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Stability analysis of planar switched linear systems

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Abstract

This paper provides stability conditions for a planar switched linear system having two subsystems that are asymptotically stable. The key issue is that even though both subsystems are stable, the switched system may be unstable by switching at particular moments. Asymptotic stability under arbitrary switching can be proved by showing the existence of a common Lyapunov function (CLF). This type of stability can also be proved when the matrices of the subsystems commute. If the switched system does not have a CLF, the system must stay 'long enough' in each location to ensure stability. This is also known as the dwell time. A formula on the minimum dwell time for the switched system is provided. It will turn out that the formula is rather conservative and restrictive. Therefore another formula on the minimum dwell time is provided.

Keywords: Switched system, stability, common Lyapunov function, commuting matrices, dwell time

1 Introduction

In the past decades, there has been growing interest in switched systems. They consist of multiple dynamical subsystems and a switching signal. Applications can be found in the area of power systems [1] [2]. For example, some control systems consist of a supervisor. Instead of a fixed controller, the supervisor is able to change the system to the most suitable controller in response to the dynamics of the plant. This is especially useful in systems with large uncertainties [3]. Interestingly, switched systems do not contain the same properties of the individual subsystems. An important property is stability. A switched system whose subsystems are stable does not guarantee that the switched system is stable. It is even possible to stabilize switched systems whose subsystems are unstable [4]. Liberzon [5] discussed the basic problems of stability in switched systems when the subsystems are stable. Conditions for stability are mainly based on the existence of a common Lyapunov function (CLF) that guarantees stability under arbitrary switching [6]. If such function does not exist, restricting the switched system to stay 'long enough' in a subsystem also guarantees stability, if the subsystems are assumed to be asymptotically stable. This concept is also known as dwell time. Karabacak [7] derives two different formulas for the minimum dwell time.

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1.1 Problem context

There is a lot of variety in subsystems in a switched system, because they can be linear or nonlinear and stable or unstable. This adds more difficulty to the stability conditions of the switched system. This paper considers a specific switched system that provides a good basis for more complicated ones. The switched system of the form

$$\dot{x} = A_\sigma x, \tag{1}$$

will be analyzed, where σ is a function that can take the values $\{1, 2\}$ which may be based on for example time, state or behaviour of another system, and A_1, $A_2 \in \mathbb{R}^{n\times n}$ are Hurwitz. This means that the eigenvalues of A_1 and A_2 lie in the left half of the complex plane. Since the eigenvalues of A_1 and A_2 lie in the left half of the complex plane, the subsystems are asymptotically stable. Moreover, A_1 and A_2 are diagonalizable. See [8] for the non-diagonalizable case. A schematic overview of the switched system in (1) is illustrated below.

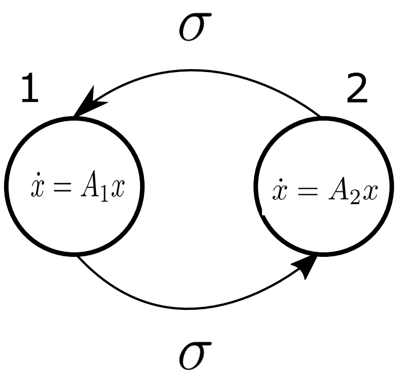

FIGURE 1: Schematic overview of the switched system

The individual subsystems are called location 1 and 2 respectively. To illustrate how instability may occur, consider the example below.

Example 1
We define a switched system as in (1) with the following phase planes of the subsystems.

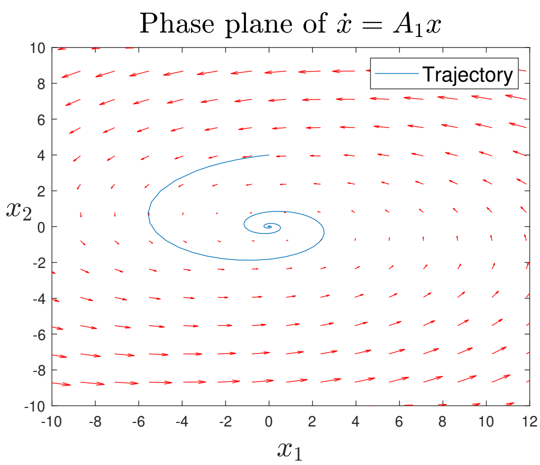

FIGURE 2: Phase plane of location 1

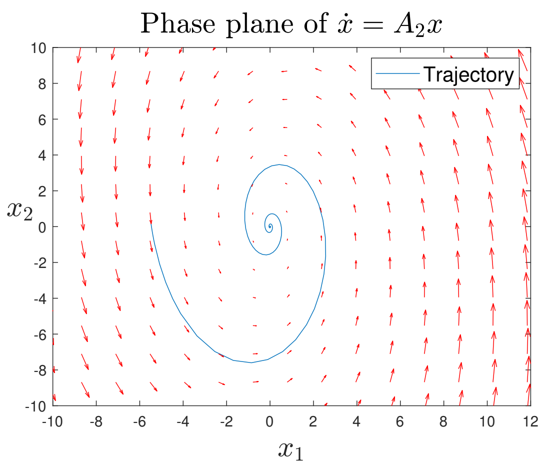

FIGURE 3: Phase plane of location 2

Switching at particular moments causes the trajectory to move away further from the origin. This is illustrated in the figure below.

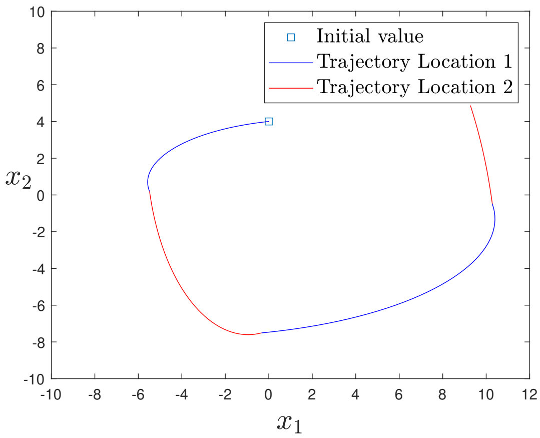

FIGURE 4: Instability in a switched system

From Figure 4 we get the impression that the solution diverges instead of converging to the origin.

This paper aims to find stability conditions for the system given by (1). Stability can be obtained in two ways. The first method is by showing that the switched system has a CLF. This means that the switching behaviour is independent on the stability and the system is therefore allowed to switch at any time. If a CLF cannot be found, we provide a minimum dwell time for each location before the system is allowed to switch.

In Section 2, the conditions for the existence of a CLF are derived. Then, a procedure is discussed that helps to determine the existence of a CLF. The special case when the matrices in the subsystems commute is also given, together with its connection to a CLF. In Section 3, dwell time will be analyzed. The first formula for the minimum dwell time given by Karabacak [7] will be revised. It will turn out that this dwell time is rather conservative. Consequently, another formula on the minimum dwell time is given.

