BSc Thesis Applied Mathematics

Joint replenishment for two non-identical companies

Femke Boelens

Supervisor: Dr. J.B. Timmer

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Department of Applied Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Science

UNIVERSITY OF TWENTE.

Preface

This thesis with the title "Joint replenishment for two non-identical companies" has been written as a completion of the bachelor programme Applied Mathematics at the University of Twente. It is the result of a period, covering April, May and June of 2019, of hard work in which I learned a lot. Before, I have not worked on my own on such an intensive and lengthy full-time project. Therefore, I not only learned about the theory included in this thesis, but also about working independently and how to set up a mathematical research.

I would first like to thank my supervisor Judith Timmer. I experienced her supervision as very pleasant. During my thesis I was given freedom but also enough guidance to accomplish this thesis successfully. She always had time for me and the meetings were inspiring, resulting in more and concrete ideas after every meeting.

In addition, I would like to thank Lotte Gerards for coming every day together with me early in the morning at the university, for the company while working on our theses and motivating each other this way. I also want to thank Annemarie Jutte and Jarco Slager for reading this thesis and providing useful comments.

Moreover, I would like to thank my friends for some distraction from my thesis and Sven Dummer for the support. Finally, I want to thank my parents for supporting me, not only during this thesis.

I hope you enjoy your reading.

Femke Boelens

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Abstract

We consider an inventory control problem of two companies facing Poisson demand with a different rate, that review their inventory continuously and replenish it by ordering different quantities. Our aim is to analytically identify conditions under which joint ordering is advantageous for both companies. Numerical experiments show that the optimal order quantities under joint ordering are lower than or equal to those under independent ordering and that joint ordering is preferred to independent ordering for two companies. In addition, we carry out a sensitivity analysis.

Keywords: Joint Replenishment Problem, non-identical companies, stochastic demand, inventory model

1 Introduction

Companies that sell products need to have a stock in order to meet their demands, meaning that their inventory needs to be replenished every now and then. Naturally, costs are associated with this replenishment. When companies decide to jointly replenish their inventories instead of independently order items, these companies can split the fixed procurement cost and therefore save costs. The problem for companies of determining a policy to replenish their inventory jointly is known as the Joint Replenishment Problem.

In this article we research the Joint Replenishment Problem. In particular, we study two companies that each have to manage the inventory level of one specific item. We consider the two companies to be non-identical, meaning that they face different demand and they can order the item in different amounts. The inventories of the two companies are reviewed continuously and they face Poisson demand. As soon as the inventory level of one of the companies drops below a prescribed level, they both place an order to replenish their inventories.

1.1 Literature Review

One of the initiators of research on the joint replenishment problem was Silver in 1965 [9]. Silver explored that the cost of joint ordering is always less than or equal to the cost of independent ordering for two items. Since this paper, much research has been done on the joint replenishment problem. Both the situations of deterministic and stochastic demand are considered. The latter is known as the stochastic joint replenishment problem, which we also do research on in this article.

For the stochastic joint replenishment problem two main policies have been researched in the literature. These policies are described in a review paper by Khouja and Goyal [6].

The first policy is the can-order policy, which can be seen as a continuous review policy. This policy was introduced by Balintfy [3]. When the inventory level one of the items drops below a certain level, the so-called reorder level, the inventory level of the other item is evaluated as well. If this is below the so-called can-order level, orders for this item are made as well. Both items will be ordered up to a fixed inventory level that may be different for each item. Federgruen et al. [4] provided an algorithm for determining the optimal can-order policy. Under the periodic review policy, every fixed period of time both items are ordered up to a fixed inventory level.

Comparisons of the performance of these policies are made by Atkins and Iyogun [2], who showed that periodic review policies outperform the can-order policy for increasing fixed cost. The comparison of performance of policies is an extensively researched topic. Much research is focused on extending and improving an existing policy and testing whether it really outperforms earlier policies. Finding exact solutions for optimal policies is really complex and results in large computational time. Özkaya et al. [7] came up with a new policy that combines aspects of both the continuous review policies and the periodic review policies and showed that this new policy outperformed existing policies. Furthermore, Tanrikulu et al. [10] proposed a new policy that involves constant size orders whenever the inventory level of one of the items drops below its reorder point. They also show that this policy performs really well, especially in situations where backorder costs are high and lead times are small. Other examples of policies that are researched in the literature are those of Roushdy et al. [8] and Fung et al. [5].

In the literature that is mentioned above, the focus is mainly on introducing and evaluating new policies for the stochastic joint replenishment problem. However, very few papers investigate how well a new policy performs relative to the policy of independent ordering. Timmer et al. [11] showed that for two companies having identical characteristics, such as demand rate and holding costs, joint ordering under a continuous review policy is better than individual ordering if the procurement cost is higher than a certain lower bound, which is a function of the holding cost and the demand rate.

1.2 Scope

Problems that are addressed in this article involve the conditions under which joint ordering is preferred to independent ordering of the items. As explained in the literature review, this is already done for companies having identical characteristics. Our goal is to identify similar conditions for companies having non-identical characteristics, such as demand and order quantities. Other important problems that we address are the number of items that is optimal to order for those non-identical companies and the amount of money that can be saved under joint replenishment, relative to independent ordering. In addition we aim to investigate how the parameters influence the optimal order quantities and costs.

1.3 Outline

We introduce our model in Section 2, including the used parameters and assumptions that we make in our analysis. We continue in Sections 3 and 4 with an analytical analysis of the cost structure of the situations of independent ordering and joint ordering, respectively, for two companies. In Section 5 we report our results. First, we provide analytical results in the form of theorems. Afterwards, the results of a numerical comparison are presented and in Section 6 a sensitivity analysis is carried out. In Section 7 we discuss our results and we conclude our research in Section 8.

2 Model

We consider two companies having non-identical characteristics and denote the set of companies by $N = \{1, 2\}$. We model the inventory control problem of a single product for the companies in this set. Our model is based on the model by Timmer et al. [11]. Their paper also includes an inventory model of a single item for multiple companies. In their paper, results are gained in the case that the companies have identical characteristics. In our problem, the companies do not have all characteristics identical to each other. Therefore, we discuss the characteristics that are involved in our model. In addition, we need to make assumptions concerning these characteristics.

The first characteristic of the two companies is the demand. For both companies, the demand of the product is according to a Poisson process with rate λ_i for company i, $i \in N$. These Poisson processes are independent of each other, and hence the demand of the companies is different. Without loss of generality, we assume that the demand rate of company 1 is smaller than the demand rate of company 2. Furthermore, we assume that the demand rate of company 2 can be expressed as a multiple of the demand rate of company 1. Therefore, we define

$$\lambda_1 = \lambda, \ \ \lambda > 0 \qquad \text{and} \qquad \lambda_2 = x\lambda, \quad x \geq 1.$$

In order to meet their demands, the companies have to replenish their inventories. The order up-to level for the replenishment of the inventory, which is the second characteristic we discuss, is denoted by Q_i for company i, $i \in N$. In the case the companies order their items independently, items are ordered up to a level of Q_i items as soon as the inventory reaches a certain level r_i ($i \in N$), the reorder level. Hence, $Q_i - r_i$ items are ordered for company $i, i \in N$. We assume that the lead time, which is the time between placing an order and the items being on stock, is zero time units and therefore, it is most optimal to choose r_i to be zero time units. Therefore, the order quantity becomes $Q_i, i \in N$. The order quantity is discrete and the reorder process is under continuous review. Since the demand rate of company 1 is smaller than the demand rate of company 2, it is plausible to consider the order quantity of company 1 to be smaller than the order quantity of company 2. If we assume otherwise, at least one of the inventories becomes out balance, since the companies always order at the same time in case of joint ordering. Therefore, we define

$$Q_1 = Q, \quad Q \in \mathbb{N} \qquad \text{and} \qquad Q_2 = yQ, \quad yQ \in \mathbb{N}, \ y \geq 1.$$

In the replenishment process of the inventory of the two companies there are also costs involved, of which the structure is as follows. First, the companies have to pay a fixed procurement cost A for every order that they place, which is independent of the order size. If the companies place a joint order instead of ordering items separately, they can save costs by splitting the procurement cost. After ordering the items, they will be on stock until they are sold. The cost of keeping one item on stock for one time unit is the holding cost and denoted by h_i for $i \in N$. In our model, we assume the holding cost for both companies to be the same, and hence we define

$$h_1 = h_2 = h, \quad h > 0.$$

Furthermore, the inventory position of company i at time t is denoted by Z_i^t. We assume the inventory position to be discrete. In addition, we assume that backorders cannot occur and therefore the inventory position is equal to the inventory level of the company.

