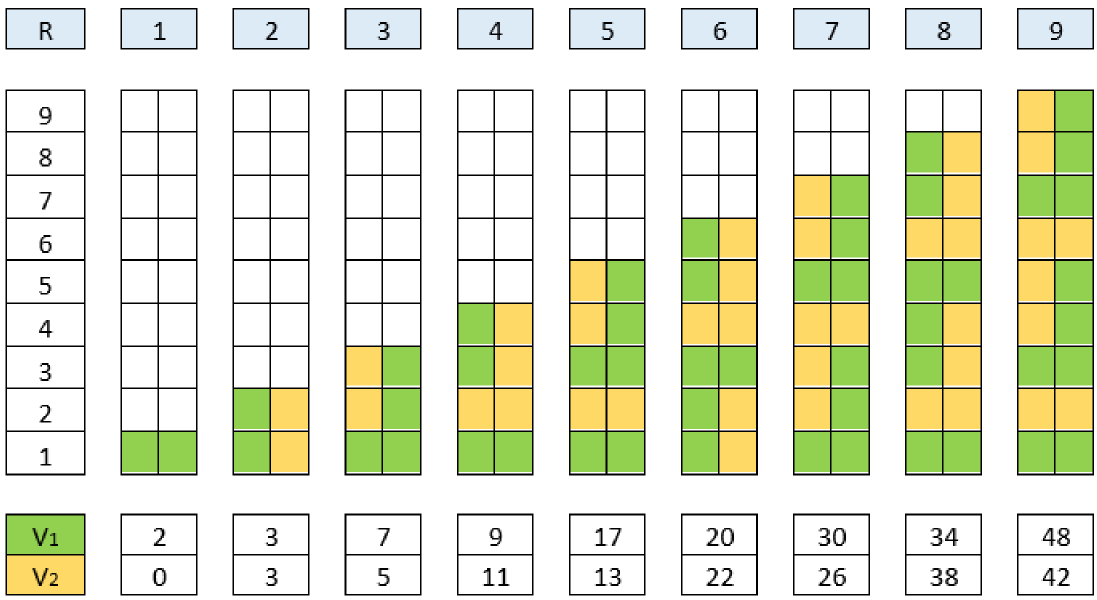
Research of Mathematics

Optimal strategies in dominogames



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Summary

Dominogames are a variant of monominogames (Timmer), the big difference being that players now play dominoes instead of monominoes. Whereas monominoes take up exactly one cell, dominoes take up two. This means they can be played horizontally or vertically. This also means that players can leave gaps in the game board that cannot be occupied, for example when a player plays horizontal over a single vertical domino. Obviously, when the game board has an odd number of cells, the game always ends up with at least one empty cell. Following up on the researches about monominogames (Timmer), and dominogames (Klomp and van Dorenvanck), this research is focussed on proving how two-column games will progress with any number of rows and what the payoffs will be. For this we assume both players play rationally. What we are looking for is what we, in this research, call Equilibrium Plays. A game can progress through different move sequences. If in a move sequence there exists no move by either player that could be changed to gain a higher payoff for the corresponding player, the move sequence is called an Equilibrium Play.

Dominogames are denoted by $D(C,R)$. Here, D stands for dominogame, which has C columns and R rows. When, for example, the bottom two rows are completely filled up (halfway through the game), and the rest isn't yet, we speak of a raised game. Let's say there are four rows left and there are only two columns. Then the game that is left is a raised two-column game with four rows. This is denoted by $D(2,4)^{+2}$. It was a six-row game, but the bottom two rows are already occupied and now there are only four rows left.

Klomp and van Dorenvanck showed us, in their research, what they thought were the Equilibrium Plays for two-column games with up to nine rows. In these nine Equilibrium Plays a certain pattern could be found that repeated itself after every four rows. With proof that this pattern holds, we could predict the final state of the game and calculate the payoffs for the players for two-column games with any number of rows. The pattern repeating itself after every four rows immediately leads to thinking about mathematical induction. If we prove the first few Equilibrium Plays correct, we can use this knowledge to prove what the Equilibrium Play of any two-column game is. For this we use a certain lemma: If we know what the Equilibrium Play of a certain $D(C,R)$ game is, then this is also the Equilibrium Play of a $D(C,R)^{+q}$ game for any q. We used this to prove the payoffs and the final state of the game for any $D(2,r)$ game, using the knowledge of the payoffs and the final state of the game for the $D(2,r-1)$ game. But, because the pattern only seemed to repeat itself after every four rows, the induction proof had to be split up into four parts, one for every group of row numbers: $R \bmod 4 = 0$, $R \bmod 4 = 1$, $R \bmod 4 = 2$ and $R \bmod 4 = 3$. Also, for the induction basis we needed the proofs for the first four two-column games because of this, instead of only the first. Using mathematical induction in this way, we proved that the payoffs of any two-column dominogame could be calculated using the formulas of table 1. Here, the functions V_1 and V_2, are the payoffs for player one and player two respectively. For any two-column game, for example the $D(2,17)$ game, one must determine in what row category this game falls. This would be $R \bmod 4 = 1$, meaning that $k = 4$. Now, one can use the corresponding formulas to calculate the payoffs of player one and player two for the Equilibrium Play of this game.

	$V_{1,R}$	**$V_{2,R}$**
$R = 4k$	$8k^2 + k$	$8k^2 + 3k$
$R = 4k + 1$	$8k^2 + 7k + 2$	$8k^2 + 5k$
$R = 4k + 2$	$8k^2 + 9k + 3$	$8k^2 + 11k + 3$
$R = 4k + 3$	$8k^2 + 15k + 7$	$8k^2 + 13k + 5$

Table 1: Formulas for calculating the payoffs of two-column games

The nature of this proof also tells us that the final state of $D(2,R)$ games is indeed following the pattern found in the first nine Equilibrium Plays. Also, this proof tells us that unoccupied cells never occur in the Equilibrium Plays of $D(2,R)$ games.

Also, some research on three-column games and four-column games was done. For three-column games the first four Equilibrium Plays were found, and for four-column games the first five were found. These Equilibrium Plays also gave some presumptions about how the game would progress when more rows would be added. These would be useful for further research, just like the presumptions of Klomp and van Dorenvanck were very useful for this research.

The research was concluded with some perceptions and insights acquired during this research, and a few suggestions for further research.

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1 Introduction

The Research of Mathematics is a part of the master M-ECB. The research was done under supervision of Dr. J. B. Timmer of *Mathematics of Operations Research (MOR)*. I followed the courses Graph Theory, Game Theory and Discrete optimization prior to this research.

This research is about dominogames, a variant of monominogames (Timmer, Aarts, van Dorenvanck, & Klomp, 2017). Dominogames were also previously researched (van Dorenvanck & Klomp, 2010), with a few differences to the rules. Monominogames are played with monominoes, whereas dominogames are played with dominoes. A monomino fills up only one cell of the game board, and dominoes fill up two adjacent cells. The goal of the game is to try to maximize the payoff. Units of payoff are earned by playing domino's in certain rows or columns. Each cell occupied by a player on row 1 earns him one unit of payoff. A cell occupied on row 2 is worth two units of payoff, and so on.

The main goal of this type of research is to find out what moves the players make if they play rationally. If both players played a game rationally, and every move was made to fully maximize each player's own payoff, the move sequence is called an Equilibrium Play. In Equilibrium Plays, there exists no move that could have been played differently to increase the payoff of the corresponding player. In dominogames, players have extra options each turn, because they may choose to play their domino horizontally or vertically. Sometimes, playing horizontally may leave a gap that cannot be filled anymore during the remainder of the game. These cells will be left unoccupied, and the payoff units of these cells are then wasted. In games with an odd number of cells, there is always at least one empty cell at the end of a game, because players fill up two cells each turn. These extra choices and the possibility of leaving gaps make dominogames a bit more complicated than monominogames. This research was started by finding Equilibrium Plays for Dominogames with two columns and any number of rows. After that we found some perceptions and insights for games with three and four columns as well.

Any perceptions and insights gained during this research will be written down. Also, recommendations will be given for potential students that want to follow up this research.

2 Theory

This chapter will cover monominogames and dominogames, as well as some games that have similarities to monominogames and dominogames. Also, some relevant theory that will be useful for the research will be covered here. This chapter will conclude with the research question and a paragraph about the setup of the research.

2.1 Monominogames

The dominogame that will be researched is a variant of similar monominogame. That is why this monominogame will first be introduced.

The monominogame (Timmer, Aarts, van Dorenvanck, & Klomp, 2017) is a game that is played by two players. It has some similarities to the game 'four in a row', but a big difference is that the monomino game only ends when the board is full and there are no more moves possible. Instead of trying to be the last one to make a move ('four in a row' or 'chess'), both players are only interested in optimizing their own payoffs. The game is played on a rectangular board of *C* columns and *R* rows, so the board consists of *C x R* cells. The notation $M(C, R)$ indicates a monominogame with *C* columns and *R* rows. The game is played with pieces called monominoes. These can cover exactly one cell of the board. The columns are numbered *1* to *C*, with the leftmost column being column 1. The rows are numbered from *1* to *R*, with the bottom row being row *1*. Having a monomino in a row $i \in \{1, 2, \ldots, R\}$ gives the player a payoff *i*. The game is played by two players, player 1 and player 2. Player 1 always begins. Each turn the players pick a column and place one monomino at the lowest available cell. When a player has a choice between cells of the same row the player chooses the column with the lowest number. The game only ends when all cells on the board are covered, which happens after *C x R* moves. After this each player counts their total payoff. The goal of the game is for each player to maximize their own payoff. The players are not interested in having a greater payoff than their opponent, only in maximizing their own payoff.

2.2 Dominogames by Klomp and van Dorenvanck

Dominogames (van Dorenvanck & Klomp, 2010) are an extension of monominogames. Instead of playing with monominoes, players play with dominoes. Dominoes are game pieces that cover more than one cell of the game board. In the article by Klomp dominoes were described as game pieces with lengths equal to the number of columns. This means the dimensions of dominoes are *1xC*. Because of the rectangular form of the dominoes, players have to take into account the position as well las the orientation of the dominoes they place. The players can either play their dominoes vertically or horizontally. The notation $D(C, R)$ indicates a dominogame with *R* rows and *C* columns.

2.3 Other games with similarities to monomino- and dominogames

2.3.1 Four in a row

Four in a row is a well-known game for two players. The rules are quite different, but the game basically uses a square board and both players play with single-cell chips (this is basically what monominoes are). The chips also fall down to the lowest possible row in a column, just like with monominogames. Both players use different colours so that they can distinguish each other's chips. In four in a row the size of the board is fixed (R and C are fixed) and players are not obliged to place their chip in the left most column if multiple cells on the same row are free. For visualization this game can be a basis for monomino games. In fact, two people could actually play a monominogame with a four in a row board and its chips. The rules of four in a row are quite different from monominogames though. In four in a row, the players don't try to score points, but they try to make the last move. Each player tries to get four chips in a row, either horizontally, vertically or diagonally. When one player places a chip and gets four in a row, the game is finished and this player wins the game. This also means the board rarely ever gets completely filled with chips. As a result of these rules the players always try to work against each other, whereas in monominogames players are only interested in their own gains. In short, the mechanics of four in a row are the same, but the rules are different.

2.3.2 Tetris

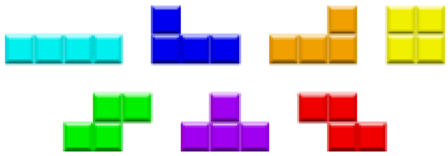


Figure 1: the seven kinds of tetriminoes

Tetris is a well-known computer game. It is played by only one player. Tetris is played on a rectangular board like in monominogames. The board may be any size, but the standard size is 10 cells wide and 20 cells high. The game is played with game pieces called tetriminoes. Tetriminoes are pieces that take up four cells and are shaped differently as shown in figure 1. One piece at a time will show up at the top side of the game board. After it appears it will start to fall down. While the tetrimino is falling, the player can move it left and right, or he can rotate it. After the tetrimino hits the bottom of the board or another tetrimino that was placed before, the next tetrimino will appear. When all cells of a row are filled the row is emptied and all cells above this row that were filled will move down by one. Clearing a row grants points to the player. The goal is to try to fit the tetriminoes together as tightly as possible. When a player doesn't manage to clear enough lines, and the game board is filled up so that no other tetrimino fits, he loses. Tetris can never end in the player's victory. The player can only hold on for so long before an inevitable loss, so the goal is to try and get the highest score before losing.

2.3.3 Pentominoes

Pentominoes (Orman & Hilarie, 1996) is a game for two players that is played on an 8x8 board. The game is played with pieces called pentominoes, which are pieces that cover 5 cells of the board. Pentominoes come in different shapes. The first player that is unable to make a move loses the game. The game ends when no other move is possible. There is no scoring system like in monominogames or dominogames. The pentominoes can be placed anywhere on the board, and don't necessarily have to be placed on the bottom row.

2.3.4 Dominono

Dominono (Gardner, 2000) is a game that resembles tic-tac-toe. It can be played on a sheet of paper. Both players pick a symbol to play with. For example, one player uses crosses and the other uses circles. Unlike tic-tac-toe, where players try to get three in a row, players now try to avoid making a domino. A domino is created by having two of your symbols adjacent to each other horizontally or vertically, but not diagonally. The players try to avoid creating a domino, hence the name dominono. Whoever creates a domino loses the game. In this game players do not necessarily have to mark the bottom row first. The size of the board is not fixed. Also, just like in four in a row, players try their best to work against each other instead of only playing for their own gains.

