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Investigating a Zero-One Law for Linear Temporal Logic Statements in Large Random Graphs

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Abstract

This article investigates the existence of a Zero-One law for Linear Temporal Logic in random graphs. A sample statement, *Always*, formulated in Linear Temporal Logic is tested in four different regimes of directed as well as undirected random graphs. For two of the regimes, the ones with the highest edge probability, the probability of an *Always* statement to hold tends to zero when the number of nodes of the graph tends to infinity. A Zero-One Law could not be disproved. For the remaining two regimes, when the graph has a tree-like structure, a counterexample of a Zero-One law is found for the undirected random graph. For the directed random graph a counterexample in this regime is assumed but not yet proven. In conclusion, a Zero-One law does not hold for Linear Temporal Logic in at least some regimes of random graphs.

Keywords: Binomial Random Graph, Linear Temporal Logic, Zero-One Law

1 Introduction

Linear Temporal Logic statements for graphs are widely used for model checking purposes [1]. It is used in model checking, when properties of a system, e.g. a software or a hardware system, needs to be expressed. If a property of these system needs to be checked, the system is modeled as a graph where a node is assigned to every state of the system. Properties are assigned to every node, for example one can assign a property 'critical' or 'non-critical' to every node, representing whether the corresponding state of the system is critical or non-critical. The property is tested on the graph. Linear Temporal Logic covers statements as 'Always' and 'Eventually'. For example, one wants to check whether a system fulfils the statement 'Eventually critical', to check whether the system can reach a 'critical' state. Linear Temporal Logic formulae are mainly checked using algorithms on existing graphs. However, due to the so called *state-explosion problem*, it is hard to model large systems by deterministic graphs. To still get an idea about the behaviour of these large systems, one uses random graphs with a large number of nodes. The random graphs are build on a vertex set n. Every possible edge between every pair of vertices exists with a given probability $p(n)$. If one choses a random graph in the same regime, i.e. choose a matching edge probability, as the system, the random grahp has the same behavior as the modeled system. The properties of these large random graphs can be an indication for the behavior of the large systems.
We know by [4, 9] that a Zero-One law for first order logic holds. That is, for a graph with

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a number of nodes tending to infinity, the probability that a first order logic statement holds will either tend to zero or to one. First order logic statements are of the form '*Is the graph connected?*' or '*Is there a triangle in the graph?*'. A more precise definition of the Zero-One law and first order logic can be found in the remainder of this article. Of interest is whether Linear Temporal Logic, as described in [1], satisfies a Zero-One law as well. If this is the case, all Linear Temporal Logic statements will either occur almost surely or almost never in large random graphs and one could draw interesting conclusions for systems with a large number of states.
In this article we will not focus on an entire overview of Linear Temporal Logic and a Zero-One law for this logic and we do not try to prove a Zero-One law for Linear Temporal Logic in general. Instead, we will focus our attention on a specific example statement in Linear Temporal Logic, the *Always* statement. We will investigate whether the *Always* statement meets a Zero-One law in certain regimes of random graphs. Herewith, we hope to achieve some insight in the behaviour of Linear Temporal Logic statements in large random graphs and the existence of a possible Zero-One law.

Problem Description Consider a random graph with n vertices, where n tends to infinity. Between each of the vertices there exists an edge with probability $0 < p(n) < 1$, independently of other edges. Assign to each of the vertices in the graph a *red* label with probability $0 < q(n) < 1$ or a *blue* label with probability $1 - q(n)$. What is the probability, for a random vertex S, that all possible paths starting at S only contain *red* vertices? That is, what is the probability of the vertex S satisfying *Always Red* when the number of nodes, n, of the graph tends to infinity? We will investigate the probability of *Always Red* for four different edge probabilities in undirected as well directed random graphs. The four edge probabilities are:

- $p(n) = c$, where $0 < c < 1$,
- $p(n) = c\frac{\log(n)}{n}$, where $0 < c \leq 1$,
- $p(n) = \frac{\lambda}{n}$ where $0 < \lambda < 1$
- $p(n) = \frac{\lambda}{n}$ where $\lambda > 1$.

In Section 2 we define structures used in the article. In Section 3 we will state preliminary results regarding the Zero-One law for first order logic, the behaviour of an undirected random graph varying its number of edges and some results on branching processes. Section 4 we will classify each of our edge probability regimes into different asymptotic behaviours. In Section 5 we determine the probability that the *Always Red* statement holds in the undirected random graph for all the edge probability regimes. In Section 6 we will give insight in the *Always Red* statement in directed random graphs.

2 Definitions

In this section we will give an overview of the formal definitions of the regularly used expressions in this article. We give the random graph definition described in [6].

Definition 2.1 (Binomial Undirected Random Graph). *The binomial undirected random graph $G(n, p)$ is constructed on the vertex set $\{1, ..., n\}$. Between every pair of vertices, an undirected edge is included with probability $p(n) \in (0, 1)$ and no edge is included with probability $1 - p(n)$. The $\binom{n}{2}$ possible edges are placed independently. Note that in such random graphs parallel edges are not allowed.*

Definition 2.2 (Binomial Directed Random Graph). *The binomial directed random graph $G(n,p)$ is constructed on the vertex set $\{1,...,n\}$. Between every ordered pair of vertices, a directed edge is included with probability $p(n) \in (0,1)$ and no edge is included with probability $1-p(n)$. The $n(n-1)$ possible edges are placed independently. [6]*

Given a random graph we can assign a degree value to each of the vertices.

Definition 2.3 (Vertex Degree). *For an undirected graphs the degree of a vertex, v, denoted by $d(v)$, is the number of adjacent vertices of v. This number is equal to the number of edges incident to v. For a directed graph one distinguishes between the in-degree of a vertex, denoted by $d_{in}(v)$, and the out-degree of a vertex, denoted by $d_{out}(v)$. The in-degree equals the number of ingoing edges, e.g. edges of the form $w \to v$, where w is any other vertex. The out-degree equals the number of edges of the form $v \to w$, that is edges starting at vertex v and ending at any other vertex w.*

To every vertex of a graph labels can be assigned. We only consider the labels *red* and *blue*. The labels are assigned using random vertex colouring as defines as follows.

Definition 2.4 (Random Vertex Colouring). *Consider a random graph $G(n,p)$. We say, $G(n,p)$ is a coloured graph with colour probability $q(n)$, if each vertex of $G(n,p)$ receives a* red *label with probability $q(n) \in (0,1)$ and a* blue *label with probability $1-q(n)$.*

To check whether a system holds a given property, we have to consider all possible states that can be reached from a starting state. Corresponding to a graph, we have to consider all possible paths starting at S. A path is defined as follows.

Definition 2.5 (Path). *A path starting at vertex v_1 is a sequence of states $< v_1, v_2, \cdots v_k >$, where k may be infinite, such that (v_i, v_{i+1}) is in the edge set for all $1 \leq i \leq k-1$.*

Aim of this article is to investigate whether a Zero-One law holds for Linear Temporal Logic. Linear Temporal Logic is defined in [1].

Definition 2.6 (Linear Temporal Logic). *Linear Temporal Logic (LTL) is constructed from atomic propositions AP, Boolean connectors (conjunction and negation) and two basic temporal modalities, $\bigcirc$ (pronounced 'next') and $\cup$ (pronounced 'until'). The statements are build using the following grammar rules:*

$$\varphi ::= true \quad | \quad a \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \bigcirc \varphi \quad | \quad \varphi_1 \cup \varphi_2$$

where $a \in AP$. A vertex S meets a statement φ, denoted by $S \vDash \varphi$, if all vertices in all possible paths, starting at S, meet the statement φ.

Statements in Linear Temporal Logic are of the form $S \vDash \varphi$, where S is a vertex of a graph. For example, the statement $S \vDash a$ holds if S has the property a. Thus $S \vDash red$ holds if the vertex S has the label *red*. The statement $S \vDash \bigcirc\varphi$ holds if the successors of S has the property φ. The statement $S \vDash \varphi_1 \cup \varphi_2$ holds if, starting in S, S and all it's successor states have the property φ_1 till the path has a vertex with property φ_2. For a more detailed description of Linear Temporal Logic we refer the reader to [1, Chapter 5]. We will investigate the Linear Temporal Logic statement *Always* in different random graph regimes. The *Always* statement is defined in [1].

Definition 2.7 (Always). *The Linear Temporal Logic statement '*Always*' is denoted by $\Box$. The statement '$S \vDash \Box\varphi$' holds if S and all its successor states have the property φ. The property '$S \vDash \Box\varphi$' fails if there is at least one vertex in a path starting at S that does not have the property φ.* Always *φ is satisfied if and only if it is not the case that $(\neg\varphi)$ holds at some point.*

The *Always Red* statement for a vertex S thus describes the property that all possible paths starting in S only pass through red vertices. The property fails if there exists a *blue* vertex in a path starting at S.

3 Background

In this section we will state some preliminary results content-covering the Zero-One law for first order logic, the random graph regimes of the Erdős-Rényi random graph and some useful theorems on branching processes. The Zero-One law for first order logic is of interest since we try to investigate a Zero-One Law for Linear Temporal Logic. The random graph regimes of the Erdős-Rényi random graph are our object of investigation, and branching properties help to approximate the structure of random graph in some regimes.

3.1 Zero-One Law

Fagin proved in [4] a Zero-One law for first order logic statements for graphs, as in Theorem 3.1. Of interest is whether the Linear Temporal Logic satisfies a Zero-One law, too. First order logic is defined as in [8].

Definition 3.1 (First Order Logic for Graphs). *The first order logic for graphs is build up from atomic formulae of the form $v_i = v_j$ and $v_i \sim v_j$. Here, v_i denote the underlying vertices of the graph. The two binary predicates, equality (=) and adjacency (∼), are assumed to be symmetric. Furthermore, adjacency is assumed to be anti-reflexive. A first order logic statement may contain the logical connectives conjunction, disjunction, negation and implication ($\wedge, \vee, \neg, \rightarrow$) and quantifiers ($\exists, \forall$).*

The Zero-One law for first order logic is defined as follows.