3 Independent Ordering

In this section we consider the two companies introduced in Section 2 in the case that they replenish their inventories independently from each other. Therefore, the inventory level processes $\{Z_i^t\}_{i \in N}$ are independent processes. Since these processes are independent, we derive for each of the two companies an expression for the total costs per time unit of replenishing their inventory. However, since the cost structure is the same for both companies, we focus on determining the cost structure of company 1. This is the company with demand rate λ and order quantity Q, as defined in the previous section. When we arrive at an expression for the cost structure, we can modify this expression in order to derive an expression for the costs for company 2.

In order to derive an expression for the costs, we model the inventory process as a Markov Chain, which is shown in Figure 1. The state space of this process is $S = \{n \mid 1 \leq n \leq Q\}$. The transition probability from state n to $n-1$ equals the probability of selling an item and therefore equals the demand rate. When there is only one item left in the inventory, this last item is sold with probability λ and immediately the inventory is replenished.

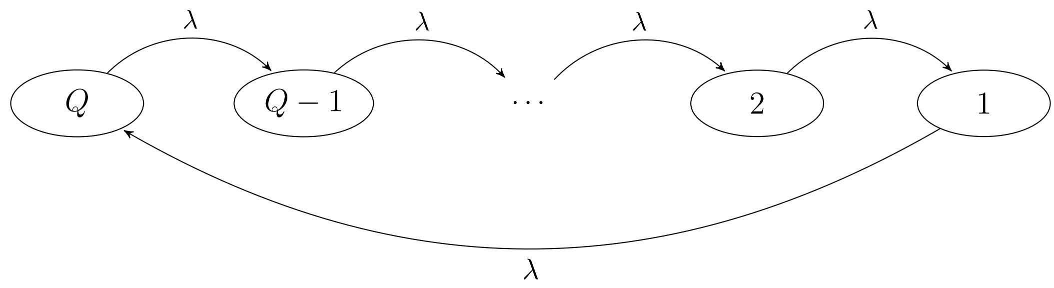


Figure 1: Markov Chain of the inventory process of company 1, having demand rate λ and order quantity Q

Let π_n be the stationary probability that the Markov Chain is in state n, for $n = 1, 2, \ldots, Q$, meaning that the inventory consists of n items. For this Markov Chain, we can formulate the following balance equations.

$$\begin{aligned} &\lambda\pi_{n+1} = \lambda\pi_n, \qquad n = 1, 2, \ldots, Q-1; \\ &\lambda\pi_Q = \lambda\pi_1. \end{aligned}$$

Therefore, the equilibrium distribution is $\pi_n = \frac{1}{Q}$ for $n = 1, 2, \ldots, Q$.

Using this equilibrium distribution, Timmer et al. [11] derive an expression for the total costs per time unit. From this, we obtain that in the situation of independent ordering the total expected costs for company 1 per time unit equal

$$K_1^{\text{ind}}(Q) = \frac{A\lambda}{Q} + \frac{1}{2}h(Q+1). \tag{1}$$

As a consequence, we find a similar expression for the total costs per time unit for company 2, by substituting λx for the demand rate and yQ for the order quantity. Hence, we obtain

$$K_2^{\text{ind}}(yQ) = \frac{A\lambda x}{yQ} + \frac{1}{2}h(yQ+1). \tag{2}$$

In these expressions the superscript 'ind' indicates independent ordering.

4 Joint Ordering

In this section we consider the two companies introduced in Section 2 in the case that they jointly replenish their inventory. A joint order is placed as soon as one of the two companies runs out of inventory. We can also model this situation as a Markov Chain. Here, the state space is two-dimensional and represents the inventory level of both companies. Therefore, the state space is given by $S = \{(n_1, n_2 \mid 1 \leq n_1 \leq Q, 1 \leq n_2 \leq yQ\}$. Since the Markov Chain is more complicated than the one of the previous section due to the two-dimensional state space, we first show the general structure in the middle of the Markov Chain. In addition, we study the special case of the Markov Chain that includes the state (Q, yQ) and the cases that involve either the state (n_1, yQ) for $n_1 < Q$ or the state (Q, n_2) for $n_2 < yQ$. We let $\pi(n_1, n_2)$ denote the stationary probability that the Markov Chain is in state (n_1, n_2) for $n_1 = 1, 2, \ldots, Q$ and $n_2 = 1, 2, \ldots, yQ$.

The structure of the Markov Chain around a general state (n_1, n_2), where $n_1 < Q$ and $n_2 < yQ$, is given in Figure 2. The transition probability from state (n_1, n_2) ($n_1 = 1, 2, \ldots, Q$ and $n_2 = 1, 2, \ldots, yQ$) to state $(n_1 - 1, n_2)$ equals the probability that company 1 sells an item before company 2 does, and hence equals $\frac{\lambda}{\lambda + x\lambda}$.

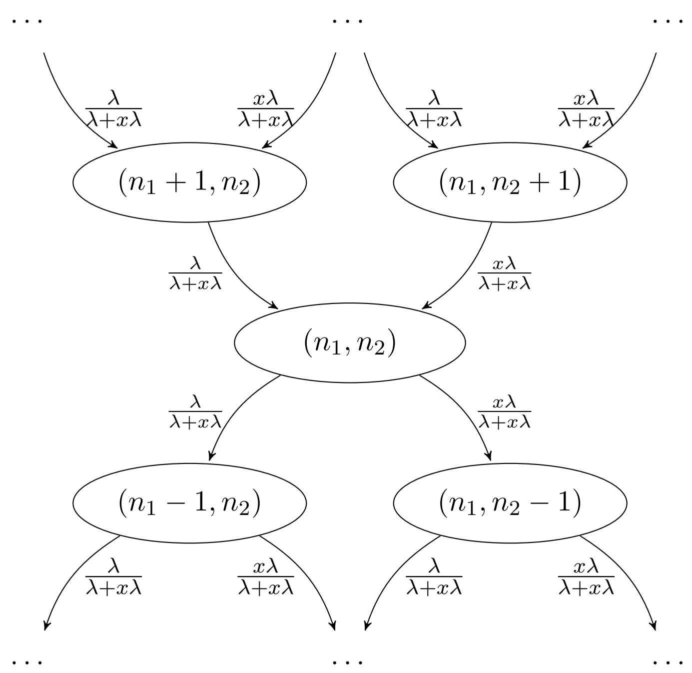


Figure 2: The inventory process of two companies, company 1 having demand rate λ and company 2 having demand rate $x\lambda$

For $n_1 = 1, 2, \ldots, Q - 1$ and $n_2 = 1, 2, \ldots, yQ - 1$ the balance equations for this Markov Chain are given by

$$(\lambda + x\lambda)\pi(n_1, n_2) = \lambda\pi(n_1 + 1, n_2) + x\lambda\pi(n_1, n_2 + 1). \tag{3}$$

The part of the Markov Chain that involves the state (Q, yQ) is given in Figure 3.

