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Proof of All

Verifiable Computation in a Nutshell

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Abstract Recent advances in the cryptographic field of "Zero-Knowledge Proofs" have sparked a new wave of research, giving birth to many exciting theoretical approaches in the last few years. Such research has often overlapped with the need for private and scalable solutions of Blockchain-based communities, resulting in the first practical implementations of such systems. Many of these innovative constructions have developed in parallel, using different terminologies and evolving into a fragmented ecosystem, calling for their consolidation into the more stable domain of "Verifiable Computation". In this master thesis I propose a unifying Verifiable Computation model for the simplification and efficient comparison of all cryptographic proof systems. I take advantage of this model to analyse innovative technologies (Homomorphic Authenticators, Verifiable Delay Functions) which developed into their own specialised domains, and I attempt to make them more accessible for newcomers to the field. Furthermore, I expand on the future of Verifiable Computation, Universal proof compilers and "Proofs of All", by approaching the state-of-the-art zk-STARK construction from a more accessible and informal design perspective.

Thanks To those who supported me in times of need, and to those who gave me technical advice. In particular: Mamma e Papà, prof. Ligeti, Mathijs, Eszter, István.

Let's shoot for the moon. If we miss, we'll still land amongst the stars.

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Objectives

The objective of this MSc thesis is to tackle innovative technologies from a unifying perspective, prioritising simplicity and understanding over obscure constructions, and benefitting further popularisation of the cryptographic proofs domain. My main incentive for writing this thesis is the conflict of interest that exists in the security field between the desire to exploit innovations in cryptography for building disruptive technologies (e.g. the "blockchain revolution"[1]), and the complexity and knowledge barrier required to understand them, which leads to consistent misinformation in the market (e.g. the "Bitconnect scandal" [2]) as well as in developer channels, and fragmentation within the research community.

The domain I target is that of cryptographic proof systems, and, specifically, I gather them under the umbrella term *"Verifiable Computation"*. There are **3 thesis objectives** which this work hopes to achieve:

1. *A Unifying Model*

   for the cryptographic Verifiable Computation domain. The idea is to select and define the most important and comprehensive properties that have been spread out over various VC technologies in the course of more than 3 decades, sometimes using different names and definitions, By using a standardised model for defining protocols, researchers can attempt to merge the fragmented domain of cryptographic proofs, and thus unite their efforts under a single research domain.

2. *Technical Analysis*

   of VC technologies, revisiting exciting and correlated protocols using the unifying model. While the focus is kept only on more favoured constructions of recent years, I wish to help new researchers get quickly acquainted with the VC landscape, hopefully leading to further popularisation and systemisation of the domain.

3. *A Simplified Guide*

   to understanding the VC domain and its prominent technologies. This "layman's view" of core cryptographic properties is achieved through uncompromisingly logical and verbose debates on each technical design, where no assumptions are left to the imagination. While there are many formal definitions used to organise technical details, they are always accompanied by informal descriptions. I believe in simplifying technical constructions as much as possible, as it is paramount to their implementation and diffusion in the engineering field; it also stimulates further research, as the cryptocurrency community has proven[3].

[1] for a list of companies currently investing in blockchain, see [1]

This thesis is divided into 3 chapters, which reflect the objectives set forth during the development of this work. **Chapter I, *"Introduction and VC Model"***, discusses the history behind proof protocols while giving a gentle introduction to the topic, progressively presenting my systemising model for understanding and analysing VC protocols and properties. **Chapter II, *"Non-universal VC Protocols"***, introduces and gathers currently expanding and innovative VC technologies (i.e. HAUTHs, VDFs) which have previously been considered within separate domains, offering a comprehensive analysis under the model provided in the first chapter. **Chapter III, *"Universal VC Compilers"***, introduces to our model and breaks down state-of-art prominent VC technology (i.e. STARKs) which has yielded groundbreaking results, with the potential to revolutionise the cryptographic community as well as disrupt the cryptocurrency market itself.

1 Introduction and VC Model

Throughout the past few decades, our society has put a great deal of effort into developing technologies upon which to build *trusted platforms* and services. Along with this explosion of services, the Internet has brought digital freedom into our daily lives. The latest example of this is distributed networking (e.g. Bitcoin, Ethereum), which aims to replace important societal functions. This latest trend marks an important milestone of our globalised society: the time has come to build *trustless platforms*, built upon technologies we can all indiscriminately trust.

The services we are speaking of take electronic form and exist only in the realm of Computer Science, which imposes restrictions on what security and trust really mean, expressed fundamentally in the form of Cryptography. Cryptography the art, Cryptography the science, has been developing at an accelerated rate of research ever since human conflict,[1] and the need for trusted communication, has existed. What we deal with in this thesis is "how to trust *someone* doing *something* with *some secret data*". That phrase might seem a little vague, but I promise: it encompasses so many notions of computer science and cryptography that it extends to virtually any computation on paper or silicon. To define this domain, I'll use the term "Verifiable Computation" (or VC).

In order to explain what it means to *prove computation*, I would like to start by taking a brief look at the most common form of provably secure computation since the birth of the Web: Authentication (Section 1.1). Afterwards, I'll move on to define formally (and informally) what generic cryptographic proofs entail (Sections 1.2, 1.3); then how to perform privacy-friendly computations of hidden variables using Zero-Knowledge Proofs (Section 1.4); then, how to do this without multiple rounds of communication (Section 1.5), more efficiently and with less communication overheads (Section 1.6), as well as other VC properties and open questions (Section 1.7). For a discussion on how to put it all together in a single convenient package, please check Section 3.1.2 in the chapter on Universal Verifiable Computation.

1.1 Identification Schemes and Authentication

Before we talk about Verifiable Computation, let's scale down a bit and talk about a simpler concept: Identification Schemes. The process of authenticating a user, which simply defines a Prover showing to a Verifier that he knows a specific non-deterministic secret relating to his publicly known identifier, has roughly evolved under the three following cryptographic constructions.

[1] it appears that Caesar was also a big fan of cryptography :)

1.1.1 Simple Password-based Authentication

In this naïve approach the Verifier doesn't trust the Prover, so he asks him to send over the secret password (i.e. "knowledge witness") so that he may verify it.

$$Prover \overset{password}{\rightarrow} Verifier$$

This is an extremely flawed protocol because:

1. the whole secret is revealed to the Verifier (if the Prover actually knows the password, that is);
2. and anybody else looking at this conversation;
3. the secret password needs to be well protected and stored by both parties.

The following solutions have been devised to improve this method:

1. N/A (it is a requirement of the protocol);
2. secure unicast communication channels, e.g. HTTPs;
3. the Server stores the password in Hash+Salt format, or Encrypted format. This helps take stress off hacked servers whose database is compromised, as long as the hack is detected (otherwise, the webpage can be modified to redirect login attempts). Another solution is to have an auxiliary check (called Two Factor Authentication, or 2FA) using one-time tokens sent via SMS (or an app); unfortunately this is mainly a convenient hack invented by the industry to patch up the inherent weaknesses of this authentication approach, and it is as secure as the device and communication used to receive the token, as well as the token generation process itself.

Even though Simple Password-based authentication is a very straightforward interaction (the user types into a text box) it still conveys a false sense of security, leads to many failed login attempts (due to typing mystakes) as well as poor password generation habits by lazy or misinformed users.

1.1.2 Challenge-Response Authentication Protocols

In this approach neither party can trust the other, or the communication channel may be unsafe, so they take advantage of any "hard cryptographic problem" to challenge knowledge of the secret. Essentially, Asymmetric Encryption and Signature schemes help provers avoid revealing their secret to malicious verifiers.

$$Prover \overset{challenge}{\leftarrow} Verifier$$
$$Prover \overset{response}{\rightarrow} Verifier$$

(where the challenge is a random number chosen by the Verifier, which the Prover must use to generate as response a unique signature or reveal a random message previously encrypted by the Verifier.)

This method is already a huge improvement over the previous one, and yet it has received very little adoption amongst the most popular Web services, even though it could easily be implemented through browser plugins and software wallets. In fact, its most widespread adoption seems to be physical authentication cards, used for traditional banking transactions at ATMs or for authorising entry to company offices.

There is still one small issue: Challenge-Response protocols still reveal some information, such as unique signatures or decrypted cipher-texts. While Encryption and Signature schemes are chosen to leak as little information as possible (e.g. Computationally Indistinguishable from random values), there is still something to be learned from selective forgery attacks (for signatures) and chosen cipher-text attacks (for encryption); should the underlying cryptographic scheme be broken, the credentials and privacy of the users might be compromised.

1.1.3 Zero-Knowledge Identification Protocols

Neither party trusts the other. With this technique exactly nothing about the secret is revealed to the Verifier, except that it is valid. The secret to achieving this marvellous result lies within Interactive Proof Systems and their properties, which we will discuss in the rest of this chapter. A common approach to such protocols is through one or more rounds of interactivity:

$$\textit{commit step}\left\{ Prover \overset{statement}{\rightarrow} Verifier \right.$$
$$\textit{round } i \begin{cases} Prover \overset{challenge}{\leftarrow} Verifier \\ Prover \overset{proof}{\rightarrow} Verifier \end{cases}$$

Alternatively, there is a field of protocols which performs transparent preprocessing and then sends off a single large proof to be probabilistically checked offline:

$$Prover \overset{large\_proof}{\rightarrow} Verifier$$

1.2 Theorem Proving and Interactive Proofs

The roots of Verifiable Computation extend all the way to Theorem Proving, when mathematicians still wrote their proofs on paper. If we wish to convert mathematical theorems to the domain of computer science, we should take a look at the well fleshed out theories of NP complexity classes; here is an informal definition of NP Theorem Proving:

NP Languages $Th \in NP \iff \exists$ *"witness" w : Th is "easy" to verify using w*

Note: You should look at the witness as a sequence of logical deductions which start from truthful statements and lead all the way to the theorem claim:

$$\textit{axiom(s)} \Longrightarrow \overbrace{... \Longrightarrow ... \Longrightarrow ...}^{witness} \Longrightarrow Th$$

If one part of the sequence is already known to the Prover, the witness represents the part which is not known.

This definition was extended in 1985 by [4] to represent an Interactive Proof System IP:

IP Languages

$$\textit{Given}\begin{cases} P_{\text{UNBOUNDED}}, V_{\text{POLY}} \in ITM \textit{ (Interactive Turing Machine)} \\ L \subseteq \{0,1\}^* \in NP - lang \\ n \textit{ input size}, c \textit{ large constant} \\ w \textit{ secret witness of } P \end{cases}$$

$$\textit{Then } X \in L \iff$$

$$\wedge \begin{cases} \textbf{Completeness} \iff \forall \textit{ input } X \in L \textit{ to } (P,V): Pr[V \textit{ accepts } X] \geq 1 - \frac{1}{n^c} \\ \textbf{Soundness} \iff \forall P'_{POLY} \in ITM \wedge X \notin L \textit{ input to } (P',V): Pr[V \textit{ accepts } X] \leq \frac{1}{n^c} \\ \quad \textit{or} \iff \forall P'_{POLY} \in ITM \wedge X \notin L \textit{ input to } (P',V): \\ \quad\quad \Big(Pr[V \textit{ accepts } X] \geq 1 - \frac{1}{n^c} \implies \\ \quad\quad\quad \exists \textit{ "Extractor" } E_{POLY} \in ITM : \exists R \subseteq \{0,1\}^* : E(X) = R(w) \Big) \end{cases}$$

(please note that we've defined the language as NP, but IP protocols have been shown to support even more expressive spaces such as PSPACE or even NEXP)

These systems are often called “Proofs (or Protocols) of Knowledge”, because the Completeness property defines a protocol (i.e. set of rules) to follow in order to accept a given statement, and the Soundness property implies instead the existence of some sort of “knowledge” (also known as “witness”), needed to distinguish right from wrong. The alternative definition of Soundness, which makes use of an Extractor machine, is typically used to single out unique Knowledge which is possessed by the Prover, and is useful for understanding “Zero-Knowledge Proofs of Knowledge”. Here is a more intuitive definition of those properties:

Completeness if the statement X is valid, then there is an “easy” way to prove it using the protocol. The Verifier will be able to efficiently check this in polynomial time. In order words: *all valid statements are always accepted.*

Soundness if the statement X is false, then there is “almost” no way to prove it. The Verifier only needs to trust its own knowledge and randomness to disprove false proofs from an all-powerful Prover. In other words: *all invalid statements are always rejected.*

(When discussing the Extractor, the key to understanding the definition is that it shouldn't be possible to accept false statements, unless the illegitimate Prover was somehow capable of extracting the witness, or a relationship on the witness, from the statement X itself in order to use it.)

An important security observation to make, especially when considering the second definition for Soundness, is that there is no restriction to the amount of "knowledge leaked" by the execution of an IP instance. This essentially means that the Prover could naïvely just send his secret password (i.e. witness) over, and the protocol might still be valid. Restrictions on such flaws, as well as the importance of Interactivity, will be added by the Zero-Knowledge definition.

A second important security observation is that the Soundness property is so vague that it does not really provide any security guarantee that the statement $X \in L$ will be hard to prove for cheaters, only that it should not be possible to prove $X \notin L$. In fact, if the language L trivial enough, it might even be possible to randomly choose any $X \in L$ and extract the witness required to prove it. Thus, the security of our proof lies entirely within the chosen language L, which is typically based on some hard cryptographic problem such as finding the prime factors of a large number.

NOTE: While we've defined soundness based on negligible probability, practical constructions only require $Pr < \frac{1}{2}$*, and repetition is employed to achieve the definition above. Also, almost all practical systems have perfect completeness (*$Pr = 1$*).*

1.2.1 Probabilistically Checkable Proofs

There is an alternative field of cryptographic proof systems that is roughly equivalent to IPs, but uses different constructions: Probabilistically Checkable Proofs (PCPs). We will not go into detail regarding PCPs, but suffice to say that they share a lot of similarities with IPs. The main differences in the definition are minor details regarding specific cryptographic properties:

- *Soundness*: it is always computational, since the Prover is computationally (P_{POLY}) bounded, just like the Verifier. This is due to the fact that PCPs are not technically "proof" systems, but "argument"-based systems.
- *Non Interactivity*: such protocols don't require any interactivity by default (IPs need the Fiat-Shamir extension discussed later); instead, the Prover preprocesses the original language statement to generate a (typically large) proof to send off to the Verifier for inspection. This notion will be defined in Section 1.5.
- *Transparency*: such systems do not employ interactivity because everything is prepared in a trustless fashion. The Verifier will be able to use (public) randomness to analyse a few elements of the given proof. This notion will be defined in Section 1.7.
- *Verifier Efficiency*: these protocols are required to be efficient by default. This is due to the groundbreaking results emerging from the PCP Theorem [5]–[8] finalised in '98 by Aurora et al., which led to the conferring of the Gödel Prize for multiple cryptographers having worked on it throughout the 90s. This notion is defined in Section 1.6 through *proof succinctness* and *verifier scalability*.

In practice, PCP systems make heavy use of polynomial arithmetisation, making them better suited for *Universal* VC systems, as seen in Chapter 3; IPs, instead, typically focus on constructions based on specific problem isomorphisms. An extension of PCPs called

Interactive Oracle Proofs (IOPs) [9], which combines them with IPs, can be found in state-of-the-art Universal VC systems and I mention it in Section 3.1.7).

1.3 Types of Knowledge

The issue with most weak cryptographic authentication methods is that some uniquely identifiable knowledge about the secret is somehow "leaked" during the authentication process. Intuitively, we would like to reduce this "knowledge leakage" as much as possible. In order to do so, we must first understand what possessing "knowledge" truly means. Let us define two major scenarios where knowledge is typically conveyed:

1. *Communication*:

   the Prover has chosen (or is in possession of) some non-deterministic private value which the Verifier needs to solve some other publicly known problem (likely published by the Prover and verified by a Trusted Authority). The only way for this value to be known is through the Prover himself.

2. *Computation*:

   the Verifier would like to extract some knowledge from a given hard problem, but is too computationally bounded to be able to do so. Given enough computational power, any Prover would be able to extract the required knowledge and convey it to the Verifier.

Knowledge seems to be strictly related to the act of communicating some value which is the result of a computation that was either too difficult or even impossible for the Verifier to perform. In other words, knowledge is transferred between two communicating parties if and only if the output of their interaction was the result of an infeasible computation for one or both of the parties.[2] Here is an informal definition:

Knowledge Complexity KC

$$\textit{Given} \begin{cases} P_{\text{UNBOUNDED}}, V_{\text{POLY}} \in ITM \\ L \in IP(P,V) \\ f : \mathbb{N} \to \mathbb{N} \wedge f \textit{ non-decreasing} \\ n \textit{ input size} \end{cases}$$

$$\textit{Then } KC_L(f(n)) \iff \wedge \begin{cases} \textbf{i. } X \in L, X \ \underline{\textit{only}} \textit{ input to} (P,V) \\ \textbf{ii. } P \textit{"communicates"} \leq f(n) \textit{ bits of "knowledge"} \end{cases}$$

Whenever we have $KC_L(0)$, that means we can only convey one bit of knowledge with our protocol: $X \overset{?}{\in} L$.

[2] An interesting note here is that transferring random bits does not typically convey any information, since any party can generate randomness by itself (being an ITM). This may seem counterintuitive, but those random bits would only convey knowledge if related to some pre-defined public statement or problem which the Verifier cannot solve by himself.

1.4 Zero-Knowledge Interactive Proofs

If we embed $KC_L(0)$ into the notion of IP, we get the following:

ZKIP Languages

$$\textit{Given} \begin{cases} P_{\text{UNBOUNDED}}, V_{\text{POLY}} \in ITM \textit{ (Interactive Turing Machine)} \\ L \subseteq \{0,1\}^* \in NP - lang \end{cases}$$

$$\textit{Then } X \in L \iff$$

$$\wedge \begin{cases} \textbf{Completeness} \\ \textbf{Soundness} \\ \textbf{Zero-Knowledge} \\ \iff \forall V'_{\text{POLY}} \in ITM : \exists \textit{ "Simulator" } S_{\text{POLY}} \in ITM : Tx(S(V')) \approx Tx(P,V) \\ \quad \implies \textbf{Deniability} \end{cases}$$

Or, more intuitively:

Zero-Knowledge The idea is that no extra knowledge can be extracted from a legitimate valid interaction (i.e. leading to an accepting state), as long as it is "indistinguishable" from a forged valid interaction. In fact, there should be an efficient Simulator algorithm to simulate a valid interaction's transaction record Tx even when the simulating Verifier V' doesn't have access to the Prover's real witness. The Simulator can generate $Tx(S(V'))$ either by executing many protocol runs until an accepting state is met, or just by deducing the correct statement X starting from any final accepting state (known as "rewind-ability"). I will elaborate later on what "indistinguishable" (i.e. $\approx$) means in Section 1.4.2.

The reason that this simulator-based definition leads to privacy-friendly (i.e. non witness-leaking) protocols is because there can be no witness Extractor for legitimate transcripts, since they're indistinguishable from forged transcripts, which are assumed to lack any witness at all. In other words: *the protocol's soundness needs to rely entirely on interactivity and randomness!*

Zero-Knowledge also implies Deniability:

Deniability A Transaction record from a valid ZKIP interaction does not constitute an independent proof of knowledge. No external third parties can watch (or be given) a valid ZKIP communication and infer that the Prover really has a witness for $X \in L$, because the interaction may have been simulated. Only the original parties of the ZKIP communication can verify that it is indeed legitimate, because they know that the messages were not forged when challenging each other.

The trick to actually achieving Zero-Knowledge in a meaningful manner lies within the combination of Interactivity and Randomness. The two parties cannot use a Challenge-Response protocol, because the response to the challenged question is rather unique, regardless of the chosen challenge. However, if the response were to be randomly selected (i.e. challenged) out of a random distribution of values selected by the Prover, it would not contain any meaningful information. In order for such a protocol to be sound, only a legitimate Prover would be **always** able to calculate the required response: a deterministic relationship (selected using the Verifier's random challenge) on a statement Y, randomly derived from the original statement X.[3] Interactivity is required because, regardless of the chosen challenge, the Prover's set needs to be random for each protocol execution.