Theorem 3.1 (Zero-One Law for First Order Logic statements for Graphs). *Let $G(n,p)$ be a random graph with a number of vertices n and edge probability $p \in [0,1]$. Let $\mathcal{A}$ be a first order logic statement for graphs. Then, for all fixed p and statement $\mathcal{A}$*

$$\lim_{n\to\infty} P(G(n,p) \models \mathcal{A}) = 0 \, or \, 1 \quad .$$

That is, every first order sentence is either almost always true or almost always false for a random graph. In our set up, the *Always Red* statement defined in Section 2 cannot be expressed as a first order logic statement. Indeed, we divide the vertices in two distinct sets, a set with *red* vertices and a set with *blue* vertices. Therefore, we need a higher order logic to describe the *Always Red* property. In conclusion, Theorem 3.1 does not apply to Linear Temporal Logic statement.

3.2 Evolution of the Erdős-Rényi Random Graph

Erdős and Rényi described the process of evolution of the uniform random graph [3]. The uniform random graph is given by $G(n, M(n))$ where n is the number of vertices and $M(n)$ is the number of edges of the graph. The random graph is chosen uniformly out of all possible graphs with n vertices and $M(n)$ edges. Erdős and Rényi distinguished five phases through which the graph passes when $M(n)$ grows from 1 to $\binom{n}{2}$, the maximum number of edges. The thresholds for the five phases are:

Phase I : $M(n) \sim o(n) \quad ,$

Phase II : $M(n) \sim \tilde{c}\, n$ with $0 < \tilde{c} < \frac{1}{2} \quad ,$

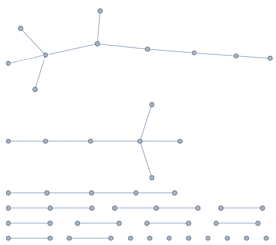
Phase III : $M(n) \sim \tilde{c}\, n$ with $\tilde{c} \geq \frac{1}{2} \quad ,$

Phase IV : $M(n) \sim \tilde{c}\, n \log n$ with $\tilde{c} \leq \frac{1}{2} \quad ,$

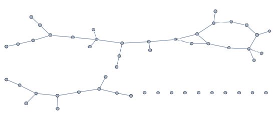
Phase V : $M(n) \sim (n \log n)\omega(n)$ with $\omega(n) \to \infty \quad .$

The graph changes throughout the phases from having mostly single isolated points in Phase I, to being completely connected in Phase V. We will describe the different Phases by explaining the behaviour of graphs in the different Phases and determining the average vertex degree, $\bar{d}$. Observe, that adding an edge to the graph increases the summed vertex degree of the graph, $\sum_{i=1}^{n} d(v_i)$, by two. We conclude $\sum_{i=1}^{n} d(v_i) = 2M(n)$. The average vertex degree is given by $\bar{d} = \frac{\sum_{i=1}^{n} d(v_i)}{n} = \frac{2M(n)}{n}$. In Figure 1 graphs in the different regimes, i.e. in different Phases, are illustrated.

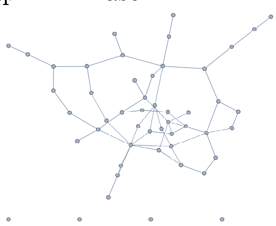
(A) $G(n, M(n))$ with $n = 50$ and $M(n) = 5$. The graph is an example of a graph in Phase I.



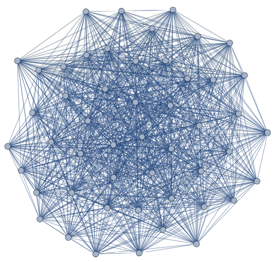
(B) $G(n, M(n))$ with $n = 50$ and $M(n) = 20$. The graph is an example of a graph in Phase II.



(C) $G(n, M(n))$ with $n = 50$ and $M(n) = 40$. The graph is an example of a graph in Phase III.



(D) $G(n, M(n))$ with $n = 50$ and $M(n) = 60$. The graph is an example of a graph in Phase IV.



(E) $G(n, M(n))$ with $n = 50$ and $M(n) = 1000$. The graph is an example of a graph in Phase V.

FIGURE 1: Randomly generated Uniform Random Graphs, $G(n, M(n))$, with different values of $M(n)$.

Phase I : A graph in Phase I is a graph with mostly isolated points. The average degree of a vertex tends to zero as the number of vertices tends to infinity. All vertices with degree larger than zero are with probability tending to one in a tree. In Figure 1a a typical graph in Phase I is shown. Most of the vertices are isolated. The vertices having edges are in components containing just two vertices.

Phase II : In Phase II the graph contains trees and cycles. However, almost all vertices will be in components which are trees. The average degree of the vertices is equal to $\bar{d} = 2\tilde{c}$. Consequently, $\bar{d} \in (0, 1)$. Figure 1b displays a random graph in Phase II with $\tilde{c} = 0.4$. Note, that isolated points can be seen as trees with a progeny of one. The graph in Figure 1b therefore consist of only trees.

Phase III : When $M(n)$ passes the threshold of Phase III the structure of the graph changes abruptly. The trees present in Phase II merge to one giant component. The graph now contains one giant connected component with a complex structure and some small trees. The giant component will contain about $G(\tilde{c})n$ vertices where

$$G(\tilde{c}) = 1 - \frac{1}{2\tilde{c}} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (2\tilde{c}e^{-2\tilde{c}})^k \quad .$$

The average degree of a vertex is equal to $\bar{d} = 2c$. Consequently, $\bar{d} \geq 1$. Figure 1c displays a random graph in Phase III with $\tilde{c} = 0.8$. In Figure 1c one can distinguish one connected component including cycles, one tree and some remaining isolated points.

Phase IV : A graph in Phase IV contains one giant connected component and some smaller components. The vertices which lie outside of the giant component are either isolated points or are in small trees. With probability tending to one, the graph consist of one giant connected component and some isolated points when the number of nodes of the graph tends to infinity. The average degree of a vertex is equal to $\bar{d} = 2\tilde{c} \log n$. In Figure 1d a graph in Phase IV with $\tilde{c} = 0.3$ is displayed. One can clearly distinguish one giant component and and some isolated points.

Phase V : When $M(n)$ is in the range of Phase V, the entire graph is almost surely connected. The average degree of a vertex equals $\bar{d} = (2 \log n)\omega(n)$ and $\bar{d} \to \infty$ when $n \to \infty$. Figure 1e displays a graph in Phase V. The graph consists of one giant component.

The results of this section are used in Section 4 to classify the binomial random graphs in different phases in order to determine their different behaviours. For a more detailed description of the phases and proof of results stated in the section, we refer the reader to [3].

3.3 Branching Process

As we will conclude later, in some regimes a random graph can be closely approximated by a Poisson branching process. We therefore will state some useful results on branching processes. The following section in mainly based on [10]. For an entire overview on branching processes we therefore refer the reader to [10, Chapter 3].
Consider a branching process where each individual has independent offspring with offspring distribution X. The branching process will either die or survive infinitely long. Denote the extinction probability of a branching process by η and the survival probability

by ζ, where $\zeta = 1 - \eta$. The total progeny of the branching process is denoted by T. The following results are used to understand the size of components in random graphs. For a random graph with an infinite number of nodes it is, for example, of interest whether all vertices are part of one giant connected component or the graph is split up in different components. When comparing to a branching process, the graph will have just one giant component if the corresponding branching process survives almost surely. To determine the extinction probability of a branching process, we make use of Theorem 3.2.

Theorem 3.2 (Extinction Probability). *The extinction probability η is the smallest solution in $[0, 1]$ of*

$$\eta = G_X(\eta) \quad ,$$

with $G_X(s)$ being the probability generating function of the offspring density X, i.e., $G_X(s) = E[s^X]$.

Observe that if $E[X] < 1$ then $\eta = 1$. A branching process with expected offspring smaller than on will die out almost surely. If $E[X] > 1$ then $\eta < 1$. Prove of this results is given in [10, Chapter 3]. I we want to compute the component size in a random graph, the total progeny of a branching process is of interest. Given a certain offspring distribution X, we can determine the probability a branching process having a progeny of n.

Theorem 3.3 (Law of Total Progeny). *For a branching process with i.i.d. offspring distribution X,*

$$P(T = n) = \frac{1}{n} P(X_1 + X_2 + \ldots + X_n = n - 1),$$

where $(X_i)_{i=1}^n$ are i.i.d. copies of X.

Proof of this results can be found in [10]. When it is known that a random graph consists of trees with a finite number of nodes we compare the random graph with a branching process based on extinction. The distributions $(p_k)_{k\geq 0}$ and $(p'_k)_{k\geq 0}$ are called conjugate pairs if $p'_k = \eta^{k-1} p_k$.

Theorem 3.4 (Duality principle for Branching Processes). *Let $(p_k)_{k\geq 0}$ and $(p'_k)_{k\geq 0}$ be a conjugate pair of offspring distributions. The branching process with distribution $(p_k)_{k\geq 0}$ conditioned on extinction, has the same distribution as the branching process with offspring distribution $(p_k)'_{k\geq 0}$.*

The proof of this Theorem and the definition of conjugate pairs can be found in [10]. As we will conclude later in this article, we can compare a random graph in some regimes to a Poisson branching process. We therefore will state some specific results for Poisson branching processes. If X is Poisson distributed, the probability distribution of the total progeny is given in Theorem 3.5.

Theorem 3.5 (Total Progeny for Poisson Branching Process). *For a branching process with i.i.d. offspring X, where X has a Poisson distribution with mean λ,*

$$P_\lambda(T = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n} \quad (n \geq 1).$$

Proof. By Theorem 3.3

$$P_\lambda(T = n) = \frac{1}{n} P(X_1 + X_2 + \ldots + X_n = n - 1).$$

Since $(X_i)_{i+1}^n$ are independently Poisson distributed with mean λ the equation can be rewritten by

$$\frac{1}{n}P(X_1 + X_2 + \ldots + X_n = n-1) = \frac{1}{n}P(\hat{X} = n-1),$$

where $\hat{X}$ is a Poisson distributed random variable with mean $n\lambda$. Thus

$$P(T = n) = \frac{1}{n}\frac{(n\lambda)^{n-1}e^{-\lambda n}}{(n-1)!} = \frac{(\lambda n)^{n-1}}{n!}e^{-\lambda n} \quad (n \geq 1).$$

□

For a Possion branching process conditioned on extinction we formulate the following theorem.

