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MSc Applied Mathematics

Master's Thesis

Cost allocation and bargaining strategy for an n nodes information network

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Contents

1 Introduction

This thesis was written while working at APG, a service provider for pension fund of the Netherlands. I was part of the Groeie Fabriek, the business development department. I was sta ed in the Pension infrastructure (PI) project that aims to optimise the management of the process involved in maintaining the participants of a pension fund. I was not directly involved in writing the software and the database structure. as not directly involved in writing the software and the database structure. In order to maximize the pro t while expanding as much as possible the sales departent's network of clients, my task was to study the pricing strategy that the department should follow, analyzing the incentives created by di erent cost alllocation methods and the optimal sequence of bid.

2 Problem description

Each pension fund in the Netherlands manage the data of its own participants. This system creates redundancies that do not allow the full exploitation of the bene t granted by an economy of scale. Furthermore the participants have the option to switch pension provider when they change job, and therefore it is necessary to transfer the data regarding that person from the old fund to the new one. Since each fund uses its own data structure and has di erent internal processes, this transfer can be costly and time consuming. The process usually involves sending and receiving multiple letters, that have to be manually processed by the employees of both pension funds.

In order to solve these problems a third party system, the seller PI, created a centralized database that can host the data of each service provider's participants. This means that there is the possibility of exploiting the economy of scale that comes with a centralized data management and reduce the cost of transferring participants between funds that use the system, since the network will use standardized data and can automatically perform all the controls needed to validated an operation. The results will be a network of fund that are able to communicate faster with less possibility of human error.

The seller is interested in licensing the use of this centralized database using a Software as a Service (SaaS) business model. The provider will licence the use of this software to each pension fund and will receive a periodic fee for services o ered. In this thesis I will answer the following questions:

1. Assuming complete information and that the player are willing to cooperate, which is the best way to allocate the maintenance cost of the network infrastructure to each fund?

2. Assuming incomplete information what is the best sequence of bid that PI can make to maximize its expected revenue?

In section 3 I analyze the problem as a cooperative game , assuming that all the buyers are willing to disclose any information required to compute the fair cost allocation. This section is an overview of the existing literature on cost allocation methods, where I also discuss the incentive that each one gives to di erent type of funds. Section 4 will give an historical overview of literature about bargaining problems, with a focus on recent developments that are particularly useful for this thesis. In section 5 and 6 I describe the assumptions needed to analyze this particular bargaining scenario and the model used. In section 7 and 8 I analyze the problem of bargaining the licence price of the data management services, starting from the simple case of one possible buyer and deriving an approximated optimal strategy for the general case with any number n of possible buyers.
In section 9 and 10 I will explain why the results of the two previous sections can not be directly applied to bargain the data transfer services and how to modify the Markov Decision Process to this new scenario. In section 11 I propose an

approximated greedy strategy to avoid the curse of dimensionality. In section 12 I draw the conclusion of my dissertation and suggest further research in the eld.

3 Cost allocation methods from cooperative game theory

In order to use a cooperative game theory framework I have to introduce the assumption that the players are willing to collaborate with one another both by sharing all private information that are necessary for computing a cost allocation scheme and by subscribing the proposed fee. The solution concept I am interested in must have the following characteristics

1. Each fund spends less being in the coalition than being on his own.
2. If a fund i has an higher cost than fund j , i will contribute more than j to the cost of the network.
3. No fund in a coalition should prefer a smaller coalition.
4. No fund is subsidizing another

Condition 1 is given by the individual rationality of each fund, nobody is willing to switch to a new system if this decision causes an increase in cost without additional service. Condition 2 is necessary because the funds are in competition in other business areas, and the new system should preserve the current relative strength of each player. The fund will be less willing to join a coalition if this decision will give a competitive advantage to an adversary. Condition 3 and 4 are necessary to guarantee the stability and scalability of the network. If some of the current participants see an increase in cost when the network expands they will be hostile to new joiners and might veto them or leave the network.

The main issue with cost allocation rules is that there is no perfect notion of fairness that can be used to allocate the cost, and every method used will inevitably favour some business more than another, focusing only on certain aspects. In the next section of the thesis I will present an overview of 4 different methods, and a brief discussion on the consequences of using one rather than the other. In conclusion I make a proposal on what I believe is the best option.

3.1 The cost of managing a coalition and parameters of the problem

Before discussing the cost allocation methods it is necessary to define which parameters influence the cost incurred by all the participants of the game, both the vendor and the buyers The seller cost of managing a coalition depends on the number of participants in it. The allocation of this cost will depend on the rule used and the current cost of each fund. The first part of this section is dedicated to analyze the cost allocation when there are no transfer between funds. The parameters of this problem are

- n: The number of funds that can join the network
- x_i: The number of participants in each fund i
- ${}_i$: The current fixed cost incurred by each fund i during a billing period
- ${}_0$: The fixed cost incurred by the seller during a billing period
- $a; b$: The parameter to compute the cost incurred by the seller to manage a certain coalition

The cost of managing the participants data of a coalition S are computed as

$$c(S) = {}_0 + b x_S^a \tag{1}$$

where $x_S := \sum_{i \in S} x_i$ is the total number of participants in the coalition. Given that, for any coalition, the cost $c(s)$ is known, each cost allocation rule will assign a weight w_{iS} to each player, in order to compute the licence fee $\phi_{iS} = w_{iS}c(S)$.

Theorem: The corresponding cost allocation game is convex iff $a < 1$.

Proof. A necessary and sufficient condition [1] to have a convex game is that

$$\forall i \in S \subseteq T \subseteq N \setminus \{i\} \rightarrow c(S \cup i) - c(s) \geq c(T \cup i) - c(T) \tag{2}$$

Let $x_T = \sum_{j \in S} x_j; x_T = \sum_{j \in T} x_j$ and define the auxiliary function $g(x) := c(x + x_i) - c(x) = b(x + x_i)^a - bx^a$. Since S is a subset of T and each fund has a non-negative number of participants I have that $x_S \leq x_T$. Without loss of generality the game is convex iff $\forall x_1 \leq x_2; g(x_1) \geq g(x_2)$ or equivalently $g'(x) < 0$.

$$\begin{aligned} g'(x) &\leq 0 \\ ab(x + x_i)^{(a-1)} - abx^{(a-1)} &\leq 0 \\ (x + x_i)^{(a-1)} - x^{(a-1)} &\leq 0 \\ \left(\frac{x + x_i}{x}\right)^{(a-1)} &\leq 1 \end{aligned} \tag{3}$$

Given that $x_i \geq 0$ for all possible fund, the base $(\frac{x+x_i}{x})$ is greater than 1 for all possible funds. This means that the game is convex only if the economy of scale factor a is strictly smaller than 1. If the scale factor is equal to one there is no economy of scale and the game is inessential, that is $c(s \cup i) = c(s) + c(\{i\}; \forall S; i$. In this situation each player is indifferent in either joining the coalition or in being on its own. □

The computation of the cost of managing the value transfers between players in the coalition S requires to define some additional parameters.

r_i: the average leave rate from fund i

p_{ij}: The probability that a transfer from i arrives to fund j.

$B; A$: The scale parameter to compute the cost of managing the transfers

The probability that a transfer goes to a fund is directly proportional to the number of participants. it follows that the probability is calculated as

$$p_{ij} = \frac{x_j}{x_N - x_i} \tag{4}$$

In the current system both parties in a value transfer face administrative cost in the process,since the request must be approved and then registered on both ends. Therefore even a fund with no outgoing transaction will have to bear some of the cost. This means that the total number of transactions to manage amounts to the sum of all out going and in going value transfers, costing

$$t(S) = B\left(\sum_{i \in S} r_i x_i \sum_{i \in S; i \neq j} p_{ij} + \sum_{i;j \in S} r_j x_j p_{ji}\right)^A \tag{5}$$

Given that the transactions to manage are only the one that completely belongs in the network, the number of total outgoing transactions is the same of total ingoing transactions. This means that the cost function in (5) can be simplified to

$$t(S) = B\left(2 \sum_{i \in S} r_i x_i \sum_{i \in S; i \neq j} p_{ij}\right)^A \tag{6}$$

The cost of maintaining the information centers and the cost of managing the value transfers are allocated separately.

3.1.1 Allocation proportional to use

Each fund will use the service provided by PI at a different rate. The IT cost will depend on the current number of participants in a given coalition, while the cost of the value transfer depends on the number of ingoing and out going jobs. One way to allocate the cost is proportional to the labor required by each fund. Therefore the portion of c(S) allocated to each fund will be

$$w_{ic(S)} = \frac{x_i}{x_S} \tag{7}$$

Similarly the portion of t(S) paid by each fund will be

$$w_{it(S)} = \frac{r_i x_i \sum_{j\in S} p_{ij} + \sum_{j\in S} r_j x_{ji}}{2 \sum_{i\in S} r_i x_i \sum_{i\in S, i\neq j} p_{ij}} \tag{8}$$

This weighting system assigns a fixed cost per participants and a fixed cost per job to each fund. The cost grows linearly with the number of participants and transfers, while the current management cost grows sublinearly with the same parameters. This means that larger funds, with an efficient economy of scale, are penalized with this cost allocation rule. The smaller funds on the other hand will be the main beneficiaries, since they will have a higher percentual saving. The cost of the coalition is smaller than the sum of the individual cost, thanks to the concavity of the cost function, and therefore each fund has an incentive to join in. Given that the cost grows sublinearly adding funds to the coalition reduce the average cost per user and therefore reduces the fee that each fund has to pay and therefore every player in the network will always be favourable to a bigger coalition. However this allocation rule does not guarantee the preservation of the relative cost order since a large efficient fund might end up paying more than a smaller but more inefficient competitor.

3.1.2 Allocation proportional to current management cost

The weight of each fund is proportional to its current management cost

$$w_{ic(S)} = \frac{{}_i}{\sum_{j\in S} {}_j} \tag{9}$$

and

$$w_{it(S)} = \frac{{}_i\left(r_i x_i \sum_{j\in S} p_{ij} + \sum_{j\in S} r_j x_j p_{ji}\right)^{{}_i}}{\sum_{k\in S} {}_k\left(r_k x_k \sum_{j\in S} p_{kj} + \sum_{j\in S} r_j x_j p_{jk}\right)^{{}_k}} \tag{10}$$

Given that large funds are usually more efficient this cost allocation penalize smaller funds, that will save less switching to the new system. The allocation is individually rational, provided that the sum of the individual cost is greater than the cost of managing the coalition S. Given that each fund pays proportionally to its current cost condition 2 follows directly. The third condition is guaranteed by the concavity of the cost structure. This cost allocation presents the opposite problem of the allocation proportional to the number of user. It only rewards the efficiency of larger funds, proportionally charging the smaller funds more. Furthermore since the cost are allocated based on individual cost it does not take into account the price of managing each fund in S.

3.1.3 Shapley allocation

The Shapley value [2] is a well known method to allocate the cost of a product or share the profit in a joint venture. In order to allocate to each player a fair share of the cost the weigth is given by the average of the marginal cost created by joining any possible ordered coalition.

$$= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(N - |S| - 1)!}{N!}\left(c(S \cup \{i\}) - v(S)\right) \tag{11}$$

It is worth noting that the first fund to join the system in any possible coalition brings a management cost of at least $_0$, that is the fix cost of maintaining the system. This means that each agent has to pay at least $\frac{_0}{n}$, that is the fixed cost are allocated per capita, ignoring the size of the fund. This minimum allocation hits the smaller fund particularly hard. Furthermore the complexity of compute the Shapley value grows factorially with the maximum possible size of the coalition, therefore an exact computation of this value becomes unfeasible for a realistically sized network of around 150 customers.

3.1.4 Separable Cost Remaining Benefit allocation

The three previous cost allocation methods focus mostly on the cost of managing the coalition S but fails to capture another important aspect of the problem: some funds will save more than others when switching to a centralized data management, therefore have a greater incentive to join and are willing to pay more.The Separable Cost Remaining Benefit allocation method, historically used to allocate the price of building water distribution infrastructures [3, 4], incorporate this savings in the computation of the allocated cost.

While it is not possible to define how much managing a fund will cost exactly, since the same fund in different coalitions has different marginal cost, there is a part of the total cost involved in managing a coalition that can be directly assigned to each fund. This is the marginal cost of managing the fund when it enters the great coalition, that is

$$m_i := c(N) - c(N \setminus \{i\}) \tag{12}$$

This cost is a direct consequence of the fund i entering the coalition, and therefore will be allocated to the fund. Since the cost function is not linear the sum of the total marginal cost $\sum_i m_i$ will be lower that the total cost $c(N)$. The difference

$$g(N) := c(N) - \sum_{i \in N} m_i \tag{13}$$

is called non separable cost, and can not be attributed directly to any particular fund. Each player has a potential saving (or benefit) joining the grand coalition given by the difference of its current cost $c\{i\}$ and the its marginal cost m_i, that is the minimum charge it will incur by joining the grand coalition,($r_i = c\{i\} - m_i$).

As a matter of fact a player will join a coalition only if it is charged less than its current cost and this is possible only if its marginal cost is lower than its current cost. If a player join a coalition and is charged less than the its marginal cost it means that the fund is subsidized by the others, and no other funds would accept this condition. This means that the remaining benefit of a player is always non negative.

The non separable cost are allocated in proportion to the benefit it brings to each player, that is

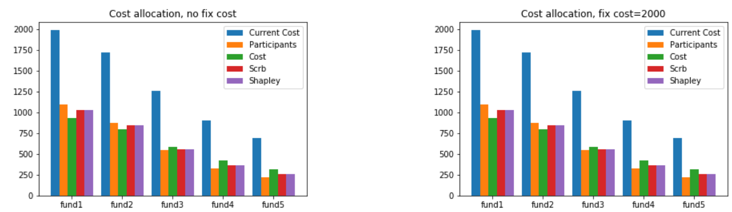
$$ {}_i = m_i + \frac{r_i}{\sum_{j \in N} r_j} g(N) $$

The computation of this allocation is more efficient than the Shapley value, since it is only necessary to compute n marginal cost, instead of $n!$. However it has the downside of considering only the great coalition and all the single funds. This does not guarantee that there are no subcoalitions preferred by some funds.

While this is a well known issue of the SCRB allocation method in this particular application it can be neglected, at least in the allocation of the database management cost. Since the game is convex, and therefore semiconvex, the separable cost allocation coincide with the cost gap allocation and with the value defined by Tijs in [5] . In this case the player does not have any credible threat to force a subcoalition S smaller than the grand coalition N, since the marginal cost used to compute their fair share of the total cost is already the smallest possible.

3.1.5 Conclusion and recommendation

The graphs in figure 1 show the effects of different cost allocations on the 5 different funds, ranging from 30000 to 100.000 participants. The larger funds prefer allocation based on current cost, while the smaller prefer an allocation based on the number of participants. The Shapley value and the SCRB allocation are between the two other methods and are not the best cost allocation for anyone. As mentioned in section 3.1.3 the smaller fund are particularly impacted by the per capita allocation of the fixed cost ${}_0$. I believe that the The SCRB is a good compromise to allocate the cost between the funds that combines well the main characteristic of the three other methods. Part of the cost is allocated on the basis of the cost of managing the fund, and is roughly proportional to the size of a fund, On the other hand, when sharing the non separable cost, the e cient funds that will not save much capital by switching system are rewarded with a low weight. Furthermore since the game is convex it implicitly takes into account all possible subcoalitions, without the added factorial complexity of the Shapley value.