Finally, the same security assumption of IPs apply to ZKIPs: the difficulty of proving $X \in L$ lies in the chosen language L and its cryptographic hardness assumptions.

NOTE: alternative definitions have been used in the past to describe zero-knowledge, such as "witness preservation" and "witness indistinguishability", but the one given here is the strongest one and the current standard.

NOTE2: if you would like an alternative informal explanation of ZKIP protocols, I highly recommend the beautiful paper by Quisquater et al. [10] on the metaphor of the "Ali Baba Cave". One important feasibility result for ZKIP proofs, based on finding Hamiltonian cycles in graphs, was given by Blum in 1986 [11].

1.4.1 ZKIP as a solution to malicious actors

Zero-Knowledge proofs are regarded as being an extremely powerful tool to convert malicious actors into semi-honest actors. A researcher first builds a protocol which is shown to be secure when all parties (or eavesdroppers) are semi-honest (i.e. they always follow the protocol's rules); then, any party sending messages is required to provide proofs that they were generated following protocol requirements. Since each proof is Zero-Knowledge, the security of the original protocol is not compromised. Because each message must be accompanied by a proof, malicious attackers have no choice but follow the rules of the protocol, or just abort. While there is a computational cost to be paid per proof, Universal VC systems (discussed in Chapter 3) are a convenient and efficient solution for adding such capabilities.

1.4.2 Types of Zero-Knowledge

We previously defined a Simulator capable of generating fake valid protocol which are also indistinguishable from legitimate valid runs: $Tx(S(V')) \approx Tx(P,V)$.

There are currently 4 different classifications of indistinguishability ($\approx$):

1. *Perfect*: there exists a Simulator which produces communication transcripts *identically distributed* to the legitimate distribution of valid transcripts between (P,V).
2. *Statistical*: there exists a Simulator which produces communication transcripts *identically distributed* to the legitimate distribution of valid transcripts between (P,V), except for a constant (i.e. "small") number of exceptions.

[3] there can also be multiple Y statements derived from X at the same time, for efficiency purposes.

3. *Computational* (default): there exists a Simulator which produces communication transcripts *not-identically distributed* to the legitimate transcripts produced between (P,V), but it is believed to be *computationally infeasible* to detect such differences.
4. *Not Known (No Use)*: there does not exist a Simulator but the communication Transactions are still believed to leak nothing about the witness.

*NOTE: the "Not Known" type of indistinguishability **does not** satisfy full Zero-Knowledge requirements, and it also implies Non-Deniability. See the next section.*

1.5 Non Interactivity and Digital Signature Algorithms

We have covered the basics of Zero-Knowledge Proofs, and we've seen that two essential aspects are *randomness and interactivity*. Well, what if the Prover and Verifier's interactivity in the real world is effectively limited? For example, they may not be online at the same moment, or the Prover might want to pre-process multiple proofs by himself. Is it even possible to have "Non-Interactive" Zero-Knowledge Proofs (NIZK)?

In 1987 an article by Israeli researchers Fiat and Shamir [12] proposed a heuristic to solve the aforementioned problem. The key takeaway here is that, while Zero-Knowledge is not deemed to exist without Interactivity, we can adopt the famous Random Oracle Model (ROM) [13] assumption to make use of an interacting "oracle" party which will supply us with "public-coin" random challenges for our protocol. If we assume that cryptographic Hash functions correctly implement a Random Oracle, we can employ them as universal and passive Verifiers to participate in our proof, thus obtaining:

Fiat-Shamir Heuristic the Verifier selects a challenge $e = H(pp)$, where H is a strong cryptographic hash function implementing a Public-Coin Random Oracle, and pp are the public parameters of the problem and the current protocol execution (including the Prover's randomness)

As can be easily be understood, everyone with the same Hash function also has access to the same challenges. Which means that they can validate the lack of bias within the selection of random challenges, hence the legitimacy of the proof. Since this check can be performed independently after the execution of a NIZK, once the recorded communication trace is given the proof can essentially be verified by anybody. This is what it means for the protocol to become "**Non-Interactive**", while still retaining Interactivity in the ROM model.

On a final security note, while the lack of bias in the selection of challenges is apparent, the challenges are still selected based on the Prover's random inputs. These can be biased and, since interactivity is outsourced to the Oracle, an illegitimate Prover can mount an **offline attack** to keep simulating protocol runs until he finds lucky challenges he can satisfy. Therefore, to prevent cheating from the Prover, we must exponentially decrease the error rate for Soundness to brute-force levels (i.e. $\epsilon = 2^{-256} \iff Pr[V\ accepts\ X] \leq \frac{1}{2^{256}}$). This would imply that if a Prover does not have unlimited resources (like in a real-life scenario, but unlike the formal ZKIP definition), then he should not be able to come up with a simulated NIZK valid protocol run.

1.5.1 Flawed NIZK Zero-Knowledge and Non-Deniability

The last remark noted that we're preventing Provers from being able to simulate protocol runs. Does this also mean that the Zero-Knowledge property is broken? Well, yes, but actually no. There does not seem to be a definitive answer in the cryptographic community as to whether Zero-Knowledge is truly preserved for the Fiat-Shamir heuristic (a good debate on this can be found in [14]), but $KC_L(0)$ is believed to hold as long as the ROM model holds. The Zero-Knowledge property for a Fiat-Shamir NIZK is currently classified as "Not Known" (see the relevant subsection).

As an important consequence of the fact that NIZK proofs can be validated by anybody with a transcript of the communication, the Deniability property is broken:

$$NIZK \implies \textbf{Non-Deniability} \not\Rightarrow \textbf{Deniability}$$

This has the downside that any third parties can detect whether a proof was legitimate or not. While this may not seem like such a big deal, uniquely identifying logins (i.e. Identity Proofs) in censorship states can pose a real threat to human rights. It is best to use pseudonymous identities and de-anonymising networks when using NIZK technology (e.g. ZCash) under such harsh regimes.

1.5.2 Digital Signature Algorithm construction

An important upside of NIZKs of Knowledge is that they can be extended from one-shot Identification Schemes to one-shot Digital Signature Algorithms!

DSA Fiat-Shamir Heuristic the Verifier selects a challenge $e = H(pp, m)$, where all parameters are the same as the standard Fiat-Shamir Heuristic, and m is the message that the Prover wants to sign.

The "signature" is, of course, actually just a proof of Knowledge which ties the Identification proof to the presence of a specific message in the Verifier's challenge. This suffices to show that the Prover knows the witness for his identity, and that he is committing to using randomness (i.e. challenges) derived from a specific message.

1.6 Performance through Scalability

Over the years, as cryptographers struggled to develop Zero-Knowledge VC protocols for practical use cases, a few more properties on **performance requirements** were devised[4]. These properties are especially relevant for comparison of recent Universal VC systems, such as the ones mentioned in Section 3.2, which tend to make compromises in the name of expressiveness.

If outsourcing verifiable computations is to be seen as a commodity, then they have to be fast to verify. This requirements boils down to two main properties: there should be an exponential gap between the protocol execution complexity of the Prover and Verifier

[4] A result of this approach can be seen in the the PCP field of proof protocols.

(where the Verifier takes less time), and the proof size should be small enough that the Verifier can read it. Furthermore, Provers of the past often required hundreds of gigabytes and weeks just to process simple proofs, we'd like to avoid that as well. Formally:

Fully Scalable Proof

$$Given \begin{cases} P_{\mathrm{POLY}}, V_{\mathrm{POLY}} \in ITM \text{ (Interactive Turing Machine)} \\ (x,y,f) = X \in L, \\ y = f(x),\ O_y(\Delta) \end{cases},$$

$$Then\ X \in L \implies \wedge \begin{cases} \textbf{Completeness} \\ \textbf{Soundness} \\ \textbf{Prover Scalability} \iff O_P(\Delta + polylog(\Delta)) \\ \textbf{Verifier Scalability} \iff O_V(polylog(\Delta)) \\ \textbf{Proof Succinctness} \iff \forall \pi = Tx(P,V) : O_{|\pi|}(polylog(|x|)) \end{cases}$$

Intuitively, we want to validate proofs π much faster than it takes the Verifier to actually check the statement himself, and without excessive overhead for the Prover. Also, the communication complexity for such protocols should be always be well within acceptable standards.

It is important to note that most protocols don't achieve such results, so the actual definitions are typically relaxed based on the current best solution in that field. For the STARK protocol analysed in this thesis I'll use as satisfying prover-scalability requirement a quasilinear Prover $O_P(\Delta \cdot polylog(\Delta))$, which can yield acceptable concrete performance results in most cases.

NOTE: often verifier-scalability and proof-succinctness are regarded together as "verifier efficiency". For extra confusion, sometimes researchers also use the term "succinctness" to refer to one or both properties, or just scalability in general. Sometimes a fully scalable system is called doubly scalable, or just scalable.

1.7 Other VC Properties

Finally, some further non-essential but **highly appreciated properties** are added, which increase the reliability and flexibility of a proof system, allowing it to be used in more demanding use-cases:

Transparency $Tx(P \leftarrow V) \in$ *public random coins ; i.e. the Verifier only ever sends messages taken from a randomness source that is also available to the Prover.*

This property was first conceived with Arthur-Merlin (AM) protocols ([15], [16]), which were proven to be equivalent in expressiveness to IP protocols that had separate randomness sources for the two parties; it was first called "transparency" in [17]. Transparency is typically present in all PCP-family protocols.

The reason that this property is "transparent" is because as long as the Prover and Verifier have access to the same randomness source, there can be no trusted or trapdoor-derived setup for the underlying protocol[5]. Because trust is eliminated, the security of the protocol cannot be compromised as it does not depend on any specific party, only mathematics. Transparency has also become a matter of interest lately, due to the increased popularity of zk-SNARK constructions (see Section 3.2), which are infamous for their trusted setups and less suitable for decentralised, trustless settings.

Universality $L \iff NP - lang$ *; i.e. the protocol language supports statements taken from any NP computation.*

This property is extremely useful for implementing basic cryptographic proving primitives that can be applied to computations of Turing-complete machines. The utility of such Universal VC protocols lies with the convenience of being able to freely design an application, and then automatically generate proofs for the actions performed by said application. This topic is widely discussed in Chapter 3.

Post-Quantum Safety *the protocol makes use of cryptographic assumptions which are not shown to be compromised by Quantum algorithms.*

Finally, a couple of **open questions** which have been less (if at all) studied in popular VC constructions:

Composition *the proofs of different statements can be efficiently combined, or extended into more complex ones.*

Multi-Party *a single proof can be generated using multiple inputs taken from different Provers.* Achieving such a property for Zero-Knowledge protocols would be akin to achieving Multi-Party Computation.

[5] all setups are either deterministic, or public-coin non-deterministic.

2 Non-universal VC Protocols

In the decades leading up to the introduction of practical Universal VCs, most protocols only dealt with either secret proving or specialised computational proofs. Common tools to achieve this were either Interactivity (and isomorphic problems) or Homomorphic Encryption, or both. In this chapter I will evaluate two **innovative fields** of cryptography which have a strong correlation with universal VC solutions: Homomorphic Authenticators and Verifiable Delay Functions. While they have mostly been developed under different contexts, they share many of the fundamental properties introduced in the VC Model chapter. Homomorphic Authenticators deal with outsourced (homomorphic) computation and VDFs deal with scalable computation; both yield interesting protocols which can be adapted, with a little expertise, into practical ad-hoc applications.

I will carefully evaluate the properties achieved by each construction, trying to understand the cryptographic design behind it without sacrificing the simplicity of our VC Model. This aim of this chapter is to alleviate the fragmentation and complexity of the fields which stand below Universal VC solutions, showing that they can be useful starting points for achieving richer VC constructions. Further non-universal VC protocols will be discussed in the conclusive remarks of this chapter.

2.1 Homomorphic Authenticators

Homomorphic Authenticators (HAUTHs) stem from an interesting and active area of research, with recent publications being in 2018. The incipit of this field lies with Homomorphic Signature schemes, which were originally rejected by cryptographers due to their implicit susceptibility to forgery attacks. These schemes were brought up again by Rivest, and formalised by Johnson et al in 2002[18], to include better definitions for security against forger (i.e. Random Forgery attacks).

Multiple researchers followed down this path, coming up with innovative constructions for validating **outsourced computations** (e.g. cloud computing). An initially successful design [19] (in terms of VC features) relied on fully homomorphic MAC constructions using polynomials. These MACs would then be sent to a server (i.e. Prover), which would leverage their homomorphic properties to generate computational proofs on the given data. This way, homomorphism can be used to yield valid isomorphic problems, akin to the concepts introduced in Chapter 1. The big advantages of this technique, compared to IPs, are: outsourced proving, proof composition, and efficiency for really big problem sizes. In fact, a core feature of Hom.Authenticators is that a Prover can upload really large databases to the Verifier, and then delete them; the Authenticators themselves will be sufficient to verify the validity of any computation on this data.

The polynomial-based construction was then extended by [20] to support inputs for multiple clients, which would grant a simil multi-party property. This was achieved by adding a form of homomorphism to the keys themselves, and then allowing the Verifiers to merge them during the verification phase. Other forms of publicly verifiable schemes were provided (based on lattices), but they proved to be very complex and inefficient. In order to make the construction more practical, Fiore et al.[21] managed to achieve verifier scalability, albeit sacrificing the universality of the protocol. This modification used a combination of polynomials and additive group schemes with bilinear pairings for multiplication. Finally, public verifiability and zero-knowledge were added just now in 2018 by Schabhüser et al.[22]. Public verifiability is achieved by building a homomorphic signature scheme out of the homomorphic MACs; simil ZK was achieved through a property known as "context-hiding".

In this section I will provide the following content: an overview of Homomorphic Authenticator protocols and commonly used syntax for this field (Section 2.1.1); the basic homomorphic components behind them (Section 2.1.2); a basic MAC construction construction (Section 2.1.3); an extension to multiple clients participating in the protocol (Section 2.1.4); and finally support for verifier scalability (Section 2.1.5).

2.1.1 Protocol Syntax

Let's consider the protocol as a 3-step proof, which makes it simpler to compare it with other VC technologies. We wish to prove $f(x) = y$, the construction is as follows:

$$client\ Verifier \xrightarrow{\sigma_x} server\ Prover$$
$$client\ Verifier \xleftarrow{\sigma_y} server\ Prover$$
$$check(\sigma_y) \stackrel{?}{=} True$$

Here is the sequence of steps which the parties go through, in a basic construction:

1. **Preparing the Authenticator:** The Verifier needs to convert his message x into a Homomorphic Authenticator σ_x, which is a MAC or Signature made using a secret key sk. First, the client generates a unique label L relating to the message x, e.g. "message #1" or "message x on time 10:54". The label is then converted into a random value r, required for the security of the scheme, using a keyed one-way PRF[1]

$$r \leftarrow PRF_K(L)$$

A Homomorphic MAC (HMAC) is typically built using polynomial interpolation:

$$\sigma_x \leftarrow p = (p_0, p_1) = (x, (r-x)/sk) = Interpolate((0,x),(sk,r))$$
$$with\ p(i) = p_0 + p_1 i$$

[1]for example, a seeded PRNG constructed from a keyed cryptographic hash function such as Keccak256[23]

Since it's more common to define a function as composition of multiple inputs, i.e. $f(x_1, x_2, ...) = y$, then this HMAC interpolation process can be repeated for each input message, and each message x_i will be associated with a different label L_i:

$$\sigma_x \leftarrow (\sigma_1, \sigma_2, ...) = (p_1, p_2, ...)$$

2. **Generating an Authenticator-based proof**: The Prover then uses the MAC/Signature scheme to convert function f into a sequence of homomorphic operations on σ_x. After these operations have been performed, they will yield a valid MAC/Signature σ_y, which is considered as proof for this protocol. First, the server converts the function f into a Turing-complete sequence of HMAC operations, e.g. f_+ or $f_\times$ for the polynomial construction.

$$f \implies (f_+, f_\times, ...)$$

These operations are applied in sequence to σ_x, with a resulting Authenticator polynomial called σ_y:

$$\sigma_y \leftarrow (f_+, f_\times, ...)(\sigma_x)$$

3. **Verifying the proof's validity**: The Verifier uses the protocol's verification function, this is a crucial step in the construction of the protocol which establishes our soundness property. To verify whether a given σ_x is a valid Authenticator polynomial, the client needs to check whether evaluation on the secret key sk yields a value consistent with the input label(s):

$$\sigma_x(sk) \stackrel{?}{=} r$$

(we will discuss why in detail later.) Which means that any homomorphic derivate of σ_x will necessarily yield an equivalent homomorphic derivate of r when evaluated on sk:

$$\sigma_y(sk) \stackrel{?}{=} f(r)$$

For compatibility's sake, and to help newcomers to the field follow the original papers better, let us also display the **domain-specific syntax** for the Homomorphic Authenticators domain[2]:

$$(sk, ek) \leftarrow Keygen(\lambda)$$
$$\sigma_i \leftarrow Auth(sk, m_i, L_i)$$
$$\sigma \leftarrow Eval(ek, f, \sigma_i ...)$$
$$\{0,1\} \leftarrow Ver(sk, f, L_i ..., \sigma, m_i ...)$$

In the next subsections, we will see how to enhance this simple protocol to include multiple VC features.

[2] ek is a protocol-abstracted evaluation key, but it is not typically present in most constructions and it can be representative of the public scheme parameters; σ_i is computed for each input message m_i.

2.1.2 Adding Completeness

As described in the previous section, we seek full homomorphism in order to achieve completeness:

Homomorphism a Signature/MAC scheme Sig is operator $\odot$ homomorphic

$$\iff \exists \, operator \otimes : y = Sig(x) \wedge y' = Sig(x') \implies y \otimes y' = Sig(x \odot x')$$

Full Homomorphism a signature/MAC scheme Sig is fully homomorphic

$$\iff additively\ homomorphic \wedge multiplicatively\ homomorphic$$
$$\iff \exists \oplus \ operator\ , \odot \ operator : y = Sig(x) \wedge y' = Sig(x')$$
$$\implies y \odot y' = Sig(x \cdot x') \wedge y \oplus y' = Sig(x + x')$$

Finding a fully homomorphic scheme is essential to achieve *universality*, so the researchers found a mathematical object (polynomials) which supported both additive and multiplicative composition, and then built an authentication scheme on top of it. Given $c \in \mathbb{Z}$ and polynomials p and q such that

$$\begin{cases} p = (p_0, p_1, \ldots) \in F[x] \ \left(with\ n = degree(p),\ |p| = n+1 \right) \iff p(x) = \sum_{i=0}^{n} p_i \cdot x^i \\ q \in F[x] \ \left(with\ m = degree(q) \right) \end{cases}$$

, the following additive and multiplicative polynomial operators are given:

$$p + q \stackrel{def}{=} \forall_{i=0}^{max(n,m)} p_i + q_i$$
$$p + c \stackrel{def}{=} (p_0 + c,\ \ \forall_{i=1}^{n} p_i)$$
$$p \times q \stackrel{def}{=} \forall_{i=0}^{n+m} \sum_{j=0}^{i} p_j \cdot q_{i-j}$$
$$p \times c \stackrel{def}{=} \forall_{i=0}^{n} pi \cdot c$$

Of course, these two operations are only a minor part out of all those which have been defined by mathematicians in the coming ages; however, it is a widely known Computer Science fact that these two operations suffice to describe any boolean circuit, and thus fully homomorphic signature schemes are Turing-complete.

