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A new concept to classically account for varying particle numbers in general relativity

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ABSTRACT

A single classical massive point particle in general relativity is described by a timelike worldline, which plays a key role in the interpretation of spacetime curvature and the modern Hawking-Penrose definition of singularities. The very concept of such worldlines describing one single particle, however, is at severe odds with the key relativistic feature of energy-mass equivalence, since the latter allows for the annihilation of classical particles in favour of the creation of others. This report develops the kinematical set-up to remedy this problem by superseding the concept of a timelike worldline by the new concept of a history. The latter is one single object that captures all worldlines in a spacetime and also allows for the creation and annihilation of particles subject to energy-momentum conservation. This framework is then employed to generalize the free dynamics for massive particles to free dynamics for a history that describes a variable particle number and to discuss possible future applications in general relativity.

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I. INTRODUCTION

Conventionally, a classical point particle on a Lorentzian spacetime is described by a worldline, which is a piecewise smooth curve on that manifold. Massive particles are described by curves whose tangent vectors lie within the open convex cones defined by a Lorentzian metric at each point along the curve, while the worldlines of massless particles have non-zero tangent vectors that are have vanishing length with respect to the Lorentzian metric [1]. In this work, we consider only massive particle worldlines for simplicity.

A cornerstone of relativity, both special and general, is that (from the point of view of any particular oberserver) the energy E of a particle with rest mass m is related to the spatial momentum $\mathbf{p}$ by the relation $E^2 = m^2 + \mathbf{p}^2$, in locally inertial coordinates and in units where the speed of light $c = 1$. This is more aptly expressed as the coordinate and observer-independent statement

$$g^{-1}(p, p) = m^2$$

for the so-called four-momentum p of the particle and g^{-1} denotes the inverse spacetime metric. The connection to one, or another, particular observer is then made by realizing that the components of the four-momentum related to the energy E and momentum $\mathbf{p}$ measured by the chosen observer are given by $p_0 = E$ and $p_\alpha = \mathbf{p}_\alpha$ for $\alpha = 1, 2, 3$. In any case, particles are kinematically allowed to be annihilated or created as long as the total four-momentum is conserved. For the decay of an unstable particle into two particles, for instance, the four-momentum balance

$$p_{\text{before}} = p_{1,\text{after}} + p_{2,\text{after}}$$

must hold at the spacetime point of decay, where p_{before} denotes the four-momentum of the unstable particle right before the decay and $p_{1,\text{after}}$ and $p_{2,\text{after}}$ the four-momentum of the first and second decay product right after the decay. It is clear that while each single of these particles can be described by a worldline, the entire process cannot.

To remedy this the introduction of one single new mathematical object, namely a so-called history, which captures all point particles including their potential annihilation and creation and all thus ensuing patterns, is the aspiration of this thesis. In order to technically implement the four-momentum conservation into this construction, we lift the entire setting to the cotangent bundle of the spacetime manifold, which contains the phase space of massive and massless particles. In order to accommodate the changing particle numbers, we devise a suitably classical analogue of the Fock space used in quantum mechanics. These classical Fock space differ, as they must from the quantum mechanical ones. Histories are curves on this classical Fock space that satify necessary physical conditions. Much of the mathematical constructions are concerned with establishing the correct topology on this Fock space.

The conventionally used particle worldlines are among the fundamentally most important objects in general relativity. The physical interpretation of spacetime curvature, for instance, is directly related to the behaviour of neighbouring geodesics [2] where geodesics, which represent free falling particles under the influence of gravity only. Another instance, where geodesic worldlines play a key role is in the Hawking-Penrose approach to the study of spacetime singularities [2, 3].

This focus on worldlines in the study of fundamental aspects to general relativity deserved scrutiny in the face of the possibility for particle creation and annihilation. Relativistic quantum field theory was born out of precisely the need to accommodate varying particle numbers [4]. The failure of any one-particle interpretation of the relativistic Klein-Gordon equation for particles without spin and the Dirac equation for particles with spin $\frac{1}{2}$ was ultimately recognized to be due to precisely the utter inconsistency of any relativistic theory that assumes a constant number of particles.