2.4 Dominogames in this research

The dominogames that have been researched here are again a variant of the before mentioned dominogames, only now the dominoes always have dimension *1x2* regardless of the number of columns. This means that dominoes always cover 2 cells of the game board. Also, when players have a choice between cells of the same row, they do not have to pick the leftmost column(s) anymore. This can sometimes lead to situations where one player can block certain options for the other player. For example, in the game $D(3,2)$, there is a difference in placing the first domino vertically in column 1 or column 2. Even though both options give player 1 the same amount of points, the latter deprives the other player of the option to place their domino horizontally. Whether this is advantage for player 1 or not may depend on the exact situation.

2.5 Game Theory

Game Theory will be used to study these dominogames. Game Theory is used for analysing situations and predicting outcomes in situations with strategic interactions between decisionmakers/players. In Game Theory we look at the choices each player has and how these choices affect the choices of each other. Many types of games can be analysed using Game Theory. In some games the players have to cooperate, and in some games, they play against each other. Dominogames are non-cooperative games (Nash, 1951) because the players play against each other for themselves. In such games each player has a payoff function that each player wants to maximize. Players are rational, selfish, for they only care about maximizing their own payoff. Unlike games like Rock-Paper-Scissor, dominogames consist of a series of decisions. The players make moves in turn, instead of simultaneously. At every point in the game both players know what moves the other player made. This information can be used to decide what move to make next. In such games we speak of 'perfect information' (Peters, 2008). Both players are fully informed on their opponents' moves. An example of a game where this is not the case is a card game like poker, because each player hides his cards from his opponent. The best strategy of each player depends on the strategy the other player uses. When the game is finished and neither player could have increased their payoff by changing strategies, then this set of moves (by both players) is called a 'Nash- Equilibrium' (Peters, 2008) (Groenewoud, 2011). The Nash-Equilibrium is one of the most central concepts in game theory.

2.6 Nash Equilibria

The goal of the game is for each player to maximize his payoffs. When a player covers a cell in row *i*, this generates a payoff of *i* units. The goal is not to defeat the other player, but instead each player tries to maximise their own total payoff, regardless of the other player's payoff. This means that each player doesn't try to minimise the other player's payoff and cares only for his own payoff. An interesting question is how the game will progress assuming both players do the best possible move every turn. Playing an optimal move means the player could not have improved his payoff by playing another move. In game theory we call this concept a 'Nash Equilibrium' (Peters, 2008) (Groenewoud, 2011).

2.7 Extensive form games and backwards induction

A game in extensive form is described by a game tree. Game trees exist of nodes and edges. Each node is either a decision node or an end node. Each edge corresponds to an action of a player. Monominogames and dominogames are extensive form games. They can be described by game trees. Figure 2 shows the game tree of an $M(2,3)$ game for three players (Prakken, 2018). Here, the rules say that when a player can play a monomino on the same row of two or more columns, he always has to play the leftmost column. Next to each node we see which player has the next move, and what the state of the game is. For example, 3:[1,1,0] means that it is now player three's turn, and column 1 and 2 both have one cell occupied. Under each end node the payoffs of each player are given. Basically, the game tree describes every decision each player can make and what the payoffs for each player will be for every move sequence.

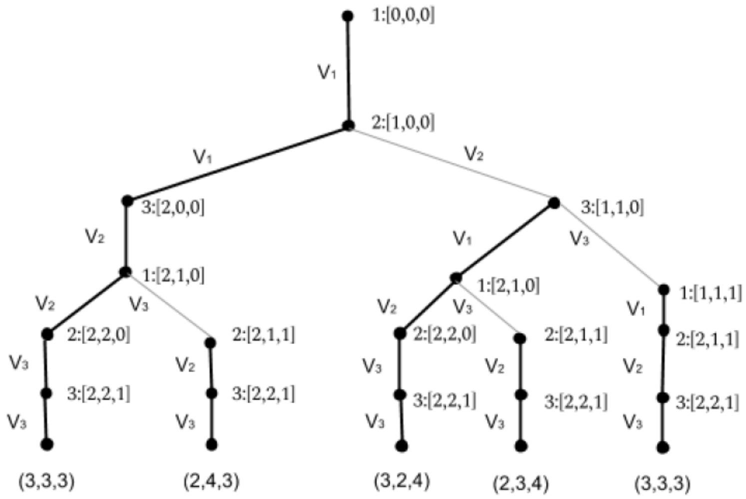


Figure 2: Game Tree of an M(2, 3) game for three players

Using backwards induction on an extensive form game, one can find the Nash Equilibrium of the game. All thick edges together describe this Nash Equilibrium. This method is called backwards induction because when applied, one starts at the end nodes and works his way to the top node. This means one starts by looking at the last move of the game, and then works backwards through the sequence of moves that are played each turn. Starting at an end node, you thicken the next edge if this decision is optimal for the player's payoff at this point. If this is done for the whole game tree there may be one or more routes of thick edges that go from the starting node all the way to an end node. If all players play rationally, they should always play along these thick routes. This way, all players try to maximize their own payoff. A move sequence along only thick edges will be called an Equilibrium Play. In figure 2, we can clearly see the difference between a Nash Equilibrium and an Equilibrium Play. All thick edges together describe the Nash Equilibrium, while the Equilibrium Play is only described by the leftmost path of thick edges that goes all the way from the top to the bottom. Only the Equilibrium Plays will be researched, since it saves us from a lot of unnecessary work.

Figure 3 shows what the board would look like in the end (green: player 1, yellow: player 2, blue: player 3). By backwards induction we have proved that this is what a finished $M(2,3)$ game for three players looks like when all players play rationally.

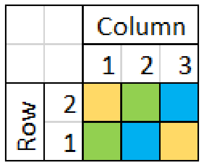


Figure 3: The Equilibrium Play of an M(2, 3) game for three players

Of course, the more rows and/or columns a game has, the more moves there are to be made, and the bigger the game tree grows. This research is about dominogames, and dominoes can be played horizontally or vertically. Because of this extra decision each player has to make, game trees grow even bigger. Drawing game trees to apply backwards induction to can get (too) elaborate very quickly. This is why this is only useful for games with a relatively small game board.

2.8 Research Questions

During this research the main question is:

> *What are the payoffs for each player in dominogames of the form $D(C,R)$ for two players, when both players play Equilibrium Plays and $C \leq 4$?*

This question will be split up into sub-questions:

- *What are the payoffs for each player in $D(2,R)$ games for two players, when both players play Equilibrium Plays?*
- *What are the payoffs for each player in $D(3,R)$ games for two players, when both players play Equilibrium Plays?*
- *What are the payoffs for each player in $D(4,R)$ games for two players, when both players play Equilibrium Plays?*

3 Model

The dominogames studied in this research are extensive form games played by two players. The game is played on a game board existing of C columns and R rows. The notation for dominogames is $D(C, R)$. This describes the number of columns and rows of the game. The players play with pieces called dominoes, game pieces that fill up two adjacent cells of the board. The dominos are inserted at the top of the game board, after which the domino falls down to the bottom or on top of another domino. Since dominoes fill up two cells, players can either choose to insert it horizontally or vertically (not diagonally). Each player plays one domino per turn and then his turn is immediately over. When no more dominos fit anywhere, the game ends. A domino only fits when it covers two cells of the game board.

A cell occupied on row r generates r payoff units for the player owning that domino. This means that, for example, cells on row 5 generate 5 payoff units each and cells on row 37 generate 37 payoff units each. A domino played vertically occupies two cells on two different rows, which means these two cells generate different payoffs. The payoffs for player one and player two are denoted by v_1 and v_2 respectively. The goal for each player is to maximise his own payoff. The players don't care for the payoff of the opponent, only for their own payoff. So, this game is not about defeating the opponent.

If, for example, a player plays horizontal over another single vertical domino, this leaves a gap under the horizontal domino. Such a move is allowed, but it means that these cells will remain unoccupied for the rest of the game. Leaving such gaps shortens the game and reduces the total payoff that can be generated.

In this research, dominogames will be visualised with green and yellow colours for player one and player two respectively. We will denote the payoff earned by player one and two as v_1 and v_2 respectively. In figure 4 we see two Equilibrium Plays of $D(2, 2)$ games. Since both are basically mirrored variants of each other, we will just view these two as one and the same move sequence. We will do this whenever mirrored variants of move sequences exist.

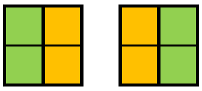


Figure 4: Two mirrored variants of an Equilibrium Play

4 $D(1, R)$ games

Because these games have only one column, players cannot play their dominoes horizontally, which makes these games rather easy to research. In fact, there exists only one move sequence for each kind of $D(1, R)$ game. Still, to lay a foundation for this research, we will show how these games are played out.

When there is only one column, players can only play vertical in this column. This means that the players have to keep stacking dominoes on top of each other until no more dominoes fit the game board. Figure 5 illustrates this.

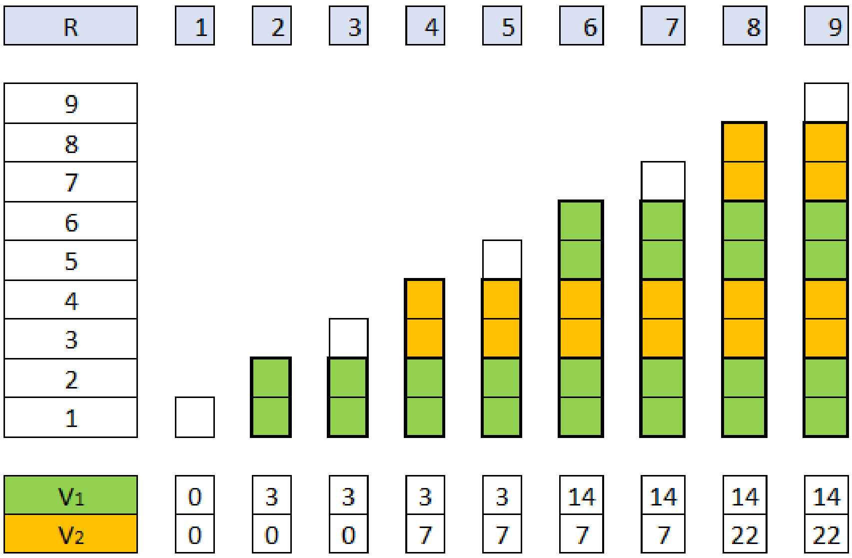


Figure 5: the Equilibrium Plays of D(1, R) *games*

Of course, we can imagine how $D(1, R)$ games with ten or more rows will be played out. These Equilibrium Plays are trivial. Now we will try to formulate the payoff functions for both players so that we can calculate the payoff of either player for any number of rows.

For player one we see that any time when $R \bmod 4 = 2$ ($R = 4k + 2$) and $R \bmod 4 = 3$, he gets to play $k + 1$ dominoes. For $R \bmod 4 = 0$ and $R \bmod 4 = 1$ player one gets to play only k dominoes. Player one's first domino is worth 3 units of payoff. The second is worth 11, the third 19, and so on. The next domino is always worth 8 more than the last domino. This is because the next domino is raised by four rows relative to the last domino, so both of the cells it occupies earn 4 extra units of payoff. That makes an extra payoff of 8 units. For $R \bmod 4 = 0$ and $R \bmod 4 = 1$, the payoff function for player one is then:

$$v_1 = \sum_{i=0}^{k-1} (3 + 4 \cdot 2 \cdot i) = \sum_{i=0}^{k-1} (3 + 8i)$$

Here, 3 is the payoff of the first domino (the one on the bottom two rows). $4 \cdot 2 \cdot i$ is the extra payoff that the other dominoes are generating. For summations we know that:

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{1}{2} n(n + 1)$$

Therefore, the payoff function V_1 can be written as:

$$\sum_{i=0}^{k-1}(3+8i) = 3k + \sum_{i=0}^{k-1} 8i = 3k + 8\sum_{i=0}^{k-1} i = 3k + 8\left(\frac{1}{2}(k-1)k\right) = 4k^2 - k$$

When $R \; mod \; 4 = 2$ or $R \; mod \; 4 = 3$, player one gets to play one more domino, so the same function cannot hold. The payoff function for these two row groups is:

$$v_1 = 3 + \sum_{i=1}^{k}(3 + 4 \cdot 2 \cdot i) = 3 + 3k + 8\sum_{i=1}^{k} i = 3 + 3k + 8(\frac{1}{2}\big(k(k+1)\big) = 4k^2 + 7k + 3$$

Player two always gets to play k dominoes, which is why there is only one payoff function for him:

$$v_2 = \sum_{i=0}^{k-1}(7 + 4 \cdot 2 \cdot i) = 7k + 8\sum_{i=0}^{k-1} i = 7k + 8\left(\frac{1}{2}(k-1)k\right) = 4k^2 + 3k$$

The payoff functions for both players are shown in table 2:

Row group	v_1	v_2
$R \; mod \; 4 = 0$ and $R \; mod \; 4 = 1$	$4k^2 - k$	$4k^2 + 3k$
$R \; mod \; 4 = 2$ and $R \; mod \; 4 = 3$	$4k^2 + 7k + 3$	$4k^2 + 3k$

Table 2: The payoff functions for D(1, R) *games*

Now, to calculate the payoff for player one for any $D(1, R)$ game, you have to decide in what row group the number of rows falls. Then you can decide the value of k, and you substitute this value in the correct payoff function for player one. For player two you only have to decide the value of k and then substitute this in the payoff function for player two.