Now that we've defined the basic building blocks for our protocol, let's show that these two operations hold for any polynomial evaluation:

Completeness

$$p(x) + q(x) = \sum_{i=0}^{n} p_i x^i + \sum_{i=0}^{m} q_i x^i = \sum_{0}^{min(n,m)} p_i x^i + \sum_{min(n,m)+1}^{max(n,m)} p_i x^i + \sum_{0}^{min} q_i x^i + \sum_{min+1}^{max} q_i x^i$$
$$= \sum_{0}^{min} p_i x^i + q_i x^i + \sum_{min+1}^{max} p_i x^i + q_i x^i = \sum_{0}^{max} p_i x^i + q_i x^i = \sum_{0}^{max} (p_i + q_i) x^i = (p+q)(x)$$

and

$$p(x)\cdot q(x)=\sum_0^n p_i x^i+\sum_0^m q_i x^i=\sum_0^n\sum_0^m p_i x^i\cdot q_j x^j=\sum_0^n\sum_0^m p_i q_i x^{i+j}$$
$$=\ldots=\sum_0^{n+m}\left(x^i\cdot\sum_0^i p_j\cdot q_{i-j}\right)=(pq)(x)$$

2.1.3 Adding Soundness

Our interest here lies in two considerations:

1. building a MAC out of polynomials
2. making sure that we can take advantage of the operations explained in the previous section to achieve a fully homomorphic MAC

Since preserving the full homomorphism is important, we can start from step (2) and build our way towards step (1). Let's consider the following relationships on polynomials:

$$\begin{array}{c} p(x)+q(x)=(p+q)(x) \\ p(x)\cdot q(x)=(pq)(x) \end{array} \iff \begin{array}{c} Eval(x,p)+Eval(x,q)=Eval(x,p+q) \\ Eval(x,p)\cdot Eval(x,q)=Eval(x,pq) \end{array}$$

If we tried to represent this as a more traditional Encryption/Signature scheme, it might look like this:

$$Sig(sk,p)+Sig(sk,q)=Sig(sk,p+q)$$
$$Sig(sk,p)\cdot Sig(sk,q)=Sig(sk,pq)$$

We can, therefore, understand that we should use the secret key instead of the x-coordinate, and the homomorphisms should still hold. Thanks to the Completeness properties achieved above, operating on a polynomial has the effect of operating on all its points at the same time; which means that interpolating two polynomials on the same x-coordinates allows us to combine them to operate on their y-coordinates:

$$\begin{array}{r} p=Interpolate((0,m_1)(1,m_2)(2,m_3)) \\ q=Interpolate((0,m_4)(1,m_5)(2,m_6)) \\ (p+q)(0)=m_1+m_4 \\ (p+q)(1)=m_2+m_5 \\ (p\cdot q)(2)=m_3\cdot m_6 \end{array} \implies \begin{array}{r} \sigma_p=Interpolate((sk,m_1)) \\ \sigma_q=Interpolate((sk,m_2)) \\ (\sigma_p+\sigma_q)(sk)=\sigma_p(sk)+\sigma_q(sk)=m_1+m_2 \\ (\sigma_p\cdot\sigma_q)(sk)=\sigma_p(sk)\cdot\sigma_q(sk)=m_1\cdot m_2 \end{array}$$

The polynomial σ_p is already very close to a homomorphic MAC (sk is the secret key, and m_1 is the message being signed), but we mustn't disclose the sk x-coordinate to anyone. There are two problem:

1. if we disclose the message being signed, something usually allowed by signature and MAC schemes, then someone could figure out our secret key sk.[3]
2. to protect against oracle attacks we should add randomness to the scheme (as well as prove that it is secure against Random Forgery).

To address both problems at the same time, we will move the message m to a known x-coordinate, while using a random value for our sk x-coordinate:

$$\sigma = Interpolate((0,m),(sk,r))$$

It is paramount to avoid disclosing r, so it must always be stored privately by the signer and associated with the message being signed. Since this is often an inconvenient constraint for the user of MAC, the user instead chooses a unique label L associated with the signature on m at that specific point in time (e.g. "m || time"), and randomises it using a keyed one-way PRF (e.g. a cryptographic PRNG or a keyed cryptographic hash function). The following is the **HMAC construction** for signing (m, L) using private keys (sk, K):

$$r = PRF_K(L)$$
$$\sigma = Interpolate((0,m),(sk,r))$$

This construction is also shown in [19] to be secure (*sound*) against Random Forgery attacks, as long as the label L is never re-used.

Please note that, since our signatures are still just polynomials, our completeness property from the previous section still holds:

$$\left.\begin{array}{l}\sigma_1 \leftarrow Interpolate((0,m_1),(sk,r_1)) \wedge r_1 \leftarrow PRF_K(L_1)\\ \sigma_2 \leftarrow Interpolate((0,m_2),(sk,r_2)) \wedge r_2 \leftarrow PRF_K(L_2)\end{array}\right\}$$
$$\implies \begin{array}{l}(\sigma_1+\sigma_2)(0) = \sigma_1(0)+\sigma_2(0) = m_1+m_2\\ (\sigma_1\cdot\sigma_2)(sk) = \sigma_1(sk)+\sigma_2(sk) = r_1\cdot r_2\end{array}$$

2.1.4 Adding Multiple Clients

In order to support multiple clients, we will have to change both the homomorphism and the MAC constructions. For the new homomorphism, we will take advantage of another property about polynomials: they can be multi-variate. In fact, a polynomial $p(x)$ can support full-homomorphism just as much as $p(x,y)$ can. This is intuitive if you remap x as a composition between two other variables. Given $c \in \mathbb{Z}$ and univeriate polynomials p and q

$$\begin{cases} p = (p_0, p_1, \ldots) \in F[x] \ \left(with\ n = degree(p),\ |p| = n+1\right) \iff p(x) = \sum_{i=0}^{n} p_i \cdot x^i \\ q \in F[y] \ \left(with\ m = degree(q)\right) \end{cases}$$

[3] as long as the degree of the polynomial is higher than 0, which we will see is a useful thing to have

, the following additive and multiplicative polynomial operators are given:

$$
\begin{aligned}
p+q &\overset{def}{=} \forall_{i=0}^{max(n,m)} p_i + q_i \textit{ (for missing values, } p_i = 0 \textit{ and } q_i = 0\textit{)}, \\
&\quad \textit{with } max(m,n) = degree(p+q),\ |p+q| = 2max(m,n)) \\
p+c &\overset{def}{=} \textit{same as univariate homomorphism} \\
p \times q &\overset{def}{=} \forall_{i=0}^{n+m} \forall_{j=0}^{i} p_j \cdot q_{i-j} \textit{ (coefficients for } x^j y^{I-j}\textit{)}, \\
&\quad \textit{with } m+n = degree(p \times q), |p \times q| = |m+n|^2 \\
p \times c &\overset{def}{=} \textit{same as univariate homomorphism}
\end{aligned}
$$

Note: the size of the multiplicative homomorphism result can be further compressed down to $|p \times q| = \sum_0^{m+n} i + 1 = m^2 + n^2$, and even more using techniques described in [20]

Now that we have modified our polynomials (while still retaining completeness) we can construct a Multi-Key Fully-Homomorphic MAC out of different separate keys:

$$
\begin{aligned}
\sigma_p &= Interpolate_X((sk_1, m_1)) \\
\sigma_q &= Interpolate_Y((sk_2, m_2)) \\
(\sigma_p + \sigma_q)(sk_1, sk_2) &= \sigma_p(sk_1) + \sigma_q(sk_2) = m_1 + m_2 \\
(\sigma_p \sigma_q)(sk_1, sk_2) &= \sigma_p(sk_1) * \sigma_q(sk_2) = m_1 \cdot m_2
\end{aligned}
$$

Clearly both keys are required for the final evaluation step, hence, verification of any signature requires that the Verifiers share their secret keys, or perform a MPC computation; Fiore et al. [20] take the simpler approach, and have the parties share all the secrets. Because of this, scheme is actually a MAC and not a digital signature, just like the previous construction. If we group the keys like $sk = (sk_1, sk_2)$, we can perform the evaluation on the keys exactly like the main protocol syntax requires.

Of course, we should harden our primitive MAC using the same randomisation process as before, revealing only the signed message for the 0 x-coordinate:

$$
\begin{aligned}
r &= PRF_{K_i}(L) \\
\sigma_i &= Interpolate((0,m),(sk_i, r))
\end{aligned}
$$

If we wish to adjust the syntax to the final step of the protocol, it'll look like this:

$$
\begin{aligned}
sk &= (sk_0, sk_1, \ldots, sk_{last\ party}) \textit{ (all participants)} \\
r &= (r_0, r_1, \ldots, r_{last\ message}) \textit{ (all messages)} \\
\sigma_y(sk) &\overset{?}{=} f(r)
\end{aligned}
$$

This construction is also shown to be sound in [20], in a way that is similar to the previous one.

2.1.5 Adding Verifier Scalability

The scheme obtained so far has a lot of nice properties, such as outsourced proving and support for large inputs, but it imposes a big toll on the Verifier: the client must compute the function on an alternative set of inputs (the labels) each time he wishes to validate a computation. In short, the scheme is not *verifier scalable*. In this step, we will essentially change the construction of our Hom.MAC into [21], incorporating polynomials into additive groups. Before we do that, however, let's consider what we're going to need: amortisation.

2.1.5.1 What is Amortisation?

The final verification step, essentially, requires receiving the evaluated Authenticator σ_y from the Prover, and then checking it against a constant $f(r)$ evaluated by the Verifier. Wouldn't it be nice to re-use $f(r)$ for multiple executions of the protocol?

Unfortunately, security assumptions from [19] for the soundness of our basic HMAC require that r always be randomly chosen, even for multiple signatures on the same message — therefore, L needs to be randomly chosen as well. What we can do is split L into a changing part Δ, and a constant part l: $L = (l, \Delta)$; this is also called a "Multi-Label" by the authors. These multi-labels might look a little like this:

- $L =$ (*"message m"*,*"at time 12:54"*), so that f can be computed on messages of the same nature (i.e. index in a database), but changing over time; or
- $L =$ (*"message m at time 8am"*,*"on day 08/12/2019"*), so that f can be computed on the same set of messages (i.e. a single row indexed in a database), but changing over dates.

While $f(r) = f(PRF_K(L))$ will still change across multiple execution runs, we might find a way to precompute $C = f(PRF_K(l))$, and then efficiently add the component Δ later on:

$$f(r) = Load(C, \Delta)$$

Assuming the function $Load$ has an exponentially lower complexity than f, the check should also be *verifier scalable*.

In order to actually build the $Load$ function, we'll have to somehow pull the Δ out of f:

$$f(r) = f(PRF_K(L)) = f(PRF_K((l, \Delta))) \implies \exists f' : f'(PRF_K(l), \Delta) = Load(f(PRF_K(l), \Delta)$$

This act of "pulling out" a value is exactly what full homomorphism allows us to achieve for a function g:

$$\exists g' : E(g(x, y)) = g'(E(x), E(y))$$

Unfortunately, while f may operate on polynomials, L is not one. In fact, even PRF operates on specific values (you may think of them as numbers, but a string is also valid input to a hash function), and returns a value as well. In order to "pull Δ out", we will perform two tricks:

1. transform l into a 1st degree polynomial, whose variable represents Δ
2. convert PRF into its equivalent sequence of operators PRF' for the homomorphic signature scheme

We can then evaluate this polynomial on Δ:

$$Load(f(PRF_K(l),\Delta) \stackrel{def}{=} f(PRF'_K(l))(\Delta)$$

While this approach certainly works, PRF' would probably be cumbersome to evaluate on polynomials, especially when PRF is actually a keyed hash-function such as Keccak256[23]. Instead, the researchers came up with a more efficient construction, which manages to first evaluate the PRF on simple values, and then convert it into a polynomial:

1. $$r_1 = PRF1_K(l)$$ $$r_2 = PRF2_K(\Delta)$$ $$PRF_K((l,\Delta)) \stackrel{def}{=} r_1 \oplus r_2$$

   [4] $PRF1$ and $PRF2$ are defined similarly to the original PRF
2. transform r_1 into a 1st degree polynomial whose variable represents Δ
3. convert PRF into its equivalent sequence of operators PRF' for the homomorphic signature scheme
4. The new check becomes (after amortisation):

$$r_1 \stackrel{amortised}{=} PRF1_K(l)$$
$$r_2 = PRF2_K(\Delta)$$
$$f(r) = Load(r_1, r_2) = PRF'(r_1, r_2)$$
$$\sigma_y(sk) \stackrel{?}{=} f(r)$$

The construction provided by the authors for PRF' requires the use of additive groups, therefore we will adapt the rest of our homomorphism construction to this requirement.

2.1.5.2 Amortized Completeness

Now that we have obtained amortisation, we just need to move our previous HMAC, based on polynomials, to an additive group $\mathbb{G}$:

$$\sigma \stackrel{def}{=} Interpolate((0,m)(sk,r)) = p = (p_0, p_1) = (m, (r-m)/sk)$$
$$\iff \sigma_{\mathbb{G}} \stackrel{def}{=} Interpolate_{\mathbb{G}}((0,m)(sk,r)) = p_{\mathbb{G}} = (g^{p_0}, g^{p_1}) = (g^m, g^{(r-m)/sk})$$

[4] the actual operation to merge $PRF1$ and $PRF2$ is not really a XOR, but another trickery defined on top of additive groups. The cost for PRF' is $O_{f\ amortised}(|r_1|)$, so $O(1)$ for the 2nd degree restriction that was added by the authors, as we will see later.

As can be seen, we just simply move all the polynomial coefficients into the group generator's exponent. All polynomial homomorphisms only need to work on the exponents; given $c \in \mathbb{Z}$ and polynomials p and q:

$$\begin{cases} p_{\mathbb{G}} = (g^{p_0}, g^{p_1}, \ldots) \in G[x] & \big(with\ n = deg(p_{\mathbb{G}}),\ |p_{\mathbb{G}}| = n+1\big) \\ q_{\mathbb{G}} \in G[x] \ \big(with\ m = deg(q_{\mathbb{G}})\big) \end{cases}$$

the following additive and multiplicative group-polynomial operators are given:

$$\begin{aligned} p + q &\stackrel{def}{=} \forall_{i=0}^{max(n,m)} p_i + q_i \\ p + c &\stackrel{def}{=} (p_0 + c,\ \forall_{i=1}^{n} p_i) \\ p \cdot q &\stackrel{def}{=} \forall_{i=0}^{n+m} \sum_{j=0}^{i} p_j \cdot q_{i-j} \\ p \times c &\stackrel{def}{=} \forall_{i=0}^{n} pi \cdot c \end{aligned} \implies \begin{aligned} p_{\mathbb{G}} + q_{\mathbb{G}} &\stackrel{def}{=} \forall_{i=0}^{max(n,m)} g^{p_i + q_i} = \forall_{i=0}^{max(n,m)} (g^{p_i} \cdot g^{q_i}) \\ p_{\mathbb{G}} + c &\stackrel{def}{=} (g^{p_0 + c}, \forall_{i=1}^{n} g^{p_i}) = ((g^{p_o})^c, \forall_{i=1}^{n} g^{p_i}) \\ p_{\mathbb{G}} \cdot q_{\mathbb{G}} &\stackrel{def}{=} \forall_{i=0}^{n+m} \sum_{j=0}^{i} g^{p_j \cdot q_{i-j}} = \forall_{i=0}^{n+m} \sum_{j=0}^{i} (g^{p_j})^{q_{i-j}} \\ &= \forall_{i=0}^{n+m} \sum_{j=0}^{i} (g^{p_i})^{dlog(g^{q_{i-j}})} \\ p_{\mathbb{G}} \times c &\stackrel{def}{=} \forall_{i=0}^{n} g^{p_i \cdot c} = \forall_{i=0}^{n} (g^{p_i})^c \end{aligned}$$

Completeness is straightforward and leverages the same concepts mentioned previously. Evaluation is also pretty simple:

$$p_{\mathbb{G}} = (g^{p_0}, g^{p_1}, g^{p_2}) \in G[x]$$
$$p_{\mathbb{G}}(x) = g^{p_0 + p_1 x + p_2 x^2} = g^{\sum_0^n p_i x^i} = \prod_0^n g^{p_i \cdot x^i} = \prod_0^n (g^{p_i})^{x^i} = g^{p_0} \cdot (g^{p_1})^x \cdot (g^{p_2})^{x^2}$$

As can be noticed, the multiplicative homomorphism requires that we use $dlog$ to compute the multiplication between any two elements of $\mathbb{G}$. However, for security purposes, the authors decided to integrate our polynomial-based fully-homomorphic scheme into groups where the Discrete Logarithm Problem would hold. The alternative is to apply a bilinear mapping in order to simulate (and obtain) up to one multiplicatively homomorphic operation:

$$e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T,\ e(g^a, g^b) = e(g,g)^{ab} = g_t^{ab},\ g_t = e(g,g),$$
$$\langle g \rangle = \mathbb{G},\ \langle g_t \rangle = \mathbb{G}_T$$

In particular, two choices were made:

1. Use an additive group with just one bilinear mapping. This effectively limits f to only functions of 2nd degree, thus also eliminating the scheme's previous *universality* claim.
2. There are a couple of small changes to the group-polynomials' definition when applied to our HMAC scheme:
   - r is actually calculated a little differently, as its $dlog$ (calculated by the client) is used instead; it should still hold as valid entropy source, see the paper for more details[21].

- the very fist coefficient of any polynomial $p_{\mathbb{G}}$ (i.e. g^{p_0}), is actually set to p_0. This makes multiplication for two polynomials of first degree a little more efficient, because

$$p_{\mathbb{G}} \stackrel{def}{=} (g^{p_0},\ g^{p_1})$$
$$q_{\mathbb{G}} \stackrel{def}{=} (g^{q_0},\ g^{q_1})$$
$$\begin{aligned} p_{\mathbb{G}} \times q_{\mathbb{G}} &= (g^{p_0 q_0},\ g^{p_1 q_0 + q_1 p_0},\ g^{p_1 q_1}) = ((g^{p_0})^{q_0},\ (g^{p_1})^{q_0} \cdot (g^{q_1})^{p_0},\ (g^{p_1})^{q_1}) \\ &= ((g^{p_0})^{dlog(g^{q_0})},\ (g^{p_1})^{dlog(g^{q_0})} \cdot (g^{q_1})^{dloq(g^{q_0})},\ (g^{p_1})^{dlog(g^{q_1})}) \\ &= (e(g^{p_0},\ g^{q_0}),\ e(g^{p_1},\ g^{q_0}) \cdot e(g^{q_1},\ g^{q_0}),\ e(g^{p_1},\ g^{q_1})) \end{aligned}$$

becomes

$$\begin{aligned} p_{\mathbb{G}} \times q_{\mathbb{G}} &= (p_0 q_0,\ g^{p_1 q_0 + q_1 p_0},\ g^{p_1 q_1}) = ((p_0 q_0,\ (g^{p_1})^{q_0} \cdot (g^{q_1})^{p_0},\ (g^{p_1})^{q_1}) \\ &= ((p_0 q_0,\ (g^{p_1})^{q_0} \cdot (g^{q_1})^{p_0},\ (g^{p_1})^{dlog(g^{q_1})}) \\ &= ((p_0 q_0,\ (g^{p_1})^{q_0} \cdot (g^{q_1})^{p_0},\ e(g^{p_1},\ g^{q_1})) \end{aligned}$$

2.1.5.3 Amortized Soundness and Scalability

The construction of the HMAC follows the same idea as in the previous ones, so I will be brief.

$$\begin{aligned} r &= PRF_K(L) \quad \textit{(L is a full multi-label)} \\ \sigma_x &= Interpolate_{\mathbb{G}}((0,m)(sk,r)) \end{aligned}$$

Then, $f(x)$ gets evaluated by mapping f to its counterpart f' using the group homomorphisms: $\sigma_y = f'(\sigma_x)$. And, finally, the check is the same because it leverages polynomial evaluation within the additive group-polynomials:

$$\sigma_y(sk) \stackrel{?}{=} f(r)$$

Scalable Verifier *in the multi-client or the multi-message construction, the idea is that all L_i have the same Δ, the Load function only takes 1 value, so its complexity is $O_V(1)$*

2.2 Verifiable Delay Functions

Verifiable Delay Functions (VDFs) are currently a very active research area in the cryptocurrency community, but they have actually been around for a long time, with a formal definition given only in 2018 by [24]. Until recently, researchers had been toying with many different constructions, trying to find adequate "time-lock puzzles". In 1996 Rivest et al. [25] introduced a mathematical problem which seemed to exhibit interesting properties, with relation to time delaying functionality, previously only briefly considered in naïve PoW-like schemes by researchers such as Merkle [26].