(a) No x vost

(b) x cost = 2000

Figure 1: E ect of the 4 cost allocation methods discussed on 5 di erent funds

4 Literature Review on bargaining problems

Modelling bargaining has always been a challenge for mathematicians and economists. The rst solution for a mathematical approach to bargaining theory is due to Nash [6], that proposed an axiomatic approach to the propriety that a bargaining solution should have. Although discussed in the appendix B of the thesis this approach has limited application to its central question, since it relies on complete information about the players default status in case of a failed negotiation and it is implicitly relying on a one turn bargaining game. The original paper describes a simple two player game, but it can be generalized to any number n of players. Since no buyer is willing to share its current data management cost (the default status) this method is of limited use in this thesis.

Rubinstein also tackled the problem[7], adding the possibility for the players to continue bargaining for an in nite time. Provided that there is a discount factor greater than $\frac{1}{2}$ or that delays are costly the article shows that a deal between the parties will be reached immediately, if these information are public. The immediate reach of an equilibrium is a natural consequence of public information, since each player can foresee the strategy of the other and the first player can force its preferred solution.

The existence of incomplete information and the possibility of making multiple bids adds new strategic options to the decision makers, as discussed in [8], where the case of infinite horizon bargaining with incomplete information is analyzed. Furthermore it provides a useful justification of the assumption that only the seller can make offers and ignore all counteroffers. The article also assumes that the seller has a sunk cost of production and a null reservation price, an a priori that can not be made for this thesis. The seller will face an increased network maintenance cost after the buyers agreed to the bid and therefore there is a lower bound strictly greater than 0 on the possible bid that a rational seller can actually propose.

The work discussed in the previous paragraphs offers an interesting insight on the nature of incomplete information bargaining, and why sequential games naturally arise when some necessary information are not publicly disclosed. However they only described a very limited class of bargaining problems: a two person game with one buyer and one seller. This can be useful to model the extremely specific case of trying to sell the software licence to only one fund, but is of limited use for a general number of funds n.

Given that PI will face recurring cost to manage the funds in a coalition, such as server cost, research cost and employee salary, it needs a recurring source of revenue to remain profitable. This constraint lead to the decision of adopting as Software as a Service business model [9]. The funds that decide to outsource the management of their data and their value transfers to PI will pay a periodic fee, to cover the cost and allow the seller to be profitable. The computation of the optimal licence cost is the main focus of this thesis.

The rise in popularity of SaaS business model[10] ignited an interest in researching the price equilibria between buyers and sellers. The article [11] study a static game with complete information, where the service provider and the receiver have to decide their privacy preference at the same time. The article [12] introduces a 1-to-1 bargaining for a cloud computing service, where the seller can quote the price of services and the buyer decides which percentage of its service will outsource to the provider. Furthermore the article discusses 1-to-many bargaining scenarios and the e ect of economies of scale on the resulting pro t. However this model assumes a one turn game and a continuous spectrum of choice available to the buyer, namely which percentage of service will be made in house. The Service o ered by PI gives a binary choice to the buyer: either accepting the o er and outsource all its service or maintain the status quo.

The widespread of online marketplace and complex environments lead to an increased interest in automated decision making, since the number of choices to make and the speed required represent an unfeasible challenge for human decision makers. The game theoretic approach is discussed Jenning et al [13] and its limitation are highlighted: the computation required for the optimal solution are often long and expensive, the fact that a solution exist is no guarantee that is actually achievable. The heuristic approach, relying on a realistic assumption on the behavior of the opponents, are presented as an approximation of game theoretic models, able to reach a "*good* rather than *optimal*" solution. An example of the application of a Markov Decision Process to a bargaining games is described in [14]. The aim of this thesis is to expand on the existing literature on the application of Markov Decision Process and game theory to bargaining problems, nding the optimal bids in a multiple bids 1-to-many bargaining scenario.

5 Bargaining with private information and game description

The cost allocation methods discussed in the previous sections rely on the implicit assumptions that all funds join the PI network at the same time and are willing to disclose information regarding their management cost, both fixed and variables. This is not always a realistic assumption, since most buyers are unwilling to disclose their reservation price, hoping to get a better price if the seller is forced to make an offer. In this section I will discuss the model used to study this case and an optimal strategy to maximize the expected profit obtained from licensing the SaaS.

Given that PI is the only agent that sells this service it has a good bargaining power, since it is not competing with other players to license its product to the possible buyers. This does not mean that it can behave as a monopoly, setting any possible price, since it is still competing against the current database management system of each individual buyer. The pension funds need to manage their IT infrastructure and the value transfer, but still have the possibility of doing it in-house if the price quote by PI is deemed to be excessive. This bargaining advantage allows the seller to be the only player that is allowed to make an offer, while the buyer only have the options to accept the quoted price or refuse if they know that is cheaper to maintain the status quo. The seller will continue to quote offer, provided that the price does not fall below the marginal cost of managing the new fund.

The decision to accept or decline the offer depends on the risk preference of the player. A risk seeker or a risk neutral player is inclined to refuse an offer even if it is below their current cost, if the decision maker believes that the seller is above its marginal cost. A completely risk averse player will accept the first offer that achieves a positive saving, since it is not willing to risk not receiving a new offer and to be stuck with their default cost. In this thesis the buyer are assumed to be completely risk averse.

This scenario can be modelled as a 1-to-many bargaining game, with a finite number of turns T. The players are divided in two categories:

- 1 seller who developed the software and intend to licence it for a fixed fee.
- n possible buyers of licence

At the beginning of the turns $0, 1, \ldots T - 1$ the seller proposes a licence fee to each buyer that is not licencing the service, that will be accepted of refused. The vendor will stop making offers if it is forced to quote a fee lower than the marginal cost of managing the new fund. After an offer is accepted the seller start to collect the revenue and incurr in the cost associated with the new coalition. No party can renegotiate a licencing agreement after the acceptance.

6 The Markov decision process for data management bargaining

Since the buyers can only accept or refuse the bid proposed by PI, the evolution of the system is uniquely determined by the sequence of the price quoted by the seller to each player. Furthermore the decision of joining depends uniquely on the last price quoted. This assumption allows to model the system as a Markov chain and therefore study the optimal strategy using Markov Decision Theory. In order to describe the Markov process it is necessary to define the state space, the decision space, the transition probability matrix, the reward and the parameters of the problem. The parameters of this problem are:

- n: Number of possible seller
- N: The grand coalition

T: Number of times it is possible to quote an offer

: discount factor to actualize future cash flow

$[\ _{iL};\ _{iH}]$: the lower and upper bound of the reservation price of fund i

$_i$: Current administrative cost of fund i, this parameter is a private information known only to the buyer

$F(\ )$: The probability distribution function of the reservation price.

x_i: Number of participants managed by the fund i

$1 \quad k$: The minimum saving that a fund need to obtain to accept the offer

$a; b$: The parameters to estimate the cost of managing the participants in the network. These parameters are private information, only known to the seller.

$_0$: Fixed cost of operating the network.

The state of the network is uniquely identified by the players that accepted the offer, the coalition S. To ease the notation I introduced the binary vector s defined as

$$\begin{cases} s_i = 1 & \text{if} \quad i \in S \\ s_i = 0 & \text{if} \quad i \notin S \end{cases}$$

While the vector s identifies each possible state it is also necessary to keep track of the licence fee of each fund in the system, to compute the future revenue, and the last refused price quoted to funds that are not in the network, to have a new upper bound to the possible future quote. These information are stored in the vector , that is

$$\begin{cases} \ _i & \text{license fee if} \quad s_i = 1 \\ \ _i & \text{last quote if} \quad s_i = 0 \end{cases}$$

Given that the state space is described by a binary vector the dimension of the state space grows as 2^n.

The only decision available to the seller is which price to quote to each possible buyer at the beginning of the period t. Theoretically the seller can quote any price $_i \in [_{iL}; _{iH}]$ to all buyers. However it is pointless to quote a price to buyers already licensing the services, since the price can not be renegotiated, or to quote a price $_i$ $_i$, since the buyer was already not willing to buy at $_i$. This means that the seller will quote a price $_i \in [_{iL}; _i]$ to each player that is not in the network. This implies that the decision space shrinks after each turn, no matter the outcome of the bid: if a fund decides to join there is one less decision variable, if it refuses the bid then the range of viable bid is reduced.

The transition probability is determined by the price quoted at a certain turn. The probability of going from the state s to the state s' is equivalent to the probability that the offers are accepted by the funds $i \in S' \setminus S$ and refused by the funds $i \in N \setminus S'$. Each fund decision depends only on its current cost $_i$ and the proposed cost $_i$, and therefore is independent from the action of all the other funds. Calling $P(_i)$ the probability that a fund accept the price $_i$ quoted the transition probability can be written as

$$P(s'|s;) = \prod_{i \in S' \setminus S} P(_i) \prod_{i \in N \setminus S'} (1 \quad P(_i)) \tag{14}$$

Given that the accepted offers cannot be renegotiated by either party the state $s = 1_N$ is an absorption state, since there are no more decisions to be made, and the fee received and the cost incurred at each turn are known and constant.

Finally the reward of a particular state is given by the sum of the total fee minus the cost of managing the participants in the system.

$$R(s;\ \ ) = \sum_{i \in N} (s_i \ \ _i) \ \ b(\sum_{i \in N} s_i x_i)^a \ \ \ _0 \tag{15}$$

The expected reward at the end of a turn is given by the weighted sum of the reward in each possible state after the bargaining turn.

$$\mathrm{E}[R_t(s(t);\ \ (t))] = \sum_{s' \ S} R(s')P(s'|s(t);\ \ (t)) \tag{16}$$

Given that there is a finite number of bargaining turns T the goal of the decision maker is to maximize the discounted expected total reward,

$$\mathrm{E}[R_{tot}] = \sum_{t=1}^{T} \ ^t\mathrm{E}[R_t] \tag{17}$$

Before going into an in depth analysis of this Markov Decision Problem, it is useful to know which kind of optimal policy should be expected. The state space S is countable, there are 2^n possible subcoalition. The decision space in any given state is finite, since each possible $\ _i$ belongs in $[\ _{iL};\ \ _{iH}]$. This two conditions guarantee the existence of an optimal deterministic Markov policy[15]. It is worth noting that the existence of an optimal policy does not guarantee it is possible to find it. In the next sections of this thesis I show that finding an optimal strategy is possible in the special case with $n = 1$ and that for larger network the curse of dimensionality makes it infeasible to compute the best solution.

7 Optimal bidding strategy for one fund

The simplest scenario to study is the particular case in which there is only 1 possible fund that can join in the network. While this case will never be encountered in the real world it is useful to study as a basis for a more general analysis of the problem. In this scenario the state space is composed by only two elements, $s = 0$ the fund does not licence the software and $s = 1$, the fund accepted the offer. This also means that the state 1 coincides with the grand coalition and is therefore an absorption state for the corresponding Markov chain. The total discounted reward is just the sum of the discounted cash flow from the turn t that the fund accept the price quoted $\ _t$ and to the dismissal of the software at time T. Using the notation $R_t(\ _t)$ I identify the value of the reward obtained when the fund joins at t and it is possible to write:

$$R_t(\ _t) = \sum_{t'=1}^{t} {}^{t'}\ _0 + \sum_{t'=t+1}^{T} {}^{t'}(\ _t \quad c) \tag{18}$$

where c is the cost incurred in managing the fund. The seller incurs in this cost only after the the fund agrees to buy its service, and therefore it is only subtracted after t. To simplify the notation in the future calculation it is useful to define the factor $D_{t_1 t_2}$, the factor to compute the total Net Present Value of a cash flow received from $t_1 + 1$ to the period t_2.

$$\begin{aligned} D_{t_1 t_2} &= \sum_{t=t_1+1}^{t_2} {}^{t} \\ &= \sum_{t=0}^{t_2} {}^{t} \quad \sum_{t=0}^{t_1} {}^{t} \\ &= \frac{1 \quad ^{t_2+1}}{1} \quad \frac{1 \quad ^{t_1+1}}{1} \\ &= \frac{^{t_1+1} \quad ^{t_2+1}}{1} \end{aligned} \tag{19}$$

Using (19) it is possible to rewrite (18)

$$\begin{aligned} R_t(\ _t) &= \sum_{t'=1}^{t} \ _0 {}^{t'} + \sum_{t'=t+1}^{T} {}^{t}(\ _t \quad c) \\ R_t(\ _t) &= \ _0 \sum_{t'=1}^{t} {}^{t'} + (\ _t \quad c) \sum_{t'=t+1}^{T} {}^{t'} \\ R_t(\ _t) &= \ _0 D_{0T} + (\ _t \quad c) D_{tT} \end{aligned}$$

If I call P_t the probability that the buyer joins at time t I have that expected reward can be written as

$$\mathrm{E}[R(\ )] = P_0 R_0(\ _0) + (1 \quad P_0) P_1 R_1(\ _1) + \cdots \prod_{t=0}^{T \ 2} (1 \quad P_t) P_{T \ 1} R_{T \ 1}(\ _{T \ 1}) \tag{20}$$

A fund will accept the price quoted if and only if $< k$, this mean that the possibility of accepting the rst o er can be written as

$$P_0 = P(\ _0 < k\ ) = 1 \quad P(\ \ < \frac{_0}{k}) = 1 \quad F(\frac{_0}{k})$$

If the rst bid is not accepted it gives a new upper bound to the problem, I know that since the bid was declined the reservation price must be smaller than the bid, and I have

$$P_1 = P(\ _1 < k\ j\ _0 > k\ ) \quad = \frac{P(\ _1 < k \ \ < \ _0)}{P(k \ \ < \ _0)} = \frac{F(\frac{_0}{k}) \quad F(\frac{_1}{k})}{F(\frac{_0}{k})}$$

$$\vdots$$

$$P_t = P(\ _t < k\ j\ _{t\ 1} > k\ ) = \frac{P(\ _t < k \ \ < \ _{t\ 1})}{P(k \ \ < \ _{t\ 1})} = \frac{F(\frac{_{t\ 1}}{k}) \quad F(\frac{_t}{k})}{F(\frac{_{t\ 1}}{k})}$$

Since the probability of refusing an o er is the complimentary case I can write

$$1 \quad P_t = 1 \quad \frac{F(\frac{_{t\ 1}}{k}) \quad F(\frac{_t}{k})}{F(\frac{_{t\ 1}}{k})} = \frac{F(\frac{_t}{k})}{F(\frac{_{t\ 1}}{k})}$$

and the possibility of obtaining the reward $R(t)$ can be rewritten as

$$\prod_{t'=0}^{t\ 1} (1 \quad P_{t'}) P_t = F(\frac{_0}{k}) \frac{F(\frac{_1}{k})}{F(\frac{_0}{k})} \frac{F(\frac{_2}{k})}{F(\frac{_1}{k})} \cdots \frac{F(\frac{_{t\ 1}}{k}) \quad F(\frac{_t}{k})}{F(\frac{_{t\ 1}}{k})}$$

$$= F(\frac{_{t\ 1}}{k}) \quad F(\frac{_t}{k})$$

The function in (20) can therefore be write as

$$\mathrm{E}[R(\ )] = (1 \quad F(\frac{_0}{k}))R_0(\ _0) + \sum_{t=1}^{T\ 1} (F(\frac{_{t\ 1}}{k}) \quad F(\frac{_t}{k}))R_t(\ _t) \qquad (21)$$

