BSc Thesis Applied Mathematics

A study on backstepping boundary control for two classes of linear port-Hamiltonian systems

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January 22, 2020

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Preface

This paper has been written as a Bachelor thesis for the study Applied Mathematics at the University of Twente. I've enjoyed studying Applied Mathematics at the University of Twente immensely and I am proud of this final product.

First, I would like to thank prof. dr. H.J. Zwart for his help in finding a suitable bachelor project and for supervising it. This project fit my preferences beautifully and was challenging and rewarding to do.

Second, I would like to thank Lilian Spijker for her support throughout my bachelor. I hope she will help many more students obtain their degree.

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January 22, 2020

Abstract

For two classes of linear port-Hamiltonian systems with constant parameters backstepping boundary control is investigated. An exponentially stable port-Hamiltonian system with homogeneous boundary conditions is for both classes the target system and the goal is to use linear multiplicative coordinate transformations. For the first class a coordinate transformation based on a multiplicative operator suffices and the condition for its existence is algebraic. Using this a boundary controller is constructed. For the second class it shown that a multiplication mapping does not suffice.

Keywords: Backstepping, Boundary control, Linear port-Hamiltonian systems, Multiplicative coordinate transformation.

1 Introduction

Many physical systems described by partial differential equations (PDE's) are controlled through their boundaries. To stabilize these systems, backstepping boundary control has been successfully applied to, among other examples, the wave equation, the slender Timoshenko beam and the linearized Saint-Venant–Exner model [1], [4], [5], [7]. The construction of a stabilizing controller has been done by mapping the controlled system to an exponentially stable target system. To achieve this mapping from the PDE into a target PDE, an invertible Volterra integral coordinate transformation has to be found. The finding of such a Voltera integral mapping has been proven to be quite difficult and is usually done on specific cases for different classes of PDE's.

In the study of physical systems the theory of port-Hamiltonian systems has been developed and due to the advantages of this framework research has focused on the control of port-Hamiltonian systems [9]. A backstepping approach has been used to develop a boundary controller for a linear port-Hamiltonian system with constant parameters [6]. Again an exponentially stable target system is used but a linear multiplicative coordinate transformation suffices to achieve an invertible mapping. The condition for the existence of this transformation is an algebraic equation depending on the boundaries. Hence this transformation is simpler than the Volterra integral mapping and still allows the computation of the desired boundary controller.

The objective of this paper is to extend the use of linear multiplicative coordinate transformations to obtain stabilizing boundary feedback for port-Hamiltonian systems. To achieve this the linear port-Hamiltonian systems studied by H. Ramirez et al. [6] will be expanded with (anti)damping terms, leading to two classes of linear port-Hamiltonian systems with constant parameters to be studied. Both classes will have an exponentially stable target system and a general class of coordinate transformations is considered. For the first class of port-Hamiltonian systems it is deduced that a linear multiplicative coordinate transformation suffices to achieve an invertible mapping and obtain a boundary controller. The condition for the existence of this transformation is an algebraic equation on the boundaries. For the second class of port-Hamiltonian systems it is deduced that a linear multiplicative coordinate transformation cannot lead to the desired result. Instead, a short study on the mapping shows that a Volterra integral coordinate transformation is likely to be necessary.

This paper is organized as follows. In Chapter 2 the first class of linear port-Hamiltonian systems is introduced, after which in Section 2.1 the backstepping boundary controller is presented. In Chapter 3 the second class of linear port-Hamiltonian systems is introduced, after which in Section 3.1 it is shown that a multiplication mapping cannot give the desired result. In Section 3.2 the form of the mapping is further studied. Finally, in Chapter 4 are some concluding remarks and future directions.

2 First class of port-Hamiltonian systems

Let $M_n(\mathbb{R})$ denote the space of real $n \times n$ matrices and let $M_n(\mathbb{C})$ denote the space of complex $n \times n$ matrices. The first class of port-Hamiltonian systems under study is described by the following partial differential equation (PDE):

$$\frac{\partial \tilde{x}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\tilde{x}(t,\zeta)) + P_0 \tilde{x}(t,\zeta), \tag{1}$$

$\zeta \in (0,1)$, where $P_1 \in M_n(\mathbb{R})$ is non-singular and symmetric, $P_0 \in M_n(\mathbb{R})$ and $\tilde{x}$ takes values in $\mathbb{R}^n$. The variable t represents time. Furthermore, P_1 and P_0 are simultaneously diagonalizable.

There are homogeneous and controlled boundary conditions. That is, there are matrices of appropriate sizes such that:

$$u(t) = \widetilde{W}_{B,1} \begin{bmatrix} \tilde{x}(t,1) \\ \tilde{x}(t,0) \end{bmatrix} \tag{2}$$

and

$$0 = \widetilde{W}_{B,2} \begin{bmatrix} \tilde{x}(t,1) \\ \tilde{x}(t,0) \end{bmatrix}. \tag{3}$$

Here (2) represents the controlled boundary conditions, thus $u(t)$ is the control, while (3) represents the homogeneous boundary conditions.

Define the state space X as $X = L_2((0,1);\mathbb{R}^n)$ with the standard inner product $\langle \tilde{x}_1, \tilde{x}_2 \rangle$ and norm $||\tilde{x}||^2 = \langle \tilde{x}, \tilde{x} \rangle$. The Sobolev space of order k is denoted by $H^k((0,1),\mathbb{R}^n)$, which is the subspace of the state space X where the elements $\tilde{x}$ and their derivatives up to order k have a finite norm.

Associated with the homogenous PDE we define the operator $E\tilde{x} = P_1 \frac{d}{d\zeta}(\tilde{x})$ with domain

$$D(E) = \left\{ \tilde{x} \in H^1((0,1),\mathbb{R}^n) \mid \begin{bmatrix} \tilde{x}(1) \\ \tilde{x}(0) \end{bmatrix} \in \ker \widetilde{W}_B \right\}, \tag{4}$$

where $\widetilde{W}_B = \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix}$.

Assumption 2.1. *We assume that for the operator E the following hold:*

1. *The matrix $\widetilde{W}_B$ is an $n \times 2n$ matrix of full rank.*
2. *For $\tilde{x}_0 \in D(E)$ we have $\langle E\tilde{x}_0, \tilde{x}_0 \rangle \leq 0$.*

The first assumption is to ensure that all boundary conditions are known. The second assumption is made to ensure that the target system will be exponentially stable.

2.1 The Backstepping Controller

Now a stabilizing controller for the port-Hamiltonian system (1) will be constructed. For this the port-Hamiltonian system will be written into a different form, after which it is mapped onto an exponentially stable target system. Then a coordinate transformation will be derived based on a multiplicative operator.

Lemma 2.1. *The linear port-Hamiltonian system (1) can be rewritten into the form*

$$\frac{\partial x(t,\zeta)}{\partial t} = \Lambda \frac{\partial}{\partial \zeta}(x(t,\zeta)) + Mx(t,\zeta), \tag{5}$$

$\zeta \in (0,1)$, *where* $\Lambda = diag\,(\lambda_i) \in M_n(\mathbb{R})$ *with non-zero elements on the diagonal and* $M = diag\,(\mu_i) \in M_n(\mathbb{C})$. *The boundary conditions are*

$$u(t) = W_{B,1} \begin{bmatrix} x(t,1) \\ x(t,0) \end{bmatrix} \tag{6}$$

and

$$0 = W_{B,2} \begin{bmatrix} x(t,1) \\ x(t,0) \end{bmatrix}. \tag{7}$$

Proof. Since P_1 is a real, symmetric, nonsingular matrix, there exists a matrix $J \in M_n(\mathbb{C})$ such that

$$\Lambda = J^{-1} P_1 J = J^T P_1 J \tag{8}$$

where Λ is diagonal with non-zero elements on the diagonal [8]. The columns of J consists of the eigenvectors of P_1 and the eigenvalues of P_1 form the diagonal of Λ [2]. Since P_1 is a real symmetric matrix, all eigenvalues are real, thus $\Lambda = \text{diag}\,(\lambda_i) \in M_n(\mathbb{R})$ [8]. Notice that the matrix J need not be unique, for example this matrix is not unique when P_1 equals the identity matrix.