The main objective is to come up with a cryptographic proof of elapsed time, i.e. a delay. Researchers figured that a universal reference for measuring the passage of time could be represented by the maximum speed at which a single operation can be processed on a circuit (of any kind), so they set out to find "sequential functions" – i.e. which could only be computed on a single cpu core one operation at a time. This idea can be seen as a PoSW, "Proof of Sequential Work"; we will discuss later the implications for this construction.

Once such a "time-lock puzzle" (or PoSW) was found, the need emerged for an *efficient verification* mechanism, to relieve the Verifier from the burden of wasting the same amount of time as the Prover just to check that he did indeed compute the right result. This would allow for efficient outsourcing of elapsed time, which may sound like a useless tool, but it can lead to surprisingly innovative solutions in the time-agnostic world of computer science. Attempts to find such *verifier scalable* PoSWs lasted for years, with some improved but incomplete results in 2015 [27] and 2018 [24], culminating with two complete solutions that same year by Wesolowski [28] and Pietrzak [29].

The recent rush of new research in this field is probably due to the increased popularity of Blockchain technology (see use-cases in Section 2.2.2), and a formal definition for these systems was finally given by [24] under the new name "Verifiable Delay Functions". We will be mainly considering Wesolowski's scheme in this section, with some references to Pietrzak's. A good comparison of the two schemes is also provided in [30].

2.2.1 Utility

The issue of time synchronisation has long plagued electronic computers which interact on the Internet. To solve synchronisation between honest parties, a hardware clock (or constant delay networks) might suffice, but malicious parties would still be able to report incorrect timestamps. Most importantly, the issue of time synchronisation also extends to time delay proving. There are two main ways to detect such malicious attempts:

1. *Distributed consensus mechanism.*

   This idea basically adapts the concept of a trusted third party to a scenario where no such party exists (or at least it is not recognised as such by all honest parties).Trust is distributed amongst all the parties (according to some satisfactory proportion or relation), and the validity of a claim is based on whether it is the most supported one by the network.

In order for this system to work there needs to be a majority of parties incentivised to act honestly, which is commonly achieved by distributing trust amongst a large number of independently motivated parties, all interested in using the same protocol (e.g. pseudonymous Bitcoin users participating from all around the world). Also, the network itself needs to always be available to all parties (i.e. censorship resistant), otherwise honest parties might be unable to stave off false claims by supporting only the correct ones.

2. *Use a universal time delay measurement reference.*

   This would be some sort of event occurring in our world which can be universally verified just based on the laws of physics. The Prover would perform some sort of action or operation over a period of time, and it would automatically reflect on some object in the universe in such a way that it would be infeasible to replicate the same exact object without the same period of time having elapsed.

Of course, these two systems can be combined into a single solution – Bitcoin makes both use of the PoW system as well as the distributed consensus model – VDFs, instead, take the second approach and try to find a trustless and convenient solution to measure the passage of time.

A commonly proposed universal time delay source is "maximum computation speed", as in the fastest way that a specific computation can be performed in any implementation of any computational model in the world. This is deemed as "universal" because, if *no participant in the world* can perform a certain computation faster than the expected amount of time, it can be used as a universal time delay reference for humans.

2.2.2 Use-Cases

VDF schemes are particularly interesting because reliable and efficient time delay outsourcing leads to innovative computer science applications. The original applications were of cryptographic value:

- **timed encryption**: also known as "time capsules", they would allow for self-decrypting messages through a "timed key escrow" ([31], [32]) mechanism, where a Trapdoor-VDF would reveal the key after some elapsed time. Timed encryption can be leveraged to build **scheduled payments**: one could prepare multiple transactions in advance, and they would self-decrypt in due time. At any moment prior to the deadline, the owner can invalidate the payments. Timed encryption can be used for many scenarios, such as **timed top secret archives**, in order to guarantee security and transparency for a country's intelligence services.
- **timed commitments**: using timed encryption as a building block, one can build self-revealing commitment schemes[33], which can be used for lots of protocols, including **auction bidding**: everyone commits during the first phase, and in the second phase the bids self-reveal. Timed commitments can also be used for other **voting protocols**, where the vote is revealed after the voting has taken place.

- **slow-timed hash functions**: delay functions are interesting alternatives to classic iterated hashing techniques and Key Derivation Functions, with the advantage of being *scalable* and *sequential*. They can be used for **password storage**, when the password are generated by humans, in order to stave off brute-force pre-image attacks. Compared to classic techniques (such as *scrypt*), VDFs do not leverage memory, instead relying on sequentiality. Initial slow-timed hash function constructions were the precursors to what eventually became "Verifiable Delay Functions" in [24].

In practice, the ability for VDFs to generate public random numbers (when given a biased entropy source) can be the basis for achieving Transparency in many other protocols, such as a lottery. Over the last few years, there has also been increased interest in adapting VDFs to cryptocurrencies, where the lack of a trusted third party is a common assumption:

- **transparent public PRNG beacon**: The main properties of random numbers is that they're both *unpredictable* and their generation is *unbiased*. Classic solutions to generating public random numbers on the blockchain have been to either: take block hashes, or use MPC computations [34]. The problem is that repeating MPC computations is highly inconvenient (all parties must be online at the same time and perform hefty computations), and that block hashes are subject to the biased selection of transactions by PoW miners.

  Using VDFs as "slow-timed hash functions", we can generate random numbers on the blockchain which remain secret for a short period of time. The main idea is that we can use transaction history as an (unpredictable) entropy source[5], and then remove the bias introduced by miners by using VDFs. The VDF inputs are block hashes, and the outputs are our random numbers: the miners can bias the inputs only up to the block's confirmation time (not just its publication), after which they cannot be changed. If the VDF delay is longer than an input block's confirmation time, then the outputs will become unbiased because the miners won't be able to evaluate and change them at the same time. Of course, it is important to accurately measure the maximum block confirmation time for the blockchain at hand, after which any miner attack becomes infeasible, and set it to be smaller than the VDF delay.

- **transparent lottery systems**: much like Randomness Beacons, lottery systems require the selection of a random number only after all relevant actions (i.e. the betting) have taken place. The trick is the same, and players are given less time to bet than it takes to figure out the random number, fixed at the beginning of the computation by using block hashes. I've implemented a prototype trapdoor version of such a protocol myself, on Ethereum Kovan[36]. Of course, the lottery game could take advantage of a Randomness Beacon and just give players n blocks' time to bet, taking as winning number the Beacon's output of the n-th block's hash after the start of the lottery.

[5] the entropy for a Bitcoin block hash (approx 10 minutes of transaction time) was estimated to be ≈ 70 bits in 2015 by [35], and it is based directly on the difficulty of the mining problem. On Ethereum, blocks are published ≈ 40 times more frequently (i.e. around 15 seconds per transaction) [34], which entails lower entropy for each block hash.

- **improved blockchain efficiency**: arguably one of the biggest issues cryptocurrencies have right now is the incredible waste of resources used for mining in PoW-based blockchains. There are entire mining farms which combined consume as much energy as a small country, all for the purpose of making Bitcoin run. The Ethereum2.0 research team is experimenting with PoS (Proof of Stake) consensus protocols, where a new leader is randomly chosen to publish each block, without the need for wasteful mining. VDFs can be used for this purpose because their output is deterministic, but can still be used to choose leaders in a fair (pseudo-random) fashion.

  Comparing the Nakamoto hash inversion puzzle (used in Bitcoin, BitGold, and others) with VDFs, leader selection would be akin to fixing a PoW output from the start, and then running many parallel processes to brute force the input space. The advantage for PoW schemes is that they are *verifier scalable*, because other users can quickly check that the correct pre-image was found. However, the price to pay for fair currency distribution starting from a given biased state (i.e. the previous block's hash) is a non-deterministic search by exhaustion, which results in huge energy consumptions. VDFs can remove the same bias present in block hashes, while still being *verifier scalable* ***and deterministic***. That's because we fix the input instead of the output, and then proceed sequentially with the computation: only one person needs to calculate and publish the proof. This leads to a drastic reduction in resource wastage, and is a highly anticipated feature of the Ethereum blockchain.[6]

Other interesting applications of VDFs have been identified for more ambitious scenarios, such as Web3 and SWARM-like [37] solutions. An example is **proof of "age"**, where the minimum age of a given file can be proven to show that some information was indeed known ahead of time[7].

2.2.3 A Language for Time (Delays)

But how to measure time in the time-agnostic world of computers? We could try to equate cpu cycles and operations to the flow of time, measuring them with a *real-time* $\rightarrow$ *μ-time* formula. There are a couple of **issues** with this approach:

1. *μ-time* $\neq$ *real-time*

   Algorithms are not an immediately useful tool for measuring time, since time runs by itself and they don't. Users might use our protocol for measuring time, but we cannot ask them to run it indefinitely. This means that we need to adapt our language to measuring time delays, and not time; as long as the algorithm runs, a delay will

[6] of course, the consensus protocol also requires incentivising users towards a unified blockchain state. In PoW consensus protocols, miners are incentivised to mine for the longest chain, or they risk wasting time and money; but in PoS consensus protocols, leaders can generate multiple chains without wasting any time. One of the suggested solutions is to force cheating leaders to lose money, but the details are still being fleshed out for Ethereum2.0.

[7] this can be regarded as the opposite of showing that a given file is recent, such as what what abductors used to do when taking a photo of their captives along with the daily newspaper.

be measured! We won't be able to prove something like "this message at 13:54 on 1/1/2019", but we might be able to prove something like "this message took one week to process". As long as the message is unique, we can also prove "this message is from more than a week ago".

2. *μ-time $\neq$ universal*

   The flow of time has the nice property of being the same for everyone: nobody can speed it up or slow it down! However, this does not apply to computers – anyone with more money can buy more processors, and then use them to parallelise and speedup most computations.

   For this reason, we aim to find sequential computations which cannot be parallelised, such that money will not be a factor when measuring the flow of time; thus making our protocol fair and transparent for all users.

Now that we've identified the main issues, let's discuss the **main properties** that we want for a statement $X \in L$:

1. *Sequential*

   In order to build a reliable language for measuring time delays, we wish to base it on sequential computations. This is because the time spent computing paralelisable algorithms varies wildly according to the amount of money invested: a poor individual with a single 10€ processor will run a Bitcoin mining algorithm orders of magnitude slower than a rich company with 1000x times the same amount of processors; this makes for unreliable time delay measurement, hence unfair VDF protocols.The same does not occur when comparing processor frequencies: average processors on the market lie at around 1GhZ speeds, while the fastest ones in the world at 9GhZ – just a factor of ten! As long as we account for maximum 10GhZ speeds in our sequential computation, our delay measurements should apply to everyone: nobody will be able to complete the VDF faster than the expected amount of time, although some might take a little longer.

   This solution, however, is not without flaws. An alternative to sequentiality, frequently used in Cryptography for KDFs and password storage, has been to employ algorithms which require using large amounts of memory in order to greatly increase the cost for achieving parallelised computation. Two successful examples of this are *scrypt*[38], commonly used for password-based key-derivation-functions, and *Ethash*[39], used for the Ethereum cryptocurrency's PoW. Another issue is with the assumption that the difference between the world's average processor speed and the world's fastest one is "small", and that specialised hardware implementations (e.g. ASICs) cannot improve this margin by a substantial amount. These assumptions are currently being researched by the Chia Foundation and the Ethereum foundation ([28], [40]).

2. *Deterministic*

Since we're trying to measure effective time delays, and not average ones, our scheme cannot rely on well studied PoW protocols. The issue being that they're typically probabilistic (as well as paralelisable): a problem with an estimated difficulty of 1 hour might end up taking 1 second, just out of sheer luck! A deterministic computation would give us a guarantee as to the number of performed computations, hence, the minimum elapsed time.

2.2.4 Building a SPoW Protocol

The major idea behind the success of current VDFs is the specific protocol language designed by Rivest et al. in [25], based on repeated squaring in RSA groups. This time-delay language will become the basis for improved VDF protocols, here is its definition:

Time-Lock Puzzle $TL(\Delta, \lambda, \mu)$

$$\Big\{(x,y) \mid y \leftarrow \overbrace{(\mu \circ \mu \circ \ldots \circ \mu)}^{T\ times}(x), T \leftarrow \Delta \cdot \frac{sec}{\Omega_{\mu_\lambda}},\ \Omega_y(\Delta),$$
$$T \in \mathbb{Z}, \Delta \in seconds, \mu : D \to C, x \in D_\lambda,\ y \in C_\lambda \Big\}$$

[8]

Rivest et al. [25] believed their language contained intrinsic sequentiality properties, and based their "time-lock" protocol on it. Given the difficulty of estimating a function $(\Delta, \lambda) \to T$[9], the puzzle was simply based on any T directly:

RSW96 $TL(T, \lambda, \mu)$

$$\Big\{(x,y,N) \mid y \leftarrow x^{2^T} \pmod N,\ \mu = x \mapsto x^2 \pmod N,\ \Omega_y(\Delta),$$
$$T \in \mathbb{Z},\ N \in_R RSA_\lambda\ modulus,\ x \in \mathbb{Z}_N^*,\ \lambda_{RSA}\ derived\ from\ \lambda \Big\}$$

[10]

Clearly, calculating a power 2^T which is huge (e.g. $T = 2^{40}$) is not feasible, so we will not be able to employ classic modular exponentiation techniques; two known techniques are shown in the *completeness* proof. Here is the protocol we can derive from the language:

[8] in the practical scenarios, T is typically determined heuristically according to the specific implementation, or based on concrete metrics of the basic μ operation. A typical example provided by most researchers in their academic articles is $T \leftarrow 2^{40}$, however, concrete time measurements are not typically discussed.

[9] since there can be many other costs associated with usage of SPoWs (such as network transmission), they are not well suited for precise time measurements. It's best to choose delays which range from a few minutes to hours or days.

[10] λ is the security parameter in bits for the RSA group. From λ we typically derive λ_{RSA}, according to conventions based on statistical brute-force attacks shared by the cryptographic community. Today it is believed that $\lambda = 100 \implies \lambda_{RSA} = 2048$, but this assumption may change in the future, or have changed already.

SPoW

$$Given \begin{cases} L \equiv TL(T,\lambda,\mu) \subseteq \{0,1\}^* \in NP \\ T \in \mathbb{Z} \text{ timing parameter} \\ \lambda \text{ security parameter in bits} \\ \mu : \text{squaring in } RSA_\lambda \\ N \text{ RSA modulus} \end{cases}$$

$$And\ X \in L \iff \wedge \begin{cases} \textbf{Complete} \\ \textbf{Sound} \end{cases}$$

The protocol is clearly complete, since the repeated μ operation does yield a correct y:

Completeness

$$\forall X \in L,\ (x,y,N) = X : Pr[y = x^{2^T} \pmod N] = 1$$

with the algorithm for computing y being:

$$\begin{cases} \overbrace{x \to x^2 \to x^{2^2} \to x^{2^3} \to \dots \to}^{T \text{ group squarings}} x^{2^T} \pmod N \textit{ (order is unknown)} \\ e \leftarrow 2^T \pmod{\phi(N)} \wedge x^e = x^{2^T} \pmod N \textit{ (order is known)} \end{cases}$$

[11]

*(i.e. $X \in L$ means that (x,y) are correct, **and** that $\Omega(T)$ time was spent. The second algorithm for calculating y is particularly important, since it implies knowledge of the RSA trapdoor and private key).*

The actual soundness for this language's claim to being a universal time-delay reference (i.e. sequential and deterministic computation) is not proven, but it does rely on two assumptions:

Soundness

$$\forall X \notin TL(x) : Pr[y = x^{2^T}] = negl(\lambda)$$

*if cracking RSA is hard **and** all TL puzzles can only be solved in minimum T time without $\phi(N)$ **and** $X \notin TL$ but the puzzle solved means that it took less than T time, **then** the puzzle was solved with $\phi(N)$, **then** the prover had to have cracked RSA,*

[11] in a typical SPoW scenario, the RSA group order $\phi(N)$ is not known to the Prover.

which has negligible probability! In order words:

$$\left.\begin{array}{r} Pr[\exists\text{"extractor"}\ E_{POLY} : \phi(N) \leftarrow E(N)] = negl(\lambda) \\ \forall x \in \mathbb{Z}_N^* : \Omega_{x^{2^T}\ w/out\ \phi(N)}(T \cdot \mu_\lambda) \\ \left(X \notin TL \wedge y = x^{2^T} \implies \Omega_y(< T \cdot \mu_\lambda)\right) \end{array}\right\} \wedge \implies$$

$$\Omega_y = \Omega_{x^{2^T}\ w/\ \phi(N)}(< T \cdot \mu_\lambda) \implies \exists\text{"extractor"}\ E_{POLY} \iff$$

$$negl(\lambda) = Pr[\exists E : \phi(N) \leftarrow E(N)] = Pr[y = x^{2^T} \wedge w/out\ \phi(N)] = Pr[y = x^{2^T} \wedge X \notin L]$$

The soundness of the protocol relies on the usage of groups of unknown order, and the inability to reverse a cryptographic one-way function.[12] In particular, two assumptions are required for this protocol to work:

1. Cracking RSA is hard (i.e. extracting $\phi(N)$ from N)

   This assumption has been upheld by the cryptographic community for decades, only being proven invalid within the still developing context of Quantum Computers.

2. There is no faster way to solve the puzzle w/out $\phi(N)$ than with $\Omega(T)$ sequential RSA_λ group squarings (i.e. μ_λ)

   This was not proven in th original '96 paper by Rivest et al.[25], but it is believed to be true by all subsequent authors (inluding [27], [24], [29], [28] and others).

No specific construction for the protocol is provided[13], but you could think of it as something similar to Homomorphic Authenticators:

$$client\ Verifier \xrightarrow{\sigma_x} server\ Prover$$

$$client\ Verifier \xleftarrow{\sigma_y} server\ Prover$$

$$check(\sigma_y) \stackrel{?}{=} True$$

or

$$Verifier \xrightarrow{TL(T,\lambda,\mu),\ N,\ x} Prover$$

$$Verifier \xleftarrow{y} Prover$$

$$y \stackrel{?}{=} x^{2^T}$$

2.2.4.1 A note on Trapdoor-SPoWs and Trapdoor-VDFs

So, whoever owns the RSA private key also knows $\phi(N)$, and can therefore invalidate the protocol and spoof proofs at will. This does not preclude utility:

[12] we will discuss another groups of unknown order $\mathbb{G}(\sqrt{q})$ in the subsection on transparency Section 2.2.7.

[13] actually, [25] only states the language TL, and assumes that protocols based on its time-delaying sequentiality will be sound. A construction is only provided for a *timed encryption* use case.

the Prover may not be given the key anyway, or the trapdoor may be used to generate "time-capsules". In "time-capsule" constructions the owner of the private key can take advantage of his **Trapdoor-SPoW** to quickly calculate the output of a unique $X \in L$, and use it as OTP key to encrypt some secret message: all others will have to wait before they can decipher his message.

At the same time, using RSA groups necessarily requires us to generate private keys. If we wish for others to use our SPoW in a trustless (*transparent*) fashion, what can we do to prove we did not keep nor use the private keys? We will delve deeper into this topic in Section 2.2.7.