Theorem 3.6 (Duality Principle for Poisson Branching Process). *The branching process with Poisson distributed offspring with mean λ conditioned on extinction has the same distribution as a branching process with Poisson offspring distribution with mean $\mu = \eta\lambda$.*

Proof. Let p_k be the offspring distribution of a Poisson branching process with mean λ. Then the conjugate pair of p_k is given by

$$p_k' = \eta^{k-1}p_k = \eta^k e^{-\lambda(\eta-1)}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda\eta}\frac{(\lambda\eta)^k}{k!}$$

since p_k is Poisson distributed and $\eta = e^{\lambda(\eta-1)}$, obtained from Theorem 3.2. The offspring distribution p_k' is again Poisson distributed with mean $\mu = \eta\lambda$. Applying Theorem 3.4 completes the prove. □

4 Evolution of a Binomial Undirected Random Graph

To use the results of Erdős and Rényi described in Section 3.2, we have to reformulate the five phases to apply them to the binomial random graph model. Thus, instead of finding a threshold for the number of edges of the graph $M(n)$, we try to find a threshold for the edge probability $p(n)$. Since we consider large random graphs where $n \to \infty$, we assume the uniform random graph $G_u(n, M(n))$ to have the same behaviour as the binomial random graph $G_b(n, p(n))$ with $p(n) = \frac{M(n)}{\binom{n}{2}}$. Here, $p(n)$ is equal to the ratio between the number of edges in the graph and the total number of possible edges. Theorem 4.1, as stated in [5, p. 8-9], confirms this assumption.

Theorem 4.1 (Equivalence of Uniform Random Graph and Binomial Random Graph). *Let $G_b(n,p)$ be a binomial random graph and let $G_u(n,M)$ be a uniform random graph. Let $0 \leq p_0 \leq 1$, $s(n) = n\sqrt{p(1-p)} \to \infty$ and $\omega(n) \to \infty$ arbitrarily slowly as $n \to \infty$.*

(i) Suppose that $\mathcal{P}$ is a graph property such that $P(G_u(n,M) \in \mathcal{P}) \to p_0$ for all

$$m \in \left[\binom{n}{2}p - \omega(n)s(n), \binom{n}{2}p + \omega(n)s(n)\right].$$

Then $P(G_b(n,p) \in \mathcal{P}) \to p_0$ as $n \to \infty$.

(ii) Let $p_- = p - \omega(n)s(n)/n^3$ and $p_+ = p + \omega(n)s(n)/n^3$. Suppose that $\mathcal{P}$ is a monotone graph property such that $P(G_b(n,p_-) \in \mathcal{P}) \to \infty$ and $P(G_b(n,p_+) \in \mathcal{P}) \to p_0$. Then $P(G_u(n,M) \in \mathcal{P}) \to p_0$, as $n \to \infty$, where $m = \lfloor\binom{n}{2}p\rfloor$.

Let property $\mathcal{P}$ be the property of a graph being in a given Phase. By Section 3.2, $P(G_u(n, M(n)) \in \mathcal{P}) \to 1$ when $n \to \infty$ whenever $M(n)$ is in the corresponding range of the Phase. If we let $p(n) = \frac{M(n)}{\binom{n}{2}}$, we can conclude from the first part of Theorem 4.1 that $P(G_b(n, p(n)) \in \mathcal{P}) \to 1$ when $n \to \infty$. The binomial graph $G_b(n, p(n))$ with $p(n) = \frac{M(n)}{\binom{n}{2}}$ is almost surely in the same Phase as the uniform random graph $G_u(n, M(n))$. By rephrasing the thresholds for the different Phases using this result, we obtain:

Phase I : $p(n) \sim o(n^{-1})$,

Phase II : $p(n) \sim \frac{\lambda}{n}$ with $0 < \lambda < 1$,

Phase III : $p(n) \sim \frac{\lambda}{n}$ with $\lambda > 1$,

Phase IV : $p(n) \sim c\frac{\log n}{n}$ with $c \leq 1$,

Phase V : $p(n) \sim \omega(n)\frac{\log n}{n}$ with $\omega(n) \to \infty$.

We can classify each of the scaling regimes for $p(n)$ in one of the phases.

$p(n) = c$ with $0 < c < 1$: Let $w(n) = c\frac{n}{\log n}$. As $n \to \infty$, $w(n) \to \infty$. Now, $p(n)$ can be rewritten as $p(n) = w(n)\frac{\log n}{n}$ and thus the graph $G(n, p)$ with $p = c$ will be in the regime of Phase V.

$p(n) = \frac{c \log n}{n}$ with $0 < c \leq 1$: A graph $G(n, p)$ with $p(n) = \frac{c \log n}{n}$ will be in Phase IV.

$p(n) = \frac{\lambda}{n}$ with $0 < \lambda < 1$: The graph of $G(n, p)$ with $p = \frac{\lambda}{n}$, $0 < \lambda < 1$ will behave like a graph of Phase II

$p(n) = \frac{\lambda}{n}$ with $\lambda > 1$: The graph of $G(n, p)$ with $p = \frac{\lambda}{n}$, $\lambda > 1$ will behave like a graph of Phase III.

Using this results and the phases described in [3], a graph $G(n, p)$ with $p = c$ will be almost surely connected and will consist of one giant component. A graph $G(n, p)$ with $p(n) = \frac{c \log n}{n}$ and $c \leq 1$ contains one giant component and a few isolated points. For a graph $G(n, p)$ with $p = \frac{\lambda}{n}$ and $0 < \lambda < 1$, almost all vertices will be part of components which are trees. If $\lambda > 1$ the structure of the graph changes abruptly. There will be one giant component and some small trees. The giant component will contain about $G(\lambda)n$ vertices where

$$G(\lambda) = 1 - \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (\lambda e^{-\lambda})^k \quad . \tag{1}$$

Consequently, the fraction of vertices lying in the giant component will be equal to $G(\lambda)$. The other vertices will lie in small trees.

5 Probability of *Always Red* in Undirected Random Graph

In this section the probabilities of *Always Red* for a coloured binomial undirected random graph, $G(n, p)$, in different regimes are computed. That is, starting at a vertex S, what is the probability that all possible paths starting at S will contain only *red* vertices when the number of vertices of the random graphs tends to infinity. Aim is to check a possible Zero-One law for Linear Temporal Logic. If the probability of the *Always Red* statement equals zero or one in all cases, we cannot draw any conclusions about a Zero-One law. If, however, the probability is strictly bounded by zero and one, a counterexample of a Zero-One law for Linear Temporal Logic is found. We know that a random graph passes through different phases as one increases the edge probability. We therefore believe that the *Always Red* statement occurs with different probabilities in different random graph regimes. We will compute the probability of *Always Red* for different edge probabilities. We will use the results of Section 4 to determine the behaviour of a random graph in the different regimes. We will start with the $p(n) = c$ case, followed by the $p(n) = \frac{c\log(n)}{n}$ and $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$ and $\lambda > 1$ cases. For the $p(n) = c$ and the $p(n) = \frac{c\log(n)}{n}$ case we will find analytical results of the *Always Red* property. For the $p(n) = \frac{\lambda}{n}$ with $0 < \lambda < 1$ and $\lambda > 1$ regimes we will find bounds for the *Always Red* probability as well compute numerical results. The *Always Red* statement is defined as in Definition 2.7. The vertex colouring is defined as in Definition 2.4, where $q(n)$ is the probability of a vertex being *red*. Let S denote the starting vertex, chosen at random.

5.1 *Always Red* in coloured undirected random graph $G(n, p)$ with $p(n) = c$, $0 < c < 1$

A random graph with edge probability $p(n) = c$ is almost surely connected. A typical graph with colour probability $q = 0.3$ can be found in Figure 2. From any starting vertex S all other vertices can be reached. Since the property *Always Red* fails if we can find a path with just one *blue* vertex, all vertices of the graph have to be red for the property to hold. Since we consider graphs with an infinite number of nodes, the probability of having only *red* vertices will tend to zero. We expect the probability of *Always Red* to tend to zero. This corresponds to the results found in Theorem 5.1.

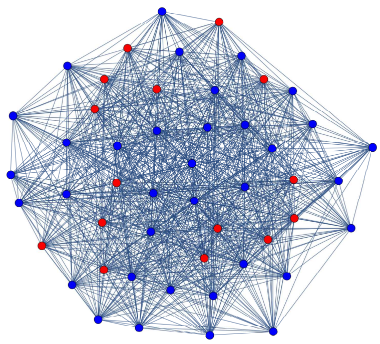


FIGURE 2: Coloured Binomial Undirected Random Graph $G(n,p)$ with $n = 50$, $p = 0.7$ and colour probability $q = 0.3$. The Graph is an example of an undirected random graph with edge probability $p = c$, $0 < c < 1$.

Theorem 5.1 (*Always Red* in coloured undirected random graph $G(n, p)$ with $p(n) = c$)**.** *For a coloured undirected random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = c$, $c \in (0,1)$,*

$$P(S \vDash \Box \textit{ red}) = 0 \textit{ when } n \to \infty \quad .$$

Proof. The graph $G(n, p)$ is connected with high probability, as described in Section 4. That is

$$P(G(n,p) \text{ is conncected}) = 1 \text{ for } n \to \infty$$
$$P(G(n,p) \text{ is not conncected}) = 0 \text{ for } n \to \infty \quad .$$

By conditioning on $G(n,p)$ being connected

$$\begin{aligned} P(S \vDash \Box\, red) &= P(S \vDash \Box\, red | G(n,p) \text{ is conncected}) P(G(n,p) \text{ is conncected }) \\ &\quad + P(S \vDash \Box\, red | G(n,p) \text{ is not conncected}) P(G(n,p) \text{ is not conncected}) \\ &= P(S \vDash \Box\, red | G(n,p) \text{ is conncected}) \text{ for } n \to \infty \quad . \end{aligned}$$

If $G(n,p)$ is connected, all states can be reached from a starting vertex S. Thus, if there is at least one *blue* state we can find a path for which the *Always Red* property does not hold. The property only holds if there are only *red* vertices. So

$$P(S \vDash \Box\, red | G(n,p) \text{ is conncected}) = P(\forall\ v : v \text{ is } red) = q^n$$

which converges to zero as n tends to infinity, since $q < 1$. Concluding,

$$P(S \vDash\ red) = 0 \text{ for } n \to \infty \quad .$$

$\square$

5.2 *Always Red* in coloured undirected random graph $G(n,p)$ with $p(n) = c\frac{\log n}{n}$, $0 < c \leq 1$

When $p = c\frac{\log n}{n}$ with $0 < c \leq 1$, the graph consist of one giant connected component and some isolated points. A typical graph in this regime with colour probability $q = 0.3$ can be found in Figure 3. If S lies in the giant connected component, we expect a equal behaviour of the *Always Red* statement as in the $p(n) = c$ edge probability regime. If S is an isolated point, the probability of the *Always Red* property to hold is equal the colour probability q. Either, S itself is *red*, in which case the property holds, or S is *blue*, in which case the property fails. Of interest is with what probability S is an isolated point. For the *Always Red* property is this regime we state Theorem 5.2.