2 Common Lyapunov function

Lyapunov functions are widely used to prove stability of dynamical systems described by differential equations. They can also be applied to prove stability of the switched system in (1). First, we look at asymptotic stability in a subsystem of (1) and then we provide a theorem on asymptotic stability on the switched system in (1). Some examples are worked out to find a CLF using the developed procedure. In the end, the special case is given when the matrices of the subsystems of (1) commute.

2.1 Lyapunov Stability

The definiteness of matrices plays a key role in Lyapunov stability of linear systems. For that reason, we shall give the definitions of positive and negative matrices. Next, some properties of a symmetric matrix are considered. These are needed for the theorem on asymptotic stability of the switched system of (1).

Definition 2.1. [9, p. 407] *Let $P \in \mathbb{R}^{n\times n}$ and symmetric: $P = P^{\mathrm{T}}$. Then, P is positive definite if and only if*

$$x^{\mathrm{T}}Px > 0 \quad \forall x \in \mathbb{R}^n \setminus 0. \tag{2}$$

Similarly, P is negative definite if and only if

$$x^{\mathrm{T}}Px < 0 \quad \forall x \in \mathbb{R}^n \setminus 0. \tag{3}$$

Some properties of a symmetric matrix are given in the following Lemma.

Lemma 2.2. [9, pp. $399, 407 - 408$] *Consider the symmetric matrix $P \in \mathbb{R}^{n\times n}$.*

1. *The eigenvalues of P are real*
2. *P is positive definite if and only if all eigenvalues of P are positive.*
3. *$\lambda_{\min}(P)x^{\mathrm{T}}x \leq x^{\mathrm{T}}Px \leq \lambda_{\max}(P)x^{\mathrm{T}}x \quad \forall x \in \mathbb{R}^n \setminus 0$,*
   where $\lambda_{\min}(P)$ (or $\lambda_{\max}(P)$) denotes the minimum (or maximum) eigenvalue of P.

Proof. 1. Let $Px = \lambda x$ for some $x \in \mathbb{R}^n \setminus 0$. It follows that

$$\lambda\bar{x}^{\mathrm{T}}x = \bar{x}^{\mathrm{T}}(\lambda x) = \bar{x}^{\mathrm{T}}Px = (P^{\mathrm{T}}\bar{x})^{\mathrm{T}}x = (\bar{P}\bar{x})^{\mathrm{T}}x = (\bar{\lambda x})^{\mathrm{T}}x = (\bar{\lambda}\bar{x})^{\mathrm{T}}x = \bar{\lambda}\bar{x}^{\mathrm{T}}x.$$

Since $x \neq 0$, $\bar{x}^{\mathrm{T}}x \neq 0$ so $\lambda = \bar{\lambda}$, hence λ is real.
2. $\Rightarrow$ Let λ be an eigenvalue of matrix P and x the corresponding eigenvector. This means that

$$Px = \lambda x$$

and multiplying both sides by x^{T} gives

$$x^{\mathrm{T}}Px = \lambda||x||^2. \tag{4}$$

Matrix P is positive definite so (4) must be positive. The norm $||x||^2$ is strictly positive since it is a nonzero vector. It follows that $\lambda > 0$.
$\Leftarrow$ Since P is symmetric, it is orthogonally diagonalizable by the spectral theorem [9, p. 399]. This means that P can be written in the form

$$P = MDM^{\mathrm{T}}, \tag{5}$$

where M is an orthogonal matrix and D a diagonal matrix containing the eigenvalues of P. For arbitrary $x \in \mathbb{R}^n \setminus 0$ and using (5) we get that

$$x^{\mathrm{T}}Px = x^{\mathrm{T}}MDM^{\mathrm{T}}x = y^{\mathrm{T}}Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots\lambda_n y_n^2, \tag{6}$$

where $y = M^{\mathrm{T}}x$. By hypothesis, all eigenvalues of P are positive so it follows that (6) is also positive. Hence, by Definition 2.1, P is positive definite.

3. By taking the maximum or minimum eigenvalue in (6), the result in 3 of Lemma 2.2 follows. $\square$

The following lemma provides more insight in the theorem on asymptotic stability of linear systems.

Lemma 2.3. [10, pp. 266 − 267] *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz.*

1. *For any $Q = Q^{\mathrm{T}}$, there exists a unique solution $P = P^{\mathrm{T}}$ such that*
$$A^{\mathrm{T}}P + PA = -Q \tag{7}$$
2. *If $Q > 0$, then $P > 0$*

Proof. 1. Define
$$P = \int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}dt. \tag{8}$$
This integral converges because A is Hurwitz. Substituting (8) into (7) gives
$$\begin{aligned}
A^{\mathrm{T}}P + PA &= A^{\mathrm{T}}\int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}dt + \int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}dt \cdot A \\
&= \int_0^\infty A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}Adt \\
&= \int_0^\infty \frac{d}{dt}\left(e^{A^{\mathrm{T}}t}Qe^{At}\right)dt \\
&= \left[e^{A^{\mathrm{T}}t}Qe^{At}\right]_0^\infty \\
&= -Q,
\end{aligned}$$
where we use the fact that $\lim_{t\longrightarrow\infty} e^{A^{\mathrm{T}}t}Qe^{At} = 0$. For the uniqueness, consider two solutions P_1 and P_2 such that
$$A^{\mathrm{T}}P_1 + P_1A = -Q \tag{9}$$
$$A^{\mathrm{T}}P_2 + P_2A = -Q. \tag{10}$$
We shall show that $P_1 = P_2$. Subtracting (10) from (9) gives
$$\begin{aligned}
0 &= A^{\mathrm{T}}(P_1 - P_2) + (P_1 - P_2)A \\
&= e^{A^{\mathrm{T}}t}(A^{\mathrm{T}}(P_1 - P_2) + (P_1 - P_2)A)e^{At} \\
&= e^{A^{\mathrm{T}}t}A^{\mathrm{T}}(P_1 - P_2)e^{At} + e^{A^{\mathrm{T}}t}(P_1 - P_2)Ae^{At} \\
&= \frac{d}{dt}\left(e^{A^{\mathrm{T}}t}(P_1 - P_2)e^{At}\right).
\end{aligned}$$
This means that $e^{A^{\mathrm{T}}t}(P_1 - P_2)e^{At}$ is a constant function. Therefore, evaluating at $t = 0$ gives
$$\left(e^{A^{\mathrm{T}}t}(P_1 - P_2)e^{At}\right) = P_1 - P_2 \quad \forall\, t \geq 0. \tag{11}$$
Letting $t \longrightarrow \infty$ in (11) we get that $P_1 - P_2 = 0$, so $P_1 = P_2$, hence the solution is unique.
2. For a vector $x \in \mathbb{R}^n \setminus 0$ and using (8), we get that
$$\begin{aligned}
x^{\mathrm{T}}Px &= \int_0^\infty x^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}xdt \\
&= \int_0^\infty (e^{At}x)^{\mathrm{T}}Q(e^{At}x)dt.
\end{aligned} \tag{12}$$
Therefore, if $Q > 0$ the right side of (12) is positive, hence P is positive definite. □

Below follows the Lyapunov stability theorem on asymptotic stability of linear systems.