2.2.5 "Compressing" Time

Now that we've build our SPoW universal time-delay reference, we can proceed towards refining it. While the protocol does allow us to prove time delays, it also requires the Verifier to wait the same amount of time as the Prover (unless he has access to the RSA trapdoor, which is not a scenario we wish to focus on). This can be a limiting factor for computer applications, where performance is essential. If an auction lasting 1 hour takes place between 100 participants we want the bids to be revealed as soon as the auction ends, but our SPoW protocol requires each participant to wait 100 hours before they can be sure of the winning party. Research into *(verifier) scalable* SPoW protocols, also known as VDF protocols, has recently resulted in two competing approaches:

1. *Cut-&-Choose*:

   Wesolowski came up with a proof [28] which shares some similarities with the Schnorr Σ-Protocol. The idea is to "generate"[14] many problems isomorphic to the original SPoW, and then solve the one chosen randomly by the Verifier. Concretely, the isomorphic problems are derived from the SPoW output, making sure that they still preserve the protocol's witness (i.e. that there have been numerous sequential squarings starting form the input). This way, each available isomorphic problem will be made dependent on the original SPoW problem, while having the property of being much faster to check. When the Verifier randomly selects and validates one of these isomorphic problems he can be confident that, as long as the check succeeds, the Prover has surely waited the correct amount of time calculating the SPoW. The "Cut & Choose" approach was perhaps first employed, informally, by Rabin in 1979 [41].

2. *Recursive Cut-&-Choose*:

   Pietrzak came up with another innovative protocol [29] that instead shares similarities with the famous Graph-Isomorphism ZKIP feasibility result, in that soundness is improved over multiple rounds of the protocol. The protocol makes use of an intermediate execution trace derived from the SPoW computation, where each state

[14] we don't actually generate all the possible problems in practice, but only the one that will be needed. For the purposes of this discussion, however, it does not make any difference.

is made dependent not on the input and output, but on another state that occurred just a little earlier within the computation. When the Verifier recursively validates these intermediate results, reaching the very first one, the proof becomes sound. The protocol is still Cut-&-Choose, because during each round the Verifier can select amongst many problems isomorphic to the current state. While this technique requires multiple rounds, increasing the verification costs, the prover overhead is almost completely gone (unlike in the Wesolowski protocol). The use of an "execution trace" to prove computations is also found in STARKs, analysed in Chapter 3.

We will be discussing only the Wesolowski VDF due to its *verifier scalability*; however, the second one is just as valid due to its additional *prover scalability* improvement. The interesting aspect of Pietrzak's approach is that it is better suited for time-sensitive scenarios (e.g. PRNG beacons), where the Prover wants to submit his result quickly and the Verifier can spare extra storage for longer proofs. Other interesting but less successful approaches are discussed in [24], and include making use of SNARKs (see Section 3.2), as well as inversion of permutation polynomials, and modular square root constructions.

2.2.6 Building a VDF Protocol

To build an efficient protocol we need to find a problem that is isomorphic to our SPoW, but which is also fast to check, or *verifier scalable*. Consider the main delaying factor in the SPoW problem, $x^{2^T} \pmod N$: the exponent 2^T is way too large to compute and store (for values such as $T = 2^{40}$), so we need to use our sequential repeated squaring method. Likewise, if we choose another large exponent, the problem will stay the same. Say we decompose our exponent into

$$\exists r \in_R Z_\lambda : 2^T = q \cdot r$$

Then, if we assume $\lambda \approx 100$, the value q is still very large[15], and $x^q \pmod N$ is nearly just as hard to compute as the original problem. So the new problem is of similar nature as the old one, but is it isomorphic? Here's the twist, we will use a different check function from the one in our SPoW: instead of

$$x^{2^T} \stackrel{?}{=} y \pmod N$$

We'll use

$$(x^q)^r \stackrel{?}{=} y \pmod N$$

The idea is to have the Verifier randomly choose r *after* y has been submitted by the Prover, and then wait for (x^q) to be submitted by the Prover. Thus, the Verifier only needs to perform one efficient calculation, and he can check whether the two values x^q and y are consistent with each other. This also has the effect of suggesting that a T-time time-puzzle was solved, as we will discuss later in the *soundness* proof, but there are still a few details to be ironed out before we can be satisfied.

[15] the symbol q is chosen in [28] because it the "quotient" for $2^T/r$.

Two more security devices need to be added to this construction. The first deals with a detail in the *soundness* proof, whereby we choose a prime random value $r \in_R PRIME_{2\lambda}$ instead of a normal integer, which also changes the construction slightly because $2^T = q \cdot r + residue$. Intuitively, this gives the Prover more control over the check because he is not only bound to values that might be unrelated to the SPoW (if the Prover is malicious), but he can also factor in the input (e.g. $x^{residue}$, see below) of the SPoW to force x^q to be chosen correctly. For related reasons, [28] also requires that we modify our RSA group to be $RSA_\lambda/\{\pm 1\}$. The second observation is more of a practical requirement: since inputs for the SPoW should be unique and randomly chosen across different protocol executions, it is better to remove the bias of the input x by calculating a new input $x' \leftarrow H(x)$, with hash function $H: \{0,1\}^* \rightarrow RSA_\lambda/\{\pm 1\}$.[16] If this is not performed, any biased input $x_2 = x_1^\alpha$ can be exploited to speed-up the computation by using a previous SPoW output and taking advantage of the group's commutative properties: $x_2^{2^T} = (x_1^\alpha)^{2^T} = (x_1^{2^T})^\alpha = y_1^\alpha \pmod N$.

Finally, a note on efficiency — is it possible for the Prover to generate the auxiliary value x^q faster than using again the repeated squaring method, such as by taking advantage of the relationship it has with x^{2^T}? In fact, it is, and [28] takes $x^q \leftarrow x^{\lfloor \frac{2^T}{r} \rfloor}$ (the flooring is due to the prime divisor); algorithms are discussed in the *completeness* proof. Now that we've discussed how to build the protocol, here is the construction for a $VDF(x, y, N)$:

$$Verifier \xleftarrow{y} Prover$$
$$Verifier \xrightarrow{r} Prover$$
$$Verifier \xleftarrow{\pi} Prover$$
$$\pi^r x'^{\,residue} \stackrel{?}{=} y$$
$$where \begin{cases} x' = H(x) \\ y = (x')^{2^T} \pmod N \\ r \in_R PRIME_{2\lambda} \\ \pi = x'^{\lfloor 2^T/r \rfloor} \\ residue = 2^T \pmod r \end{cases}$$

Completeness

$$\forall (x, y, N) = X \in VDF : Pr[\pi^r x'^{\,residue} \pmod N) = y] = 1$$

This is straightforward when expanding the formula:

$$\pi^r x'^{\,residue} = (x'^{\,\lfloor 2^T/r \rfloor})^r x'^{\,residue} = (x'^{\,q})^r x'^{\,residue}$$
$$= x'^{\,qr+residue} = x'^{\,2^T} = y$$

[16]the hash function can be something like Keccak256[23], adapted to provide enough λ_{RSA} bits, such as by iterating inputs through a counter in a PRNG-like construction.

And $x'^{\,q}$ is calculated using

$$\begin{cases} x'^{\,q} = x'^{\,\frac{2^T - residue}{r}} = x'^{\,\frac{2^T - (2^T \mod r)}{r}} = x'^{\,\lfloor 2^T/r \rfloor} \leftarrow \mathscr{A}(x', r, T) \text{ (order is unknown)} \\ x'^{\,q},\ q \leftarrow (2^T - residue) r^{-1} \pmod{\phi(N)} \text{ (order is known)} \end{cases}$$

$\mathscr{A}$ is chosen to be the "on-the-fly long division algorithm", with worst complexity of $O(2T)$, but an improved algorithm in [28] reaches $O(T/log(T))$, and can also be parallelised.

Just as the SPoW, this protocol can be broken if the order of the group $\phi(N)$ is known to the Prover. Trapdoor-VDFs are still useful, as mentioned before, when the owner of the private RSA key is not the VDF's Prover.

Soundness

$$\forall X \notin VDF : Pr[\pi^r x'^{\,residue} = y] = negl(\lambda)$$

if *cracking RSA is hard,* ***and*** *breaking the Adaptive Prime Roots assumption is hard,* ***and*** *$X \notin VDF$ but the check succeeds means that either the Prover spent less than $\Omega(T)$ time or he chose the wrong language parameters for (y, π),* ***then*** *one of* ***two cases*** *holds:*

1. *if the problem was solved in less than T steps then RSA was broken, which is assumed to be improbable;*
2. *if the problem was solved with wrong protocol parameters (y, π) and they're correlated by an exponent r in the check, then π had to have been based off of y*[17] *(see below for exact relationship) and the exponent removed (i.e. an exponent-root was calculated), but removing any prime exponent requires calculating any prime root for a value in the group, which breaks the Adaptive Prime Roots assumption, which is assumed to be of negligible probability.*

[17] π was necessarily chosen after y because the relationship between the two requires a parameter r that needs to be given by the Verifier, and an honest Verifier wouldn't have continued the protocol without having been given a y first. So y cannot be based off of π in an attack against our adaptive protocol.

In other words:

$$\left.\begin{array}{r}
Pr[\exists\text{"extractor"}\ E_{POLY} \in ITM : \phi(N) \leftarrow E(N)] = negl(\lambda) \\
\textbf{Adaptive Prime Roots Assumption} \\
\iff \forall \alpha \in RSA_\lambda(N),\ \alpha \notin \{0, \pm 1\}, r \in_R PRIME_{2\lambda} : \\
Pr[\exists\text{"Extractor"}E'_{POLY} \in ITM : \sqrt[r]{\alpha} \pmod N \leftarrow E'(\alpha)] = negl(\lambda) \\
\forall x \in \mathbb{Z}_N^* : \Omega_{x^{2^T}\ w/out\ \phi(N)}(T\mu_\lambda) \\
\Big[\forall x', y, \pi \in RSA_\lambda(N)/\{\pm 1\}, r \in PRIME_{2\lambda} : \\
\pi^r x'^{\ residue} \pmod N = y \implies \pi = \sqrt[r]{yx'^{\ -residue}} \pmod N\Big] \\
\Big[X \notin TL \wedge \pi^r x'^{\ residue} = y \implies \textbf{(c1)}\ \Omega_y(< T\mu_\lambda) \vee \textbf{(c2)}\ \neg(y = x'^{\ 2^T} \wedge \pi = x'^{\ q})\Big]
\end{array}\right\} \wedge$$

$$\implies \begin{cases}
\Omega_y = \Omega_{x^{2^T}\ w/\ \phi(N)}(< T\mu_\lambda) \xLeftrightarrow{\text{same as SPoW soundness}} Pr = negl(\lambda)\ \textit{\textbf{(case 1)}} \\
(y \neq x'^{\ 2^T} \vee \pi \neq x'^{\ q}) \wedge \pi = \sqrt[r]{yx'^{\ -residue}} \pmod N\ \textit{\textbf{(case 2)}} \\
\implies \exists \alpha \in RSA_\lambda(N)/\{\pm 1\} : y \leftarrow x^{2^T} \alpha \wedge \pi \leftarrow x^q \sqrt[r]{\alpha} \implies \exists\text{"Extractor"}E'_{POLY} \\
\iff negl(\lambda) = Pr[\exists E'_{POLY}] = Pr[\pi^r x'^{\ residue} = y \wedge (y \neq x'^{\ 2^T} \vee \pi \neq x'^{\ q})] \\
\qquad = Pr[\pi^r x'^{\ residue} = y \wedge X \notin L]
\end{cases}$$

The soundness of the protocol relies on the same assumptions as the SPoW protocol, as well as the inability to find prime roots in groups of unknown order. In particular, two assumptions are required for this protocol to work:

1. Cracking RSA is hard
2. There is no faster way to solve the puzzle w/out either breaking the underlying SPoW time-lock puzzle, or breaking the Adaptive Roots Assumption. *It is an open question whether this assumption can be reduced directly to the RSA hardness one, but it feels like a natural outcome and it would be a nice security improvement.*

   There are two main attacks that the Adaptive Prime Roots Assumption takes care of, when an attacker is targeting our VDF protocol. Let us also omit any residues for the sake of keeping things simple. First, the attacker could guess the expected Verifier's choice of r, and subsequently choose a random value π and set $y = \pi^r$. This is easily staved off by using a bruteforce-resistant security parameter (e.g. 2^{256}), for example based off of our RSA parameter $2\lambda \approx 200$. The second attack deals with the reason as to why we choose r to be prime, and not just any random number from $\mathbb{Z}_{2\lambda}$. The reason is that if r turns out to be a smooth-integer, then the attacker could choose $y = \alpha^B$, for random α and B the product of many prime powers (up to some limit); then, $\pi = \sqrt[r]{y} = \alpha^{B/r} \pmod N$ with $B/r \pmod{\phi(N)} = B/r$ if there is no residue, hence there is no need to know the group order to calculate the root because we don't need to work within the group. If the r is prime, then there is a really high probability that there will be a residue left, with an exception for the unlikely scenario where it is chosen to be equal to one of $\phi(N)$'s factors.

Scalability This protocol is fully succinct, because it is both *verifier scalable*

$$O_V(polylog(T)) \Longleftarrow O_V(2\cdot\lambda) = O_V(2)$$

and it has a *succinct proof*

$$O_{|\pi|}(|x| = \lambda_{RSA})) \Longleftarrow O_{|\pi|}(1\cdot\lambda_{RSA} + 1\cdot\lambda) = O_{|\pi|}(1\cdot\lambda_{RSA})$$

Specifically, the checking algorithm for the Verifier V only requires two (π^r and $x'^{\ residue}$) small RSA_λ group exponentiations, which require respectively $|r|$ and $|residue|$ group-squarings using the "square-and-multiply" algorithm, and a group multiplication to put them together for the equality check. As for the messages required to complete the proof, only π and r need to be transferred, the first one is just a group element and the second one is much smaller (due to the fact that λ_{RSA} is derived from λ)

2.2.7 Eliminating Trust Issues

Now that we've achieved such a cool protocol, let's address a final issue: how to make multiple parties use the same VDF without fear of cheating? Alternatively, how to achieve **transparency**?

The only construction we've mentioned so far, using RSA groups, clearly requires someone to generate private keys which, as we've seen earlier, can be used to break the protocol. What we really need are techniques to prevent anyone from owning the private key, so that nobody even has the chance to cheat. Here are a three strategies discussed by [28]:

1. **Alternative modulus generation**: There is an approach to generating RSA groups, presented by [42], which aims to completely skip the private key generation by randomly selecting a large modulus which can satisfy RSA requirements with high probability. If this modulus is indeed large and chosen randomly, nobody should be able to extract $\phi(N)$ from it. While this method is the simplest and most efficient way to patch our protocol, it does not always lead to correct RSA groups and it is believed to severely damage VDF sequentiality requirements, leading to more efficient μ_λ implementations. Thus, it might break SPoW soundness assumptions and cannot be used reliably.

2. **MPC-based RSA setup**: a popular solution to solving trust issues is, as we've discussed already, distributing trust. As it so happens, secure Multi-Party Computation protocols (e.g. Yao's garbled circuits [43] and secret sharing with arithmetic circuits[44]) would allow multiple participants to jointly generate a provably valid RSA modulus, without leaking the private key to anyone; a technique for this is presented in [45]. Such MPC-based approaches are practical enough that they were also employed by the ZCash cryptocurrency [46] for its setup.

Unfortunately, this method is secure only if at least one party in the computation is honest, which means that all (independent) parties interested in using the *transparent* VDF protocol should participate in the setup phase, to be sure that it is trust-less. However, in blockchain scenarios multiple parties join the protocol long after the initial setup phase — which means that some degree of trust is involved. As long as the number of (independent) parties participating in the MPC is a significant portion of the total number of VDF users, this method is convenient and reliable.

Since MPC setups do involve the generation of secret random values, they cannot be considered strictly *transparent*, according to the definition we gave in our VC model. However, they do provide a strong form of trust reduction through distribution — which is commonly cited as being the core of Bitcoin's "trustless" design. We can consider them to be a weaker form of *transparency*, perhaps called "trustlesness".

3. **Alternative Groups**: a newer approach, given by [28] in his VDF construction, has been to replace RSA with a trapdoor-free multiplicative group, also called "Class Group of an Imaginary Quadratic Field" [47] (adaptation to RSA presented in [48]). This approach promises to be uncompromising, since the group order is not known even to the party setting it up. However, this method still requires someone to generate the public parameters, so there is an assumption to be made: no other setup procedure for these groups needs to allow for a Trapdoor, so either the known setup procedure is the only one available, or any other procedure cannot leak the group order. Given that these groups have not yet been sufficiently studied by the cryptographic community, this method can be considered to be less reliable. It is, however, a very interesting topic for future research, and it provides an innovative solution to the problem at hand.

2.2.7.1 A note on VDFs as transparency enablers

Given that VDFs can be used to build trustless Randomness Beacons, and that these short-term[18] random numbers can be used to setup other cryptographic protocols in a transparent fashion, it is imperative that VDFs themselves be transparent as well. However, one does not need to go overkill with the setup procedure — if the protocols which make use of the Randomness Beacon have less or equal security requirements than that of the VDF (i.e. the number of participants in the MPC phase is still significant compared to the total number of users of the protocol), the MPC setup procedure is still good enough for their purpose. In fact, there are current plans [40] to implement a blockchain-wide VDF for the Ethereum cryptocurrency, rather than as an isolated third-party instance, such that all the "smart contracts" (i.e. subprotocols) running within Ethereum already lie within its security requirements.

[18] i.e. they cannot be used, once revealed, for new protocol executions; for example, in a lottery system you may only use numbers which have will be revealed after the bidding phase is over. Because of this, you will need to keep using "fresh" beacon outputs.

2.3 Conclusion

We've seen two very powerful techniques for building VC protocols: (1) arithmetisation, used in Homomorphic Authenticators for outsourcing small-degree computations; (2) interactivity and randomness, used in VDFs for "compressing" computations and measuring time. It is my belief that while many non-universal VC protocols have been considered as part of separate fields, we should try to converge them under the domain of Verifiable Computation, to compare them and understand the most efficient designs behind specific cryptographic VC properties. Such designs can later be abstracted and employed for building more expressive protocols, as demonstrated by the execution trace idea found in both Pietrzak's VDF protocol and many other Universal VC protocols (e.g. STARKs), discussed in the next chapter.

Unfortunately, and due to time constraints, this chapter only barely scratches the surface of all non-universal proof protocols that have been built over the past decades, so I leave as an open question the analysis and unification of the remaining ones. To motivate the reader in that direction, allow me to acknowledge other very interesting and influential systems:

- **Sigma Protocols**: Σ-Protocols is the field representative of traditional Zero-Knowledge Interactive Proof systems, which were developed decades ago with the intent of building more practical constructions through the use of relaxed VC properties. This is an extensive field, focusing primarily on public key authentication, seeing the likes of the famous Schnorr Identification and Signature scheme. Protocols from this field share many properties with the universal proof systems discussed in the next chapter. The Ring-based Learning With Errors scheme found in [49] is an interesting recent development in this field.
- **Proof of Work**: this field was made popular by the deployment of the Bitcoin cryptocurrency, which uses it for its transactions (i.e. proofs). A lot of research from the cryptocurrency communities has gone into extending this field with more efficient constructions, resulting in improved consensus solutions for decentralised-trust protocols. A notable evolution of this field is VDF protocols, which we analysed in this chapter.
- **(RSA) Accumulators**[19]: this interesting field, whose protocols implement very efficient operations for checking membership of an element in a set, typically makes use of hidden order groups (e.g. RSA). Such constructions can also support other set operations, such as union and intersection. This field comes closest to implementing the *universality* property found in Universal VC schemes presented in the next chapter. Fun fact: Zerocoin, the precursor to the Zerocash protocol that the Zcash [46] cryptocurrency implements, was actually based upon RSA Accumulators.
- **Attribute Based Encryption (ABE)**: such systems take advantage of user identities to establish public key pairs, which offers the big advantage of being able to send a single message to a specific hierarchy of users without needing to collect many

[19] an interesting starting point for the reader might be [50], which offers a systemisation of this field, including constructions not based on hidden order groups.

different keys. The use of a public authority (i.e. Trusted Third Party) is typically required, and almost all such systems employ homomorphic encryption, used to build complex relationships between messages and identity attributes.

