum_{j=1}^{4} x_j^{S_1}} \tag{3.6}$$

$$y_i^{S_2} = \frac{x_i^{S_2}}{\sum_{j=5}^{8} x_j^{S_2}} \tag{3.7}$$

Which then are used in order to calculate Q.

$$Q = \frac{\sum_{i=5}^{8} y_i^{S_2} p_i}{\sum_{i=1}^{4} y_i^{S_1}(1-p_i) + \sum_{i=5}^{8} y_i^{S_2} p_i} \tag{3.8}$$

The result of Equation 3.8 is Q, and in this situation Q stands for the long term relative frequency of play taking place in State 1. $(1-Q)$ therefore is the frequency of play taking place in State 2.

[7] Also known as ergodic Markov chains.

Algorithm 10 Balance Equation Function.

Input: Frequency vector x, transition probabilities p

Output: Frequency vector x adjusted to stationary distribution

1: Initialize Q and y
2: Calculate y
3: Calculate Q with y and p
4: Calculate new frequency vector x based on Q and y
5: Return new frequency vector x adjusted to the stationary distribution

This new equation can then be used in order to calculate the threat point for (non)-FD Type II games. We incorporate it within Algorithm 5, the result is the following algorithm:

Algorithm 11 (Non)-FD Type II threat point algorithm.

Input: Type II Game, total number of points, activate FD payoff function

Output: Threat point

1:
2: If v^1 is calculated, then $X = 1$, $Y = 2$,
3: If v^2 is calculated, then $X = 2$, $Y = 1$
4:
5: Turn reward matrix Player X into flattened reward vector
6: Threatening strategies Player Y = Draw Random Strategies
7: Best response Player X frequency vector = Create Frequency Vector
8: Calculate adjusted frequency vector by computing the balance equation
9: **if** Activate FD payoff function = True **then**
10: Activate and Calculate FD Function result
11: **end if**
12: Payoffs Player X = Sum over all columns of: (Frequency Vector per row $\cdot$ flattened vector Player X)
13: **if** FD Function is active **then**
14: Element wise multiplication of FD function result with Payoffs Player X
15: **end if**
16: Sorted payoff matrix = Payoff sort
17: Pick the maximum value of each row of the sorted payoff matrix as best response of Player X
18: Pick the minimum value over all rows as the result of v^X
19:
20: Return threat point $v = (v^1, v^2)$

For the non-FD game we are able to find a threat point of $v = (13.7501, 13.75001)$ within roughly 2 seconds by generating $200,000$ points. By activating the FD-function we find a threat point of $v = (8.02105, 8.02128)$, again in roughly 2 seconds. The algorithm works as expected, but should we favor the Relative Value Iteration (RVI) algorithm over the JCPS algorithm? When looking at FD-games we only have the option to use the JCPS algorithm. But for non-FD stochastic games we could use both.

3.6.3 Visualizing the results

But before we compare both algorithms we look at the visualizations of the example game. We again use the algorithm by Joosten and Samuel (2018) to retrieve the set of rewards. The threat point in the non-FD game is retrieved with the RVI algorithm, for the stochastic game with the FD payoff function we determine the threat point with the JCPS algorithm.

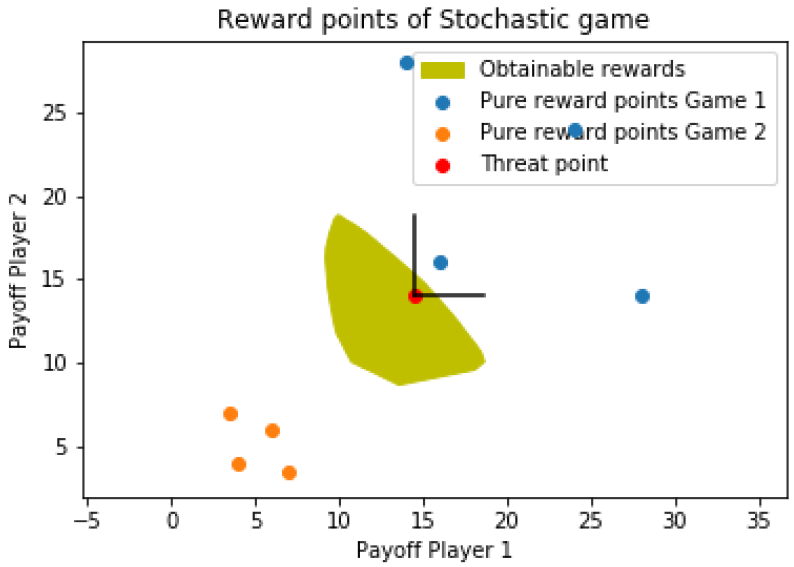


Figure 3.5: Payoffs including threat point of Type II non-FD example game.

In Figure 3.5 we see that the set of obtainable rewards is between the pure rewards of State 1[8] and State 2. The threat point is relative close to the Pareto frontier[9]. The players always receive at least the threat point, which in this case seems to be a reasonable solution, they can however improve to equilibria to the north-east of the threat point. Next we are going to look at the effect of the FD payoff function on the threat point and the obtainable rewards.

[8]Denoted in the Figure as 'Game 1'.
[9]The Pareto frontier is the set of all Pareto optimal allocations.

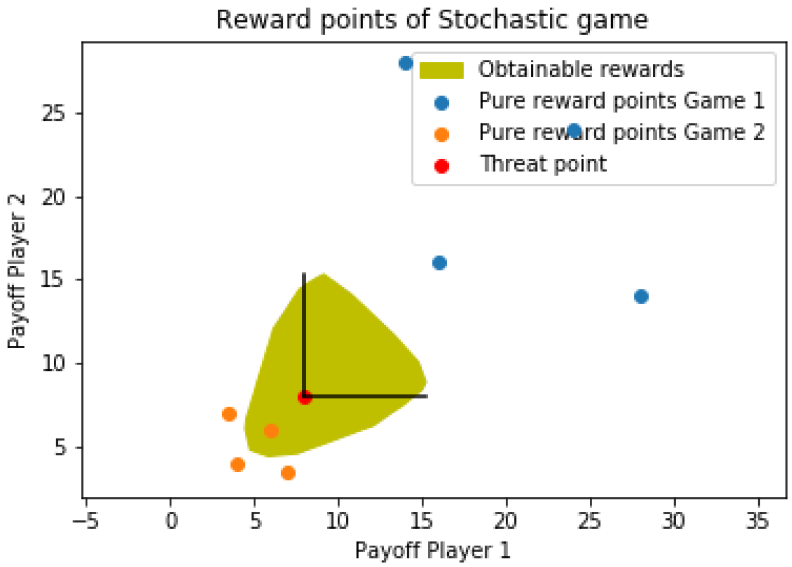


Figure 3.6: Payoffs including threat point of Type II FD example game.

Figure 3.6 shows that payoffs have shifted dramatically to the bottom-left corner. This means that the payoff of the players are reduced due through the activation of the FD payoff function. The common-pool resource is reduced in its availability as can be seen by the negative shift of the payoffs. The threat point is also much lower than in the non-FD version of the game, including more obtainable rewards north-east of the threat point. This gives players more possibilities in earning certain payoffs involving threats.

3.6.4 Comparing the two algorithms

In case of the Type II FD game it is only possible now to use the JCPS algorithm. But in case of the Type II non-FD game we could use both the JCPS and RVI algorithm. Therefore we compare the two algorithms for two terms, speed versus accuracy. We again look at the example game but also at two additional games. On basis of this we hope to give an useful advice in the application of both algorithms.

1. The example game
2. A random two state stochastic game in which each state is of size 4x4
3. A random two state stochastic game in which each state is of size 10x10

We start with the example stochastic game without FD payoff function. Theoretically the Relative Value Iteration algorithm is capable of computing ϵ-optimal values. For the JCPS algorithm we only know that accuracy increases when more points are being

generated. Therefore we look at the example game first, we know that the threat point is $v = (13.75, 13.75)$. We compute the threat point for an increasing number of points to be generated and state the decimal accuracy and run-time.

# points	**JCPS Decimal Accuracy**	**Run-time JCPS**	**RVI Decimal Accuracy**	**Run-time RVI**
100	0 decimals accurate	0.004 seconds	2 decimals accurate	0.4 seconds
10,000	2 decimals accurate	0.328 seconds	4 decimals accurate	36.36 seconds
1,000,000	4 decimals accurate	22.95 seconds	6 decimals accurate	3,250.78 seconds
10,000,000	5 decimals accurate	255.3 seconds	-	-

Table 3.4: Accuracy versus run-time in example stochastic game with the algorithms.

Table 3.4 clearly shows that the RVI algorithm is capable of calculating a result with a higher accuracy while needing fewer points. A remark on the side should be made here, because the RVI algorithm uses a local search within a rough estimation. Applying a local search within a rough estimation of the JCPS algorithm could theoretically also increase accuracy while having a low increase in run-time. However, looking at a larger number of points generated it is clear that the JCPS algorithm is much more efficient in computational time due to vectorization. Aiming for 4 decimal accuracy of the threat point means that the JCPS algorithm can do the job roughly 37% quicker than the RVI algorithm.

With this advantage for the JCPS algorithm we look at a 4x4 stochastic game in which we generate random payoff and transition matrices. We do not know what the exact threat point is we search for, but we check for an approximation of the lower boundary by calculating the maximin result. The following results are obtained from running the algorithms.

# points	**JCPS Algorithm Result**	**Run-time JCPS**	**RVI Algorithm Result**	**Run-time RVI**	**Difference JCPS - RVI**
100	(12.1794, 10.5948)	0.006 seconds	(11.5734, 10.5313)	0.391 seconds	(0.606, 0.0635)
10,000	(11.1956, 10.3064)	1.342 seconds	(10.7620, 10.1679)	39.48 seconds	(0.4336, 0.1385)
1,000,000	(10.8054, 10.2205)	118.125 seconds	(10.6523, 10.1564)	4,323.83 seconds	(0.1531, 0.0641)

Table 3.5: Threat point results on random 4x4 stochastic game from the two algorithms.

Again Table 3.5 shows clearly that the JCPS algorithm is much faster and therefore has a shorter run-time. We were able to find a best approximation of maximin of $(10.4126, 9.9896)$. The RVI algorithm seems to be more accurate than the JCPS algorithm but still lacks to find a decimal accurate threat point. The JCPS algorithm however gets close to the RVI algorithm when $1,000,000$ points are generated in less computational time necessary. In order to find gather more information about the accuracy of the algorithms we now test both algorithms for a random 10x10 bimatrix game.

# points	**JCPS Algorithm Result**	**Run-time JCPS**	**RVI Algorithm Result**	**Run-time RVI**	**Difference JCPS - RVI**
100	(12.0658, 11.1363)	0.272 seconds	(11.6290, 10.4773)	0.70 seconds	(0.4368, 0.659)
10,000	(11.3764, 10.6083)	19.9735 seconds	(10.8377, 9.3613)	56.66 seconds	(0.5387, 1.247)
1,000,000	Memory Error	Memory Error	(10.4810, 9.2542)	5,548.46 seconds	

Table 3.6: Threat point results on 10x10 stochastic game from the two algorithms.

The results in Table 3.6 again are in favor of the RVI algorithm when we focus on the accuracy of both algorithms. The JCPS algorithm is clearly the faster algorithm, but also finds less accurate points. When we try to run the JCPS algorithm for $1,000,000$ points we run into a memory error. The best maximin result found was $(9.0553, 7.8528)$, which shows that both algorithms are in the 10x10 stochastic game case not even close to an accurate threat point. The JCPS algorithm has a difficulty with the storage of larger stochastic games. Therefore the vectorization applied in the JCPS algorithm is a double-edged sword. On one hand it could produce identical accurate results compared to the RVI algorithm within a lower amount of computational time required. But it fails to produce accurate results when the game is getting too large and therefore runs into memory issues. The JCPS algorithm is the best option when looking at simple stochastic games with or without a FD payoff function. The RVI algorithm is superior for non-FD stochastic games in terms of accuracy but this required a significantly higher amount of computational time.

3.7 Type III games

The last category of games are Type III games. These games are also known as ETP games. We introduce a new function in which the transition probabilities are affected by the frequency of play. In Type I and Type II games the transition probabilities are constant as denoted by p_0. However now we have a situation in which they are dependent on the frequency of play in the past, denoted by $p(x)$. This increase in complexity requires an alteration of the earlier algorithms. We have looked into multiple options to incorporate endogenous transition probabilities within a Markov Decision Process but have not succeeded. The JCPS algorithm based on earlier work by Joosten and Samuel (2018) is capable of handling these endogenous transition probabilities. Therefore we again incorporate new functionalities within the Type II JCPS algorithm in order to cope with Type III games.

We start by stating that we test and run this algorithm only on the example game of Samuel (2017). Samuel (2017) introduces matrix A which states the dependency of the frequency vector with the transition probabilities such that:

$$p(x) = p_0 - [x \cdot A] \tag{3.9}$$

She introduces an ETP matrix A specified and built for the example game such that non eco-friendly strategies have a negative impact on the transition probabilities by ending up more in the generally lower second state. The ETP matrix A defined for the example game is:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .35 & .30 & .30 & .25 & .20 & .15 & .15 & .05 \\ .35 & .30 & .30 & .25 & .20 & .15 & .15 & .05 \\ .70 & .60 & .60 & .50 & .40 & .30 & .30 & .10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .35 & .30 & .30 & .25 & .20 & .15 & .15 & .05 \\ .35 & .30 & .30 & .25 & .20 & .15 & .15 & .05 \\ .70 & .60 & .60 & .50 & .40 & .30 & .30 & .10 \end{bmatrix}$$

The big question is, how does this adaptation influence the algorithm and more importantly, what effect does this have on the stationary distribution? Joosten and Samuel (2018) state in their paper: “determining a stationary distribution is like shooting at a moving target". We determine the stationary distribution with the balance Equation 3.3 and with help of Equations 3.6, 3.7 & 3.8. However, determining the stationary distribution results in a new frequency vector x^* which again has an effect on the transition probabilities. Therefore the system of Equation 3.3 cannot be seen as balanced as long as a new frequency vector results in a new stationary distribution.

Algorithm wise this means we now have to compute y, Q and x^* multiple times as long as the system of balance equations has not converged into a steady state. Because we apply vectorization there is an inherent risk in computing all of these equations until all have converged. It would be much more efficient to only compute the frequency vectors for which the system of balance has not ended up at a stationary distribution. We therefore look after each iteration which frequency vector has converged into a stationary distribution and which not. All that have converged are removed from the computations by removing their index values. We continue calculations for all other vectors until we have convergence on a broad scale.

In order to incorporate this we alter the function that computes the balance equation and gives back an adjusted frequency vector. We now incorporate the recalculation of the balance equation based on the endogenous transition probabilities. We built the following function:

Algorithm 12 Balance Equation ETP function.

Input: Frequency vector x, transition probabilities p, ETP matrix A
Output: Frequency vector x adjusted to stationary distribution

Initialize Q and y
Calculate y

while Not all rows in frequency vector x have converged **do**
    Calculate new $p(x)$ with $x \cdot A$
    Calculate Q with y and $p(x)$
    Calculate new frequency vector x based on Q and y
    Check whether Q has converged based on earlier result

    if Q has converged **then** Remove frequency vector x from calculating a new Q
    end if
end while
Return new frequency vector x adjusted to the stationary distribution

The question with Algorithm 12 is, how long does it take for all frequency vectors to converge? We have calculated this for 65 iterations. The result shows us that roughly 99.5% has converged after 65 iterations. Joosten and Samuel (2018) have examined the number of iterations required for general convergence, they state that after roughly $50 - 60$ iterations most frequency vectors have reached a stationary state. Our own tests confirm this in Figure 3.7. However, even $50 - 60$ iterations implies still quite a lot of time when running vectorized code. Also some entries of Q refuse to converg, even for a large number of iterations between $100 - 120$.

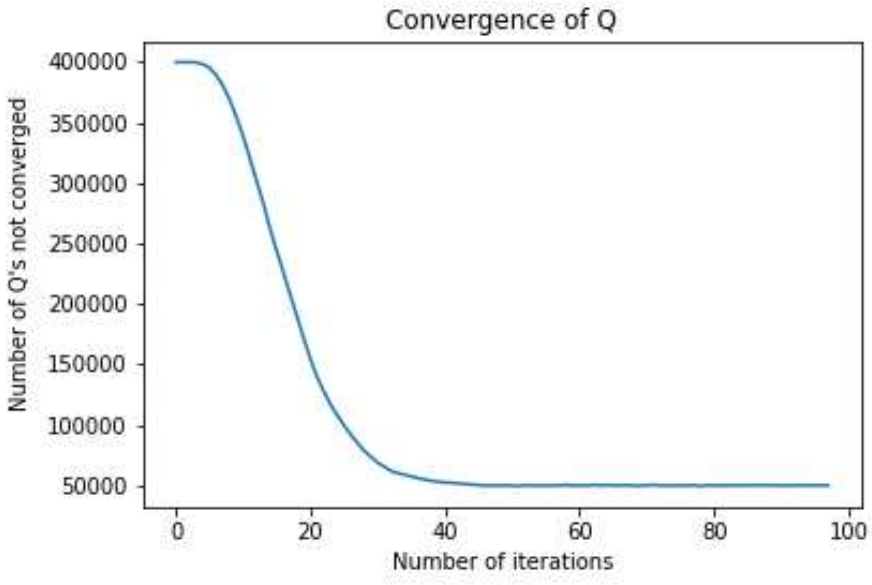


Figure 3.7: Convergence of Q.

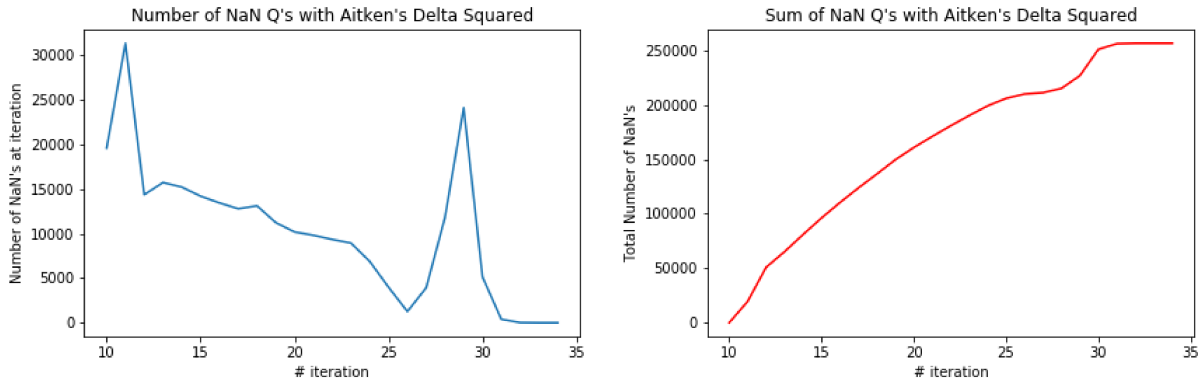
A solution to this is the use of a so-called accelerator. Joosten and Samuel (2018) use Aitken's Δ^2 method. An accelerator which converges a sequence to a certain value as long as long as this sequence in it self is linearly convergent (Burden & Faires, 2010). They describe the method with the assumption that if the sequence is described by $\{p_n\}_{n=0}^{\infty}$, then Aitken's Δ^2 is given as:

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \tag{3.10}$$

The problem with Equation 3.10 is that it is not always numerically stable. To improve numerical stability, one should use the following adaptation of Aitken's Δ^2:

$$\hat{p}_n = p_{n+2} - \frac{(p_{n+2} - p_{n+1})^2}{(p_{n+2} - p_{n+1}) - (p_{n+1} + p_n)} \tag{3.11}$$

Applying this accelerator on convergence of Q improves the necessary computational time. Joosten and Samuel (2018) even apply the accelerator on the accelerator. Applying Aitken's Δ^2 only once decreases the general iterations needed to only $25 - 30$ (Joosten & Samuel, 2018). The use of the accelerator is not without any risk. During the programming of the algorithm we encountered a potential problem with using Aitken's Δ^2 method. Due to the rapid convergence of Q the accelerator could run into floating-point errors. These floating-point errors are caused by the computer being unable to recognize the change in Q due to the representation of the smallest possible number within a computer.[10] The result of a floating-point error regarding Aitken's Δ^2 is a Not a Number (NaN).