Theorem 2.4 (Lyapunov stability theorem)**.** [10, pp. 263 – 264] *Consider the system* $\dot{x} = Ax$, $A \in \mathbb{R}^{n \times n}$. *Let* A, $P = P^{\mathrm{T}}$ *and* $Q = Q^{\mathrm{T}}$ *satisfy*

$$A^{\mathrm{T}}P + PA = -Q.$$

If $P > 0$ *and* $Q > 0$, *then the system is asymptotically stable. The corresponding Lyapunov function is* $V(x) = x^{\mathrm{T}}Px$.

The subsystems in the switched system (1) satisfy Theorem 2.4. The question now is how Theorem 2.4 can be extended to the switched system in (1). The idea is to have a matrix P that satisfies the Lyapunov equation in both subsystems. This is given in the following theorem:

Theorem 2.5. *The switched system* $\dot{x} = A_{\sigma}x$, *where* σ *can take the values* $1, 2$ *and* $A \in \mathbb{R}^{n \times n}$ *is asymptotically stable if there exist matrices* $P = P^{\mathrm{T}}$, $Q_1 = Q_1^{\mathrm{T}} > 0$ *and* $Q_2 = Q_2^{\mathrm{T}} > 0$ *such that the following is satisfied:*

$$P > 0, \tag{13}$$

$$A_1^{\mathrm{T}}P + PA_1 = -Q_1 \tag{14}$$

$$A_2^{\mathrm{T}}P + PA_2 = -Q_2 \tag{15}$$

The corresponding Lyapunov function is $V(x) = x^{\mathrm{T}}Px$ *and is called a common Lyapunov function (CLF).*

Proof. Let P satisfy the inequalities (13), (14) and (15) for some positive definite matrices Q_1 and Q_2. Then, the time derivative of $V(x)$ is negative in both locations 1 and 2:

$$\frac{d}{dt}V(x(t)) = \begin{cases} -x(t)^{\mathrm{T}}Q_1x(t), & \text{if in location 1} \\ -x(t)^{\mathrm{T}}Q_2x(t), & \text{if in location 2} \end{cases} \tag{16}$$

The idea is to find an upper bound for (16) and use the same procedure of proof as in Theorem 2.4. Using 3 in Lemma 2.2 for Q_1 and Q_2, there exists a positive definite symmetric matrix Q such that $Q \leq Q_1$ and $Q \leq Q_2$:

$$Q = \alpha I, \tag{17}$$

where $\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}$. It follows that

$$\frac{d}{dt}V(x(t)) \leq -x(t)^{\mathrm{T}}Qx(t). \tag{18}$$

Now, we use 3 in Lemma 2.2 again to get that

$$x(t)^{\mathrm{T}}Qx(t) \geq \lambda_{\min}(Q)x(t)^{\mathrm{T}}x(t) \quad \text{and} \quad x(t)^{\mathrm{T}}Px(t) \leq \lambda_{\max}(P)x(t)^{\mathrm{T}}x(t). \tag{19}$$

The inequalities in (19) imply that

$$\frac{x(t)^{\mathrm{T}}Qx(t)}{V(x(t))} \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} := \beta. \tag{20}$$

Using (18) and (20), it follows that

$$\frac{d}{dt}V(x(t)) \leq -x(t)^{\mathrm{T}}Qx(t) \leq -\beta V(x(t)),$$

which means that

$$\frac{d}{dt}V(x(t)) \leq -\beta V(x(t)). \tag{21}$$

Integrating (21) gives that

$$V(x(t)) \leq e^{-\beta t}V(x(0)). \tag{22}$$

As $t \longrightarrow \infty$ in (22), it follows that $V(x(t)) \longrightarrow 0$. Moreover,

$$V(x(t)) \geq \lambda_{\min}(P)x(t)^{\mathrm{T}}x(t) = \lambda_{\min}||x(t)||_2^2, \tag{23}$$

where $||x(t)||_2$ denotes the Euclidean length, so as $t \longrightarrow \infty$, (23) implies that $||x(t)||_2 \longrightarrow 0$. Therefore every solution of the system converges to 0, hence it is asymptotically stable. □

We want to emphasize that the choice of P does not matter, as long as it satisfies the conditions (13), (14) and (15). Moreover, the existence of a CLF is only a sufficient condition. In [11] Dayawansa gives an example of a switched system that does not have a CLF but is asymptotically stable.

Given a switched system of the form in (1), we want to check if there exists a matrix P such that conditions (13), (14) and (15) are satisfied. The conditions are based on the definiteness of the matrices. Checking the definiteness is not convenient to do using Definition 2.1. Instead, we shall use Sylvester's criterion. This is based on the definiteness of the principal minors.

Definition 2.6. [12] *Let $A \in \mathbb{R}^{n\times n}$. For $1 \leq k \leq n$, the kth principal submatrix of A is the $k \times k$ submatrix by taking the first k rows and columns of A. Its determinant is the kth principal minor.*

Theorem 2.7 (Sylvester's criterion [12]). *A real symmetric matrix P is positive definite if and only if all its principal minors are positive.*

Remark 1. *A real symmetric matrix P is negative definite if $-P$ is positive definite.*

Finding a suitable $n \times n$ matrix P quickly becomes complicated because each entry of matrix P is considered to be a variable. For that reason, this paper shows a procedure to check the existence of a matrix P in the case when 2×2 matrices are considered. Since $P = P^{\mathrm{T}}$, the most general form would be

$$P = \begin{bmatrix} p & \tilde{q} \\ \tilde{q} & \tilde{r} \end{bmatrix},$$

where $p, \tilde{q}, \tilde{r} \in \mathbb{R}$, but it is possible to reduce the number variables. Matrix P must be positive definite, so by Sylvester's criterion $p > 0$ and $\det(P) > 0$. Dividing each entry of P by the scalar p gives

$$P = \begin{bmatrix} 1 & q \\ q & r \end{bmatrix}, \tag{24}$$

where $q, r \in \mathbb{R}$. Since the scalar p is positive, it does not change the sign of the determinant. Therefore, without loss of generality, we can consider (24).

We have developed a procedure in *Mathematica* that shows the region of the existence of matrix P satisfying (13), (14) and (15) in the rq-plane, given two 2×2 Hurwitz matrices A_1 and A_2 of the switched system in (1). The procedure is as follows. Using the form of P as in (24), we expand (13), (14) and (15) in terms of r and q using Sylvester's criterion. For (13), this gives 1 inequality, since the upper left entry is already greater than 0. Inequalities (14) and (15) give 4 inequalities, so in total we have 5 inequalities. The regions for which the respective inequalities hold are drawn in the rq-plane. The intersection of those regions represents the possible values for r and q. If the intersection is empty, matrix P does not exist. This means that we cannot conclude if the switched system is asymptotically stable. If the intersection is non-empty, we pick a point in the intersection such that P satisfies (13), (14) and (15) to conclude that the switched system is stable. In the next section, the procedure is illustrated using some examples.

2.2 Illustrative examples

Example 2
Consider the system (1) where

$$A_1 = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{25}$$

The goal is to check whether or not a matrix P satisfies (13), (14) and (15). The inequalities that follow are

$$q^2 + r > 0 \tag{26}$$
$$-6 + 8q < 0 \tag{27}$$
$$-4 + 16q - 36q^2 + 4r + 32qr - 16r^2 > 0 \tag{28}$$
$$-2 < 0 \tag{29}$$
$$-9q^2 + 8r > 0 \tag{30}$$

The regions for which the above respective inequalities hold are depicted in Figure 5 below.