It is given that P_1 and P_0 are simultaneously diagonalizable. Thus the matrix J can be chosen such that

$$M = J^{-1} P_0 J \tag{9}$$

is also diagonal. Let $M = \text{diag}\,(\mu_i) \in M_n(\mathbb{C})$.

Defining $x = J^{-1}\tilde{x}$ and using (8) and (9) we can rewrite (1) to (5). The boundary conditions also become of the desired form with $W_{B,1} = \widetilde{W}_{B,1} \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$ and $W_{B,2} = \widetilde{W}_{B,2} \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$. □

Theorem 2.1. *Beside the notation as introduced above, we define for* $c \in \mathbb{R}, c > 0$

$$\Upsilon(\zeta) = \begin{bmatrix} e^{\left(\frac{c+\mu_1}{\lambda_1}\right)\zeta} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\left(\frac{c+\mu_n}{\lambda_n}\right)\zeta} \end{bmatrix}. \tag{10}$$

If there exists an invertible, diagonal matrix $A \in M_n(\mathbb{C})$ *such that or all* $p, q \in \mathbb{R}^n$ *satisfying*

$$\begin{bmatrix} p \\ q \end{bmatrix} \in \ker(\widetilde{W}_{B,2}) \tag{11}$$

we have that

$$\begin{bmatrix} JA\Upsilon(1)J^{-1}p \\ JAJ^{-1}q \end{bmatrix} \in \ker \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix}, \tag{12}$$

then the boundary controller

$$u(t) = -\widetilde{W}_{B,1} \begin{bmatrix} J(A\Upsilon(1) - I)J^{-1}\widetilde{x}(t,1) \\ J(A-I)J^{-1}\widetilde{x}(t,0) \end{bmatrix} \tag{13}$$

stabilizes the port-Hamiltonian system (1). Furthermore, there exists a (positive) constant m_0 such that solutions of the closed loop system satisfy

$$||\widetilde{x}(t)|| \le m_0 e^{-ct}||\widetilde{x}(0)||. \tag{14}$$

Proof. In order to prove that the given boundary controller exponentially stabilizes the port-Hamiltonian system (1), it will be mapped onto the following target system

$$\frac{\partial \widetilde{z}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\widetilde{z}(t,\zeta)) - cI\widetilde{z}(t,\zeta), \tag{15}$$

where $c \in \mathbb{R}, c > 0$ and with homogeneous boundary conditions

$$0 = \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix} \begin{bmatrix} \widetilde{z}(t,1) \\ \widetilde{z}(t,0) \end{bmatrix}. \tag{16}$$

Using the assumptions, it can be proven that the target system is exponentially stable, which has been done in Appendix A.1. Furthermore, for smooth initial conditions

$$||\widetilde{z}(t)|| \le e^{-ct}||\widetilde{z}(0)||. \tag{17}$$

Defining $z = J^{-1}\widetilde{z}$ and using (8) the target system can equivalently be formulated as the diagonal target system

$$\frac{\partial z}{\partial t} = \Lambda \frac{\partial}{\partial \zeta}(z) - cIz, \tag{18}$$

with homogeneous boundary conditions

$$0 = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} z(t,1) \\ z(t,0) \end{bmatrix}. \tag{19}$$

Now the rewritten port-Hamiltonian system (5) will be mapped to the rewritten target system (18). Consider the coordinate transformation

$$z(t,\zeta) = x(t,\zeta) + Q(x(t,\zeta)), \tag{20}$$

where Q is a bounded linear mapping from $L_2((0,1);\mathbb{R}^n)$ to $L_2((0,1);\mathbb{R}^n)$ independent of t. By taking the partial derivative with respect to time one obtains

$$\frac{\partial z}{\partial t} = \frac{\partial x}{\partial t} + Q\left(\frac{\partial x}{\partial t}\right) = \Lambda \frac{\partial x}{\partial \zeta} + Mx + Q\left(\Lambda \frac{\partial x}{\partial \zeta}\right) + Q(Mx). \tag{21}$$

On the other hand

$$\frac{\partial z}{\partial t} = \Lambda \frac{\partial z}{\partial \zeta} - cz = \Lambda \frac{\partial x}{\partial \zeta} + \Lambda \frac{\partial Q(x)}{\partial \zeta} - cx - cQ(x). \tag{22}$$

Thus the following equation should be satisfied

$$\Lambda\frac{\partial Q(x)}{\partial\zeta} = Q\left(\Lambda\frac{\partial x}{\partial\zeta} + (c+M)x\right) + (c+M)x. \tag{23}$$

In order to solve the equation above, we first try to solve it for special x of the form $x(\zeta) = e^{s\zeta}v$ with $s \in \mathbb{C}, v \in \mathbb{C}^n$. Thus one obtains

$$\Lambda\frac{\partial Q(e^{s\zeta}v)}{\partial\zeta} = Q\left((s\Lambda + c + M)e^{s\zeta}v\right) + (c+M)e^{s\zeta}v. \tag{24}$$

Consider the particular v given by $v_1 = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}^T$, then the equation above becomes

$$\Lambda\frac{\partial Q(e^{s\zeta}v_1)}{\partial\zeta} = Q\left((s\lambda_1 + c + \mu_1)e^{s\zeta}v_1\right) + (c+\mu_1)e^{s\zeta}v_1. \tag{25}$$

To compute $Q(e^{s\zeta}v_1)$ we write

$$Q(e^{s\zeta}v_1) = \begin{bmatrix} q_{s,1}(\zeta) \\ p_{s,1}(\zeta) \\ \vdots \\ p_{s,n-1}(\zeta) \end{bmatrix}. \tag{26}$$

Using this notation the differential equation (25) gives for the top element of $Q(e^{s\zeta}v_1)$:

$$\lambda_1\frac{\partial q_{s,1}(\zeta)}{\partial\zeta} = (s\lambda_1 + c + \mu_1)q_{s,1}(\zeta) + (c+\mu_1)e^{s\zeta}, \tag{27}$$

which is equivalent to

$$\frac{\partial q_{s,1}(\zeta)}{\partial\zeta} = \left(s + \frac{c+\mu_1}{\lambda_1}\right)q_{s,1}(\zeta) + \frac{c+\mu_1}{\lambda_1}e^{s\zeta}, \tag{28}$$

and thus implies that

$$q_{s,1}(\zeta) = \alpha_{s,1}e^{\left(s+\frac{c+\mu_1}{\lambda_1}\right)\zeta} - e^{s\zeta}, \tag{29}$$

with $\alpha_{s,1} \in \mathbb{C}$.