This tension between the concept of a worldline — with all its assumed fundamental relevance and practiced applications in general relativity on the one hand and the variability of particle number on the other hand — leads us to the idea of generalizing worldlines, or rather their lift to phase spacetime, to the concept of histories that naturally captures all kinematically admissible ways in which classical particles can be annihilated and created. Downwards compatibility to worldline constructions is given since the projections of 'portions' of a history recover all individual pieces worldlines one would consider conventionally.

For histories to fully supersed the concept of worldlines, not only the kinematics, but also the dynamics of the latter need to be generalized. This is done with the help of an action functional, which is a concept widely used in classical mechanics and relativistic classical mechanics [1, 5, 6]. In this thesis, only massive, uncharged and spinless particle histories will be considered.

We briefly sketch several possible applications of histories in general relativity, including their roles in a classical interpretation of spacetime curvature and singularity theory. This is only to give the reader an idea of where the constructed theory can be applied and leaves their development for future work. For an in-depth discussion of these topics in the standard worldline context, see, e.g., [2, 3].

The structure of this report is as follows. In section II, we concisely introduce the necessary apparatus of abstract topology and the theory of differentiable manifolds at precisely the level required for the foundations of this work. Moreover, an equally condensed review of general relativity, as it is required for an understanding of the present work, is given. In section III we formulate the definition of a one-particle history on phase space and extend the notion to constant particle number, these form the foundations for the theory that is introduced in this thesis. In section IV we will introduce the notion of a classical Fock space and define histories with variable particle number, which satisfy the kinematical constraints of energy-momentum conservation. This Fock space will be turned into a topological space that will make all histories continuous. In section V we formulate dynamics for free falling one particle histories from a Hamiltonian point of view. These dynamics are then extended to formulate the dynamics for free falling histories of constant particle number and, ultimately, dynamics for free falling histories with variable particle numbers. Finally, several possible applications of histories are sketched.

II. REVIEW OF DIFFERENTIAL GEOMETRY AND GENERAL RELATIVITY

A. Topological manifolds

The weakest structure one can establish on a set that allows to define convergence on a set, or indeed continuity of maps between sets, is that of a topology. Topological manifolds are a very special class of topological spaces that can be understood locally by continuous maps into some $\mathbb{R}^d$ and their inverses. We will introduce all relevant definitions and results in this subsection.

Topological space. The precise definition is the following. A *topology* $\mathcal{O}_M$ for set M is a subset of the powerset $\mathcal{P}(M)$ of M such that the following conditions are satisfied:

1. The trivial subsets of M are in the topology, $\emptyset \in \mathcal{O}_M$ and $M \in \mathcal{O}_M$.

2. The intersection of finitely many elements in a topology is again in the topology,
$$\{U_1, U_2, \ldots, U_N\} \subseteq \mathcal{O}_M \quad \Longrightarrow \quad \bigcap_{i=1}^{N} U_i \in \mathcal{O}_M .$$

3. The union of arbitrarily many elements of a topology are again in the topology,
$$\mathcal{U} \subseteq \mathcal{O}_M \quad \Longrightarrow \quad \bigcup_{U \in \mathcal{U}} U \in \mathcal{O}_M .$$

The pair $(M, \mathcal{O}_M)$ is called a *topological space.* The elements of a chosen topology $\mathcal{O}_M$ are called the *open sets* of M. Subsets of M that can be written as $M \backslash U$ for some open set U are called the closed sets of M. A particular subset of M may be open, closed, open and closed, open but not closed, closed but not open, or finally neither open nor closed.

Note that for any given set M, there are typically many possible choices for a topology. The two extreme choices, which can be made for any set M the *trivial topology* $\{M, \emptyset\}$ and the *discrete topology* $\mathcal{P}(M)$. There is an important partial order between topologies: If $\mathcal{O}_M \subset \mathcal{O}'_M$ for two topologies on the same set M, we call the former *coarser* than the latter and the latter *finer* than the former.

The standard notion of open and closed sets one defines in the real numbers is recovered, in the general topological framework, by declaring all those subsets U of $M = \mathbb{R}$ as open if for any $x \in U$ there is a positive real ϵ such that the interval $(x - \epsilon, x + \epsilon) \subset U$. The resulting topology is the so-called standard topology $\mathcal{O}_{standard}$ and this is the topology chosen on $\mathbb{R}$ if nothing to the contrary has been said.

Convergence and continuity. The primary task of a topological space is to define a notion of convergence of sequences on the one hand and the continuity of maps on the other hand.