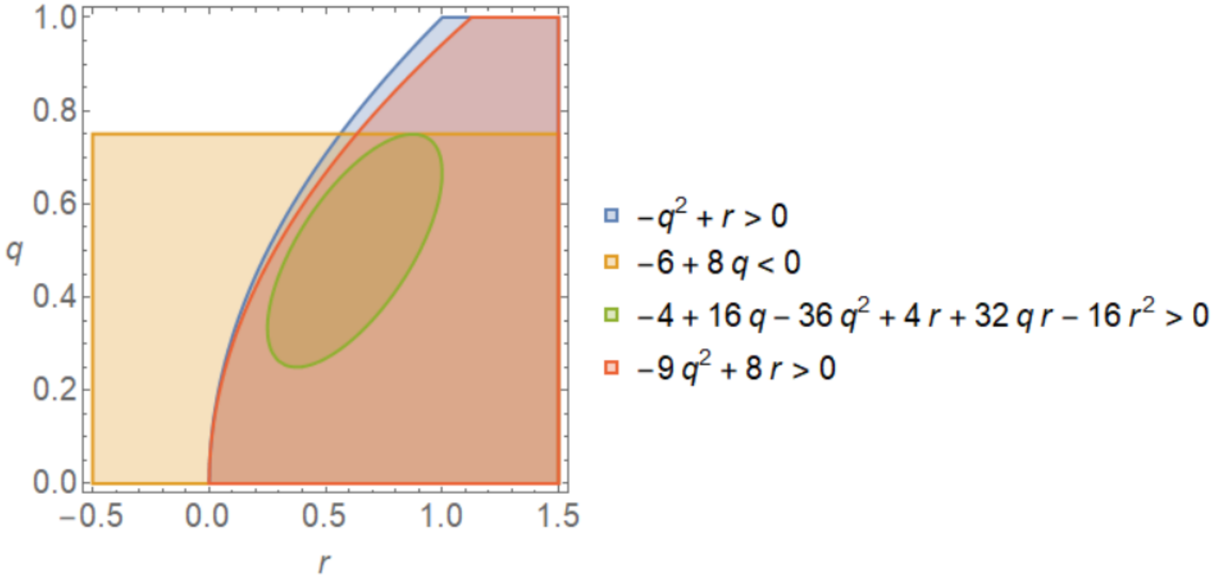

FIGURE 5: All regions where the respective inequalities hold are depicted in one figure.

Inequality (29) is automatically satisfied and therefore not included in Figure 5. The shaded areas Figure 5 represent the regions corresponding to each inequality. From Figure 5 we can see that the non-empty intersection of all inequalities is the green region. A picture does not prove that we have found a matrix P. Consequently, we pick a point inside the intersection, for example $(0.5, 0.5)$, to prove that the matrix P exists. Then it follows that

$$P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \tag{31}$$

We verify that this matrix P satisfies the inequalities (13), (14) and (15):

$$\det(P) = \det \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = 0.25 > 0 \tag{32}$$

$$\det(A_1^{\mathrm{T}} P + P A_1) = \det \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} = 1 > 0 \tag{33}$$

$$\det(A_2^{\mathrm{T}} P + P A_2) = \det \begin{bmatrix} -2 & -1.5 \\ -1.5 & -2 \end{bmatrix} = 1.75 > 0 \tag{34}$$

Using Sylvester's criterion, we conclude that (32) is positive definite, while (33) and (34) are negative definite. Hence, by Theorem 2.5, the switched system (1) described by (25) is asymptotically stable.

Example 3
Consider the system (1) where

$$A_1 = \begin{bmatrix} -0.2 & -5 \\ 1 & -0.3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -0.4 & -1 \\ 5 & -0.6 \end{bmatrix}. \tag{35}$$

This is the same switched system described by Example 1 in the introduction, for which we had the impression that the switched system is not asymptotically stable. Similar to Example 2, we obtain the following inequalities:

$$-q^2 + r > 0 \tag{36}$$
$$-0.4 + 2q < 0 \tag{37}$$
$$-25 - 1.q - 20.25q^2 + 10.24r - 0.2qr - r^2 > 0 \tag{38}$$
$$-0.8 + 10q < 0 \tag{39}$$
$$-1 - 0.4q - 21.q^2 + 10.96r - 2.qr - 25r^2 > 0 \tag{40}$$

The regions where the respective inequalities hold, are depicted in the Figure 6 below.

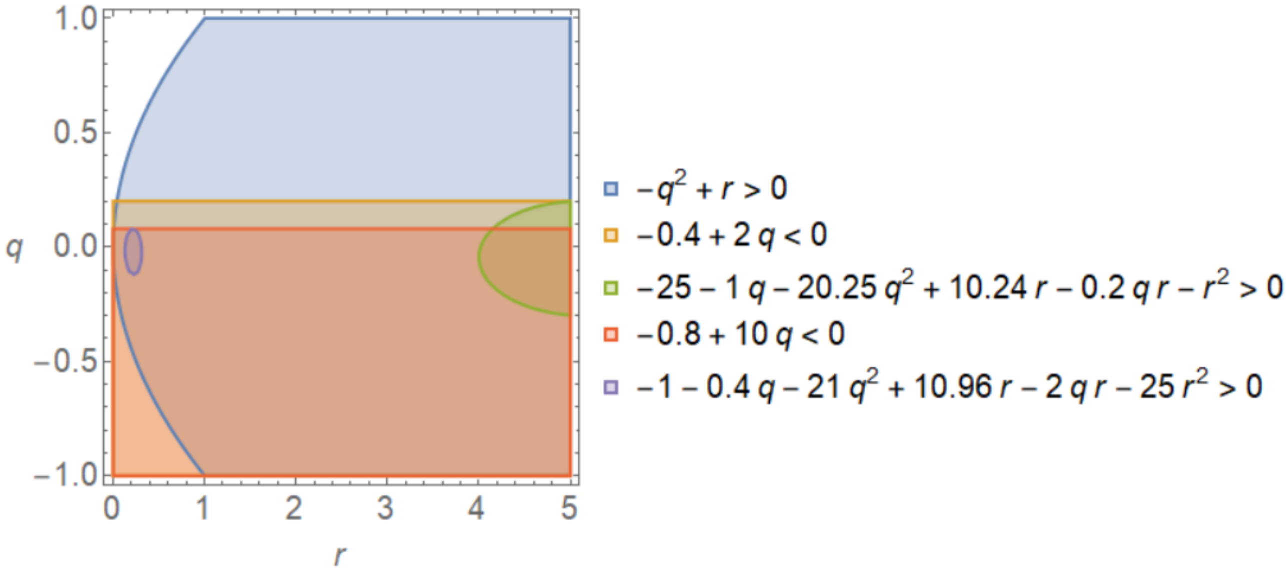

FIGURE 6: All regions where the respective inequalities hold are drawn in one figure.