Equation (25) implies that the second element of $Q(e^{s\zeta}v_1)$ satisfies

$$\lambda_2\frac{\partial p_{s,1}(\zeta)}{\partial\zeta} = (s\lambda_1 + c + \mu_1)p_{s,1}(\zeta) \tag{30}$$

which admits as solution $p_{s,1} = 0$. By (25) the following elements of $Q(e^{s\zeta}v_1)$ admit the same solution, thus

$$Q(e^{s\zeta}v_1) = \begin{bmatrix} q_{s,1}(\zeta) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{31}$$

We can argue similarly for the other basis vectors of $\mathbb{R}^n$. Thus, using the linearity of Q, we obtain

$$Q(e^{s\zeta}v) = \begin{bmatrix} q_{s,1}(\zeta) & 0 & \dots & 0 \\ 0 & q_{s,2}(\zeta) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & q_{s,n}(\zeta) \end{bmatrix} v \tag{32}$$

with $q_{s,i}(\zeta) = \alpha_{s,i} e^{\left(s+\frac{c+\mu_i}{\lambda_i}\right)\zeta} - e^{s\zeta}$ for $i = 1,\dots,n$. Since Q is a bounded linear mapping from $L_2((0,1);\mathbb{R}^n)$ to $L_2((0,1);\mathbb{R}^n)$, it is necessary that the mapping does not depend on s. Thus we impose that $\alpha_{s,i} \equiv \alpha_i \in \mathbb{C}$ and it follows that

$$Q(e^{s\zeta}v) = \begin{bmatrix} \alpha_1 e^{\left(\frac{c+\mu_1}{\lambda_1}\right)\zeta} - 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_n e^{\left(\frac{c+\mu_n}{\lambda_n}\right)\zeta} - 1 \end{bmatrix} e^{s\zeta} v. \tag{33}$$

So for $x(\zeta) = e^{s\zeta} v$ we see that

$$Q(x(\zeta)) = \operatorname{diag}\left(\alpha_i e^{\left(\frac{c+\mu_i}{\lambda_i}\right)\zeta} - 1\right) x(\zeta). \tag{34}$$

Since Q is linear and the subspace spanned by exponential functions lies dense in $L_2((0,1);\mathbb{R}^n)$, Q must be a multiplicative operator. This has been proven in Appendix A.2. Thus the coordinate transformation becomes

$$z(t,\zeta) = Y(\zeta)x(t,\zeta), \tag{35}$$

with

$$Y(\zeta) = \begin{bmatrix} \alpha_1 e^{\left(\frac{c+\mu_1}{\lambda_1}\right)\zeta} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_n e^{\left(\frac{c+\mu_n}{\lambda_n}\right)\zeta} \end{bmatrix}. \tag{36}$$

It remains to show that $Y(\zeta)$ is invertible and that the boundary conditions can be satisfied. For this we write Y as

$$Y(\zeta) = A\Upsilon(\zeta) \tag{37}$$

with $A = \operatorname{diag}(\alpha_i)$ and

$$\Upsilon(\zeta) = \begin{bmatrix} e^{\left(\frac{c+\mu_1}{\lambda_1}\right)\zeta} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\left(\frac{c+\mu_n}{\lambda_n}\right)\zeta} \end{bmatrix}. \tag{38}$$

Now Y is invertible if and only if all alpha's are nonzero or equivalently A is invertible. The boundary conditions of the target system are given by

$$0 = \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix} \begin{bmatrix} \widetilde{z}(1) \\ \widetilde{z}(0) \end{bmatrix}. \tag{39}$$

Recalling that $\tilde{z} = Jz$, $z = A\Upsilon x$ and $x = J^{-1}\tilde{x}$, we can rewrite the above as

$$0 = \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix} \begin{bmatrix} JA\Upsilon(1)J^{-1}\tilde{x}(1) \\ JAJ^{-1}\tilde{x}(0) \end{bmatrix}. \tag{40}$$

By the assumption in Theorem 2.1, we can transform, boundedly and invertible, our system into the target system. From the first line of (40) we have that

$$0 = \widetilde{W}_{B,1} \begin{bmatrix} JA\Upsilon(1)J^{-1}\tilde{x}(t,1) \\ JAJ^{-1}\tilde{x}(t,0) \end{bmatrix} \tag{41}$$

$$= \quad \widetilde{W}_{B,1} \begin{bmatrix} \tilde{x}(t,1) \\ x(t,0) \end{bmatrix} + \widetilde{W}_{B,1} \begin{bmatrix} J(A\Upsilon(1)-I)J^{-1}\tilde{x}(t,1) \\ J(A-I)J^{-1}\tilde{x}(t,0) \end{bmatrix} \tag{42}$$

$$= \quad u(t) + \widetilde{W}_{B,1} \begin{bmatrix} J(A\Upsilon(1)-I)J^{-1}\tilde{x}(t,1) \\ J(A-I)J^{-1}\tilde{x}(t,0) \end{bmatrix} \tag{43}$$

from which the stabilizing input follows. Since only invertible state transformations were used between the port-Hamiltonian systems (1) and the target systems (15), the closed loop has the same decay rate as the target systems. Thus the theorem has been proven. □

By Theorem 2.1, the port-Hamiltonian system (1) can be exponentially stabilized if the conditions in the theorem are met. Notice that if the port-Hamiltonian system (1) has n controls, then there are only controlled boundary conditions and the homogeneous boundary conditions (3) are absent. It follows that in this case the conditions in the theorem are satisfied, thus such a system is exponentially stabilizable.

Unfortunately, many linear port-Hamiltonian systems are not covered by Theorem 2.1. Consider for example the linear port-Hamiltonian system for the uniform Timoshenko beam where the constant coefficients have been chosen equal to 1. This example is described by the following PDE [3]:

$$\frac{\partial x(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(x(t,\zeta)) + P_0 x(t,\zeta) \tag{44}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} x(t,\zeta) + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t,\zeta). \tag{45}$$

Since P_1 and P_0 do not commute, that is $P_0 P_1 \neq P_1 P_0$, they are not simultaneously diagonalizable [2]. Therefore Theorem 2.1 cannot be applied. Notice though that P_0 is skew symmetric. The general class of linear port-Hamiltonian systems where P_0 is skew symmetric will be studied in Chapter 3.

3 Second class of port-Hamiltonian systems

For the second class of port-Hamiltonian systems we will focus on the two dimensional case. The port-Hamiltonian systems under study are described by the following partial differential equation (PDE):

$$\frac{\partial \tilde{x}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\tilde{x}(t,\zeta)) + P_0 \tilde{x}(t,\zeta), \tag{46}$$

$\zeta \in (0,1)$, where $P_1 \in M_2(\mathbb{R})$ is non-singular and symmetric, $P_0 \in M_2(\mathbb{R})$ is skew symmetric and $\tilde{x}$ takes values in $\mathbb{R}^2$. Since P_0 is skew symmetric, by definition $P_0^T = -P_0$. The boundary conditions and the state space are defined the same as for the first class of port-Hamiltonian systems (1). Also the operator E and the corresponding Assumption 2.1 are defined the same.

3.1 The absence of a multiplication mapping

By trying to map the port-Hamiltonian system (46) to a generalized form of exponentially stable target systems via a multiplicative operator, it will be shown that a multiplication mapping is not possible. Thus constructing a stabilizing controller of the form found in Chapter 2 is not possible.

Lemma 3.1. *The linear port-Hamiltonian system (46) can be rewritten into the form*

$$\frac{\partial x(t,\zeta)}{\partial t} = \Lambda \frac{\partial}{\partial \zeta}(x(t,\zeta)) + Mx(t,\zeta), \tag{47}$$

$\zeta \in (0,1)$, *where* $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \in M_2(\mathbb{R})$ *with non-zero elements on the diagonal and* $M = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} \in M_2(\mathbb{R})$. *The boundary conditions are*

$$u(t) = W_{B,1} \begin{bmatrix} x(t,1) \\ x(t,0) \end{bmatrix} \tag{48}$$

and

$$0 = W_{B,2} \begin{bmatrix} x(t,1) \\ x(t,0) \end{bmatrix}. \tag{49}$$

Proof. Since P_1 is is a real, symmetric, nonsingular matrix, (8) holds for a matrix $J \in M_2(\mathbb{R})$ [8]. Since P_0 is skew symmetric, it follows that

$$(J^T P_0 J)^T = J^T P_0^T J = -J^T P_0 J. \tag{50}$$

Thus $J^T P_0 J \in M_2(\mathbb{R})$ also skew symmetric. Since it is a real skew symmetric matrix the entries on the diagonal equal zero. Let $M = J^T P_0 J = J^{-1} P_0 J = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}$.