A sequence $f : \mathbb{N} \to M$ on a topological space $(M, \mathcal{O}_M)$ is said to *converge* to some $p \in M$, if for every open set $U \ni p$ there is an $N \in \mathbb{N}$ such that $f(n) \in U$ for all $n > N$, which fact is expressed as the statement $\lim_{n\to\infty} f_n = p$. A particular sequence may converge or not converge, and if it converges, it may converge to multiple points.

A map $f : M \to N$ between two topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ is called *continuous* if for every U that is an open set in the target space N of the map, the pre-image

$$\mathrm{preim}_f(U) := \{m \in M \mid f(m) \in U\}$$

is an open set of the domain M of the map.

If a continuous map f is a bijection and its inverse map $f^{-1} : N \to M$ is also continuous, f is called a *homeomorphism.* If some homeomorphism between two topological space $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ exists, the topological spaces are called *homeomorphic.* By construction, homeomorphisms are the two-way structure-preserving maps between any to topological spaces.

Topological properties. Throughout this thesis, we will only consider topological spaces that have the following two additional properties.

A topological space $(M, \mathcal{O}_M)$ is said to be *Hausdorff* if for every pair of distinct points $x \neq y$ in M, there is an open set $U \ni x$ and an open set $V \ni y$ such that $U \cap V = \emptyset$. On a topological space that is Hausdorff, limits are unique.

The second property, second-countability, is based on the notion of a basis for a topology. A *topological basis* for a topological space $(M, \mathcal{O}_M)$ is a set $\mathcal{B} \subseteq \mathcal{O}_M$ such that every open set $U \in \mathcal{O}_M$ satisfies that for every $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. A topological space is *second-countable* if it admits a basis with countable elements.

Inherited toplogies There are two ways to construct a new topology from one or several given toplogies.

A subset $S \subseteq M$ of a topological space $(M, \mathcal{O})$ can be equipped with the so-called *subset topology*

$$\mathcal{O}_M|_S := \{U \cap S | U \in \mathcal{O}_M\} \ .$$

This is the topology on the subset N for which the restriction $f|_S$ of a continuous map $f : M \to N$ to S is continuous.

The Cartesian product set $M \times N$ can be equipped with the so-called *product* *topology* $\mathcal{O}_{M\times N}$, which is defined as the coarsest topology that contains all $U = V \times W$ with $V \in \mathcal{O}_M$ and $W \in \mathcal{O}_N$.

If nothing else is said then the set $\mathbb{R}^n$ is endowed with the topology $\mathcal{O}_{\mathbb{R}^n}$ that is recursively definex as the product topology on $\mathbb{R}^{n-1} \times \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$. The resulting topology is called the *standard topology* on $\mathbb{R}^n$.

Topological manifolds Now that we have some structure defined on topological spaces, we can go into the main topological structure used in geometry, namely an *n-dimensional topological manifold*, i.e. a topological space $(M, \mathcal{O}_M)$ that has the following properties:

1. $(M, \mathcal{O}_M)$ is locally Euclidean of dimension n, i.e. around every point $x \in M$, we can find an open $U \in \mathcal{O}_M$ such that U endowed with subset topology $\mathcal{O}_M|_U$ is homeomorphic to some $\Omega \in \mathcal{O}_{\mathbb{R}^n}$, endowed with the subset topology $\mathcal{O}_{\mathbb{R}}|_\Omega$.

2. $(M, \mathcal{O}_M)$ is Hausdorff.

3. $(M, \mathcal{O}_M)$ is second-countable.

Any open set $U \in \mathcal{O}_M$ of a topological manifold that is homeomorphically mapped to an open set $\Omega \in \mathcal{O}_{\mathbb{R}^n}$ by virtue of the homeomorphism $\chi : U \to \Omega$ constitutes a *chart* (U, χ). The set U is then referred to as the *domain* of the chart and χ the *chart map*. If $U \ni p$ then the chart is said to be around the point p.

An *atlas* $\mathcal{A}$ for a topological manifold $(M, \mathcal{O}_M)$ is a set of charts such that for every point $p \in M$ there is a chart $(U, \chi) \in \mathcal{A}$ such that p lies in its chart domain U.

B. Differentiable manifolds

For an in-depth analysis on differentiable manifolds, we refer the reader to [7].