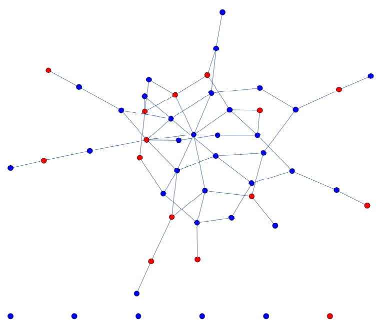


FIGURE 3: Coloured Binomial Undirected Random Graph $G(n,p)$ with $n = 50$, $p = 0.05$ and colour probability $q = 0.3$. The Graph is an example of an undirected random graph with edge probability $p = c\frac{\log(n)}{n}$ with $0 < c < 1$.

Theorem 5.2 (*Always Red* in coloured undirected random graph $G(n,p)$ with $p(n) = \frac{c \log n}{n}$). *For a coloured undirected random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = c\frac{\log n}{n}$, $0 < c \leq 1$,*

$$P(S \vDash \Box\, red) = 0 \textit{ when } n \to \infty \quad .$$

Proof. The graph is with probability tending to one a graph with one giant component and some isolated points. Thus, S lies either in the giant component, denoted by $\mathcal{C}_{max}$, or is an isolated point. By conditioning on S lying in the giant component

$$\begin{aligned} P(S \vDash \Box red) =& P(S \vDash \Box red | S \in \mathcal{C}_{max}) P(S \in \mathcal{C}_{max}) \\ &+ P(S \vDash \Box red | S \notin \mathcal{C}_{max}) P(S \notin \mathcal{C}_{max}) \quad . \end{aligned}$$

The vertex S is an isolated point if $d(S) = 0$. Since the vertex degree of S is binomial distributed with $d(S) \sim Bin(n-1, p(n))$, the probability of S being an isolated vertex decreases with the number of vertices. Indeed $P(d(S) = 0) = (1 - \frac{c \log n}{n})^n$ and $P(d(S) = 0) \to 0$ as $n \to \infty$. We therefore claim that the number of isolated points will be of order $O(1)$ when $n \to \infty$. The number of vertices in the giant component is of order $n - O(1)$. We can conclude

$$\begin{aligned} P(S \in \mathcal{C}_{max}) &= \frac{n - O(1)}{n} = 1 \quad , \\ P(S \notin \mathcal{C}_{max}) &= \frac{O(1)}{n} = 0 \end{aligned}$$

when n tends to infinity. Furthermore, if S lies in the giant component, all vertices in the giant component have to be *red* for the *Always Red* statement to hold. If S is an isolated point, only S needs to be red for the statement to hold. We conclude,

$$\begin{aligned} P(S \vDash \Box red | S \in \mathcal{C}_{max}) &= q^{n - O(1)} \quad , \\ P(S \vDash \Box red | S \notin \mathcal{C}_{max}) &= q \quad . \end{aligned}$$

Thus,

$$P(S \vDash \Box red) = q^{n - O(1)} = 0 \text{ for } n \to \infty \quad .$$

□

5.3 *Always Red* in coloured undirected random graph $G(n,p)$ with $p = \frac{\lambda}{n}, 0 < \lambda < 1$

A graph with edge probability $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$, contains components which are trees. See Figure 4 for a random graph in this regime. In this graph regime we expect the probability of the *Always Red* property to differ significant from the previous described random graph regimes. The random graph contains no giant component. Furthermore, we expect all components to have a finite number of vertices, even though the total number of vertices will tend to infinity. The probability of all vertices in the component of S are *red*, and the *Always Red* property holds, will therefore be greater than zero.

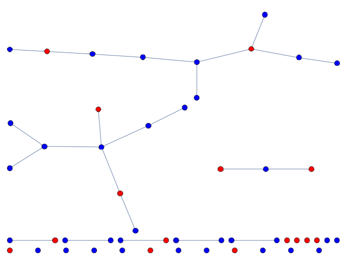


FIGURE 4: Coloured Binomial Undirected Random Graph $G(n,p)$ with $n = 50$, $p = 0.015$ and colour probability $q = 0.3$. The Graph is an example of an undirected random graph with edge probability $p = \frac{\lambda}{n}$, $\lambda = 0.75$.

We will calculate the probability that all the vertices in the component of S are *red*. Note that the set of reachable vertices from S and the set of vertices in the component of S are equivalent. Let $\mathcal{C}(S)$ denote the component of S. Then,

$$P(S \models \Box \; red) = P(\forall \; v \in \mathcal{C}(S) : v \text{ is } red).$$

We will calculate the above probability by conditioning on the component size of S, denoted by $|\mathcal{C}(S)|$. The components size can vary from 1, in case S is an isolated point, to n, in case the whole graph is connected.

$$P(S \models \Box \; red) = \sum_{i=1}^{n} P(\forall \; v \in \mathcal{C}(S) : v \text{ is } red \mid |\mathcal{C}(S)| = i) \; P(|\mathcal{C}(S)| = i). \tag{2}$$

Clearly, $P(\forall \; v \in \mathcal{C}(S) : v \text{ is } red \mid |\mathcal{C}(S)| = i) = q^i$. To calculate the component size $|\mathcal{C}(S)|$, we approximate the random graph by a branching process, since the graph will look like a collection of trees with high probability. Every vertex has with probability $p(n) = \frac{\lambda}{n}$ an edge to each of the other $n-1$ vertices. Thus, the degree of every vertex v_i is binomial distributed with

$$d(v_i) \sim Bin(n-1, p) \quad .$$

Since the binomial distribution can be approximated by the Poisson distribution when n tends to infinity, we say that the degree of every vertex is distributed in the limit by

$$d(v_i) \sim Poi(\lambda) \quad .$$

Since S lies in a tree, we claim that the probability $P(|\mathcal{C}(S)| = i)$ is equal to the probability of a branching process with Poisson offspring distribution with mean λ has a progeny of i. Thus, by using the results form section 3.3, Theorem 3.3,

$$P(|\mathcal{C}(S)| = i) = \frac{(\lambda i)^{i-1}}{i!} e^{-\lambda i} \quad (i \geq 1) \quad .$$

The total probability, as given in equation (2), equals

$$\begin{aligned} P(S \models \Box red) &= \sum_{i=1}^{n} q^i \frac{(\lambda i)^{i-1}}{i!} e^{-\lambda i} \\ &= \sum_{i=1}^{n} \frac{i^{i-1}}{i!} \frac{1}{\lambda} (\lambda q)^i e^{-\lambda i} \quad . \end{aligned} \tag{3}$$

Equation (3) gives the probability for the limiting object. We will continue by giving a simpler approximation for this equation to gain more insight in it's value and it's dependency on λ and q. To approximate equation (3) we use Stirling's inequality given in Lemma 5.3. We obtained Lemma 5.3 by simplifying the error bounds given in [7].

Lemma 5.3 (Stirlings Inequality). *For $n \in \mathbb{N} \setminus \{0\}$, Stirling's inequality is given by*

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq n^{n+\frac{1}{2}} e^{-n+1} \quad . \tag{4}$$

Proof. By the error bounds given in [7] we know

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \cdot e^{r_n} \text{ , where } \frac{1}{12n+1} < r_n < \frac{1}{12n} \quad .$$

It remains to prove that

$$1 < e^{r_n} < \frac{e}{\sqrt{2\pi}} \qquad \Longleftrightarrow \qquad 0 < r_n < 1 - \log(\sqrt{2\pi}) \quad ,$$

by monotonicity of the exponential function. Clearly, $r_n > 0$ for all n, wherefore the correctness of the lower bound is proven. For the upper bound observe that the maximum value of r_n is given at $n = 1$, and thus,

$$r_n < r_1 = \frac{1}{12} < 1 - \log(\sqrt{2\pi}) \quad ,$$

which proves the upper bound. □

By substituting the results of Lemma 5.3 into equation (3) we can bound the probability by

$$\sum_{i=1}^{n} \frac{i^{i-1}}{i^{i+\frac{1}{2}}e^{-i+1}} \frac{1}{\lambda}(\lambda q)^i e^{-\lambda i} \leq P(S \models \Box red) \leq \sum_{i=1}^{n} \frac{i^{i-1}}{\sqrt{2\pi} i^{i+\frac{1}{2}}e^{-i}} \frac{1}{\lambda}(\lambda q)^i e^{-\lambda i} \quad .$$

This can be rewritten as

$$\frac{1}{e\lambda} \sum_{i=1}^{n} \frac{e^{-I_{\lambda,q} \cdot i}}{i^{\frac{3}{2}}} \leq P(S \models \Box red) \leq \frac{1}{\sqrt{2\pi}\lambda} \sum_{i=1}^{n} \frac{e^{-I_{\lambda,q} \cdot i}}{i^{\frac{3}{2}}} \quad , \tag{5}$$

where we define the variable $I_{\lambda,q}$ by $I_{\lambda,q} = \lambda - 1 - \log(\lambda q)$. Observe that $I_{\lambda,q}$ is positive for all $0 < q < 1$ and $\lambda > 0$. The summation in equation (5) for $n \to \infty$ is equal to the polygarithm function

$$Li_s(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^s} \quad ,$$

where $z = e^{-I_{\lambda,q}}$ and $s = \frac{3}{2}$. According to [2] the polygarithm function has a integral representation of

$$Li_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - z} dt \tag{6}$$

for all z, except z lying on the segment of the real axis from 1 to ∞. Since, in our case, $z = e^{-I_{\lambda,q}}$ is bounded by zero and one, $0 < e^{-I_{\lambda,q}} < 1$, the integral representation is valid and the summation equals

$$Li_{\frac{3}{2}}(e^{-I_{\lambda,q}}) = \frac{e^{-I_{\lambda,q}}}{\Gamma(1.5)} \int_0^{\infty} \frac{t^{0.5}}{e^t - e^{-I_{\lambda,q}}} dt \quad . \tag{7}$$