Defining $x = J^{-1}\tilde{x} = J^T \tilde{x}$ and using (8) and M as above we can rewrite (46) as (47). For the boundary conditions $W_{B,1} = \widetilde{W}_{B,1} \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$ and $W_{B,2} = \widetilde{W}_{B,2} \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$. □

We are not interested in the case when P_1 and P_0 of (46) are simultaneously diagonalizable, since this coincides with the case discussed in Chapter 2. Since P_1 is symmetric and P_0 is skew symmetric, they are both diagonalizable per definition. Here P_0 is only diagonalizable over $\mathbb{C}$, not over $\mathbb{R}$. They are simultaneously diagonalizable if and only if they commute, that is if $P_1 P_0 = P_0 P_1$ [2]. Thus we assume from now on that P_1 and P_0 do not commute. If P_1 and P_0 do not commute, then this is equivalent with that $\Lambda = J^{-1} P_1 J$ and $M = J^{-1} P_0 J$ do not commute. To ensure that Λ and M do not commute, we have to assume from now on that $\lambda_1 \neq \lambda_2$ and $\mu \neq 0$.

To show that a multiplication mapping to an exponentially stable target system is not possible, the chosen target system will be of a generalized form instead of the more specific form (15) used in Chapter 2. The target system is

$$\frac{\partial \tilde{z}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\tilde{z}(t,\zeta)) - P_2 \tilde{z}(t,\zeta), \tag{51}$$

where $P_2 \in M_2(\mathbb{R})$ is such that $x^T P_2 x > 0$ for any $x \in \mathbb{R}^2$ with $x \neq 0$ but not necessarily symmetric. It follows that $P_2 + P_2^T$ is positive definite. The target system has the homogeneous boundary conditions

$$0 = \begin{bmatrix} \widetilde{W}_{B,1} \\ \widetilde{W}_{B,2} \end{bmatrix} \begin{bmatrix} \tilde{z}(t,1) \\ \tilde{z}(t,0) \end{bmatrix}. \tag{52}$$

Using Assumption 2.1, it can proven that the target system is exponentially stable. This proof can be found in Appendix A.1.

Since $x = Jy$ defines a 1-1 correspondence between $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$, we conclude from

$$x^T P_2 x = y^T J^T P_2 J y = y^T P y \tag{53}$$

that $P = J^{-1} P_2 J = J^T P_2 J \in M_2(\mathbb{R})$ also has the property that $x^T P x > 0$ for any $x \in \mathbb{R}^2$ with $x \neq 0$. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Defining $z = J^{-1}\tilde{z} = J^T \tilde{z}$ and using (8) and P as above the target system can equivalently be formulated as the target system

$$\frac{\partial z}{\partial t} = \Lambda \frac{\partial}{\partial \zeta}(z) - Pz, \tag{54}$$

with homogeneous boundary conditions

$$0 = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} z(t,1) \\ z(t,0) \end{bmatrix} \tag{55}$$

Now the rewritten port-Hamiltonian systems (47) will be mapped to the rewritten target system (54). Consider the coordinate transformation

$$z(t,\zeta) = x(t,\zeta) + Q(x(t,\zeta)), \tag{56}$$

where Q is a bounded linear mapping from $L_2((0,1);\mathbb{R}^2)$ to $L_2((0,1);\mathbb{R}^2)$ independent of t. By taking the partial derivative with respect to time one obtains

$$\frac{\partial z}{\partial t} = \frac{\partial x}{\partial t} + Q\left(\frac{\partial x}{\partial t}\right) = \Lambda \frac{\partial x}{\partial \zeta} + Mx + Q\left(\Lambda \frac{\partial x}{\partial \zeta}\right) + Q(Mx). \tag{57}$$

On the other hand

$$\frac{\partial z}{\partial t} = \Lambda \frac{\partial z}{\partial \zeta} - Pz = \Lambda \frac{\partial x}{\partial \zeta} + \Lambda \frac{\partial Q(x)}{\partial \zeta} - Px - PQ(x). \tag{58}$$

Thus the following equation should be satisfied

$$\Lambda \frac{\partial Q(x)}{\partial \zeta} = Q\left(\Lambda \frac{\partial x}{\partial \zeta} + Mx\right) + PQ(x) + (P+M)x. \tag{59}$$

In order to solve the equation above, we first try to solve it for special x of the form $x(\zeta) = e^{s\zeta} v$ with $s \in \mathbb{C}$, $v \in \mathbb{C}^2$. Thus one obtains

$$\Lambda \frac{\partial Q(e^{s\zeta} v)}{\partial \zeta} = Q\Big((s\Lambda + M)e^{s\zeta} v\Big) + PQ(e^{s\zeta} v) + (P+M)e^{s\zeta} v. \tag{60}$$

Consider the particular solutions given by $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and write:

$$Q(e^{s\zeta} v_1) = \begin{bmatrix} f_{s,1}(\zeta) \\ g_{s,1}(\zeta) \end{bmatrix}. \tag{61}$$

$$Q(e^{s\zeta} v_2) = \begin{bmatrix} f_{s,2}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix}. \tag{62}$$

Using this notation it follows that for $x(\zeta) = e^{s\zeta} v_1$

$$\Lambda \frac{\partial Q(e^{s\zeta} v_1)}{\partial \zeta} = Q(s\lambda_1 e^{s\zeta} v_1 - \mu e^{s\zeta} v_2) + PQ(e^{s\zeta} v_1) + ae^{s\zeta} v_1 + (c-\mu)e^{s\zeta} v_2 \tag{63}$$

and thus

$$\lambda_1 \frac{\partial f_{s,1}(\zeta)}{\partial \zeta} = (s\lambda_1 + a) f_{s,1}(\zeta) - \mu f_{s,2}(\zeta) + b g_{s,1}(\zeta) + ae^{s\zeta} \tag{64}$$

and

$$\lambda_2 \frac{\partial g_{s,1}(\zeta)}{\partial \zeta} = (s\lambda_1 + d) g_{s,1}(\zeta) - \mu g_{s,2}(\zeta) + c f_{s,1}(\zeta) + (c-\mu)e^{s\zeta}. \tag{65}$$

Similarly for $x(\zeta) = e^{s\zeta} v_2$

$$\Lambda \frac{\partial Q(e^{s\zeta} v_2)}{\partial \zeta} = Q(s\lambda_2 e^{s\zeta} v_2 + \mu e^{s\zeta} v_1) + PQ(e^{s\zeta} v_2) + (b+\mu)e^{s\zeta} v_1 + de^{s\zeta} v_2 \tag{66}$$

and thus

$$\lambda_1 \frac{\partial f_{s,2}(\zeta)}{\partial \zeta} = (s\lambda_2 + a) f_{s,2}(\zeta) + \mu f_{s,1}(\zeta) + b p_{s,2}(\zeta) + (b+\mu)e^{s\zeta} \tag{67}$$

and

$$\lambda_2 \frac{\partial g_{s,2}(\zeta)}{\partial \zeta} = (s\lambda_2 + d) g_{s,2}(\zeta) + \mu g_{s,1}(\zeta) + c f_{s,2}(\zeta) + de^{s\zeta}. \tag{68}$$