We now know what the payoff functions for $D(1, R)$ games are, and how the move sequences progress. This concludes the research for $D(1, R)$ games.

5 $D(2, R)$ games

5.1 The first nine Equilibrium Plays of *D(2, R)* games (Klomp and van Dorenvanck)

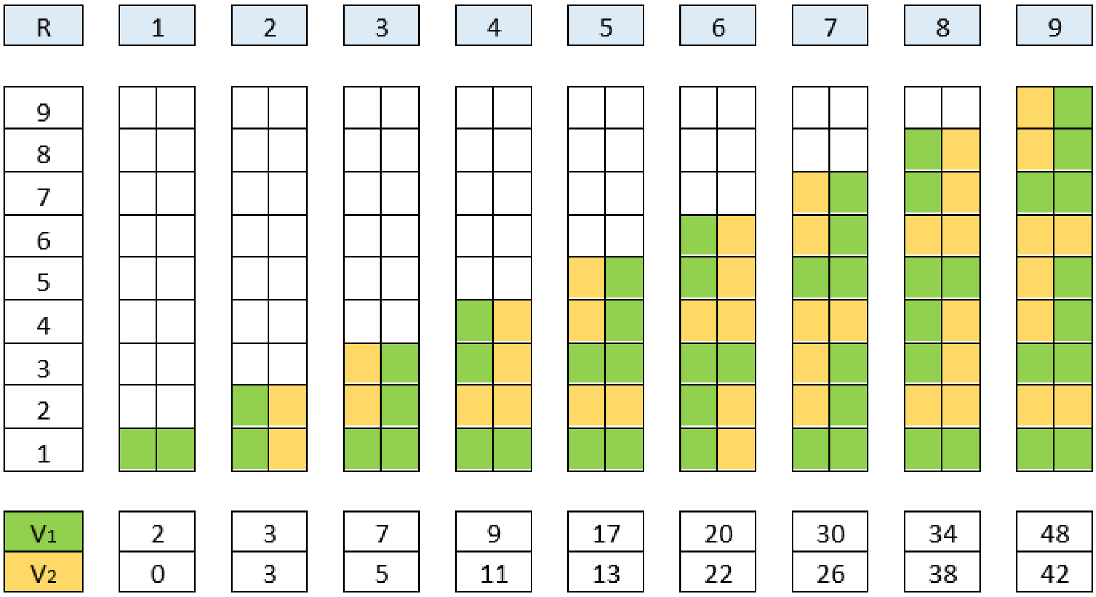


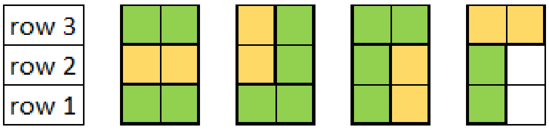
Figure 6: the first nine Equilibrium Plays of D(2, R) *games*

The unique Equilibrium Plays of the first nine $D(2, R)$ games are as shown in figure 6 (van Dorenvanck & Klomp, 2010). If we follow the pattern of these nine games, we could practically predict how subsequent games would play out. The specific pattern of the $D(2,4)$ game seems to be repeated a lot (two vertical dominoes on top of two horizontal ones). When $R = 5$, we see that player one plays horizontal first, and then the 2x4 pattern of the $D(2,4)$ game is played again. Only now player two initiated the 2x4 pattern. When $R = 9$, we see that another 2x4 pattern is stacked on top of the Equilibrium play of the $D(2,5)$ game. The pattern seems to be repeating with every four rows added. The first four Equilibrium Plays of $D(2, R)$ games will now be proved.

For $R = 1$, player one can only play horizontal, and then the game is finished. Player one gets two payoff units and player two gets zero.

For $R = 2$, player one will play vertical, and player two will then play vertical in the other column. This generates 3 units of payoff for both players. If player one would play horizontal, then player two could only play horizontal as well, but then player one would give one unit of payoff over to player two.

For $R = 3$, the following move sequences are possible (figure 7):

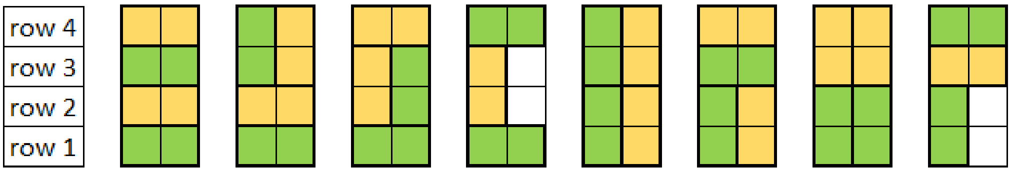


V1	8	7	9	3
V2	4	5	3	6

Figure 7: move sequences for D(2, 3) *games*

Green indicates dominoes played by player one and yellow indicates dominoes played by player two. Using backwards induction on the corresponding game tree, we can easily see that the second move sequence is the Nash Equilibrium. This generates a payoff of 7 for player one and a payoff of 5 for player two.

For $R = 4$, the following move sequences are possible (figure 8):



V1	8	9	7	10	10	9	6	11
V2	12	11	13	5	10	11	14	6

Figure 8: move sequences for D(2, 4) *games*

Again, using backwards induction on the corresponding game tree, we can see that the second move sequence is the Nash Equilibrium. This generates a payoff of 9 for player one and a payoff of 11 for player two.

These four equilibrium plays match the Equilibrium Plays of Klomp and van Dorenvanck in figure 6 perfectly.

5.2 The presumed optimal payoff functions

Because the pattern seems to be repeating itself after every four games, we formulate the payoff functions in four groups:

- *R mod 4 = 0* (or: *R = 4k* with $k \in \mathbb{N}$)
- *R mod 4 = 1* (or: *R = 4k+1* with $k \in \mathbb{N} \cup \{0\}$)
- *R mod 4 = 2* (or: *R = 4k+2* with $k \in \mathbb{N} \cup \{0\}$)
- *R mod 4 = 3* (or: *R = 4k+3* with $k \in \mathbb{N} \cup \{0\}$)

Now, for each of these four groups, payoff functions for player one and player two, V_1 and V_2 respectively, will be formulated.

R=4k:

It looks like the 2x4 pattern of the $D(2,4)$ game is being repeated over and over. Looking at the $D(2,8)$ game, we see two of these patterns on top of each other. Therefore, at the $D(2,12)$ game, we expect three of these patterns on top of each other, and so on. The presumed optimal payoffs functions V_1 and V_2 of player one and two can therefore be formulated as following:

$$V_1 = \sum_{i=0}^{k-1}(9 + 4 \cdot 4i) = 9k + \sum_{i=0}^{k-1} 16i = 9k + 16\sum_{i=0}^{k-1} i$$

$$V_2 = \sum_{i=0}^{k-1}(11 + 4 \cdot 4i) = 11k + \sum_{i=0}^{k-1} 16i = 11k + 16\sum_{i=0}^{k-1} i$$

In the summation each term represents the payoff of one 2x4 pattern. For $i = 0$, the payoff for player one is 9 and the payoff for player two is 11 (figure 6). The next 2x4 pattern (for $i = 1$) is raised by four rows. Then each cell is worth 4 extra units of payoff. So, for each player, the payoff for this 2x4 pattern is raised by the number of cells occupied multiplied by the number of rows the pattern is raised. In the 2x4 pattern both players occupy 4 cells, and in the summation 4i indicates by how many rows the pattern is raised.

Recall that:

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

Therefore, the payoff functions V_1 and V_2 can be written as:

$$V_1 = 9k + 16\left(\frac{1}{2}(k-1) \cdot k\right) = 8k^2 + k$$

$$V_2 = 11k + 16\left(\frac{1}{2}(k-1) \cdot k\right) = 8k^2 + 3k$$

For $R \ mod \ 4 = 1$, $R \ mod \ 4 = 2$ and $R \ mod \ 4 = 3$, the number of rows doesn't fit an exact number of repeating 2x4 patterns. However, we see in figure 6 that in these cases some initial moves are played, after which a raised game with $R' \ mod \ 4 = 0$ is left These initial moves match the first three Equilibrium Plays. From that point on the 2x4 pattern will be repeated again. The player starting this pattern varies, which is important because the payoff for the player that initiates this pattern is lower.

The payoff functions for $R \ mod \ 4 = 1$, $R \ mod \ 4 = 2$, and $R \ mod \ 4 = 3$ can be formulated similarly to the payoff functions for $R \ mod \ 4 = 0$:

R=4k+1: After player one plays horizontal, player two initiates the 2x4 pattern. Player one gets a payoff of 2 for the first horizontal play. Since it's now player two's turn, the players swap roles in the 2x4 pattern. This means player one now gets a base of 11 payoff units per 2x4 pattern and player two gets a base of 9 payoff units. The number of rows the 2x4 pattern is raised is now *4i-3*.

$$V_1 = 2 + \sum_{i=1}^{k} (11 + 4 \cdot (4i - 3)) = 8k^2 + 7k + 2$$

$$V_2 = \sum_{i=1}^{k} \big(9 + 4 \cdot (4i - 3)\big) = 8k^2 + 5k$$

R=4k+2: First both players play vertical in separate columns, generating 3 payoff units for both players. The base payoffs of 9 and 11 units are swapped again because player one now initiates the 2x4 pattern. The number of rows the 2x4 pattern is raised is now *4i-2*.

$$V_1 = 3 + \sum_{i=1}^{k} (9 + 4 \cdot (4i - 2)) = 8k^2 + 9k + 3$$

$$V_2 = 3 + \sum_{i=1}^{k} \big(11 + 4 \cdot (4i - 2)\big) = 8k^2 + 11k + 3$$

R=4k+3: First player one plays horizontal, and then both players play vertical in separate columns. This generates 7 payoff units for player one and 5 payoff units for player two. The base payoffs of 9 and 11 units are swapped again because player two now initiates the 2x4 pattern. The number of rows the 2x4 pattern is raised is now *4i-1*.

$$V_1 = 7 + \sum_{i=1}^{k} (11 + 4 \cdot (4i - 1)) = 8k^2 + 15k + 7$$

$$V_2 = 5 + \sum_{i=1}^{k} \big(9 + 4 \cdot (4i - 1)\big) = 8k^2 + 13k + 5$$

	$V_{1,R}$	$V_{2,R}$
$R = 4k$	$8k^2 + k$	$8k^2 + 3k$
$R = 4k + 1$	$8k^2 + 7k + 2$	$8k^2 + 5k$
$R = 4k + 2$	$8k^2 + 9k + 3$	$8k^2 + 11k + 3$
$R = 4k + 3$	$8k^2 + 15k + 7$	$8k^2 + 13k + 5$

Table 3: Presumed Optimal Payoff Functions

Table 3 shows the presumed optimal payoff functions that we just formulated. Still assuming the pattern holds for any number of rows, one can calculate the optimal payoff of either player one or player two by checking the number of rows of a certain game, deciding in which category it falls, and picking the correct payoff function for either player one or player two. For example, if you want to know the optimal payoff for player two in a $D(2, 15)$ game, you start by looking at the number of rows. A $D(2, 15)$ game has 15 rows, which falls in the category $R = 4k + 3$, because $4 \cdot 3 + 3 = 15$. Note that in this case $k = 3$. Then we pick the payoff function from the column V_2, to calculate the optimal payoff for player two in this game.

5.3 Proving correctness of the Presumed Optimal Payoff functions correct:

These optimal payoff functions all hold for the first four $D(2, R)$ games. But what about subsequent games? If we want to calculate the optimal payoff for any $D(2, R)$ game, we have to prove that these formulas hold for any $k \in \mathbb{N}$. For this we need Lemma 1.

Lemma 1: Any proven Equilibrium Play, for example the Equilibrium Play of the $D(2, 7)$ game, would also be the Subgame Equilibrium Play for any $D(2, 7)^{+q}$ game. Here *q* signifies that a game is reduced to another $D(2, R)$ game when *q* rows have already been filled.

Lemma 1 must be true because in any raised game, the payoffs of all cells are raised linearly. If we look closely at the opening moves of the first nine Equilibrium Plays, we see that it looks like when $R \bmod 4 = 2$, player one plays vertical, and player two plays vertical in the other column. This leaves a $D(2, R-2)^{+2}$ game. By Lemma 1, players should then play the same moves as they would in a $D(2, R-2)$ game. For $R \bmod 4 = 0$, $R \bmod 4 = 1$ and $R \bmod 4 = 3$ it looks like player one always plays horizontal first, leaving a $D(2, R-1)^{+1}$ game immediately, with player two having the next move. Since these opening moves leave a raised game with one or two less rows, we can formulate new situational payoff functions W consisting of the payoff of these opening moves and the payoff of the raised games and try to prove that these equal the presumed optimal payoff functions V from table 3. This means we are using induction to prove the next theorem:

Theorem 1:

- If $R \bmod 4 = 2$, player one will play vertical in the first column and player two will play vertical in the right column, leaving a $D(2, R-2)^{+2}$ game with player one having the first move.
- If $R \bmod 4 = 0$, $R \bmod 4 = 1$, or $R \bmod 4 = 3$, player one will play horizontal first, leaving a $D(2, R-1)^{+1}$ game with player two having the first move.
- For the situations mentioned in the first two parts of this theorem, the situational payoff functions W equal the presumed optimal payoff functions V.

5.4 Proof of Theorem 1:

Induction will be used to prove theorem 1.