3 Universal VC Compilers

A revolution in the applicative world of cryptography, first with Blockchain technology and now with Zero-Knowledge proofs, has been developing over the last decade. The success achieved by these protocols is starting to spark excitement, in the hopes that it could change not just our societal functions (e.g. cryptocurrencies vs traditional fiat money), but also the way we interact online and develop software. The main goal of these efforts has been that of developing innovative cryptography to help us regain trust in a trustless world, to help us base all our communications on verifiable statements: to build a **"Proof of All"**.

So far we've discussed the basic building blocks for cryptographic proofs, and how specialised protocols can be designed to handle private or computationally-sensitive scenarios. One common characteristic, and potential downside, of using such non-universal protocols is that adapting them to particular use-cases requires technical know-how; compromises (in terms of VC properties) are often also required to retain efficiency or privacy. In this chapter we will be taking a step towards an uncompromising solution, and the jack-of-all-trades when it comes to Zero-Knowledge proofs, *Universal* VC protocols. These systems do not only protect the privacy of their users, but they can also guarantee the integrity of any computation. Because such protocols can be used to generate proofs based on any other program, automatically and without much technical know-how, I call them **Universal Proof Compilers**.

The focus of the chapter will be understanding and designing the fundamentals of a protocol which marks a breakthrough in the field of VC technology: *zk-STARKs*. While this construction is fairly recent, it is the result of many years of research and has already been well received by the cryptographic community. This system is the first **concretely efficient** (i.e. suited for realistic usage) Universal VC compiler that is also post-quantum safe, and does not require any form of trusted setup (i.e. it can be used out-of-the-box, unlike zk-SNARKs). Towards the end of the chapter we will also mention alternative systems to zk-STARKs that have been developed in recent years.

3.1 zk-STARKs

What are zk-STARKs? Glad you asked:

- *zk*, as in zero-knowledge and privacy-preserving;
- *Scalable*, or efficient, as proving requires little increased overhead, generated proofs are relatively "small" (or acceptable) in size, and verification takes exponentially less time than executing computations naïvely (i.e. almost instantly, even for very heavy ones);

- *Transparent*, as in there is no requirement for a trusted setup, like in zk-SNARK systems;
- *ARgument*, as in a computationally secure cryptographic proving scheme achieving completeness and soundness for a specific language;
- of *Knowledge*, as in based on statements with relation to publicly known information (see more in Section 3.1.2.1).

But most importantly of all, (zk)STARKs are *Universal* Verifiable Computation systems. Unfortunately, these definitions are not sufficient to build or understand such systems. The construction by Ben-Sasson et al. presented in [51] is fairly complex, filled with engineering-specific details (the protocol was designed to be concretely efficient), and overall tough to digest even for cryptography students. For these reasons, and following the goals of the thesis, I chose to focus on design principles, foregoing formal proofs in favour of a simplified understanding. In this section I will break down the core concepts of zk-STARKs, showing how a general purpose computational-integrity statement can be converted into a proof.

In Section 3.1.1 I will give an overview of how we're going to approach building a STARK. Each step of our design represents a problem instance that abstracts the following step, thus providing a useful overview for breaking down STARKs. By taking a stricter mathematical formalisation of the initial problem statement and following the given reduction steps, it is possible to synthesise the protocol into a single statement, comparable to that provided within the original paper.[1]

Each later subsection will present: the main objective of a universal VC system (Section 3.1.2); the intermediate arithmetisation process required to break down normal computations into usable components (Section 3.1.3); $2POLY$, the name I give to the core subprotocol used by STARKs to implement a VC proof (Section 3.1.4); concrete performance results achieved by STARKs through interactivity, and security (i.e. soundness) assumptions (Section 3.1.5); the privacy extension to convert STARKs into zk-STARKs (Section 3.1.6); FRI, the subprotocol used by $2POLY$ for probabilistic degree testing (Section 3.1.7).

3.1.1 The Main Design

The single most important tool which is used by all known Universal V.C. systems is that of **arithmetisation**. It is the process of converting a question on the integrity of a general purpose calculation into a mathematical statement which we can manipulate through cryptographic means. For zk-STARKs this is the *polynomial comparison* problem, for their zk-SNARk predecessors it is *quadratic arithmetic programs*, but similar ideas arise in all the competing universal VC protocol systems, exhibiting varying approaches: homomorphic cryptography, multiparty computation, probabilistic checkable proofs, and interactive proofs. The original problem statement provided here is reduced (not without any assumptions, as we will see later) to an algebraic statement on polynomials, for which

[1] the original paper in [51] also formalises and takes care of multiple engineering optimisations, which I only briefly touch upon later on. These details can be considered to be essential for the implementation of a practical STARK and are an important contribution to the achievements of the paper, as well as the basis for the official open-source implementation provided by the authors in [52].

we actually have a working cryptographic protocol. It is the final problem of polynomial comparison on which we will focus our cryptographic tools deriving from PCPs and IPs, the rest is mostly arithmetisation.

First, we will informally introduce the main problem of Computational Integrity and Privacy which we are trying to solve (Section 3.1.2); then we will perform a few arithmetisation steps which bring us closer to a formal statement on polynomials (Section 3.1.3); then, we will present the core polynomial comparison and proximity testing protocols used by zk-STARKs (Section 3.1.4).

Here is an overview of the problems addressed by our design, in increasing order of specialisation:

Arithmetisation Step	Problem	Description	VC Benefit
1	Generic Statement	*Was the output of this computation, within the specified time-frame, correct?*	*Universality*
2	Computational Integrity&Privacy Statement	*Is it true that Output=Program(Input) within T steps?*	*Universality*
3	Algebraic Problem	$f(x) \stackrel{?}{=} y$, $O_f(T)$	*Universality*
4	Execution Trace Algebraic Problem	$ee \stackrel{?}{\in} \mathscr{C}$, *ee execution trace*, $\mathscr{C}$ *constraints*, $\lvert ee \rvert = T+1$	*Soundness* (2*POLY* call format), *Scalability*
5	Polynomial Comparison Problem	$f(x) \stackrel{?}{=} g(x)$, $(f,g) \in \mathbb{F}[x]$, $deg(f) = deg(g)$	*Soundness* (check), *Zero-Knowledge*, *Scalability* (engineering optimisations), *Transparency*

3.1.2 Original Problem Statement

We're looking to build a system which can represent any VC problem, i.e. a "Proof of All" system; before building it we need to define it, in order to state our requirements and boundaries. The researchers behind zk-STARKs provide a language to define any trustless computation, called **Computational Integrity and Privacy** (CIP) problem statements. Such statements represent the state of the art of what current cryptographic proof systems can achieve in any computational model.

First off, let's clarify the name "computational". With this name, we simply wish to allow the system's users to make statements regarding any sort of general purpose computation.

The *universality* property of our VC model suffices to satisfy this requirement. Here are the main properties defining CIP problem statements:

1. *Integrity*

   In order to trust the output of a specific computation, we need to consider that a Prover may be incentivised to cheat. We can think of income tax statements, for example, where a citizen is trying to perform tax evasion by submitting false claims regarding his income. To prevent this, and to trust the validity of the Prover's claim, we need to somehow "bind" the computation's output to the actual requirements of the computation. I also informally consider this to be the binding property of a CIP statement, and it can be accomplished through the *completeness* and *soundness* properties which we defined in our VC model.

2. *Privacy*

   What happens if the output of a specific computation can be revealed, but not its input? Consider a scenario in which I'm buying drinks at a bar and I need to provide identification to the bartender, so that he may check that I am of legal age to drink alcohol, but I do not want to reveal anything else about my age, name, nationality, height, or gender. To allow a Prover to make such a privacy-friendly claim, we need to somehow "hide" our computation's inputs (i.e. my personal details, in the given example) from the Verifier. I informally consider this to be the hiding property of a CIP statement, and it can be accomplished through the VC model's *zero knowledge*, *transparency*, and *post-quantum safety* properties.

   The transparency requirement is necessary in contexts where I want anybody to be able to verify my claim, at any point in time, without the need for a trusted setup phase. Post-quantum resistance is also important in any cryptographic system meant to stand the test of time, thus becoming a reliable standard for the protection of data many decades (if not centuries) down the road.[2]

3. *Efficiency*

   Along with the previous two fundamental properties, Ben-Sasson et al. mention this additional and more practical requirement. We are concerned with the realisation of concrete systems, which can be used under realistic and fair conditions, using hardware that is commonly available to any average Prover or Verifier.

   Assume you're tasked with extracting all the facial images of people passing through an airport on a specified date, and then matching them against a known-criminals' database, as part of a police investigation. You wish to provide the list of matches as evidence for a court hearing, but the court is skeptical of your work and wishes to double-check the results. If the computation took 20 hours to complete, will the court need to take just as long to verify your statement?

[2] note, the main assumptions made by zk-STARKs are: (1) the existence of cryptographic One-Way hash functions; (2) the Random Oracle Model. These assumptions are amongst the oldest to exist in cryptography, and they have defied all sorts of cryptanalysis, including recent Quantum computer developments.

Universal zero-knowledge proof systems of the past were actually very burdensome in this regard, easily requiring terabytes of memory for even the simplest calculations; the most efficient proof systems were Sigma protocols, but they were only appropriate for very specialised computations. To make our system actually usable, we need to somehow reduce its impact to be minimal for the Prover, and actually even convenient (i.e. much faster than normal execution) for the Verifier. With regards to the communication complexity of our system, it should stay within acceptable levels of Internet communication.[3] To realise this CIP property, we will have to implement multiple properties of our VC model: *prover scalability*, *verifier scalability*, *proof succinctness*, and *non-interactivity*[4]. With regards to non-interactivity, it is an important efficiency measure because it allows using our system even when neither party can communicate at the same moment. The proofs can be batched in advance, and sent off for inspection at a later time.

Finally, we can formalise our system's language to be

CIP *Is it true that Output=Program(Input) within T steps?*

$$\iff \Big\{(P,T,x,y)\,\big|\, y = P(x),\;\; \text{“Program” } P \in ITM,\; O_P(T \text{ steps})\Big\}$$

[5]

A state-of-the-art system proving that $X \in CIP$ needs to implement the following VC properties:

- universality (it's intrinsic)
- completeness
- soundness
- zero-knowledge
- scalability

Additionally, zk-STARKs also implement

- non-interactivity
- transparency
- post-quantum safety

[3]for example, proving a single CIP statement typically requires the transfer of a few hundred bytes with zk-SNARK systems, and a few hundred kilobytes with zk-STARK systems. Considering that a CIP statement can be used to further compress other CIP statements, both complexities are acceptable even for repeated use in space-sensitive environments, such as decentralised Blockchains.

[4]the main strategies we will use to get all these properties are: (1) arithmetisation; (2) random querying. In alternative systems, such as zk-SNARKs, homomorphic encryption is also used, with a boost in efficiency but a loss in privacy (specifically, transparency).

[5]"steps" here represents a state change within the program. When comparing with other systems, it is useful to convert a step to the number of CPU cycles that are required for each state change after converting the program into a binary circuit. If it is hard to identify a specific state, then every single instruction can be interpreted as a step, with the state being the totality of the program's variables.

3.1.2.1 A few notes on Program Specifications

I would like to take a moment to debate on the utility of CIP statements in real-life scenarios. In general, the problem we're trying to solve is not always related to a specific program, as much as it is to a specific computation:

Generic Statement *was the output of this computation, within the specified time-frame, correct?*

Given such a generic requirement, it may not always be necessary to start execution of a STARK from the CIP statement of a binary program. The requirements for the computation may, in fact, already be defined by human generated **program specifications**, which document the desired functionality of the program being analysed. While such documentation is often seen underdeveloped (or lacking) even in the biggest software projects – since unit test-cases are a cheaper alternative – it can still serve as a concise and efficient definition for the core functionality of a specific computation. In fact, it allows us to skip the CIP's binary program conversion phase, and directly use our program specification for the intermediate arithmetisation phases.

Here are two example scenarios, one which appeals to program specifications and one which appeals to CIP statements on binary programs:

- *Copyright-Protected Streaming*: a cloud provider's technician is tasked with adding DRM to their video streaming service, in order to comply with a recently approved European Copyright Directive. The only issue is that the service specialises on streaming encrypted videos, as an added privacy benefit. The technician immediately thinks to use his favourite zero-knowledge universal VC proof system, zk-STARKs, so as to retain privacy of the streams and minimise the impact of the new feature on the service's performance. In this scenario the constraints are very simple: each source file needs to be checked against a list of blacklisted files, then it is encrypted and checked against the corresponding stream.

  While the technician could write a program to do this, compile it, and send it over to the clients so that they can generate CPI-based proofs, there are several downsides: (1) waste of resources, as developing and deploying consumer-level applications can take a lot of man hours; (2) bugs, as traditional testing does not typically guarantee that the security specifications are met by the program with a high degree of certainty; (3) performance hit for the clients, because full arithmetisation of a stateful binary program can lead to much more complex constraints and larger execution traces than is really necessary, also leading to bloated proof sizes; (4) last but not least is security, because the users are asked to trust that executing a binary program will not compromise the confidentiality of their files[6].

  A much smarter solution is that of taking advantage of appropriately documented program specifications for the requirements of the DRM feature, and sending those

[6] note that even if the program was released as open source, it still takes a much longer time to analyse thousands of lines of code (also including libraries) rather than just a few lines of specification requirements.

off to the clients in a standardised format. The streaming service's users can then take advantage of trusted zk-STARK implementations to build proofs based upon a very small set of constraints.

- *CTF Challenge*: in "Capture The Flag" competitions, participants typically take part in jeopardy-style cybersecurity games where they must solve multiple challenges to score points. One popular category of these challenges, known as "Pwning", requires that participants discover a vulnerability hidden somewhere within a given program; to verify that a player has exploited the program successfully, instead of just manually bypassing the security checks through binary editing, the program is uploaded to a sandboxed server and players are restricted to feeding it input through an internet socket.

  In all common recurrences of this scenario, a few issues arise: (1) a server with high computational and bandwidth capacities needs to be rented to host the vulnerable program; (2) the vulnerable program also needs to be sandboxed or virtualised to protect the server's integrity, leading to further impacts on computational requirements; (3) in case of oversights made during setup of the sandbox, the server itself may become vulnerable, leading to a potential compromise of the whole competition[7]; (4) some malicious actors may choose to carry out a DoS attack on the server by overloading the vulnerable program with continuous inputs, leading to an abrupt end of the whole competition.[8]

  In this specific scenario, applying STARKs using the CIP binary program statement makes perfect sense. There is no need to apply zero-knowledge (HTTPs is probably sufficient), but there is still a desire to verify knowledge of the vulnerability in a very short time-frame, and without potential compromise of the server. Furthermore, the requested knowledge directly relates to a specific stateful program execution, so it makes sense to arithmetise that same program along with its every nook and cranny. Thanks to STARKs, CTF competition maintainers could host challenges at a fraction of previous costs, without worrying too much about security of the hosting server.

3.1.2.2 A note on Zero-Knowledge Statements

In this section, we regarded the input x of a program as part of the CIP language statement. Truthfully, things are a little different when we consider the need for zero-knowledge. With zero-knowledge we actually aim to hide the input x, so it cannot be part of the statement, it will instead be part of the witness. At the same time, having a statement of the form

zk-CIP

$$\Big\{(P,T,y)\,\Big|\,\exists x : y = P(x),\ \text{“Program” } P \in ITM,\ O_P(T \text{ "steps"})\Big\}$$

[7] a well justified concern when dealing with participants whose expertise is cybersecurity and penetration testing, actually!

[8] another common recurrence in CTF competitions. . .

does not always equate to a proof of knowledge. For example, proving that a number is composed of prime factors and proving that these factors are known constitute two entirely different ordeals. Because of this, typical zero-knowledge CIP statements aim to prove knowledge of a secret through a publicly known element (that we'll call h instead of x). This public input should uniquely identify the secret, without revealing any information. The most popular way to do this, with an computationally indistinguishable amount of information revealed, is through a cryptographic hash function H:

zk-CIP of knowledge

$$\Big\{(P,T,h,y)\,\Big|\,\exists x : y = P(x) \wedge h = H(x),\ \text{“Program” } P \in ITM,\ O_P(T\ \text{"steps"})\Big\}$$

[9]

This simple solution is also useful for authenticating users based on their public keys or other forms of public data that constitute a unique reference to the secret.

NOTE: when performing such proofs, it's very important to take into consideration hash functions that are better suited to the algebraic nature of STARKs. This is due to the complexity that arises when arithmetising such "functions" which are typically optimised for real processors. Because of this, the authors of the paper opted to make use of a Davies-Meyer AES-based hash construction ([51], [53]), which offered better performance compared to $SHA2$ when used in the binary fields that their polynomials were based upon. This concept also applies to the $2POLY$ and FRI protocols that we will see later, due to the requirement of a commitment scheme.

3.1.3 Intermediate Arithmetisations

The first important step to take is that of turning our problem statement on binary inputs, outputs, and programs into a statement on algebraic objects. In particular, the most difficult aspect of this transition is that of converting a stateful binary program into a function. In the original paper this is performed through a series of complex engineering steps called APR (Algebraic Placement and Routing) reduction, where the whole state of the program is also abstracted, including RAM and networking[10]. For our purposes it will suffice to assume that we have already converted, perhaps thanks to well-defined program specifications mentioned in Section 3.1.2.1, CIP statements into the following Algebraic Problem:

AP1

$$\Big\{(x,f,y,T)\,\Big|\,f(x) = y,\ O_f(T),\ f : D \to C\Big\}$$

[9] formally, it might actually be more correct to place all constraints inside the program, so that they can be considered to be part of the arithmetisation steps needed for a STARK: $P'(x,h) \overset{def}{=} \{y = P(x) \wedge h = H(x)\}$.

[10] of course, applying the APR to real programs running on MacOS/Linux operating systems and Intel/AMD processors is not yet realistic, so the researchers provided a proof of concept in [52] using a simple RISC virtual machine called TinyRAM [54].

We can now stop our process for a moment, to reflect on what it means to achieve a protocol with *scalability*. In the context of IP proofs, the Prover comes up with a randomised problem that is isomorphic to the original one, which allows revealing the witness under masked disguise. But even if we were to forego *zero-knowledge* and reveal our witness directly (the input x), it would still take the Verifier $O_f(T)$ steps to check our $AP1$ problem statement, which is just as long as naïve execution and so it precludes *verifier scalability*.

VDFs are another family of protocols which is similar to STARKs, because they are also trying to make really long computations become efficient to verify. For the VDF construction by Wesolowski [28], the choice of an isomorphic problem is justified because the specific algebraic properties of the chosen problem lead to powerful relationships that are efficient to verify, leveraging the security provided by RSA groups. But in our scenario, we are dealing with generic computations which may not possess such neat algebraic properties, so we cannot take advantage of such shortcuts. The construction by Pietrzak [29], instead, offers an approach that is a step closer towards the right direction. The idea is to explicitly expand the witness of the given problem into an execution trace, resulting in an isomorphic problem that allows the Verifier to randomly and efficiently inspect parts of the computation. Each state within the trace can be queried, and, with some randomness and smart recursion, the boundaries of the trace are checked without relying on clever isomorphism assumptions. We will take a similar approach of expanding our witness input x into an execution trace leading up to y, and our burdensome computation will be reduced to a few constraints with complexity much lower compared to that of the original execution. Because of this, I call the technique **witness expansion** and **constraint compression**. We will leave the "randomness and smart recursion" counterpart to the $2POLY$ subprotocol, which will leverage interpolation and proximity testing to obtain prover and verifier *scalability*.