Since a topological manifold $(M, \mathcal{O}_M)$ is a special kind of topological space, one can decide, by construction, whether a curve $\gamma : I \to M$ for some interval $I \subset \mathbb{R}$ is continuous. Similarly, one can decide whether a function $f : M \to \mathbb{R}$ is continuous. In order to define the differentiability of curves and functions in particular, and maps in general, however, we now need to specialize the concept of a topological manifold further to a differentiable manifold. This is done by way of choosing special atlases.

C^k-compatibility of charts, C^k-atlases and C^k-manifolds. Two charts (U,χ) and (V,ψ) for an n-dimensional topological manifold $(M,\mathcal{O}_M)$, are C^k-*compatible* if the *change of coordinates map* from χ to ψ given by

$$\psi\circ\chi^{-1}:\chi(U\cap V)\to\psi(U\cap V)$$

is C^k with the standard definition in $\mathbb{R}^n$ (c.f. [7]). Note that if $U\cap V=\emptyset$, this is trivially true. Two charts (U,χ) and (V,ψ) are *smoothly compatible* is they are C^k-compatible for every $k\in\mathbb{N}$. A C^k-*Atlas* for $(M,\mathcal{O}_M)$ is an atlas $\mathcal{A}$ for $(M,\mathcal{O}_M)$ that consists of only C^k-compatible charts. A *smooth atlas* for $(M,\mathcal{O}_M)$ is an atlas $\mathcal{A}$ that consists of only smoothly compatible charts. A maximal C^k-atlas is a C^k-atlas such that there is no C^k-atlas $\mathcal{A}'$ that strictly contains $\mathcal{A}$ and a *maximal smooth atlas* is a smooth atlas $\mathcal{A}$ such that no smooth atlas $\mathcal{A}'$ strictly contains $\mathcal{A}$.

A triple $(M,\mathcal{O}_M,\mathcal{A})$ is called an *n-dimensional C^k-manifold* if M is an n-dimensional topological manifold $(M,\mathcal{O}_M)$ and $\mathcal{A}$ is a maximal C^k-atlas.

Smooth product manifolds. For two smooth manifolds $(M,\mathcal{O}_M,\mathcal{A}_M)$ and $(N,\mathcal{O}_N,\mathcal{A}_N)$, we can equip the Cartesian product set $M\times N$ with the product topology $\mathcal{O}_{M\times N}$ and the *smooth product atlas* $\mathcal{A}_{M\times N}$ that is defined as the maximal atlas that contains all charts $(U\times V,\xi_{\chi,\psi})$ constructed from charts $(U,\chi)\in\mathcal{A}_M$ and $(V,\psi)\in\mathcal{A}_N$, where the chart map

$$\xi_{\chi,\psi}:U\times V\to\mathbb{R}^{\dim M}\times\mathbb{R}^{\dim N}\,,\qquad (p,q)\mapsto(\chi(p),\psi(q))$$

and its inverse are continuous since the respective charts on M and N are. This indeed defines a smooth atlas [7].

Smooth maps. A map $F:M\to N$ between two smooth manifolds $(M,\mathcal{O}_M,\mathcal{A}_M)$ and $(N,\mathcal{O}_N,\mathcal{A}_N)$ is called *smooth* if for every point $p\in M$, there is a chart $(U,\chi)\in\mathcal{A}_M$ with $x\in U$ and a chart $(V,\psi)\in\mathcal{A}_N$ with $F(p)\in V$ such that the map

$$\psi\circ F\circ\chi^{-1}:\chi(U\cap\operatorname{preim}_F(V))\to\psi(F(U)\cap V)$$

is smooth as a map from $\mathbb{R}^{\dim M}$ to $\mathbb{R}^{\dim N}$ (c.f. [7]). Note that since all charts in $\mathcal{A}_M$ are smoothly compatible and the same holds for charts in $\mathcal{A}_N$, the above definition is independent of choice of charts, as the composition of smooth maps is again smooth

[7]. A similar definition can be obtained for C^k-maps between smooth manifolds. The set of smooth functions $f : M \to \mathbb{R}$ on a smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ is denoted by $C^\infty(M)$ and is made into a real vector space $(C^\infty(M), +, \cdot)$ by pointwise definition of the addition $+$ and scalar multiplication $\cdot$. The set of C^k-functions $f : M \to \mathbb{R}$ on M is denoted by $C^k(M)$.