In order to maximize the expected value of this function I can solve the following optimization problem

$$\begin{array}{ll} \text{minimize} & \mathrm{E}[R(\ )] \\ \text{subject to} & \ _L \quad \ _t \quad \ _H;\ t = 0; \ldots; T \quad 1. \end{array} \qquad (22)$$

Given that most of literature on continuous optimization is dedicated to minimization problems it is useful to rewrite a maximization problem as the minimization of the opposite. In order to nd a candidate maximum of the reward function it is necessary to nd its critical points, that is the values $_0; \ _1; \dots \ _{T \ 1}$ such that $r R = 0$. Using the fact that

$$\frac{@F(\frac{t}{k})}{@} = \frac{1}{k} f(\frac{t}{k}) \quad \frac{@R(t)}{@ \ _t} = \ _t D_{tT}$$

the gradient can be written as

$$\frac{@E[R(\ )]}{@ \ _0} = \frac{1}{k} f(\frac{0}{k})(\ _0 \ c) D_{0T} \quad (1 \ F(\frac{0}{k})) D_{0T} \quad \frac{1}{k} f(\frac{0}{k})(\ _1 \ c) D_{1T}$$

$$\frac{@E[R(\ )]}{@ \ _1} = \frac{1}{k} f(\frac{1}{k})(\ _1 \ c) D_{1T} \quad (F(\frac{0}{k}) \quad F(\frac{1}{k})) D_{1T} \quad \frac{1}{k} f(\frac{1}{k})(\ _2 \ c) D_{2T}$$

$$\vdots$$

$$\frac{@E[R(\ )]}{@ \ _t} = \frac{1}{k} f(\frac{t}{k})(\ _t \ c) D_{tT} \quad (F(\frac{t \ 1}{k}) \quad F(\frac{t}{k})) D_{tT} \quad \frac{1}{k} f(\frac{t}{k})(\ _{t+1} \ c) D_{(t+1)T}$$

$$\vdots$$

$$\frac{@E[R(\ )]}{@ \ _{T \ 1}} = \frac{1}{k} f(\frac{T \ 1}{k})(\ _t \ c) D_{(T \ 1)T} \quad (F(\frac{T \ 2}{k}) \quad F(\frac{T \ 1}{k})) D_{(T \ 1)T}$$

Since the x expenses $_0$ can not be changed they do not have to be considered when studying the optimal strategy, without loss of generality I will consider $_0 = 0$ for the rest of the thesis. The number of solutions and the type of critical points obviously depends on the distribution function of the reservation prices.

7.1 Uniform distribution of the reservation prices

The uniform distribution function is the most simple that cna be used to describe the reservation prices. In this section I nd an analytical solution of the optimization problem(22). If I assume that the distribution of reservation prices follows a uniform distribution in the support $[\ _L;\ _H]$ I have that

$$F(\frac{t}{k}) = \frac{\frac{t}{k}\quad L}{H\quad L} \qquad \frac{@F(\frac{t}{k})}{@\ t} = \frac{1}{k(\ H\quad L)}$$

$$F(\frac{t}{k}) = \frac{t\quad k\ L}{kd} \qquad \frac{@F(\frac{t}{k})}{@\ t} = \frac{1}{kd}$$

where $d := \ _H \quad _L$ is de ned to simplify future notation. Using these results the gradient of $\ \mathrm{E}[R(\ )]$ can be written as

$$\begin{aligned}
\frac{@\mathrm{E}[R(\ )]}{@\ _0} &= \frac{1}{kd}(\ _0 \quad c)D_{0T} \quad \frac{k\ _h \quad _0}{kd}D_{0T} \quad \frac{1}{kd}(\ _1 \quad c)D_{1T} \\
&= \frac{2\ _0}{d}D_{0T} \quad \frac{_1}{kd}D_{0T} \quad \frac{D_{0T} \quad D_{1T}}{kd}c \quad \frac{_H}{d}D_{0T} \\
&\vdots \\
\frac{@\mathrm{E}[R(\ )]}{@\ _t} &= \frac{1}{kd}(\ _t \quad c)D_{tT} \quad (\frac{_{t\ 1} \quad _L}{kd} \quad \frac{_t \quad _L}{kd})D_{tT} \quad \frac{1}{kd}(\ _{t+1} \quad c)D_{(t+1)T} \\
&= \quad \frac{_{t\ 1}}{kd}D_{tT} + \frac{2\ _t}{kd}D_{tT} \quad \frac{_{t+1}}{kd}D_{(t+1)T} \quad \frac{D_{tT} \quad D_{(t+1)T}}{kd}c \\
&\vdots \\
\frac{@\mathrm{E}[R(\ )]}{@\ _{T\ 1}} &= \frac{1}{kd}(\ _{T\ 1} \quad c)D_{tT} \quad (\frac{_{T\ 2} \quad _L}{kd} \quad \frac{_{T\ 1} \quad _L}{kd})D_{tT} \\
&= \quad \frac{_{T\ 2}}{kd}D_{(T\ 1)T} + \frac{2\ _{(T\ 1}}{d}D_{(T\ 1)T} \quad \frac{D_{(T\ 1)T}}{kd}c
\end{aligned}$$

The system of linear equations can be rewritten in matrix form as

$$\frac{1}{kd}\begin{bmatrix} 2D_{0T} & D_{1T} & & \\ D_{1T} & 2D_{1T} & D_{2T} & \\ & & \ddots & \\ & & D_{(T\ 1)T} & 2D_{(T\ 1)T} \end{bmatrix}\begin{bmatrix} _0 \\ _1 \\ \vdots \\ _{T\ 1} \end{bmatrix} = \frac{1}{kd}\begin{bmatrix} c(D_{0T} \quad D_{1T}) + k\ _H D_0 \\ c(D_{1T} \quad D_{2T}) \\ \vdots \\ cD_{(T\ 1)T} \end{bmatrix}$$

$$A \quad = b \tag{23}$$

The critical point is a minimum i the matrix A is Positive semide nite (PSD). A su cient condition to prove that a tridiagonal matrix is PSD can be found in [16]. The matrix must be diagonally dominant, that is

$$|A_{jj}| > |A_{(j\quad 1)(j)}| + |A_{(j+1)j}|; \forall j \quad T \quad 1 \tag{24}$$

With the parameters provided this reduces to the inequality

$$2D_{tT} > D_{tT} + D_{(t+1)T}$$

$$D_{tT} > D_{(t+1)T}$$

$$\frac{\ ^{t+1} \quad ^{T+1}}{1} > \frac{\ ^{t+2} \quad ^{T+1}}{1}$$

$$^{t+1} > \ ^{t+2}$$

$$1 >$$

Since I am only discussing scenarios with < 1 the last inequality always holds. This means that the matrix A is PSD for any possible combination of input parameters and the vector that solve the system of equation (23) is a minimum of the objective function.

Another advantage of working with a PSD matrix is that it is possible to use Cholesky decomposition using the algorithm described in [17] and rewrite the matrix as

$$A = LL^T$$

With the matrix rewritten in this new form it is possible to solve the linear system $A \quad = b$ becomes $LL^T \quad = b$. This new form allows to compute the optimal strategy vector solving the system $Ly = b$ using forward substitution and $L^T \quad = y$ using backward substitution.[18].

The results of the previous discussion can be summarized in the following theorem.

Theorem 1: Given a buyer that has a uniform distribution of reservation price in the range $[\ _L; \ _H]$, a cost of service c, a discount factor and T bargaining turns, the vector of that describe the optimal bid at each bargaining turn can be obtained by solving the linear system

$$A \quad = b$$

The matrix A and the vector b are de ned as

$$A := \begin{bmatrix} 2D_{0T} & D_{1T} & & \\ D_{1T} & 2D_{1T} & D_{2T} & \\ & & \ddots & \\ & & D_{(T\ \ 1)T} & 2D_{(T\ \ 1)T} \end{bmatrix}$$

$$b := \begin{bmatrix} c(D_{0T} \quad D_{1T}) + k \ \ _H D_0 \\ c(D_{1T} \quad D_{2T}) \\ \vdots \\ cD_{(T\ \ 1)T} \end{bmatrix}$$

$$D_{tT} := \frac{\ ^{t+1} \qquad ^{T+1}}{1 \qquad}$$

The pseudocode to solve the optimization problem stated in 22 is the following.

```
function find_optimal_psi(gamma_H,c,k,DF,T)
#DF=discount factor
t=0
#compute D_tT
while t<T:
    D(t)=(DF^(t+1)-DF^(T+1))/(1-DF)
    t=t+1
#initialize matrix A
diagonal(A)=2*D
diag_inf(A)=-D(1:T-1)
diaf_sup(A)=-D(1:T-1)
```

```
#compute L such that A=LL^T
L=cholesky(A)
#initialize column vector b
b(0)=c*(D(0)-D(1))+k*gamma_H*D(0)
t=1
while t<T:
    b(t)=c*(D(t)-D(t+1))
    t=t+1
#solve A*psi=b using backward and forward substitution
y=inv(L)*b
psi=inv(L^T)*y
return psi
```

At each turn the decision maker has the choice to quote a price, if the fund is not licencing the services, or do nothing, if the fund is already in the network. The vector $\psi = (\psi_0; \psi_1 \ldots \psi_{T-1})$ is used to determine the optimal offer at any time $t = 0; 1; \ldots; T - 1$. The optimal Markov Decision Strategy is shown in table 1.

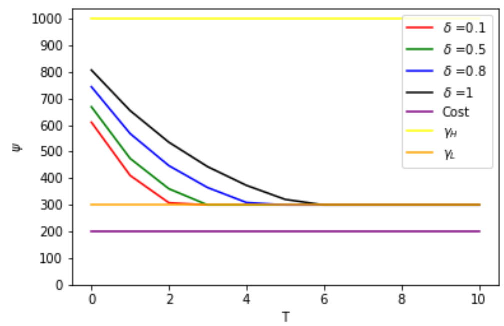
State	Action
$s_t = 0$	Bid $\max(\psi_t; \ _L)$
$s_t = 1$	Do nothing

Table 1: Optimal Markovian Deterministic Strategy

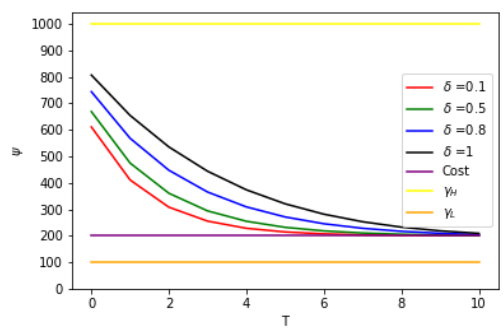
It is not necessary to recompute the optimal bidding strategy at each turn since all the relevant information (the price distribution and administrative cost c) are already known at the beginning of the the bargaining period. The only information that can modify the strategy is that the fund accepted the bid, in this case the bargaining ends and there is no need to make any new offer.

If the marginal cost of managing the fund is lower that the lower bound of their administrative cost $_L$ there is a probability that no mutually beneficial agreement will be reached. PI is not willing to provide services at a loss, since it would be be better off only paying the fix maintenance cost $_0$ instead of $_0 + (\quad c)$ since if is smaller than c accepting a client lead to an even larger loss per turn. The probability of not reaching an agreement is simply the probability that the reservation price of the fund is below the marginal cost c, that is $P_{fail} = F(c)$. In the opposite case, if the marginal cost is below the minimum reservation price I are sure that an agreement will be reached, since even bidding the minimum amount $_L$ will lead to a certain agreement that saves capital to the fund and increase APG's future cash flow. Beside being influenced by the range of reservation prices, the cost of managing a fund and the number of turns, the optimal bid is also determined by the discount factor . If $\to 0$ the value of future cash flow becomes very small and therefore the decision maker has an incentive to make lower offer, since it needs to start licensing the software as soon as possible.

In order to show the effect of different input parameters on the optimal bidding strategy I computed the optimal for different scenarios. In figure 2a PI licence the software to a fund with a minimum reservation price $_L$ above the marginal cost c. In this scenario the bid at each turn decreases, until it reaches $_L$, at which point the seller is certain to reach an agreement offering a price $_L$. In figure 2b PI licence the software to a fund with a minimum reservation price $_L$ above the marginal cost c. In this case the bid decreases and goes to c when the bargaining is close to the end turn T. Provided that all the other parameters are equal the bid at any given turn the optimal bid is higher the closer the discount factor is to 1.



(a) Optimal strategy to bargain with a fund with $_L$ = 300



(b) Optimal strategy to bargain with a fund with $_L$ = 100

Figure 2: The graphs show the optimal bid at the beginning of each bargain turn, for di erent discount rate . The purple line is the cost of managing the fund in the system.

7.2 Truncated normal distribution

The assumption of a uniform distribution of reservation prices yields an elegant analytic solution but it is not realistic. The chance that a fund spends all its administrative costs in database management (! $_H$) or does not spend anything in information management (! $_L$) are highly unlikely. A more suited distribution to describe the reservation prices is the truncated normal distribution. It allows to give more weight to the best estimate of the actual management cost, that is the mean of the normal distribution, and to quantify the uncertainty of this estimation, using the variance ^{2}. It is necessary to use a truncated distribution, since there is no chance that the reservation price can be below $_L$ or above the total administrative cost of the fund, and therefore a nite support is needed. A normal distribution has an unlimited support, which would allow the unrealistic cases of a negative reservation price or one higher than the current management cost.

Given a truncated normal distribution with mean and variance ^{2}, and support $[\ _L;\ _H]$ the probability distribution function and the cumulative distribution function are respectively

$$f(\ ) = \frac{(\frac{}{})}{(\ (\frac{H}{})\quad (\frac{L}{}))}$$

$$F(\ ) = \frac{(\frac{}{})\quad (\frac{L}{})}{(\frac{H}{})\quad (\frac{L}{})}$$

The resulting system of equation $rR = 0$ does not have an analytical solution and it is necessary to use numerical methods to nd the root. I decided to nd the minimum with a iterative approximation algorithm, the gradient descent method as described in described in [19]. This method is only applicable provided that both $F(\ )$ and $f(\ )$ are continuous, as in our case.

It is worth noting that also the triangular distribution would be a suitable candidate to describe the reservation prices, since it has a nite support and is possible to give more weight to certain values. However the fact that the cdf is not continuously di erentiable makes it impossible to use most of the continuos optimization approximation methods.

Using the analytical function described in section 7 it is possible to compute the gradient in any given point of the decision space, and use this value to apply the gradient descent method. The pseudo code of the algorithm is written below.

```
fuction find_optimal_trunc_norm(psi0,c,gamma_L,gamma_H,mu,sigma)
next_psi=psi0
i=0
while i< max_iters:
    curr_psi =next_psi
    next_psi = curr_psi - learning_rate * gradient(curr_psi)
    step = next_psi - curr_psi
    if norm(gradient(next_psi)) <= precision:
        break
        return next_psi
    i=i+1
```

Using the norm of the gradient as the stopping criterion it is possible to approximate the optimal results as much as possible. As a stopping criterion I used norm(r (E[R] < 0:001). It is interesting to show the e ects that di erent values of and have on the optimal strategies. I am mainly interested in three situation

1. < 0:5($_h$ + $_L$): The best estimation of the reservation price is below the average of the uniform distribution (Fig.11a)

2. = 0:5($_h$ + $_L$):The best estimation of the reservation price is equal to the average of the uniform distribution (Fig.11b)

3. > 0:5($_h$ + $_L$):The best estimation of the reservation price is above the average of the uniform distribution(Fig.11c)

With low values of the bid changes slowly from one turn to the other. This is justi ed, since if the estimation of the reservation price is good, there is no point in lowering signi cantly the previous bid,given that the changes in the probability of making a successful o er are negligible when is far from . For relatively small the bid is always lower than the on obtained assuming uniform distribution, while for higher the opposite is true. Qualitatively it is also possible to notice that for high uncertainty the optimal bids are almost equal to the optimal bid obtained assuming a uniform distribution of reservation prices. This is a natural consequence of the fact that for large the truncated normal distribution approximates the uniform distribution on the same support, as shown in fig.12. This means that if the decision maker of the seller has no accurate estimate of the buyer reservation price (! 1) it is equivalent to playing using a uniform distribution of the reservation prices.