5.4.1 The induction basis

The functions shown in table 3 will be proved correct using induction. The first step is to show that these functions are true for $k = 0$. Because for $R \; mod \; 4 = 0$ there exists no game for $k = 0$, the functions will have to be shown true for $k = 1$ in this case.

	$V_{1,R}$	$V_{2,R}$
$R = 4 \cdot 1 = 4$	$8 \cdot 1^2 + 1 = \mathbf{9}$	$8 \cdot 1^2 + 3 \cdot 1 = \mathbf{11}$
$R = 4 \cdot 0 + 1 = 1$	$8 \cdot 0^2 + 7 \cdot 0 + 2 = \mathbf{2}$	$8 \cdot 0^2 + 5 \cdot 0 = \mathbf{0}$
$R = 4 \cdot 0 + 2 = 2$	$8 \cdot 0^2 + 9 \cdot 0 + 3 = \mathbf{3}$	$8 \cdot 0^2 + 11 \cdot 0 + 3 = \mathbf{3}$
$R = 4 \cdot 0 + 3 = 3$	$8 \cdot 0^2 + 15 \cdot 0 + 7 = \mathbf{7}$	$8 \cdot 0^2 + 13 \cdot 0 + 5 = \mathbf{5}$

Table 4: Payoffs of the first four D(2, R) *games*

These payoffs match the payoffs of the first four Equilibrium Plays in figure 6 perfectly, giving us an induction basis for every group of row numbers. Also, the opening moves fit theorem 1 perfectly.

5.4.2 The situational payoff functions *W*

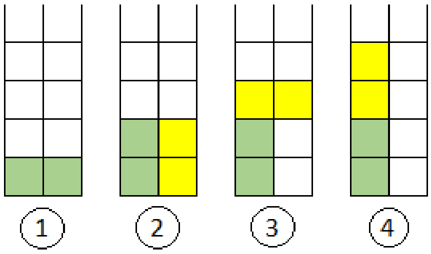


Figure 9: the four different opening move situations

Of course, there are more ways to open a game than the two situations mentioned in section 1.3 (these are situations 1 and 2). Figure 9 shows four different situations for a start of any $D(2, R)$ game. Green still represents the occupied cells for player one, and yellow represents occupied cells for player two. The first situation leaves a $D(2, R-1)^{+1}$ game, where player two has the first move. The second and third situations leave $D(2, R-2)^{+2}$ and $D(2, R-3)^{+3}$ games respectively, with player one having the first move. The fourth is the situation does not leave a raised game immediately because column 2 is not occupied. This makes situation 4 a bit more difficult, so it has to be split up into different variations later on.

For each of the four starting situations, payoff functions *W* will be formulated. They will generally exist of the direct payoff the situation generates and the payoff of the $D(2, R-q)^{+q}$ game that is left. If the optimal situational payoff functions *W* match the presumed optimal payoff functions *V* from table 3, the proof is complete. This will have to be done for all four groups of row numbers separately, because we already know that situations 1 and 2 both occur.

Explanation of the situational payoff functions *W*:

Example (situation 1): $W_{1,R} = 2 + V_{2,R-1} + 1(R - 1 - a) = V_{2,R-1} + R + 1 - a$

When situation 1 is played, player one plays horizontal. This generates the first two units of payoff. Then, players are playing a $D(2, R-1)^{+1}$ game with player two having the next move. This means that the move sequence of a $D(2, R-1)$ game will be played, with player one and player two switching roles. Therefore, on top of the 2 units already generated, player one gets the payoff that player two would get in a $D(2, R-1)$ game ($V_{2,R-1}$). However, because this game was raised by 1, for every cell occupied by player one, he generates 1 extra unit of payoff. When $R-1$ is even, both players occupy $R-1$ cells of the raised game. When $R-1$ is odd, player one occupies two less cells than player two in the raised game. In this case, the number of cells occupied by player one should be reduced by one. Therefore, the functions contain a parameter $a \in \{0, 1\}$, with *a* being 0 if R is even (because then $R-1$ is odd), and *a* being 1 if R is odd. So, $R - 1 - a$ signifies the number of cells occupied by player one in a 1-raised game where player two has the first move. All these cells generate 1 extra unit of payoff, hence the term $1(R - 1 - a)$. After this the function $W_{1,R}$ was just reduced to a simpler form.

The function $W_{2,R}$ for situation 1 and the functions W of situation 2, 3 and 4 for both players are built in a similar way. Only for situation 4 in $R \bmod 4 = 2$, the payoff functions are formulated directly.

The situational payoff functions:

Situation 1:

$$W_{1,R} = 2 + V_{2,R-1} + 1(R - 1 - a) = V_{2,R-1} + R + 1 - a$$

$$W_{2,R} = 0 + V_{1,R-1} + 1(R - 1 + a) = V_{1,R-1} + R - 1 + a$$

Situation 2:

$$W_{1,R} = 3 + V_{1,R-2} + 2(R - 2 + (1 - a)) = V_{1,R-2} + 2R + 1 - 2a$$

$$W_{2,R} = 3 + V_{2,R-2} + 2(R - 2 - (1 - a)) = V_{2,R-2} + 2R - 3 + 2a$$

Situation 3:

$$W_{1,R} = 3 + V_{1,R-3} + 3(R - 3 + a) = V_{1,R-3} + 3R - 6 + 3a$$

$$W_{2,R} = 6 + V_{2,R-3} + 3(R - 3 - a) = V_{2,R-3} + 3R - 3 - 3a$$

Situation 4:

Situation 4 does not immediately leave a raised game, so we cannot build the optimal payoff functions for this situation in the same way as the other situations. Therefore, the payoff function for this situation will now be explained and given separately for $R \bmod 4 = 0$, $R \bmod 4 = 1$, $R \bmod 4 = 2$ and $R \bmod 4 = 3$. Figure 10 shows variations of a few moves played after the move sequence of situation 4 is played.

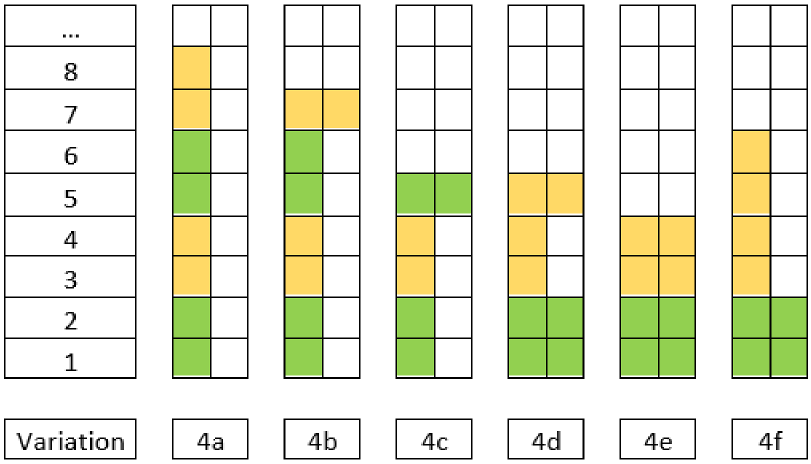


Figure 10: Variations of situation 4

$R \bmod 4 = 0$:

Variation 4a is very disadvantageous for player one, because player two constantly takes the maximum payoff in every set of four rows. He can avoid this variation by playing horizontal like in variation 4c, or playing vertical in column 2 like in variations 4d, 4e and 4f. If player one would again play vertical in column 1, player two would just play along and do the same, so 4b would never happen. Variation 4d is bad for player two because he will turn the 2x4 stacking pattern to the advantage of player one. Looking at variations 4e and 4f, player 2 is choosing between going to the repeating pattern immediately (4e), or prolonging the vertical stacking game (4f). Assuming both players keep stacking vertically, player one takes the upper hand in one column and player two then takes the upper hand in the other column. From row 5 and up the payoff is divided equally between the players. When playing variation 4e, player 2 has the advantage of the stacking pattern that follows, because he will always have the upper of the two horizontal dominoes in each 2x4 pattern. This means player two prefers variation 4e over 4f. Variation 4e is favorable over variation 4c for player one because they will basically play out the same, with variation 4e generating 3 more units of payoff for player 1. This means the optimal variation of situation 4 is variation 4e. When $R = 4k$ and the game starts with variation 4e, the payoff functions are:

- Player 1:

$$W_{1,R} = 6 + V_{1,R-4} + 4(R-4) = V_{1,R-4} + 4R - 10$$

- Player 2:

$$W_{2,R} = 14 + V_{2,R-4} + 4(R-4) = V_{2,R-4} + 4R - 2$$

$R \bmod 4 = 1$:

When $R \bmod 4 = 1$, and players keep stacking vertically in one column, one cell will be left on row R. Player one will then choose to play horizontal on the top row, ending the game. This leaves a lot of empty cells in column 2. We assume that this is not favorable for either player. When situation 4 occurs, players will want to stop stacking vertically in column 1. Player 1 favours 4c over 4b, because 4b will leave a raised game with $R' \bmod 4 = 2$ (here R' is the number of rows left in the raised game). This gives player two the more advantageous 2x4 repeating pattern. Player 1 gets this pattern if he plays horizontal himself like in variation 4c, leaving a raised game with $R' \bmod 4 = 0$. If player one would play vertical in column 2, player two would play horizontal like in variation 4d. This would leave player two with the advantageous repeating pattern. This is why player one prefers to play variation

4c when $R \ mod\ 4 = 1$. This leaves a raised game with $R' \ mod\ 4 = 0$, where player two has the first move, giving player one the advantage of the repeating pattern.

- Player 1:

$$W_{1,R} = 13 + V_{2,R-5} + 5(R-5) = V_{2,R-5} + 5R - 12$$

- Player 2:

$$W_{2,R} = 7 + V_{1,R-5} + 5(R-5) = V_{1,R-5} + 5R - 18$$

$\boldsymbol{R\ mod\ 4 = 2}$:

Variation 4a represents the move sequence where players keep stacking vertically in one column until it is full, after which they well start stacking in the other column until that one is also full. Since R is even, an exact number of vertical dominoes fits in each column. Since $R \ mod\ 4 = 2$, player one will come out on top in column 1. But then player two will start in column 2, and will come out on top there. Their payoffs will both be equal (or half the total payoff). If player two plays horizontal at some point, like in variation 4b, he will give away the advantage of the 2x4 pattern that will follow up on this. This means his payoff will be less than half the total payoff. If player one decides to play horizontal at some point like in variation 4c, he will give away the advantage of the 2x4 pattern and generate less than half the total payoff. Player one could also play vertical in column 2 at some point, instead of playing vertical in column 1, like in variations 4d, 4e, and 4f. At this point, player two will keep stacking in column 1, because this way he will come out on top in both columns. Variation 4e also gives the advantage to player two. So, the best outcome of situation 4 for player one is to get half the total payoff, because diverting from this strategy at this point will give player two the advantage. When player two diverts, he gives away the advantage in the same way. So, situation 4 will play out like variation 4a. When the payoff of both players is half of the total payoff of this game, their functions W are:

$$W = \frac{1}{2} \cdot 2 \sum_{1}^{4k+2} i = \frac{1}{2}(4k+2)(4k+3) = \frac{1}{2}(16k^2 + 20k + 6) = 8k^2 + 10k + 3$$

$\boldsymbol{R\ mod\ 4 = 3}$:

If both players keep stacking in column 1 (variation 4a), in the end there will be one row left for player two to play horizontal. Player two would rather play horizontal (variation 4b) at some earlier point, because this would leave a raised game with $R' \ mod\ 4 = 0$, with player one starting. This gives player two the advantage of the repeating 2x4 pattern. This also leaves fewer empty cells. But then player one would rather play horizontal (variation 4c) earlier than player two, because this gives the advantage of the repeating 2x4 pattern over to him. Player one would also rather do this this than play vertical in the other column, like variations 4d, 4e and 4f. 4d gives player two the advantage player one could have in 4c. 4e plays out like 4c, with an extra payoff of 7 units, player one still having the advantage. If player two tries to keep stacking in column 1, and tries to draw player one with him into this strategy, player one comes out on top with the horizontal domino. Because variation 4d is bad for player one, he will play horizontal himself, like in variation 4c. After this a raised game will be played with $R' \ mod\ 4 = 2$, with player two having the first move. We know that this is advantageous for player one.

The payoff functions in this situation are:

- Player 1:

$$W_{1,4k+3} = 13 + V_{2,R-5} + 5(R-5) = V_{2,R-5} + 5R - 12$$

- Player 2:

$$W_{2,4k+3} = 7 + V_{1,R-5} + 5(R-5) = V_{1,R-5} + 5R - 18$$

The functions W will now be written as functions of k, replacing R with either $4k$, $4k+1$, $4k+2$ or $4k+3$, depending on the group of row numbers.

5.4.3 The Induction Hypothesis

We assume that the presumed optimal payoff functions V from table 3 are correct for a number of rows up to $R-1$. Now we try to prove that these functions are also correct when the number of rows is R. We can do this by proving that the situational payoff functions W that match the opening moves of the group of row numbers are equal to the presumed optimal payoff functions V.

5.4.4 The optimal situational payoff function W for $R \bmod 4 = 0$

We want to prove that $W = V$ only for situation 1, because figure 6 shows us that when $R \bmod 4 = 0$, player one starts by playing horizontal. The functions W will now be written as functions of k for each situation. Note that $a = 1$ because R is even. Also, a star above an equality sign ($\stackrel{*}{=}$) means that the induction hypothesis is plugged in.