Here, $\Gamma(s)$ is the Gamma-function with $\Gamma(1.5) = \frac{\sqrt{\pi}}{2}$. With equation (5), equation (7) and $I_{\lambda,q} = \lambda - 1 - \log(\lambda q)$ we can formulate following Lemma:

Lemma 5.4 (*Always Red* in coloured undirected random graph $G(n,p)$ with $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$). *For a coloured undirected random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$,*

$$\frac{2q}{\sqrt{\pi}} e^{-\lambda} \int_0^{\infty} \frac{\sqrt{t}}{e^t - \lambda q e^{1-\lambda}} dt \leq P(S \models \Box red) \leq \frac{2q}{\sqrt{2\pi}} e^{1-\lambda} \int_0^{\infty} \frac{\sqrt{t}}{e^t - \lambda q e^{1-\lambda}} dt \quad . \tag{8}$$

Observe that the upper and lower bound only differ by a factor of $k = \frac{e}{\sqrt{2\pi}} \approx 1.0844$, obtained by Stirling's approximation. We cannot solve the integral of equation (8) analytically. Therefore, we will calculate the integral numerically as well find an analytical bound for the integral.

Numerical Solution of Equation (8) To solve equation (8) numerically, we use MATLAB's implemented *integral* function. Graphs of the numerical results of equation (8) can be found below. The zero and one bounds are displayed blue. The Lower bound is displayed green, the upper bound is displayed red.

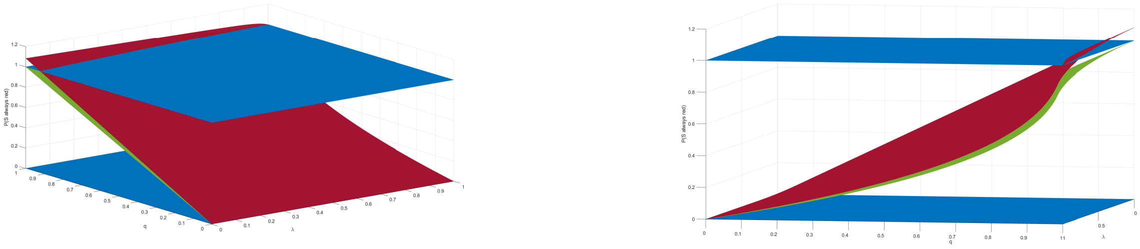


FIGURE 5: Numercial Results for $P(S \models \Box\, red)$ in the coloured undirected random graph, $G(n,p)$, with $p = \frac{\lambda}{n}$ for $0 < \lambda < 1$. The Zero and One bounds are displayed blue. The lower bound is displayed green, the upper bound is displayed red.

The probability is increasing with q. As one can conclude from the numerical data, both, upper and lower bound are larger than zero for all values of q and λ. The lower bound does not exceed one. The upper bound, however, exceeds one for large values of q.

Analytical Solution of Solving Equation (8) We will bound the integral of equation (8), $\int_0^\infty \frac{\sqrt{t}}{e^t - e^{-I_{\lambda,q}}} dt$, where $I_{\lambda,q} = \lambda - 1 - \log(\lambda, q)$. To estimate the integral we use Taylor expansion on $\frac{1}{e^t - e^{-\lambda,q}}$. For convenience let $x = e^t$. Observe that $x > 1$. Then,

$$\frac{1}{x - e^{-I_{\lambda,q}}} = \frac{1}{x} \frac{1}{(1 - \frac{e^{-I_{\lambda,q}}}{x})} = \frac{1}{x} \sum_{j=0}^{\infty} \frac{e^{-jI_{\lambda,q}}}{x^j} \tag{9}$$

since $0 < \frac{e^{-I_{\lambda,q}}}{x} < 1$. For the lower bound observe that

$$\frac{1}{x} \sum_{j=0}^{\infty} \frac{e^{-jI_{\lambda,q}}}{x^j} = \frac{1}{x}(1 + \frac{e^{-I_{\lambda,q}}}{x} + \frac{e^{-2I_{\lambda,q}}}{x^2} \cdots) > \frac{1}{x}$$

since all terms of the sum are positive. We will estimate the lower bound of the integral by

$$\int_0^\infty \frac{\sqrt{t}}{e^t - e^{-I_{\lambda,q}}} dt > \int_0^\infty \frac{\sqrt{t}}{e^t} dt \quad . \tag{10}$$

For the upper bound observe that the tail of the sum of equation 9 consist of smaller order terms, since $\frac{e^{-I_{\lambda,q}}}{x} < 1$. Moreover, $(\frac{e^{-I_{\lambda,q}}}{x} + \frac{e^{-2I_{\lambda,q}}}{x^2} + \frac{e^{-3I_{\lambda,q}}}{x^3} + \cdots)$ obtains its maximum for a minimum value of x. By substituting $x = 1$, the minimum value of e^t in $t \in (0, \infty)$, we obtain

$$e^{-I_{\lambda,q}} + e^{-2I_{\lambda,q}} + e^{-3I_{\lambda,q}} + \cdots = e^{-I_{\lambda,q}}(1 + e^{-2I_{\lambda,q}} + e^{-3I_{\lambda,q}} + \cdots) = e^{-I_{\lambda,q}}(\frac{1}{1 - e^{-I_{\lambda,q}}})$$

since $0 < e^{-I_{\lambda,q}} < 1$. Therefore,

$$\frac{1}{x - e^{-I_{\lambda,q}}} < \frac{1}{x}(1 + \frac{e^{-I_{\lambda,q}}}{1 - e^{-I_{\lambda,q}}}) = \frac{1}{x} \frac{1}{(1 - e^{-I_{\lambda,q}})} \quad .$$

We will estimate the upper bound of the integral by

$$\int_0^\infty \frac{\sqrt{t}}{e^t - e^{-I_{\lambda,q}}} dt < \frac{1}{(1 - e^{-I_{\lambda,q}})} \int_0^\infty \frac{\sqrt{t}}{e^t} dt \quad . \tag{11}$$

By substituting this bounds in inequality (8) and solving the integral $\int_{-}^{\infty} \frac{\sqrt{t}}{e^t}$, we obtain

$$\frac{2}{\sqrt{\pi}} \frac{1}{e\lambda} e^{-I_{\lambda,q}} \int_0^\infty \frac{\sqrt{t}}{e^t} dt \leq P(S \models \Box\, red) \leq \frac{\sqrt{2}}{\pi} \frac{1}{\lambda} e^{-I_{\lambda,q}} \frac{1}{(1 - e^{-I_{\lambda,q}})} \int_0^\infty \frac{\sqrt{t}}{e^t} dt \quad ,$$

$$\frac{e^{-I_{\lambda,q}}}{e\lambda} \leq P(S \models \Box\, red) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda} \frac{1}{(e^{I_{\lambda,q}} - 1)} \quad .$$

By substituting $I_{\lambda,q} = \lambda - 1 - \log(\lambda q)$ we obtain

$$\frac{q}{e^\lambda} < P(S \models \Box\, red) < \frac{1}{\sqrt{2\pi}} \frac{q}{e^{\lambda - 1} - \lambda q} \quad .$$

First observe that the lower bound of $P(S \models \Box\, red)$ is larger than zero for all $\lambda > 0$ and $q > 0$. However, for particular values of λ, it is possible for the upper bound to exceed one. The upper bound for $P(S \models \Box\, red)$ is smaller than one whenever

$$\frac{1}{\sqrt{2\pi}} \frac{1}{e^{I_{\lambda,q}}\lambda - \lambda} < 1 \qquad \Longleftrightarrow \qquad 0 < e^\lambda - eq\lambda - \frac{eq}{\sqrt{2\pi}} \quad .$$

Now let

$$f_q(\lambda) = e^\lambda - eq\lambda - \frac{eq}{\sqrt{2\pi}} \quad .$$

From the first and second derivative of $f_q(\lambda)$ we can conclude that $f_q(\lambda)$ has a local minimum at $\lambda = \log(eq)$. From the definition of $f_q(\lambda)$ we can conclude that

$$f_q(\log(eq)) = eq(1 - \log(eq) - \frac{1}{\sqrt{2\pi}})$$

which is larger than zero whenever $e^{-\frac{1}{\sqrt{2\pi}}} \approx 0.67 < q$. Whenever the local minimum of $f_q(\lambda)$ is greater than zero, $f_q(\lambda)$ is greater than zero for all values of $0 < \lambda < 1$. Thus,

$$0 < e^\lambda - eq\lambda - \frac{eq}{\sqrt{2\pi}} \text{ whenever } e^{-\frac{1}{\sqrt{2\pi}}} < q \quad .$$

We can conclude that

$$\frac{1}{e^\lambda} < P(S \models \Box\, red) < \frac{1}{\sqrt{2\pi}} \frac{q}{e^{\lambda-1} - \lambda q} \quad \text{for} \quad 0 < \lambda < 1 \quad . \tag{12}$$

Furthermore

$$0 < P(S \models \Box\, red) < 1 \quad \text{for} \quad 0 < \lambda < 1 \quad \text{and} \quad e^{-\frac{1}{\sqrt{2\pi}}} < q \quad . \tag{13}$$

Observe that these results correspond to the numerical results found in Figure 5 since the lower bound is always greater than zero and the upper bound only exceeds one for large values of q.

Using the results stated in equation (12) and Lemma 5.4 we formulate the following Theorem.