It follows that to find the desired mapping the following inhomogeneous linear system of differential equations of dimension 4 and order 1 has to be solved:

$$\frac{\partial}{\partial \zeta} \begin{bmatrix} f_{s,1}(\zeta) \\ g_{s,1}(\zeta) \\ f_{s,2}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} = \begin{bmatrix} \frac{s\lambda_1+a}{\lambda_1} & \frac{b}{\lambda_1} & \frac{-\mu}{\lambda_1} & 0 \\ \frac{c}{\lambda_2} & \frac{s\lambda_1+d}{\lambda_2} & 0 & \frac{-\mu}{\lambda_2} \\ \frac{\mu}{\lambda_1} & 0 & \frac{s\lambda_2+a}{\lambda_1} & \frac{b}{\lambda_1} \\ 0 & \frac{\mu}{\lambda_2} & \frac{c}{\lambda_2} & \frac{s\lambda_2+d}{\lambda_2} \end{bmatrix} \begin{bmatrix} f_{s,1}(\zeta) \\ g_{s,1}(\zeta) \\ f_{s,2}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} + \begin{bmatrix} \frac{a}{\lambda_1} \\ \frac{c-\mu}{\lambda_2} \\ \frac{b+\mu}{\lambda_1} \\ \frac{d}{\lambda_2} \end{bmatrix} e^{s\zeta} \tag{69}$$

This linear system can be rewritten as follows:

$$\begin{aligned} \frac{\partial q_s(\zeta)}{\partial \zeta} &= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda_1}{\lambda_2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{\lambda_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s + \begin{bmatrix} \frac{a}{\lambda_1} & \frac{b}{\lambda_1} & \frac{-\mu}{\lambda_1} & 0 \\ \frac{c}{\lambda_2} & \frac{d}{\lambda_2} & 0 & \frac{-\mu}{\lambda_2} \\ \frac{\mu}{\lambda_1} & 0 & \frac{a}{\lambda_1} & \frac{b}{\lambda_1} \\ 0 & \frac{\mu}{\lambda_2} & \frac{c}{\lambda_2} & \frac{d}{\lambda_2} \end{bmatrix} \right) q_s(\zeta) + \begin{bmatrix} \frac{a}{\lambda_1} \\ \frac{c-\mu}{\lambda_2} \\ \frac{b+\mu}{\lambda_1} \\ \frac{d}{\lambda_2} \end{bmatrix} e^{s\zeta} && (70) \\ &= (Ks + A) q_s(\zeta) + Be^{s\zeta} && (71) \end{aligned}$$

where $q_s = \begin{bmatrix} f_{s,1} & g_{s,1} & f_{s,2} & g_{s,2} \end{bmatrix}^T$, $K, A \in M_4(\mathbb{R})$ and $B \in \mathbb{R}^4$. We want to know whether there exists a mapping that is of a similar form as (34), so a mapping such that $Q(e^{s\zeta} v) = \Upsilon(\zeta) e^{s\zeta} v$, where $\Upsilon(\zeta)$ is a function in $M_2(\mathbb{C})$ and does not depend on s. This is necessary to find a linear multiplicative coordinate transformation. Thus for the linear system (70) we want a solution of the form $q_s(\zeta) = Y(\zeta)e^{s\zeta}$, with the function $Y(\zeta) = \begin{bmatrix} y_1(\zeta) & y_2(\zeta) & y_3(\zeta) & y_4(\zeta) \end{bmatrix} \in \mathbb{C}^4$ not depending on s. Imposing $q_s(\zeta) = Y(\zeta)e^{s\zeta}$ yields

$$\frac{dY(\zeta)}{d\zeta}e^{s\zeta} + sY(\zeta)e^{s\zeta} = (Ks + A)Y(\zeta)e^{s\zeta} + Be^{s\zeta} \tag{72}$$

thus

$$\frac{dY(\zeta)}{d\zeta} + sY(\zeta) = ((Ks + A)Y(\zeta) + B. \tag{73}$$

Since $Y(\zeta)$ is not allowed to depend on s, it follows that we require that

$$Y(\zeta) = KY(\zeta) \tag{74}$$

and this statement is only true if $\lambda_1 = \lambda_2$ or if $y_2 = y_3 = 0$. If $\lambda_1 = \lambda_2$, then P_1 and P_0 from the port-Hamiltonian system (46) are simultaneously diagonalizable and this case has already been solved in Chapter 2. This is why we have assumed that $\lambda_1 \neq \lambda_2$. If $y_2 = y_3 = 0$, then (73) reduces to

$$\frac{d}{d\zeta}\begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix} = \begin{bmatrix} \frac{a}{\lambda_1} & \frac{b}{\lambda_1} & \frac{-\mu}{\lambda_1} & 0 \\ \frac{c}{\lambda_2} & \frac{d}{\lambda_2} & 0 & \frac{-\mu}{\lambda_2} \\ \frac{\mu}{\lambda_1} & 0 & \frac{a}{\lambda_1} & \frac{b}{\lambda_1} \\ 0 & \frac{\mu}{\lambda_2} & \frac{c}{\lambda_2} & \frac{d}{\lambda_2} \end{bmatrix}\begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix} + \begin{bmatrix} \frac{a}{\lambda_1} \\ \frac{c-\mu}{\lambda_2} \\ \frac{b+\mu}{\lambda_1} \\ \frac{d}{\lambda_2} \end{bmatrix}. \tag{75}$$

Thus there are two differential equations and two regular equations:

$$\frac{dy_1}{d\zeta} = \frac{a}{\lambda_1}y_1 + \frac{a}{\lambda_1} \tag{76}$$

$$\frac{dy_4}{d\zeta} = \frac{d}{\lambda_2}y_4 + \frac{d}{\lambda_2} \tag{77}$$

$$0 = \frac{c}{\lambda_2}y_1 - \frac{\mu}{\lambda_2}y_4 + \frac{c-\mu}{\lambda_2} \tag{78}$$

$$0 = \frac{\mu}{\lambda_1}y_1 + \frac{b}{\lambda_1}y_4 + \frac{b+\mu}{\lambda_1} \tag{79}$$

Remember that the values a, b, c and d from the matrix P are not defined other than the property that $x^T Px > 0$ for any $x \in \mathbb{R}^2$ with $x \neq 0$ must hold. Thus it is possible to take $b = c = 0$, since if $a > 0$ and $d > 0$ that property still holds. Since we have assumed that $\mu \neq 0$ to ensure that P_1 and P_0 are not simultaneously diagonalizable, it follows from (78) and (79) that $y_1 = y_4 = -1$, which also satisfies (76) and (77). Thus

$$Y(\zeta) = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \tag{80}$$

Returning to the bounded linear mapping Q, it follows that for this particular solution

$$Q(e^{s\zeta}v_1) = \begin{bmatrix} f_{s,1}(\zeta) \\ g_{s,1}(\zeta) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{s\zeta} \tag{81}$$

$$Q(e^{s\zeta}v_2) = \begin{bmatrix} f_{s,2}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{s\zeta}. \tag{82}$$

Using the linearity of Q we obtain

$$Q(e^{s\zeta}v) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} e^{s\zeta}v. \tag{83}$$

Since Q is linear and the subspace spanned by exponential functions lies dense in $L_2((0,1);\mathbb{R}^2)$, Q must be a multiplicative operator (see Appendix A.2), thus

$$Q(x(t,\zeta)) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t,\zeta) = -Ix(t,\zeta). \tag{84}$$

The coordinate transformation becomes

$$z(t,\zeta) = x(t,\zeta) + Q(x(t,\zeta)) = x(t,\zeta) - Ix(t,\zeta) = 0. \tag{85}$$

Clearly this is not the desired result, since this does not give a useful mapping to determine the stabilizing input. Thus when mapping the port-Hamiltonian system (46) to the exponentially stable target system (51), it is impossible to find a multiplication mapping.