Situation 1:

- Player 1:

$$W_{1,R} = W_{1,4k} = V_{2,R-1} + R + 1 - a = V_{2,4k-1} + 4k + 1 - 1 = V_{2,4(k-1)+3} + 4k \stackrel{*}{=} 8(k-1)^2 + 13(k-1) + 5 + 4k = 8k^2 + k$$

- Player 2:

$$W_{2,R} = W_{2,4k} = V_{1,R-1} + R - 1 + a = V_{1,4k-1} + 4k - 1 + 1 = V_{1,4(k-1)+3} + 4k \stackrel{*}{=} 8(k-1)^2 + 15(k-1) + 7 + 4k = 8k^2 + 3k$$

Situation 2:

- Player 1:

$$W_{1,R} = V_{1,R-2} + 2R + 1 - 2a = V_{1,4k-2} + 2 \cdot 4k + 1 - 2 \cdot 1 = V_{1,4(k-1)+2} + 8k - 1$$

$$\stackrel{*}{=} 8(\mathrm{k}-1)^2 + 9(k-1) + 3 + 8k - 1 = 8k^2 + k + 1$$

- Player 2:

$$W_{2,R} = V_{2,R-2} + 2R - 3 + 2a = V_{2,4k-2} + 2 \cdot 4\mathrm{k} - 3 + 2 \cdot 1 = V_{2,4(k-1)+2} + 8k - 1$$

$$\stackrel{*}{=} 8(k-1)^2 + 11(k-1) + 3 + 8k - 1 = 8k^2 + 3k - 1$$

Situation 3:

- Player 1:

$$W_{1,R} = V_{1,R-3} + 3R - 6 + 3a = V_{1,4k-3} + 3 \cdot 4\text{k} - 6 + 3 \cdot 1 = V_{1,4(k-1)+1} + 12\text{k} - 3$$

$$\stackrel{\star}{=} 8(k-1)^2 + 7(k-1) + 2 + 12k - 3 = 8k^2 + 3k$$

- Player 2:

$$W_{2,R} = V_{2,R-3} + 3R - 3 - 3a = V_{2,4k-3} + 3 \cdot 4\text{k} - 3 - 3 \cdot 1 = V_{2,4(k-1)+1} + 12\text{k} - 6$$

$$\stackrel{\star}{=} 8(k-1)^2 + 5(k-1) + 12k - 6 = 8k^2 + k - 3$$

Situation 4:

- Player 1:

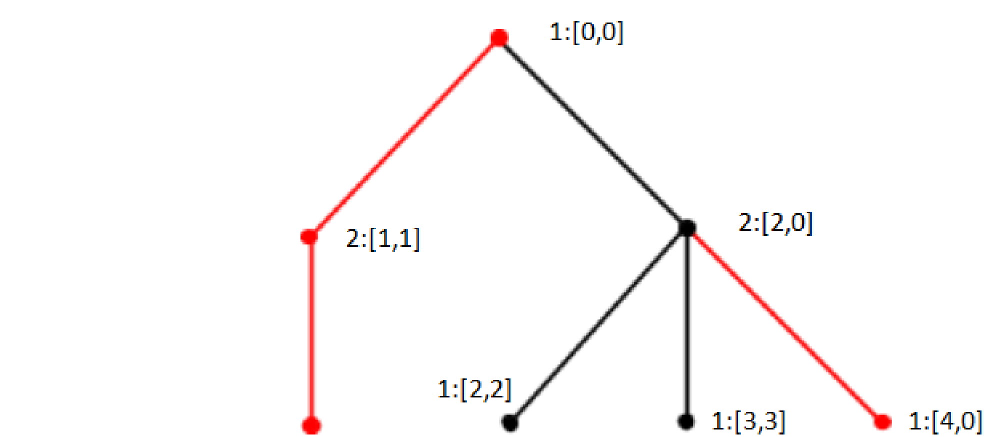
$$W_{1,R} = V_{1,R-4} + 4R - 10 = V_{1,4k-4} + 16\text{k} - 10 = V_{1,4(\text{k}-1)} + 16\text{k} - 10$$

$$\stackrel{\star}{=} 8(\text{k}-1)^2 + (\text{k}-1) + 16\text{k} - 10 = 8\text{k}^2 + k - 3$$

- Player 2:

$$W_{2,R} = V_{2,R-4} + 4R - 2 = V_{2,4k-4} + 16\text{k} - 2 = V_{2,4(k-1)} + 16k - 2$$

$$\stackrel{\star}{=} 8(k-1)^2 + 3(k-1) + 16\text{k} - 2 = 8k^2 + 3k + 3$$



Situation	1	2	3	4
$W_{1,4k}$	$8k^2 + k$	$8k^2 + k + 1$	$8k^2 + 3k$	$8\text{k}^2 + k - 3$
$W_{2,4k}$	$8k^2 + 3k$	$8k^2 + 3k - 1$	$8k^2 + k - 3$	$8\text{k}^2 + 3k + 3$

Figure 11: Reduced tree diagram for R mod $4 = 0$

Figure 11 proves that situation 1 always occurs when $R \bmod 4 = 0$. These functions W equal the functions V for $R \bmod 4 = 0$.

5.4.5 The optimal situational payoff function W for $R \ mod\ 4 = 1$

We want to prove that $W = V$ only for situation 1, because figure 6 shows us that when $R \ mod\ 4 = 1$, player one starts by playing horizontal. The functions W will now be written as functions of k for each situation. Note that $a = 0$ because R is odd.

Situation 1:

- Player 1:

$$W_{1,R} = V_{2,R-1} + R + 1 - a = V_{2,4k} + 4k + 2 \stackrel{*}{=} 8k^2 + 3k + 4k + 2 = 8k^2 + 7k + 2$$

- Player 2:

$$W_{2,R} = V_{1,R-1} + R - 1 + a = V_{1,4k} + 4k \stackrel{*}{=} 8k^2 + k + 4k = 8k^2 + 5k$$

Situation 2:

- Player 1:

$$W_{1,R} = V_{1,R-2} + 2R + 1 - 2a = V_{1,4k-1} + 2(4k + 1) + 1 = V_{1,4(k-1)+3} + 8k + 3$$
$$\stackrel{*}{=} 8(k-1)^2 + 15(k-1) + 7 + 8k + 3 = 8k^2 + 7k + 3$$

- Player 2:

$$W_{2,R} = V_{2,R-2} + 2R - 3 + 2a = V_{2,4k-1} + 2(4k + 1) - 3 = V_{2,4(k-1)+3} + 8k - 1$$
$$\stackrel{*}{=} 8(k-1)^2 + 13(k-1) + 5 + 8k - 1 = 8k^2 + 5k - 1$$

Situation 3:

- Player 1:

$$W_{1,R} = V_{1,R-3} + 3R - 6 + 3a = V_{1,4k-2} + 3(4k + 1) - 6 = V_{1,4(k-1)+2} + 12k - 3$$
$$\stackrel{*}{=} 8(k-1)^2 + 9(k-1) + 3 + 12k - 3 = 8k^2 + 5k - 1$$

- Player 2:

$$W_{2,R} = V_{2,R-3} + 3R - 3 - 3a = V_{2,4k-2} + 3(4k + 1) - 3 = V_{2,4(k-1)+2} + 12k$$
$$\stackrel{*}{=} 8(k-1)^2 + 11(k-1) + 3 + 12k = 8k^2 + 7k$$

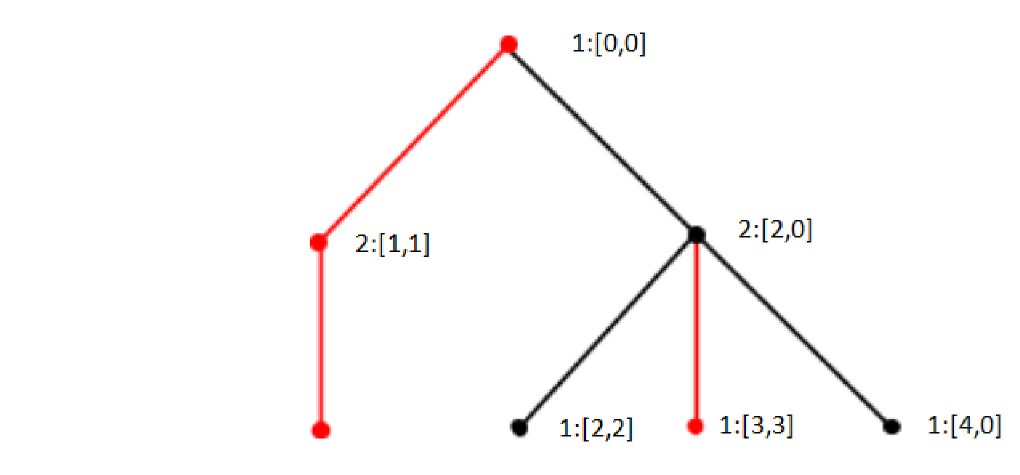
Situation 4:

- Player 1:

$$W_{1,R} = V_{2,R-5} + 5R - 12 = V_{2,4k-4} + 5(4k + 1) - 12 = V_{2,4(k-1)} + 20k - 7$$
$$= 8(k-1)^2 + 3(k-1) + 20k - 7 = 8k^2 + 7k - 2$$

- Player 2:

$$W_{2,R} = V_{1,R-5} + 5R - 18 = V_{1,4k-4} + 5(4k + 1) - 18 = V_{1,4(k-1)} + 20k - 13$$
$$= 8(k-1)^2 + (k-1) + 20k - 13 = 8k^2 + 5k - 6$$



Situation	1	2	3	4
$W_{1,4k+1}$	$8k^2+7k+2$	$8k^2+7k+3$	$8k^2+5k-1$	$8k^2+7k-2$
$W_{2,4k+1}$	$8k^2+5k$	$8k^2+5k-1$	$8k^2+7k$	$8k^2+5k-6$

Figure 12: Reduced tree diagram for R mod $4=1$

Figure 12 proves that situation 1 always occurs when $R \bmod 4 = 1$. These functions W equal the functions V for $R \bmod 4 = 1$.

5.4.6 The optimal situational payoff function W for $R \bmod 4 = 2$

We want to prove that $W = V$ only for situation 2, because figure 6 shows us that when $R \bmod 4 = 2$, players start by playing vertical in one column each. The functions W will now be written as functions of k for each situation. Note that $a = 1$ because R is even.

Situation 1:

- Player 1:

$$W_{1,R} = V_{2,R-1} + R + 1 - a = V_{2,4k+1} + 4\text{k} + 2 \triangleq 8\text{k}^2 + 5k + 4\text{k} + 2 = 8\text{k}^2 + 9k + 2$$

- Player 2:

$$W_{2,R} = V_{1,R-1} + R - 1 + a = V_{1,4k+1} + 4\text{k} + 2 \triangleq 8\text{k}^2 + 7\text{k} + 2 + 4\text{k} + 2 = 8\text{k}^2 + 11k + 4$$

Situation 2:

- Player 1:

$$W_{1,R} = V_{1,R-2} + 2R + 1 - 2a = V_{1,4k} + 2(4\text{k} + 2) - 1 = V_{1,4k} + 8\text{k} + 3$$
$$\triangleq 8k^2 + k + 8k + 3 = 8k^2 + 9k + 3$$

- Player 2:

$$W_{2,R} = V_{2,R-2} + 2R - 3 + 2a = V_{2,4k} + 2(4\text{k} + 2) - 1 = V_{2,4k} + 8\text{k} + 3$$
$$\triangleq 8k^2 + 3k + 8k + 3 = 8k^2 + 11k + 3$$

Situation 3:

- Player 1:

$$W_{1,R} = V_{1,R-3} + 3R - 6 + 3a = V_{1,4k-1} + 3(4\text{k} + 2) - 3 = V_{1,4(\text{k}-1)+3} + 12\text{k} + 3$$
$$\triangleq 8(\text{k} - 1)^2 + 15(k - 1) + 7 + 12k + 3 = 8k^2 + 11k + 3$$

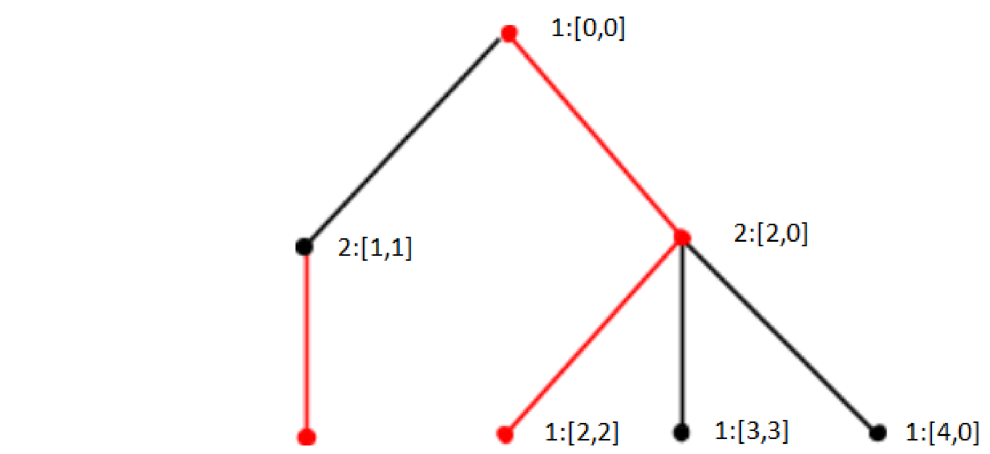
- Player 2:

$$W_{2,R} = V_{2,R-3} + 3R - 3 - 3a = V_{2,4\text{k}-1} + 3(4\text{k} + 2) - 6 = V_{2,4(k-1)+3} + 12\text{k}$$
$$\triangleq 8(k - 1)^2 + 13(k - 1) + 5 + 12k = 8k^2 + 5k$$

Situation 4:

When the payoff of both players is half of the total payoff of this game, their functions W are:

$$W = \frac{1}{2} \cdot 2 \sum_{1}^{4k+2} i = \frac{1}{2}(4k + 2)(4k + 3) = \frac{1}{2}(16k^2 + 20k + 6) = 8k^2 + 10k + 3$$



Situation	1	2	3	4
$W_{1,4k+2}$	$8k^2 + 9k + 2$	$8k^2 + 9k + 3$	$8k^2 + 11k + 3$	$8k^2 + 10k + 3$
$W_{2,4k+2}$	$8k^2 + 11k + 4$	$8k^2 + 11k + 3$	$8k^2 + 5k$	$8k^2 + 10k + 3$

Figure 13: Reduced tree diagram for R mod $4 = 2$

Figure 13 proves that situation 2 always occurs when $R \; mod \; 4 = 2$. These functions W equal the functions V for $R \; mod \; 4 = 2$.