The new problem can be regarded as checking an execution trace against one or more constraints:

AP2

$$\left\{(ee, \mathscr{C}, T) \,\middle|\, ee \in \mathscr{C},\ \mathscr{C}\ \text{"Constraints"}, |ee| = T+1, deg(\mathscr{C}) \ll T\right\}$$

To define what constraints are, and for the sake of simplicity, we can consider two case scenarios:

1. *Domain-based constraints*: each of the elements of the trace must satisfy a specific set-membership condition. This can be useful in very simple scenarios where we just want to check whether each element of a list lies within a given domain.
2. *Polynomial-based constraints*: this scenario is more realistic, and it considers the requirements that a normal program would have. They can be represented as polynomials, taking as input one or more execution states.

We will elaborate on reducing these two scenarios to a $2POLY$ problem in the following subsections.

NOTE: a single state of the execution trace defined above can be composed of multiple variables, especially when extracted from a binary program. The authors of the paper handled this case efficiently by considering each variable separately, splitting a single execution trace into multiple Reed-Solomon codes; this allows for notable space savings after interpolation, and the trace evaluations can later be joined through a linear combination.

3.1.3.1 Domain-based Constraints

Let's assume to have been given the following problem:

$$\forall x \in D : f(x) \overset{?}{\in} \mathscr{C},$$
$$with\ |\mathscr{C}| \ll deg(f),\ f : D \to C$$

Where $\mathscr{C}$ is precisely the domain constraint, end f is the function indexing an execution trace. Our main objective is to reduce the original statement to a comparison between two polynomials (f', g'):

$$f \in \mathscr{C} \overset{?}{\iff} f' = g'$$

First, we shall convert the set membership constraint to a vanishing polynomial, where $True$ values for the membership relationship end up evaluating to zero:

$$\forall x \in D : C(f(x)) = 0$$
$$C(y) \overset{def}{=} (y - C(0))(y - C(1))...(y - C(|\mathscr{C}| - 1))$$

Unfortunately, this equation is not yet sufficient, because it is bound to a specific domain. We will see later on that the $2POLY$ subprotocol needs to work on domains which can be extended, so we must expand the domain of our inputs to span over all the integers.[11] To help us do this, we can recall a useful theorem:

Th. Vanishing Polynomial Composition *it is always possible to extend the domain of a univariate vanishing polynomial through (de)composition*:

$$\forall x \in D : P(x) = 0$$
$$\iff \exists P' : \wedge \begin{cases} P(x) = Z_D(x)P'(x) \\ deg(P) = |D| + deg(P') \end{cases}$$
$$with\ Z_D(x) \overset{def}{=} \prod_{i \in D} (x - i)$$

(note: Z_D is also common notation to denote a polynomial vanishing on all of the domain D. The polynomial P' can be extracted by the prover by interpolating P and calculating P/Z_D.)

[11] as mentioned previously, the optimised variant of STARKs actually be works with specific fields. More on this later.

We can now extract the full polynomial:

$$\exists P' : C(f(x)) = Z_D(x)P'(x) \wedge deg(P') = deg(C) - |D|$$

And reduce to a $2POLY$ problem:

$$\exists P' : \begin{cases} f'(x) = g'(x) \\ deg(f') = deg(g') \\ \\ f'(x) = C(f(x)) \\ g'(x) = Z_D(x)P'(x) \end{cases}$$

We've discussed reduction to polynomial comparison, and we know that the $2POLY$ protocol will take care of efficient comparison. However, the the scalable version of $2POLY$ would have the Verifier defer querying (for a point i) $f'(i)$ and $g'(i)$ to the Prover, which would only guarantee that two random polynomials given by the prover are equivalent. To make sure that the domain (i.e. Z_D) and codomain (i.e. C) constraints are respected, and to retain **constraint soundness** of our CIP problem, the Verifier needs to call the $2POLY$ protocol using a special format:

$$C(\underline{f(i)}) = Z_D(i) \cdot \underline{P'(i)}$$

where the underlined parts are provided by the Prover, and the rest is calculated by the Verifier. This type of check still retains scalability, because C is of low-degree by assumption, and we now show an efficient technique for evaluating Z_D.

NOTE: in the original paper, the authors actually work with polynomials with domain taken from multiplicative field subgroups, in order to optimise vanishing polynomial evaluations:

$$Z_D(x) = \prod_{i \in D}(x - i) \;\wedge D \subseteq (\mathbb{F}, \times)$$
$$\overset{Th.Lagrange}{\Longrightarrow} Z_D(x) = x^{|D|} - 1 \;\wedge x \in \mathbb{F}$$
$$\left(\iff \forall i \in \mathbb{Z}_{|D|} : Z_D(g^i) = 0 \;\wedge \langle g \rangle = D \right)$$

with

Th. Lagrange

$$(\mathbb{G}, \times) \implies \forall x \in \mathbb{G} : x^{|\mathbb{G}|} = 1$$

This improves polynomial evaluation times from $O(|D|)$ to $O(log(|D|))$ thanks to the square-and-multiply algorithm for multiplicative field exponentiation.

3.1.3.2 Polynomial-based Constraints

In this more realistic scenario, we will check whether a specific execution trace ee follows the given program constraints:

$$ee \in \mathscr{C},\ ee : D \to C$$
$$\mathscr{C} = \{\mathscr{C}_{BOUNDARY}, \mathscr{C}_{EXECUTION}\}$$
$$deg\mathscr{C} \ll deg(ee)$$

$\mathscr{C}_{BOUNDARY}$ can be considered to be the **boundary constraint** polynomial (or list), which identifies the value that specific elements of the execution trace need to have; for example, the first/last elements might correspond to specific input/output values for a given CIP statement. $\mathscr{C}_{EXECUTION}$ can be considered to be one or more **execution constraint** polynomials, which define relationships between intermediate execution trace states; typically, they are one or more state changing functions.[12] Let's start with the necessary definitions for these constraint functions.

$$\mathscr{C}_{BOUNDARY} : D_B \to C$$
$$D_B \subseteq D$$

$$\mathscr{C}_{EXECUTION} : C_E \to C$$
$$C_E \subseteq C^{deg(C_{EXECUTION})}$$

And we can now start defining the constraint relationships:

$$\forall i \in D_B : \mathscr{C}_{BOUNDARY}(i) = ee(i)$$

$$\forall (i_{prev}, i_{next}) \in D_E : C_{EXECUTION}(i_{prev}) = ee(i_{next})$$
$$D_E \subseteq D^{deg(\mathscr{C}_{EXECUTION})+1}$$

As can be noted, the domains for these functions only apply to a subset of the execution trace. This is evident when we consider that boundaries apply typically only to specific elements of the trace, and state changing functions apply to a specific pattern of elements in the trace (e.g. subsequent states).

Just like we did for domain-based constraints, we can convert the relationships to vanishing polynomials:

$$\forall i \in D_B : C_B(i) = 0$$
$$C_B(i) \stackrel{def}{=} \mathscr{C}_{BOUNDARY}(i) - ee(i)$$

[12] I will only consider one state changing function, but there can be multiple; I will later make considerations on joining two (or more) constraint polynomials.

$$\forall i \in D_E : C_E(i) = 0$$

$$C_E(i) \stackrel{def}{=} C_E(i_{prev}, i_{next}) = C_E(i_1, \ldots, i_{\deg(\mathscr{C}_{EXECUTION})}, i_{next}) = \mathscr{C}_{EXECUTION}(ee(i_1), \ldots, ee(i_{\deg(\mathscr{C}_{EXECUTION})})) - ee(i_{next})$$

We can now expand the domain using the same theorem on vanishing polynomials that we used previously:

$$\exists P' : C_B(i) = Z_{D_B}(i)P'(i) \wedge deg(P') = deg(C_B) - |D_B|$$

$$\exists P'' : C_E(i) = Z_{D_E}(i)P''(i) \wedge deg(P'') = deg(C_E) - |D_E|$$

Thus, obtaining two distinct $2POLY$ problems:

$$\exists P' : \begin{cases} f'(x) = g'(x) \\ deg(f') = deg(g') \\ \\ f'(x) = C_B(x) \\ g'(x) = Z_{D_B}(x)P'(x) \end{cases}$$

$$\exists P'' : \begin{cases} f''(x) = g''(x) \\ deg(f'') = deg(g'') \\ \\ f''(x) = C_E(x) \\ g''(x) = Z_{D_E}(x)P''(x) \end{cases}$$

NOTE: the authors of the paper don't actually call $2POLY$ twice for these two statements, but they define a randomised (by the Verifier) linear combination to join them all together and check them at once. This is especially useful, considering that there may be multiple execution constraints, each detailing different conditions on successive execution trace indexes.

Finally, the same domain and codomain **constraint soundness** considerations mentioned in Section 3.1.3.1 apply here. The call formats for $2POLY$ are:

$$\mathscr{C}_{BOUNDARY}(i) - \underline{ee(i)} = Z_B(i) \cdot \underline{P'(i)}$$

$$\mathscr{C}_{EXECUTION}(\underline{ee(i_1)}, \underline{ee(i_2)}, \ldots) - \underline{ee(i_{next})} = Z_{D_E}(i)\underline{P''(i)}$$
$$with\ i = (i_1, \ldots, i_{next}) \in D_E$$

3.1.4 The Polynomial Comparison Problem

All of our efforts so far can be seen as having one main goal: reducing everything to the 2 polynomials' comparison protocol presented in this subsection. This is because this protocol satisfies two main properties that we're after, and that are typically harder to achieve in a universal proof system: *scalability* and *zero knowledge*[13]

How does this problem take form? Essentially, the Verifier is given (the evaluations of) two polynomials and asked to verify whether they're equal or not. We want to do this in the fastest way possible. Here is our typical language definition:

2POLY($\mathbb{F}$)

$$\Big\{(f,g) \,\Big|\, f(x) = g(x) \wedge deg(f) = deg(g) = d \wedge f,g \in \mathbb{F}[x]\Big\}$$

[14] or just

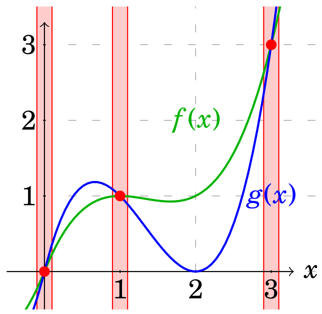
$$\Big\{(f,g) \,\Big|\, \forall i \in D : f(i) = g(i),\ f : D \to C,\ g : D \to C,\ |D| = d+1\Big\}$$

For now, we will assume that the polynomials (i.e. lists) are of the same degree, and we will focus on building a protocol to check their equality; Section 3.1.7 will take care of the degree check. The first approach the Verifier can take to solving this problem is just naïve comparison:

$$\forall i \in D : f(i) \stackrel{?}{=} g(i)$$

This method gives us **perfect soundness**, but it also takes $O(d+1)$ steps to run. Assuming that each polynomial is an extremely long execution trace, this check would force the Verifier to waste too much time, thus precluding *verifier scalability* from our final solution.

We can do much better by **slightly increasing the soundness error**, the same way that PCPs (Section 1.2.1) employ "probabilistic checks". Let us, then, consider the error probability of checking the polynomials against just a single element of the domain, which I will call "succinct query". The error occurs on any index which makes our check succeed, in spite of having two different polynomials; here is an example with the errors highlighted in red:[15]



[13] *universality* is another important one, but that's precisely what we achieve through our problem reductions!

[14] You will notice that we're using polynomials with coefficients taken from a field, this is useful for efficiency optimisations that we will outline later. For now, just consider all elements to be integers.

[15] again, note that in this plot the polynomials map to real numbers, but they will be part of a field when used for real programs.

We can see by the plot that the size of the errors space is much smaller compared to the rest of the domain, but is it really the worst case scenario? Thankfully, there is a well-known theorem regarding polynomials which can answer this question:

Th. Polynomial Comparison *two differing univariate polynomials of degree d are equal in at most d evaluation points.*

Now we have all the necessary information to calculate the error rate of a succinct query:

$$\begin{aligned} Pr[error] &\iff Pr[X \notin L \wedge check(X) = True] \\ &\iff Pr[(f,g) \notin 2POLY \wedge prob\_check(f,g) = True] \\ &\iff Pr[f(x) \neq g(x) \wedge \exists x_0 . f(x_0) = g(x_0)] = \frac{d}{|D|} \end{aligned}$$

NOTE: an alternative way to visualise this problem, and leading to the same probability, can be seen as the application of the Schwartz–Zippel Lemma [55]–[57] to probabilistic polynomial identity testing.

So, our error rate is dependent on the degree of the given polynomials, and the size of the domain they're evaluated on. In order to decrease this ratio, we have two available methods:

1. *Compress the polynomials*: to decrease d, we need to replace our lists with equivalent alternatives of lower degree. In the given $2POLY$ problem this is not possible because the lists are given as is, but within the STARK context the polynomials actually relate to execution traces. Each element of a trace can be anything, as long as it complies with the given constraints – it may also contain irrelevant local variables, after being extracted from a complex program! Carefully crafting such execution traces can result in a reduction of their size, and of our polynomials' degrees. Another technique is that of carefully interpolating the execution traces: the authors of the paper convert an execution trace to multiple Reed Solomon codes, obtaining further compressions because each local variable is considered separately!
2. *Add Redundancy*: to increase $|D|$, we need to increase the space from which we can pick our succinct queries. To do this, we can simply have the Prover give polynomial evaluations over a domain that is much larger than their degree, and this easily be obtained through interpolation.

Since the second method can always be applied to our $2POLY$ problem, we can always apply it to obtain any desired soundness error ϵ:

$$\epsilon = \frac{d}{|D|} \implies |D| = \frac{d}{\epsilon}$$

This protocol will be completed with the FRI protocol in Section 3.1.7 for checking our original assumption that the two polynomials have the same degree.

NOTE: the approach of adding redundancy can also be applied to to probabilistic polynomial comparison using directly coefficients instead of evaluation points, with an even better *soundness error. The basic idea is to multiply both polynomials with a random polynomial of large degree, thereby spreading out single coefficients across multiple ones.*

NOTE2: it is also possible to use multiple dependent queries to further improve the accuracy. Care must be taken to respect the zero-knowledge requirements described in Section 3.1.6.

3.1.5 Scalability through Interactivity

We've discussed just how we can have precise but succinct polynomial identity tests with just a single evaluation point, but how do we evaluate this point? The polynomials need to be interpolated to add redundancy, how much is this going to cost us? It's time to reveal the trick that has made so many proof protocols successful: interactivity. Thanks to interactivity, the Verifier can ask the Prover for auxiliary information with regards to the original problem, without compromising the actual integrity or privacy of the statement at hand. Any good interactive protocol makes use of a *Cut&Choose* technique (like the one discussed in Section 2.2.5), where the Prover sends over an alternative representation of his original problem, after the Verifier has made his choice. In the scenarios discussed within STARKs, the original problem is typically a polynomial evaluated on a specific domain, and the choice of the Verifier is a single point within that polynomial. In traditional non-universal proof schemes, it is common for the researchers to seek out an alternative representation of the original problem that is: (1) isomorphic to the original one, (2) randomise-able, and (3) that does not reveal anything about the problem's witness (whenever it is a non-deterministic secret fixed by a public key or hash function). An example of this can be, in Schnorr protocols, the task of finding a masked private key dependent on the original secret, or, in the Wesolowski VDF, the adaptive prime roots assumption used to request a randomised exponentiation strongly coupled with the original one.

In STARKs, however, we do not have access to such isomorphic problems for universal CIP statements that **also** inherently bind the Verifier's choice to the original statement. Because of this, to make sure that the Prover does not cheat based on the Verifier's selection, we ask the Prover to make use of a Commitment Scheme to bind the original problem statement to the Verifier's choices:

1. *Commit*

   the Prover commits to each possible evaluation of the interpolated polynomial on the required domain. Each evaluated point will be kept hidden by the Commitment Scheme (due to its "hiding" property), which is useful for the Zero-Knowledge extension discussed later. This step is the "cut" part of the *Cut&Choose* technique.

2. *Query*

   the Verifier chooses one (or more) point(s) from the polynomial that he would like to query. This step is the "choose" part of the *Cut&Choose* technique.

3. *Reveal*

the Verifier opens the commitment for the requested points, revealing the requested evaluation; because a Commitment Scheme is “binding”, he will not be able to change the value of the evaluations that were committed in the first step (as the Verifier would notice and abort the protocol).

Thanks to this neat trick, the full domain of the evaluated polynomial will be kept consistent by the Prover, otherwise either the reveal step or the subsequent soundness check required by the protocol will fail.[16]

There are still a few details to iron out:

- *Who interpolates the polynomials?* The Prover interpolates the polynomials using their original domain (e.g. the execution trace) and evaluates them on the domain defined in the $2POLY$ subprotocol (which in practice can be quite large, and at least 100 times larger than the original domain). Interpolation and evaluation was combined into a single process using a state-of-the-art quasi-linear time algorithm for Reed-Solomon codes based on additive-FFT techniques, described in [58]. This also produces our quasilinear *scalable prover* VC property, having $O_P(T \log^2 T)$[17].

- *What is the communication complexity?* While the Verifier only needs to request a few points to be evaluated on a specific domain (with values at most of size $|\mathbb{F}| = 64\ bits$), the commitments made by the Prover take up as a large amount of such values. This can lead up to as many as $|D'| \cdot 64\ bits$, with | D' | being easily x100 or more times the size of the original domain (such as the size of the execution trace). This is not practical for the Verifier to store, and imposes a huge strain on communications.

  Because of such issues, the authors decided to rely on the Kilian-Micali ([59], [60]) “argument compiler” for PCPs[18], which basically uses a Merkle-Tree [61] (whose leaves are the Prover’s evaluation points) as basis for the Commitment Scheme, sending just a single hash value (i.e. the tree’s root) as commitment. However, each revealed evaluation (i.e. leaf of the tree) also needs to verify the commitment using an “authentication path”, which is basically a tuple of the necessary hash values required to traverse and validate the Merkle-Tree from the revealed leaf up to the tree’s root. If we assume that we’re using a cryptographic hash function $H : \{0,1\}^\lambda \times \{0,1\}^\lambda \to \{0,1\}^\lambda$, the final proof ends up becoming a *succinct proof* for its size complexity: $O_{|pi|}(\#queries \cdot \log(|\mathbb{F}|)\ bits + pathlen \cdot \lambda\ bits) \approx O_{|\pi|}(\log^2 T)$.[19]

  Finally, in the case of the FRI protocol for low degree testing, there will be multiple polynomials to commit to, so multiple Merkle-Tree roots will have to be used and authenticated.

[16] that is, as long as the check is truly sound. See the bottom of this subsection for a discussion on soundness assumptions for STARKs.

[17] T is the same as the one discussed in the CIP problem statement.

[18] the improvement made by Micali was for the non-interactive version of the protocol.

[19] the left summand refers to the size of each queried and revealed point, the right summand refers to the size of each authentication path required to validate the revealed point (the path is logarithmic with relation to the total number of elements).

- *What Commitment Scheme to use?* Commitment Schemes are typically built using some randomness and a cryptographic hash function, $SHA2$ or *Keccak* is a typical choice. In STARKs, it turns out that using using $SHA2$ was too costly for the arithmetisation, so they used the same Davies-Meyer AES-based construction [53] that we mentioned earlier (Section 3.1.2.2).

Thanks to all of these efforts, as well the requirement on low-degree constraints $\mathscr{C}$ (Section 3.1.3), and taking into consideration the FRI subprotocol discussed later, we are also able to achieve *verifier scalability* for $O_V(\log^2 T)$. For a concrete comparison with zk-SNARKs, we have approximately $1/10th$ proving time, half verification time, and 100 to 1000 times the proof length.

On a final note, let us consider the **security assumptions** that we require for a full zk-STARK proof, leading to a *transparent* and *post-quantum safe* system:

1. Existence and availability of cryptographic one-way Hash Functions
2. Validity of the Random Oracle Model (ROM) (only for the *non-interactive* variant)
3. Existence of a ZK Argument of Knowledge Statement (Section 3.1.2.2)
4. Public Randomness Source (for *transparency*)

3.1.5.1 The Non-Interactive variant

Non-interactive STARKs were proven to exist using the ROM model, which is performed using the traditional Fiat-Shamir heuristic [12] shown in our model ([Section sec. 1.5); but keep in mind that it reduces our perfect *zero-knowledge* scheme to a **computational *zero-knowledge*** scheme.