3.2 The form of the mapping

As we have seen in the previous section, a multiplication mapping from the port-Hamiltonian system (46) to the exponentially stable target system (51) is not possible if P_1 and P_0 do not commute and thus are not simultaneously diagonalizable. This is equivalent with $\lambda_1 \neq \lambda_2$ and $\mu \neq 0$. In this section the mapping will be further investigated. To solve the linear system (69), the Laplace transformation is used. Due to the complexity of this solution, we will focus on the target system where $P_2 = P = aI$ with $a > 0$, thus $a = d > 0$ and $b = c = 0$. For this target system the inhomogeneous linear system (69) reduces to the two following inhomogeneous linear systems of dimension 2 and order 1

$$\frac{\partial}{\partial\zeta}\begin{bmatrix} f_{s,1}(\zeta) \\ f_{s,2}(\zeta) \end{bmatrix} = \begin{bmatrix} \frac{s\lambda_1+a}{\lambda_1} & -\frac{\mu}{\lambda_1} \\ \frac{\mu}{\lambda_1} & \frac{s\lambda_2+a}{\lambda_1} \end{bmatrix} \begin{bmatrix} f_{s,1}(\zeta) \\ f_{s,2}(\zeta) \end{bmatrix} + \begin{bmatrix} \frac{a}{\lambda_1} \\ \frac{\mu}{\lambda_1} \end{bmatrix} e^{s\zeta} \tag{86}$$

and

$$\frac{\partial}{\partial\zeta}\begin{bmatrix} g_{s,1}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} = \begin{bmatrix} \frac{s\lambda_1+a}{\lambda_2} & -\frac{\mu}{\lambda_2} \\ \frac{\mu}{\lambda_2} & \frac{s\lambda_2+a}{\lambda_2} \end{bmatrix} \begin{bmatrix} g_{s,1}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} + \begin{bmatrix} -\frac{\mu}{\lambda_2} \\ \frac{a}{\lambda_2} \end{bmatrix} e^{s\zeta}. \tag{87}$$

To solve the two linear systems, the Laplace transform will be used, transforming the variable ζ to the variable ϵ. Thus the two linear systems (86) and (87) are transformed into

$$\epsilon\begin{bmatrix} F_{s,1}(\epsilon) \\ F_{s,2}(\epsilon) \end{bmatrix} = \begin{bmatrix} \frac{s\lambda_1+a}{\lambda_1} & -\frac{\mu}{\lambda_1} \\ \frac{\mu}{\lambda_1} & \frac{s\lambda_2+a}{\lambda_1} \end{bmatrix} \begin{bmatrix} F_{s,1}(\epsilon) \\ F_{s,2}(\epsilon \end{bmatrix} + \begin{bmatrix} \frac{a}{\lambda_1} \\ \frac{\mu}{\lambda_1} \end{bmatrix} \frac{1}{\epsilon - s} + \begin{bmatrix} f_{s,1}(0) \\ f_{s,2}(0) \end{bmatrix} \tag{88}$$

and

$$\epsilon\begin{bmatrix} G_{s,1}(\epsilon) \\ G_{s,2}(\epsilon) \end{bmatrix} = \begin{bmatrix} \frac{s\lambda_1+a}{\lambda_2} & -\frac{\mu}{\lambda_2} \\ \frac{\mu}{\lambda_2} & \frac{s\lambda_2+a}{\lambda_2} \end{bmatrix} \begin{bmatrix} G_{s,1}(\epsilon) \\ G_{s,2}(\epsilon) \end{bmatrix} + \begin{bmatrix} -\frac{\mu}{\lambda_2} \\ \frac{a}{\lambda_2} \end{bmatrix} \frac{1}{\epsilon - s} + \begin{bmatrix} g_{s,1}(0) \\ g_{s,2}(0) \end{bmatrix} \tag{89}$$

The above can be rewritten into:

$$\begin{bmatrix} F_{s,1}(\epsilon) \\ F_{s,2}(\epsilon) \end{bmatrix} = \frac{1}{(\epsilon - \frac{s\lambda_1+a}{\lambda_1})(\epsilon - \frac{s\lambda_2+a}{\lambda_1}) + \frac{\mu^2}{\lambda_1^2}} \begin{bmatrix} \epsilon - \frac{s\lambda_2+a}{\lambda_1} & -\frac{\mu}{\lambda_1} \\ \frac{\mu}{\lambda_1} & \epsilon - \frac{s\lambda_1+a}{\lambda_1} \end{bmatrix} \begin{bmatrix} \frac{a}{\lambda_1(\epsilon-s)} + f_{s,1}(0) \\ \frac{\mu}{\lambda_1(\epsilon-s)} + f_{s,2}(0) \end{bmatrix} \tag{90}$$

and

$$\begin{bmatrix} G_{s,1}(\epsilon) \\ G_{s,2}(\epsilon) \end{bmatrix} = \frac{1}{(\epsilon - \frac{s\lambda_1+a}{\lambda_2})(\epsilon - \frac{s\lambda_2+a}{\lambda_2}) + \frac{\mu^2}{\lambda_2^2}} \begin{bmatrix} \epsilon - \frac{s\lambda_2+a}{\lambda_2} & -\frac{\mu}{\lambda_2} \\ \frac{\mu}{\lambda_2} & \epsilon - \frac{s\lambda_1+a}{\lambda_2} \end{bmatrix} \begin{bmatrix} -\frac{\mu}{\lambda_2(\epsilon-s)} + g_{s,1}(0) \\ \frac{a}{\lambda_2(\epsilon-s)} + g_{s,2}(0) \end{bmatrix} \tag{91}$$

Solving $(\epsilon - \frac{s\lambda_1+a}{\lambda_1})(\epsilon - \frac{s\lambda_2+a}{\lambda_1}) + \frac{\mu^2}{\lambda_1^2} = 0$ gives:

$$\alpha_+/\alpha_- = \frac{1}{2\lambda_1}\Big(s(\lambda_1+\lambda_2) + 2a \pm \sqrt{s^2(\lambda_1-\lambda_2)^2 - 4\mu^2}\Big). \tag{92}$$

Similarly, solving $(\epsilon - \frac{s\lambda_1+a}{\lambda_2})(\epsilon - \frac{s\lambda_2+a}{\lambda_2}) + \frac{\mu^2}{\lambda_2^2} = 0$ gives:

$$\beta_+/\beta_- = \frac{1}{2\lambda_2}\Big(s(\lambda_1+\lambda_2) + 2a \pm \sqrt{s^2(\lambda_1-\lambda_2)^2 - 4\mu^2}\Big). \tag{93}$$

Notice that α_+, α_-, β_+ and β_- are the eigenvalues of the two square matrices in (86) and (87). Also notice that if $\lambda_1 = \lambda_2$ or $\mu = 0$, these eigenvalues could be further simplified, allowing the existence of a multiplication mapping.

Now, using partial fraction decomposition, after tedious but straightforward calculation the following is obtained:

$$F_{s,1}(\epsilon) = \frac{-1}{\epsilon - s} + \frac{h_{f,1}}{\epsilon - \alpha_+} + \frac{h_{f,2}}{\epsilon - \alpha_-}, \tag{94}$$

$$F_{s,2}(\epsilon) = \frac{h_{f,3}}{\epsilon - \alpha_+} + \frac{h_{f,4}}{\epsilon - \alpha_-}, \tag{95}$$

$$G_{s,1}(\epsilon) = \frac{h_{g,1}}{\epsilon - \beta_+} + \frac{h_{g,2}}{\epsilon - \beta_-}, \tag{96}$$

$$G_{s,2}(\epsilon) = \frac{-1}{\epsilon - s} + \frac{h_{g,3}}{\epsilon - \beta_+} + \frac{h_{g,4}}{\epsilon - \beta_-}, \tag{97}$$

where $h_{f,i}$ depends on $s, f_{s,1}(0)$ and $f_{s,2}(0)$ for $i = 1,2,3,4$ and $h_{g,i}$ depends on $s, g_{s,1}(0)$ and $g_{s,2}(0)$ for $i = 1,2,3,4$.