(a) Iterations of Q with NaN

(b) Sum of Q with NaN

Figure 3.8: Occurence of NaN's during iterations of Q with Aitken's Δ^2.

Figure 3.8 shows the effect of the floating-point error on the NaN's generated. As can be seen in Figure 3.8(a) is that large proportions of NaN's occurs at the 11th iteration of Q[11] and the 28th iteration. A reason for this could be that a large number of Q's converges and the computer cannot recognize this convergence because of the smallest possible number within the computer. A total of 62.5% of the total number of Q's will be stated as a NaN by the computer. We have build in a safeguard against this problem by registering the last known Q before a NaN was given as the definite value for that Q. But the problem of the NaN still occured when multiplying Q with frequency vector x as done in Algorithm 14. Approximately a mean of 0.7% of the multiplications ended up in a NaN. Therefore we had to remove roughly 3.1%[12]of the payoffs from the algorithm in order to retrieve accurate results.

[10]Usually a 64 bit modern computer is able of recognizing numbers as small as approximately 2.22 * $10^{-}308$.

[11]This is right after we apply the accelerator.

[12]Each generated Q is multiplied with a certain x and later on sorted by the sort payoffs function. As a result of the NaN a whole row of payoffs has to be deleted, in case of a 2x2 game each row contains 4 payoffs

Another important issue faced was with regards to the number of normal iterations of Q required before applying Aitken's Δ^2. Testing this suggested that the algorithm would be numerically slightly unstable for less than 10 iterations of Q. This resulted in a slight difference for which maximin would be larger than minimax. Therefore we opted to start using Aitken's Δ^2 after ten iterations, the result of the rapid convergence of Q thereafter is visible in Figure 3.9.

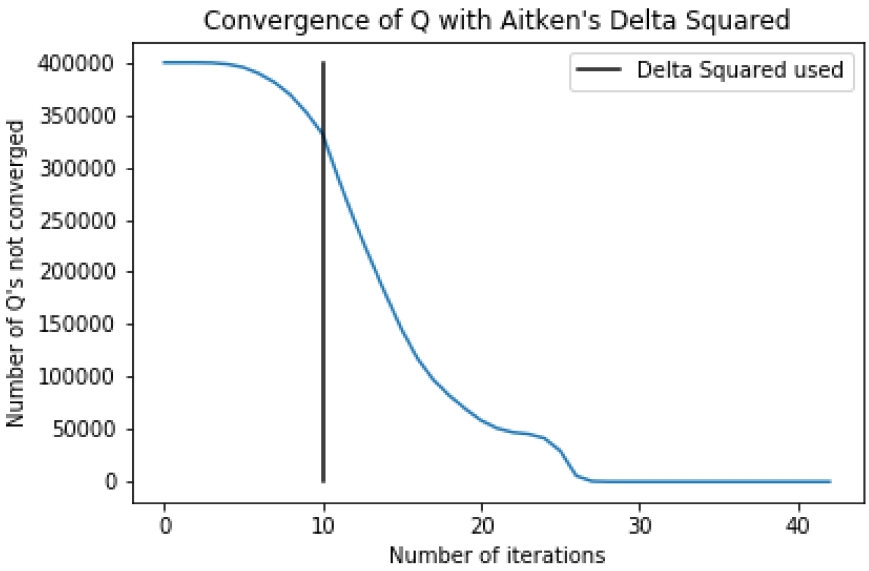


Figure 3.9: Convergence of Q with Aitken's Δ^2.

We now incorporate the numerically stable version of Aitken's Δ^2 on the convergence of Q. We adapt Algorithm 12 such that convergence of Q is sped up by using Aitken's Δ^2 while safeguarding for possible NaN's. The result is the following adaptation of the algorithm.

Algorithm 13 Balance Equation ETP Aitken's function.

Input: Frequency vector x, transition probabilities p, ETP matrix A
Output: Frequency vector x adjusted to stationary distribution

Initialize Q and y
Calculate y

for ten iterations **do**
    Calculate new $p(x)$ with $x \cdot A$
    Calculate Q with y and $p(x)$
    Calculate new frequency vector x based on Q and y

    Return Q
end for

while Not all rows in frequency vector x have converged **do**
    Calculate new Q with Aitken Δ^2
    Compare new Q with old Q for convergence
    if Q has converged **then** Remove frequency vector x from calculating a new Q
    end if
end while
Return new frequency vector x adjusted to the stationary distribution

We now incorporate Algorithm 13 into the Type II version of the JCPS algorithm to create a Type III version.

Algorithm 14 (Non)-FD Type III threat point algorithm.

Input: Type III Game, total number of points, activate FD payoff function
Output: Threat point

If v^1 is calculated, then $X = 1$, $Y = 2$,
If v^2 is calculated, then $X = 2$, $Y = 1$

Turn reward matrix Player X into flattened reward vector
Threatening strategies Player Y = Draw Random Strategies
Best response Player X frequency vector = Create Frequency Vector
Calculate Q based on function 13 with Aitken's Δ^2
Calculate adjusted frequency vector with help of Q
if Activate FD payoff function = True **then**
    Activate and Calculate FD Function result
end if
Payoffs Player X = Sum over all columns of: (Frequency Vector per row $\cdot$ flattened vector Player X)
if FD payoff function is active **then**
    Element wise multiplication of FD payoff function result with Payoffs Player X
end if
Sorted payoff matrix = Payoff sort
Pick the maximum value of each row of the sorted payoff matrix as best response of Player X
Pick the minimum value over all rows as the result of v^X

Return threat point $v = (v^1, v^2)$

We now try Algorithm 14 by running it with the example game. Running the non-FD version by generating $100,000$ points generates a threat point of $v = (8.27508, 11.01180)$. When applying the maximin algorithm and looking for the lower boundary we find a result of $(8.27501, 11.01176)$. Therefore we can say with certainty that our algorithm is capable of finding a solution by generating $100,000$ points that is up to three decimals accurate. The time necessary for finding the threat point is only roughly 39.03 seconds. A significant improvement when looking at earlier calculations by Samuel (2017). We also find a threat point for the FD version with $100,000$ points of $v = (4.45906, 7.55750)$. The maximin value found is $(4.45870, 7.55619)$. The ETP-ESP version of the game therefore has a two decimal accuracy and finds this within roughly 35.51 seconds.

3.7.1 Visualizing the results

Now that we have a working algorithm for finding the threat point we want to see what this means for the games visually. We again plot the set of rewards with the algorithm by Joosten and Samuel (2018) and the threat point with the JCPS algorithm.

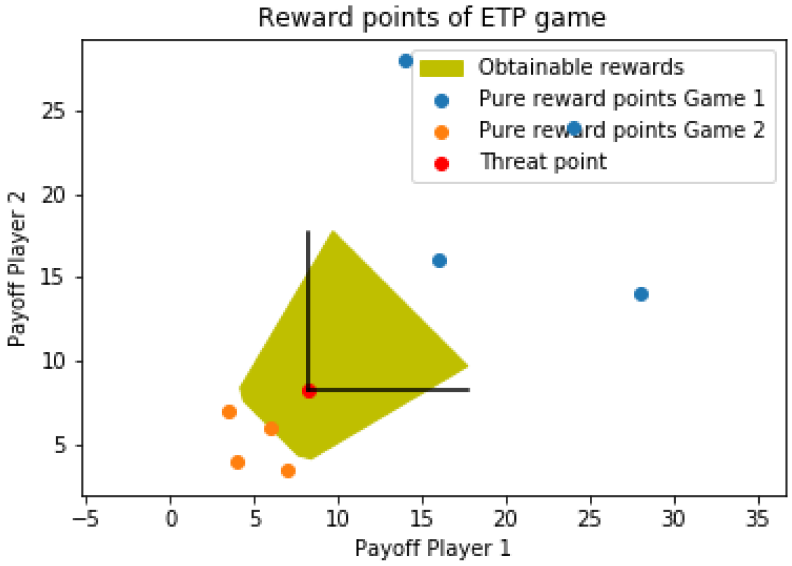


Figure 3.10: Set of rewards of Type III non-FD example game including threat point.

Figure 3.10 shows that the set of obtainable payoffs is larger than the set of obtainable payoffs in Figure 3.5 which is the effect of the ETP matrix on the payoffs.

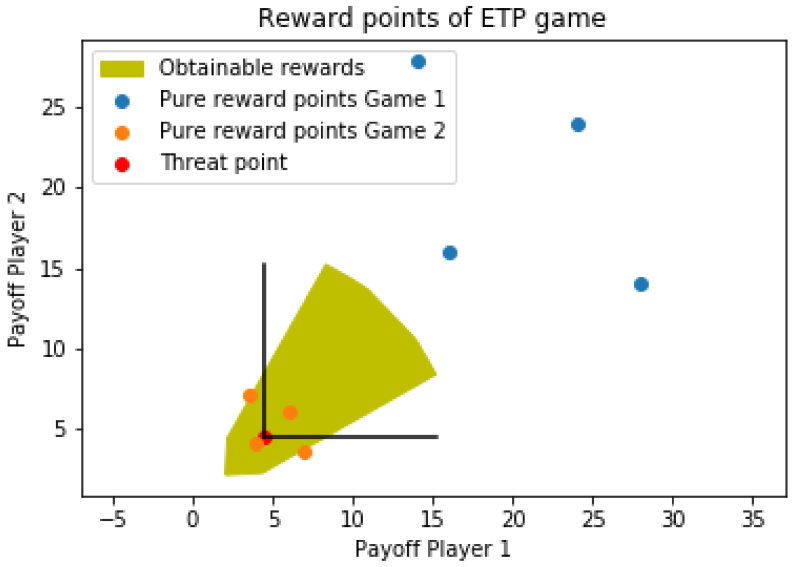


Figure 3.11: Set of rewards of Type III FD example game including threat point.

In Figure 3.11 it is clearly visible what the effect is of the ETP matrix and the FD payoff function. The threat point is almost near the lowest possible pure reward point for both players, but now even rewards that were not obtainable without the ETP matrix are obtainable. The threat point is close to the lowest pure reward possible. This gives players the opportunity to reach a multitude of equilibria involving threats.

We have shown that it is possible to construct an algorithm which can cope with calculating threat points in Type III games, especially ETP-ESP games, which can be regarded as games with a high complexity. However, simplifying this down to games for which the stationary distribution is guaranteed, our algorithm finds an accurate value within reasonable time. Also the JCPS algorithm can be used on the broad spectrum of games, not only does it apply to Type III games, but it is also useable on Type I and Type II games.

3.8 Room for further improvements

There are still possibilities for further improvements. Due to limited time one cannot explore all options, but we think three paths are worth investing in.

Adapting the SciPy optimizer to work on (FD) Type II and Type III games. We have used the SciPy optimizer in non-FD Type I games with good results. The SciPy optimizer was able to find an ϵ-accurate result within a fraction of the time needed for the JCPS algorithm. We think that it is able to deliver accurate results in Type II and Type III games, with our without FD function, but currently we have not found a working way to incorporate the SciPy optimizer at the basis of calculating the threat point in these games. We have tried to use it on the MDP algorithm for determining the minimizing strategy for the threatening player, but attempts at this were unsuccessful. More research into the SciPy optimizer is needed in order to make it work for Type II and Type III games.

Adjusting the MDP algorithm such that it can cope with ETP and ESP games. We applied the Relative Value Iteration algorithm which is an algorithm for a Markov Decision Process. Currently it is unable to handle endogenous transition probabilities and endogenous stage payoffs. There is research on uncertainty in an MDP (Liu & Sukhatme, 2018), (Delgado, Sanner, & de Barros, 2011). However most of these adaptations to the MDP work with methods which are simulating a model-free MDP and therefore have an approximation as a result. We think that it should be able to adapt the model-based MDP algorithms such that they provide an exact solution to ETP or ESP games.

Further optimization of the JCPS algorithm. Currently we have made a step in the optimization of the general algorithm by introducing vectorization. In comparison to Mahohoma (2014) and Samuel (2017) we have made a significant improvement in terms of speed. However, during the research we also had a hard-coded version of a 2x2 stochastic game, which was even faster than the current version. Due to making the code scalable to larger games we had to make a concession in terms of speed. There are possibilities to make another increase in speed by applying Aitken's Δ^2 method on convergence of the rewards. We are currently looking into possibilities for this, but we are facing problems in retrieving the rewards in such a manner that linear convergence is guaranteed. Once we overcome this problem, we think that we can reduce computational time a lot while retrieving accurate results. An alternative to this could be to apply local search as has been done with the RVI algorithm. Last but not least, we should further investigate the option of a n-player algorithm.

3.9 Practical implications

During the writing of and investigations for this thesis we went from a description of the origin of Game Theory to Frequency-Dependent Games. This chapter contains a more technical approach into the algorithm building which has been done in this thesis. We think that it is essential to put this algorithm into a broader practical perspective. Currently, there is a lot of debate and maybe even hype surrounding algorithms in general. They are seen as secret algorithms which control information and money and because of the black box mechanism incorporated it is unclear how an algorithm comes to a certain decision (Pasquale, 2015).

Partly this is true in our opinion due to increasingly complex deep learning algorithms which seem to exihibit characteristics of black boxes. They can only be analyzed based on their results, the inner workings of these systems are almost inscrutable, exceptions aside. We want to mention this because we think that the algorithm produced in this thesis is not a black box. Our vision is that an algorithm is an arithmetic representation of a more complex mathematical problem. Our algorithm is a simple artihmetic representation that follows game theoretical logic. Algorithms are necessary because computers only can do simple calculations on the lowest level. Therefore a lot depends on how the mathematics are defined surrounding the algorithm.

Game theory works based on the assumptions that players are rational and that under perfect information they are able to maximize their own interests. In practice we think that actors in everyday life are rational in general. Surely, humans are not completely selfish, but even when they cooperate they will try to make the most out of a situation in their own interest. A bigger problem in practice lies within the perfect information assumption. In everyday life it seems to us that it is almost impossible to retrieve perfect information. The complex world we live in makes it hard to retrieve all the data necessary, and when all data is retrieved it is still hard to analyze this in a game-theoretical sense. Analysis of game-theoretical problems which have a high complexity is extremely time-consuming when done by hand.

Therefore we think that this algorithm has an important role. When simplifying a complex real-life problem in such a matter that it can be analyzed as a game theoretical problem, this algorithm will prove to be useful in order come to well-found conclusions or interpretations. In literature a lot of practical situations have been derived in order to fit a game theoretical context. Well-known examples are Fishery games (e.g. (Joosten, 2007b)), economic applications (e.g. (Aumann & Hart, 1994)), advertisement games (e.g. (Joosten, 2015)) and political games (e.g. (Finus, 2002)) . The range of applicability is broad and should definitely not be limited to these known situations, but as Einstein once said “Logic will take you from A to B, imagination will take you everywhere."

Chapter 4

Conclusions and recommendations

In this section we conclude our research, refer to the progress made and do recommendations for future research.

4.1 Conclusion

Our research goal was to develop an algorithm which calculates threat points in Type I, Type II and Type III games. We have shown that there are even several algorithms to calculate threat points in these types of games. For Type I non-FD games the SciPy optimizer is a faster and more accurate option than the Jointly-Convergent Pure-Strategy algorithm. The Relative Value Algorithm can cope with ϵ-accurate solutions in theory, but in practice fails to cope with larger games and cannot find exact solutions. However the algorithm still performs better in terms of accuracy in non-FD Type II games in comparison to the Jointly-Convergent Pure-Strategy algorithm. In general, the Jointly-Convergent Pure-Strategy algorithm has proven to be fitting in all cases while also generating obtainable rewards. By applying vectorization and programming optimization we have also been able to increase the speed of the algorithm. For 2x2 stochastic games the algorithm is able to find very accurate solutions in reasonable time. Larger games seem to be problematic for all algorithms, but even more problematic for the Jointly-Convergent Pure-Strategy algorithm. These problems could be dealt with by employing a local search. Additionally there are limitations on the characteristics of the games in order to protect the guaranteed working of the algorithm. In practice, the total structure of the game should be taken into consideration in order to retrieve an optimal working of the algorithm.

4.2 Recommendations

As stated in this research, there are many ways to skin a cat. Therefore our research is just a limited view on an endless sea of possibilities. First we advise for future research to look at a way to cope with the SciPy optimizer in terms of Type II and Type III games. In this research we were not able to find a programming logic which could cope with the optimizer approach, but we think that there could be an enormous improvement in speed. Secondly, we think that there should be a more theoretical approach to incorporate endogenous transition probabilities and endogenous stage payoffs. This research could then lead to well-founded algorithms based on a Markov Decision Process for Type II and Type III games. Last but not least, the Jointly-Convergent Pure-Strategy algorithm can be optimized by looking for ways to guarantee linear convergence of the payoffs and then calculating the payoff with Aitken's Δ^2 method. However, one should keep in mind that this method is not without risks and that floating-point errors are prone to occur. Another option would be to apply the Jointly-Convergent Pure-Strategy algorithm with a global search and then search further locally.

For those who want to use this algorithm in finding a threat point in one of the three types of games we advise the following. When using this algorithm for a broad spectrum of games we should use the code as it is. If it is clear that games for a certain reason are only applied on for example 2x2 stochastic games, then one should try to alter the code by hard-coding the frequency vector. In this case the improvement in speed is significant in favor of scalability.

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Appendix A

Python Code containing the Algorithms

A.1 How to use the Python code - A short manual on the usage

A.1.1 Using Python

Python is an interpreted language, therefore it has to run the code with help of the Python interpreter. The Python interpreter is widely available, but for people with limited knowledge on Python we would advice to download the Anaconda package at www.anaconda.com. Anaconda also includes a lot of the packages necessary, like NumPy and SciPy. The MDP Toolbox, if not available in Anaconda, can be found on https://pymdptoolbox.readthedocs.io/en/latest/api/mdptoolbox.html.