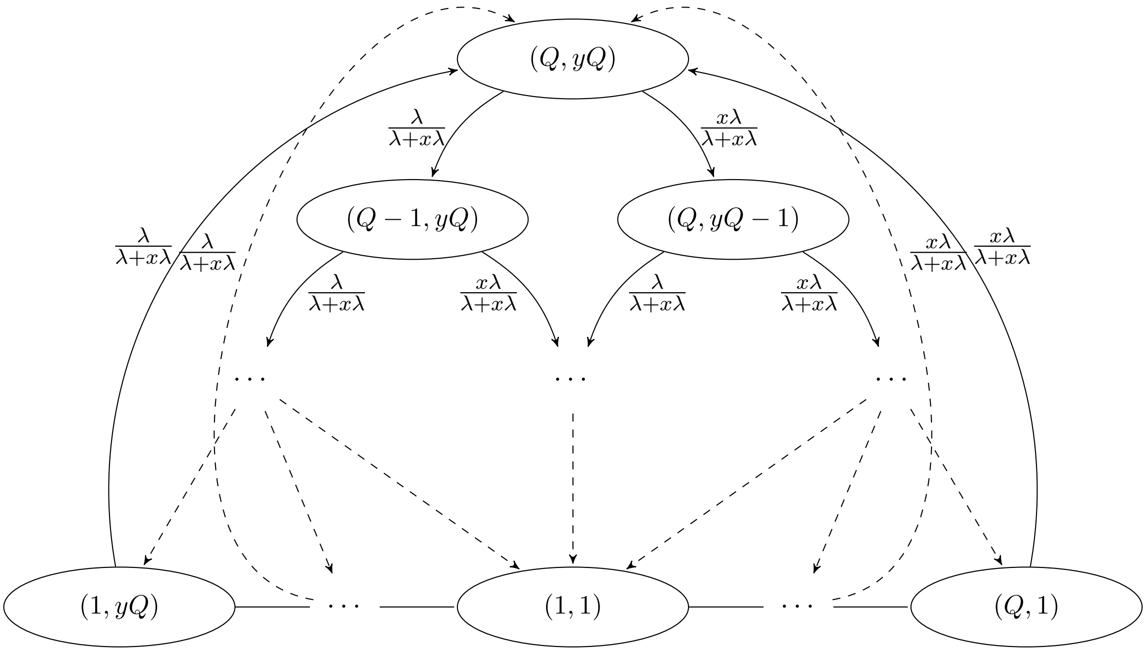


Figure 3: Part of the Markov Chain of the inventory process, involving the node (Q, yQ) and all its in- and outgoing edges.

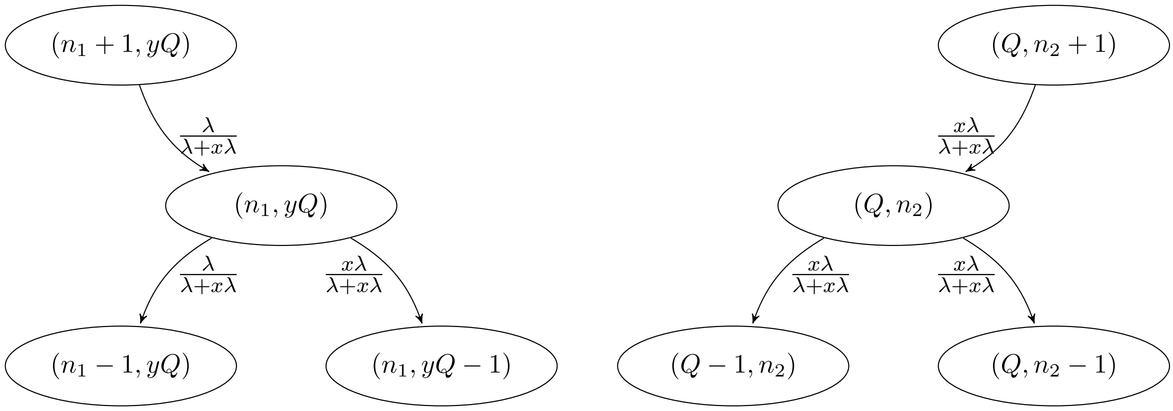
Around the state (Q, yQ), the outgoing arrows are similar to the previous Markov Chain. However, the ingoing arrows are different. As soon as one company runs out of stock, the stock of both companies is replenished up to the maximum number of items, which is Q items for company 1 and yQ items for company 2. This is the case for states where either n_1 or n_2 equals one, and thus for yQ states there is an arrow with probability $\frac{\lambda}{\lambda+x\lambda}$ to the state (Q, yQ) and for Q states there is an arrow with probability $\frac{x\lambda}{\lambda+x\lambda}$ to (Q, yQ). Hence, the balance equation corresponding to this part of the Markov Chain is as follows:

$$(\lambda + x\lambda)\pi(Q, yQ) = \lambda \sum_{n_2=1}^{yQ} \pi(1, n_2) + x\lambda \sum_{n_1=1}^{Q} \pi(n_1, 1). \tag{4}$$

The parts of the Markov Chain that involve either the state (Q, n_2), where $n_2 < yQ$ or the state (n_1, yQ), where $n_1 < Q$ are given in Figure 4. For simplicity of the figures, we did not draw the ingoing edges of the states on top of the figures and the outgoing edges for the states on the bottom of the figures. For these Markov Chains, the balance equations are similar to each other and are given by

$$(\lambda + x\lambda)\pi(n_1, yQ) = \lambda\pi(n_1 + 1, yQ), \quad n_1 = 1, 2, \dots Q - 1; \tag{5}$$
$$(\lambda + x\lambda)\pi(Q, n_2) = x\lambda\pi(Q, n_2 + 1), \quad n_2 = 1, 2, \dots, yQ - 1. \tag{6}$$

In the paper of Timmer et al. [11] the equilibrium distribution, corresponding to the balance equations in Equations (3), (4), (5) and (6) is determined. Before stating this equilibrium distribution, we first define the generalized incomplete beta function to be $I_q(a, b) = \sum_{s=0}^{b-1} \binom{s+a-1}{s} q^a (1-q)^s$[1]. We use this function because this expression occurs



(a) Part of the Markov Chain that involves the state $(n_1, yQ), n_1 < Q$, and all its in- and out-going edges.

(b) Part of the Markov Chain that involves the state $(Q, n_2), n_2 < yQ$, and all its in- and outgoing edges.

Figure 4: Two parts of the Markov Chain of the inventory process.

several times in the equation of the equilibrium distribution. In the equilibrium distribution we also use a normalizing constant which is given by

$$\begin{aligned} G(Q, yQ) &= \frac{yQ(1+x)}{x} I_{\frac{x}{1+x}}(yQ+1, Q) + Q(1+x) I_{\frac{1}{1+x}}(Q+1, yQ) \\ &= Q(1+x)\left(\frac{y}{x} I_{\frac{x}{1+x}}(yQ+1, Q) + I_{\frac{1}{1+x}}(Q+1, yQ)\right). \end{aligned} \tag{7}$$

Now that we defined the generalized incomplete beta function and the normalizing constant, we can state the computed equilibrium distribution:

$$\pi(n_1, n_2) = \frac{1}{G(Q, yQ)} \binom{Q-n_1+yQ-n_2}{Q-n_1} \left(\frac{1}{1+x}\right)^{Q-n_1} \left(\frac{x}{1+x}\right)^{yQ-n_2}. \tag{8}$$

Using this equilibrium distribution we wish to derive an expression for the total joint cost of replenishment for the two companies under cooperation. This cost is as follows.

Theorem 1. *In case of cooperation, for two companies, having demand rates λ and $x\lambda$ and up-to order levels Q and yQ, respectively, and both having holding cost h per item per time unit, the expected joint cost per time unit is given by*

$$\begin{aligned} K(Q, yQ) = {} & \frac{1}{2} h(yQ+1-xQ)\frac{x-1}{x} \\ & + \frac{A\lambda x/Q + \frac{1}{2}h\left[yQ(1+x) + (1-x^2) I_{\frac{1}{1+x}}(Q+1, yQ)\right]}{y - y\binom{Q+yQ}{Q}\frac{x^{yQ}}{(1+x)^{yQ+Q}} + (x-y) I_{\frac{1}{1+x}}(Q+1, yQ)}. \end{aligned} \tag{9}$$

Proof. Using the equilibrium distribution in Equation (8), Timmer et al. derive an expression for the total joint cost, which we use as the basis of this proof. We modify this expression to the case where the order quantity of company 1 equals Q, the order quantity of company 2 equals yQ and the demand rates of company 1 and 2 are λ and $x\lambda$, respectively. Let z_i denote the difference between Q_i and the current inventory level of company

$i, i \in N$. With $G(Q, yQ)$ defined as in Equation (7) the expression of the total joint cost is

$$
\begin{aligned}
K(Q, yQ) = & \frac{A(\lambda + x\lambda)}{G(Q, yQ)} \\
& + \frac{yQ}{G(Q, yQ)} \sum_{z_1=0}^{Q-1} [h(Q - z_1/2) + h(yQ + 1)/2] \binom{z_1 + yQ}{z_1} p^{z_1} (1-p)^{yQ} \\
& + \frac{Q}{G(Q, yQ)} \sum_{z_2=0}^{yQ-1} [h(Q+1)/2 + h(yQ - z_2/2)] \binom{z_2 + Q}{z_2} p^{Q} (1-p)^{z_2}. \quad (10)
\end{aligned}
$$

In order to derive a convenient expression for the cost for investigating its behaviour, we study this function in parts.