In Figure 6 we see that the regions described by inequalities (38) and (40) do not intersect. Therefore the intersection of all regions where the respective inequalities hold, is empty. This means that there does not exist a matrix P satisfying the inequalities (13), (14) and (15). This confirms the impression of the switched system being unstable.

2.3 Commuting matrices

A special case for which the switched system (1) is stable under arbitrary switching is when the matrices A_1 and A_2 commute. The proof is based on a property of matrix exponentials when the matrices commute. First, the definition of the matrix exponential is given and after that the corresponding property.

Definition 2.8. [13, p. 417] *For any $n \times n$ matrix A, the matrix exponential is defined as*

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} \tag{41}$$

Lemma 2.9. [13, p. 420] *Let A and B matrices in $\mathbb{R}^{n\times n}$. If $AB = BA$, then*

1. $e^A e^B = e^{A+B}$

2. $e^A e^B = e^B e^A$

Proof. 1. We expand the infinite sum of the matrix exponential and use that A and B commute:

$$\begin{aligned}
e^A e^B &= \left(\sum_{m=0}^{\infty} \frac{A^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right) \\
&= \left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...\right)\left(I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + ...\right) \\
&= I + (A+B) + \frac{1}{2!}(A^2 + 2AB + B^2) + \frac{1}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + ... \\
&= I + (A+B) + \frac{1}{2!}(A^2 + AB + BA + B^2) \\
&+ \frac{1}{3!}(A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3) + ... \\
&= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + ... \\
&= e^{A+B}
\end{aligned}$$

2. Using relation 1 in Lemma 2.9 it follows that

$$e^A e^B = e^{A+B} = e^{B+A} = e^B e^A.$$

□

Theorem 2.10. [6] *If the matrices A_1 and A_2 in (1) commute, that is $A_1A_2 = A_2A_1$, then the switched system is stable under arbitrary switching.*

Proof. The general solution of the switched system in (1) is given by

$$x(t) = e^{A_1t_1}e^{A_2s_1}e^{A_1t_2}e^{A_2s_2}...x_0,$$

where t_i and s_i $(i = 1, 2, ...)$ denote the time durations that the system is in location 1 and 2 respectively. Since the matrices A_1 and A_2 commute, use Lemma 2.9 to get

$$\begin{aligned}
x(t) &= e^{A_1t_1}e^{A_1t_2}...e^{A_2s_1}e^{A_2s_2}...x_0 \\
&= e^{A_1(t_1+t_2+...)}e^{A_2(s_1+s_2+...)}x_0
\end{aligned}$$

At least one series $t_1 + t_2 + ...$ or $s_1 + s_2 + ...$ converges to infinity as t goes to infinity. Since both subsystems are asymptotically stable, the matrix exponential corresponding to the series that converges to infinity goes to zero, which means that $x(t)$ converges to 0. Hence the system is stable under arbitrary switching. □

Alternatively, we can use common Lyapunov functions to prove that (1) is stable under arbitrary switching when the matrices in the subsystems commute.

Theorem 2.11. [14] *Consider the switched system in* (1) *and $A_1A_2 = A_2A_1$. Given a symmetric positive definite matrix P_0, let P_1 and P_2 be unique symmetric positive definite matrices that satisfy*

$$A_1^{\mathrm{T}}P_1 + P_1A_1 = -P_0 \tag{42}$$

$$A_2^{\mathrm{T}}P_2 + P_2A_2 = -P_1. \tag{43}$$

Then $A_1^{\mathrm{T}}P_2 + P_2A_1$ is negative definite.

Proof. Substituting P_1 from (43) into (42) and using the fact that A_1 and A_2 commute, we get

$$\begin{aligned} P_0 &= -A_1^{\mathrm{T}} P_1 - P_1 A_1 \\ &= A_1^{\mathrm{T}} \left(A_2^{\mathrm{T}} P_2 + P_2 A_2\right) + \left(A_2^{\mathrm{T}} P_2 + P_2 A_2\right) A_1 \\ &= A_1^{\mathrm{T}} A_2^{\mathrm{T}} P_2 + A_1^{\mathrm{T}} P_2 A_2 + A_2^{\mathrm{T}} P_2 A_1 + P_2 A_2 A_1 \\ &= A_2^{\mathrm{T}} \left(A_1^{\mathrm{T}} P_2 + P_2 A_1\right) + \left(A_1^{\mathrm{T}} P_2 + P_2 A_1\right) A_2. \end{aligned}$$

Since A_2 is Hurwitz and P_0 is positive definite, it follows from Lemma 2.3 that $A_1^{\mathrm{T}} P_2 + P_2 A_1 < 0$. This means that $V(x) = x^{\mathrm{T}} P_2 x$ is a CLF. $\square$

3 Dwell time

From the previous paragraph, it could be seen that (1) is asymptotically stable under arbitrary switching if a common Lyapunov function exists. The question now arises how asymptotic stability is preserved when there does not exist a common Lyapunov function. Since both subsystems of (1) are asymptotically stable, it is possible to guarantee asymptotic stability given that the system stays 'long enough' in both locations. This concept is also known as the dwell time.

Definition 3.1. *For a system of the form $\dot{x} = Ax$, $A \in \mathbb{R}^{n \times n}$ and Hurwitz, the minimum dwell time is given by*

$$\tau_A = \max_{||x_0||=1} \{\min\{t_0 \geq 0 \mid \forall\ t \geq t_0 \quad ||\underbrace{e^{At} x_0}_{x(t)}|| \leq ||x_0||, ||x_0|| = 1\}\}. \tag{44}$$

In Definition 3.1 the maximum and minimum of a set are considered. To show that the minimum exists, the idea is to consider a fixed initial condition x_0. Then there always exists a t_0 since $e^{At} x_0 \longrightarrow 0$ as $t \longrightarrow \infty$. In addition, the set $\min\{t_0 \geq 0 \mid \forall\ t \geq t_0 \quad ||\underbrace{e^{At} x_0}_{x(t)}|| \leq ||x_0||, ||x_0|| = 1\}$ is bounded below by 0 and this value is contained in the set, so the minimum exists. It remains to show that the set is compact. Then, it follows that a maximum exists. To explain the concept of the minimum dwell time in more detail, consider Figure 7 below.