In order to read the Python code an application has to be used on Windows computer. In this thesis we use Jupyter Notebook. If necessary, contact the author of this thesis to receive the source code and Jupyter Notebook file which has been used.

A.1.2 Using the Python Code

One should first load the code described at the 'Import packages' section. This section contains code which states the necessary Python packages for the computations done in the algorithms. After this code has been loaded by the interpreter the user has the option to choose from three types of game codes which are all accompanied by example games. The game code per type of game contains all required coding in order to receive visual results in this thesis. However, one could choose to only use a subset of the functions within the code. We will describe them briefly in the next section. All games are created by use of a class. Therefore we have programmed

the games as a class, we will elaborate on this after the description of the different functions with an example.

A.1.3 Functions

We introduce the functions of all the games. We state their names, describe them and also state which input is necessary from the user.

Type of Game	Function Name	Description	Input necessary
Type I	plot_pure_reward_points	Plots the pure reward points in a two dimensional figure	-
Type I	plot_all_reward_point	Plot all possible rewards points in a Type I game	FD function –>Yes = active, No = deactive
Type I	plot_threat_point	Plot's the threat point (only after it has been found)	-
Type I	plot_threat_point_lines	Plot the lines which define the limits for the NE	-
Type I	threat_point_algorithm	Find the threat point with the SciPy algorithm	-
Type I	maximin_algorithm	Find the maximin result with the SciPy algorithm	-
Type I	threat_point_optimized	Find the threat point with the JCPS algorithm	# of points to generate, show strategy of player 1/player 2, print text FD function –>Yes = active, No = deactive
Type II	plot_single_period_pure_rewards	Plots the pure reward points in a two dimensional figure	-
Type II	plot_convex_hull_pure_rewards	Plot a convex hull around the pure reward points	-
Type II	plot_threat_point	Plot's the threat point (only after it has been found)	-
Type II	plot_threat_point_lines	Plot the threat point lines defining the NE borders	-
Type II	plot_all_reward_points	Plot all reward points obtainable in the game	FD function –>Yes = active, No = deactive
Type II	threat_point_algorithm	Find the threat point with the RVI algorithm	T –>Number of points to generate sensi –>Sensitivity of the local search
Type II	maximin_point	Find the maximin point with the RVI algorithm	T –>Number of points to generate
Type II	optimized_maximin	Find the maximin point with the JCPS algorithm	# of points to generate, show strategy of player 1/player 2, print text FD function –>Yes = active, No = deactive
Type II	threat_point_optimized	Find the threat point with the JCPS algorithm	# of points to generate, show strategy of player 1/player 2, print text FD function –>Yes = active, No = deactive
Type III	plot_single_period_pure_rewards	Plots the pure reward points in a two dimensional figure	-
Type III	plot_convex_hull_pure_rewards	Plot a convex hull around the pure reward points	-
Type III	plot_all_rewards	Plot all reward points obtainable in the game	FD function –>Yes = active, No = deactive
Type III	plot_threat_point	Plot the threat point if found	-
Type III	plot_threat_point_lines	Plot the threat point lines defining the NE borders	-
Type III	optimized_maximin	Find the maximin point with the JCPS algorithm	# of points to generate, show strategy of player 1/player 2, print text FD function –>Yes = active, No = deactive
Type III	threat_point_optimized	Find the threat point with the JCPS algorithm	# of points to generate, show strategy of player 1/player 2, print text FD function –>Yes = active, No = deactive

A.1.4 An example of computations of a Type II game

We explain the usage of our code with a Type II non-FD game example. We need to construct a game before we can apply functions. The Type II game is programmed in a class called 'StochasticGame'. The Type II game needs two types of input: The transition probabilities and the payoffs of the game in both states. Our games are limited to two-player, two-state games. But in general the states can contain n actions. We define the transition probabilities and the payoffs of the game with a NumPy matrix.

See the following code:

```
    encoding=*-60
p1_1 = np.matrix('16 14; 28 24')
p2_1 = np.matrix('16 28; 14 24')
```

```
p1_2 = np.matrix('4 3.5; 7 6')
p2_2 = np.matrix('4 7; 3.5 6')

trans1_1 = np.matrix('0.8 0.7; 0.7 0.6')
trans2_1 = np.matrix('0.5 0.4; 0.4 0.15')

trans1_2 = np.matrix('0.2 0.3; 0.3 0.4')
trans2_2 = np.matrix('0.5 0.6; 0.6 0.85')
```

Then we load these transition probabilities and payoffs by creating a new class which we attach to the variable 'FirstTry'.

```
    encoding=*-60
FirstTry = StochasticGame(p1_1,p2_1,p1_2,p2_2,trans1_1,trans2_1,trans1_2,trans2_2)
```

As a result, Python now has created a class assigned to the variable FirstTry. We can access functions from this class by stating 'FirstTry.functionname'. We now want to plot all single period pure rewards, all reward points and find the maximin point and threat point with the RVI algorithm in 10000 points with a sensitivity of 0.025. The result is the following code:

```
    encoding=*-60
FirstTry.plot_single_period_pure_rewards()
FirstTry.plot_all_reward_points(False)
FirstTry.maximin_point(10000)
FirstTry.threat_point_algorithm(10000,0.025)
```

As can be seen, we set the FD function to false in the $plot\_all\_rewards\_points$ function. We also fill in 10000 points to be generated in both the $maximin\_point$ and $threat\_point\_algorithm$ function. Another way to have done this was by using the JCPS algorithm, which needs some more input. Both the maximin point and the threat point can be found by the following code.

```
    encoding=*-60
FirstTry.threat_point_optimized(10000,False,False,True,False)
FirstTry.optimized_maximin(10000,False,False,False)
```

The result again is a maximin point and a threat point. But now we have to state some more input, based on the input necessary described in the function table in the previous section. Last but not least we are able to plot this found threat point. We can do this with the $plot\_threat\_point$ function and also plot the threat point lines, the following code should be used in this example:

```
    encoding=*-60
FirstTry.plot_threat_point()
FirstTry.plot_threat_point_lines()
```

And the result should be a visualization of the Type II game without FD-payoffs.

A.2 Import packages

```
encoding=*-60
import numpy as np  #import numpy
import scipy as sp  #import scipy
from scipy import optimize #import the scipy optimize part
from scipy import linalg #import the linear algebra section
from scipy.spatial import ConvexHull #import scipy convex hull package
import matplotlib.pyplot as plt #import package to plot stuff
import msmtools as msm          #import package for markov chains
from msmtools.analysis import stationary_distribution #import for calculating stationary
    distributions
import mdptoolbox #import toolbox for MDP's
import time #import time package
from itertools import permutations #import package for permutations
```

A.3 Type I Game Code

```
encoding=*-60
class RepeatedGame:
    """In this type of game we model a repeated type of game. It means that we play a
        repeating game
    in which the payoffs are repeated for a certain (or unlimited) amount of time."""

    def __init__(self,payoff_p1,payoff_p2):
        "Here we initialize the game with respective playoffs for both players"

        # here below we set the payoffs as an aspect of the game, for both players each
        self.payoff_p1 = payoff_p1
        self.payoff_p2 = payoff_p2

        # we define a set of best pure strategies
        self.best_pure_strategies = np.array([[1,0,1,0],[0,1,1,0],[1,0,0,1],[0,1,0,1]])

    def plot_pure_reward_points(self):
        "This function plots the pure reward points in a two dimensional figure"

        # set the payoffs as an array
        payoff_p1_array = self.payoff_p1.A1
        payoff_p2_array = self.payoff_p2.A1


        plt.figure()   #create a figure
        plt.scatter(payoff_p1_array,payoff_p2_array, label="Pure reward points", zorder=15,
            color='b')   #simple plot which plots the possible rewards

        # label the x- and y-axis
        plt.xlabel("Payoff Player 1")
        plt.ylabel("Payoff Player 2")


    def plot_all_reward_points(self,FD_yn):
        "Here we plot all reward points possible in this repeated game."

        # number of times the game is played (large number)
        T = 100000
```

```
        # create payoff vectors based on the number of times the game is played
        Payoff1 = np.zeros(T)
        Payoff2 = np.zeros(T)

        x = np.zeros(4)      #initialize x for frequency vector
        r = np.zeros(4)      #initialize r for random drawing


        #loop over the number of periods and length of x
        for v in range(0,T):
            for i in range(0,4):
                r[i] = np.random.beta(0.5,0.5)     #draw r from beta distribution
                norm_val = np.sum(r)

            for i in range(0,4):
                x[i] = r[i]/norm_val               #normalize r and put it in as x

            # if there is a FD type of game, then active the FD function
            if FD_yn == True:
                FD = 1-0.25*(x[1]+2*x[2])-(2/3)*x[3]
            else:
                FD = 1

            # calculate the first payoffs for p1 and p2
            V_p1 = x*np.transpose(self.payoff_p1.flatten())
            V_p2 = x*np.transpose(self.payoff_p2.flatten())

            # calculate the payoff based on the FD function
            Payoff1[v] = FD*np.sum(V_p1)
            Payoff2[v] = FD*np.sum(V_p2)

        # store the maximal payoffs
        self.maximal_payoffs = np.zeros(2)
        self.maximal_payoffs[0] = np.max(Payoff1)
        self.maximal_payoffs[1] = np.max(Payoff2)

        all_payoffs = np.array([Payoff1,Payoff2])  #payoffs player 1 and and p2 merging
        all_payoffs = np.transpose(all_payoffs)        #transpose for use in convex_hull
        Convex_Hull_Payoffs = ConvexHull(all_payoffs)  #calculate convex_hull of the payoffs

        # here below we plot the convex hull
        plt.fill(all_payoffs[Convex_Hull_Payoffs.vertices,0], all_payoffs[Convex_Hull_Payoffs
            .vertices,1], color='y', zorder=5, label="Obtainable rewards")

#         plt.plot(Payoff1,Payoff2, color ='y', zorder=2, label="Reward points") # disabled
    for now

        # do some nice plotting
        plt.title("Reward points of Repeated game")
        plt.xlabel("Payoff Player 1")
        plt.ylabel("Payoff Player 2")


    def plot_threat_point(self):
        "Plot the threat point found (only working after the threat_point_algorithm function
            has been applied)"

        # here below the threat point is plotted
        plt.scatter(self.threat_point[0],self.threat_point[1], color='r', zorder=17, label="
            Threat point")
        plt.legend()
```

```
    def plot_threat_point_lines(self):
        "Plot␣the␣lines␣which␣define␣the␣limits␣for␣the␣NE␣reachable␣under␣Folk␣Theorem."

        # plot based on the maximum payoffs and threat point
        plt.plot([self.threat_point[0],self.threat_point[0]],[self.threat_point[1],self.
            maximal_payoffs[1]], color='k', zorder=16)
        plt.plot([self.threat_point[0],self.maximal_payoffs[0]],[self.threat_point[1],self.
            threat_point[1]], color='k', zorder=16)
        plt.axis('equal')


    def threat_point_p1(self,x):
        "Function␣in␣order␣to␣determinate␣the␣threat␣point␣for␣p1"

        return np.max(np.dot(self.payoff_p1,x))

    def threat_point_p2(self,x):
        "Function␣in␣order␣to␣determinate␣the␣threat␣point␣for␣p2"

        return np.max(np.dot(x,self.payoff_p2))

    def maximin_p1(self,x):
        "Function␣in␣order␣to␣determine␣the␣maximin␣strategy␣for␣p1"

        return np.max(-np.dot(x,self.payoff_p1))

    def maximin_p2(self,x):
        "Function␣in␣order␣to␣determine␣the␣maximin␣strategy␣for␣p2"

        return np.max(-np.dot(self.payoff_p2,x))

    def threat_point_algorithm(self):
        "This␣is␣the␣much␣shorter,␣optimized␣search␣for␣the␣threat␣point,␣not␣depending␣on␣
            the␣number␣of␣actions."

        start_time = time.time() # start a timer for speed measures

        print("Threat␣point␣algorithm␣initiated")

        p_initialize = np.zeros(np.size(self.payoff_p1,0))     #initialize an array for p
        q_initialize = np.zeros(np.size(self.payoff_p2,1))     #initialize an array for q
```

A.4 Type I Example Games Code

```
encoding=*-60
# Here below we create a repeated game from the Llea thesis
a = RepeatedGame(np.matrix('16␣14;28␣24'),np.matrix('16␣28;␣14␣24'))
a.threat_point_algorithm()
a.plot_pure_reward_points()
a.plot_all_reward_points(True)
a.maximin_algorithm()
a.threat_point_optimized(100000,True,True,True,True)
a.plot_threat_point()
a.plot_threat_point_lines()
```