Using the inverse Laplace transformation, it follows that all possible solutions are of the following form:

$$f_{s,1}(\zeta) = -1e^{s\zeta} + h_{f,1}e^{\alpha_+\zeta} + h_{f,2}e^{\alpha_-\zeta}, \tag{98}$$

$$f_{s,2}(\zeta) = h_{f,3}e^{\alpha_+\zeta} + h_{f,4}e^{\alpha_-\zeta}, \tag{99}$$

$$g_{s,1}(\zeta) = h_{g,1}e^{\beta_+\zeta} + h_{g,2}e^{\beta_-\zeta}, \tag{100}$$

$$g_{s,2}(\zeta) = -1e^{s\zeta} + h_{g,3}e^{\beta_+\zeta} + h_{g,4}e^{\beta_-\zeta}. \tag{101}$$

In this form the parts $-1e^{s\zeta}$ in $f_{s,1}(\zeta)$ and $g_{s,2}(\zeta)$ correspond to the solution if the mapping has to be independent of s, see (81) and (82).

Thus one has to find a mapping which gives results of the form

$$Q(e^{s\zeta}v_1) = \begin{bmatrix} f_{s,1}(\zeta) \\ g_{s,1}(\zeta) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{s\zeta} + \begin{bmatrix} h_{f,1}e^{\alpha_+\zeta} + h_{f,2}e^{\alpha_-\zeta} \\ h_{g,1}e^{\beta_+\zeta} + h_{g,2}e^{\beta_-\zeta} \end{bmatrix} \tag{102}$$

and

$$Q(e^{s\zeta}v_2) = \begin{bmatrix} f_{s,2}(\zeta) \\ g_{s,2}(\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{s\zeta} + \begin{bmatrix} h_{f,3}e^{\alpha_+\zeta} + h_{f,4}e^{\alpha_-\zeta} \\ h_{g,3}e^{\beta_+\zeta} + h_{g,4}e^{\beta_-\zeta} \end{bmatrix}. \tag{103}$$

Such a mapping will probably contain integrals, since we have assumed that $\lambda_1 \neq \lambda_2$ and $\mu \neq 0$, and a Volterra integral coordinate transformation is a good candidate. If a mapping is found, invertibility of the mapping and whether the boundary conditions of the port-Hamiltonian system (46) and the target system (51) can be simultaneously satisfied has to be verified.

4 Conclusion

A backstepping approach has been used to construct a boundary controller for a class of linear port-Hamiltonian systems (1), which has constant parameters and the property that the matrices P_1 and P_0 are simultaneously diagonalizable. To construct this boundary controller, an exponentially stable target system (15) has been used. From this a multiplication mapping has been deduced after diagonalizing both the port-Hamiltonian system and the target system. The condition for the existence of this mapping, and thus of the constructed boundary controller, is an algebraic equation on the boundaries.

A similar approach has been used on a second class of linear port-Hamiltonian systems (46) in order to investigate whether a multiplication mapping could again yield the desired result. This class also has constant parameters, but now P_0 is skew symmetric and it is assumed that P_1 and P_0 are not simultaneously diagonalizable. A more general exponentially stable target system (51) has been used and again the port-Hamiltonian system and the target system have been diagonalized. It has been shown that a multiplication mapping does not yield a coordinate transformation to an exponentially stable target system.

Further work will deal with whether an invertible Volterra integral coordinate transformation can give the desired mapping for the second class of linear port-Hamiltonian systems in order to construct a boundary controller.

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A Appendix

A.1 Exponential stability of the target systems

Consider the system

$$\frac{\partial \tilde{z}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\tilde{z}(t,\zeta)) - cI\tilde{z}(t,\zeta), \tag{104}$$

$\zeta \in (0,1)$, where $P_1 \in M_n(\mathbb{R})$ is non-singular and symmetric, $\tilde{z}$ takes values in $\mathbb{R}^n$ and $c \in \mathbb{R}, c > 0$. It has homogeneous boundary conditions

$$0 = \widetilde{W}_B \begin{bmatrix} \tilde{z}(t,1) \\ \tilde{z}(t,0) \end{bmatrix} \tag{105}$$

where the matrix $\widetilde{W}_B$ is an $n \times 2n$ matrix of full rank.

Define the state space X as $X = L_2((0,1);\mathbb{R}^n)$ with the standard inner product $\langle \tilde{x}_1, \tilde{x}_2 \rangle$. The Sobolev space of order k is denoted by $H^k((0,1),\mathbb{R}^n)$. We define the operator $E\tilde{x} = P_1 \frac{d}{d\zeta}(\tilde{x})$ with domain

$$D(E) = \left\{ \tilde{x} \in H^1((0,1),\mathbb{R}^n) \mid \begin{bmatrix} \tilde{x}(1) \\ \tilde{x}(0). \end{bmatrix} \in \ker \widetilde{W}_B \right\}. \tag{106}$$

We will only consider strong or classical solutions of the system (104), thus $\tilde{z} \in D(E)$ for all $\tilde{z}$.

Theorem A.1. *If $\langle E\tilde{z}, \tilde{z} \rangle \le 0$ for $\tilde{z} \in D(E)$, then system (104) is globally exponentially stable.*

Proof. Consider the following candidate Lyapunov function, which is continuously differentiable and positive definite:

$$V(t) = \frac{1}{2}\int_0^1 \tilde{z}(t,\zeta)^2 d\zeta = \frac{1}{2}||\tilde{z}(t)||^2. \tag{107}$$

For $\tilde{z} \in D(E)$ the time derivative of V is

$$\frac{dV}{dt} = \frac{1}{2}\int_0^1 2\tilde{z}^T \frac{\partial \tilde{z}}{\partial t} d\zeta \tag{108}$$

$$= \int_0^1 \tilde{z}^T \Big(P_1 \frac{\partial \tilde{z}}{\partial \zeta} - cI\tilde{z}\Big) d\zeta \tag{109}$$

$$= \int_0^1 \tilde{z}^T P_1 \frac{\partial \tilde{z}}{\partial \zeta} d\zeta - c\int_0^1 \tilde{z}^2 d\zeta \tag{110}$$

$$= \langle E\tilde{z}, \tilde{z} \rangle - c||\tilde{z}(t)||^2 \tag{111}$$

$$\le -c||\tilde{z}(t)||^2 = -2cV < 0. \qquad \text{if } \tilde{z} \neq 0 \tag{112}$$

Thus $\frac{dV}{dt}$ is globally negative definite and V is a Lyapunov function. Furthermore we have that

$$\frac{dV}{dt} \le -2cV \tag{113}$$

As a consequence of Grönwalls lemma

$$V(t) \le e^{-2ct} V(0) \tag{114}$$

which is equivalent to

$$||\tilde{z}(t)||^2 \le e^{-2ct} ||\tilde{z}(0)||^2 \tag{115}$$

thus

$$||\tilde{z}(t)|| \le e^{-ct} ||\tilde{z}(0)||. \tag{116}$$

and the system is globally exponentially stable. □

Now consider the system

$$\frac{\partial \tilde{z}(t,\zeta)}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\tilde{z}(t,\zeta)) - P_2 \tilde{z}(t,\zeta), \tag{117}$$

where $P_2 \in M_2(\mathbb{R})$ is such that $x^T P_2 x > 0$ for any $x \in \mathbb{R}^2$ with $x \neq 0$ and otherwise the same as system (104).