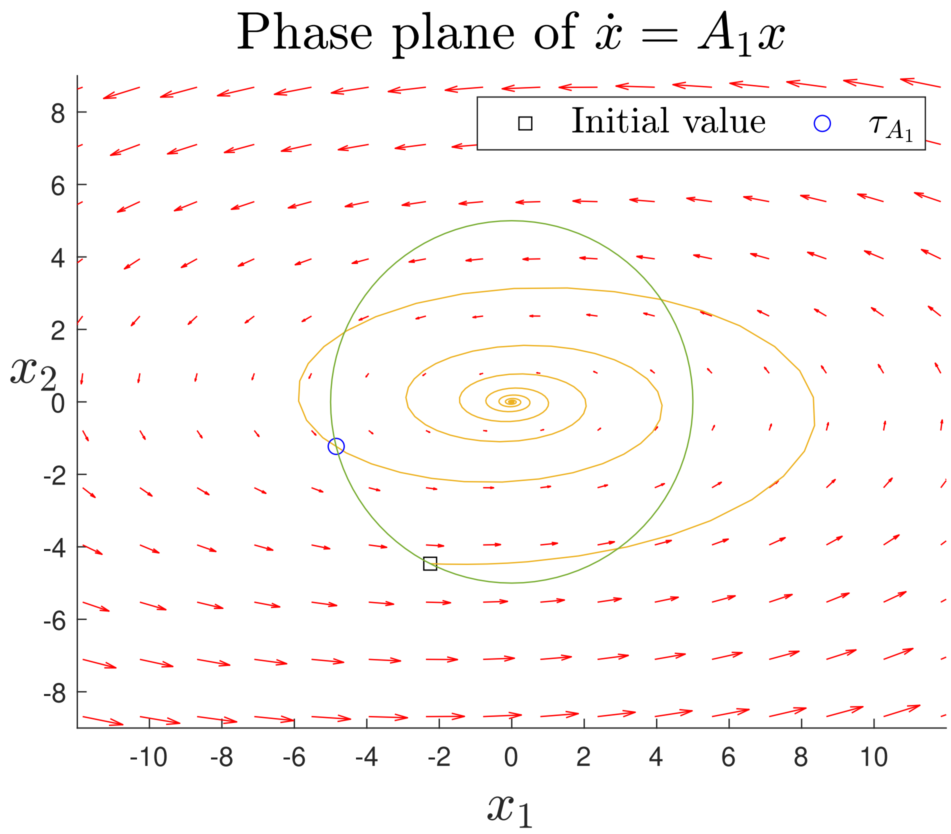

FIGURE 7: Phase plane of a system of the form $\dot{x} = A_1 x$

In Figure 7, the trajectory of the solution of the system with initial conditions at the black square is drawn in yellow. The green circle indicates the distance from the initial condition to the origin. The dwell time of this system is given by the blue circle. As can be seen from the figure, after time τ_A the solution stays inside the green circle.

Considering switched systems, this minimum dwell time is very important. After the dwell time, the distance from the trajectory to the origin will always be smaller than the distance from the initial condition to the origin. For a switched system in (1), there are two dwell times, one for each location. If we wait at least the corresponding dwell time in each location, it is impossible to create a trajectory that is greater - in the sense of the distance to the origin - than the initial condition. Consequently, each trajectory converges to 0. The goal is find a formula for the minimum dwell time. Before giving the formula, some definitions and lemmas shall be discussed. Consider the following matrix norm:

Definition 3.2. [15, p. 343] *For any $n \times n$ matrix A and $x \in K^n$, the induced Euclidean norm is given by*

$$||A||_2 = \max_{||x||_2 = 1} ||Ax||_2,$$

where $||x||_2 = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ is the Euclidean norm in $\mathbb{R}^n$.

This is also known as the spectral norm. The property that is needed for the formula on the minimum dwell time is given in the following lemma.

Lemma 3.3. [15, p. 344] *The induced Euclidean norm is a submultiplicative matrix norm: for any $n \times n$ matrices A and B,*

$$||AB||_2 \leq ||A||_2 \cdot ||B||_2$$

Proof. From the definition of the induced Euclidean norm, it follows that

$$||A||_2 = \max_{||x||_2=1} ||Ax||_2 \geq ||Ax||_2 \quad \forall\, ||x||_2 = 1. \tag{45}$$

Using (45), it follows that

$$\begin{aligned} ||AB||_2 &= \max_{||x||_2=1} ||ABx||_2 \leq \max_{||x||_2=1} ||A||_2 \cdot ||Bx||_2 \\ &\leq \max_{||x||_2=1} ||A||_2 \cdot ||B||_2 \cdot ||x||_2 = ||A||_2 \cdot ||B||_2. \end{aligned}$$

This means that $|AB||_2 \leq ||A||_2 \cdot ||B||_2$. □

However, the definition of the induced Euclidean norm is not very convenient to use when calculating a matrix norm. Consequently, the following lemma removes this difficulty. The proof makes use the singular value decomposition [15, pp. 150 – 151].

Lemma 3.4. [15, p. 346] *Let $A \in \mathbb{R}^{n\times n}$. The induced Euclidean norm reduces to*

$$||A||_2 = \sqrt{\lambda_{\max} A^{\mathrm{T}} A}, \tag{46}$$

where $\lambda_{\max}$ denotes the greatest eigenvalue of matrix $A^{\mathrm{T}}A$.

Proof. The singular value decomposition allows us to decompose matrix A as follows:

$$A = U\Sigma V^{\mathrm{T}},$$

where $U, \Sigma, V^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. More specifically, U and V^{T} are unitary matrices and Σ is a diagonal matrix containing the singular values of A, which are denoted by $\sigma_i(A) = \sqrt{\lambda_i(A^{\mathrm{T}}A)}$. The singular value decomposition is not unique, so for convenience, choose Σ such that the diagonal entries are in descending order. This means that the greatest eigenvalue is the first diagonal entry. Since U is unitary, note that

$$||Ux||_2 = \sqrt{\langle Ux, Ux\rangle} = \sqrt{\langle x, U^{\mathrm{T}}Ux\rangle} = \sqrt{\langle x, x\rangle} = ||x||_2. \tag{47}$$

Using Definition 3.2 and (47), it follows that

$$\begin{aligned} ||A||_2 &= \max_{||x||_2=1} ||Ax||_2 = \max_{||x||_2=1} ||U\Sigma V^{\mathrm{T}}x||_2 \\ &= \max_{||x||_2} ||\Sigma V^{\mathrm{T}}x||_2 = \max_{||y||_2=1} ||\Sigma y||_2 = \sigma_1(A) = \sqrt{\lambda_{\max} A^{\mathrm{T}} A}, \end{aligned}$$

for $y = [1 \;\; 0 \;\; ... \;\; 0]^{\mathrm{T}}$. □

The last lemma shows that the spectral norm of a matrix exponential of a diagonal matrix can be calculated.

Lemma 3.5. *Consider the diagonal Hurwitz matrix $D \in \mathbb{C}^{n\times n}$. It follows that*

$$||e^D||_2 = e^{-\lambda^*},$$

where $\lambda^* = |\mathrm{Re}(\lambda_{\max}(D))|$.

Proof. Consider the matrix

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{48}$$

where

$$\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq ... \leq \mathrm{Re}(\lambda_n) < 0. \tag{49}$$

A similar proof holds when the order of eigenvalues in (49) is different . The eigenvalues of $(e^D)^* e^D$ (in this case we consider the conjugate transpose denoted by an $*$, since $D \in \mathbb{C}^{n\times n}$) are given by:

$$(e^D)^* e^D = \begin{bmatrix} e^{\bar{\lambda_1}} & & 0 \\ & \ddots & \\ 0 & & e^{\bar{\lambda_n}} \end{bmatrix} \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} = \begin{bmatrix} e^{2\mathrm{Re}\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\mathrm{Re}\lambda_n} \end{bmatrix}$$

The eigenvalues of the above matrix are the diagonal entries. From (49), we conclude that the largest eigenvalue is given by $e^{2\mathrm{Re}(\lambda_n)}$, so it follows that

$$||e^D||_2 = \sqrt{\lambda_{\max}\left((e^D)^* e^D\right)} = \sqrt{e^{2\mathrm{Re}(\lambda_n)}} = e^{-\lambda^*}.$$

□

We are now ready to state the theorem of the minimum dwell time of a linear system.