A.5 Type II Game Code

```
encoding=*-60
class StochasticGame:
    [U"+FFFD]"""In this class we model Stochastic Games, also known in the thesis as Type II games"""

    def __init__(self,payoff_p1_game1,payoff_p2_game1,payoff_p1_game2,payoff_p2_game2,
        trmatrixg1,trmatrixg2,trmatrixg3,trmatrixg4):
        "Here below we initialize the game by storing payoff and transition matrices
            according to the upper input."
        self.payoff_p1_game1 = payoff_p1_game1                  #payoff p1 in game 1
        self.payoff_p2_game1 = payoff_p2_game1                  #payoff p2 in game 1

        self.payoff_p1_game2 = payoff_p1_game2                  #payoff p1 in game 2
        self.payoff_p2_game2 = payoff_p2_game2                  #payoff p2 in game 2

        self.transition_matrix_game1_to1 = trmatrixg1           #transition matrix from game
            1 to game 1
        self.transition_matrix_game2_to1 = trmatrixg2           #transition matrix from game
            2 to game 1

        self.transition_matrix_game1_to2 = trmatrixg3           #transition matrix from game
            1 to game 2
        self.transition_matrix_game2_to2 = trmatrixg4           #transition matrix from game
            2 to game 2

        self.printing = False   #set printing to False

    def plot_single_period_pure_rewards(self):
        "Here we plot the pure rewards possible for a single period"

        plt.figure()                                            #create a figure
        payoff_p1_g1_flat = self.payoff_p1_game1.A1             #create a flattend payoff of
            p1 in game 1
        payoff_p2_g1_flat = self.payoff_p2_game1.A1             #create a flattend payoff of
            p2 in game 1
        plt.scatter(payoff_p1_g1_flat,payoff_p2_g1_flat, label="Pure reward points Game 1",
            zorder = 15) #plot payoffs game 1

        payoff_p1_g2_flat = self.payoff_p1_game2.A1             #create a flattend payoff of
            p1 in game 2
        payoff_p2_g2_flat = self.payoff_p2_game2.A1             #and for p2 in game 2
        plt.scatter(payoff_p1_g2_flat,payoff_p2_g2_flat, label="Pure reward points Game 2",
            zorder = 15)  #plotting this again

        plt.xlabel("Payoff Player 1")                           #giving the x-axis the label
            of payoff p1
        plt.ylabel("Payoff Player 2")                           #and the payoff of the y-axis
             is that of p2
        plt.title("Reward points of Stochastic game")      #and we give it a nice titel
        plt.legend()

    def plot_convex_hull_pure_rewards(self):
        "Here we plot a convex hull around the pure reward point, therefore resulting in the
            total possible reward space"

        payoff_p1_g1_flat = self.payoff_p1_game1.A1       #store the flattend payoff of p1
            game 1
        payoff_p2_g1_flat = self.payoff_p2_game1.A1       #store the flattend payoff of p2
            game 1
```

```
        payoff_p1_g2_flat = self.payoff_p1_game2.A1       #store the flattend payoff of p1
            game 2
        payoff_p2_g2_flat = self.payoff_p2_game2.A1       #store the flattend payoff of p2
            game 2

        payoff_p1_merged = np.concatenate((payoff_p1_g1_flat,payoff_p1_g2_flat))  #merge p1
            payoffs
        payoff_p2_merged = np.concatenate((payoff_p2_g1_flat,payoff_p2_g2_flat))  #merge p2
            payoffs

        all_payoffs = np.array([payoff_p1_merged,payoff_p2_merged])  #create one array of
            payoffs
        all_payoffs = np.transpose(all_payoffs)                      #and rotate this one

        rewards_convex_hull = ConvexHull(all_payoffs)                #retain the convex hull
            of the payoffs
        plt.fill(all_payoffs[rewards_convex_hull.vertices,0], all_payoffs[rewards_convex_hull
            .vertices,1], color='k')
        plt.title("Convex␣hull␣of␣payoffs")

        #here above we fill the convex hull in black

    def plot_threat_point(self):
        "This␣function␣plots␣the␣threat␣point␣of␣the␣game"
        plt.scatter(self.threat_point[0],self.threat_point[1], zorder=10, color = 'r', label=
            'Threat␣point')
        plt.legend()


    def plot_threat_point_lines(self):
        "This␣function␣plots␣the␣threat␣point␣lines␣defining␣the␣NE␣borders"

        plt.plot([self.threat_point[0],self.threat_point[0]],[self.threat_point[1],self.
            maximal_payoffs[1]], color='k', zorder=15)
        plt.plot([self.threat_point[0],self.maximal_payoffs[0]],[self.threat_point[1],self.
            threat_point[1]], color='k', zorder=15)

    def plot_all_reward_points(self,FD_yn):
        "Here␣we␣use␣the␣algorithm␣developed␣in␣the␣thesis␣of␣Llea␣with␣supervision␣of␣
            Joosten␣for␣Type␣2␣games"

        ###Payoffs and probabilitys
        payoff_p1_g1_flat = self.payoff_p1_game1.A1          #flatten payoff p1 game 1
        payoff_p2_g1_flat = self.payoff_p2_game1.A1          #flatten payoff p2 game 1

        payoff_p1_g2_flat = self.payoff_p1_game2.A1          #flatten payoff p1 game 2
        payoff_p2_g2_flat = self.payoff_p2_game2.A1          #flatten payoff p2 game 2

        A1 = np.concatenate((payoff_p1_g1_flat,payoff_p1_g2_flat))   #A1 is payoff Player 1
        B1 = np.concatenate((payoff_p2_g1_flat,payoff_p2_g2_flat))   #B1 is payoff Player 2

        trans_game1_1_flat = self.transition_matrix_game1_to1.A1  #flatten transition
            probabilities game 1 to 1
        trans_game2_1_flat = self.transition_matrix_game2_to1.A1  #flatten transition
            probabilities game 2 to 1

        p = np.concatenate((trans_game1_1_flat,trans_game2_1_flat)) #merge them into one
            transition array


        ###Intialization of variables
```

```
T = 100000                          #number of points to generate

x = np.zeros(8)                     #preallocate frequency array
xstar = np.zeros(8)                 #preallocate converged frequency array
r = np.zeros(8)                     #preallocate random samples from beta distribution
y = np.zeros(8)                     #preallocate intermediate vector
yp = np.zeros(4)                    #preallocate intermediate vector with transition
yp_not = np.zeros(4)                #preallocate intermediate vector with transition
v_p1 = np.zeros(8)                  #preallocate intermediate payoffs p1
v_p2 = np.zeros(8)                  #preallocate intermediate payoffs p2
payoff_p1 = np.zeros(T)             #preallocate definite payoffs p1
payoff_p2 = np.zeros(T)             #preallocate definite payoffs p2

for t in range(0,T):                #sum over the number of points to generate
    for i in range(0,8):
        r[i] = np.random.beta(0.5,0.5)      #generate random frequency pair

    for i in range(0,8):
        x[i] = r[i]/np.sum(r)               #normalize the frequency pair so it sums
            to 1

###intermediate calculations (flow equations)
    for i in range(0,4):
        y[i] = x[i]/np.sum(x[0:4])          #calculate intermediate vector

    for i in range(4,8):
        y[i] = x[i]/np.sum(x[4:8])          #see above

    for i in range(0,4):
        yp_not[i] = y[i] * (1-p[i])         #prepare for Q calculations
        yp[i] = y[i+4]*p[i+4]

    Q = np.sum(yp)/(np.sum(yp)+np.sum(yp_not))   #calculate Q and Qnot
    Q_not = 1-Q

###Solve for X
    for i in range(0,4):
        xstar[i] = Q*y[i]                   #now calculate converged frequency pairs

    for i in range(4,8):
        xstar[i] = Q_not*y[i]               #and for second game

    if FD_yn == True:
        FD = 1-0.25*(xstar[1]+xstar[2])-(1/3)*xstar[3]-(1/2)*(xstar[5] + xstar[6]) -
            (2/3) * xstar[7]
    else:
        FD = 1

    for i in range(0,8):
        v_p1[i] = xstar[i]*A1[i]            #calculate payoffs player1
        v_p2[i] = xstar[i]*B1[i]            #calculate payoffs player2


### Stage payoff vectors
    payoff_p1[t] = FD*np.sum(v_p1)            #result is one payoff of player 1
    payoff_p2[t] = FD*np.sum(v_p2)            #result is one payoff of player 2

all_payoffs = np.array([payoff_p1,payoff_p2])  #payoffs player 1 and and p2 merging
all_payoffs = np.transpose(all_payoffs)        #transpose for use in convex_hull
Convex_Hull_Payoffs = ConvexHull(all_payoffs)  #calculate convex_hull of the payoffs
```

```
        self.maximal_payoffs = np.zeros(2)
        self.maximal_payoffs[0] = np.max(payoff_p1)
        self.maximal_payoffs[1] = np.max(payoff_p2)

        #here below we fill the convex_hull of the payoffs and plot it
        plt.fill(all_payoffs[Convex_Hull_Payoffs.vertices,0], all_payoffs[Convex_Hull_Payoffs
            .vertices,1], color='y', zorder=5, label="Obtainable rewards")

    def markov_decision_process_p1(self,x):
        "Here we run the threat point algorithm for P1 with fixed strategy for the punisher
            and the punished one maximizes this MDP"

        def chance_multiplication(self,x):
            "In this function we compute the new transition probabilities based on chosen
                actions"

            # store the actions of the players
            actions_p1_game1 = self.payoff_p1_game1.shape[0]
            actions_p1_game2 = self.payoff_p1_game2.shape[0]
            actions_p2_game1 = self.payoff_p1_game1.shape[1]
            actions_p2_game2 = self.payoff_p1_game2.shape[1]

            length_of_actions = np.max([actions_p1_game1,actions_p1_game2]) # determine the
                maximum length

            p1_combined = np.zeros(length_of_actions, dtype=object) # prepare an object for
                the probabilities

            # in the loop below we calculate the new transition probabilities
            for i in np.nditer(np.arange(length_of_actions)):
                working_probs = np.zeros((2,2))

                working_probs[0,0] = np.asscalar(np.dot(self.transition_matrix_game1_to1[i
                    ,:],x[0:actions_p2_game1]))
                working_probs[0,1] = np.asscalar(np.dot(self.transition_matrix_game1_to2[i
                    ,:],x[0:actions_p2_game1]))
                working_probs[1,0] = np.asscalar(np.dot(self.transition_matrix_game2_to1[i
                    ,:],x[actions_p2_game1:actions_p2_game1+actions_p2_game2]))
                working_probs[1,1] = np.asscalar(np.dot(self.transition_matrix_game2_to2[i
                    ,:],x[actions_p2_game1:actions_p2_game1+actions_p2_game2]))

                p1_combined[i] = working_probs/np.sum(working_probs,1)

            return p1_combined

        def reward_multiplication(self,x):
            "This function calculates the adjusted rewards"

            # initialize empty arrays
            rewards_game1 = []
            rewards_game2 = []

            # store the actions for p2
            actions_p2_game1 = self.payoff_p1_game1.shape[1]
            actions_p2_game2 = self.payoff_p1_game2.shape[1]

            # calculate the rewards for game 1
            for i in np.nditer(np.arange(self.payoff_p1_game1.shape[0])):
                rewards_game1.append(np.asscalar(np.dot(self.payoff_p1_game1[i,:],x[0:
                    actions_p2_game1])))
```

```
            # calculate the rewards for game 2
            for i in np.nditer(np.arange(self.payoff_p1_game2.shape[0])):
                rewards_game2.append(np.asscalar(np.dot(self.payoff_p1_game2[i,:],x[
                    actions_p2_game1:actions_p2_game1+actions_p2_game2])))

            Reward = np.array([rewards_game1,rewards_game2]) # Combine the rewards in one
                array

            return Reward

        p1_combined = chance_multiplication(self,x) # run the transition probability function
        Reward_matrix = reward_multiplication(self,x) # run the rewards function

        # Run the MDP process to find the threat point for P1
        mdp_threatpointp1 = mdptoolbox.mdp.RelativeValueIteration(p1_combined, Reward_matrix,
             epsilon=0.00000000001)
        mdp_threatpointp1.setSilent()
        mdp_threatpointp1.run()

        self.threatpoint_p1_policy = mdp_threatpointp1.policy # store the policy (strategy)
            found

        if self.printing == True:
            print("Running MDP")
            print("With X as",x)
            print("Resulting in:")
            print(mdp_treatpointp1.average_reward)
            print("With policy:")
            print(mdp_treatpointp1.policy)


        return mdp_threatpointp1.average_reward


    def markov_decision_process_p2(self,x):
        "This function creates a MDP for the second player to determine the threat point"

        def chance_multiplication(self,x):
            "Calculates the transitions probabilities for the MDP"

            # store the actions of the players
            actions_p1_game1 = self.payoff_p1_game1.shape[0]
            actions_p1_game2 = self.payoff_p1_game2.shape[0]
            actions_p2_game1 = self.payoff_p2_game1.shape[1]
            actions_p2_game2 = self.payoff_p2_game2.shape[1]

            length_of_actions = np.max([actions_p2_game1,actions_p2_game2]) # determine the
                maximum length of the actions

            p2_combined = np.zeros(length_of_actions, dtype=object) # initialize an object
                for storing the probabilities

            # in this loop we calculate the probabilities and return them
            for i in np.nditer(np.arange(length_of_actions)):
                working_probs = np.zeros((2,2))

                working_probs[0,0] = np.asscalar(np.dot(self.transition_matrix_game1_to1[:,i
                    ],x[0:actions_p1_game1]))
                working_probs[0,1] = np.asscalar(np.dot(self.transition_matrix_game1_to2[:,i
                    ],x[0:actions_p1_game1]))
                working_probs[1,0] = np.asscalar(np.dot(self.transition_matrix_game2_to1[:,i
```

```
                ],x[actions_p1_game1:actions_p1_game1+actions_p1_game2]))
            working_probs[1,1] = np.asscalar(np.dot(self.transition_matrix_game2_to2[:,i
                ],x[actions_p1_game1:actions_p1_game1+actions_p1_game2]))

            p2_combined[i] = working_probs/np.sum(working_probs,1)

        return p2_combined

    def reward_multiplication(self,x):
        "Calculate the rewards for the game"

        # initialize and store the rewards
        rewards_game1 = []
        rewards_game2 = []

        # store the actions of p2
        actions_p1_game1 = self.payoff_p2_game1.shape[0]
        actions_p1_game2 = self.payoff_p2_game2.shape[0]

        # calculate the rewards of the first game
        for i in np.nditer(np.arange(self.payoff_p1_game1.shape[0])):
            rewards_game1.append(np.asscalar(np.dot(self.payoff_p2_game1[:,i],x[0:
                actions_p1_game1])))

        # calculate the rewards of the second game
        for i in np.nditer(np.arange(self.payoff_p1_game2.shape[0])):
            rewards_game2.append(np.asscalar(np.dot(self.payoff_p2_game2[:,i],x[
                actions_p1_game1:actions_p1_game1+actions_p1_game2])))

        Reward = np.array([rewards_game1,rewards_game2]) # combine the rewards of both
            games

        return Reward

    p2_combined = chance_multiplication(self,x) # run the chance function
    Reward2 = reward_multiplication(self,x) # run the rewards function

    # Run the MDP for P2 and calculate his threat point
    mdp_threatpointp2 = mdptoolbox.mdp.RelativeValueIteration(p2_combined, Reward2,
        epsilon=0.00000000001)
    mdp_threatpointp2.setSilent()
    mdp_threatpointp2.run()

    self.threatpoint_p2_policy = mdp_threatpointp2.policy # store the policy of p2 (best
        strategy)

    if self.printing == True:
        print("Running MDP")
        print("With X as",x)
        print("Resulting in:")
        print(mdp_threatpointp2.average_reward)
        print("With policy:")
        print(mdp_threatpointp2.policy)

    return mdp_threatpointp2.average_reward

def threat_point_algorithm(self,T,sensi):
    "This algorithm calculates the threat point in a two-player game with"

    # set the shape of the actions for p1
    x_shape_p1_g1 = self.payoff_p1_game1.shape[0]
```

```
        x_shape_p1_g2 = self.payoff_p1_game2.shape[0]

        # set the shape of the actions for p2
        x_shape_p2_g1 = self.payoff_p2_game1.shape[1]
        x_shape_p2_g2 = self.payoff_p2_game2.shape[1]

        # combine the shapes
        x_shape_p1 = x_shape_p1_g1 + x_shape_p1_g2
        x_shape_p2 = x_shape_p2_g1 + x_shape_p2_g2

        # initialiaze the strategies
        xtry = np.zeros(4)
        xtry_p1 = np.zeros(x_shape_p1)
        xtry_p2 = np.zeros(x_shape_p2)

        # initialize the MDP values
        tried_mdp_p1 = 0
        stored_value_p1 = 0

        tried_mdp_p2 = 0
        stored_value_p2 = 0

        # initialize storage of best threat strategy
        x_best_p1 = np.zeros(x_shape_p1)
        x_best_p2 = np.zeros(x_shape_p2)

        new_time = time.time() # start the timer!

        for i in np.nditer(np.arange(T)): #loop over the number of to generate points

            # generate random strategies for p1 and p2
            xtry_p1 = np.random.beta(0.5,0.5,x_shape_p1)
            xtry_p2 = np.random.beta(0.5,0.5,x_shape_p2)

            # normalize these strategies for both p1 and p2
            xtry_p1[0:x_shape_p1_g1] = xtry_p1[0:x_shape_p1_g1]/np.sum(xtry_p1[0:
                x_shape_p1_g1])
            xtry_p1[x_shape_p1_g1:x_shape_p1] = xtry_p1[x_shape_p1_g1:x_shape_p1]/np.sum(
                xtry_p1[x_shape_p1_g1:x_shape_p1])

            xtry_p2[0:x_shape_p2_g1] = xtry_p2[0:x_shape_p2_g1]/np.sum(xtry_p2[0:
                x_shape_p2_g1])
            xtry_p2[x_shape_p2_g1:x_shape_p2] = xtry_p2[x_shape_p2_g1:x_shape_p2]/np.sum(
                xtry_p2[x_shape_p2_g1:x_shape_p2])

            # run the MDP for both P1 and P2
            tried_mdp_p1 = self.markov_decision_process_p1(xtry_p1)
            tried_mdp_p2 = self.markov_decision_process_p2(xtry_p2)

            if i == 0: # if it is the first run, just store the values
                stored_value_p1 = tried_mdp_p1
                stored_value_p2 = tried_mdp_p2
                x_best_p1 = xtry_p1
                x_best_p2 = xtry_p2

            else: # if not, check if we have found lower values for both MDP's and store them
                 if so
                if tried_mdp_p1 < stored_value_p1:
                    stored_value_p1 = tried_mdp_p1
                    x_best_p1 = xtry_p1
```

```
        elif tried_mdp_p2 < stored_value_p2:
            stored_value_p2 = tried_mdp_p2
            x_best_p2 = xtry_p2

# print the preliminary results
print("Rough value found =",stored_value_p1)
print("With best threaten strategy:",x_best_p1)

print("For p2 =",stored_value_p2)
print("With strategy",x_best_p2)
print("Found within (seconds): ",time.time() - new_time)

print("")
print("")

print("Now let's find a more precise point by generating more precise points in an
    interval")
print("")

# again initialize the best threat strategies
updatex_p1 = np.zeros(x_shape_p1)
updatex_p2 = np.zeros(x_shape_p2)

# loop again for better results
for i in np.nditer(np.arange(T)):

    # calculate updates to the found best threat strategies
    updatex_p1 = x_best_p1-sensi+((x_best_p1+sensi)-x_best_p1)*np.random.beta
        (0.5,0.5,x_shape_p1)
    updatex_p2 = x_best_p2-sensi+((x_best_p2+sensi)-x_best_p2)*np.random.beta
        (0.5,0.5,x_shape_p2)

    # fail safe in case any action becomes negative
    for j in np.nditer(np.arange(x_shape_p1)):
        if updatex_p1[j] < 0:
            updatex_p1[j] = -updatex_p1[j]

    for j in np.nditer(np.arange(x_shape_p2)):
        if updatex_p2[j] < 0:
            updatex_p2[j] = -updatex_p2[j]

    # normalize the strategies
    updatex_p1[0:x_shape_p1_g1] = updatex_p1[0:x_shape_p1_g1]/np.sum(updatex_p1[0:
        x_shape_p1_g1])
    updatex_p1[x_shape_p1_g1:x_shape_p1] = updatex_p1[x_shape_p1_g1:x_shape_p1]/np.
        sum(updatex_p1[x_shape_p1_g1:x_shape_p1])

    updatex_p2[0:x_shape_p2_g1] = updatex_p2[0:x_shape_p2_g1]/np.sum(updatex_p2[0:
        x_shape_p2_g1])
    updatex_p2[x_shape_p2_g1:x_shape_p2] = updatex_p2[x_shape_p2_g1:x_shape_p2]/np.
        sum(updatex_p2[x_shape_p2_g1:x_shape_p2])

    # run the MDP for new chosen strategies
    new_try_mdp_p1 = self.markov_decision_process_p1(updatex_p1)
    new_try_mdp_p2 = self.markov_decision_process_p2(updatex_p2)

    # if new found values are lower then store them
    if new_try_mdp_p1 < stored_value_p1:
        stored_value_p1 = new_try_mdp_p1
        x_best_p1 = updatex_p1
```

```
            elif new_try_mdp_p2 < stored_value_p2:
                stored_value_p2 = new_try_mdp_p2
                x_best_p2 = updatex_p2


        # print the threat point
        print("New value found =",stored_value_p1)
        print("With best strategy =",x_best_p1)
        print("")
        print("For p2 =",stored_value_p2)
        print("With strategy",x_best_p2)
        print("")
        print("")
        print("End of algorithm")
        print("")
        print("")

        self.threat_point = [stored_value_p1,stored_value_p2] # store the threat point

    def markov_try_out_max_p1(self,x):
        "Maximize the payoff by using negative values"

        def chance_multiplication(self,x):
            "This function calculates the transition probabilities"

            # this function stores the actions of the players
            actions_p1_game1 = self.payoff_p1_game1.shape[0]
            actions_p1_game2 = self.payoff_p1_game2.shape[0]
            actions_p2_game1 = self.payoff_p2_game1.shape[1]
            actions_p2_game2 = self.payoff_p2_game2.shape[1]

            length_of_actions = np.max([actions_p2_game1,actions_p2_game2]) # determine the
                maximum length of the actions

            p2_combined = np.zeros(length_of_actions, dtype=object) # initialize an empty
                object

            # calculate the transition probabilities based on chosen actions
            for i in np.nditer(np.arange(length_of_actions)):
                working_probs = np.zeros((2,2))

                working_probs[0,0] = np.asscalar(np.dot(self.transition_matrix_game1_to1[:,i
                    ],x[0:actions_p1_game1]))
                working_probs[0,1] = np.asscalar(np.dot(self.transition_matrix_game1_to2[:,i
                    ],x[0:actions_p1_game1]))
                working_probs[1,0] = np.asscalar(np.dot(self.transition_matrix_game2_to1[:,i
                    ],x[actions_p1_game1:actions_p1_game1+actions_p1_game2]))
                working_probs[1,1] = np.asscalar(np.dot(self.transition_matrix_game2_to2[:,i
                    ],x[actions_p1_game1:actions_p1_game1+actions_p1_game2]))

                p2_combined[i] = working_probs/np.sum(working_probs,1)

            return p2_combined


        p2_combined_max = chance_multiplication(self,x) # run the chance multiplication
            function

        def reward_multiplication(self,x):
            "This function computes the reward matrix based on the input strategy"
```

```
            # initialize the reewards for game 1 and game 2
            rewards_game1 = []
            rewards_game2 = []

            # store the actions for p1 in game 1 and 2
            actions_p1_game1 = self.payoff_p1_game1.shape[0]
            actions_p1_game2 = self.payoff_p1_game2.shape[0]

            # calculate the new rewards for game 1
            for i in np.nditer(np.arange(self.payoff_p1_game1.shape[0])):
                rewards_game1.append(np.asscalar(-np.dot(self.payoff_p1_game1[:,i],x[0:
                    actions_p1_game1])))

            # calculate the new rewards for game 2
            for i in np.nditer(np.arange(self.payoff_p1_game2.shape[0])):
                rewards_game2.append(np.asscalar(-np.dot(self.payoff_p1_game2[:,i],x[
                    actions_p1_game1:actions_p1_game1+actions_p1_game2])))

            Reward = np.array([rewards_game1,rewards_game2]) # store them in one array

            return Reward

        Reward2_max = reward_multiplication(self,x) # run the reward function

        # run the MDP which computes the minimax for p1
        mdp_threatpoint_minp1 = mdptoolbox.mdp.RelativeValueIteration(p2_combined_max,
            Reward2_max, epsilon=0.00000000001)
        mdp_threatpoint_minp1.setSilent()
        mdp_threatpoint_minp1.run()

        self.threatpoint_p1_min_policy = mdp_threatpoint_minp1.policy # store the policy

        # print the result
        if self.printing == True:
            print("Running␣MDP")
            print("With␣X␣as",x)
            print("Resulting␣in:")
            print(mdp_treatpoint_minp1.average_reward)
            print("With␣policy:")
            print(mdp_treatpoint_minp1.policy)

        return mdp_threatpoint_minp1.average_reward

    def markov_try_out_max_p2(self,x):
        "The␣maximin␣version␣of␣the␣MDP␣for␣P2"

        def chance_multiplication(self,x):
            "The␣transition␣probability␣function"

            # store the actions for both players
            actions_p1_game1 = self.payoff_p1_game1.shape[0]
            actions_p1_game2 = self.payoff_p1_game2.shape[0]
            actions_p2_game1 = self.payoff_p1_game1.shape[1]
            actions_p2_game2 = self.payoff_p1_game2.shape[1]

            length_of_actions = np.max([actions_p1_game1,actions_p1_game2]) # select the
                maximal length of the actions

            p1_combined = np.zeros(length_of_actions, dtype=object) # initialize the p1
                transition probability
```

```
        # calculate the transition probabilities for p1
        for i in np.nditer(np.arange(length_of_actions)):
            working_probs = np.zeros((2,2))

            working_probs[0,0] = np.asscalar(np.dot(self.transition_matrix_game1_to1[i
                ,:],x[0:actions_p2_game1]))
            working_probs[0,1] = np.asscalar(np.dot(self.transition_matrix_game1_to2[i
                ,:],x[0:actions_p2_game1]))
            working_probs[1,0] = np.asscalar(np.dot(self.transition_matrix_game2_to1[i
                ,:],x[actions_p2_game1:actions_p2_game1+actions_p2_game2]))
            working_probs[1,1] = np.asscalar(np.dot(self.transition_matrix_game2_to2[i
                ,:],x[actions_p2_game1:actions_p2_game1+actions_p2_game2]))

            p1_combined[i] = working_probs/np.sum(working_probs,1)

        return p1_combined

    def reward_multiplication(self,x):
        "Multiply the rewards for p2's maximin"

        # initialize the array for storage
        rewards_game1 = []
        rewards_game2 = []

        # store the actions for p2 in both games
        actions_p2_game1 = self.payoff_p2_game1.shape[1]
        actions_p2_game2 = self.payoff_p2_game2.shape[1]

        # compute the rewards for game 1
        for i in np.nditer(np.arange(self.payoff_p2_game1.shape[0])):
            rewards_game1.append(-np.asscalar(np.dot(self.payoff_p2_game1[i,:],x[0:
                actions_p2_game1])))

        # compute the rewards for game 2
        for i in np.nditer(np.arange(self.payoff_p1_game2.shape[0])):
            rewards_game2.append(-np.asscalar(np.dot(self.payoff_p2_game2[i,:],x[