5.4.7 The optimal situational payoff function W for $R \; mod \; 4 = 3$

We want to prove that $W = V$ only for situation 1, because figure 6 shows us that when $R \; mod \; 4 = 3$, player one starts by playing horizontal. The functions W will now be written as functions of k for each situation. Note that $a = 0$ because R is odd.

Situation 1:

- Player 1:

$$W_{1,R} = V_{2,R-1} + R + 1 - a = V_{2,4k+2} + 4\text{k} + 4$$
$$\stackrel{*}{=} 8\text{k}^2 + 11\text{k} + 3 + 4\text{k} + 4 = 8\text{k}^2 + 15k + 7$$

- Player 2:

$$W_{2,R} = V_{1,R-1} + R - 1 + a = V_{1,4k+2} + 4\text{k} + 2$$
$$\stackrel{*}{=} 8k^2 + 9k + 3 + 4\text{k} + 2 = 8\text{k}^2 + 13k + 5$$

Situation 2:

- Player 1:

$$W_{1,R} = V_{1,R-2} + 2R + 1 - 2a = V_{1,4k+1} + 2(4\text{k} + 3) + 1 = V_{1,4\text{k}+1} + 8\text{k} + 7$$
$$\stackrel{*}{=} 8\text{k}^2 + 7\text{k} + 2 + 8\text{k} + 7 = 8\text{k}^2 + 15k + 9$$

- Player 2:

$$W_{2,R} = V_{2,R-2} + 2R - 3 + 2a = V_{2,4k+1} + 2(4\text{k} + 3) - 3 = V_{2,4\text{k}+1} + 8\text{k} + 3$$
$$\stackrel{*}{=} 8\text{k}^2 + 5\text{k} + 8\text{k} + 3 = 8\text{k}^2 + 13k + 3$$

Situation 3:

- Player 1:

$$W_{1,R} = V_{1,R-3} + 3R - 6 + 3a = V_{1,4k} + 3(4\text{k} + 3) - 6 = V_{1,4\text{k}} + 12\text{k} + 3$$
$$\stackrel{*}{=} 8\text{k}^2 + \text{k} + 12\text{k} + 3 = 8\text{k}^2 + 13k + 3$$

- Player 2:

$$W_{2,R} = V_{2,R-3} + 3R - 3 - 3a = V_{2,4\text{k}} + 3(4\text{k} + 3) - 3 = V_{2,4k} + 12\text{k} + 6$$
$$\stackrel{*}{=} 8\text{k}^2 + 3\text{k} + 12\text{k} + 6 = 8\text{k}^2 + 15k + 6$$

Situation 4:

- Player 1:

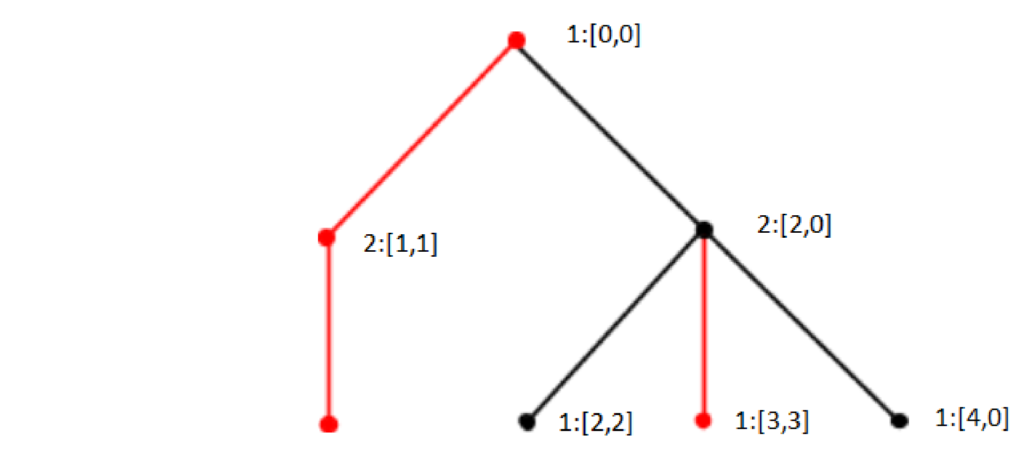
$$W_{1,4k+3} = V_{2,R-5} + 5R - 12 = V_{2,4k-2} + 5(4k+3) - 12$$

$$\stackrel{*}{=} 8(k-1)^2 + 11(k-1) + 3 + 20k + 3 = 8k^2 + 15k + 3$$

- Player 2:

$$W_{2,4k+3} = V_{1,R-5} + 5R - 18 = V_{1,4k-2} + 5(4k+3) - 18$$

$$\stackrel{*}{=} 8(k-1)^2 + 9(k-1) + 3 + 20k - 3 = 8k^2 + 13k - 1$$



Situation	1	2	3	4
$W_{1,4k+3}$	$8k^2 + 15k + 7$	$8k^2 + 15k + 9$	$8k^2 + 13k + 3$	$8k^2 + 15k + 3$
$W_{2,4k+3}$	$8k^2 + 13k + 5$	$8k^2 + 13k + 3$	$8k^2 + 15k + 6$	$8k^2 + 13k - 1$

Figure 14: Reduced tree diagram for R mod $4 = 3$

Figure 14 proves that situation 1 always occurs when $R \bmod 4 = 3$. These functions W equal the functions V for $R \bmod 4 = 3$.

5.4.8 Concluding the proof of theorem 1

For each of the four groups of row numbers we have shown what the optimal opening situation is. These optimal opening situations match theorem 1. Also, we have shown that the payoff functions W of these situations equal the presumed optimal payoff functions V from table 3. This completes the proof of theorem 1. The presumed optimal payoff functions V were indeed optimal. This means that in any $D(2,R)$ game, given that the players play rationally, we can calculate the payoff of each player whatever the number of rows is.

This also proves that when both players think rationally, they will never leave empty cells in $D(2,R)$ games by playing horizontal over a single vertical domino.

6 $D(3, R)$ Games

6.1 The Equilibrium Plays of the first four $D(3, R)$ games

The proof of the continuing pattern in $D(2, R)$ games makes us believe there may be a repeating pattern every four rows in games with more columns as well. Because of this, let's first try to find the Equilibrium Plays of the first four $D(3, R)$ games and see what we find.

$D(3,1)$ games:

This is trivial. Player 1 plays horizontal and then the game is over.

$D(3,2)$ games:

There are six possible move sequences for $D(3,2)$ games. Figure 15 shows these move sequences, and backwards induction on the corresponding game tree shows that the fourth and the sixth move sequence are Equilibrium Plays. These two Equilibrium Plays are very similarly played out. The difference is that when player 1 plays vertical in column two, he deprives player two of the opportunity to play horizontal. As we can see in the fifth move sequence, this move wouldn't be favorable for player two anyway. Therefore player 1 has no reason to play vertical in column 2 rather than column 1. Because of this, we take the fourth move sequence as the main Nash Equilibrium.

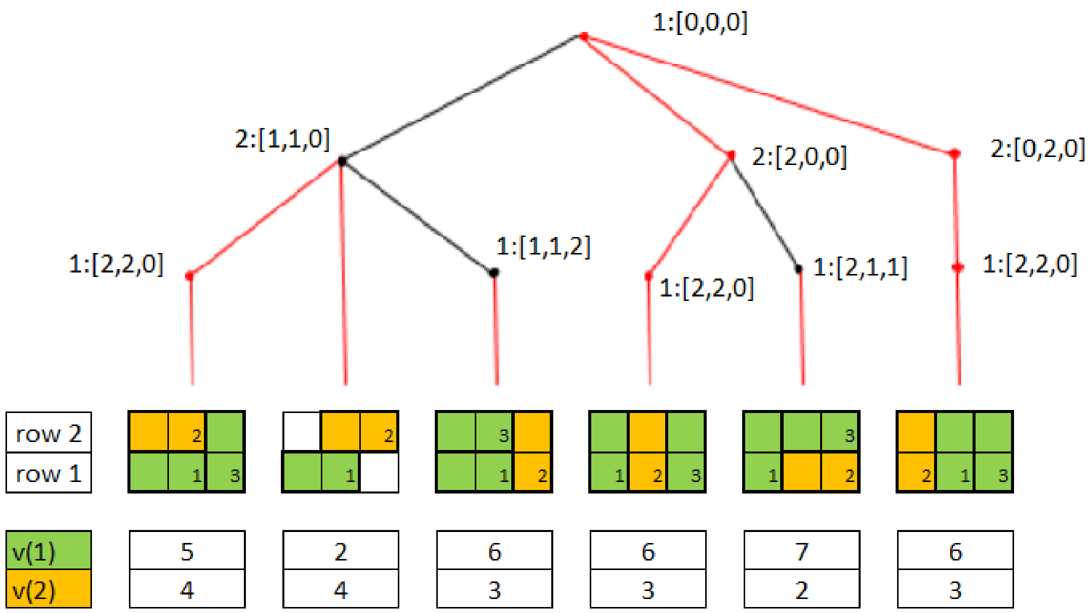


Figure 15: the game tree of D(3, 2) *games*

$D(3,3)$ games:

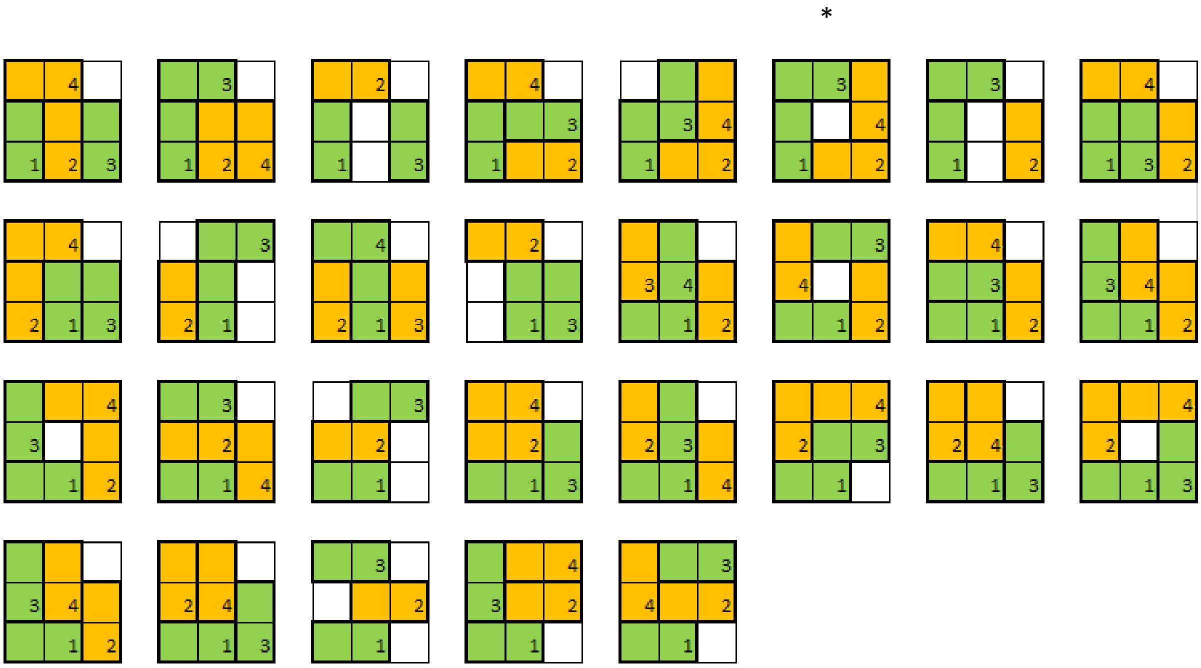


Figure 16: move sequences of D(3, 3) *games*

In figure 16, move sequences of $D(3,3)$ games are shown. As we can see, things just got a lot more elaborate by just adding one row. These 29 move sequences were all found manually, working very systematically. After these 29 move sequences no more could be found, but in this way, we cannot prove that there are no other possible move sequences. What we can do is prove which of these move sequences is an Equilibrium Play. When you look at the game tree (not shown), the sixth move sequence of the first row is the only Equilibrium Play (it is marked by a '*' above it). But because we might have forgotten one or more move sequences (one of which might be an Equilibrium Play), we'd still like to prove that the Equilibrium Play found with the game tree is actually an Equilibrium Play.