Theorem 3.6. [7] *For a system of the form $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$ Hurwitz and diagonalizable, the minimum dwell time is given by*

$$\tau_A = \frac{\log\left(||H||_2 \cdot ||H^{-1}||_2\right)}{\lambda^*}, \tag{50}$$

where H denotes the modal matrix of A and $\lambda^* = |Re(\lambda_{\max}(A))|$.

Proof. The general solution $\dot{x} = Ax$ is given by

$$x(t) = e^{tA}x_0 = He^{tD}H^{-1}x_0,$$

where H is the modal matrix of A, D the diagonal matrix containing the eigenvalues of matrix A and x_0 the initial condition. Taking the Euclidean norm gives the following:

$$\begin{aligned} ||x(t)||_2 &= ||He^{tD}H^{-1}x_0||_2 \\ &\leq ||H||_2 \cdot ||e^{tD}||_2 \cdot ||H^{-1}||_2 \cdot ||x_0||_2 \\ &= ||H||_2 \cdot ||H^{-1}||_2 \cdot e^{-t\lambda^*} \cdot ||x_0||_2, \end{aligned}$$

where in the second line, Lemma 3.3 is used. In the last line Lemma 3.5 is used. Substituting this relation in the definition of the dwell time (44) gives

$$||H||_2 \cdot ||H^{-1}||_2 \cdot e^{-t\lambda^*} \leq 1. \tag{51}$$

The norm of the modal matrix H and the inverse are strictly positive, so the left part of (51) is a strictly decreasing function of t. Extract the variable t to get

$$e^{-t\lambda^*} \leq \frac{1}{||H||_2 \cdot ||H^{-1}||_2}$$

$$-t\lambda^* \leq \log\left(\frac{1}{||H||_2 \cdot ||H^{-1}||_2}\right)$$

$$t \geq \frac{\log\left(||H||_2 \cdot ||H^{-1}||_2\right)}{\lambda^*}.$$

□

Now that we have a minimum dwell time for a linear system, the goal is to apply this to the switched system in (1). The general solution of the switched system is given by

$$x(t) = e^{A_1 t_1} e^{A_2 s_1} e^{A_1 t_2} e^{A_2 s_2} \ldots x_0,$$

where t_i and s_i denote the time durations that the switched system is in location 1 and 2 respectively. We can calculate the dwell times of the subsystems of (1). Let's call these τ_{A_1} and τ_{A_2}. Then, if

$$t_i \geq \tau_{A_1} \quad \text{and} \quad s_i \geq \tau_{A_2} \quad \forall \quad i = 1, 2, \ldots \tag{52}$$

the switched system is asymptotically stable, since in each location we wait at least a period of time such that the solution is less than or equal to the start.

3.1 Illustrative examples

Example 4

Consider the system

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} x.$$

The corresponding dwell time is $\tau_A = 0.9624$. For two different initial conditions, we have drawn the solution in the phase plane. The circle indicates the distance from the origin to the initial condition.

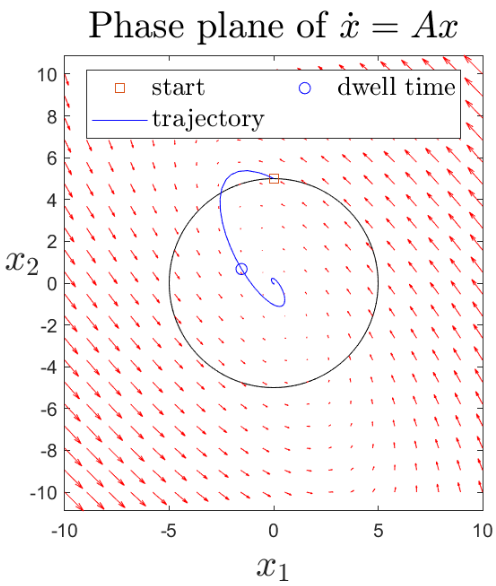

FIGURE 8: Solution to the system when the initial condition is $(0, 5)$

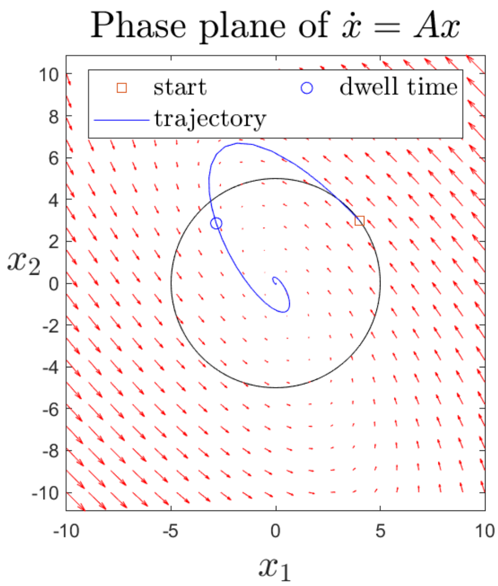

FIGURE 9: Solution to the system when the initial condition is $(4, 3)$

In Figure 8 it can be seen that the trajectory stays inside the circle a lot earlier than the calculated dwell time. In Figure 9, the calculated dwell time corresponds better to when the solution stays inside the circle. This shows that the dwell time heavily depends upon the initial condition. Therefore the formula for the dwell time is rather conservative. This comes down to the fact that we maximize over the initial condition. For some initial conditions, the dwell time is smaller than the calculated minimum dwell time.

3.2 A less restrictive constraint

As could be seen from Example 4, the dwell time formula is rather conservative. In addition, in each location we have to wait at least the dwell time before switching, which is quite restrictive. Consequently, we want to look at a less conservative dwell time with less restrictions. Instead of looking at the minimum dwell times of the individual subsystems of the switched system, we consider the minimum dwell time of the switched system. Suppose we start in location 1, distance r_1 from the origin and switch to the other location at a period of time earlier than the dwell time of location 1. Then, we end up at distance r_2 from the origin, $r_1 < r_2$. To ensure stability, we must stay at least a period of time in location 2 such that we end up at distance r_1 or less from the origin. This is depicted in the figure below.