8 Multiple funds

If there is more than one possible fund that can join the system, optimizing the expected total revenue becomes more challenging. The main issue is that the number of possible states, and therefore the complexity of the function that represents the expected reward, grows exponentially with the number of funds. In the next sections I will study thoroughly the case with 2 and 3 funds, then propose an approximated solution for the general problem with n funds.

Due to the curse of dimensionality it is unfeasible to have an analytical function to compute the expected reward. Therefore it is necessary to use numerical experiment to compute the results obtained using different strategies. The testing method is described in appendix A.

8.1 2 Funds

Even the second simplest system, with only two possible clients, prove to be computationaly challenging to extend to a multiple turn bargaining. In this section I find an analytical solution to a one turn bargain game. When the number of possible funds is 2 there are 4 possible states of the systems, identified by the binary vectors s

$$s = \begin{cases} (0;0) & \text{Both funds are outside the network} \\ (1;0) & \text{Fund 1 joined, fund 2 didn't} \\ (0;1) & \text{Fund 2 joined, fund 1 didn't} \\ (1;1) & \text{Both funds joined} \end{cases}$$

Using the notation π_t^i to indicated the offer made to fund i at time t and defining the cost

$$\begin{cases} c_1 = bx_1^a \\ c_2 = bx_2^a \\ c_{12} = b(x_1 + x_2)^a \end{cases}$$

it is possible to write the reward received in each state as

$$\begin{cases} R(0;0) & = 0 \\ R(1;0) & = (\pi_0^1 - c_1) \\ R(0;1) & = (\pi_0^2 - c_2) \\ R(1;1) & = (\pi_0^1 + \pi_0^2 - c_{12}) \end{cases}$$

If $a \neq 1$ then $c_{12} \neq c_1 + c_2$ the total reward is not the sum of the individual rewards and it is not possible to study the Markov Decision problem as two separate optimization problems. Restricting the analysis to a one turn bargaining game the expected total reward can be written as

$$\begin{aligned} \mathrm{E}[R] = \; & [(1 - P(\pi_0^1))(1 - P(\pi_0^2))R(0;0) + P(\pi_0^1)(1 - P(\pi_0^2))R(1;0) + \\ & + (1 - P(\pi_0^1))P(\pi_0^2)R(0;1) + P(\pi_0^1)1P(\pi_0^2)R(1;1)] \\ = \; & [P(\pi_0^1)(1 - P(\pi_0^2))(\pi_0^1 - c_1) + (1 - P(\pi_0^1))P(\pi_0^2)(\pi_0^2 - c_2) + P(\pi_0^1)P(\pi_0^2)(0^1 + \pi_0^2 - c_{12})] \\ = \; & [P(\pi_0^1)(\pi_0^1 - c1) + P_0^2(\pi_0^2 - c_2) - P(\pi_0^1)P(\pi_0^2)\, c12] \end{aligned}$$

where $_{c12} := c_{12} \quad c_1 \quad c_2$ takes into account the di erence in managing the funds separately and combined. Provided that $\neq 0$ I can x it to 1 without loss of generality. Assuming again that the reservation price of both funds follows two uniform distributions $[_{1L};\ _{1H}][_{2L};\ _{2H}]$ I obtain the optimization problem

$$\text{minimize} \quad E[R] = \frac{_{1H} \quad ^1_0}{d_1}(_1 \quad c1) \quad \frac{_{2H} \quad ^2_0}{d_2}(_2 \quad c_2) + \frac{_{1H} \quad ^1_0}{d_1}\frac{_{2H} \quad ^2_0}{d_2} \quad _{c12}$$

$$\text{subject to} \quad _{1L} \quad ^1_0 \quad _{1H};$$

$$_{2L} \quad ^2_0 \quad _{2H};$$

The gradient necessary to nd the critical points is

$$\begin{cases} \frac{\partial R}{\partial ^1_0} = \frac{1}{d_1}(^1_0 \quad c_1) \quad \frac{_{1H} \quad ^1_0}{d_1} \quad \frac{1}{d_1}\frac{_{2H} \quad ^2_0}{d_2} \quad _{c12} = \frac{2}{d_1} \ ^1_0 + \frac{_{c12}}{d_1 d_2} \ ^2_0 \quad \frac{c_1 + \ _{1h}}{d_1} \quad \frac{_{c12}}{d_1 d_2} \ _{2h} \\ \frac{\partial R}{\partial ^2_0} = \frac{1}{d_2}(^2_0 \quad c_2) \quad \frac{_{2H} \quad ^2_0}{d_2} \quad \frac{1}{d_2}\frac{_{1H} \quad ^1_0}{d_1} \quad _{c12} = \frac{_{c12}}{d_1 d_2} \ ^1_0 + \frac{2}{d_2} \ ^2_0 \quad \frac{c2 + \ _{2h}}{d_2} \quad \frac{_{c12}}{d_1 d_2} \ _{1h} \end{cases}$$

This can be written in matrix from as

$$\begin{bmatrix} \frac{2}{d1} & \frac{_{c12}}{d_1 d_2} \\ \frac{_{c12}}{d_1 d_2} & \frac{2}{d_2} \end{bmatrix} \begin{bmatrix} ^1_0 \\ ^2_0 \end{bmatrix} = \begin{bmatrix} \frac{c1 + \ _{1h}}{d1} + \frac{_{c12}}{d_1 d_2} \ _{2h} \\ \frac{_{c12}}{d_1 d_2} \ _{1h} + \frac{c2 + \ _{2h}}{d2} \end{bmatrix}$$

Using the condition (24) the critical point is the optimal solution i

$$\begin{cases} |\frac{2}{d_1}| > |\frac{c_{12}}{d_1 d_2}| \\ |\frac{2}{d_2}| > |\frac{c_{12}}{d_1 d_2}| \end{cases}$$

$$\begin{cases} 2d_2 > c_1 + c_2 \quad c_{12} \\ 2d_1 > c_1 + c_2 \quad c_{12} \end{cases}$$

A priori it is not possible to guarantee this condition and it is necessary to check it before solving the linear system. It is worth noting that when the parameter $_{c12} \to 0$ the solution converge to

$$\begin{cases} ^1_0 = \frac{c_1 + \ _{1H}}{2} \\ ^2_0 = \frac{c_2 + \ _{2H}}{2} \end{cases}$$

which is the sum of the optimal bidding strategy of two independent funds when there is only one bidding turn. This means that the closer this parameter is to 0 the more this solution resembles the case of two independent funds. The difference in cash flow of the two bidding strategies becomes important the closer the upper bound on the funds reservation prices is to the maximum marginal cost. Figure 3 shows the results of this analytical optimal bid compared to the ones obtained using the naive strategy described in section 8.3.1 with $= 0$. The maximum reservation price of the two funds is a multiple m of their maximum marginal cost c_i. For low multiple m the exact bidding strategy clearly outperforms the Naive strategy. For greater values of m the strategy leads to practically the same results. This is due to the fact that for high upper bound $_i$ the correction introduced by $\frac{c12}{d_1 d_2}$ becomes negligible and the optimal bid goes to $^i_0$ $\frac{c_i + \ _{iH}}{2}$ $\frac{\ _{iH}}{2}$.

8.2 3 funds

With three funds there are 8 possible states and studying a solution for the general problem with t turns becomes unfeasible. In this section I study the simplified case with only one possible bargaining turn. This bidding strategy can be the base for a greedy approximation of the complete case with n turn.

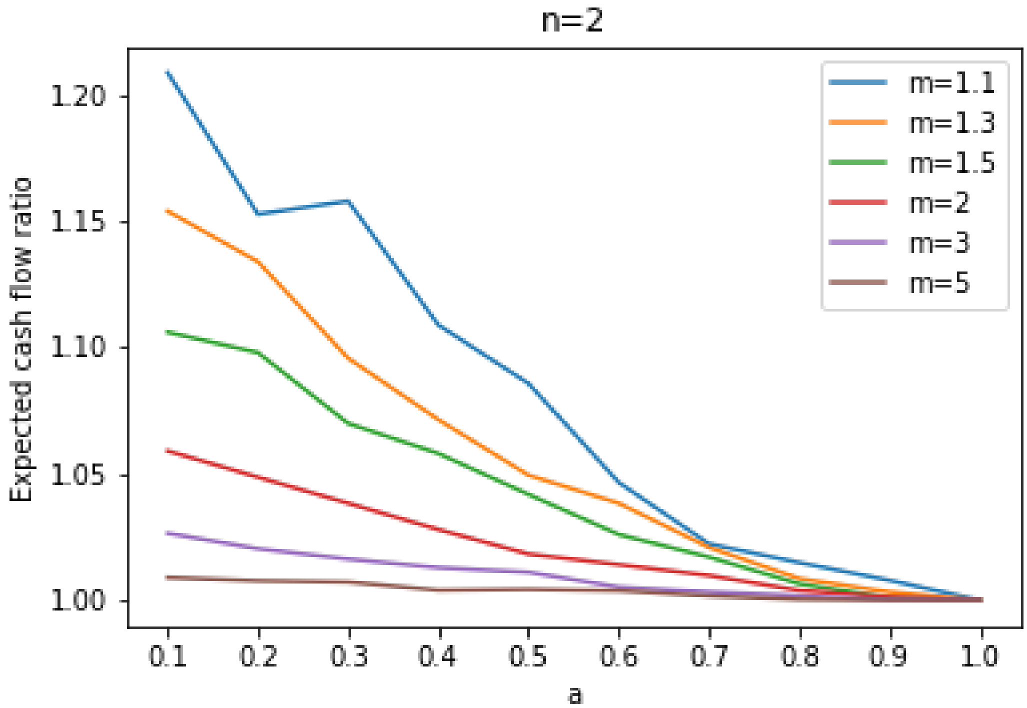


Figure 3: 2 funds $b = 1; x_1 = x_2 = 100{:}000; c_i = bx_i^a; \ _{iH} = mc_i; \ _{iL} = 0$. The results are expressed as a multiple of the expected cash obtained using the naive strategy with $= 0$

With three funds the expected value of the total reward can be written as

$$
\begin{aligned}
E[R] =& P(\tfrac{1}{0})1(1 \quad P(\tfrac{2}{0}))(1 \quad P(\tfrac{3}{0}))(\tfrac{1}{0} \quad c_1) + (1 \quad P(\tfrac{1}{0}))P(\tfrac{2}{0})(1 \quad P(\tfrac{3}{0}))(\tfrac{2}{0} \quad c_2) + \\
& (1 \quad P(\tfrac{1}{0}))(1 \quad P(\tfrac{2}{0}))P(\tfrac{3}{0})(\tfrac{3}{0} \quad c_3) + P(\tfrac{1}{0})P(\tfrac{2}{0})(1 \quad P(\tfrac{3}{0}))(\tfrac{1}{0} + \tfrac{2}{0} \quad c_{12}) + \\
& P(\tfrac{1}{0})(1 \quad P(\tfrac{2}{0}))P_3(\tfrac{1}{0} + \tfrac{3}{0} \quad c_{13}) + (1 \quad P(\tfrac{1}{0}))P(\tfrac{2}{0})P(\tfrac{3}{0})(\tfrac{2}{0} + \tfrac{3}{0} \quad c_{23}) + \\
& P(\tfrac{1}{0})P(\tfrac{2}{0})P(\tfrac{3}{0})(\tfrac{1}{0} + \tfrac{2}{0} + \tfrac{3}{0} \quad c_{123}) \\
=& :::::: \\
=& P(\tfrac{1}{0})(\tfrac{1}{0} \quad c_1) + P(\tfrac{2}{0})(\tfrac{2}{0} \quad c_2) + P_3((\tfrac{3}{0}) \quad c_3) \quad P(\tfrac{1}{0})P(\tfrac{2}{0}) \ _{c12} \\
& P(\tfrac{1}{0})P(\tfrac{3}{0}) \ _{c13} \quad P(\tfrac{2}{0})P(\tfrac{3}{0}) \ _{c23} \quad P(\tfrac{1}{0})P(\tfrac{2}{0})P(\tfrac{3}{0}) \ _{c123}
\end{aligned} \tag{25}
$$

The critical points can be found by solving the following non linear system

$$\begin{bmatrix} \frac{2}{d_1} & \frac{c12}{d_1 d_2} + \frac{3h \quad c123}{d_1 d_2 d_3} & \frac{c13}{d_1 d_3} + \frac{2h \quad c123}{d_1 d_2 d_3} \\ \frac{c12}{d_1 d_2} + \frac{3h \quad c123}{d_1 d_2 d_3} & \frac{2}{d_2} & \frac{c23}{d_2 d_3} + \frac{2h \quad c123}{d_1 d_2 d_3} \\ \frac{c13}{d_1 d_3} + \frac{2h \quad c123}{d_1 d_2 d_3} & \frac{c23}{d_2 d_3} + \frac{2h \quad c123}{d_1 d_2 d_3} & \frac{2}{d_3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \quad 3 \frac{c123}{d_1 d_2 d_3} \\ 1 \quad 3 \frac{c123}{d_1 d_2 d_3} \\ 1 \quad 2 \frac{c123}{d_1 d_2 d_3} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1H + c_1}{d_1} + \frac{2H \quad c12}{d_1 d_2} + \frac{3H \quad c13}{d_1 d_3} + \frac{2H \quad 3H}{d_1 d_2 d_3} \quad c123 \\ \frac{1H \quad c12}{d_1 d_2} + \frac{2H + c_2}{d_2} + \frac{3H \quad c23}{d_2 d_3} + \frac{1H \quad 3H}{d_1 d_2 d_3} \quad c123 \\ \frac{1H \quad c13}{d_1 d_3} + \frac{2H \quad c23}{d_2 d_3} + \frac{3H + c_3}{d_3} + \frac{1H \quad 2H}{d_1 d_2 d_3} \quad c123 \end{bmatrix}$$

$$B \quad \omega = b$$

The additional term ω contains multiplication between the variables and introduces non linear element in the system. This system of equation does not have an analytical solution. In the particular case that the input parameters are such that $\frac{c123}{d_1 d_2 d_3} \quad 1$ it is possible to neglect the non-linear term ω. Therefore it is possible to solve the simpler linear system

$$\begin{bmatrix} \frac{2}{d_1} & \frac{c12}{d_1 d_2} & \frac{c13}{d_1 d_3} \\ \frac{c12}{d_1 d_2} & \frac{2}{d_2} & \frac{c23}{d_2 d_3} \\ \frac{c13}{d_1 d_3} & \frac{c23}{d_2 d_3} & \frac{2}{d_3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1H + c_1}{d_1} + \frac{2H \quad c12}{d_1 d_2} + \frac{3H \quad c13}{d_1 d_3} \\ \frac{1H \quad c12}{d_1 d_2} + \frac{2H + c_2}{d_2} + \frac{3H \quad c23}{d_2 d_3} \\ \frac{1H \quad c13}{d_1 d_3} + \frac{2H \quad c23}{d_2 d_3} + \frac{3H + c_3}{d_3} \end{bmatrix}$$

The results for this situation are comparable to the case with only two funds. When $a \to 1$ the optimal bid becomes similar to the optimal bid for independent funds while correcting the bid to take into account the economy of scale is more important when the upper bounds on the reservation price is close to the maximum marginal cost. The ratio of expected cash flows between the corrected optimal bid and the naive strategy is show in figure 4.