Proof: In a $D(3,3)$ game, both players get to play up to two moves. Both moves of player one are optimal for him, so he actually has the maximal payoff he could get. Because of this, player one will definitely want to go down this road. Player two could have chosen to play vertical on his first move. Doing this in column 3 would result in move sequence #7 of figure 16, not an Equilibrium Play (by backwards induction). Doing this in column 2 would result in move sequence #2, also not an Equilibrium play. Playing horizontal on top of player one's first vertical domino leaves the players with move sequence #3, again not an Equilibrium Play. The only option left for player two's first move, is to play horizontal in columns 2 and 3. After this, player one plays his optimal second move, and player two plays the only move left to do.

$D(3,4)$ games:

We just saw how quickly the number of different move sequences escalated by just adding one row. We can expect there to be a lot more move sequences with $D(3,4)$ games. The harder this task becomes to do manually, the more unreliable the results are. This is why we tried to prove an Equilibrium Play for $D(3,4)$ games in a different method. Assuming player one doesn't want to start with playing vertical (because player two will just play vertical on top of him), he will start horizontal. Playing vertical anywhere will now be bad for player two, because player one will play either horizontal or vertical on top of his last domino. The best thing to do now is to play horizontal on the same columns as player one just did. Now both players play vertical on top of the horizontal dominoes to claim cells of the highest row. The moves played until now form the 2x4 repeating pattern of $D(2, R)$ games. Now there's just one empty column left. Player one has to play vertical, and player two plays vertical on top of him. In figure 17 we see the result. This move sequence is definitely to the advantage of player two, because he has the advantage in the 2x4 pattern, and also in the single column next to it. After the first two horizontal plays, player one could have played vertical in the only empty column left. But then a $D(3, 2)$ game would be left, for three vertical dominoes to be played. This leaves exactly the same result as the afore mentioned move sequence. So, what if player one actually does start vertical? He has to do so in one of the side columns, or else he would have the lowest payoff possible with three dominoes. Then player two immediately plays vertical on top of him, and then a 2x4 empty space is left. We know what happens from $D(2, R)$ games. In the next four plays the 2x4 repeating pattern will be played, ending the game. We see that no matter what, the result will be a 2x4 repeating pattern started by player one, and one column where player one has the bottom vertical domino and player two has the upper vertical domino. We just described three paths to this end result. These are three routes on the game tree, making them three different Equilibrium Plays in reality. But because the pattern is exactly the same in all three, let's just see them as one and the same Equilibrium Play.

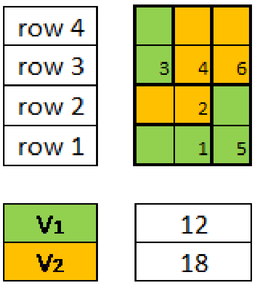


Figure 17: the Equilibrium Play of D(3, 4) *games*

$D(3, 5)$ games:

After a quick look into these games we came to the following Equilibrium Play (figure 18).

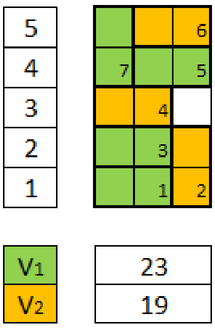


Figure 18: Potential Equilibrium Play of D(5, 3) *games*

6.2 The results

In figure 19 we see the Equilibrium Plays of $D(3, R)$ games with up to four rows. For two rows, only the most straightforward Equilibrium Play is shown. The Equilibrium Play on the right-hand side of figure 14 is not shown, because it is so similar.

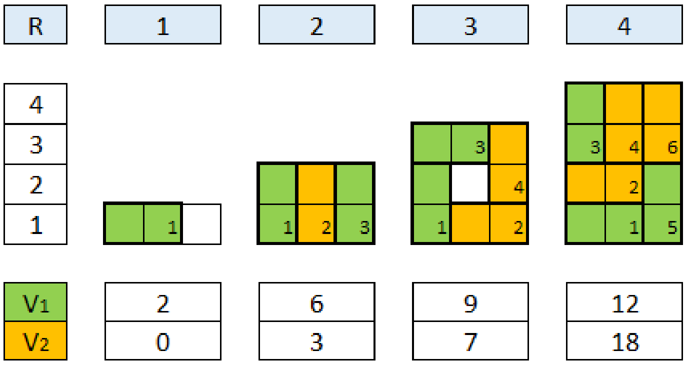


Figure 19: The first four Equilibrium Plays of D(3, R) *games*

7 $D(4, R)$ Games

For dominogames with four columns we will do the same as we did for dominogames with two columns. We try to formulate payoff functions for calculating the payoffs of the Equilibrium Plays for each player. We suspect that with four columns the pattern of the $D(2, R)$ Equilibrium Plays will be doubled.

7.1 The Equilibrium Plays of the first four $D(4, R)$ games

Let's first try to find the Equilibrium Plays of the first four $D(4, R)$ games to see if these are doubled versions of the Equilibrium Plays of $D(2, R)$ games. For each number of rows an explanation will be given. A visualisation can each time be found in figure 20. Each domino is again marked by a number, indicating the turn in which the domino was played.

1 row:

The first two are pretty trivial. When there is only one row, both players just play horizontal. Another Equilibrium Play is when player one plays horizontal in columns 2 and 3. Player two would not be able to make a move in this case, but it gives player one the same payoff as the first Equilibrium Play. However, we will dismiss this Equilibrium Play because the only reason for player one to play it is to oppose player two, and defeating the opponent is not the purpose of this game.

2 rows:

When there are two columns the players only play vertical. This is because if any player plays horizontal, the other player just reacts by playing horizontal on top of that to steal one unit of payoff. This is disadvantageous for the player who played horizontal first, so neither player will want to do this.

3 rows:

This is where things get interesting. We found three Equilibrium Plays for $D(4, 3)$ games (again, see figure 20). The first one is where both players fill up row 1 by playing horizontal next to each other. This leaves a $D(4, 2)^{+1}$ game, and we just showed what the Equilibrium Play of that game is. The second is actually the doubled version of the Equilibrium Play of a $D(2, 3)$ game. We already suspected this would be an Equilibrium Play. These first two Equilibrium Plays are a lot like each other, as you can clearly see in the figure. The third however, is a little different. Here, player one starts by playing vertical instead of horizontal. If player two would now play vertical in column 2, player 1 would play horizontal in columns 1 and 2, and would gain the upper hand of the game. Instead, player two plays vertical in column 3 (or 4). This makes player one play a vertical domino first, after which player two also plays vertical. Then both players play horizontal. As is visual in the figure, Player one occupies all cells in columns 1 and 2, and player two occupies all cells in columns 3 and 4. The players basically tell each other: "Let's just share the total payoff equally by each minding our own half of the game board."

4 rows:

When there are four rows players will never want to play vertical in an empty column. If a player does this, the other player will just counter by playing vertical in the same column. This is very disadvantageous for the player with the bottom vertical domino, so he will never do this. So, player one has to start by playing horizontal. If he plays horizontal in columns 2 and 3, player two will do the same. Now Both players have to play vertical in columns 2 and 3, because they want to occupy these cells of the most valuable rows. Two non-adjacent columns are left now, in which player two both ends up on top. So, player one cannot start horizontal in columns 2 and 3. The only opening move for player one that is left is playing horizontal in columns 1 and 2 (not counting the mirrored variant where he plays horizontal in columns 3 and 4).

In figure 20 we see the results that we have found so far.

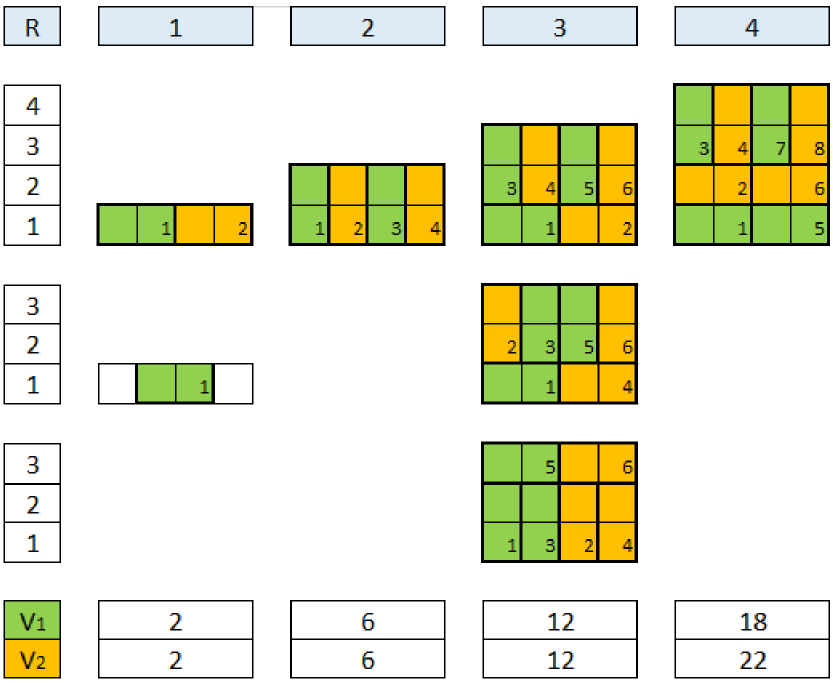


Figure 20: Equilibrium Plays for D(4, R) *games with 1-4 rows*

7.2 $D(4, 5)$ games

We have found the Equilibrium Plays of $D(4, R)$ games with a number of rows up to four. For each of these four types of games an Equilibrium Play was found that was in fact a doubling of the Equilibrium Play of the corresponding 2-column game. However, in $D(4, 3)$ games, more Equilibrium Plays were found. Especially the third one was interesting because there, player one starts the game with a different opening move. What impact will these extra Equilibrium Plays have on games with more rows? Lets first look at $D(4, 5)$ games. According to our assumption that any equilibrium play in a game with 4 columns is a doubling of the Equilibrium Play of the game with 2 columns and the same number of rows. This is shown in figure 21 as example 1. Both players have equal payoff. However, we see that if player two changes his first move from horizontal in columns 1 and 2 to horizontal in columns 3 and 4, this leaves a $D(4, 4)^{+1}$ game, in which player two is able to get more than half of the total payoff. This is shown in example 2. This means that our assumption was not correct, because the pattern is not the Equilibrium Play pattern of a $D(2, 5)$ game doubled. But it gets even more interesting. If player one opens with a different move, namely vertical in column 1, he is still able to get half the total payoff. This move sequence is shown in Example 5. Note that after the first four moves in this Equilibrium Play

(Ex. 4), a $D(4,3)^{+2}$ is left. We already saw that $D(4,3)$ games had three different Equilibrium Plays, which means $D(4,5)$ games must have at least a many.

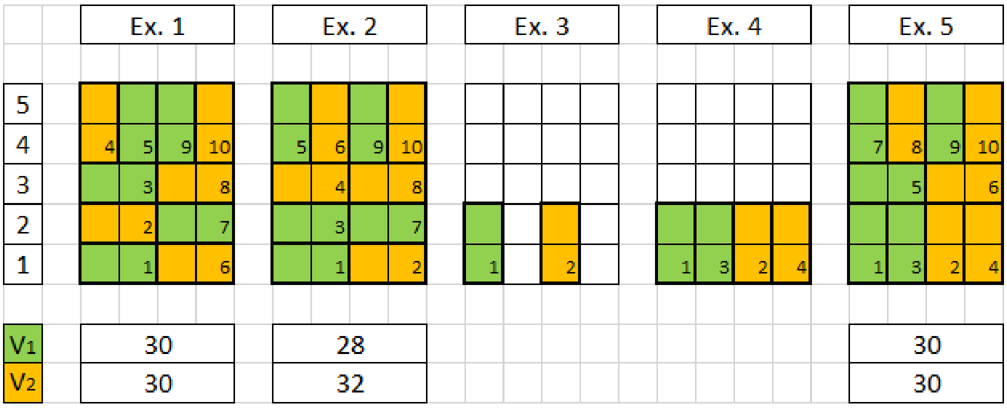


Figure 21: Examples of D(4, 5) *games*

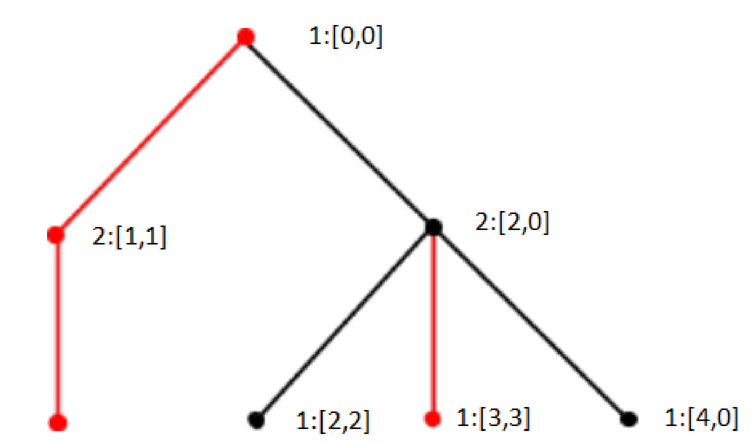
8 Perceptions

This chapter will contain a few perceptions and insights acquired during this research. These are mostly presumptions that haven't been proved. This chapter is meant for giving a head start to another student that potentially want to continue this research.