                actions_p2_game1:actions_p2_game1+actions_p2_game2])))

        Reward = np.array([rewards_game1,rewards_game2]) # combine the rewards in one
            array

        return Reward

    p1_combined = chance_multiplication(self,x) # compute the new transition
        probabilities
    Reward = reward_multiplication(self,x) # compute the new rewards

    # run the MDP to compute the maximin for p2
    mdp_threatpointp1 = mdptoolbox.mdp.RelativeValueIteration(p1_combined, Reward,
        epsilon=0.00000000001)
    mdp_threatpointp1.setSilent()
    mdp_threatpointp1.run()

    self.threatpoint_p1_policy = mdp_threatpointp1.policy #store the policy

    # print some blabla (wonder if someone ever reads this)
    if self.printing == True:
        print("Running MDP")
        print("With X as",x)
        print("Resulting in:")
        print(mdp_treatpointp1.average_reward)
```

```
            print("With policy:")
            print(mdp_treatpointp1.policy)


        return mdp_threatpointp1.average_reward

    def maximin_point(self,T):
        "This part tries to find the maximin point for T points"
        print("Trying to find the maximin values for both players")

        # store the shape of the games for both players
        x_shape_p1_g1 = self.payoff_p1_game1.shape[0]
        x_shape_p1_g2 = self.payoff_p1_game2.shape[0]

        x_shape_p2_g1 = self.payoff_p2_game1.shape[1]
        x_shape_p2_g2 = self.payoff_p2_game2.shape[1]

        # add up the shapes for both players
        x_shape_p1 = x_shape_p1_g1 + x_shape_p1_g2
        x_shape_p2 = x_shape_p2_g1 + x_shape_p2_g2

        # initialize for trying strategies
        x_try_p1_max = np.zeros(x_shape_p2)
        x_try_p2_max = np.zeros(x_shape_p1)

        # set all 'found' values to zero
        tried_mdp_p1_max = 0
        stored_value_p1_max = 0

        tried_mdp_p2_max = 0
        stored_value_p2_max = 0

        # loop over the number of points to generate
        for i in range(0,T):

            # draw new strategies
            x_try_p1_max = np.random.beta(0.5,0.5,x_shape_p2)
            x_try_p2_max = np.random.beta(0.5,0.5,x_shape_p1)

            # normalize these strategies
            x_try_p1_max[0:x_shape_p2_g1] = x_try_p1_max[0:x_shape_p2_g1]/np.sum(x_try_p1_max
                [0:x_shape_p2_g1])
            x_try_p1_max[x_shape_p2_g1:x_shape_p2] = x_try_p1_max[x_shape_p2_g1:x_shape_p2]/
                np.sum(x_try_p1_max[x_shape_p2_g1:x_shape_p2])

            x_try_p2_max[0:x_shape_p1_g1] = x_try_p2_max[0:x_shape_p1_g1]/np.sum(x_try_p2_max
                [0:x_shape_p1_g1])
            x_try_p2_max[x_shape_p1_g1:x_shape_p1] = x_try_p2_max[x_shape_p1_g1:x_shape_p1]/
                np.sum(x_try_p2_max[x_shape_p1_g1:x_shape_p1])

            # run the markov maximin problems
            tried_mdp_p1_max = -self.markov_try_out_max_p1(x_try_p1_max)
            tried_mdp_p2_max = -self.markov_try_out_max_p2(x_try_p2_max)

            # if it is the first result, store it
            if i == 0:
                stored_value_p1_max = tried_mdp_p1_max
                stored_value_p2_max = tried_mdp_p2_max

                x_best_p1_max = x_try_p1_max
                x_best_p2_max = x_try_p2_max
```

```
            # if a higher value is found and it is not the first result, then store again
            else:
                if tried_mdp_p1_max > stored_value_p1_max:
                    stored_value_p1_max = tried_mdp_p1_max
                    x_best_p1_max = x_try_p1_max
                if tried_mdp_p2_max > stored_value_p2_max:
                    stored_value_p2_max = tried_mdp_p2_max
                    x_best_p2_max = x_try_p2_max

        print("Maximin value found for P1",stored_value_p1_max)
        print("With best maximization strategy:",x_best_p1_max)
        print("")
        print("Maximin value found for P2",stored_value_p2_max)
        print("With best maximization strategy:",x_best_p2_max)
        print("")
        print("")

    def optimized_maximin(self,points,show_strat_p1,show_strat_p2,FD_yn):
        "This algorithim is a more optimized way of calculating the maximin results "
        print("Start of the maximin algorithm")

        def random_strategy_draw(points,number_of_actions):
            "This function draws random strategies from a beta distribution, based on the
                number of points and actions"

            # draw the strategies and normalize them
            strategies_drawn = np.random.beta(0.5,0.5,(points,number_of_actions))
            strategies_drawn = strategies_drawn/np.sum(strategies_drawn, axis=1).reshape([
                points,1])

            return strategies_drawn

        def frequency_pairs_p1(points,p2_actions,p1_actions,strategies_drawn):
            "This function sorts the strategies that punish based on the best replies for p1"

            # store the size of the game
            game_size_1 = self.payoff_p1_game1.size
            game_size_2 = self.payoff_p1_game2.size

            # store the actions of the game
            p1_actions_game1 = self.payoff_p1_game1.shape[0]
            p1_actions_game2 = self.payoff_p1_game2.shape[0]

            # calculate the combination of actions and set it in a range
            p1_actions_combi = p1_actions_game1*p1_actions_game2
            p1_action_range = np.arange(p1_actions_combi)

            # initialize a frequency pair
            frequency_pairs = np.zeros((points*(p1_actions_game1*p1_actions_game2),
                game_size_1+game_size_2))

            # set ranges for game 1 and 2
            p1_act_game1_range = np.arange(p1_actions_game1)
            p1_act_game2_range = np.arange(p1_actions_game2)

            # set the frequency pairs for game 1 based on best replies
            for i in np.nditer(p1_action_range):
                for j in np.nditer(p1_act_game1_range):
                    mod_remain = np.mod(i,p1_actions_game1)
                    frequency_pairs[i*points:(i+1)*points,p1_actions_game1*mod_remain+j] =
```

```
                strategies_drawn[:,j]

        # set the frequency pairs for game 2 based on best replies
        for i in np.nditer(p1_action_range):
            for j in np.nditer(p1_act_game2_range):
                floor_div = np.floor_divide(i,p1_actions_game2)
                frequency_pairs[i*points:(i+1)*points,j+game_size_1+(p1_actions_game1*
                    floor_div)] = strategies_drawn[:,p1_actions_game1+j]

        return frequency_pairs

    def balance_equation(self,tot_act_ut,tot_act_thr,tot_payoffs_game1,tot_payoffs,
        frequency_pairs):
        "Calculates the result of the balance equations in order to adjust the frequency
            pairs"

        # store the game size
        game_size_1 = self.payoff_p1_game1.size
        game_size_2 = self.payoff_p1_game2.size

        # initialize yi and Q
        yi = np.zeros((points*(tot_act_thr*tot_act_ut),game_size_1+game_size_2))
        Q = np.zeros((1,points*(tot_act_thr*tot_act_ut)))

        # calculate yi
        yi[:,0:tot_payoffs_game1] = frequency_pairs[:,0:tot_payoffs_game1]/np.sum(
            frequency_pairs[:,0:tot_payoffs_game1], axis=1).reshape([points*
            tot_payoffs_game1,1])
        yi[:,tot_payoffs_game1:tot_payoffs] = frequency_pairs[:,tot_payoffs_game1:
            tot_payoffs]/np.sum(frequency_pairs[:,tot_payoffs_game1:tot_payoffs], axis=1)
            .reshape([points*(tot_payoffs-tot_payoffs_game1),1])

        # store px
        p1_px_between = np.asarray(px)
        p1_px = p1_px_between[0]

        # calculate Q
        Q[0,:] = (np.sum(yi[:,tot_payoffs_game1:tot_payoffs]*p1_px[tot_payoffs_game1:
            tot_payoffs],axis=1))/((np.dot(yi[:,0:tot_payoffs_game1],(1-p1_px[0:
            tot_payoffs_game1]))+np.dot(yi[:,tot_payoffs_game1:tot_payoffs],p1_px[
            tot_payoffs_game1:tot_payoffs])))

        # calculate new frequency pairs based on Q
        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q.transpose(),yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q.transpose()),
            yi[:,tot_payoffs_game1:tot_payoffs])

        return frequency_pairs

    def frequency_pairs_p2(points,p2_actions,p1_actions,strategies_drawn):
        "This function sorts the punish strategies based on the best replies of p2"

        # store the game size
        game_size_1 = self.payoff_p2_game1.size
        game_size_2 = self.payoff_p2_game2.size

        # create a range of actions of both players
        p1_actions_range = np.arange(p1_actions)
        p2_actions_range = np.arange(p2_actions)
```

```
        # store the shape of the both games
        p2_actions_game1 = self.payoff_p2_game1.shape[1]
        p2_actions_game2 = self.payoff_p2_game2.shape[1]

        # calculate the range of possible actions
        p2_actions_combo = p2_actions_game1*p2_actions_game2
        p2_action_range = np.arange(p2_actions_combo)

        # initialize the frequency pairs
        frequency_pairs = np.zeros((points*(p2_actions_game1*p2_actions_game2),
            game_size_1+game_size_2))

        # sort the frequency pairs for the first game
        for i in np.nditer(np.arange(p2_actions_game1)):
            for j in np.nditer(p2_action_range):
                modul = np.mod(j,p2_actions_game1)
                frequency_pairs[j*points:(j+1)*points,p2_actions_game1*i+modul] =
                    strategies_drawn[:,i]

        # sort the frequency pairs for the second game
        for i in np.nditer(np.arange(p2_actions_game2)):
            for j in np.nditer(p2_action_range):
                divide = np.floor_divide(j,p2_actions_game2)
                frequency_pairs[j*points:(j+1)*points,p2_actions_combo+divide+(i*
                    p2_actions_game2)] = strategies_drawn[:,i+p2_actions_game1]

        return frequency_pairs

    def payoffs_sorted(points,payoffs,actions):
        "Sort the payoffs for use on the max and min functions"

        # create two ranges based on the number of points and actions
        points_range = np.arange(points)
        actions_range = np.arange(actions)

        payoffs_sort = np.zeros((points,actions)) # initialize the payoff_sort array

        # sort the payoffs
        for x in np.nditer(points_range):
            for i in np.nditer(actions_range):
                payoffs_sort[x,i] = payoffs[points*i+x]

        return payoffs_sort

    ## Start of p1 maximin ##

    start_time = time.time() # start the time!

    # flatten the transition matrices
    flatten1_1 = self.transition_matrix_game1_to1.flatten()
    flatten2_1 = self.transition_matrix_game2_to1.flatten()

    # save and compute the total number of actions
    actions_p2_game1 = self.payoff_p1_game1.shape[1]
    actions_p2_game2 = self.payoff_p1_game2.shape[1]
    total_actions_p2 = actions_p2_game1 + actions_p2_game2

    actions_p1_game1 = self.payoff_p1_game1.shape[0]
    actions_p1_game2 = self.payoff_p1_game2.shape[0]
    total_actions_p1 = actions_p1_game1 + actions_p1_game2
```

```
# flatten the payoffs
payoff_p1_game_1flatten = self.payoff_p1_game1.flatten()
payoff_p1_game_2flatten = self.payoff_p1_game2.flatten()

total_payoffs_p1_game1 = payoff_p1_game_1flatten.size
total_payoffs_p1_game2 = payoff_p1_game_2flatten.size
total_payoffs_p1 = total_payoffs_p1_game1 + total_payoffs_p1_game2

# initialize and assign the payoffs for p1
payoff_p1 = np.zeros(total_payoffs_p1)
payoff_p1[0:total_payoffs_p1_game1] = payoff_p1_game_1flatten
payoff_p1[total_payoffs_p1_game1:total_payoffs_p1] = payoff_p1_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # create px

y_punisher = random_strategy_draw(points,total_actions_p1) # draw random strategies
    for the punisher

frequency_pairs = frequency_pairs_p2(points,total_actions_p1,total_actions_p2,
    y_punisher) # sort these

# run the balance equation
frequency_pairs = balance_equation(self,actions_p2_game1,actions_p2_game2,
    total_payoffs_p1_game1,total_payoffs_p1,frequency_pairs)

# if the FD_function is available, run this (note: Only the FD function from the
    thesis)
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# calculate the payoffs with multiplication of payoffs and Fd function
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p1),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p2_game1*actions_p2_game2)) #
    sort the payoffs

print("")
print("")
minimax_found = np.max(np.min(max_payoffs,axis=1))
print("Maximin value for P1 is",minimax_found)
print("")
print("")

# print the results
if show_strat_p1 == True:
    minimax_indices_p2 = np.where(max_payoffs == minimax_found)
    found_strategy_p2 = y_punisher[minimax_indices_p2[0]]
    fnd_strategy_p2 = found_strategy_p2.flatten()
    fnd_strategy_p2[0:2] = fnd_strategy_p2[0:2]/np.sum(fnd_strategy_p2[0:2])
    fnd_strategy_p2[2:4] = fnd_strategy_p2[2:4]/np.sum(fnd_strategy_p2[2:4])
    print("Player 1 plays stationary strategy:", fnd_strategy_p2)
    print("While player 2 replies with a best pure reply of:", self.
        best_pure_strategies[minimax_indices_p2[1]])

end_time = time.time() # stop the time!
```

```
print("Seconds done to generate", points, "points", end_time-start_time)

## End of P1 maximin algorithm ##

start_time_p2 = time.time() # start the time (Again!)

#flatten the payoffs
payoff_p2_game_1flatten = self.payoff_p2_game1.flatten()
payoff_p2_game_2flatten = self.payoff_p2_game2.flatten()

# store and compute the total payoffs
total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

# initialize and assign the payoffs for p2
payoff_p2 = np.zeros(total_payoffs_p2)
payoff_p2[0:total_payoffs_p2_game1] = payoff_p2_game_1flatten
payoff_p2[total_payoffs_p2_game1:total_payoffs_p2] = payoff_p2_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # store px

x_punisher = random_strategy_draw(points,total_actions_p2) # draw punisher strategies

frequency_pairs = frequency_pairs_p1(points,total_actions_p1,total_actions_p2,
    x_punisher) # best replies

# do the balance equation trick
frequency_pairs = balance_equation(self,actions_p1_game1,actions_p1_game2,
    total_payoffs_p1_game1,total_payoffs_p1,frequency_pairs)

# if the FD function is on, then run it
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# compute the payoffs with the payoffs and FD function
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p2),axis=1)

payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p1_game1*actions_p1_game2))
    # sort the payoffs

print("")
print("")
minimax_found_p2 = np.max(np.min(max_payoffs,axis=1)) # find the maximin value
print("Maximin value for P2 is",minimax_found_p2)
print("")
print("")

# print the strategies
if show_strat_p2 == True:
    maximin_indices_p2 = np.where(max_payoffs == minimax_found_p2)
    found_strategy = x_punisher[maximin_indices_p2[0]]
    fnd_strategy = found_strategy.flatten()
    fnd_strategy[0:2] = fnd_strategy[0:2]/np.sum(fnd_strategy[0:2])
```

```
            fnd_strategy[2:4] = fnd_strategy[2:4]/np.sum(fnd_strategy[2:4])
            print("Player 2 plays stationairy strategy:", fnd_strategy)
            print("While player 2 replies with a best pure reply of:", self.
                best_pure_strategies[maximin_indices_p2[1]])

        end_time_p2 = time.time() # stop the time
        print("Seconds done to generate", points, "points", end_time_p2-start_time_p2)
        print("")
        print("")

    def threat_point_optimized(self,points,show_strat_p1,show_strat_p2,print_text,FD_yn):
        "The optimized, super awesome threat point algorithm!"

        def random_strategy_draw(points,number_of_actions):
            "This function draws random strategies from a beta distribution, based on the
                number of points and actions"

            # draw the strategies and normalize them
            strategies_drawn = np.random.beta(0.5,0.5,(points,number_of_actions))
            strategies_drawn = strategies_drawn/np.sum(strategies_drawn, axis=1).reshape([
                points,1])

            return strategies_drawn

        def frequency_pairs_p1(points,p2_actions,p1_actions,strategies_drawn):
            "This function sorts the punisher strategies based on the replies for p1"

            # store the game sizes
            game_size_1 = self.payoff_p1_game1.size
            game_size_2 = self.payoff_p1_game2.size

            # store the actions of p1 for both games
            p1_actions_game1 = self.payoff_p1_game1.shape[0]
            p1_actions_game2 = self.payoff_p1_game2.shape[0]

            # set both actions within a certain range
            p1_actions_combi = p1_actions_game1*p1_actions_game2
            p1_action_range = np.arange(p1_actions_combi)

            # initialize the frequency pair
            frequency_pairs = np.zeros((points*(p1_actions_game1*p1_actions_game2),
                game_size_1+game_size_2))

            # set action ranges for both games
            p1_act_game1_range = np.arange(p1_actions_game1)
            p1_act_game2_range = np.arange(p1_actions_game2)

            # set the best replies for the first game
            for i in np.nditer(p1_action_range):
                for j in np.nditer(p1_act_game1_range):
                    mod_remain = np.mod(i,p1_actions_game1)
                    frequency_pairs[i*points:(i+1)*points,p1_actions_game1*mod_remain+j] =
                        strategies_drawn[:,j]

            # set the best replies for the second game
            for i in np.nditer(p1_action_range):
                for j in np.nditer(p1_act_game2_range):
                    floor_div = np.floor_divide(i,p1_actions_game2)
                    frequency_pairs[i*points:(i+1)*points,j+game_size_1+(p1_actions_game1*
                        floor_div)] = strategies_drawn[:,p1_actions_game1+j]
```

```
        return frequency_pairs

    def balance_equation(self,tot_act_ut,tot_act_thr,tot_payoffs_game1,tot_payoffs,
        frequency_pairs):
        "Calculates the result of the balance equations in order to adjust the frequency
            pairs"

        # store size of game 1 and 2
        game_size_1 = self.payoff_p1_game1.size
        game_size_2 = self.payoff_p1_game2.size

        # initialize yi and Q
        yi = np.zeros((points*(tot_act_thr*tot_act_ut),game_size_1+game_size_2))
        Q = np.zeros((1,points*(tot_act_thr*tot_act_ut)))

        # compute Yi
        yi[:,0:tot_payoffs_game1] = frequency_pairs[:,0:tot_payoffs_game1]/np.sum(
            frequency_pairs[:,0:tot_payoffs_game1], axis=1).reshape([points*
            tot_payoffs_game1,1])
        yi[:,tot_payoffs_game1:tot_payoffs] = frequency_pairs[:,tot_payoffs_game1:
            tot_payoffs]/np.sum(frequency_pairs[:,tot_payoffs_game1:tot_payoffs], axis=1)
            .reshape([points*(tot_payoffs-tot_payoffs_game1),1])

        p1_px_between = np.asarray(px) # some tricks with px  (ha-ha)
        p1_px = p1_px_between[0]

        # compute Q
        Q[0,:] = (np.sum(yi[:,tot_payoffs_game1:tot_payoffs]*p1_px[tot_payoffs_game1:
            tot_payoffs],axis=1))/((np.dot(yi[:,0:tot_payoffs_game1],(1-p1_px[0:
            tot_payoffs_game1]))+np.dot(yi[:,tot_payoffs_game1:tot_payoffs],p1_px[
            tot_payoffs_game1:tot_payoffs])))

        # adjust the frequency pairs based on Q
        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q.transpose(),yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q.transpose()),
            yi[:,tot_payoffs_game1:tot_payoffs])

        return frequency_pairs

    def frequency_pairs_p2(points,p2_actions,p1_actions,strategies_drawn):
        "This function sorts the punisher strategies based on the replies for p2"

        # store the sizes of both games
        game_size_1 = self.payoff_p2_game1.size
        game_size_2 = self.payoff_p2_game2.size

        # make a nice range of the actions for both players and store them
        p1_actions_range = np.arange(p1_actions)
        p2_actions_range = np.arange(p2_actions)

        p2_actions_game1 = self.payoff_p2_game1.shape[1]
        p2_actions_game2 = self.payoff_p2_game2.shape[1]

        p2_actions_combo = p2_actions_game1*p2_actions_game2
        p2_action_range = np.arange(p2_actions_combo)

        # initialize the frequency pairs
        frequency_pairs = np.zeros((points*(p2_actions_game1*p2_actions_game2),
            game_size_1+game_size_2))
```

```
    # loop over the first games best responses
    for i in np.nditer(np.arange(p2_actions_game1)):
        for j in np.nditer(p2_action_range):
            modul = np.mod(j,p2_actions_game1)
            frequency_pairs[j*points:(j+1)*points,p2_actions_game1*i+modul] =
                strategies_drawn[:,i]

    # loop over the second games best responses
    for i in np.nditer(np.arange(p2_actions_game2)):
        for j in np.nditer(p2_action_range):
            divide = np.floor_divide(j,p2_actions_game2)
            frequency_pairs[j*points:(j+1)*points,p2_actions_combo+divide+(i*
                p2_actions_game2)] = strategies_drawn[:,i+p2_actions_game1]

    return frequency_pairs

def payoffs_sorted(points,payoffs,actions):
    "This function sorts the payoffs for use on the threat point function"

    # set a points and actions range
    points_range = np.arange(points)
    actions_range = np.arange(actions)

    payoffs_sort = np.zeros((points,actions)) # initialize the payoffs sorted

    # and sort them in a loop
    for x in np.nditer(points_range):
        for i in np.nditer(actions_range):
            payoffs_sort[x,i] = payoffs[points*i+x]

    return payoffs_sort

if print_text == True:
    print("The start of the algorithm for finding the threat point")
    print("First let's find the threat point for Player 1")

# flatten the transition probabilities
flatten1_1 = self.transition_matrix_game1_to1.flatten()
flatten2_1 = self.transition_matrix_game2_to1.flatten()

# store the actions of both players
actions_p2_game1 = self.payoff_p1_game1.shape[1]
actions_p2_game2 = self.payoff_p1_game2.shape[1]
total_actions_p2 = actions_p2_game1 + actions_p2_game2

actions_p1_game1 = self.payoff_p1_game1.shape[0]
actions_p1_game2 = self.payoff_p1_game2.shape[0]
total_actions_p1 = actions_p1_game1 + actions_p1_game2

# Start of algorithm for player 1

start_time = time.time() # and start the time!

# flatten the payoffs of both players
payoff_p1_game_1flatten = self.payoff_p1_game1.flatten()
payoff_p1_game_2flatten = self.payoff_p1_game2.flatten()

total_payoffs_p1_game1 = payoff_p1_game_1flatten.size
total_payoffs_p1_game2 = payoff_p1_game_2flatten.size
total_payoffs_p1 = total_payoffs_p1_game1 + total_payoffs_p1_game2
```

```
# initialize and assign the payoffs for p1
payoff_p1 = np.zeros(total_payoffs_p1)
payoff_p1[0:total_payoffs_p1_game1] = payoff_p1_game_1flatten
payoff_p1[total_payoffs_p1_game1:total_payoffs_p1] = payoff_p1_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # create px

y_punisher = random_strategy_draw(points,total_actions_p2) #  draw strategies for the
     punisher

frequency_pairs = frequency_pairs_p1(points,total_actions_p2,total_actions_p1,
    y_punisher) # sort based on best replies

# calculate the adjustments based on the balance equation
frequency_pairs = balance_equation(self,actions_p1_game1,actions_p1_game2,
    total_payoffs_p1_game1,total_payoffs_p1,frequency_pairs)

# if FD function is activated, activate it
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# calculate the payoffs based on the frequency pairs and FD function
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p1),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p1_game1*actions_p1_game2)) #
    sort the payoffs

threat_point_p1 = np.min(np.max(max_payoffs,axis=1)) # determine the threat point

if print_text == True:
    print("")
    print("")
    print("Threat point value is",threat_point_p1)
    print("")
    print("")

if show_strat_p1 == True:
    threat_point_indices_p1 = np.where(max_payoffs == threat_point_p1)
    found_strategy_p1 = y_punisher[threat_point_indices_p1[0]]
    fnd_strategy_p1 = found_strategy_p1.flatten()
    fnd_strategy_p1[0:2] = fnd_strategy_p1[0:2]/np.sum(fnd_strategy_p1[0:2])
    fnd_strategy_p1[2:4] = fnd_strategy_p1[2:4]/np.sum(fnd_strategy_p1[2:4])
    print("Player 2 plays stationary strategy:", fnd_strategy_p1)
    print("While player 1 replies with a best pure reply of:", self.
        best_pure_strategies[threat_point_indices_p1[1]])

end_time = time.time() # stop the time!

if print_text == True:
    print("Seconds done to generate", points, "points", end_time-start_time)
    print("")

# End of algorithm player 1

# Start of algorithm player 2
```

```
if print_text == True:
    print("")
    print("")
    print("First␣start␣the␣threat␣point␣for␣player␣2")
start_time_p2 = time.time() # new timer, new times

# flatten the payoffs for p2
payoff_p2_game_1flatten = self.payoff_p2_game1.flatten()
payoff_p2_game_2flatten = self.payoff_p2_game2.flatten()

# assign tha payoffs for p2
total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

# initialize payoffs p2 and assign them
payoff_p2 = np.zeros(total_payoffs_p2)
payoff_p2[0:total_payoffs_p2_game1] = payoff_p2_game_1flatten
payoff_p2[total_payoffs_p2_game1:total_payoffs_p2] = payoff_p2_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # px in the mix

x_punisher = random_strategy_draw(points,total_actions_p1) # draw strategies for the
    punisher

frequency_pairs = frequency_pairs_p2(points,total_actions_p2,total_actions_p1,
    x_punisher) # sort based on best replies

# adjust based on the balance equation
frequency_pairs = balance_equation(self,actions_p2_game1,actions_p2_game2,
    total_payoffs_p2_game1,total_payoffs_p2,frequency_pairs)

# Activate the FD function, or not
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# determine the payoffs based on the frequency pairs and FD function
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p2),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p2_game1*actions_p2_game2)) #
    sort the payoffs

threat_point_p2 = np.min(np.max(max_payoffs,axis=1)) # determine the threat point

if print_text == True:
    print("")
    print("")
    print("Threat␣point␣value␣is",threat_point_p2)
    print("")
    print("")

if show_strat_p2 == True:
    threat_point_indices_p2 = np.where(max_payoffs == threat_point_p2)
    found_strategy = x_punisher[threat_point_indices_p2[0]]
```

```
            fnd_strategy = found_strategy.flatten()
            fnd_strategy[0:2] = fnd_strategy[0:2]/np.sum(fnd_strategy[0:2])
            fnd_strategy[2:4] = fnd_strategy[2:4]/np.sum(fnd_strategy[2:4])
            print("Player 1 plays stationairy strategy:", fnd_strategy)
            print("While player 2 replies with a best pure reply of:", self.
                best_pure_strategies[threat_point_indices_p2[1]])

        end_time_p2 = time.time() # stop the time!

        if print_text == True:
            print("")
            print("Seconds done to generate", points, "points", end_time_p2-start_time_p2)
            print("")
            print("")

        self.threat_point = [threat_point_p1,threat_point_p2] # store the threat point

        return [threat_point_p1,threat_point_p2]
```