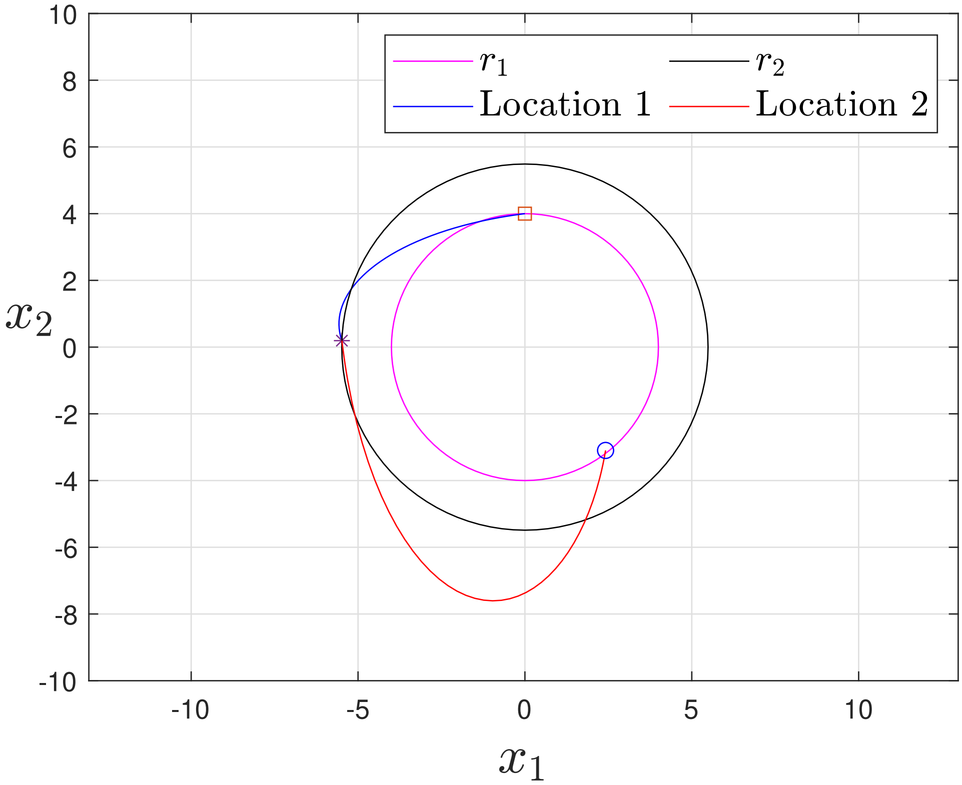

FIGURE 10: A possible trajectory of the switched system.

In Figure 10, the initial condition is given by a small red square. Starting in location 1, the system switches to location 2 at the asterisk in the figure. When the trajectory is at the blue small circle, the system is allowed to switch again. Therefore, if we find a lower bound on the sum of the time periods in location 1 and 2, we can establish asymptotic stability. This allows the system to switch earlier than the dwell time, making switching less restrictive. This is given in the theorem below.

Theorem 3.7. *Consider the switched system given in* (1)*. Let* t_i *and* s_i *(*$i = 1, 2, ...$*) denote the time durations in location* 1 *and* 2 *respectively and* $T := t_i + s_i$*. If*

$$T = t_i + s_i \geq \frac{\log\left(||H_1||_2 \cdot ||H_1||_2^{-1} \cdot ||H_2||_2 \cdot ||H_2||_2^{-1}\right)}{\min_{i=1,2}\left(\lambda_i^*\right)} \quad \forall i = 1, 2, ...$$

then (1) *is asymptotically stable.*

Proof. The general solution of the switched system (1) is given by

$$x(t) = e^{A_1 t_1} e^{A_2 s_1} e^{A_1 t_2} e^{A_2 s_2} ... x_0,$$

where t_i and s_i denote the time durations that the switched system is in location 1 and 2 respectively. Using Lemma 3.3, it follows that

$$\begin{aligned} ||x(t)||_2 &= ||e^{A_1 t_1} e^{A_2 s_1} e^{A_1 t_2} e^{A_2 s_2} ... x_0||_2 \\ &\leq ||e^{A_1 t_1} e^{A_2 s_1} x_0||_2 \cdot ||e^{A_1 t_2} e^{A_2 s_2}||_2 \cdot .... \end{aligned} \tag{53}$$

Consider the part $||e^{A_1 t_1} e^{A_2 s_1} x_0||_2$ in (53). Similar to the proof of Theorem 3.6, we get that

$$\begin{aligned} ||x(t)|| &= ||e^{A_1 t_1} e^{A_2 s_1}||_2 \\ &= ||H_1 e^{D_1 t_1} H_1^{-1} H_2 e^{D_2 s_2} H_2^{-1}||_2 \\ &\leq e^{-\lambda_1^* t_1} \cdot e^{-\lambda_2^* s_1} ||H_1||_2 \cdot ||H_1^{-1}||_2 \cdot ||H_2||_2 \cdot ||H_2^{-1}||_2 \\ &= e^{-(\lambda_1^* t_1 + \lambda_2^* s_1)} \cdot ||H_1||_2 \cdot ||H_1^{-1}||_2 \cdot ||H_2||_2 \cdot ||H_2^{-1}||_2 \\ &\leq e^{-\min_{i=1,2}(\lambda_i^*)(t_1 + s_1)} \cdot ||H_1||_2 \cdot ||H_1^{-1}||_2 \cdot ||H_2||_2 \cdot ||H_2^{-1}||_2, \end{aligned}$$

so it follows that if

$$t_1 + s_1 \geq \frac{\log\left(||H_1||_2 \cdot ||H_1^{-1}||_2 \cdot ||H_2||_2 \cdot ||H_2^{-1}||_2\right)}{\min_{i=1,2}(\lambda_i^*)}.$$

the part $||e^{A_1 t_1} e^{A_2 s_1}||_2$ is always smaller than the initial condition x_0. By similar reasoning, we can show that the other parts $||e^{A_1 t_i} e^{A_2 s_i}||_2, \ \ i = 2, 3, ...$ in (53) are always smaller than or equal to x_0. Therefore if

$$t_i + s_i \geq \frac{\log\left(||H_1||_2 \cdot ||H_1||_2^{-1} \cdot ||H_2||_2 \cdot ||H_2||_2^{-1}\right)}{\min_{i=1,2}\left(\lambda_i^*\right)} \quad \forall i = 1, 2...$$

then $||x(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$. Hence, (1) is asymptotically stable. □

Theorem 3.7 allows us to switch earlier than the restrictions given in (52), but it is still dependent on the initial condition of the switched system.

4 Conclusion

In this paper we developed stability criteria for a specific switched system described by (1). Asymptotic stability can be proved by showing the existence of a common Lyapunov function (CLF). When the matrices A_1 and A_2 commute, we have shown that a common Lyapunov function exists. Another method to prove that the switched system is asymptotically stable under the assumption that A_1 and A_2 commute is given. When a CLF does not exist, one way to ensure asymptotic stability is to wait a minimum amount of time in each location before switching. This is also known as the dwell time and a formula on the dwell time is given. It turned out that this formula is conservative and restrictive, because we maximize over the initial condition. For that reason, we developed another minimum dwell time with less restrictions.

This paper only discussed a specific switched system described by (1). The given procedure to check the existence of a matrix P for the CLF only works for 2×2 matrices. This is more complicated when the matrices are $n \times n$, since there are more variables in P. A method to check the existence of matrix P is to view (13), (14) and (15) as linear matrix inequalities (LMI). There exist efficient methods to solve the LMI or show infeasibility. [16]. The switched system could also contain more locations, say n. Theorem 2.5 can be altered to show when a switched system of n locations is asymptotically stable. The minimum dwell time given in Theorem 3.6 can readily be applied to such switched system. The formula on the minimum dwell time given in Theorem 3.7 is harder to apply since the general solution of such switched system is more complicated.

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