Theorem A.2. *If $\langle E\tilde{z}, \tilde{z}\rangle \leq 0$ for $\tilde{z} \in D(E)$, then system (117) is globally exponentially stable.*

Proof. Consider the following candidate Lyapunov function, which is continuously differentiable and is positive definite:

$$V(t) = \frac{1}{2}\int_0^1 \tilde{z}(t,\zeta)^2 d\zeta = \frac{1}{2}||\tilde{z}||^2. \tag{118}$$

Now the time derivative of V is

$$\frac{dV}{dt} = \frac{1}{2}\int_0^1 2\tilde{z}^T \frac{\partial \tilde{z}}{\partial t} d\zeta \tag{119}$$

$$= \int_0^1 \tilde{z}^T \Big(P_1 \frac{\partial \tilde{z}}{\partial \zeta} - P_2 \tilde{z}\Big) d\zeta \tag{120}$$

$$= \int_0^1 \tilde{z}^T P_1 \frac{\partial \tilde{z}}{\partial \zeta} d\zeta - \int_0^1 \tilde{z}^T P_2 \tilde{z} d\zeta \tag{121}$$

$$= \langle E\tilde{z}, \tilde{z}\rangle - \int_0^1 \tilde{z}^T P_2 \tilde{z} d\zeta \tag{122}$$

$$\leq -\int_0^1 \tilde{z}^T P_2 \tilde{z} d\zeta. \tag{123}$$

By the given property of P_2, there exists a constant $c > 0$ such that $x^T P_2 x \geq x^T c I x = cx^2$ for all $x \in \mathbb{R}^n$. It follows that

$$\frac{dV}{dt} = \leq -\int_0^1 \tilde{z}^T P_2 \tilde{z} d\zeta \tag{124}$$

$$\leq -c\int_0^1 \tilde{z}^2 d\zeta = -c||\tilde{z}(t)||^2 = -2cV < 0. \qquad \text{if } \tilde{z} \neq 0 \tag{125}$$

Thus $\frac{dV}{dt}$ is globally negative definite and V is a Lyapunov function. Similarly as in the proof of Theorem A.1, we have that

$$\frac{dV}{dt} \leq -2cV \tag{126}$$

thus

$$||\tilde{z}(t)|| \leq e^{-ct}||\tilde{z}(0)||. \tag{127}$$

and the system is globally exponentially stable. □

A.2 The multiplication mapping Q

Before it is proved that the mapping derived in Section 2.1 is a multiplication mapping, it will be proven that the subspace spanned by exponential functions lie dense in $L^2((0,1);\mathbb{R}^n)$. This means that the closure of the subspace of all exponential functions is the whole space.

Lemma A.1. *The subspace spanned by exponential functions lie dense in* $L^2((0,1);\mathbb{R}^n)$.

Proof. It can be seen by the Fourier series that the subspace spanned by $e^{i\pi k\zeta}$ with $k \in \mathbb{Z}$ lie dense in $L^2(0,1)$, since the Fourier series states that for every function $f(\zeta) \in L^2(0,1)$

$$f(\zeta) = \sum_{k\in\mathbb{Z}} f_k e^{i\pi k\zeta} = \lim_{N\to\infty} \sum_{k=-N}^{N} f_k e^{i\pi k\zeta} \tag{128}$$

where f_k are the Fourier coefficients.

Now consider a function $f(\zeta) = \begin{bmatrix} f_1(\zeta) & f_2(\zeta) & \dots & f_n(\zeta)\end{bmatrix}^T \in L^2((0,1);\mathbb{R}^n)$. By the Fourier series, there exist Fourier coefficients $f_{i,k}$ with $i = 1,2,...,n$ such that for each element of $f(\zeta)$

$$f_i(\zeta) = \sum_{k\in\mathbb{Z}} f_{i,k} e^{i\pi k\zeta} = \lim_{N\to\infty} \sum_{k=-N}^{N} f_{i,k} e^{i\pi k\zeta}. \tag{129}$$

By taking $v_k = \begin{bmatrix} f_{1,k} & f_{2,k} & \dots & f_{3,k}\end{bmatrix}^T$, it follows that

$$f(\zeta) = \sum_{k\in\mathbb{Z}} e^{i\pi k\zeta} v_k = \lim_{N\to\infty} \sum_{k=-N}^{N} e^{i\pi k\zeta} v_k. \tag{130}$$

with $v_k \in \mathbb{C}^n$. Thus the subspace spanned by $e^{i\pi k\zeta} v$, $v \in \mathbb{C}^n$, lie dense in $L^2((0,1);\mathbb{R}^n)$. Since the subspace spanned by $e^{i\pi k\zeta} v$ is contained in the subspace spanned by $e^{s\zeta} v$, the subspace spanned by $e^{s\zeta} v$ must also lie dense in $L^2((0,1);\mathbb{R}^n)$. □

Theorem A.3. *If the mapping Q from $L^2((0,1);\mathbb{R}^n)$ to $L^2((0,1);\mathbb{R}^n)$ is linear and bounded and there exists a function $Y(\zeta) \in M_n(\mathbb{C})$ such that $Q(e^{s\zeta} v) = Y(\zeta)e^{s\zeta} v$ for all $s \in \mathbb{C}$ and $v \in \mathbb{C}^n$, then Q must be a multiplication operator. That is, for any $f(\zeta) \in L^2((0,1);\mathbb{R}^n)$, $Q(f(\zeta)) = Y(\zeta)f(\zeta)$.*

Proof. First, we prove that the mapping Q must be continuous, given that it is linear and bounded. Notice that since Q is linear by definition

$$Q(\alpha x_1 + \beta x_2) = \alpha Q(x_1) + \beta Q(x_2) \tag{131}$$

for any $x_1, x_2 \in L^2((0,1);\mathbb{R}^n)$ and α, β constants. Since Q is bounded, it follows that there exists some constant $M \in \mathbb{R}$ such that

$$||Q(x)|| \leq M||x||. \tag{132}$$

It immediately follows that

$$||Q(x_1) - Q(x_2)|| = ||Q(x_1 - x_2)|| \leq M||x_1 - x_2||. \tag{133}$$

Thus for every $\epsilon > 0$ there exists an $\delta > 0$, namely $\delta = \epsilon/M$, such that for all $x_1, x_2 \in L^2((0,1);\mathbb{R}^n)$

$$||x_1 - x_2|| \leq \delta \Rightarrow ||Q(x_1) - Q(x_2)|| \leq \epsilon. \tag{134}$$

It follows that the mapping Q is continuous.

Now we will prove that the mapping must be multiplicative. Take any $f(\zeta) \in L^2((0,1);\mathbb{R}^n)$. From the lemma above we know that

$$f(\zeta) = \sum_{k\in\mathbb{Z}} e^{i\pi k\zeta} v_k = \lim_{N\to\infty} \sum_{k=-N}^{N} e^{i\pi k\zeta} v_k. \tag{135}$$

Thus

$$Q(f(\zeta)) = Q(\lim_{N\to\infty}\sum_{k=-N}^{N} e^{i\pi k\zeta} v_k) \tag{136}$$

$$= \lim_{N\to\infty} Q(\sum_{k=-N}^{N} e^{i\pi k\zeta} v_k) \qquad \text{by continuity} \tag{137}$$

$$= \lim_{N\to\infty}\sum_{k=-N}^{N} Q(e^{i\pi k\zeta} v_k) \qquad \text{by linearity} \tag{138}$$

$$= \lim_{N\to\infty}\sum_{k=-N}^{N} Y(\zeta)e^{i\pi k\zeta} v_k \qquad \text{since } Q(e^{s\zeta} v) = Y(\zeta)e^{s\zeta} v \tag{139}$$

$$= Y(\zeta)\lim_{N\to\infty}\sum_{k=-N}^{N} e^{i\pi k\zeta} v_k \tag{140}$$

$$= Y(\zeta)f(\zeta). \tag{141}$$

It follows that Q must be a multiplication mapping. □