A.6 Type II Example Games Code

```
encoding=*-60
"The first Stochastic Game is based on a Stochastic Game described in the thesis"

p1_1 = np.matrix('16 14; 28 24')
p2_1 = np.matrix('16 28; 14 24')

p1_2 = np.matrix('4 3.5; 7 6')
p2_2 = np.matrix('4 7; 3.5 6')

trans1_1 = np.matrix('0.8 0.7; 0.7 0.6')
trans2_1 = np.matrix('0.5 0.4; 0.4 0.15')

trans1_2 = np.matrix('0.2 0.3; 0.3 0.4')
trans2_2 = np.matrix('0.5 0.6; 0.6 0.85')

FirstTry = StochasticGame(p1_1,p2_1,p1_2,p2_2,trans1_1,trans2_1,trans1_2,trans2_2)
FirstTry.plot_single_period_pure_rewards()
FirstTry.plot_all_reward_points(True)
FirstTry.maximin_point(100)
timing = time.time()
FirstTry.threat_point_algorithm(10000,0.025)
now = time.time()
print(now-timing)
FirstTry.threat_point_optimized(10000,False,False,True,True)
FirstTry.plot_threat_point()
FirstTry.plot_threat_point_lines()
FirstTry.optimized_maximin(100,False,False,True)
```