We start with rewriting the normalizing constant. The properties $I_q(a, b) = 1 - I_{1-q}(b, a)$ and $I_q(a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} q^a (1-q)^{b-1} + I_q(a+1, b-1)$ [1] of the generalized incomplete beta function give us that

$$
I_{\frac{x}{1+x}}(yQ + 1, Q) = 1 - \binom{Q + yQ}{Q} \frac{x^{yQ}}{(1+x)^{Q+yQ}} + I_{\frac{1}{1+x}}(Q + 1, yQ).
$$

And therefore the normalizing constant can be written as

$$
G(Q, yQ) = Q \frac{1+x}{x} \left[y - y \binom{Q + yQ}{Q} \frac{x^{yQ}}{(1+x)^{Q+yQ}} + (x - y) I_{\frac{1}{1+x}}(Q + 1, yQ) \right].
$$

For the further investigation of the changing expression for the costs we consider the three terms of Equation (10) separately. We start with the term on the first line, which we call $K_1(Q, yQ)$:

$$
\begin{aligned}
K_1(Q, yQ) &= \frac{A(\lambda + x\lambda)}{G(Q, yQ)} \\
&= \frac{A\lambda x / Q}{y - y\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{Q+yQ}} + (x - y) I_{\frac{1}{1+x}}(Q + 1, yQ)}. \quad (11)
\end{aligned}
$$

The term on the second line of Equation (10), which we call $K_2(Q, yQ)$ can be expressed as

$$
K_2(Q, yQ) = \frac{\frac{1}{2} hy \left(\frac{x}{1+x}\right) \sum_{z=0}^{Q-1} (2Q + yQ - z + 1) \binom{z+yQ}{z} \frac{x^{yQ}}{(1+x)^{z+yQ}}}{y - y\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{Q+yQ}} + (x - y) I_{\frac{1}{1+x}}(Q + 1, yQ)}
$$

Using the properties of the generalized incomplete beta function [1] and basic algebra, we obtain

$$
K_2(Q, yQ) = \frac{1}{2} h \frac{\left[2Q + (yQ + 1)\left(\frac{x-1}{x}\right)\right] I_{\frac{x}{1+x}}(yQ + 1, Q) + Q\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{yQ+Q}}}{y - y\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{yQ+Q}} + (x - y) I_{\frac{1}{1+x}}(Q + 1, yQ)}. \quad (12)
$$

If we perform similar operations on the third term of Equation (10), which we denote by $K_3(Q, yQ)$, we can derive the following expression for this term

$$
K_3(Q, yQ) = \frac{1}{2} hx \frac{[2yQ + (Q + 1)(1 - x)] I_{\frac{1}{1+x}}(Q + 1, yQ) + yQ\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{yQ+Q}}}{y - y\binom{Q+yQ}{Q} \frac{x^{yQ}}{(1+x)^{yQ+Q}} + (x - y) I_{\frac{1}{1+x}}(Q + 1, yQ)}. \quad (13)
$$

We see that the denominators of the three parts of the cost, in Equations (11), (12) and (13), are the same and that the numerators of the second and the third part also show similarities both with each other and with the denominator of all three expressions. In order to give the second term and the third term the same structure, we rewrite the generalized incomplete beta function of the second term, $I_{\frac{x}{1+x}}(yQ+1,Q)$, to the form of this function in both the third term and the denominator ($I_{\frac{1}{1+x}}(Q+1,yQ)$) using the properties of the generalized incomplete beta function. This yields

$$K_2(Q,yQ) = \frac{1}{2}hy\frac{\left[2Q+(yQ+1)\frac{x-1}{x}\right] - \left[2Q+(yQ+1)\frac{x-1}{x}\right] I_{\frac{1}{1+x}}(Q+1,yQ)}{y - y\binom{Q+yQ}{Q}\frac{x^{yQ}}{(1+x)^{yQ+Q}} + (x-y)I_{\frac{1}{1+x}}(Q+1,yQ)} - \frac{\left[(2Q+(yQ+1))\frac{x-1}{x} - Q\right]\binom{Q+yQ}{Q}\frac{x^{yQ}}{(1+x)^{yQ+Q}}}{y - y\binom{Q+yQ}{Q}\frac{x^{yQ}}{(1+x)^{yQ+Q}} + (x-y)I_{\frac{1}{1+x}}(Q+1,yQ)}.$$

Now that the denominators of the all three terms are the same and also the numerators of the second and third term have the same structure, we add all three terms, resulting in Equation (9). □

5 Results

In this section, we first analytically determine under which conditions on the fixed procurement cost joint ordering is preferred to independent ordering for the companies. We do this for two specific cases. After presenting these analytical results, we numerically determine the optimal order quantities. We continue with numerically comparing the total costs under joint and independent ordering. We conclude this section by comparing our numerical results to our analytical results.

5.1 Analytical comparison of the total costs

We start with the following lemma, which provides us useful insight on which we can base our analytical results.

Lemma 1. *When both companies order one item, joint ordering has the same cost as independent ordering, for any difference in demand rate of the companies (so for all values of x).*

Proof. Since both companies order one item, we take $y = Q = 1$. We first determine the cost of independent ordering, using Equations (1) and (2), yielding

$$K_1^{\text{ind}}(1) + K_2^{\text{ind}}(1) = A\lambda(1+x) + 2h. \tag{14}$$

Furthermore, by Theorem 1 we find that the joint cost in this case is

$$K(1,1) = \frac{1}{2}h(2-x)\frac{x-1}{x} + \frac{A\lambda x + \frac{1}{2}h\left[1+x+(1-x^2)I_{\frac{1}{1+x}}(2,1)\right]}{1 - 2\frac{x}{(1+x)^2} + (x-1)I_{\frac{1}{1+x}}(2,1)}.$$

The value of the generalized incomplete beta function that appears in the above expression is $\frac{1}{(1+x)^2}$. Now multiplying both the numerator and the denominator of the second term in the expression by $(1+x)^2$ (and effectively thus multiplying by 1) yields

$$K(1,1) = 2h + A(1+x).$$

Using basic algebra it can be shown that this expression equals the cost of independent ordering, in Equation (14). □

Since the costs of joint and independent ordering are equal to each other when both companies order one item, a good strategy would be to order a quantity slightly above this order quantity more frequently. This strategy is effective under the condition that the procurement cost is low enough, because then replenishing the inventory is not really expensive. This way, by ordering often a low amount of items, the companies can save on holding costs. Since we assume that the order quantity can only be integer valued, this strategy means that at least one of the two the companies should order 2 items per time in order to gain benefits from joint ordering.

As mentioned, this strategy is effective for values of the procurement cost that are low enough. However, the most obvious reason for ordering items jointly is to save on the fixed procurement cost, because under joint ordering, the companies split this cost and only pay part of it. But if it is too low, there is not much money to save. Therefore, our aim is to determine a lower bound for this procurement cost in two different cases. In the first case, both companies order two items. The condition on the procurement cost under which joint ordering is preferable is stated in the following theorem.

Theorem 2. *In the case that both companies order two items, joint ordering is preferred to independent ordering if and only if* $A > \frac{h}{\lambda}\frac{x^2+1}{x^2+x}$.