In case of more than 3 funds there will be even more mixed terms between decision variable and the system will be even more non linear. In general the expected reward function of a single turn bargaining for a network with n funds can be written as

$$\mathrm{E}[R] = \sum_{i=1} P(\,^{i}_{0})(\,^{i}_{0} \quad c_i) \qquad \prod_{(S \in N; |S|>1)} P_S \quad _S \tag{26}$$

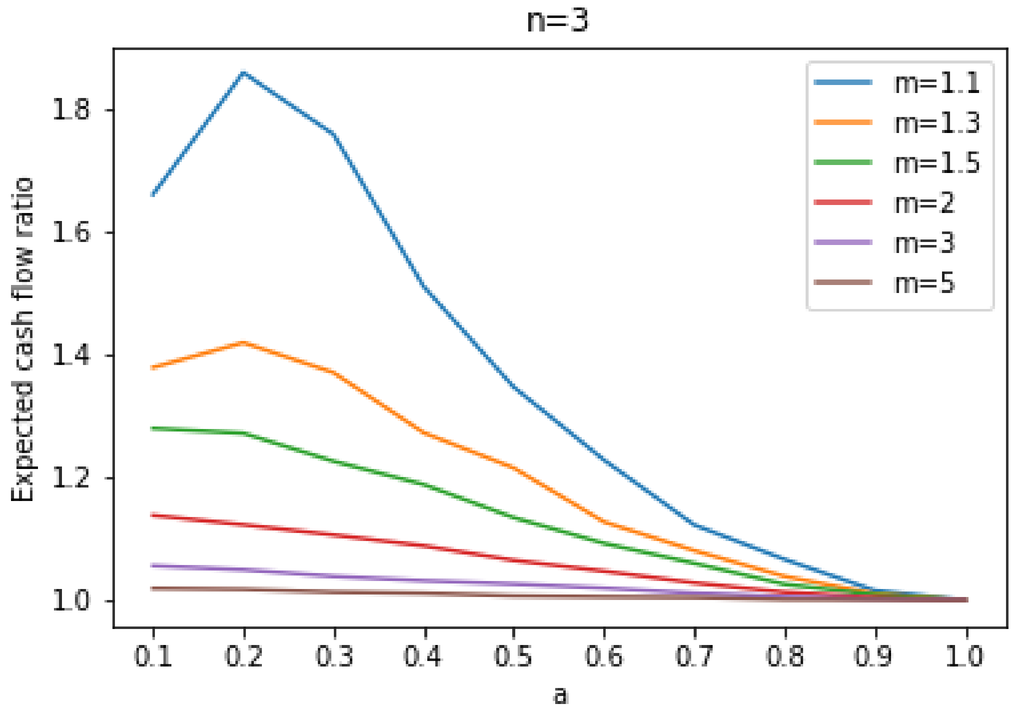


Figure 4: 3 funds $b = 1; x_1 = 100{:}000; c_i = bx^a; \ _{iH} = mc_i; \ _{iL} = 0$. The results are expressed as a multiple of the expected cash obtained using the naive strategy with $\ = 0$

where the last term takes into account all the possible coalition of 2 or more funds and the corrective term associated with managing the players in the coalition jointly or individually. The probability $P_S = \prod_{i_i nx} P_i$ roughly goes as $\propto \frac{1}{\prod_{i \in S} d_i}$. This means that the greater the coalition is, the smaller is the associated correction factor. If it is possible to ignore the corrective term for $n = 3$ it should also be possible the corrective term of higher, since the product at the denominator is even bigger. Neglecting all correction terms above the 2 players coalition the sub optimal bids are given solving the linear system $B \quad = b$ where

$$B_{ii} = \frac{2}{d_i}$$

$$B_{ij} = B_{ji} = \frac{c_{ij}}{d_i d_j}$$

$$b_i = \frac{{}_{iH} + c_i}{d_i} + \sum_{j \neq i} \frac{{}_{jH} \quad c_{ij}}{d_i d_j}$$

Fig 5 and g 6 show the results of this strategy applied to networks with 2 to 5 players, for scale factor $a > o{:}5$. These results show that the quality of the approximation deteriorates even for a small network of 5 funds. This means that higher order corrective term introduce non linearity to big to be ignored.

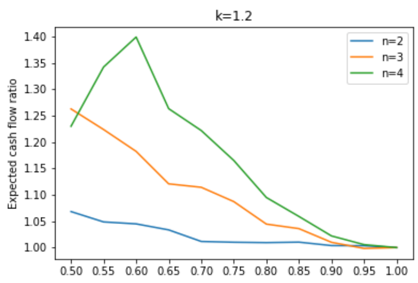


Figure 5: 4 equal funds

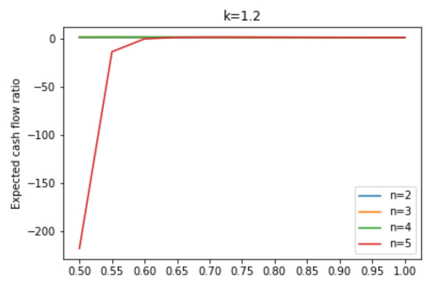


Figure 6: 5 equal funds

8.3 Approximation for n funds

The two previous sections showed that it is not feasible to nd an analytical solution when the possible size of the network is considerably larger, like in a realistic scenario where $n > 50$. It is necessary to nd a method to approximate the optimal strategy.

8.3.1 Naive approximated strategy

The main issue in nding an analytical solution for the optimal bidding strategy is that the cost of managing funds in a coalition grows non linearly and therefore it is not possible to compute independently the optimal strategy for each possible participants in the coalition.

No matter the size of the possible grand coalition it is possible to define an upper and a lower bound for the cost of managing a particular fund: the maximum marginal cost MC_i and the minimum marginal cost mc_i. Given the concavity of the cost function they correspond respectively to the cost of managing the fund i in the smallest possible coalition $s = i$ and to the extra cost added by the fund when completing the grand coalition. That is

$$MC_i = bx_i^a \quad mc_i = bx_N^a - b(\sum_{j \neq i} x_j)^a \tag{27}$$

Since it is certain that the actual marginal cost of managing a fund in any subcoalition lies between this two extreme values I can introduce a fictitious management cost

$$c_i = \lambda mc_i + (1 - \lambda)MC_i; \quad \lambda \in [0;1] \tag{28}$$

This ficticious cost can be used to approximate the real marginal cost of managing a fund. The value of λ used depends on the decision maker's desire to take into consideration the economy of scale of the cost function. Qualitatively I can say that

- $\lambda \to 1$: the approximated cost c_i tends to the minimum marginal cost mc_i. The seller decides to fully take advantage of the economy of scale to offer the lowest possible price to each possible buyer. This increases the chance that each fund accepts the bid as soon as possible. However the cost estimation might be to optimistic, if not enough funds decide to licence the software the seller might faces losses, since the bid is low and few licences were sold.

- $\lambda \to 0$: The seller ignores the benefits of the economy of scale. The bid obtained using this approxiamted strategy are high and might not be a sufficient incentive to convince the buyers to switch to the new system. However the seller is guaranteed to have a positive cash flow, since the licence revenue is never below the highest possible marginal cost.

This approximated cost can be used to reduce the dimensionality of the problem from 2^n to a much more manageable $2n$. Using this approximated cost it is possible to rewrite equations 18 and 20

$$R_t^i(\ {}_t^i) = \sum_{t'=1} {}^{t'}\ {}_0 + \sum_{t'=t+1} {}^{t'}(\ {}_t^i \quad c_i)$$

$$\mathrm{E}[R(\ {}^i)] = P_0^i R_0^i(\ {}_0^i) + (1 \quad P_0^i)P_1^i R_1^i(\ {}_1^i) + \dots \prod_{t=0}^{T\ 2}(1 \quad P_t^i)P_{T\ 1}^i R_{T\ 1}^i(\ {}_{T\ 1}^i)$$

for all funds $i = 1, 2 \dots n$. Using the analytical method developed to solve the optimization problem (22) it is possible to solve n optimization problems, one for each possible client. The pseudocode of this approximation algorithm is written below

```
function find_naive_psi(x,a,b,gamma_H,k,lambda,DF,T)
for i =1:n :
    c_lam=lambda*mc+(1-lambda)*MC
    psi(i)=find_optimal_psi(gamma_H(i),c_lam,k,DF,T)
return psi
```

The results of the approximated strategies are tested with the numerical experiments described in appendix A for different m to see which weight yields the best results. The best to compute the optimal strategy depends on the number of funds. The graphs in figure 13 and 14 show the results for different maximum sizes of grand coalition.In order to have a clear comparison of the different strategies, the expected cash flows are expressed as multiples of the expected results obtained with $= 0$. For small coalitions the best option is to use the maximum marginal cost as the approximated cost. Using an approximated cost close to the minimum marginal cost opens up to a high probability of bidding below the actual cost of managing the resulting coalition and therefore it does not yield acceptable results. When the coalition size becomes significant, (n 20) the probability of managing a large coalition increases and it becomes beneficial to consider the lowest marginal cost mc_i as the approximated managing cost. Another interesting result is that the performances of the strategy is influenced by the width of the price distribution. If the maximum reservation price is much higher than the maximum marginal cost ($\frac{iH}{MC_i}$ 1) the importance of the weight becomes negligible. When $m = 1.2$ using the appropriate weight can increase the expected profit more than 200%, provided that n is large. Putting to much enphasis on the economy of scale $\to 1$ also creates a significat chance of losing money at the end of the bargaining games, especially for small network. On the other hand with $m = 5$ the difference between the expected profits is less than 4% and the possibility of having a negative cash flow is negligible for any network of realistic size.

There is no method that I am aware of to estimate a priori what is the value that yields the highet expected profit, given the number of funds n and the probability distribution of their reservation prices. However it is possible to find the optimal weight a posteriori, using a Monte Carlo simulation. The time needed to run the simulations necessary for an accurate estimation is negligible compared to the time the decision makers has to decide the optimal bids.

While it is possible to use Monte Carlo simulations to find the weight that yield the highest expected reward, I am not able to find a boundary of this approximation. That is I cannot know how close the result of this approximated strategy is from the true optimum.

8.3.2 Refined Approximated Strategy

The naive approximated strategy is easy to compute and the systems of equations need to be solved only once, that is at the beginning of the bargaining turn $t = 0$. The simplicity of this strategy comes at a cost, since it ignores any new information revealed after each offer. Since at the beginning of each turn there is the possibility that new funds join the system the decision makers have more elements to decide the better strategy.

The maximum marginal cost of managing new participants might change at each turn. Given a certain coalition S at a turn t the maximum marginal cost of managing the fund $i \in S$ can be computed as

$$MC_{iS} = bx^a_{S \cup i} - bx^a_S$$

This allows to have a more accurate approximated cost $c_{iS} = \ mc_i + (1 - \ )MC_{iS}; \ \in [0;1]$. Since the effort of computing n independent strategies is negligible the decision makers can recompute the optimal bid at each turn, solving (22) and act accordingly. The differences between this refined behaviour and the naive approximated strategy become negligible when $\to 1$, since the minimum marginal cost is not influenced by the current state of the network. The main advantage of this strategy is that it allows to use $\to 0$, which decreases the chance of a negative outcome, while still exploiting the benefit of the economy of scale, since the maximum marginal cost is updated with new information after each turn. Figure 15 show the result obtained using the refined and the naive strategy on the same set of randomly generated funds. It is clear that for any given weigth the refined strategy outperforms the naive strategy, being up to 3 times more profitable for large network. However looking at the absolute performance it is possible to notice that the best results are obtained for $\to 1$, where the two approximated strategies converge to the same optimal bids.

8.4 Conclusion and recommendation

The two strategies proposed avoid the curse of dimensionality by splitting a single Markov Decision Problem with 2^n possible state into n problem with a know analytical solution. The quality of this approximation depends on the weight used to compute the weighted average of the maximum and minimum marginal cost of managing each possible fund. The optimal value depends on the contest and can be estimated a posteriori using numerical experiments on ctitious funds with the reservation prices randomly drawn from their probability distribution. Usually for large network the best results are obtained using a ctitious cost close to the minium marginal cost, that is ! 1. The added complexity of using the re ned approximation compared to the naive approximation is worth for risk averse vendor that want to use the maximum marginal cost as a ctitious cost while partially exploiting the economy of scale granted by the centralization of the data management services.

9 Markov Decision Process for value transfer bargaining

This bargaining problem will be tackled using Markov Decision Theory. It is necessary to introduce the additional parameters:

r_i: The average leave rate of fund i.

p_{ij}: The probability that a transaction from i goes to j.

$A; B$: The economy of scale parameter of the vendor.

$A_i; B_i$: The economy of scale parameters for fund i

$[A_{iL}; A_{iH}]; [B_{iL}; B_{iH}]$: The support of the distribution functions $F(A); F(B)$

$_i;\ _i$: The economy of scale parameters quoted by PI to fund i.

There are several major di erences between this game and the one described in section 6

The seller must quote two parameters $_i;\ _i$ instead of a single one $_i$

The decision to accept or refuse the price offered by PI now depends on the state of the network: the same offer can lead to different outcome based on the subcoalition that is currently licencing PI services.

It is not possible to study simple case such as $n = 2$ or $n = 3$ since in these scenarios there is no network effect.

The binary vector s identifies the state of the system also in this Markov chain. It is also necessary to store the last offer ($^{t\ 1}$; $^{t\ 1}$), to keep track of the revenue from each fund and to know the current shape of the decision space. There are two decision variables for each fund that is not in the network, the economy of scale parameters ($_i$; $_i$) offered by the vendor. Since the leave rate r and the transition probabilities p_{ij} are public information it is possible to compute z_i, the total number of value transfer that a fund has to manage and $_{iS}$, the number of transaction that are managed by PI if the fund i joins the current coalition S, as

$$z_i := r_i x_i \sum_{j \neq i} p_{ij} + \sum_{j \neq i} r_j x_j p_{ji}$$
$$_{iS} := r_i x_i \sum_{j \in S} p_{ij} + \sum_{j \in S} r_j x_j p_{ji}$$

Since only the transfers that are completely in the coalition S can be managed, the buyer will pay

$$_{iS} = \ _i \ _{iS}^{\ i} \qquad (29)$$

If the coalition is not complete the fund will also have to bear the cost of all the transactions with funds that are not in S. The buyers must choose between two cash flows:

1. CF_{def}: The current cost of managing the system, its default situation
2. CF_{PI}: The cost of managing the value transfers in a mixed system.