A.7 Type III Game Code

```
encoding=*-60
class ETPGame:
    "The ETP Game class represents the Type III games from the thesis, with or without ESP."
```

```
def __init__(self,payoff_p1_game1,payoff_p2_game1,payoff_p1_game2,payoff_p2_game2,
    trmatrixg1,trmatrixg2,trmatrixg3,trmatrixg4,matrixA):
    "Here below we initialize the game by storing payoff and transition matrices
        according to the upper input."
    self.payoff_p1_game1 = payoff_p1_game1                  #payoff p1 in game 1
    self.payoff_p2_game1 = payoff_p2_game1                  #payoff p2 in game 1

    self.payoff_p1_game2 = payoff_p1_game2                  #payoff p1 in game 2
    self.payoff_p2_game2 = payoff_p2_game2                  #payoff p2 in game 2

    self.transition_matrix_game1_to1 = trmatrixg1           #transition matrix from game
        1 to game 1
    self.transition_matrix_game2_to1 = trmatrixg2           #transition matrix from game
        2 to game 1

    self.transition_matrix_game1_to2 = trmatrixg3           #transition matrix from game
        1 to game 2
    self.transition_matrix_game2_to2 = trmatrixg4           #transition matrix from game
        2 to game 2

    self.etp_matrix = matrixA

    self.printing = False   #set printing to False

    self.best_pure_strategies = np.array([[1,0,1,0],[0,1,1,0],[1,0,0,1],[0,1,0,1]])

def plot_single_period_pure_rewards(self):
    "Here we plot the pure rewards possible for a single period"

    plt.figure()                                            #create a figure
    payoff_p1_g1_flat = self.payoff_p1_game1.A1             #create a flattend payoff of
        p1 in game 1
    payoff_p2_g1_flat = self.payoff_p2_game1.A1             #create a flattend payoff of
        p2 in game 1
    plt.scatter(payoff_p1_g1_flat,payoff_p2_g1_flat, label="Pure reward points Game 1",
        zorder = 15) #plot payoffs game 1

    payoff_p1_g2_flat = self.payoff_p1_game2.A1             #create a flattend payoff of
        p1 in game 2
    payoff_p2_g2_flat = self.payoff_p2_game2.A1             #and for p2 in game 2
    plt.scatter(payoff_p1_g2_flat,payoff_p2_g2_flat, label="Pure reward points Game 2",
        zorder = 15)  #plotting this again

    plt.xlabel("Payoff Player 1")                           #giving the x-axis the label
        of payoff p1
    plt.ylabel("Payoff Player 2")                           #and the payoff of the y-axis
         is that of p2
    plt.title("Reward points of ETP game")      #and we give it a nice titel
    plt.legend()

def plot_convex_hull_pure_rewards(self):
    "Here we plot a convex hull around the pure reward point, therefore resulting in the
        total possible reward space"

    payoff_p1_g1_flat = self.payoff_p1_game1.A1       #store the flattend payoff of p1
        game 1
    payoff_p2_g1_flat = self.payoff_p2_game1.A1       #store the flattend payoff of p2
        game 1

    payoff_p1_g2_flat = self.payoff_p1_game2.A1       #store the flattend payoff of p1
```

```
        game 2
    payoff_p2_g2_flat = self.payoff_p2_game2.A1      #store the flattend payoff of p2
        game 2

    payoff_p1_merged = np.concatenate((payoff_p1_g1_flat,payoff_p1_g2_flat))  #merge p1
        payoffs
    payoff_p2_merged = np.concatenate((payoff_p2_g1_flat,payoff_p2_g2_flat))  #merge p2
        payoffs

    all_payoffs = np.array([payoff_p1_merged,payoff_p2_merged])  #create one array of
        payoffs
    all_payoffs = np.transpose(all_payoffs)                      #and rotate this one

    rewards_convex_hull = ConvexHull(all_payoffs)                #retain the convex hull
        of the payoffs
    plt.fill(all_payoffs[rewards_convex_hull.vertices,0], all_payoffs[rewards_convex_hull
        .vertices,1], color='k')
    #here above we fill the convex hull in black

def plot_all_rewards(self,FD_yn):
    "This plots all rewards and is based on the algorithm by Llea Samuel"
    "IMPORTANT NOTE!: GAME IS HARDCODED WITHIN THIS FUNCTION, DUE TO TIME ISSUES, ONE
        SHOULD TAKE THIS INTO MIND"

    np.seterr(all='warn',divide='warn') # this is to show some more information on
        possible errors, can be excluded

    ## HERE BELOW IS HARDCODED GAME INFORMATION
    A1 = np.array([16, 14, 28, 24, 4, 3.5, 7, 6])
    B1 = np.array([16, 28, 14, 24, 4, 7, 3.5, 6])
    p = np.array([0.8, 0.7, 0.7, 0.6, 0.5, 0.4, 0.4, 0.15])
    ## ABOVE IS HARDCODED GAME INFORMATION

    # here below we initialize a lot of variables
    x = np.zeros(8)
    r = np.zeros(8)
    y = np.zeros(8)
    xstar = np.zeros(8)
    x_a = np.zeros(8)
    yp = np.zeros(4)
    yp_not = np.zeros(4)
    v_p1 = np.zeros(8)
    v_p2 = np.zeros(8)
    Q_vec = np.zeros(8)

    T = 100000 # number of points to generate
    payoff_p1 = np.zeros(T) # initialize payoffs for both players
    payoff_p2 = np.zeros(T)

    # store the matrix A
    matrixA = np.matrix('0.00 0.0 0.0 0.00 0.0 0.00 0.00 0.00; 0.35 0.3 0.3 0.25 0.2 0.15
        0.15 0.05; 0.35 0.3 0.3 0.25 0.2 0.15 0.15 0.05; 0.7 0.6 0.6 0.5 0.4 0.3 0.3
        0.1; 0 0 0 0 0 0 0 0; 0.35 0.3 0.3 0.25 0.2 0.15 0.15 0.05; 0.35 0.3 0.3 0.25 0.2
        0.15 0.15 0.05; 0.7 0.6 0.6 0.5 0.4 0.3 0.3 0.1')

    # loop over the total number of points to generate
    for t in range(0,T):

        ### BEGIN Q-CHECKER #####

        for i in range(0,3):
```

```
    r[i] = np.random.beta(0.5,0.5) # generate frequencies

Norm_val = np.sum(r) # normalize them

for i in range(0,3):
    x_a[i] = r[i]/Norm_val # normalize and store

# do some more fancy X calculations
x_b1 = np.array([x_a[0], x_a[2], 0 , 0])
x_b2 = np.array([x_a[1], 0, 0, 0])
x_c1 = x_b1[np.random.permutation(x_b1.size)]
x_c2 = x_b2[np.random.permutation(x_b2.size)]

for i in range(0,8):
    if i < 4:
        x[i] = x_c1[i]
    else:
        x[i] = x_c2[i-4]

# caclulate yi and yi_not
for i in range(0,4):
    y[i] = x[i]/np.sum(x[0:4])

for i in range(4,8):
    y[i] = x[i]/np.sum(x[4:8])

px = p - np.dot(x,matrixA) # calculate px

# start with computing Q
for w in range(0,4):
    for i in range(0,4):
        yp[i] = y[i+4]*px[0,i+4]
        yp_not[i] = y[i]+(1-px[0,i])

    Q = np.sum(yp)/(np.sum(yp_not) + np.sum(yp)) # compute Q
    Q_not = 1-Q
    Q_vec[w] = Q

    # adjust frequency vector based on found Q
    for i in range(0,4):
        xstar[i] = Q*y[i]

    for i in range(4,8):
        xstar[i] = Q_not*y[i]

    px = p - np.dot(xstar,matrixA) # adjust PX

# apply Aitken's delta squared
Q_check_1 = Q_vec[0]-((Q_vec[1] - Q_vec[0])**2) / (Q_vec[2] - 2*Q_vec[1] + Q_vec
    [0])
Q_check_2 = Q_vec[1]-((Q_vec[2]- Q_vec[1])**2) / (Q_vec[3] - 2*Q_vec[2] + Q_vec
    [1])
diff = Q_check_1 - Q_check_2

# if the distribution has not settled, then infinite calculate a new Q
while diff > abs(1e-8):

    Q_check_1 = Q_check_2

    # new calculations of yi
    for i in range(0,4):
```

```
                yp[i] = y[i+4]*px[0,i+4]
                yp_not[i] = y[i]*(1-px[0,i])

            Q = (np.sum(yp)) / (np.sum(yp_not) + np.sum(yp)) # again calculate Q
            Q_not = 1-Q
            Q_vec[w+1] = Q

            # adjust frequency vector X
            for i in range(0,4):
                xstar[i] = Q*y[i]

            for i in range(4,8):
                xstar[i] = Q_not*y[i]

            px = p - np.dot(xstar,matrixA) # px calculations

            # compute new Q based on aitken's
            Q_check_2 = Q_vec[w-1] - ((Q_vec[w]-Q_vec[w-1])**2)/(Q_vec[w+1] - 2*Q_vec[w]
                + Q_vec[w-1])
            diff = Q_check_1 - Q_check_2
            w = w+1

        # apply FD function if activated
        if FD_yn == True:
            FD = 1-0.25*(xstar[1]+xstar[2])-(1/3)*xstar[3]-(1/2)*(xstar[5] + xstar[6]) -
                (2/3) * xstar[7]
        else:
            FD = 1

        # calculate payoffs
        for i in range(0,8):
            v_p1[i] = xstar[i]*A1[i]
            v_p2[i] = xstar[i]*B1[i]

        # calculate definitive rewards based on FD function
        payoff_p1[t] = FD*np.sum(v_p1)
        payoff_p2[t] = FD*np.sum(v_p2)

    # store maximal payoffs
    self.maximal_payoffs = np.zeros(2)
    self.maximal_payoffs = [np.max(payoff_p1),np.max(payoff_p2)]

    all_payoffs = np.array([payoff_p1,payoff_p2])  #payoffs player 1 and and p2 merging
    all_payoffs = np.transpose(all_payoffs)        #transpose for use in convex_hull
    Convex_Hull_Payoffs = ConvexHull(all_payoffs)  #calculate convex_hull of the payoffs

    # plot a nice convex hull
    plt.fill(all_payoffs[Convex_Hull_Payoffs.vertices,0], all_payoffs[Convex_Hull_Payoffs
        .vertices,1], color='y', zorder=5, label="Obtainable rewards")


def plot_threat_point(self):
    "This function plots the threat point if found"

    plt.scatter(self.threat_point[0],self.threat_point[1], zorder=10, color = 'r', label=
        'Threat point')
    plt.legend()

def plot_threat_point_lines(self):
    "This function plots lines around the threat point indicating the limits for the NE"
```

```
        plt.plot([self.threat_point[0],self.threat_point[0]],[self.threat_point[1],self.
            maximal_payoffs[1]], color='k', zorder=15)
        plt.plot([self.threat_point[0],self.maximal_payoffs[0]],[self.threat_point[1],self.
            threat_point[1]], color='k', zorder=15)

    def aitken_delta_squared(self,q1,q2,q3):
        "This is the Aitken's Delta Squared accelerator"

        x3_x2 = np.subtract(q3,q2)
        x2_x1 = np.subtract(q2,q1)

        x3_x2_squared = np.power(x3_x2,2)
        denominator = np.subtract(x3_x2,x2_x1)

        fraction = np.divide(x3_x2_squared,denominator)

        return np.subtract(q3,fraction)


    def optimized_maximin(self,points,show_strat_p1,show_strat_p2,FD_yn):
        "This is an optimized version for determining the maximin result"

        print("Start of the maximin algorithm")

        def random_strategy_draw(points,number_of_actions):
            "This function draws random strategies from a beta distribution, based on the
                number of points and actions"

            # draw some strategies and normalize them
            strategies_drawn = np.random.beta(0.5,0.5,(points,number_of_actions))
            strategies_drawn = strategies_drawn/np.sum(strategies_drawn, axis=1).reshape([
                points,1])

            return strategies_drawn

        def frequency_pairs_p1(points,p2_actions,p1_actions,strategies_drawn):
            "Create strategies based on the best replies for player 1"

            # store the size of the games
            game_size_1 = self.payoff_p1_game1.size
            game_size_2 = self.payoff_p1_game2.size

            # store the actions for game 1 and 2
            p1_actions_game1 = self.payoff_p1_game1.shape[0]
            p1_actions_game2 = self.payoff_p1_game2.shape[0]

            # calculate the combination of the actions and a range
            p1_actions_combi = p1_actions_game1*p1_actions_game2
            p1_action_range = np.arange(p1_actions_combi)

            # initialize a frequency pair
            frequency_pairs = np.zeros((points*(p1_actions_game1*p1_actions_game2),
                game_size_1+game_size_2))

            # create actions ranges
            p1_act_game1_range = np.arange(p1_actions_game1)
            p1_act_game2_range = np.arange(p1_actions_game2)

            # loop over best responses for game 1
            for i in np.nditer(p1_action_range):
                for j in np.nditer(p1_act_game1_range):
```

```
                mod_remain = np.mod(i,p1_actions_game1)
                frequency_pairs[i*points:(i+1)*points,p1_actions_game1*mod_remain+j] =
                    strategies_drawn[:,j]

        # loop over best responses for game 2
        for i in np.nditer(p1_action_range):
            for j in np.nditer(p1_act_game2_range):
                floor_div = np.floor_divide(i,p1_actions_game2)
                frequency_pairs[i*points:(i+1)*points,j+game_size_1+(p1_actions_game1*
                    floor_div)] = strategies_drawn[:,p1_actions_game1+j]

        return frequency_pairs

    def balance_equation(self,tot_act_ut,tot_act_thr,tot_payoffs_game1,tot_payoffs,
        frequency_pairs):
        "Calculates the result of the balance equations in order to adjust the frequency
            pairs"

        # store the game sizes
        game_size_1 = self.payoff_p1_game1.size
        game_size_2 = self.payoff_p1_game2.size

        # initialize yi, Q and Q_new
        yi = np.zeros((points*(tot_act_thr*tot_act_ut),game_size_1+game_size_2))
        Q = np.zeros((1,points*(tot_act_thr*tot_act_ut)))
        Q_new = np.zeros((1,points*(tot_act_thr*tot_act_ut)))

        # compute yi
        yi[:,0:tot_payoffs_game1] = frequency_pairs[:,0:tot_payoffs_game1]/np.sum(
            frequency_pairs[:,0:tot_payoffs_game1], axis=1).reshape([points*
            tot_payoffs_game1,1])
        yi[:,tot_payoffs_game1:tot_payoffs] = frequency_pairs[:,tot_payoffs_game1:
            tot_payoffs]/np.sum(frequency_pairs[:,tot_payoffs_game1:tot_payoffs], axis=1)
            .reshape([points*(tot_payoffs-tot_payoffs_game1),1])

        index_values = np.arange(points*(tot_act_thr*tot_act_ut)) # set a range

        # set px range
        p1_px_between = np.asarray(px)
        p1_px = p1_px_between[0]

        # loop for 35 iterations
        for i in np.arange(35):

            # in the first iteration we calculate the first Q and adjust X
            if i == 0:
                new_x = p1_px - np.dot(frequency_pairs,self.etp_matrix)

                upper_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs],
                    new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)
                leftdown_part_Q = np.sum(np.multiply(yi[:,0:tot_payoffs_game1],(1-new_x
                    [:,0:tot_payoffs_game1])),axis=1)
                rightdown_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs
                    ],new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)

                Q_between = upper_part_Q/(leftdown_part_Q+rightdown_part_Q)
                Q = Q_between

                frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q,yi[:,0:
                    tot_payoffs_game1]))
                frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q),yi
```

```
            [:,tot_payoffs_game1:tot_payoffs])

        # here we just calculate Q in order to guarantee stability
        if i > 0 and i < 10:
            new_x = p1_px - np.dot(frequency_pairs,self.etp_matrix)

            upper_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs],
                new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)
            leftdown_part_Q = np.sum(np.multiply(yi[:,0:tot_payoffs_game1],(1-new_x
                [:,0:tot_payoffs_game1])),axis=1)
            rightdown_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs
                ],new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)

            Q_between = upper_part_Q/(leftdown_part_Q+rightdown_part_Q)
            Q = np.hstack((Q,Q_between))

            frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q[:,i],yi[:,0:
                tot_payoffs_game1]))
            frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q[:,i])
                ,yi[:,tot_payoffs_game1:tot_payoffs])

        # here we calculate Q based on aitken's delta squared
        if i == 10:

            Q_new = self.aitken_delta_squared(Q[:,i-3],Q[:,i-2],Q[:,i-1])
            nan_org = np.where(np.isnan(Q_new)) # check where there are NaN's
            nan_indic = nan_org[0]
            Q_new[nan_indic,:] = Q_between[nan_indic,:] # remove NaN's with last
                known values

            Q_old = np.copy(Q_new)

            Q = np.hstack((Q,Q_new))

        # all remaining iterations are with Aitkens
        if i > 10:

            Q_new[index_values,:] = self.aitken_delta_squared(Q[index_values,i-3],Q[
                index_values,i-2],Q[index_values,i-1])
            Q_old2 = np.copy(Q_old)
            nan_res = np.where(np.isnan(Q_new,Q_old)) # check for NaN's
            nan_indices = nan_res[0] # look where NaN's are

            # delete values which are NaN for future computations
            nan_between = np.where(np.in1d(index_values,nan_indices))
            nan_cands = nan_between[0]
            index_values = np.delete(index_values,nan_cands)

            Q_new[nan_indices,:] = Q_old2[nan_indices,:]

            Q = np.hstack((Q,Q_new))

            Q_old = np.copy(Q_new)
            results = np.where(Q[index_values,i-1] == Q[index_values,i]) # check
                where convergence has occured
            remove_indices = results[0]

            # remove indices which have converged
            removal_between = np.where(np.in1d(index_values,remove_indices))
            removal_cands = removal_between[0]
            index_values = np.delete(index_values,removal_cands)
```

```
        # compute the definitive frequency pair x
        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q[:,34],yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q[:,34]),yi[:,
            tot_payoffs_game1:tot_payoffs])

        return frequency_pairs

    def frequency_pairs_p2(points,p2_actions,p1_actions,strategies_drawn):
        "Best responses for P2 based on threaten strategies drawn"

        # store the game sizes
        game_size_1 = self.payoff_p2_game1.size
        game_size_2 = self.payoff_p2_game2.size

        # store the actions for p1 and p2 and create ranges
        p1_actions_range = np.arange(p1_actions)
        p2_actions_range = np.arange(p2_actions)

        p2_actions_game1 = self.payoff_p2_game1.shape[1]
        p2_actions_game2 = self.payoff_p2_game2.shape[1]

        p2_actions_combo = p2_actions_game1*p2_actions_game2
        p2_action_range = np.arange(p2_actions_combo)

        # initialize the frequency pair
        frequency_pairs = np.zeros((points*(p2_actions_game1*p2_actions_game2),
            game_size_1+game_size_2))

        # generate best responses for game 1
        for i in np.nditer(np.arange(p2_actions_game1)):
            for j in np.nditer(p2_action_range):
                modul = np.mod(j,p2_actions_game1)
                frequency_pairs[j*points:(j+1)*points,p2_actions_game1*i+modul] =
                    strategies_drawn[:,i]

        # generate best respones for game 2
        for i in np.nditer(np.arange(p2_actions_game2)):
            for j in np.nditer(p2_action_range):
                divide = np.floor_divide(j,p2_actions_game2)
                frequency_pairs[j*points:(j+1)*points,p2_actions_combo+divide+(i*
                    p2_actions_game2)] = strategies_drawn[:,i+p2_actions_game1]

        return frequency_pairs

    def payoffs_sorted(points,payoffs,actions):
        "Sort the payoffs for determination of maximin"

        # store the range of points and actions
        points_range = np.arange(points)
        actions_range = np.arange(actions)

        payoffs_sort = np.zeros((points,actions)) # initialize the payoffs sort

        # sort the payoffs!
        for x in np.nditer(points_range):
            for i in np.nditer(actions_range):
                payoffs_sort[x,i] = payoffs[points*i+x]

        return payoffs_sort
```

```
## Start of p1 maximin ##

start_time = time.time() # START TIME

# flatten the transition matrices
flatten1_1 = self.transition_matrix_game1_to1.flatten()
flatten2_1 = self.transition_matrix_game2_to1.flatten()

# store and compute some action stuff
actions_p2_game1 = self.payoff_p1_game1.shape[1]
actions_p2_game2 = self.payoff_p1_game2.shape[1]
total_actions_p2 = actions_p2_game1 + actions_p2_game2

actions_p1_game1 = self.payoff_p1_game1.shape[0]
actions_p1_game2 = self.payoff_p1_game2.shape[0]
total_actions_p1 = actions_p1_game1 + actions_p1_game2

# flatten the payoffs
payoff_p1_game_1flatten = self.payoff_p1_game1.flatten()
payoff_p1_game_2flatten = self.payoff_p1_game2.flatten()

# compute and store some payoffs stuff
total_payoffs_p1_game1 = payoff_p1_game_1flatten.size
total_payoffs_p1_game2 = payoff_p1_game_2flatten.size
total_payoffs_p1 = total_payoffs_p1_game1 + total_payoffs_p1_game2

payoff_p2_game_1flatten = self.payoff_p2_game1.flatten()
payoff_p2_game_2flatten = self.payoff_p2_game2.flatten()

total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

# initialize the payoff stuff for p1
payoff_p1 = np.zeros(total_payoffs_p1)
payoff_p1[0:total_payoffs_p1_game1] = payoff_p1_game_1flatten
payoff_p1[total_payoffs_p1_game1:total_payoffs_p1] = payoff_p1_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # px for the first time

y_punisher = random_strategy_draw(points,total_actions_p1) # draw some strategies

frequency_pairs = frequency_pairs_p2(points,total_actions_p1,total_actions_p2,
    y_punisher) # sort them based on best replies

# do the balance equations with Aitken's
frequency_pairs = balance_equation(self,actions_p2_game1,actions_p2_game2,
    total_payoffs_p2_game1,total_payoffs_p2,frequency_pairs)

# activate FD function if necessary
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1
```

```
# calculate the payoffs
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p1),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p2_game1*actions_p2_game2)) #
    sort the payoffs

nan_delete = np.where(np.isnan(max_payoffs)) # delete results which are NaN (see
    thesis why)

max_payoffs = np.delete(max_payoffs,nan_delete[0],0) # actually delete these payoffs

print("")
print("")
minimax_found = np.nanmax(np.nanmin(max_payoffs,axis=1)) # look for maximin value
print("Maximin value for P1 is",minimax_found)
print("")
print("")

if show_strat_p1 == True:
    minimax_indices_p2 = np.where(max_payoffs == minimax_found)
    found_strategy_p2 = y_punisher[minimax_indices_p2[0]]
    fnd_strategy_p2 = found_strategy_p2.flatten()
    fnd_strategy_p2[0:2] = fnd_strategy_p2[0:2]/np.sum(fnd_strategy_p2[0:2])
    fnd_strategy_p2[2:4] = fnd_strategy_p2[2:4]/np.sum(fnd_strategy_p2[2:4])
    print("Player 1 plays stationary strategy:", fnd_strategy_p2)
    print("While player 2 replies with a best pure reply of:", self.
        best_pure_strategies[minimax_indices_p2[1]])

end_time = time.time()
print("Seconds done to generate", points, "points", end_time-start_time)

## End of P1 maximin algorithm ##

start_time_p2 = time.time() # start the time

# flatten the payoffs
payoff_p2_game_1flatten = self.payoff_p2_game1.flatten()
payoff_p2_game_2flatten = self.payoff_p2_game2.flatten()

# compute and store the payoffs
total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

# initialize the payoffs and store them
payoff_p2 = np.zeros(total_payoffs_p2)
payoff_p2[0:total_payoffs_p2_game1] = payoff_p2_game_1flatten
payoff_p2[total_payoffs_p2_game1:total_payoffs_p2] = payoff_p2_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # px store

x_punisher = random_strategy_draw(points,total_actions_p2) # generate new random
    strategies for punsher

frequency_pairs = frequency_pairs_p1(points,total_actions_p1,total_actions_p2,
    x_punisher) # best reponses p1

# balance equations with Delta Squared
```

```
        frequency_pairs = balance_equation(self,actions_p1_game1,actions_p1_game2,
            total_payoffs_p1_game1,total_payoffs_p1,frequency_pairs)

        # activate FD function if necessary
        if FD_yn == True:
            FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
                [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
                frequency_pairs[:,7]
        else:
            FD = 1

        # compute the payoffs with payoffs and FD function
        payoffs = np.sum(np.multiply(frequency_pairs,payoff_p2),axis=1)
        payoffs = np.multiply(FD,payoffs)
        payoffs = payoffs.reshape((payoffs.size,1))

        max_payoffs = payoffs_sorted(points,payoffs,(actions_p1_game1*actions_p1_game2)) #
            sort the payoffs

        nan_delete = np.where(np.isnan(max_payoffs)) # check where there are nan's

        max_payoffs = np.delete(max_payoffs,nan_delete[0],0) # delete these nan's

        minimax_found_p2 = np.nanmax(np.nanmin(max_payoffs,axis=1)) # find the maximin value
            for p2
        print("Maximin value for P2 is",minimax_found_p2)
        print("")
        print("")


        if show_strat_p2 == True:
            maximin_indices_p2 = np.where(max_payoffs == minimax_found_p2)
            found_strategy = x_punisher[maximin_indices_p2[0]]
            fnd_strategy = found_strategy.flatten()
            fnd_strategy[0:2] = fnd_strategy[0:2]/np.sum(fnd_strategy[0:2])
            fnd_strategy[2:4] = fnd_strategy[2:4]/np.sum(fnd_strategy[2:4])
            print("Player 2 plays stationairy strategy:", fnd_strategy)
            print("While player 2 replies with a best pure reply of:", self.
                best_pure_strategies[maximin_indices_p2[1]])

        end_time_p2 = time.time() # end the timer
        print("Seconds done to generate", points, "points", end_time_p2-start_time_p2)
        print("")
        print("")

    def threat_point_optimized(self,points,show_strat_p1,show_strat_p2,print_text,FD_yn):
        "OPtimized threat point algorithm for ETP games"

        def random_strategy_draw(points,number_of_actions):
            "This function draws random strategies from a beta distribution, based on the
                number of points and actions"

            # draw some strategies and normalize them
            strategies_drawn = np.random.beta(0.5,0.5,(points,number_of_actions))
            strategies_drawn = strategies_drawn/np.sum(strategies_drawn, axis=1).reshape([
                points,1])

            return strategies_drawn

        def frequency_pairs_p1(points,p2_actions,p1_actions,strategies_drawn):
            "This function sorts the strategies based on the responses"
```

```
        # store the game size
        game_size_1 = self.payoff_p1_game1.size
        game_size_2 = self.payoff_p1_game2.size

        # store the actions of p1 in both game
        p1_actions_game1 = self.payoff_p1_game1.shape[0]
        p1_actions_game2 = self.payoff_p1_game2.shape[0]

        p1_actions_combi = p1_actions_game1*p1_actions_game2
        p1_action_range = np.arange(p1_actions_combi)

        # initialize frequency pairs
        frequency_pairs = np.zeros((points*(p1_actions_game1*p1_actions_game2),
            game_size_1+game_size_2))

        # set the range for both games
        p1_act_game1_range = np.arange(p1_actions_game1)
        p1_act_game2_range = np.arange(p1_actions_game2)

        # create best response for game 1
        for i in np.nditer(p1_action_range):
            for j in np.nditer(p1_act_game1_range):
                mod_remain = np.mod(i,p1_actions_game1)
                frequency_pairs[i*points:(i+1)*points,p1_actions_game1*mod_remain+j] =
                    strategies_drawn[:,j]

        # loop for best responses for game 2
        for i in np.nditer(p1_action_range):
            for j in np.nditer(p1_act_game2_range):
                floor_div = np.floor_divide(i,p1_actions_game2)
                frequency_pairs[i*points:(i+1)*points,j+game_size_1+(p1_actions_game1*
                    floor_div)] = strategies_drawn[:,p1_actions_game1+j]
        return frequency_pairs

    def balance_equation(self,tot_act_ut,tot_act_thr,tot_payoffs_game1,tot_payoffs,
        frequency_pairs):
        "Calculates the result of the balance equations in order to adjust the frequency
            pairs"

        # store the game sizes
        game_size_1 = self.payoff_p1_game1.size
        game_size_2 = self.payoff_p1_game2.size

        # initialize yi, Q and Q_new
        yi = np.zeros((points*(tot_act_thr*tot_act_ut),game_size_1+game_size_2))
        Q = np.zeros((1,points*(tot_act_thr*tot_act_ut)))
        Q_new = np.zeros((1,points*(tot_act_thr*tot_act_ut)))

        # compute Yi
        yi[:,0:tot_payoffs_game1] = frequency_pairs[:,0:tot_payoffs_game1]/np.sum(
            frequency_pairs[:,0:tot_payoffs_game1], axis=1).reshape([points*
            tot_payoffs_game1,1])
        yi[:,tot_payoffs_game1:tot_payoffs] = frequency_pairs[:,tot_payoffs_game1:
            tot_payoffs]/np.sum(frequency_pairs[:,tot_payoffs_game1:tot_payoffs], axis=1)
            .reshape([points*(tot_payoffs-tot_payoffs_game1),1])

        index_values = np.arange(points*(tot_act_thr*tot_act_ut)) # create a range of
            index values

        p1_px_between = np.asarray(px) # set px
```

```
p1_px = p1_px_between[0]

# iterate for 35 iterations
for i in np.arange(35):

    # first iteration, just calculate Q
    if i == 0:
        new_x = p1_px - np.dot(frequency_pairs,self.etp_matrix)

        upper_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs],
            new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)
        leftdown_part_Q = np.sum(np.multiply(yi[:,0:tot_payoffs_game1],(1-new_x
            [:,0:tot_payoffs_game1])),axis=1)
        rightdown_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs
            ],new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)

        Q_between = upper_part_Q/(leftdown_part_Q+rightdown_part_Q)
        Q = Q_between

        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q,yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q),yi
            [:,tot_payoffs_game1:tot_payoffs])

    # for stability, calculate until iteration 9 normal Q
    if i > 0 and i < 10:
        new_x = p1_px - np.dot(frequency_pairs,self.etp_matrix)

        upper_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs],
            new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)
        leftdown_part_Q = np.sum(np.multiply(yi[:,0:tot_payoffs_game1],(1-new_x
            [:,0:tot_payoffs_game1])),axis=1)
        rightdown_part_Q = np.sum(np.multiply(yi[:,tot_payoffs_game1:tot_payoffs
            ],new_x[:,tot_payoffs_game1:tot_payoffs]),axis=1)

        Q_between = upper_part_Q/(leftdown_part_Q+rightdown_part_Q)
        Q = np.hstack((Q,Q_between))

        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q[:,i],yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q[:,i])
            ,yi[:,tot_payoffs_game1:tot_payoffs])

    # apply Aitken's
    if i == 10:

        Q_new = self.aitken_delta_squared(Q[:,i-3],Q[:,i-2],Q[:,i-1])
        nan_org = np.where(np.isnan(Q_new)) # check whether Nan's occur
        nan_indic = nan_org[0]

        Q_new[nan_indic,:] = Q_between[nan_indic,:] # replace NaN with last known
             value

        Q_old = np.copy(Q_new)

        Q = np.hstack((Q,Q_new))

    # and only Aitken's
    if i > 10:

        Q_new[index_values,:] = self.aitken_delta_squared(Q[index_values,i-3],Q[
```

```
                    index_values,i-2],Q[index_values,i-1])
                Q_old2 = np.copy(Q_old)
                nan_res = np.where(np.isnan(Q_new,Q_old)) # check for NaN's
                nan_indices = nan_res[0]

                nan_between = np.where(np.in1d(index_values,nan_indices))
                nan_cands = nan_between[0]

                index_values = np.delete(index_values,nan_cands) # delete NaN's after
                    being returned in last known

                Q_new[nan_indices,:] = Q_old2[nan_indices,:]

                Q = np.hstack((Q,Q_new))

                Q_old = np.copy(Q_new)
                results = np.where(Q[index_values,i-1] == Q[index_values,i]) # check
                    whether Q converged
                remove_indices = results[0]

                removal_between = np.where(np.in1d(index_values,remove_indices))
                removal_cands = removal_between[0]
                index_values = np.delete(index_values,removal_cands)

        # compute definitive x
        frequency_pairs[:,0:tot_payoffs_game1] = (np.multiply(Q[:,34],yi[:,0:
            tot_payoffs_game1]))
        frequency_pairs[:,tot_payoffs_game1:tot_payoffs] = np.multiply((1-Q[:,34]),yi[:,
            tot_payoffs_game1:tot_payoffs])

        return frequency_pairs

    def frequency_pairs_p2(points,p2_actions,p1_actions,strategies_drawn):
        "Create␣frequency␣pairs␣for␣P2␣based␣on␣best␣responses"

        # store the size of the games
        game_size_1 = self.payoff_p2_game1.size
        game_size_2 = self.payoff_p2_game2.size

        # store the ranges of the actions of both players
        p1_actions_range = np.arange(p1_actions)
        p2_actions_range = np.arange(p2_actions)

        p2_actions_game1 = self.payoff_p2_game1.shape[1]
        p2_actions_game2 = self.payoff_p2_game2.shape[1]

        p2_actions_combo = p2_actions_game1*p2_actions_game2
        p2_action_range = np.arange(p2_actions_combo)

        # initialize frequency pairs
        frequency_pairs = np.zeros((points*(p2_actions_game1*p2_actions_game2),
            game_size_1+game_size_2))

        # loop over the first game
        for i in np.nditer(np.arange(p2_actions_game1)):
            for j in np.nditer(p2_action_range):
                modul = np.mod(j,p2_actions_game1)
                frequency_pairs[j*points:(j+1)*points,p2_actions_game1*i+modul] =
                    strategies_drawn[:,i]
        # loop over the second game
        for i in np.nditer(np.arange(p2_actions_game2)):
```

```
            for j in np.nditer(p2_action_range):
                divide = np.floor_divide(j,p2_actions_game2)
                frequency_pairs[j*points:(j+1)*points,p2_actions_combo+divide+(i*
                    p2_actions_game2)] = strategies_drawn[:,i+p2_actions_game1]

        return frequency_pairs

    def payoffs_sorted(points,payoffs,actions):
        "This function sorts the payoffs in order to prepare the threat point"

        # create ranges for points and actions
        points_range = np.arange(points)
        actions_range = np.arange(actions)

        payoffs_sort = np.zeros((points,actions)) # nitialize the payoffs sort

        # sort the payoffs!
        for x in np.nditer(points_range):
            for i in np.nditer(actions_range):
                payoffs_sort[x,i] = payoffs[points*i+x]

        return payoffs_sort

    if print_text == True:
        print("The start of the algorithm for finding the threat point")
        print("First let's find the threat point for Player 1")

    # flatten the transition matrices
    flatten1_1 = self.transition_matrix_game1_to1.flatten()
    flatten2_1 = self.transition_matrix_game2_to1.flatten()

    #  store the actions for both players
    actions_p2_game1 = self.payoff_p1_game1.shape[1]
    actions_p2_game2 = self.payoff_p1_game2.shape[1]
    total_actions_p2 = actions_p2_game1 + actions_p2_game2

    actions_p1_game1 = self.payoff_p1_game1.shape[0]
    actions_p1_game2 = self.payoff_p1_game2.shape[0]
    total_actions_p1 = actions_p1_game1 + actions_p1_game2

    # Start of algorithm for player 1

    start_time = time.time() # timer start

    # flatten payoffs game 1 and 2
    payoff_p1_game_1flatten = self.payoff_p1_game1.flatten()
    payoff_p1_game_2flatten = self.payoff_p1_game2.flatten()

    # store size of the payoffs
    total_payoffs_p1_game1 = payoff_p1_game_1flatten.size
    total_payoffs_p1_game2 = payoff_p1_game_2flatten.size
    total_payoffs_p1 = total_payoffs_p1_game1 + total_payoffs_p1_game2

    # initialize and assign payoffs
    payoff_p1 = np.zeros(total_payoffs_p1)
    payoff_p1[0:total_payoffs_p1_game1] = payoff_p1_game_1flatten
    payoff_p1[total_payoffs_p1_game1:total_payoffs_p1] = payoff_p1_game_2flatten

    px = np.concatenate([flatten1_1,flatten2_1],axis=1) # store px

    y_punisher = random_strategy_draw(points,total_actions_p2) # draw strategies for the
```

```
    punisher

frequency_pairs = frequency_pairs_p1(points,total_actions_p2,total_actions_p1,
    y_punisher) # sort based on best reply

# do the balance equations calculations
frequency_pairs = balance_equation(self,actions_p1_game1,actions_p1_game2,
    total_payoffs_p1_game1,total_payoffs_p1,frequency_pairs)

# activate the FD function
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# calculate the payoffs with the frequency pairs and FD function
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p1),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p1_game1*actions_p1_game2)) #
    sort the payoffs
nan_delete = np.where(np.isnan(max_payoffs)) # delete payoffs which are a NaN

max_payoffs_p1 = np.delete(max_payoffs,nan_delete[0],0) # actually delete them
threat_point_p1 = np.nanmin(np.nanmax(max_payoffs_p1,axis=1)) # determine the threat
    point

if print_text == True:
    print("")
    print("")
    print("Threat␣point␣value␣is",threat_point_p1)
    print("")
    print("")

if show_strat_p1 == True:
    threat_point_indices_p1 = np.where(max_payoffs_p1 == threat_point_p1)
    found_strategy_p1 = y_punisher[threat_point_indices_p1[0]]
    fnd_strategy_p1 = found_strategy_p1.flatten()
    fnd_strategy_p1[0:2] = fnd_strategy_p1[0:2]/np.sum(fnd_strategy_p1[0:2])
    fnd_strategy_p1[2:4] = fnd_strategy_p1[2:4]/np.sum(fnd_strategy_p1[2:4])
    print("Player␣2␣plays␣stationary␣strategy:", fnd_strategy_p1)
    print("While␣player␣1␣replies␣with␣a␣best␣pure␣reply␣of:", self.
        best_pure_strategies[threat_point_indices_p1[1]])

end_time = time.time() # stop the time!

if print_text == True:
    print("Seconds␣done␣to␣generate", points, "points", end_time-start_time)
    print("")

# End of algorithm player 1

# Start of algorithm player 2

if print_text == True:
    print("")
    print("")
    print("First␣start␣the␣threat␣point␣for␣player␣2")
```

```
start_time_p2 = time.time() # start the time (for p2)

# flatten the payoffs of the gamew
payoff_p2_game_1flatten = self.payoff_p2_game1.flatten()
payoff_p2_game_2flatten = self.payoff_p2_game2.flatten()

# check the sizes of the total payoffs
total_payoffs_p2_game1 = payoff_p2_game_1flatten.size
total_payoffs_p2_game2 = payoff_p2_game_2flatten.size
total_payoffs_p2 = total_payoffs_p2_game1 + total_payoffs_p2_game2

# initialize the payoffs for p2 and assign them
payoff_p2 = np.zeros(total_payoffs_p2)
payoff_p2[0:total_payoffs_p2_game1] = payoff_p2_game_1flatten
payoff_p2[total_payoffs_p2_game1:total_payoffs_p2] = payoff_p2_game_2flatten

px = np.concatenate([flatten1_1,flatten2_1],axis=1) # trix with px

x_punisher = random_strategy_draw(points,total_actions_p1) # draw some awesome
    strategies

frequency_pairs = frequency_pairs_p2(points,total_actions_p2,total_actions_p1,
    x_punisher) # sort them based on best replies

# do some balance equation accelerator magic
frequency_pairs = balance_equation(self,actions_p2_game1,actions_p2_game2,
    total_payoffs_p2_game1,total_payoffs_p2,frequency_pairs)

# if the FD function must be activated, activate it
if FD_yn == True:
    FD = 1-0.25*(frequency_pairs[:,1]+frequency_pairs[:,2])-(1/3)*frequency_pairs
        [:,3]-(1/2)*(frequency_pairs[:,5] + frequency_pairs[:,6]) - (2/3) *
        frequency_pairs[:,7]
else:
    FD = 1

# payoffs are calculated
payoffs = np.sum(np.multiply(frequency_pairs,payoff_p2),axis=1)
payoffs = np.multiply(FD,payoffs)
payoffs = payoffs.reshape((payoffs.size,1))

max_payoffs = payoffs_sorted(points,payoffs,(actions_p2_game1*actions_p2_game2)) #
    awesome sorting process
nan_delete = np.where(np.isnan(max_payoffs)) # look for NaN's

max_payoffs_p2 = np.delete(max_payoffs,nan_delete[0],0) # delete them where necessary
threat_point_p2 = np.nanmin(np.nanmax(max_payoffs_p2,axis=1)) # determine the threat
    point

if print_text == True:
    print("")
    print("")
    print("Threat␣point␣value␣is",threat_point_p2)
    print("")
    print("")

if show_strat_p2 == True:
    threat_point_indices_p2 = np.where(max_payoffs_p2 == threat_point_p2)
    found_strategy = x_punisher[threat_point_indices_p2[0]]
    fnd_strategy = found_strategy.flatten()
    fnd_strategy[0:2] = fnd_strategy[0:2]/np.sum(fnd_strategy[0:2])
```

```
            fnd_strategy[2:4] = fnd_strategy[2:4]/np.sum(fnd_strategy[2:4])
            print("Player 1 plays stationairy strategy:", fnd_strategy)
            print("While player 2 replies with a best pure reply of:", self.
                best_pure_strategies[threat_point_indices_p2[1]])

        end_time_p2 = time.time() # stop the time

        if print_text == True:
            print("")
            print("Seconds done to generate", points, "points", end_time_p2-start_time_p2)
            print("")
            print("")

        self.threat_point = np.zeros(2)
        self.threat_point = [threat_point_p1,threat_point_p2] # store the threat point!

        return [threat_point_p1,threat_point_p2]
```