Proof. Both companies order two items, which means that we set $Q = 2$ and $y = 1$. Now we determine a condition for the procurement cost such that the cost of joint ordering is lower than that of independent ordering; in other words, for which values of A we have $K(2,2) < K_1^{\text{ind}}(2) + K_2^{\text{ind}}(2)$. Using Equations (1) and (2), we obtain that the right-hand side equals

$$K_1^{\text{ind}}(2) + K_2^{\text{ind}}(2) = \frac{1}{2}A\lambda(1+x) + 3h.$$

By Theorem 1 we have that the joint cost in this case is given by

$$K(2,2) = \frac{1}{2}h(3-x)\frac{x-1}{x} + \frac{\frac{1}{2}A\lambda x + \frac{1}{2}h\left[2(1+x) + (1-x^2)I_{\frac{1}{1+x}}(3,2)\right]}{1 - 6\frac{x^2}{(1+x)^4} + (x-1)I_{\frac{1}{1+x}}(3,2)}.$$

The value of the generalized incomplete beta function in this expression is $\frac{1+4x}{(1+x)^4}$. We multiply the second term of the expression for the joint cost by $\frac{(1+x)^4}{(1+x)^4}$ and perform basic algebra to the equation of the joint cost and arrive at the following expression:

$$K(2,2) = \frac{1}{2}h\frac{-x^2+4x-3}{x} + \frac{\frac{1}{2}A\lambda x(1+x)^4 + \frac{1}{2}h\left[2x^5 + 10x^4 + 16x^3 + 19x^2 + 14x + 3\right]}{x(x^3+4x^2+4x+1)}.$$

Now $K(2,2) < K_1^{\text{ind}}(2) + K_2^{\text{ind}}(2)$ yields $A > \frac{h}{\lambda}\frac{x^2+1}{x^2+x}$. □

We know by Lemma 1 that with an order quantity of one item for both companies the costs are equal. Therefore, we are looking for strategies that include order quantities that are slightly higher than one item and thus also consider the case that one of the companies orders one item and the other one two items. For this case, we again determine a lower bound for the procurement cost. We let company 1, having demand rate λ, be the company that orders one item and the other company be the one that orders two items. We have this assumption because the latter company has a higher demand and hence is more likely to order more items than the company having the lower demand rate.

Theorem 3. *In the case that the company with demand rate λ orders one item and the company with demand rate $x\lambda$ orders two items, joint ordering is preferred to independent ordering if and only if $A > \frac{h}{\lambda x}$.*

Proof. Company 1 orders one item and company 2 orders two items, which means that we set $Q = 1$ and $y = 2$. Now we solve for which value of the procurement cost A the cost of joint ordering is lower than that for independent ordering, meaning $K(1,2) < K_1^{\text{ind}}(1) + K_2^{\text{ind}}(2)$. The cost of independent ordering is given by

$$K_1^{\text{ind}}(1) + K_2^{\text{ind}}(2) = A\lambda\left(1 + \frac{1}{2}x\right) + \frac{5}{2}h.$$

Furthermore, by Theorem 1, the cost of joint ordering is given by

$$K(1,2) = \frac{1}{2}h(3-x)\frac{x-1}{x} + \frac{A\lambda x + \frac{1}{2}h\left[2(1+x) + (1-x^2)I_{\frac{1}{1+x}}(2,2)\right]}{2 - 6\frac{x^2}{(1+x)^3} + (x-2)I_{\frac{1}{1+x}}(2,2)}.$$

We obtain that the value of the generalized incomplete beta function in this equation equals $\frac{1+3x}{(1+x)^3}$. After multiplying the second term by $\frac{(1+x)^3}{(1+x)^3}$ and performing basic algebra we derive that

$$K(1,2) = \frac{1}{2}h\frac{-x^2+4x-3}{x} + \frac{A\lambda x(1+x)^3 + \frac{1}{2}h\left[2x^4 + 5x^3 + 11x^2 + 11x + 3\right]}{x(2x^2+3x+1)}.$$

Now, $K(1,2) < K_1^{\text{ind}}(1) + K_2^{\text{ind}}(2)$ yields $A > \frac{h}{\lambda x}$. □

5.1.1 Cost per company

Now that we investigated under which conditions the total cost of joint ordering is lower than the total cost of independent ordering of both companies together, it is a good question whether it is for both companies equally desirable to order items jointly in these situations. This question could be answered easily by arguing that, given that the total joint cost is less than the sum of the costs of independent ordering, there is always a way of dividing the procurement cost such that both companies pay at most as much for joint ordering as for independent ordering.

5.2 Numerical comparison of the total costs under optimal order quantities

In this subsection we compare the cost of joint and independent ordering to each other under optimal order quantities. We do this numerically, since the obtained expressions of the joint cost are too complex for analytical comparison with the cost of independent ordering. For the numerical comparison we select our parameter values similar to the values that are used in the article by Timmer et al. [11]. Each experiment is such that we first fix the values of h, λ and x and then carry out the comparison for different values of A. The exact choices of our parameters are in the following ranges.

- $A \in \{50, 100, 150, 200, 250\}$
- $h \in \{6, 10\}$
- $\lambda \in \{20, 40\}$
- $x \in \{2, 4\}$

Before we can execute this numerical comparison, we need to determine what the optimal order quantities are for both joint and independent ordering.

5.2.1 Optimal order quantities

We determine the optimal order quantities for both the situation of joint ordering and independent ordering. The order quantities of the two companies do not need to be the same. Once we determined the optimal order quantities for both situations, we can determine whether or not they differ and analyse the difference.

For the situation of independent ordering, we can calculate the optimal order quantities using the formula that was for example given in the article by Timmer et al. [11]:

$$Q_{\text{opt}} = \sqrt{\frac{2A\lambda}{h}},$$

and if Q_{opt} is not an integer, then it is rounded to the integer that results in the lowest cost, because the cost function is a convex function in Q.

For the determination of the optimal order quantities under joint ordering we use the convexity of the joint cost function. We do not show the convexity in a mathematical way. However, instead we provide a graph of the joint cost function against the order quantities of both companies in Figure 5. When we plot the cost for other parameter choices, the graph is similar. From this figure it becomes clear that we can consider the joint cost function to be convex. The reason we do this assumption is that we now can make use of an important and useful property of convex functions, namely that convex functions have a unique minimizer. Due to this property the optimization of the joint cost function becomes more efficient, as we now know that we have to search in directions of order quantities that decrease the joint cost function the most. With directions we mean the combinations of order quantities of the companies.

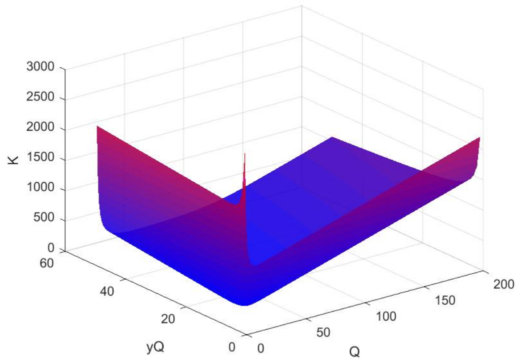


Figure 5: Joint cost for $A = 50, h = 6, \lambda = 20$ and $x = 2$.

Using a Python program we can calculate the optimal order quantities in both the situation of joint and independent ordering. For the exact implementation a request can be made at the author[1]. As the standard situation we fix the parameters to be

- $h = 6$
- $\lambda = 20$
- $x = 2$.

We first determine the optimal order quantities for both companies in this situation for different values of A, as specified before. Next, we change each of the three parameters h, λ and x separately and again determine optimal order quantities for the specified values of A. The results of the numerical calculations are shown in Table 1. Here, $C1$ denotes company 1 and $C2$ denotes company 2.

We see that the optimal order quantities for both companies under joint ordering are lower than the optimal order quantities under independent ordering for different values of the procurement cost A. We also observe this if we change the parameters h, λ and x.

[1] f.a.boelens@student.utwente.nl

Table 1: The optimal order quantities under joint and independent ordering for different values of A, h, λ and x.