A perfectly risk averse decision maker will base its decision only on facts that are certain. This means that the buyer i faced with the choice of joining the coalition S it will compute the cost CF_{PI} considering the coalition $S [i$, since it does not know what other funds will do. The two cost can be estimated as

$$CF_{def} = Bz_i^A$$
$$CF_{PI} = \quad {}_{i(S[i)} + B(z_i \quad {}_{i(S[i)})^A$$

The decision maker will accept the quote i

$$CF_{PI} < CF_{def} \tag{30}$$

Unfortunately it is not su cient that $_i < A_i; \; _i < B_i$ to ful ll (30). For any combination of $(A_i; B_i; \; _i; \; _i)$ there is a minimum $_i$ necessary to make a rational switch to licence PI service. Figure 7 shows that if $\; < \;$ the fund will pay more in PI than in the current situation. This means that if the current subcoalition does not guarantee at least transfers inside the network the buyer will decline the o er. While it is true that the switch might become pro table in the future, if enough fund will join the system, a perfectly risk averse decision makers is not willing to sustain certain losses in a period for the hope of reducing operative cost in the future.

Since the decision regarding the o er is also in uenced by the state of the network $_{i(S[i)}$ this parameter must be considered when computing the probability of making a sucessful bid and the transition probability. Similarly to the previous Markov chain the possibility of transitioning from the state s to the state $s^0 : S^0 \quad S$ is equivalent to make a successfull o er to the buyers in $S^0 \, n \, S$ and an unsucessful one to the buyers in $N \, n \, S^0$. That is

$$P(s^0 js; \; ; \; ; \;) = \prod_{i2S^0 \quad S} P(\; _i; \; _i; \; _{i(S[i)}) \prod_{NnS^0} (1 \quad P(\; _i; \; _i; \; _{i(S[i)})) \tag{31}$$

The reward function used to evaluate the vendor decision is the net pro t of the vendor that is

$$R(s; \; ; \; ; \;) = \sum_{i2S} (s_i \quad _{iS}) \quad B(\sum_{i2S} \quad _{iS})^A \tag{32}$$

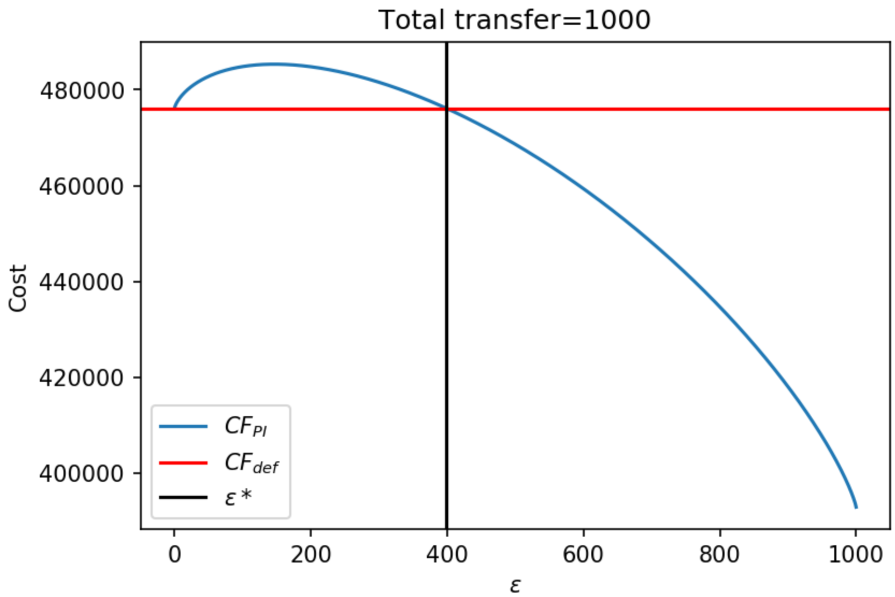


Figure 7: Comparison of cash flow, for $z_i = 1000; A_i = 0.9; B_i = 950; \ _i = 0.88; \ _i = 900$

Given that there is a finite number of turn T the objective is to maximize the total expected reward. Given that the state space is discrete and the decision space is finite there is an optimal markov strategy.

10 Bargaining Value Transfer Service licence

Before starting to discuss the bidding strategy I need to compute what what is the probability of success of an offer $(\ _i; \ _i)$, This is equivalent to computing the integral

$$I = \int_{\ _i} f(A_i) f(B_i) dA_i dB_i \tag{33}$$

where $\ _i$ is the region of the space $[A_{iL}; A_{iH}] \ [B_{iL}; B_{iH}]$ that fulfills the inequality in (30).

This inequality can be analytically written as

$$B_i z_i^{A_i} > \quad _i \ ^{i}_{i(S[i])} + B_i(z_i \quad _{i(S[i])})^{A_i} \tag{34}$$

Given that ($_i$; $_i$; $_{iS}$; z_i) are known to the vendor I can rewrite the inequality to find a condition on the variable B for any possible value of A as

$$B_i(A) > \frac{_i \ ^{i}_{i(S[i])}}{z_i^{A_i} \quad (z_i \quad _{i(S[i])})^{A_i}} \tag{35}$$

The threshold acceptance value $B_i(A)$ might be outside the support interval $[B_{iL}; B_{iH}]$. Adding the condition that $B_i(A)$ must be in this interval, it is possible to write

$$B_i(A) = \begin{cases} B_{iL} & \text{if } \frac{_{i(S[i])}}{z_i^A \quad (z_i \quad _{i(S[i])})^A} \quad B_{iL} \\ \frac{_{i(S[i])}}{z_i^A \quad (z_i \quad _{i(S[i])})^A} & \text{if } B_{iL} < \frac{_{i(S[i])}}{z_i^A \quad (z_i \quad _{i(S[i])})^A} \quad B_{iH} \\ B_{iH} & \text{if } \frac{_{i(S[i])}}{z_i^A \quad (z_i \quad _{i(S[i])})^A} > B_{iH} \end{cases}$$

Using this piecewise definition of $B_i(A)$ it is possible to write an integration domain $_i($ $_i$; $_i$; $_{i(S[i])})$ for any combination of the decision variables ($_i$; $_i$) and the network parameter $_{i(S[i])}$. Figure 8 show the integration domain $_i($ $_i$; $_i$; $_{i(S[i])})$ for different decision variables and network parameters.

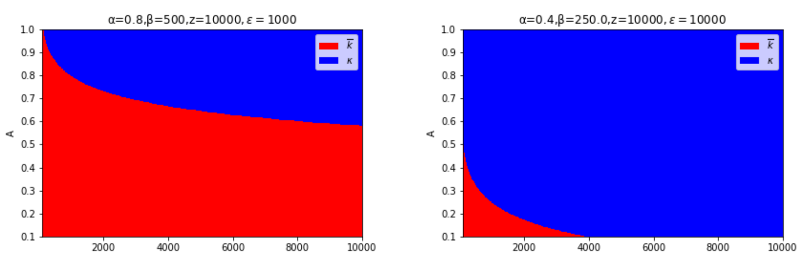


Figure 8: Integration domain for different decision variables and network parameter. An offer is accepted if the pair of private $(A; B)$ is contained in the blue region of the graph.

The unconditional probability of making a successful offer can be found computing (33) over the integration domain $_i(\ _i;\ _i;\ _{i(S[i])})$.

$$
\begin{aligned}
P(\text{Success}j\ _i;\ _i;\ _{i(S[i])}) &= P((A;B)\ 2\ \ (\ _i;\ _i;\ _{i(S[i])})) \\
&= \int_{A_{iL}}^{A_{iH}} \int_{B\ (A)}^{B_{iH}} f(A_i) f(B_i) dB_i dA_i \\
&= \int_{A_{iL}}^{A_{iH}} \frac{1}{A_{iH} \quad A_{iL}} \frac{1}{B_{iH} \quad B_{iL}} (B_{iH} \quad B\ (A)) dA_i \\
&= \ _i \int_{A_{iL}}^{A_{iH}} (B_{iH} \quad B\ (A)) dA_i
\end{aligned}
$$

where $_i := \frac{1}{A_{iH} \quad A_{iL}} \frac{1}{B_{iH} \quad B_{iL}}$ to ease the notation. Since $B\ (A)$ the last integral is de ned piecewise, the integral can be further divided in at most three integrals, de ned in the intervals $[A_{iL};A_{iL}];[A_{iL};A_{iH}];[A_{iH};A_{iH}]$, where $[A_{iL};A_{iH}]$ are the solutions of the equations

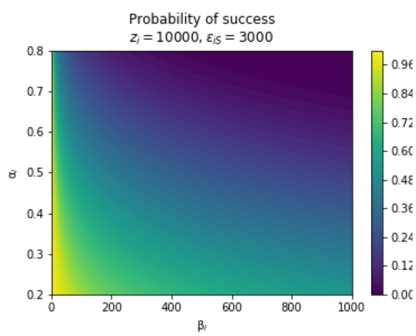
$$
\begin{aligned}
B_{iH} z_i^{A_{iL}} &= \ _i\ ^i_{i(S[i])} + B_{iH}(z_i \quad _{i(S[i])})^{A_{iL}} \\
B_{iL} z_i^{A_{iH}} &= \ _i\ ^i_{i(S[i])} + B_{iL}(z_i \quad _{i(S[i])})^{A_{iH}}
\end{aligned}
\tag{36}
$$

These equations does not have an analytical solution and will be solved using the Newton approximation method[20], as described in the appendix C. It is now possible to write the integral as

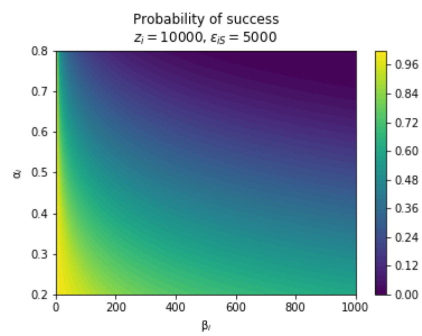
$$
\begin{aligned}
P((A;B)\ 2\ \ _i(\ _i;\ _i;\ _{i(S[i])})) &= \ _i \int_{A_{iL}}^{A_{iH}} (B_{iH} \quad B\ (A)) dA_i \\
&= \ _i(\int_{A_{iL}}^{A_{iL}} (B_{iH} \quad B_{iH}) dA_i \\
&+ \int_{A_{iL}}^{A_{iH}} (B_{iH} \quad \frac{_{iS}}{z_i^A \quad (z_i \quad _{iS})^A}) dA_i + \int_{A_{iH}}^{A_{iH}} (B_{iH} \quad B_{iL}) dA_i) \\
&= \ _i(\int_{A_{iL}}^{A_{iH}} (B_{iH} \quad \frac{_{iS}}{z_i^A \quad (z_i \quad _{iS})^A}) dA_i + (B_{iH} \quad B_{iL})(A_{iH} \quad A_{iH}))
\end{aligned}
\tag{37}
$$

The rst term of the Right Hand Side of the last equation does not have an analytical solution and is computed using the rectangle numerical integration method, with stepsize $dA = 0{:}01(A_{iH} \quad A_{iL})$.

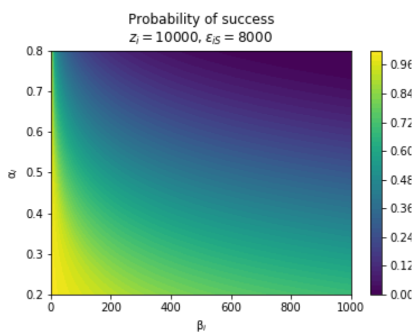
In gure 9 I show the probability of a making a successful o er for the same fund with di erent network conditions. The results clearly show that the same o er ($_i$; $_i$) has a much higher chance to be accepted when the network is close to completion ($_{i(S\,[\,i)}$! z_i). This happens because, when the network manage a small number of value transfers, the e ciency gain in outsourcing part of the transfers is negligible, and therefore a fund will required very favourable condition to join the network. On the other hand this probability highlights a major issue during the rst bargaining turn: no fund has an advantage to be the rst to join the coalition. Indeed if S = ;, the network parameter $_{i(;\,[\,i)}$ = $_{ii}$ = 0. A null network parameter lead to an empty integration domain, and therefore no o er will be accepted. I explain how this problem is goiung to be solved with a greedy bidding strategy in the next section.



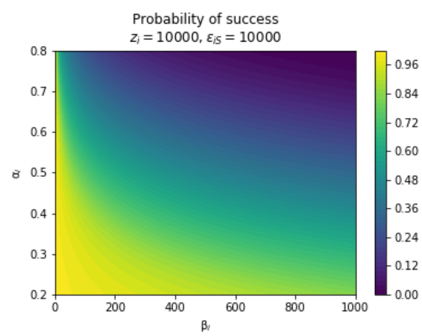
(a) Network 30% complete



(b) Network 50% complete



(c) Network 80% complete



(d) Complete network

Figure 9: Probability of making a successful bid $_i$; $_i$,for di erent network states. The completeness of a network is de ned as $\frac{_{iS}}{z_i}$

All the previous computation are needed to compute the probability of making a successful o er during the rst bargaining turn. If this o er is turned down the decision maker gain new information regarding the possible value of the buyer reservation values, namely that $(A_i; B_i) \not2 \;^t_i(\;^t_i; \;^t_i; \;^t_{i(S[i)})$. The exponent t is introduced to identify the integration domain and the decision variables at each turn t.

The conditional probability of making a sucessful o er can be written as

$$P((A_i; B_i)\ 2\ \;^t_i j (A_i; B_i) \not2\ \;^{t\ 1}_i) = \frac{P((A_i; B_i)\ 2\ \;^t_i n\ \;^{t\ 1}_i)}{(P((A_i; B_i) \not2\ \;^{t\ 1}_i)}$$

$$= \frac{P(\;^t_i)\quad P(\;^{t\ 1}_i)}{1\quad P(\;^{t\ 1}_i)}$$

The previous equation requires that $\;^{t\ 1}_i \quad \;^t_i$. This is always true, given that a rational decision maker can use the information provided by the decision of the buyer to avoid bidding in a region that guarantees a refusal.. Figure 10 show how successive bid in uence the integration domain during two bargaining turns. It also highlight another important di erence between this bargaining game and the previous one. Provided that the network parameter increase between two consecutive bargaining turn, it is possible to repeat the previous o er with a non zero probability of success.