8.1 Equilibrium Plays have at most one empty cell

As we know, players may leave empty cells by playing horizontal over a vertical domino, or by playing horizontal over one half of another horizontal domino. But every time players make one of these moves and leave empty cells that cannot be occupied anymore, the total number of moves of the game will be cut. For example, in a $D(3,4)$ game, there are 12 cells and both players have at most three moves. If player one plays vertical and player two plays horizontal on top of it, two empty cells will be left unoccupied. This means that player two has only one move left after this first move of his, instead of two. Less moves means less opportunities to score payoff units. If player one is the one to leave these two empty cells, it will still cost player two a move. But now player two has no reason not to leave empty cells, because the next time it will cost player one a move. So, player one leaves himself vulnerable as well when he leaves empty cells. The more cells left empty, the more payoff units it will cost both players. Sometimes, at the end of a game, it may be optimal for a player to leave empty cells that will cost the other player a move, but these move sequences can probably not be Equilibrium Plays. In dominogames with an odd number of rows and columns, there are obviously an odd number of cells. This means there has to be at least one cell unoccupied at the end of these games.

Now you may be wondering if, to find Equilibrium Plays manually, this means that we can just dismiss any move sequence without unoccupied cells, or with just one unoccupied cell. The answer is no. Let's have a look at a part of the proof of chapter 5 (figure 22). This is the same figure as figure 12 of section 5.4.5.



Situation	1	2	3	4
$W_{1,4k+1}$	$8k^2+7k+2$	$8k^2+7k+3$	$8k^2+5k-1$	$8k^2+7k-2$
$W_{2,4k+1}$	$8k^2+5k$	$8k^2+5k-1$	$8k^2+7k$	$8k^2+5k-6$

Figure 22: Reduced tree diagram for R mod 4 = 1

The four situations represent the four opening move sequences played in $D(2, R)$ games. Here, the number of rows is $R = 4k + 1$. Here, player one plays horizontal, to have the payoff of situation 1. If he would have played vertical, we see that player two would have played horizontal to have the payoff of situation 3. But that means player two would have left two empty cells. Because player one doesn't play vertical, situation 3 indeed doesn't lead to an Equilibrium Play. But if we had just dismissed situation 3 because of the unoccupied cells, we would assume player one would have preferred to play vertical because then player two would play vertical in the other column, ending up in situation 2. We would have found a false Equilibrium Play. This means that we cannot just dismiss any move sequence with two or more empty cells to find Equilibrium Plays manually. But, if you think you have found an Equilibrium Play that has two or more unoccupied cells, it probably isn't an Equilibrium Play.

8.2 The opening moves and vertical stacking

When the number of rows is even, you can sometimes quickly see if playing vertical is a good move. When the number of rows is $R = 4k$, for example, and player one plays vertical, player two can just play vertical in the same column. This way, player one has 7 units of payoff on the first move, and player one only has three. Player one started this move sequence, and is already 4 units of payoff behind player two. Player one could continue this strategy by playing vertical in the same column again, but then he gains 11 units of payoff, and if player two then repeats the move as well, he gains 15 units of payoff. Because $R = 4k$, player two will come out on top when this move sequence continues like this. This means that this move sequence is very advantageous for player two, but not for player one. At every move he gives two units of payoff over to player two. If player one was to start playing vertical in another column, then player two would play vertical in that column as well. Now player two dominates two columns. Just by trying out these kind of move sequences, we can see that playing vertical on the bottom row is not a very good move for either player. This is why we often see players start by playing horizontal when $R = 4k$.

Something similar goes for when $R = 4k + 2$. Now, if both players were to stack vertical dominoes in one column, player one would come out on top. This means that player one may want to start playing vertical, but player two doesn't really want to follow up on it. Instead, what we often see in Equilibrium Plays, is that players first fill up the bottom two rows with vertical dominoes.

When the number of rows is odd and players start to stack vertically on top of each other, always one empty cell will be left in that column. A player could play horizontal then to gain the maximum number of payoff units possible in one move, but that also leaves one column full of empty cells. This is bad for both players, because these units of payoff are lost for both. On the other hand, if the player skips out on this horizontal move in the top row, and plays something else instead, his opponent might make this move. Then he will at least get the units of payoff of this top row. At some point, one player wants to make this move, and both players want it for themselves. This situation is rather tricky. Instead, what we often see in Equilibrium Plays is that players play horizontal on the first move, instead of vertical.

Using these insights, we can sometimes easily tell when a move sequence is not an Equilibrium Play, and what move sequences may actually be Equilibrium Plays. Of course, you have to check the potential Equilibrium Plays still.

8.3 $D(3, R)$ games

Equilibrium Plays for $D(3, R)$ games with up to four rows were found. In search of an Equilibrium Play for $D(3, 5)$ games, we came to the conclusion shown in figure 23:

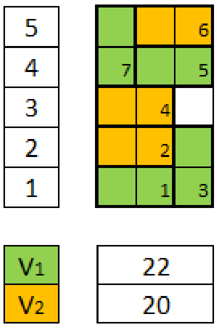


Figure 23: Presumed Equilibrium Play for D(3, 5) *games*

In this Equilibrium Play we cannot see any hints of the Equilibrium Plays for $D(3, R)$ games with up to four rows. With three rows, the cell in the middle stays empty, but with 5 rows, apparently a cell in column 3 apparently stays empty (or column 1 in the mirrored variant that is not shown). There is no sign of a pattern here, which makes it hard to find more Equilibrium Plays. Therefore, the research on $D(3, R)$ games is concluded without a guess on what the next Equilibrium Plays are going to look like.

8.4 $D(4, R)$ games

In these games we found the Equilibrium Plays of games with up to 5 rows. In $D(4, 3)$ games we found three different Equilibrium Plays, and in $D(4, 5)$ games we found that the optimal move sequence for both players started with filling up the first two rows, leaving a $D(4, 3)^{+2}$ game with again three different Equilibrium Plays. More interestingly, at the start of a $D(4, 5)$ game, we see that the players may each want to stick to their own half of the game board until there are three rows left, dividing the payoff equally as well. At the last three rows, either of the three Equilibrium Plays that were found for $D(4, 3)$ games may be played. We suspect that this concept will continue in games with an odd number of rows, as shown in figure 24. We also suspect to see this concept in other games with an even number of columns (except for games with two columns).

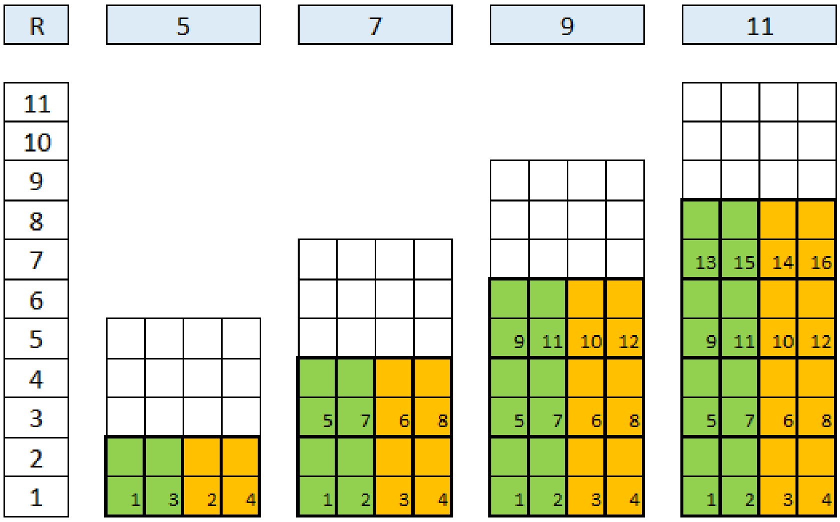


Figure 24: Suspected Equilibrium Plays of D(4, R) *games with an odd number of rows*

When there are an even number of rows, we suspect that the only Equilibrium Plays will be a doubling of their 2-column variants. This means that the payoffs are double the payoffs of the 2-column variants as well.

Also, when the number of rows is odd, the number of columns is even and $C \geq 4$, we suspect that both players will stick to their own half of the game board as well, as shown in figure 24.

9 Conclusion and Discussion

9.1 Conclusion

On dominogames with one or two columns, we have found exactly how each game will progress when players play rationally. Research was also started on games with three and four columns, but the results of these haven't exactly been proved. This is because whenever you add a column to the game, the players have so many more options each turn that there are just too many possible move sequences to research for Equilibrium Plays. Perceptions and insights about dominogames with three and four columns were still written down in chapter 8.

During the research on dominogames with one and two columns, we found that a certain pattern of dominoes repeated itself after every four rows. Because of this the proof of the correctness of the payoff functions for games with two columns had to be split up into four different parts of row groups: $R \bmod 4 = 0$, $R \bmod 4 = 1$, $R \bmod 4 = 2$ and $R \bmod 4 = 3$. Here, for example, $R \bmod 4 = 2$ means that $R = 4k + 2$ and that we are looking at games with 2, 6, 10, 14, 18, 22, … rows. The payoff functions are functions of this k.

Dominogames with one column are very straightforward. Players have only one option each turn, so they have to play it every time. This means that players keep stacking vertical dominoes on top of each other in the only column of the game board. This is a very trivial solution. Also, the payoff functions for both players were found. These are shown in table 5.

Row group	v_1	v_2
$R \bmod 4 = 0$ and $R \bmod 4 = 1$	$4k^2 - k$	$4k^2 + 3k$
$R \bmod 4 = 2$ and $R \bmod 4 = 3$	$4k^2 + 7k + 3$	$4k^2 + 3k$

Table 5: The payoff functions for D(1, R) *games*

Dominogames with two columns were a bit more complicated because the extra column means extra options to play for the players. A certain 2x4 repeating block was found, and using this, the proof was completed by splitting it up into the four mentioned parts. In table 6 we see the payoff functions for both players, for each of the mentioned row groups.

Row group	v_1	v_2
$R = 4k$	$8k^2 + k$	$8k^2 + 3k$
$R = 4k + 1$	$8k^2 + 7k + 2$	$8k^2 + 5k$
$R = 4k + 2$	$8k^2 + 9k + 3$	$8k^2 + 11k + 3$
$R = 4k + 3$	$8k^2 + 15k + 7$	$8k^2 + 13k + 5$

Table 6: The payoff functions for D(2, R) *games*

9.2 Discussion

While the payoff functions of the two-column games are proven to be correct, the proof is not 100% solid. In this proof we made a distinction between four different opening situations. The first three of these provided raised games, but the fourth didn't. Because situation 4 didn't provide a raised game, we could still not use the payoff of such a game for the payoff function. We had to split this situation up into a few variations again, so that we could manually try out how the game would progress. Sometimes certain moves could easily be dismissed because it would set the player at a great disadvantage. With trial and error, we found out how the game would probably progress until a raised game was eventually left, so that we could continue the proof in the same way we could with the first three opening situations. Even though the part about opening situation 4 is very credible, we haven't actually proven it.

On games with three and four columns we found the Equilibrium Plays for games with up to five rows. For the three-column games there is proof for games with up to three rows. For four-column games there is proof for games with up to two rows, and there is a not quite 100% solid proof for three-row games. The Equilibrium Plays for games with four and five rows were found by trial and error. Since there are so many possible move sequences, we haven't tried them all out. Still, we believe we are right about these Equilibrium Plays. These results make for a good starting point for further research.

9.3 Recommendations and suggestions for further research

In this research we started by manually finding Equilibrium Plays of $D(2, R)$ games. With up to four or five rows, this wasn't too hard to do. As the number of rows grew, however, we saw an exponential increase in the number of possible move sequences. When we started adding columns the number of possible move sequences started growing even faster. We think we managed to find all possible move sequences for $D(3,3)$ games, but if we added any more rows or columns, we found that that there would be just too many possibilities. At a certain point we just cannot prove anymore that we have found all possible move sequences. We may just miss a move sequence that turns out to be an Equilibrium Play. For this reason, if someone wants to continue this research, we highly recommend using simulations. You can find the code for similar games online (see chapter 3: Theory). From there you may be able to build a code for dominogames.

Of course, if someone wants to continue this research, it is a good idea to try to find the Equilibrium Plays for games with more columns. Maybe there is some sort of pattern between all games with an odd or even number of rows. Then you can also probably formulate the payoff functions for the Equilibrium Plays.

Another interesting thing to look at is the number of unoccupied gaps in Equilibrium Plays. In this research it is stated that there can be at most one unoccupied cell in Equilibrium Plays, namely in games with an odd number of cells. In games with an even number of cells there can be no such gaps. This is likely to be true, but it has not been proven yet.

My last suggestion also has to do with such unoccupied cells. The question is: If there are one or more unoccupied cells, where would it be then? Maybe this question is easier to answer if you have proven that there can be at most one unoccupied cell in Equilibrium Plays.

10 References

Gardner, M. (2000). Dominono. *Computers and mathematics with applications*, 3.

Groenewoud, E. (2011). *Evenwichten in de Speltheorie (in Dutch).* Amsterdam: Universiteit van Amsterdam.

Nash, J. (1951). Non-Cooperative Games. *Annals of Mathematics*.

Orman, & Hilarie, K. (1996). Pentominoes: A first player win. In *Games of no chance* (pp. 339 - 344). MSRI Publications.

Peters, H. (2008). *Game Theory - a multi-leveled approach.* Maastricht: Springer.

Prakken, K. (2018). Optimale strategieën in monominospellen (in Dutch).

Timmer, J., Aarts, H., van Dorenvanck, P., & Klomp, J. (2017). Non-cooperative Monomino Games.

van Dorenvanck, P., & Klomp, J. (2010). *Optimale strategieën in dominospellen (in Dutch).* Enschede: Universiteit Twente.