Theorem 5.5 (Always Red in coloured undirected random graph with $p(n) = \frac{\lambda}{n}$ with $0 < \lambda < 1$)**.** *For a coloured undirected random graph $G(n,p)$ with edge probability $p(n) = \frac{\lambda}{n}$ with $0 < \lambda < 1$ and colour probability $q \in (0,1)$,*

$$\frac{1}{e^{\lambda}} < P(S \models \Box \, red) < \frac{1}{\sqrt{2\pi}} \frac{q}{e^{\lambda-1} - \lambda q} \text{ when } n \to \infty \quad .$$

Moreover, $0 < P(S \models \Box \, red)$ for all values of q, and $0 < P(S \models \Box \, red) < 1$ for $e^{-\frac{1}{\sqrt{2\pi}}} < q$.

Theorem 5.5 is a powerful result in investigating a possible the Zero-One law. Theorem 5.5 contradicts a possible Zero-One law for at least some values of λ and q. Moreover, as, we can conclude from the numerical results presented earlier this section, the Zero-One law does not hold for almost all values of q and λ since the probability of $P(S \models \Box \, red)$ is strictly bounded by zero and one. Only for large values of q we cannot draw any conclusions.

5.4 *Always Red* in coloured undirected random graph $G(n,p)$ with $p = \frac{\lambda}{n}, \lambda > 1$

When $p(n) = \frac{\lambda}{n}$ with $\lambda > 1$, a random graph $G(n,p)$ consists almost surely of one giant component and some small trees. Figure 6 shows a typical graph in this regime. For the probability of *Always Red* we assume to find similar results as for $0 < \lambda < 1$ values whenever S is in one of the trees. When S is in the giant component, we expect the probability of *Always Red* to occur to tend to zero when the number of nodes tend to infinity since the number of nodes in the giant component will tend to infinity too.

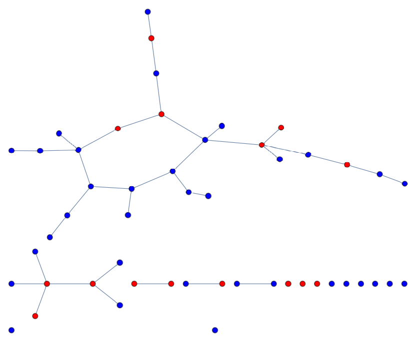


FIGURE 6: Coloured Binomial Undirected Random Graph $G(n,p)$ with $n = 50$, $p = 0.03$ and colour probability $q = 0.3$. The Graph is an example of an undirected random graph with edge probability $p = \frac{\lambda}{n}$, $\lambda = 1.5$.

For $\lambda > 1$, the starting vertex S lies either in one giant connected component or in a small tree. We will calculate $P(S \models \Box red)$ by conditioning on whether S lies in the giant component or not. Let $\mathcal{C}_{max}$ denote the giant component. Then,

$$\begin{aligned} P(S \models \Box \, red) = & P(S \models \Box \, red | S \in \mathcal{C}_{max}) \; P(S \in \mathcal{C}_{max}) \\ & + P(S \models \Box \, red | S \notin \mathcal{C}_{max}) \; P(S \notin \mathcal{C}_{max}) \quad . \end{aligned}$$

Whenever S lies in the giant component the statement $(S \models \Box \, red)$ will only hold if all vertices in the giant component are *red*. As described in Section 4, we know that the number of vertices in the giant component equals $G(\lambda)n$, where $G(\lambda)$ is given in equation (1). The number of vertices in the giant component grows linearly with the number of vertices of the graph and the number of vertices in the giant component tends to infinity whenever the number of vertices of the graph tends to infinity. The probability of all vertices in the giant component being *red*, will tend, as in the $p(n) = c$ case, to zero as $n \to \infty$. Thus

$$P(S \vDash \Box\, red) = P(S \vDash \Box\, red | S \notin \mathcal{C}_{max})\ P(S \notin \mathcal{C}_{max}) \quad . \tag{14}$$

For the probability $P(S \notin \mathcal{C}_{max})$ we state the following lemma:

Lemma 5.6 (Vertices outside of the Giant Component in an Undirected Random Graph $G(n,p)$ with $p(n) = \frac{\lambda}{n}$ and $\lambda > 1$)**.** *Given an undirected random graph $G(n,p)$ with $p(n) = \frac{\lambda}{n}$ and $\lambda > 1$ where $n \to \infty$. The probability for a vertex, v, chosen at random, to be outside of the giant component is bounded by*

$$\frac{2e^{-\lambda}}{\sqrt{\pi}} \int_0^\infty \frac{t^{0.5}}{e^t - \lambda e^{1-\lambda}} \leq P(v \notin \mathcal{C}_{max}) \leq \frac{\sqrt{2}e^{1-\lambda}}{\pi} \int_0^\infty \frac{t^{0.5}}{e^t - \lambda e^{1-\lambda}} \quad . \tag{15}$$

Proof. By [3] we know that the fraction of vertices inside of the giant component equals

$$G(\lambda) = 1 - \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (\lambda e^{-\lambda})^k \quad ,$$

and accordingly, the fraction of vertices outside of the giant component equals

$$P(S \notin \mathcal{C}_{max}) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (\lambda e^{-\lambda})^k \quad . \tag{16}$$

Using Stirling's inequality stated in Lemma 5.3, we can rewrite equation (16) by

$$\frac{1}{e\lambda} \sum_{k=1}^{\infty} \frac{(\lambda e^{1-\lambda})^k}{k^{\frac{3}{2}}} \leq P(v \notin \mathcal{C}_{max}) \leq \frac{1}{\sqrt{2\pi}\lambda} \sum_{k=1}^{\infty} \frac{(\lambda e^{1-\lambda})^k}{k^{\frac{3}{2}}} \quad .$$

We will make use of the integral representation of the poligarithm function $Li_s(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^s}$ given in equation (6) for $s = \frac{3}{2}$ and $z = \lambda e^{1-\lambda}$. The integral representation of the polygarithm function is valid since $\lambda e^{1-\lambda} < 1$. By substituting the integral representation of the polygarithm function, we obtain the required result:

$$\frac{2e^{-\lambda}}{\sqrt{\pi}} \int_0^\infty \frac{t^{0.5}}{e^t - \lambda e^{1-\lambda}} \leq P(v \notin \mathcal{C}_{max}) \leq \frac{\sqrt{2}e^{1-\lambda}}{\pi} \int_0^\infty \frac{t^{0.5}}{e^t - \lambda e^{1-\lambda}} \quad .$$

□

When S is outside of the giant component, the component of S can be represented by a branching process with mean offspring λ conditioned on extinction. With Theorem 3.4, this equals a branching process with mean offspring $\mu = \lambda\eta$, where η is the extinction probability of a branching process with mean offspring λ. Observe that $0 < \mu < 1$ and a branching process with Poisson offspring distribution μ is described in Section 5.3. By Theorem 5.5

$$\begin{aligned} \frac{2q}{\sqrt{\pi}} e^{-\eta\lambda} \int_0^\infty \frac{\sqrt{t}}{e^t - \eta\lambda q e^{1-\eta\lambda}} dt &\leq P(S \vDash \Box red | S \notin \mathcal{C}_{max}) \\ &\leq \frac{2q}{\sqrt{2\pi}} e^{1-\eta\lambda} \int_0^\infty \frac{\sqrt{t}}{e^t - \eta\lambda q e^{1-\eta\lambda}} dt \quad . \end{aligned} \tag{17}$$

To compute the probability $P(S \vDash \Box\ red)$, we substitute equation (17) and the result of Lemma 5.6 into equation (14) and obtain

$$\begin{aligned}(\frac{2q}{\sqrt{\pi}}e^{-\eta\lambda}\int_0^\infty \frac{\sqrt{t}}{e^t-\eta\lambda q e^{1-\eta\lambda}}dt)(\frac{2e^{-\lambda}}{\sqrt{\pi}}\int_0^\infty \frac{t^{0.5}}{e^t-\lambda e^{1-\lambda}}dt) &\le P(S \vDash \Box red)\\ \le (\frac{2q}{\sqrt{2\pi}}e^{1-\eta\lambda}\int_0^\infty \frac{\sqrt{t}}{e^t-\eta\lambda q e^{1-\eta\lambda}}dt)(\frac{\sqrt{2}e^{1-\lambda}}{\pi}\int_0^\infty \frac{t^{0.5}}{e^t-\lambda e^{1-\lambda}}dt)\end{aligned}$$

By simplifying aboves equation we can state follwoing Lemma:

Lemma 5.7 (*Always Red* in coloured undirected random graph $G(n,p)$ with $p(n) = \frac{\lambda}{n}$, $\lambda > 1$)**.** *For a coloured undirected random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = \frac{\lambda}{n}$, $\lambda > 1$,*

$$\begin{aligned}&\frac{4qe^{-\lambda-\eta\lambda}}{\pi}(\int_0^\infty \frac{t^{0.5}}{e^t-\lambda e^{1-\lambda}}dt)(\int_0^\infty \frac{t^{0.5}}{e^t-\eta\lambda q e^{1-\eta\lambda}}dt) \le P(S \vDash \Box red)\\ &\le \frac{2qe^{2-\lambda-\eta\lambda}}{\pi^2}(\int_0^\infty \frac{t^{0.5}}{e^t-\eta\lambda q e^{1-\eta\lambda}}dt)(\int_0^\infty \frac{t^{0.5}}{e^t-\lambda e^{1-\lambda}}dt)\ \text{as } n \to \infty \quad .\end{aligned}$$

Here, η is the extinction probability of a Poisson Branching Process with mean offspring λ.

To find η, the extinction probability of a branching process with mean offspring λ, we use the relation given in Theorem 3.2.

$$\eta = G_X(\eta) \quad , \tag{18}$$

where $G_X(s)$ is the probability generating function of the offspring distribution X. Since the offspring distribution is a Poisson distribution with parameter λ, η is given by the smallest solution of

$$\eta = e^{\lambda(\eta-1)} \quad .$$

We will obtain numerical results of the inequality in Lemma 5.7. To find η we will determine the roots of

$$f(x) = e^{\lambda(x-1)} - x$$

in the interval $(0,1)$ numerically, using MATLAB's *fzero* function. The integrals are determined numerically using MATLAB's *integral* function. Plots of the numerical results can be found in Figure 7. The zero and one bound are displayed blue, the upper bound red and the lower bound green. When λ increases, the probability approaches zero, but never reaches zero. This seems logical since the large connected component will grow when λ increases and the probability of S lying in the connected component will grow. If S lies in the connected component, the probability of *Always Red* will tend to zero when the number of states tend to infinity, since the probability of a reachable *blue* state will increase to one. For small λ and large q the probability of always red tends to one, where the upper bound exceeds one. For large λ the probability decreases with q. When λ is small, the probability of S lying in a small component grows. When S lies in a small component with finite reachable states, it is clear that the probability of always red increases with the probability of a state being *red*. In conclusion, $P(S \vDash \Box red)$ with $p = \frac{\lambda}{n}, \lambda > 1$ does not obey the Zero-One law for almost all values of λ and q. For λ close to one and large q we cannot draw any conclusions since $P(S \vDash \Box red)$ could be equal to one.