11 Greedy strategy approximation

I intend to use a two part greedy strategy to bargain this service o ering

1. Convince the two most active fund to join the network

2. Greedy optimization of the next turn expected rewards

The rst part of the strategy is necessary to kickstart the network formation process. Without a minimum number of transaction no fund is willing to accept the o er. The pathological case when there is an empty network is that $\;_{i;} =$

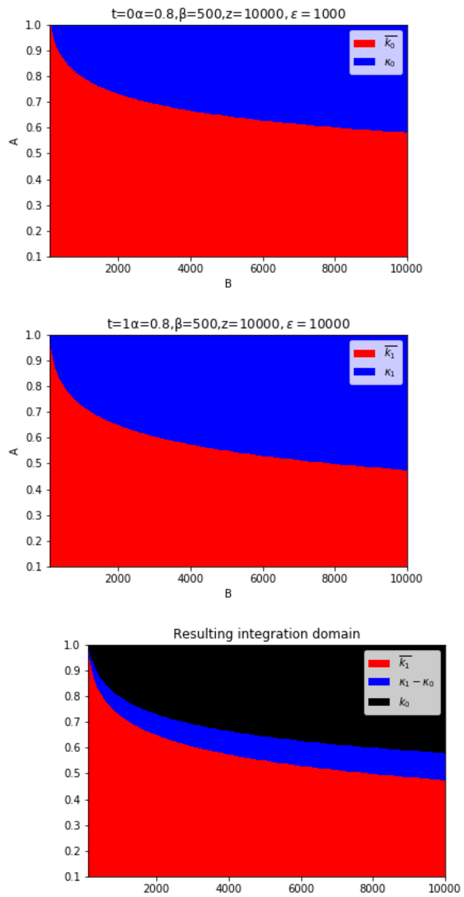


Figure 10: integration domain $^{0} n \; ^{1}$

$0; 8i$. This mean that no fund wants to be the rst to join. In order to avoid this scenario it is necessary to guarantee a minimum number of transactions to make the switch more convenient to every individual buyer. The rst part of the strategy serves this purpose. The most active funds are the pair ij with the highest number of transaction between them. That is

$$i; j = \underset{i;j}{\operatorname{argmax}} \quad _{ij} \tag{38}$$

In order to be certain that these o ers made will be accepted it is necessary to quote economy of scale parameters $(\alpha_i; \beta_i)$ and $(\alpha_j; \beta_j)$ that will lead to a cash flow CF_{PI} below the lowest possible estimate of CF_{def}, that is when both target funds operate with parameters $A_{iL}; B_{iL}$ and $A_{jL}; B_{jL}$. This means that there are two conditions to respects

$$B_{iL} z_i^{A_{iL}} > \alpha_i \delta_{ij}^{\beta_i} + B_{iL}(z_i - \delta_{ij})^{A_{iL}}$$
$$B_{jL} z_j^{A_{jL}} > \alpha_j \delta_{ij}^{\beta_j} + B_{jL}(z_j - \delta_{ij})^{A_{jL}}$$

In these two inequalities all the parameters are known and therefore it is possible to find and infinite number of pairs $(\alpha_i; \beta_i); (\alpha_j; \beta_j)$ that are guaranteed to satisfy them. I choose to use the pairs

$$\beta_i = A_{iL} \qquad \alpha_i = 0.98 \frac{B_{iL} z_i^{A_{iL}} - B_{iL}(z_i - \delta_{ij})^{A_{iL}}}{\delta_{ij}^{\beta_i}}$$
$$\beta_j = A_{jL} \qquad \alpha_j = 0.98 \frac{B_{jL} z_j^{A_{jL}} - B_{jL}(z_j - \delta_{ij})^{A_{jL}}}{\delta_{ij}^{\beta_j}}$$

These offers are not optimal and, depending on the scale parameter $A; B$ of the vendor, might even lead to a temporary negative cash flow per turn. However they are necessary to create a significant number of transactions inside the network.

The curse of dimensionality is avoided using the same method employed in the bargaining of data management service using approximated strategies (section 8). A problem with 2^n state space is divided in n optimization problems using a fictitious management cost, as a weighted average of the minimum and the maximum marginal cost of managing the new fund in the current coalition. That is the fictitious management cost c_i is computed as

$$c_{iS} = \lambda mc_i + (1 - \lambda) MC_{iS}; \quad \lambda \in [0; 1] \qquad (39)$$

The expected rewards of this new Markov problems are the expected fictitious profits, the difference between the revenue that a certain offer would bring and the ctitious cost of o ering these services, that is

$$\begin{aligned} E[R_i(\ {}^t_i)] &= P(\ {}^t_i j\ {}^{t}_{i}\ {}^{1})(\ {}^t_i\ {}^t_{i(S[i)}\quad c_{iS}) \\ &= \frac{P(\ {}^t_i)\quad P(\ {}^{t}_{i}\ {}^{1})}{1\quad P(\ {}^{t}_{i}\ {}^{1})}(\ {}^t_i\ {}^t_{i(S[i)}\quad c_{iS}) \end{aligned} \tag{40}$$

The complexity of computing the probability of making a sucessful o er means that it is not feasible to compute the optimal value of a sequence of o ers, as I did for the previous MDP. In this situation the reward are optimized greedily, that means that the bid chosen maximizes the immediate expected pro t without taking into account the long term e ects of the decision. The optimization algorithm used to nd the optimal greedy bid is described in the appendix D.

The numerical methods required to compute the sub-optimal greedy bids make this algorithm quite ine cient. While the time required to compute the decision in any given turn is still compatible with the time available to the vendor, testing this algorithm is incredibly time consuming. For these numerical results I only conducted 500 numerical experiments for each set of parameters, instead of the usual 10.000.

Figure 16 shows the results for network of 5, 10 and 20 possible buyers, when the seller is much more e cient than the buyers. The value of the weigth does not contribute to the expected pro t. For larger network the possibility of ending the game with a negative expected loss are almost 0. This is not true when considering a scenario where the vendor is only slightly more e cient than the buyers, show in gure 17. The value of is still irrelevant to the expected results. The probability of a positive pro t in this scenario is below 20%, and the expected pro t of the numerical tests is negative. This is due to the fact that the low e ciency of the vendors does not allow to recoup the loss caused by the extremely favourable o er made to the two most active fund. The network is losing capital at each turn and at the same time can not make other favourable bids, since the economy of scale is not that e ective. Unfortunately the extremely favourable bid proposed to the most active funds can not be avoided. A bid that increases the profit obtained by PI lead to a non-zero probability that it might be declined by the active funds. This makes it impossible to have the first transfers inside the network and therefore it is not possible to attract the smaller funds

12 Conclusion and further research

The aim of this thesis was two answer two research question:

1. Assuming complete information and that the player are willing to cooperate, which is the best way to allocate the maintenance cost of the network infrastructure to each fund?
2. Assuming incomplete information what is the best sequence of bid that PI can make to maximize its expected revenue?

I answered the first question with a review of cost allocation methods developed in cooperative game theory. The Separable Cost Remaining Benefit proved to be the most suited allocation method for this particular cost game. it takes into account both the cost increase due to new fund joining the network and the savings that each player can achieve outsourcing its operation to a third party software house. Furthermore it is computationally efficient.

The answer to the second research question proved to be more challenging since it depends on which bargaining game is played and on the size of the largest coalition. I found an analytical solution for the simple case of a 1-to-1 bargaining of data management licence. The more realistic case 1-to-many bargaining has been tackled with an approximated strategy that use a fictitious management cost to approximate the true marginal cost. This approximated cost is used to solve n 1-to-1 problems with knows analytical solution. This approach is suboptimal but the numerical experiments proved that it is e ective and always lead to a positive expected pro t. I was nt able to nd an analytical solution to the value transfer bargaining problem and therefore I implemented a greedy strategy to decide the bids at each bargaining turn. The numerical simulations proved that while this method is e ective when the sellers is signi cantly more e cient than the buyers it leads to an average net loss when the di erence in e ciency is small.

This thesis highlight that further research is needed in investigating 1-to-many bargaining games with one-sided o ers, especially in the case where the acceptance decision is not only in uenced by the private information of the buyer but also by the state of the system. These games are subject to the course of dimensionality and new approximated strategy need to be developed to simplify the state space and improve suboptimal decision making. The approximated strategy proposed mostly lead to an average positive net pro t but there how close these results are to the true optimal results is still an open question. Furthermore these methods relies on the assumption of a perfectly risk averse buyer which limits the general validity of the methods proposed. It is possible to study the case where the buyers have di erent risk appetite and how this fact in uence the optimal bidding strategy.

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A Testing a strategy

Given the difficulty of computing exactly the expected value of a strategy, the expected profit is computed using Monte Carlo simulations.

In every simulation each fund is assigned a reservation price drawn randomly from its distribution probability. The strategy to test is played and the fund will accept the first offer below its reservation price. Given that the joining time and the offer accepted by each fund uniquely identify the cash flow at each turn it is possible to compute the total cash discounted cash flow received by PI. The results obtained in the current thesis are tested using 10.000 trials.

B A note on Nash Equilibria

The Nash equilibria of a bargaining game with n player is the set of payoffs that maximize the function

$$(\;\;) = \prod_{i \in N} (\;_i \quad d_i)^{\;_i}$$

$$s.t. \quad \;_i \quad d_i \forall i$$

where d_i is the payoff in case of disagreement and $\;_i$ is the risk aversion coefficient of a player i. In out bargaining case the default value d_i are the current administrative cost of each player and the unavoidable cost $\;_0$ of the service provider. This means that in order to compute this value the reservation prices of each player must be common knowledge, along with its risk preference $\;_i$. Furthermore the implicit assumption is that the bargaining is a non repeated game. It is anyway interesting to solve this problem, and compare it to the cooperative solution concepts and the bargaining results.

Calling APG player 0 I have that the utility functions needed are

$$u_0 = (\sum_{j=1} \quad_i \quad c)^{\quad 0}$$

$$u_i = (\quad_i \quad_i)^{\quad i} \quad i = 1, 2, \ldots, n$$

where c is the total cost of managing the network. The corresponding maximization problems becomes

$$\text{minimize} \quad (\ ) = (\sum_{j=1} \quad_i \quad c)^{\quad 0} \prod_{i \in n} (\quad_i \quad_i)^{\quad i}$$

$$\text{subject to} \quad \sum_{i \in n} \quad c \; i \in n.$$

$$_i \; i \in n.$$

By basic property of logarithm it can be rewritten as

$$\text{minimize} \quad _0 \ln(\sum_{j=1} \quad_i \quad c) \quad \sum_{i \in n} \quad_i \ln(\quad_i \quad_i)$$

$$\text{subject to} \quad \sum_{i \in n} \quad c$$

$$_i \; i \in n.$$

The partial derivatives are

$$\frac{@}{@ \quad_1} = \frac{\quad_0}{\sum_j \quad_j \quad c} + \frac{\quad_1}{\quad_1 \quad_1} = \frac{\quad_0 \quad_1 + \quad_1 c \; (\quad_0 + \quad_1) \quad_1 \quad_1 \sum_{j \neq 1} \quad_j}{(\sum_j \quad_j \quad c)(\quad_1 \quad_1)}$$

$$\frac{@}{@ \quad_2} = \frac{\quad_0}{\sum_j \quad_j \quad c} + \frac{\quad_2}{\quad_2 \quad_2} = \frac{\quad_0 \quad_2 + \quad_2 c \; (\quad_0 + \quad_2) \quad_2 \quad_2 \sum_{j \neq 2} \quad_j}{(\sum_j \quad_j \quad c)(\quad_2 \quad_2)}$$

$$\vdots$$

$$\frac{@}{@ \quad_i} = \frac{\quad_0}{\sum_j \quad_j \quad c} + \frac{\quad_i}{\quad_i \quad_i} = \frac{\quad_0 \quad_i + \quad_i c \; (\quad_0 + \quad_i) \quad_i \quad_i \sum_{j \neq i} \quad_j}{(\sum_j \quad_j \quad c)(\quad_i \quad_i)}$$

Provided that no denominator is zero (that is $\sum \quad_i \neq c \quad_i \neq \quad_i \forall i \in n$) the critical points of this function are found by solving

$$\begin{bmatrix} {}_0 + {}_1 & {}_1 & {}_1 & \cdots & {}_1 \\ {}_2 & {}_0 + {}_2 & {}_2 & \cdots & {}_2 \\ & & \ddots & & \\ {}_n & {}_n & {}_n & \cdots & {}_0 + {}_n \end{bmatrix} \begin{bmatrix} {}_1 \\ {}_2 \\ \vdots \\ {}_n \end{bmatrix} = \begin{bmatrix} {}_0\;{}_1 + {}_1 c \\ {}_0\;{}_2 + {}_2 c \\ \vdots \\ {}_0\;{}_n + {}_n c \end{bmatrix}$$

C Solution of equation (36)

In this section I will discuss the solution of the rst equation, the second equation is solved in a similar manner. I applied Newton iterative method described in [20]. The equality can be rewritter as

$$f(A_{iL}) = B_{iH} z_i^{A_{iL}} \quad B_{iH}(z_i \quad {}_{i(S[i])})^{A_{iL}} \quad {}_i \; {}^{i}_{i(S[i])}$$

an I am interested in nding the solution $f(A_{iL}) = 0$. The derivative of this function is continuous as is computed as

$$f^0(A_{iL}) = B_{iH} z_i^{A_{iL}} \ln(Z_i) \quad B_{iH}(z_i \quad {}_{i(S[i])})^{A_{iL}} \ln(z_i \quad {}_{i(S[i])})$$

Starting from the initial guess $A_{iL}^0 := 0{:}5(A_{iL} + A_{iH})$ the solution is nd iterating, that is

$$A_{iL}^{n+1} = A_{iL}^n \quad \frac{f(A_{iL}^n)}{f^0(A_{iL}^n)}$$

The process stopp after a set number of iteration of when the error $jA_{iL}^{n+1} \; A_{iL}^n j$ is smaller then the desired precision . In this thesis I used $n_{max} = 10000$ and $= 0{:}001$

D Maximum expected profit greedy search

Given that it is not possible to nd an analytical expression of the probability of success it is not possible to use the optimization method used in section 7. I can not compute the gradient and use it to nd the critical pints where $r R = 0$.

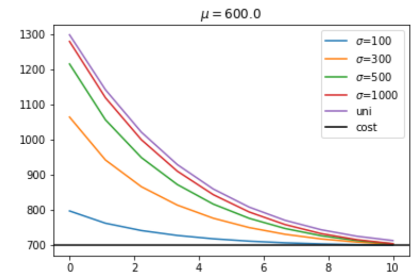
One way to nd the maximum expected pro t is to divide the support of both A_i and B_i in a given number of step n. This will give a grid with n^2 pair of coordinate and it is possible to nd the maximum with brute force, computing the expected pro t using every point in the grid as an bid $(\ ; \ )$. This is guaranteed to nd at least one maximum value but is computationally expensive. It is possible to reduce the number of grid value to evaluate using the fact that, for a xed there is only one maximum value $(\ )$. This means that, if the expected value $E[R(\ ; \ _{k+1})] < E[R(\ ; \ _k)]$ the o er $(\ ; \ _k)$ is the best possible for the given . Therefore if the expected value decline it is possible to stop iterate over and move to the next gridpoint on . the pseudocode code for this optimization algorithm is written below

```
#initialize variables
alpha=A_min
beta=B_min
step_A=(A_max-A_min)/n
step_B=(B_max-B_min)/n

#compute fictitious management cost
c=lam*min_mc+(1-lam)max_mc

while alpha<=A_max:
    max_beta_profit=-M
    while beta<=B_max:
        %compute the probability for the offer alpha beta,
        %given the parameter of the network
        p=compute_probability(alpha,beta,z,epsilon)
        expected_profit=p*beta*epsilon^alpha
        if expected_profit>max_beta_profit:
            max_beta_profit=expected_profit
            beta=beta+step_B
        else:
            %save current coording and maximum expected profit
            %of the maxium given alpha
            coord_maxes.append([alpha,beta],expected_profit)
            break
      alpha=alpha+step_A

bid=argmax(coord_maxes)
```

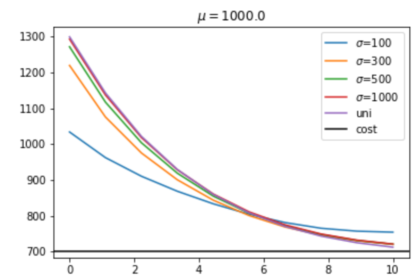
This algorithm iterate over all the discrete values of but iterates over the value of only as long that the expected value improves. After it nds the maximum of a given it stores the coordinates of this value and the value itself. Once it nishes the iteration overs the discrete value of there is a direct collection of n possible maximum points. The optimal greedy bid is the pair of coordinate that have the maximum expected value among the n candidates.