	Optimal order quantity Q			
	Independent		**Joint**	
A	**C 1**	**C 2**	**C 1**	**C 2**
50	18	25	12	21
100	25	36	17	30
150	31	44	21	37
200	36	51	24	43
250	40	57	26	47

(a) $h = 6, \lambda = 20, x = 2$.

	Optimal order quantity Q			
	Independent		**Joint**	
A	**C 1**	**C 2**	**C 1**	**C 2**
50	14	20	10	17
100	20	28	14	24
150	24	34	16	29
200	28	40	19	34
250	31	44	21	37

(b) $h = 10, \lambda = 20, x = 2$.

	Optimal order quantity Q			
	Independent		**Joint**	
A	**C 1**	**C 2**	**C 1**	**C 2**
50	25	36	17	30
100	36	51	24	43
150	44	63	29	53
200	51	73	33	60
250	57	81	37	68

(c) $h = 6, \lambda = 40, x = 2$.

	Optimal order quantity Q			
	Independent		**Joint**	
A	**C 1**	**C 2**	**C 1**	**C 2**
50	18	36	11	33
100	25	51	15	47
150	31	63	17	55
200	36	73	20	65
250	40	81	22	73

(d) $h = 6, \lambda = 20, x = 4$.

5.2.2 Comparison of the total cost

Now that we have determined the optimal order quantities for both companies under joint and independent ordering, we can use these quantities to determine the total optimal cost of joint and independent ordering. For the calculations we have the same parameter choices as for the determination of the optimal order quantities. The results of these calculations are shown in Table 2.

We see that the costs of joint ordering are lower than those of independent ordering for the tested parameter choices. We also observe this if we change the parameters h, λ and x.

Table 2: The costs of joint and independent ordering for different values of A, h, λ and x.

A	Cost Independent C 1	C 2	Total	Joint Total
50	113	158	271	219
100	158	222	380	303
150	193	271	464	367
200	222	313	535	420
250	248	349	597	467

(a) $h = 6, \lambda = 20, x = 2$.

A	Cost Independent C 1	C 2	Total	Joint Total
50	146	205	351	289
100	205	288	493	398
150	250	351	601	481
200	288	405	693	550
250	321	452	774	611

(b) $h = 10, \lambda = 20, x = 2$.

A	Cost Independent C 1	C 2	Total	Joint Total
50	158	222	380	303
100	222	313	535	420
150	271	382	654	509
200	313	441	754	584
250	349	493	842	650

(c) $h = 6, \lambda = 40, x = 2$.

A	Cost Independent C 1	C 2	Total	Joint Total
50	113	222	335	276
100	158	313	471	382
150	193	382	575	463
200	222	441	663	532
250	248	493	741	592

(d) $h = 6, \lambda = 20, x = 4$.

5.3 Comparison of of analytical and numerical results

For our numerical results of the previous subsections we chose our parameters similar to the parameters chosen by Timmer et al.[11]. However, when we look at analytical results, we see that cooperation is beneficial for a value of A that is higher than $\frac{A}{\lambda}$ multiplied by a function of x. This function of x depends on the order quantities, for which we considered two cases. For these results, see Section 5.1. These results where obtained under the assumption that for a lower procurement cost, the optimal order quantities decrease, because ordering more often less items becomes more attractive because of the holding costs. Hence, we arrived at analytical results for the cases that both companies order two items and that one company orders one item and the other company orders two items, which are order quantities that are slightly above one item, for which the costs of independent and joint ordering are equal. Both situations give another lower bound for the procurement cost for which cooperation is beneficial.

In order to see whether our numerical results are consistent with our analytical results, we investigate the behaviour of the total costs and the optimal order quantities for values of A close to $\frac{h}{\lambda}$, since this value of A is the lower bound for identical companies for which joint ordering is beneficial [11] and this fraction also appears in the lower bounds of A that we obtained. In addition, we investigate the behaviour of the costs and the optimal order quantities for values of A close to $\frac{h}{\lambda x}$ and $\frac{h}{\lambda}\frac{x^2+1}{x^2+x}$. The values of A that we chose in Section 5.2 are significantly higher than these values, which caused that joint ordering was beneficial in every situation that we investigated.

We first choose values of the parameters h, λ and x and then determine the optimal order quantities and the costs for different values of A, according to the reasoning we mentioned above. For this investigation we depict one specific situation. For this situation we choose the following values for the parameters:

- $h = 200$
- $\lambda = 2$
- $x = 2$.

Hence, we investigate the optimal order quantities and costs at least for values of A around the following values:

- $\frac{h}{\lambda} = 100$
- $\frac{h}{\lambda}\frac{x^2+1}{x^2+x} = 83\frac{1}{3}$
- $\frac{h}{\lambda x} = 50$.

In addition, we include other values of A for which we investigate the costs and order quantities. The results are in Table 3. All outcomes are rounded off to the closest integer.

Table 3: The optimal order quantities and costs for different values of A.

	Optimal order quantity Q			
	Independent		**Joint**	
A	**C 1**	**C 2**	**C 1**	**C 2**
10	1	1	1	1
20	1	1	1	1
30	1	1	1	1
49	1	1	1	1
50	1	1	1	1
51	1	1	1	2
83	1	1	1	2
84	1	1	1	2
90	1	1	1	2
99	1	1	1	2
100	1	2	1	2
101	1	2	1	2
110	1	2	2	2
150	1	2	2	2
200	2	2	2	3
250	2	3	2	3

(a) The optimal order quantities.

	Cost			
	Independent			**Joint**
A	**C 1**	**C 2**	**Total**	**Total**
10	220	240	460	460
20	240	280	520	520
30	260	320	580	580
49	298	396	694	694
50	300	400	700	700
51	302	404	706	704
83	366	532	898	819
84	368	536	904	822
90	380	560	940	844
99	398	596	994	876
100	400	500	900	880
101	402	502	904	884
110	420	520	940	915
150	500	600	1100	1014
200	500	700	1200	1113
250	550	733	1283	1203

(b) The costs.

Our first observation is that for values of A higher than 50, the optimal order quantity of company 2 under joint ordering changes. This change causes that the optimal order quantities under joint ordering become the same as the optimal order quantities that we considered in Theorem 3. In addition, up to and including $A = 50$ the costs of joint and independent ordering are equal, which can be explained by Lemma 1 since the optimal order quantities for both companies for both independent and joint ordering equal one. For higher values of A the costs of joint ordering are lower than those of independent ordering, which also follows from Theorem 3. However, in this theorem, we considered the optimal order quantities under independent ordering to be the same as those under joint ordering. Now our numerical outcomes show that even for different order quantities between the situation of independent and joint ordering, joint ordering is advantageous for values of A higher than the value we found in Theorem 3. For higher values of A, the optimal order quantities increase. When the optimal order quantities become 2 for both companies, the value of A is already higher than the lower bound that we determined in Theorem 2. Therefore, it is logical that also in the cases where the optimal order quantities are 2 for both companies joint ordering is beneficial. Our numerical results here show that for higher values of A, associated with higher optimal order quantities, joint ordering remains beneficial. This is also shown by our numerical results in Section 5.2, where relative high values of A were chosen.

Different choices for h, λ and x give similar results. Namely, up to and including $A = \frac{h}{\lambda x}$ the optimal order quantities are one for both companies under both independent and joint ordering. In addition, the costs of joint and independent ordering are equal. The closest value of A above $\frac{h}{\lambda x}$ results in an optimal order quantity of two items for company 2 under joint ordering, while the rest of the optimal order quantities stay the same. Values of A higher than this value result in lower costs for joint ordering than for independent ordering. Furthermore, in the cases that the optimal order quantities are two for both companies, as considered in Theorem 2, the value of A is significantly larger than $\frac{h}{\lambda}\frac{x^2+1}{x^2+x}$. Therefore, under these order quantities, the lower bound of A for which joint ordering is beneficial is always reached and therefore joint ordering is always beneficial under these optimal order quantities. Additionally, no choice of parameters result in a situation where independent ordering has lower costs than joint ordering. For decreasing A, the optimal order quantities decrease until they reach a value of 1 and from this moment the costs of joint and independent ordering are equal to each other.