3.1.6 Adding Zero-Knowledge

Let us now turn the page to what is probably the most captivating feature of zk-STARKs: **perfect *zero knowledge***! Shockingly, and with great distinction from previous schemes based on homomorphic cryptography (e.g. zk-SNARKs), it is actually the easiest property of the protocol to achieve. To see why, we need to pay respects to our adamant use of pure polynomials, and to the shoulders of giants on which our $2POLY$ subprotocol stands upon: Reed-Solomon codes and Shamir's Secret Sharing.

Reed-Solomon [62] codes originated in the 60s, with the objective of introducing error correction functionality to error-prone communication links. Their main success was realising that redundancy can be used to efficiently describe polynomials and detect errors with a very small overhead, a notion which granted us our *verifier scalability*. The codes relied on a well known theorem on polynomials:

Th. on Polynomial Interpolation *A univariate polynomial of degree d is uniquely defined by* $\geq d+1$ *points.*[20]

[20] The theorem is also evident when considering that $d+1$ evaluations can be put into a system of equations containing $d+1$ variables for all the polynomial's coefficients – solving the system leads to the correct solution.

At the same time, Reed-Solomon codes offered a solid theoretical foundation for secret sharing in the 80s, when Shamir's Secret Sharing [63] scheme was introduced. The main idea is to hide a secret in the very first element of a list, which is itself the evaluation of a polynomial of degree d on an arbitrarily large domain[21]. Each different element of the list, except for the first one, is distributed to a group of trusted users; when at least $d+1$ users come together, they're able to recover the secret through Lagrangian interpolation.[22]

The nice feature about Shamir's scheme is that it is information theoretically secure, because not even computationally unbounded attackers with access to $\leq d$ shares can retrieve any information on the secret. This serves as a foundation for our *zero-knowledge* property, here is the protocol extension:

1. *Deny Querying the Execution Trace*: the Verifier is not allowed to perform queries from the execution trace's original domain, as any of its values may contain traces of the original witness (i.e. the input x). Likewise, in Shamir's scheme the first element of the evaluation list, typically containing the secret, is never shared.

2. *Introduce Randomness*: the execution trace is extended with uniformly selected noise, equal to as many elements as the number of queries performed by the Verifier.

   This is performed because preventing the Verifier from querying the original domain of the execution trace is not sufficient to achieve zero-knowledge. While the same concept of Shamir's scheme applies – in that owning less than $degree + 1$ (i.e. $|ee|$) evaluations is not always sufficient to recover the full secret (i.e. ee) – the same context does not. Specifically, in Shamir's secret sharing **at least** d elements of the original polynomial are uniformly selected random values[23], so each possible interpolation of a degree d polynomial from d points is equally likely[24], making it perfectly hiding. In our scenario, instead, we cannot assume that at least $\#queries$ intermediate states are uniformly random, as the opposite is often true because these states tend to be dependent on each other or take a particular shape/form with non-uniform probability. Because of this, some possible interpolations of a degree $deg(ee)$ polynomial are more likely to occur, leading to a leakage of information for each query provided to the Verifier.

   In order to prevent these "partial interpolation" attacks, however unlikely they may be, we can simply append $\#queries$ uniformly selected random values to the original execution trace. Not only does this provide perfect zero-knowledge (as in Shamir's scheme), but it still retains soundness with regards to the given STARK execution and boundary constraints. To see why, consider that the constraints only validate the

[21] the evaluation typically starts from zero, so the coefficient of degree zero for the polynomial is simply the secret, and all other coefficients can be selected randomly from the same domain of the secret.

[22] as we discussed in Section 3.1.5 the authors of the paper actually take advantage of more efficient interpolation algorithms.

[23] this is a direct result of the fact that at least d coefficients of the degree d polynomial are uniformly selected random values.

[24] if the polynomial is part of a field $\mathbb{F}[x]$, then there are $|\mathbb{F}|$ possible polynomials, all with the same probability of being correct

domain of the original execution trace, so any noise added outside of that domain is still acceptable. This can also be easily deduced by considering the lack of restrictions on the contents of the polynomial P' found in the theorem on vanishing polynomial composition that was presented earlier (Section 3.1.3.1).

3.1.7 The (Low) Degree Testing Problem

One important condition of the $2POLY$ protocol, and the secret behind its scalability, is the requirement "$deg(f) = deg(g) = d$". In fact, knowing the degree of a specific polynomial allows for great optimisations, such as those seen in Reed-Solomon [62] error-correction codes and Shamir's Secret Sharing [63] scheme. The authors of the zk-STARK paper came up with a protocol for validating a stated degree, called *Fast Reed-Solomon Interactive Oracle Proofs of Proximity*, or FRI (presented in [64]).[25] The key innovation of this protocol is providing a concrete **Proximity Testing Protocol**, made possible through the use of Interactive Oracle Proofs (IOPs) and other engineering optimisations. In this subsection, I will discuss how a PCP for degree testing can be made scalable through interactivity.

First things first, let's start with the complex name: (1) the protocol is *fast*, in that it is concretely efficient and can be used for practical purposes (e.g. STARKs); (2) the problem statement is based on *Reed-Solomon* codes; (3) the method to solve the problem is a combination of IP and PCP proof methodologies, called *IOP*; (4) we do not test for equality with a specific degree, but for *proximity*.

The reason that the problem needs to be relaxed to proximity testing is that it is actually quite hard to achieve a concretely efficient protocol for checking degree equality, so we relax our assumptions a little bit. We transform our part of the $2POLY$ statement on degrees from something of the form $deg(f) = deg(g) = d$ to something like $deg(f) \approx deg(g) \approx d$; to be exact, through use of Reed-Solomon codes we can make our statement become $deg(f) \leq d \wedge deg(g) \leq d$ without loss of soundness precision for the $2POLY$ protocol. However, the actual result deriving from our implementation will lead us to two statements, to be checked separately, based on proximity to d: $deg(f) \leq d + d_0 \wedge deg(g) \leq d + d_0$ (for some "small" d_0 based on d). As we will discuss later, the reason that we have d_0 proximity is because FRI gets more reliable, in cases of malicious Provers, as the distance between the polynomial's real degree and d gets larger; because such proximity statements can only guarantee that the tested degree will be close to (or less than) d, they are called *"low"* degree testing problems. For practical purposes the concrete distance between d and d_0 is typically low[26], and, to accommodate for this inconvenience, we can easily increase precision of the $2POLY$ test by increasing the degree to $d + d_0$.

We can now move onto the formal problem language that this subprotocol tests:

FRI

$$\left\{(f,d) \,\middle|\, f \in RS[\mathbb{F}, D, \rho],\ f : D \rightarrow C,\ d = \rho|D|,\ (D,C) \subseteq \mathbb{F}\right\}$$

[25] a variant of FRI with improved soundness, called $DEEP\text{-}FRI$ and which we will not be discussing here, was recently published in [65]

[26] i.e. $|d_0| \approx 1 - \rho^{\frac{1}{4}}$ in FRI [64], and $\approx 1 - \rho^{\frac{1}{2}}$ in DEEP-FRI [65], with ρ being the compression rate for the RS code that is found in the FRI problem statement, mentioned formally later.

$RS[\mathbb{F}, D, \rho]$ represents all Reed-Solomon codes mapping to a field $\mathbb{F}$ and evaluated on a space D (i.e. code redundancy length is $|D|$) and whose compression rate is ρ. In short, we're stating that $deg(f) < d$, one can easily turn it into $deg(f) \leq d'$, $d' = d - 1$.

How do we go about tackling this problem? If we use a naïve check to get the degree of f we must interpolate it on the full domain D, but $O(|D|)$ complexity is far too costly. Improvements were made in the 90s [66] to bring the test complexity down to $O(d+1)$ as long as we were testing for proximity, and further improvements with regards to this problem were carried on in the field of PCP proofs. We shall further improve this result to $O(log(d))$ with 3 simple steps. The core innovation of the FRI protocol is that of improving upon the traditional PCPP (PCP of Proximity) tests through IOPP (IOP of Proximity); the main idea is to send multiple proofs (or oracles, when queried through the Prover) to the Verifier, which reduce the problem to a simpler one over time. Here are our steps:

1. Reduce f to a polynomial f' of degree $deg(f') = deg(f)/2$
2. $f \leftarrow f'$, go back to step 1 and repeat for $log(d)$ steps
3. Check that f is of degree 0

In order to reduce f to f', we take advantage of a decomposition technique that shares similarities with the Berlekamp-Welch algorithm [67] for error correction of Reed-Solomon codes, and is exactly the same one used by the *divide-et-impera* Cooley-Tukey algorithm for the (inverse) Fast Fourier Transform (FFT) [68]. The idea is to split the polynomial between odd and even coefficients, each becoming its own polynomial of degree half of the original one:[27]

$$
\begin{aligned}
f(x) &= \sum_{i=0}^{deg(f)} f_i \cdot x^i \\
&= \sum^{deg(f)/2} f_{2i} \cdot x^{2i} + \sum^{deg(f)/2} f_{2i+1} \cdot x^{2i+1} \\
&= \sum^{deg(f)/2} f_{2i} \cdot x^{2i} + x \sum^{deg(f)/2} f_{2i+1} \cdot x^{2i} \\
&= \sum^{deg(f)/2} f_{even_i} \cdot (x^2)^i + x \sum^{deg(f)/2} f_{odd_i} \cdot (x^2)^i \\
&= f_{even}(x^2) + x f_{odd}(x^2)
\end{aligned}
$$

Now, let's consider an auxiliary "composition" polynomial $g(x,y)$:

$$
\begin{aligned}
f(x) &= \forall (x^2 = y) : f_{even}(y) + x f_{odd}(y) \\
&= \forall (x^2 = y) : g(x,y)
\end{aligned}
$$

Whenever $x \in D$ is mapped onto y by squaring, we shall call that domain D', such that $y \in D'$. The polynomial g has the important property of being derived from f and being

[27] we will assume, for simplicity, that the domain D of f be 2-smooth (i.e. $\exists k \in \mathbb{N} : |D| = 2^k$); the degree of f is of the same form.

decomposable into smaller degrees, $deg_x(g) \leq deg(f)/2$ and $deg_y(g) \leq 1$, this can easily be visualised if we abstract away one of the variables:

$$\begin{cases} g_x = g_0 + x \cdot g_1 \\ g_y = f_{even_0} \cdot y^0 + \ldots + f_{even_{deg(f)/2}} \cdot y^{deg(f)/2} + g_0(f_{odd_0} \cdot y^0 + \ldots + f_{\deg(f)/2} \cdot y^{deg(f)/2}) \end{cases}$$

Because of such considerations, $|D'| = |D|/2$.

We can now generate all the polynomials for our 3-step plan:

$$f^{(i)} \stackrel{def}{=} \begin{cases} f^{(0)} \leftarrow \forall x \in D : f(x) \\ \exists x_0 \in \mathbb{F} : f^{(1)} \leftarrow \forall y \in D^{(1)} : g^{(0)}(x_0, y) \\ \exists x_1 \in \mathbb{F} : f^{(2)} \leftarrow \forall y \in D^{(2)} : g^{(1)}(x_1, y) \\ \ldots \\ \exists x_{\log d} \in \mathbb{F} : f^{(\log d)} \leftarrow \forall y \in D^{(\log d)} : g^{(\log d)}(x_{\log d}, y) \end{cases}$$

(with $g^{(i)}$ decomposition of $f^{(i)}$, $|D^{(i+1)}| = |D^{(i)}|/2$). Assuming, in the honest case, that $deg(f) = d$, clearly $f^{(\log d)}$ will be be of degree 0: a constant repeated up to $|D^{\log d}|$ times. When we apply the Kilian-Micali construction for interactive PCPs, we will have the the Verifier generate (uniformly randomly) and send points x_i, and the Prover generate and commit the list of evaluations for each $f^{(i)}$; evaluations can be calculated either by evaluating the decomposed polynomial on y for $x = x_i$, or just by interpolating the g_y shown above using $deg(f^{(i)})/2$ points of the form $(\alpha, f^{(i)}(\alpha))$. This process is also called the ***commit-phase*** *in FRI.*

Now that we've seen how to validate $deg(f)$ by checking that it reduces to a constant function after $log(d)$ steps, how dow we check consistency of that final constant value? We should check that each transition made by the Prover is actually correct, traversing through each polynomial one-by-one in a way that is totally similar to the Pietrzak VDF [29] technique that we mentioned in the Intermediate Arithmetisation section. We will also be doing so efficiently through a succinct querying of each polynomial $f^{(i)}$, called "oracle" in the IOP model that FRI is based upon; this second part of FRI is called the **query-phase**. The main idea is to check (for each round of our 3-step process) that a polynomial f reduces to a polynomial f' correctly:

$$f \in RS[\mathbb{F}, D, \rho] \stackrel{?}{\Longrightarrow} f' \in RS[\mathbb{F}, D', \rho]$$

To check for consistency, let's try to reduce f' to some other polynomial:

$$f'(y) \equiv g(x_0, y)$$

Unfortunately the Verifier cannot afford to directly interpolate $g(x, y) = f(x)$, nor can he afford to interpolate g_y, but he can afford to interpolate g_x for some value y_0:

$$\exists y_0 \in D' : g(x, y_0) \iff \exists (\alpha_0, \alpha_1) \in D, y_0 \in D' : Interpolate\Big[(\alpha_0,\ g(\alpha_0,\ y_0)),\ (\alpha_1,\ g(\alpha_1,\ y_0))\Big]$$

Which can be reduced to f (i.e. $\forall x^2 = y : g(x,y) = f(x)$) quite easily when we query two "related" points from f:

$$\iff \exists \alpha \in D, y \in D', \alpha^2 = y_0 : Interpolate\Big[(\alpha,\ g(\alpha,\ y_0)),\ (-\alpha,\ g(-\alpha,\ y_0))\Big]$$
$$\iff \exists \alpha \in D, y_0 \in D', \alpha^2 = y_0 : Interpolate\Big[(\alpha,\ f(\alpha)),\ (-\alpha,\ f(-\alpha))\Big]$$

Therefore, we end up with the following **succinct consistency check** by having the Verifier query from the Prover $f'(y_0), f(\alpha), f(-\alpha)$:

$$f'(y_0) \stackrel{?}{=} g(x_0, y_0) = Interpolate\Big[(\alpha,\ f(\alpha)),\ (-\alpha,\ f(-\alpha))\Big]\ (x_0)$$
$$\textit{(with } \alpha^2 = y_0\textit{)}$$

Finally, the actual soundness analysis (i.e. precision) for this consistency check (and the whole FRI protocol) is the toughest part of any IOPP or PCPP protocol, and something that we will not get into detail. Suffice to say that as long as the Prover is honest the protocol works just fine, and when he lies about the degree of f the protocol works very well when the real distance of $deg(f)$ from the claimed degree is large (because the soundness probability is based on a function of the distance). When this distance is small, the FRI protocol cannot reliably detect it, but it is a very small distance (which was already improved in the $DEEP-FRI$ [65] extension) and we have mentioned above how the $2POLY$ protocol can be easily adapted for this issue by increasing its precision. Furthermore, the protocol can be considered concretely efficient, with $O_P < 6 \cdot d$, $O_V < 21 \cdot \log(d)$, $O_{|\pi|} < 2 \cdot \log(d)$.

NOTE: degree testing in this section is fully pq-safe, but in other proof systems (e.g. SNARKs) it is typically based on homomorphic encryption.

NOTE2: the protocol should be applied to both $2POLY$ polynomials to check that they are of the right degree, but the authors of the paper take advantage of the fact that any linear combination of the two polynomials leads to the same degree as one of them, so they check just a single composite polynomial.

NOTE3: the authors of the paper actually discuss improving both performance and soundness of the protocol by adjusting the values in the Merkle tree of the Kilian-Micali commitment in such a way that a single subtree will contain both points $(\alpha_i, -\alpha_i)$, and that further down the tree we also find the other points $(\alpha_{i+1}, -\alpha_{i+1})$ in such a way that we can re-use $f'(y_0)$, leading to very efficient authentication paths for the commitment scheme. It is an open question whether better commitment structures than Merkle trees can lead to more improvements, such as reduced communication sizes $O_{|\pi|}$.

3.2 Conclusion

zk-STARKs are an incredibly powerful tool that can be used not only to build privacy-friendly applications, but also to drastically reduce the computational costs required to validate outsourced computations online. In short, the power of such universal compilers is that of being able to answer any sort of yes/no question:

- *Do you have the right password?*
- *Do you have the right password for user John?*
- *Do you have the authority and balance to perform a transfer of EUR 100 towards John?*
- *Were these 1000 images analysed using the Machine Learning model I gave you?*
- *Does your result comply with the Smart Contract we uploaded to Ethereum?*
- *Does my program meet security specifications, implying that it is free of bugs?*

And such answers can be checked by the Verifier in time that is much, much faster compared to simply (and naïvely) analysing all the required data himself. When the proposed question takes this binary format, the privacy of any needed data can be preserved by the protocol as long as there is access to other public and binding data published by some trusted authority (e.g. hashed citizen identities published by the government) or computationally inherent to the question's context (e.g. a known composite number uniquely defined by its prime factors).

While we've seen the current state-of-the-art in the domain of VC systems, let's take a moment to consider constructions that have been developed using alternative approaches. The majority of such systems derive from the older and quite alike fields of cryptographic protocols: IPs (Interactive Proofs) and PCPs (Probabilistically Checkable Proofs), or alternative approaches with comparable semantics. Within the context of constructions stemming from PCPs, there two common solutions for solving degree testing of arithmetic circuits: (1) multiplicatively homomorphic encryption (e.g. zk-SNARKs) and (2) proofs of proximity (e.g. zk-STARKs). While all these competing systems are part of the cryptographic VC domain (most of them also being very recent), they can be grouped into different fields according to their theoretical design:

- **MPC in the Head**: this is the only other field, apart from STARKs, which achieved both transparency and post-quantum safety. The main idea behind of such protocols is that of simulating independently (i.e. "in the head") a Multi-Party Computation (MPC), and then revealing it to the Verifier upon request. Amongst the most popular constructions there are: ZKBoo [69], ZKBoo++ [70], and Ligero [71];
- **zk-SNARKs**: this is currently the most successful field of VC protocols, with multiple open-source libraries available to the public and even a successfully deployed privacy-friendly cryptocurrency, Zcash [46]. SNARKs [72], traceable back to SNARGs [73] and typically based on QAPs (Quadratic Arithmetic Programs) and homomorphic cryptography, have received a lot of attention from researchers, giving birth to many theoretical designs such as: Geppetto [74], Pinocchio [75], Groth's [76] (most popular one), SNARKs for C [77], and very recently Aurora [78], Sonic [79], and Libra [80] (most interesting due to its linear prover complexity);
- **zk-STARKs**: this very innovative solution was groundbreaking due to all the VC features it implements, and for being valid even in realistic scenarios, it is published in [51];
- **Interactive Proofs for Muggles**: this "older" (compared with other protocols) design by [81] is one of the few which is based on IPs rather than PCPs; Hyrax [82] is an interesting recent development.

- **Linear PCPs** while this DLP-based (Discrete Logarithm Problem) field is not strictly universal, as proofs can only guarantee that given inputs lie within a specific range (e.g. boundary constraints), Bulletproofs [83] have often been compared to other universal systems because of their applicability to cryptocurrencies (they were developed for the Monero [84] cryptocurrency) and their (now outclassed) performance.

On a final note, most of the recent research in this field has been published with consideration for applicative scenarios, especially Blockchain-based solutions, by providing library implementations and concrete performance analyses. Amongst the most successful applications based on such research we find the privacy-friendly cryptocurrencies Zcash [46] and Monero [84], and privacy-friendly smart-contract (e.g. Ethereum programs) outsourcing and decentralised exchanges in ZEXE [85].

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