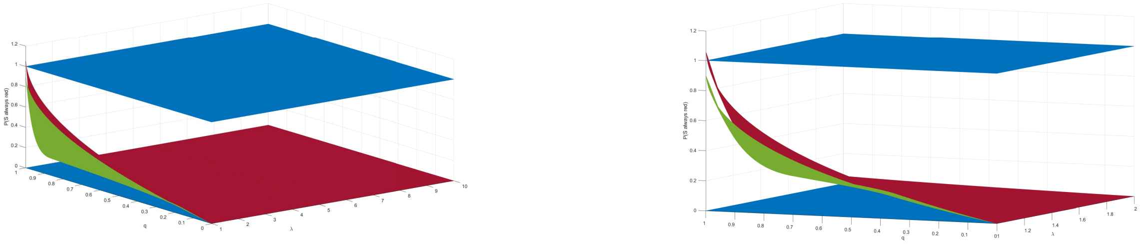


FIGURE 7: Numercial Results for $P(S \vDash \Box\, red)$ in the coloured undirected random graph, $G(n,p)$, with $p = \frac{\lambda}{n}$, $\lambda > 1$. The Zero and One bounds are displayed blue. The lower bound is displayed green, the upper bound is displayed red.

As a result of the numerical results found in this section we will state the following theorem.

Theorem 5.8 (Always Red in coloured undirected random graph with $p(n) = \frac{\lambda}{n}$, $\lambda > 1$)**.** *For a coloured undirected random graph $G(n,p)$ with edge probability $p(n) = \frac{\lambda}{n}$, $\lambda > 1$ and colour probability $q \in (0,1)$, there exists values for λ and q, such that*

$$P(S \vDash \Box\, red) \neq 0 \text{ and}$$
$$P(S \vDash \Box\, red) \neq 1 \text{ as } n \to \infty \quad .$$

The theorem states that there exist values of λ and q that refute a possible Zero-One law.

6 Probability of *Always Red* in Directed Random Graph

In the following section, let $G(n,p)$ be a coloured binomial directed random graph with colour probability $q(n)$, as described in Defintion 2.2. The probability of the *Always Red* statement will be computed for different types of edge probabilities. The difference with the previous section is the directedness of the random graph. We know that a random graph passes through different phases as one increases the edge probability. We therefore will consider different edge probabilities to determine the *Always Red* statement. Again, we will start with the $p(n) = c$ edge probability case, followed by the $p(n) = \frac{\log(n)}{n}$ and $p(n) = \frac{c\lambda}{n}$ for $0 < \lambda < 1$ and $\lambda > 1$ cases. For the $p(n) = c$ case we will find analytical results of the *Always Red* property. The $p(n) = \frac{\log(n)}{n}$, $p(n) = \frac{c\lambda}{n}$ with $0 < \lambda <$ and $\lambda > 1$ cases will need further investigation since we were not able to find any specific results in this article. However, we will try to give insight these regimes and formulate conjectures about the probability of the *Always Red* statement.

6.1 *Always Red* in coloured directed random graph $G(n,p)$ with $p(n) = c$, $0 < c < 1$

To get a better understanding of a directed random graph in this regime we will first state a theorem about strong connectivity in directed random graphs as stated in [6].

Theorem 6.1 (Strong Connectivity in Directed Binomial Random Graphs)**.** *Given a binomial random graph $G(n,p)$. For any fixed $c \in \mathbb{R}$, if $p = p(n)$ is a function of n, then $\hat{p} = \frac{\log n + c}{n}$ is a threshold for strong connectivity.*

The proof of this theorem can be found in [6]. The theorem states that

$$\lim_{n\to\infty} P(G(n,p) \text{ is strongly connected}) = \begin{cases} 0 & \text{if } p \ll \hat{p} \\ 1 & \text{if } p \gg \hat{p} \end{cases} .$$

By Theorem 6.1, a directed random graph $G(n,p)$ with $p(n) = c$ is strongly connected. Since we then, again, consider a giant connected component, we assume the *Always Red* statement to occur almost never. Figure 8 shows a typical graph in this regime.

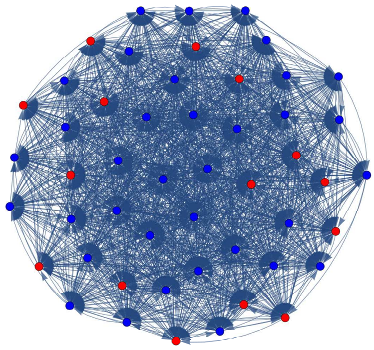


FIGURE 8: Coloured Binomial Directed Random Graph $G(n,p)$ with $n = 50$, $p = 0.7$ and colour probability $q = 0.3$. The Graph is an example of a directed random graph with edge probability $p = c$.

For the *Always Red* property in the $p(n) = c$ regime, we will prove the following theorem.

Theorem 6.2 (*Always Red* in coloured directed random graph $G(n,p)$ with $p(n) = c$)**.** *For a coloured directed random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = c$, $c \in (0,1)$,*

$$P(S \vDash \Box \ red) = 0 \text{ when } n \to \infty \quad .$$

Proof. With Theorem 6.1, we can conclude that a directed graph with $p(n) = c$ is strongly connected with high probability. That means, all vertices can be reached from a starting vertex S and the *Always Red* property does only hold if there are only *red* states.

$$P(S \vDash \Box red) = P(\forall\ v : v \text{ is } red) = q^n$$

which tends to zero as n tends to infinity, since $q < 1$. □

6.2 *Always Red* in coloured directed random graph $G(n,p)$ with $p(n) = c\frac{\log(n)}{n}$, $0 < c \leq 1$

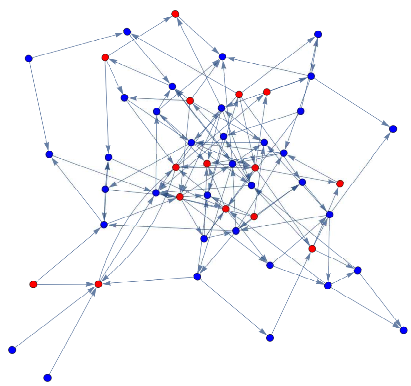


FIGURE 9: Coloured Binomial Directed Random Graph $G(n,p)$ with $n = 50$, $p = 0.05$ and colour probability $q = 0.3$. The Graph is an example of a directed random graph with edge probability $p = \frac{c\log(n)}{n}$.

As one can see in Figure 9, the graph in the regime of $p = c\frac{\log(n)}{n}$ consists of one giant, weakly connected component. However, some of the vertices have only outgoing edges and cannot be reached from any other vertex. Consider this vertices to be isolated vertices. then, the graph consists, as in the undirected case, of some isolated vertices and one giant component.
Since $p(n) = c\frac{\log(n)}{n}$ is in the regime of the threshold given in Theorem 6.1, we cannot conclude whether the graph $G(n,p)$ is strongly connected or not. Neither can we make statements about strong connectivity in the giant component. We can only conclude that the the giant connected component is weakly connected. However, by Section 3.2, we know that the average degree of a vertex is given by $\bar{d} = c\log n$. Thus, the average in-degree, as well the average out-degree grows with the number of nodes. Since this number grows, we expect the number of reachable vertices, from a starting vertex S in the giant component, to tend to infinity when the number of nodes of the graph tends to infinity. We formulate this assumption in the following conjecture.

Conjecture 6.3 (Number of reachable vertices in a directed random graph $G(n,p)$ with $p(n) = c\frac{\log(n)}{n}$). *Given a directed random graph, $G(n,p)$ with edge probability $p(n) = c\frac{\log(n)}{n}$, $0 < c \leq 1$. Let the giant component be the set of all vertices with at least one incoming edge. The number of reachable vertices from any starting vertex v in the giant component of the graph tends to infinity, whenever the number of nodes of the graph tends to infinity.*

Using this conjecture we will compute the probability of the *Always Red* statement.

Conjecture 6.4 (*Always Red* in coloured directed random graph $G(n,p)$ with $p(n) = \frac{c\log n}{n}$). *For a coloured directed random graph, $G(n,p)$, with colour probability $q \in (0,1)$ and edge probability $p(n) = c\frac{\log n}{n}$, $0 < c \leq 1$,*

$$P(S \vDash \square \text{ red}) = 0 \text{ when } n \to \infty \quad .$$

Proof. We divide the graph into two different sets. One set contains all the vertices with no incoming edge. These vertices are called isolated points. The other set, named the giant component, contains all other vertices and is denoted by $\mathcal{C}_{max}$. We condition on whether S lies in the giant component, denoted by $\mathcal{C}_{max}$, or is an isolated point.

$$\begin{aligned} P(S \vDash \square red) =& P(S \vDash \square red | S \in \mathcal{C}_{max}) P(S \in \mathcal{C}_{max}) \\ &+ P(S \vDash \square red | S \notin \mathcal{C}_{max}) P(S \notin \mathcal{C}_{max}) \quad . \end{aligned}$$

Since the in-degree of a vertex is binomial distributed with $d_in(v_i) \sim Bin(n-1, p(n)$, the number of isolated vertices decreases with the number of vertices. Indeed, $P(d_in(v_i) = 0) = (1 - \frac{c\log n}{n})^n \to 0$ as $n \to \infty$. We claim that the number of isolated points will be of order $O(1)$ when $n \to \infty$. We can conclude

$$\begin{aligned} P(S \in \mathcal{C}_{max}) &= \frac{n - O(1)}{n} = 1 \\ P(S \notin \mathcal{C}_{max}) &= \frac{O(1)}{n} = 0 \end{aligned}$$

when n tends to infinity. According to Conjecture 6.3, an infinite amount of vertices can be reached if S is part of the giant component. The property holds if all vertices reachable from S are *red*. Since an infinite amount of vertices can be reached, the property of all vertices being red tends to zero. In conclusion,

$$P(S \vDash \square red) = 0 \text{ when } n \to \infty \quad .$$

□

6.3 *Always Red* in coloured directed random graph $G(n,p)$ with $p = \frac{\lambda}{n}$, $0 < \lambda < 1$

Figure 10 shows a typical directed random graph in the $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$, regime. One can see one weakly connected component and some small trees and isolated points. The undirected graph in this regime consists exclusively of trees and a weakly connected component seems unlikely at first. If one, however, considers the possible paths from any vertex in the graph, one comes to the conclusion that the graph still has a treelike structure. We will elaborate this thought further.