E Graph

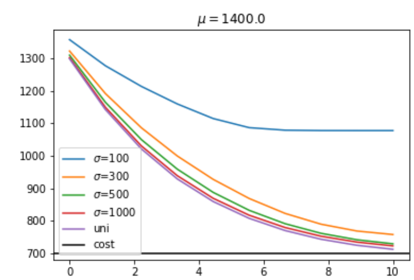
In this section are reported all the graphs that were to large to be included in the body of the thesis.



(a) Low



(b) Mid



(c) High

Figure 11: These graphs show the optimal bidding sequences for di erent values of uncertainty and how they relate to the results obtained for the uniform distributions.

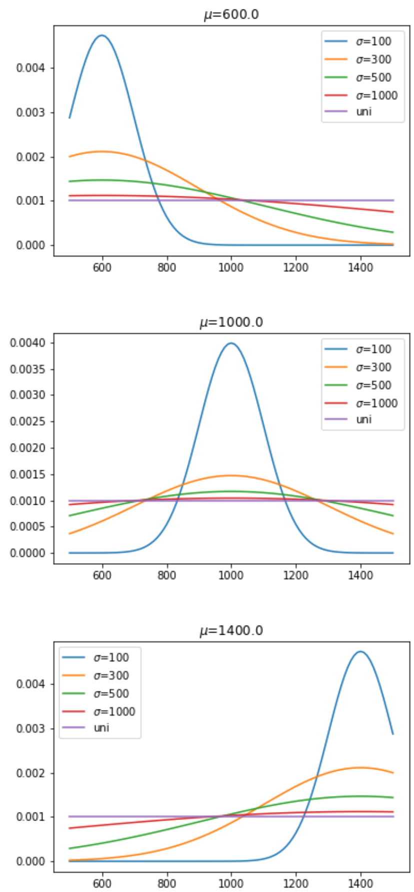


Figure 12: Comparison of truncated normal distributions with the uniform distribution on the same support

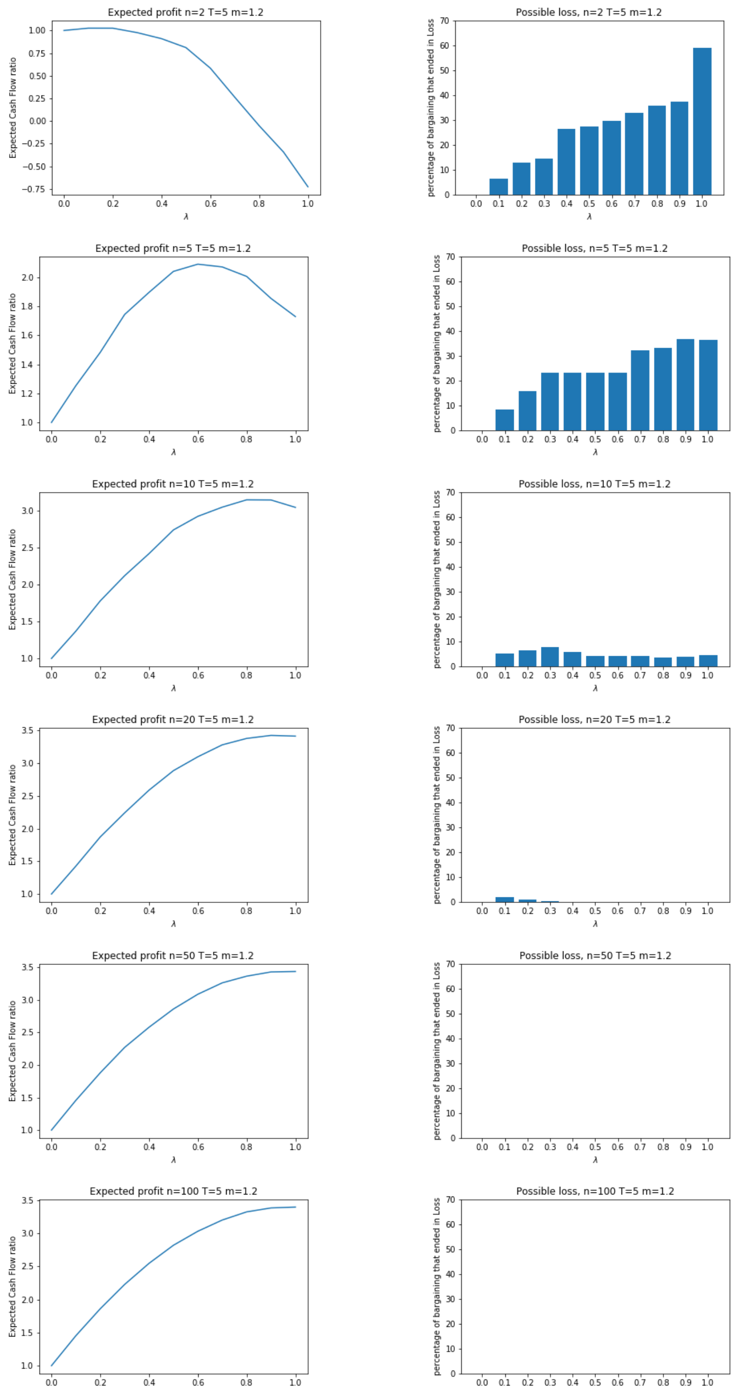


Figure 13: Strategy performance for different number of fund and weigth λ, with $_h = 1.2MC_i$. The results are expressed as a multiple of the expected profit obtained when $\lambda = 0$

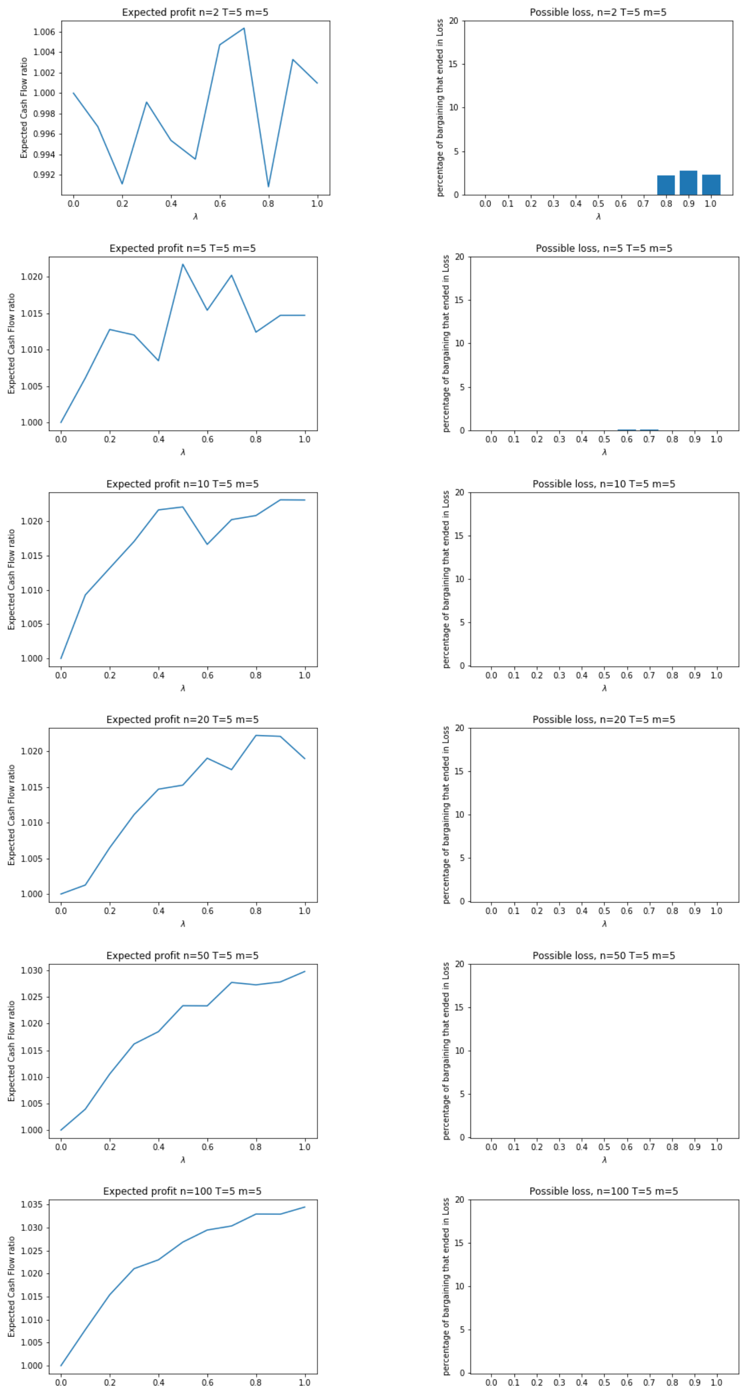


Figure 14: Strategy performance for di erent number of fund and weigth ,
with $_h = 5MC_i$. The results are expressed as a multiple of the expected pro t
obtained when = 0

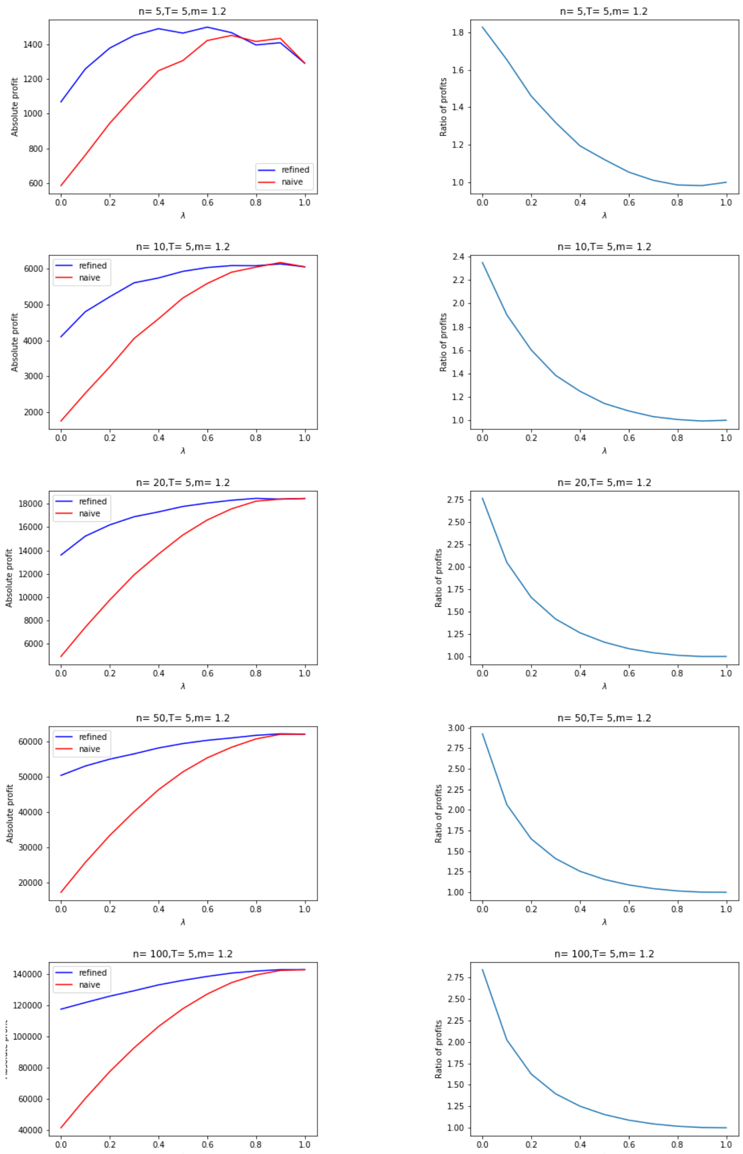


Figure 15: Strategy performance for di erent number of fund and weigth , with $_h = 5MC_i$

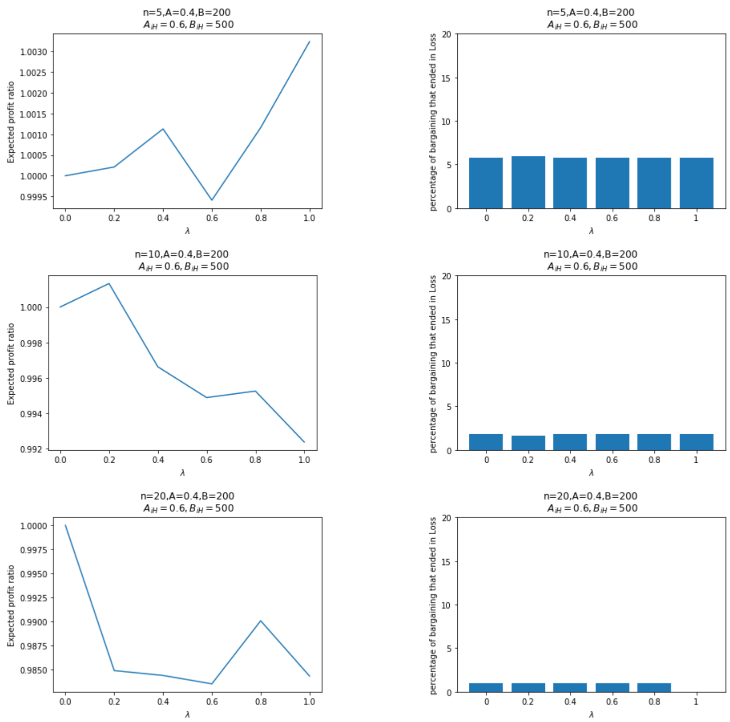


Figure 16: Results of numerical tests when the vendor is signi cantly e cient compared to the buyers. The results are expressed as a multiple of the expected pro t obtained when = 0

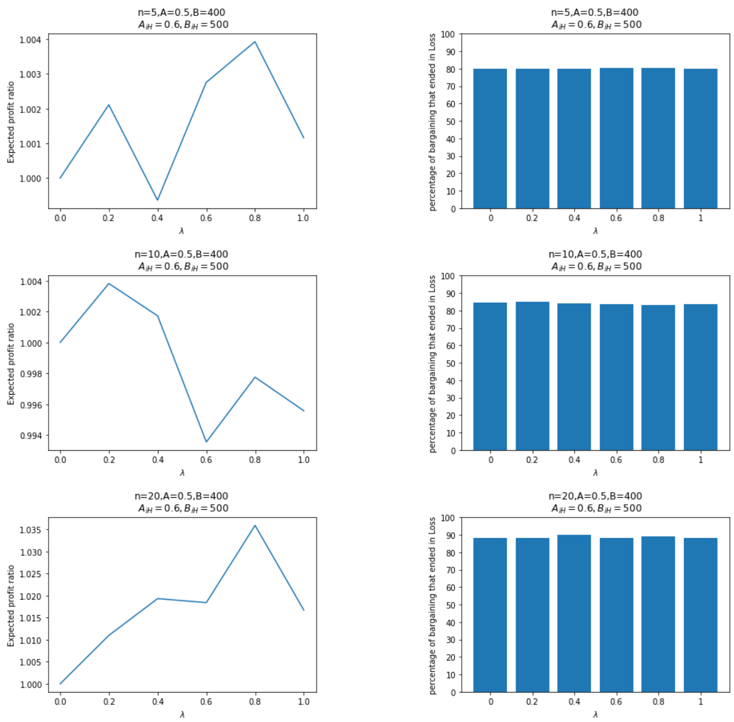


Figure 17: Results of numerical tests when the vendor is NOT significantly efficient compared to the buyers. The results are expressed as a multiple of the expected profit obtained when $\lambda = 0$