A.8 Type III Example Games Code

```
encoding=*-60
"ETP Example game as described in the thesis, based on a game developed by Llea Samuel"
p1_1 = np.matrix('16 14; 28 24')
p2_1 = np.matrix('16 28; 14 24')

p1_2 = np.matrix('4 3.5; 7 6')
p2_2 = np.matrix('4 7; 3.5 6')

trans1_1 = np.matrix('0.8 0.7; 0.7 0.6')
trans2_1 = np.matrix('0.5 0.4; 0.4 0.15')

trans1_2 = np.matrix('0.2 0.3; 0.3 0.4')
trans2_2 = np.matrix('0.5 0.6; 0.6 0.85')

matrixA = np.matrix('0.00 0.0 0.0 0.00 0.0 0.00 0.00 0.00; 0.35 0.3 0.3 0.25 0.2 0.15 0.15
    0.05; 0.35 0.3 0.3 0.25 0.2 0.15 0.15 0.05; 0.7 0.6 0.6 0.5 0.4 0.3 0.3 0.1; 0 0 0 0 0 0
    0 0; 0.35 0.3 0.3 0.25 0.2 0.15 0.15 0.05; 0.35 0.3 0.3 0.25 0.2 0.15 0.15 0.05; 0.7 0.6
    0.6 0.5 0.4 0.3 0.3 0.1')
matrixB = np.matrix('0.00 0.0 0.0 0.00 0.0 0.00 0.00 0.00; 0.0 0.0 0.0 0.00 0.0 0.0 0.0 0.00;
    0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0; 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0; 0 0 0 0 0 0 0 0; 0.0
    0.0 0.0 0.00 0.0 0.00 0.00 0.0; 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.00; 0.0 0.0 0.0 0.0 0.0 0.0 0
    0.0 0.0')

FirstTryETP = ETPGame(p1_1,p2_1,p1_2,p2_2,trans1_1,trans2_1,trans1_2,trans2_2,matrixA)
FirstTryETP.optimized_maximin(1000000,False,False,True)
FirstTryETP.threat_point_optimized(1000000,True,True,True,True)
FirstTryETP.plot_single_period_pure_rewards()
FirstTryETP.plot_all_rewards(True)
FirstTryETP.plot_threat_point()
FirstTryETP.plot_threat_point_lines()
```