6 Sensitivity Analysis

In this section, we investigate the effect of parameters on the optimal order quantities under joint and independent ordering and on the extent to which joint ordering is advantageous. In Section 5 we already showed the values of the optimal order quantities and the total costs for specific values of the parameters. Now we investigate their influence on the optimal order quantities and the costs by choosing the parameters from a significantly wider range. The parameters that we consider in our sensitivity analysis are the fixed procurement cost A, the holding cost h, the demand rate of the first company λ and the difference factor in demand rate x. Before we can perform this analysis for each of the parameters, we have to choose a basis situation of the parameters. All parameters except for the parameter that we investigate have the value defined in the basis situation. We choose the parameters to be:

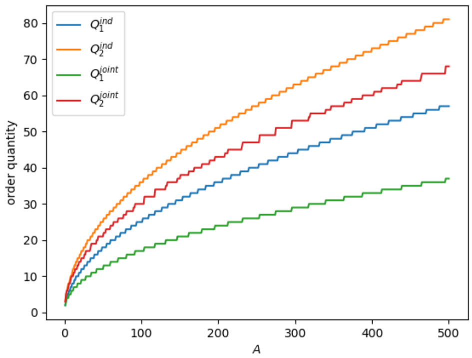
- $A = 50$
- $h = 6$
- $\lambda = 20$
- $x = 2$.

For the investigation of the effect of the parameters, we use the following ranges of the parameters:

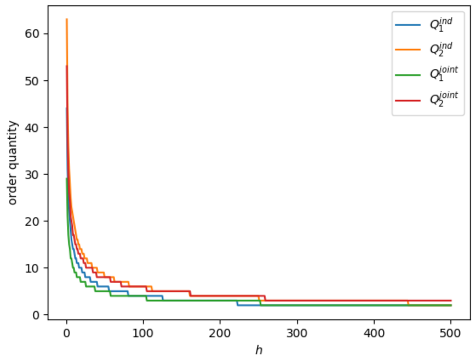
- $A \in \{1, 2, \ldots, 500\}$
- $h \in \{1, 2, \ldots, 500\}$
- $\lambda \in \{1, 2, \ldots, 100\}$
- $x \in \{1, 2, \ldots, 20\}$.

6.1 Optimal order quantities

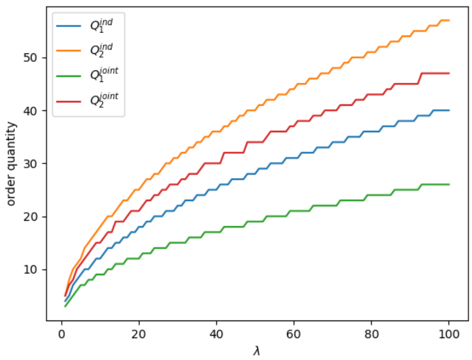
First we investigate the influence of the parameters on the optimal order quantities for both companies under both independent and joint ordering. The results of this investigation are in Figure 6. In the legends of these graphs the subscript of the Q indicates the company and the superscript indicates whether the order quantity is from the situation of independent or joint ordering.



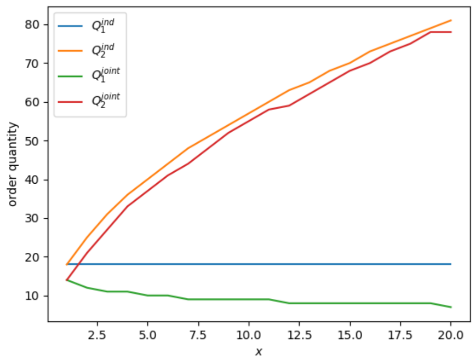
(a) The relation between A and the optimal order quantities.



(b) The relation between h and the optimal order quantities.



(c) The relation between λ and the optimal order quantities.



(d) The relation between x and the optimal order quantities.

Figure 6: The optimal order quantities for different parameter choices.

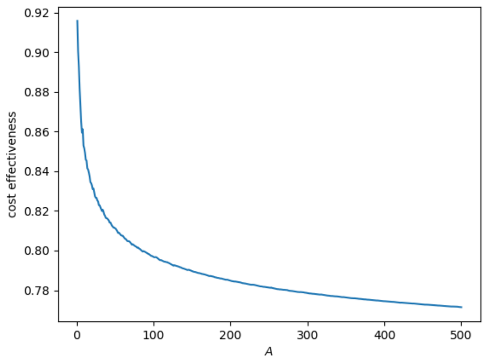
We see that the order quantities for both companies increase as the fixed procurement cost increases or the demand rate increases, in both the situation of independent and joint ordering. On the other hand, the optimal order quantities for both companies decrease as the holding cost increases, in both situations. For increasing x the effect on the optimal order quantity of company 1 under both independent and joint ordering is very limited. However, the optimal order quantity of company 2 increases as x increases, in both situations.

6.2 Cost effectiveness

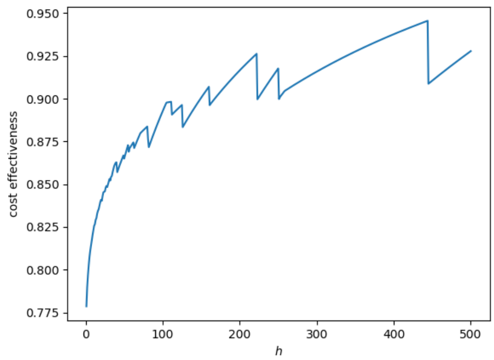
Now we investigate the influence of the parameters on the so-called cost effectiveness, which indicates the advantage of joint ordering relative to independent ordering. We define the cost effectiveness to be [11]

$$\text{cost effectiveness} = \frac{\text{cost joint ordering}}{\text{cost independent ordering}}.$$

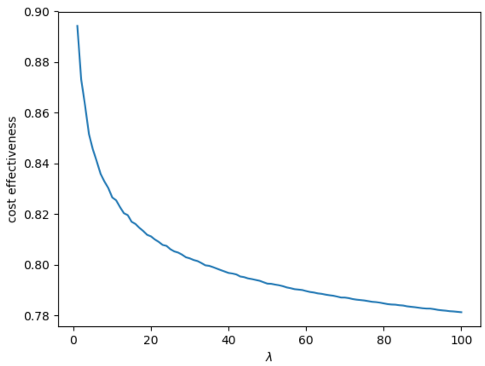
The costs in this fraction are the total costs of ordering for both companies added and not the costs per company. As long as the cost effectiveness is below 1, then the costs of joint ordering are lower than those of ordering independently and hence it is more beneficial to order jointly. According to the definition, a low value of cost effectiveness means that it is more beneficial to order jointly than to order independently for the companies. The results of the relation between the parameters and the cost effectiveness are in Figure 7.



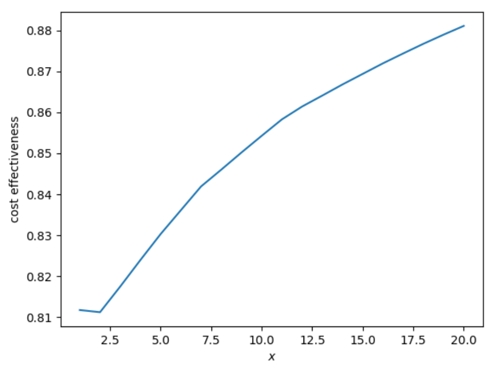
(a) The relation between A and the cost effectiveness.



(b) The relation between h and the cost effectiveness.



(c) The relation between λ and the cost effectiveness.



(d) The relation between x and the cost effectiveness.

Figure 7: The cost effectiveness for different parameter choices.

We see that the cost effectiveness increases as the h and x increase. So for increasing h and x it becomes less beneficial to order jointly for the companies. On the other hand, the cost effectiveness decreases for increasing A and λ and hence becomes more beneficial. Furthermore, we see that the cost effectiveness in all graphs remain below 1, which indicates that joint ordering yields lower costs than independent ordering.