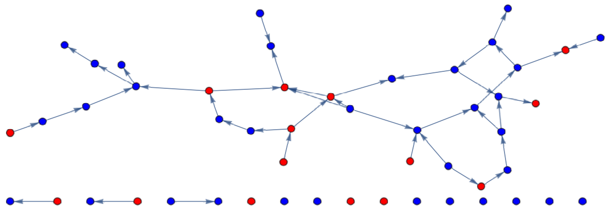


FIGURE 10: Coloured Binomial Directed Random Graph $G(n,p)$ with $n = 50$, $p = 0.015$ and colour probability $q = 0.3$. The Graph is an example of a directed random graph with edge probability $p = \frac{\lambda}{n}$ with $\lambda = 0.75$.

Every vertex i in a directed graph has incoming and outgoing edges with degrees distributed independently with

$$\begin{aligned} &\text{Incomming Degree: } d_{in}(v_i) \sim Bin(n-1, p(n)) \\ &\text{Outgoing Degree: } d_{out}(v_i) \sim Bin(n-1, p(n)) \quad . \end{aligned}$$

For $p(n) = \frac{\lambda}{n}$, this can be approximated in the limit by a Poisson distribution with

$$\begin{aligned} d_{in}(v_i) &\sim Poi(\lambda) \quad , \\ d_{out}(v_i) &\sim Poi(\lambda) \quad . \end{aligned}$$

Assume we want to explore the random graph, starting at vertex S. Starting at S, we only consider outgoing edges of S to reach it's k neighbours. In each of it's k neighbours, we again consider only outgoing edges to investigate new vertices. Continuing this step till no more new vertices can be found, equates to exploring all reachable vertices starting at vertex S. Observe that we only used outgoing edges to explore the graph. The distribution of outgoing edges is equal to the distribution of edges in a undirected random graph in the same regime. The number of reachable vertices from a starting vertex S in a directed random graph in the $p = \frac{\lambda}{n}$, $0 < \lambda < 1$, regime is equal to the number of reachable vertices in a undirected random graph in the $p = \frac{\lambda}{n}$, $0 < \lambda < 1$, regime. We assume the *Always Red* statement to hold with the same probability. Using Theorem 5.5 we formulate the following conjecture:

Conjecture 6.5 (Always Red in coloured directed random graph with $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$)**.** *For a coloured directed random graph $G(n,p)$ with edge probability $p(n) = \frac{\lambda}{n}$, $0 < \lambda < 1$, and colour probability $q \in (0,1)$,*

$$\frac{1}{e^{\lambda}} < P(S \vDash \Box\, red) < \frac{1}{\sqrt{2\pi}} \frac{q}{e^{\lambda-1} - \lambda q} \text{ when } n \to \infty \quad .$$

Moreover, $0 < P(S \vDash \Box\, red)$ for all values of q, and $0 < P(S \vDash \Box\, red) < 1$ for $e^{-\frac{1}{\sqrt{2\pi}}} < q$.

6.4 *Always Red* in coloured directed random graph $G(n,p)$ with $p = \frac{\lambda}{n}$, $\lambda > 1$

Figure 11 shows a typical random graph in the $\frac{\lambda}{n}$, $\lambda > 1$, regime. There is one weakly connected giant component. At first sight, the random graph looks nothing like the treelike structure present in the undirected graph in this regime. However, if we consider all possible path from a starting vertex S, we recognize the treelike structure.

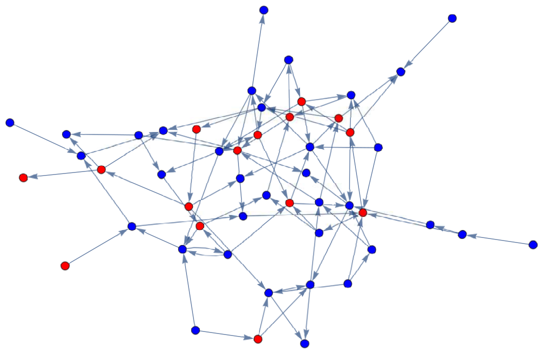


FIGURE 11: Coloured Binomial Directed Random Graph $G(n,p)$ with $n = 50$, $p = 0.03$ and colour probability $q = 0.3$. The Graph is an example of a directed random graph with edge probability $p = \frac{\lambda}{n}$ with $\lambda = 1.5$.

If we start to explore the graph, starting at a starting vertex S, we assume to find exactly the graph structure present in the undirected graph in the same regime. Then, S lies either in a tree or in a giant connected component. Using this reasoning we formulate the following conjecture.

Conjecture 6.6 (Always Red in coloured directed random graph with $p(n) = \frac{\lambda}{n}$, $\lambda > 1$)**.** *For a coloured directed random graph $G(n,p)$ with edge probability $p(n) = \frac{\lambda}{n}$, $\lambda > 1$, and colour probability $q \in (0,1)$, there exists values for λ and q, such that*

$$P(S \vDash \Box\ red) \neq 0 \text{ and}$$
$$P(S \vDash \Box\ red) \neq 1 \text{ when } n \to \infty \quad .$$

Proof. We will condition on whether S lies in the giant component or not:

$$\begin{aligned} P(S \vDash \Box red) =& P(S \vDash \Box red | S \in \mathcal{C}_{max})\ P(S \in \mathcal{C}_{max}) \\ &+ P(S \vDash \Box red | S \notin \mathcal{C}_{max})\ P(S \notin \mathcal{C}_{max}) \quad . \end{aligned}$$

By Theorem 6.1, we know that the entire graph is not strongly connected and we conclude that with positive probability, S is outside of the giant component. When S is outside of the giant component, the component of S can be described as a branching process conditioned on extinction. The probability of*Always Red* conditioned on S being outside of the the giant component is described in Conjecture 6.5. This probability is strict greater than zero. We conclude

$$P(S \vDash \Box red) \geq P(S \vDash \Box red | S \in \mathcal{C}_{max})\ P(S \in \mathcal{C}_{max}) > 0 \quad .$$

The upper bound is harder to compute, since the entire graph is at most weakly connected by Theorem 6.1. We will try to bound the probability by comparing the directed graph with it's corresponding undirected graph. Consider the undirected graph $G^{\star}{}_U(p,n)$ obtained by replacing all the directed edges of a directed random graph $G_D(p,n)$ by undirected edges. Parallel edges will be reduced to one edge. The degree of a vertex v_i of the graph $G^{\star}{}_U(p,n)$ will hold

$$d\_G_D(p,n)(v_i) \leq d\_G^{\star}{}_U(p,n)(v_i) \leq 2d\_G_D(p,n)(v_i) \quad .$$

The lower bound is obtained if every edge of vertex i in $G_D(p,n)$ has a reversed edge. The upper bound is obtained if none of the edges has a reversed edge. We claim that the number of reachable states from a starting vertex S is higher in the graph of $G^{\star}_U(p,n)$ than in the graph of $G_D(p,n)$, due to the undirectedness of the edges. Subsequent more vertices can be reached in the graph of $G^{\star}_U(p,n)$ than in the graph of $G_D(p,n)$ and $(S \vDash \Box red)$ has a lower probability in the graph of $G^{\star}_U(p,n)$ than in the graph of $G_D(p,n)$. That is

$$P_{G\_D}(S \vDash \Box red) \leq P_{G^{\star}\_U}(S \vDash \Box red) \quad .$$

Due to its average degree given by $\lambda \leq \bar{d}^{\star} \leq 2\lambda$, we claim that the graph $G^{\star}_U(p,n)$ behaves like a binomial undirected graph $G_U(p^{\star},n)$ with $p(n) < p^{\star}(n) < 2p(n)$. The probability of $(S \vDash \Box red)$ in $G_U(p^{\star},n)$ is given in Section 5.4 and Theorem 5.8 states it is strictly smaller than one for most of the values of q and λ. Thus,

$$P(S \vDash \Box red \text{ in } G_D(n,p)) \leq P(S \vDash \Box red \text{ in } G^{\star}_U(p,n)) < 1 \text{ for some values of } \lambda \text{ and } q.$$

$\Box$

7 Conclusions

For a binomial undirected random graph, the *Always Red* statement occurs with probability tending to zero for as well the $p(n) = c$ as the $p(n) = \frac{c\log(n)}{n}$ case, as n tends to infinity. In these regimes it is possible that Linear Temporal Logic obeys a Zero-One law. However, in the $p(n) = \frac{\lambda}{n}$ regime we found a counterexample for the Zero-One law for most of the values of λ and q. Theorem 5.5 and Theorem 5.8 disprove a Zero-One law for Linear Temporal Logic in undirected random graphs. For a binomial directed random graph the probability of *Always Red* will tend to zero for the $p(n) = c$ and $p(n) = \frac{c\log(n)}{n}$ regimes, as in the undirected case. A Zero-One law could exist in these regimes. For the $p(n) = \frac{\lambda}{n}$ regimes, we do not have any concrete probabilities yet, however, we have reasons to accept the probability of *Always Red* to be strictly bounded by zero and one for most values of λ and q, as stated in Conjecture 6.5 and Conjecture 6.6. We therefore assume the *Always Red* statement to be a counterexample for a Zero-One Law for Linear Temporal Logic in directed random graphs. In conclusion, we disproved a Zero-One law for Linear Temporal Logic for graphs by counterexample.

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