7 Discussion

In this section, we comment on the obtained results that are presented in the previous section. We start with discussing an important assumption that we made. Then we discuss whether the results are consistent with earlier research and analyse whether our results are how we could expect them to be. We start with our analytical results represented in Section 5.1 and afterwards we discuss our numerical results on the costs under optimal order quantities as presented in Section 5.2.

In our research we considered the holding cost of both companies to be the same. We did this for simplicity and in order to derive analytical results. However, this assumption is not really representative for a real situation. Nevertheless, the results we obtained using the assumption can give us useful understandings and knowledge which can be applied to real situations.

Our first analytical result, stated in Lemma 1, tells us that the costs of joint and independent ordering are equal when both companies order one item. This result is comparable to the result in the paper of Timmer et al. [11]. Our situation differs from their situation in the sense that in our situation the companies can have different demand. However, we found out that the difference in demand does not matter and that the same result as in the situation with equal demand still holds.

In order to get the next two results, stated in Theorems 2 and 3, we based our reasoning about order quantities on the paper of Timmer et al. [11] and considered two cases comparable to the case in this paper, where both companies order two items. In the mentioned paper the result was that for two identical companies joint ordering is preferred to independent ordering if and only if $A > \frac{h}{\lambda}$. In other words, a lower bound was found for the fixed procurement cost. In our situation we consider the companies to have different demand rate and possibly to have different order quantities. In the cases that we considered we fixed the difference in order quantity but still allowed the companies to have different demand. Therefore we expected to find a similar lower bound for the fixed procurement cost, where in addition x appears. So it is not unexpected that we obtained results in the form $A > \frac{h}{\lambda}$ multiplied by a function of x. Furthermore, when we substitute $x = 1$ in the lower bound that we obtained for the fixed procurement cost, we get the the lower bound that was known for two companies having the same demand rate. Therefore, we can conclude that these obtained results are consistent with both earlier research and our expectations, that are also partly based on earlier research.

Now that we have discussed our analytical results, we continue with our numerical results. Our first numerical result includes the optimal order quantities. As reported in Section 5.2, the optimal order quantities under joint ordering are lower than those under independent ordering, for all of our parameter choices. This result is not really surprising, as under joint ordering the companies can split the fixed procurement cost. This gives the companies an incentive to order more often, and less items per order in the case they order jointly, which results in less items on stock and hence lower holding costs per time unit. This reasoning can also be applied to the next result, namely that the cost of joint ordering is lower than the cost of independent ordering for all of our parameter choices. That is, due to the lower order size under joint ordering, the holding costs decrease when the companies choose to order jointly. This way, the total costs for joint ordering are lower than those for independent ordering. However, as we mentioned, splitting the fixed procurement cost gives the companies incentive to order more often less items. Therefore, the companies more often have to pay (part of) the fixed procurement cost. Nevertheless, this apparently does not outweigh the cost savings per time unit of the holding costs, according to our results.

When we investigated the optimal order quantities and the costs around values of A that coincide with the lower bounds that we found for A analytically, we expected that we could find a combination of A, h, λ and x such that the costs of independent ordering are lower than the costs of joint ordering. However, we do not find such a situation. Nevertheless, we discover that the lower bound of A according to Theorem 3 indeed was an important value in our numerical comparison of the costs. Up to including this value, the optimal order quantities for both companies under both independent and joint ordering are one and (in accordance with Lemma 1) the costs of independent and joint ordering are equal. Values of A higher than the obtained lower bound resulted in an increase of the order quantity of company 2 under joint ordering and even higher values of A cause increases of the order quantities of both companies in both order situations. This result is explainable in the sense that it is logical that first the optimal order quantity of company 2 increases, since this company faces a higher demand. As a consequence, the optimal order quantities under joint ordering are the same as considered in Theorem 3. Therefore, it is not surprising that joint ordering is beneficial in this situation, although the optimal order quantities under independent ordering are still equal to one item for values of A slightly higher than this value.

Our sensitivity analysis shows the influence of the parameters on the optimal order quantities and the total costs. We start by discussing the results regarding the optimal order quantities. When the fixed procurement cost A increases, then the companies have to pay more money per order and hence this gives an incentive to order less often. Since they still face the same demand, they have to order more items per order. Hence, this relation is as we could have expected. If the holding cost increases, then it becomes more expensive to have items on stock and therefore it is beneficial to order less items more often so that the average number of items on stock is minimal. Therefore, it is logical that the order quantities increase. Furthermore, if λ increases, the demand of both companies increases and hence the companies have to order more items per time unit. Therefore, it is logical that the optimal order quantities increase for increasing λ. Finally, increasing x does not have a significant effect on the order quantity under joint ordering of company 1, which is logical because an increasing value of x means that company 2 faces a higher demand. However, company 1 does not want to order more often, since they still face the same demand. Therefore, it is as expected that the optimal order quantities of company 2 increase for increasing x.

Now we continue with discussing the sensitivity analysis on the cost effectiveness. When A increases, the cost effectiveness becomes lower, meaning that joint ordering becomes more beneficial. This can be explained by the reasoning that under joint ordering the companies can share this cost. And hence, when this fixed procurement cost is higher, they relatively save more money by ordering jointly. When the holding cost increases, the cost effectiveness also increases. The reason of this could be that the companies do not want much items on stock and therefore they want to order less items per order, but order more often. We observed in our numerical results that the difference in joint and independent costs decreases for decreasing order size. Therefore, the cost effectiveness increases. We also see that the graph of the relation between h and the cost effectiveness is not smooth. This is due to the restriction that the order quantities need to be integer valued. When λ increases, then the companies have to order more items per time unit. We also expect the companies to order more often, especially in the case of joint ordering since the cost per order is lower in this case. Under independent ordering companies also order more often for higher demand, but not so much more as under joint ordering. In this case they have more incentive to order more items instead of ordering more often, which also causes higher holding costs. Therefore, it is logical that the costs of joint ordering are lower than those of independent ordering for higher λ. Finally, when x increases, then it becomes less beneficial to order jointly. This can be explained by the fact that now the difference in demand increases. Therefore, company 2 wants to order more often and/or more items. Since company 1 keeps facing the same demand, it does not want to order more often. Therefore, under joint ordering, company 2 has to order the same number of times and hence more items per time. This results in more items on stock on average and hence higher holding costs, while under independent ordering, company 2 could order less items more often. In our sensitivity analysis we observe that for a large range of parameters, the cost effectiveness never reaches the value of 1, meaning that joint ordering is beneficial in all tested situations. This is in line with our other result of not being able to find a combination of parameters for which independent ordering is beneficial, which is explained above.

8 Conclusion

We analyse the cost structure of both independent and joint ordering for two companies having different demand rate and order quantities. However, the two companies have the same holding cost. Both companies face a Poisson demand and their inventories are reviewed continuously. As soon as the inventory level drops below a prescribed level, an order is placed. In particular, we study the costs in the situation in which the companies place a joint order as soon as the inventory of one of the companies needs to be replenished, resulting in an explicit expression.

We analytically identify when joint ordering is advantageous for two companies in two specific cases, in the form of a lower bound on the fixed procurement cost, which needs to be paid per order. Numerical experiments show that the optimal order quantities are lower under joint ordering for a value of the fixed procurement cost that is higher than one of the lower bounds we derived analytically. In addition, the costs of joint ordering are lower than those for independent ordering. For values of the fixed procurement cost that are smaller or equal to this lower bound, the optimal order quantities of both companies under both joint and independent ordering are equal to one. In these cases, the costs of joint and independent ordering are equal. Hence, from our experiments we observe that the cost of independent ordering is not lower than that of joint ordering for a wide range of parameter choices. Therefore, we can conclude that in general joint ordering is preferred to independent ordering.

In further research the model can be extended to a model where the holding costs of the companies may be different. However, this extension causes the expressions for the total cost to become significantly more complex and therefore analytical comparison can become too complicated. Nevertheless, a numerical comparison can still be executed on situations including the mentioned extensions. Furthermore, in further research a situation of three or more companies can be studied. And hence, similar research in the case of a general number of companies can be done. Finally, an approach of dividing the joint costs among the companies can be studied, with as aim to find a way such that joint ordering is advantageous for both companies.

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