A set $I \subseteq \mathbb{R}$ can be turned into a smooth manifold by first equipping it with the subset topology of the standard topology on $\mathbb{R}$ and then collecting all charts that are smoothly compatible to the chart (I, id_I). In this way we can talk about smooth curves on a smooth n-dimensional manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ by defining them as smooth maps

$$\gamma : I \to M\,.$$

Tangent vectors. At any point p along a smooth curve $\gamma : I \to M$, say $p = \gamma(\lambda_p)$ for some $\lambda_p \in I$, the *tangent vector* $\dot\gamma_p$ is defined as the linear map

$$\dot\gamma_p : C^\infty(M) \to \mathbb{R}\,, \qquad \dot\gamma_p f := (f \circ \gamma)'(\lambda_p)\,.$$

The set of all such maps constructed from all smooth curves through the point p is denoted by T_pM. One can show that there is a canonical notion of addition and scalar multiplication on T_pM, which makes it into a vector space, the so-called tangent vector space to the smooth manifold at the point p.

With respect to a chart (U, χ) whose domain contains p, the action of a tangent vector $\dot\gamma_p$ on a smooth function f can be written as

$$\dot\gamma_p f = (\underbrace{(f \circ \chi^{-1})}_{\mathbb{R}^{\dim M} \to \mathbb{R}} \circ \underbrace{(\chi \circ \gamma)}_{\mathbb{R} \to \mathbb{R}^{\dim M}})'(\lambda_p) = \sum_{m=1}^{\dim M} \underbrace{(\chi^m \circ \gamma)'(\lambda_p)}_{=:\dot\gamma^m(\lambda_p)} \underbrace{\partial_m (f \circ \chi^{-1})(\chi(p))}_{=:\left(\frac{\partial}{\partial \chi^m}\right)_p f}\,,$$

which is properly interpreted as the expansion of the tangent vector $\dot\gamma_p$ in terms if its real-valued components $\dot\gamma^m$ with respect to the chart-induced basis $\left(\frac{\partial}{\partial \chi^1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi^{\dim M}}\right)_p$ of T_pM.

Smooth vector fields. *Smooth vector fields* are derivations on $C^\infty(M)$, i.e. maps $X : C^\infty(M) \to C^\infty(M)$ that satisfy

$$X(fg) = fX(g) + gX(f)\,,$$

for every $f, g \in C^\infty(M)$. These derivations give rise to a module $\mathfrak{X}(M)$ over $C^\infty(M)$.

A *smooth covector field* or *smooth one-form* on M is a $C^\infty(M)$-linear map $\omega : \mathfrak{X}(M) \to C^\infty(M)$. These one-forms then give rise to a module $\Omega^1(M)$ over $C^\infty(M)$. A *smooth (p,q)-type tensor field* T is a $C^\infty(M)$-multilinear map $T : \Omega^1(M)^{\times p} \times \mathfrak{X}(M)^{\times q} \to C^\infty(M)$.

The *contangent space* T_p^*M at a point $p \in M$ is the dual of the tangent space, i.e. space of all linear maps $\omega_p : T_pM \to \mathbb{R}$. A chart χ around p constitutes a basis for T_p^*M, by taking the basis dual to the chart induced basis $\left(\frac{\partial}{\partial \chi^1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi^{\dim M}}\right)_p$ of T_pM, that is, the set $\{(d\chi^1)_p, \ldots (d\chi^n)_p\}$ such that

$$(d\chi^i)_p \left(\frac{\partial}{\partial \chi^j}\right)_p = \delta^i_j \, .$$

where δ^i_j is the Kronecker delta, which is 0 when $i \neq j$ and 1 when $i = j$. If we have a different chart $(V, \psi) \in \mathcal{A}_M$ around p, the chart induced basis of T_p^*M relates to χ as

$$(d\psi^i)_p = \sum_{j=1}^{\dim M} \frac{\partial(\psi \circ \chi^{-1})^i}{\partial x^j}(\chi(p))(d\chi)_p$$

We can now define one of the most important structures in this thesis, the *cotangent bundle*. The cotangent bundle is the disjoint union over all cotangent spaces to the smooth manifold

$$T^*M := \coprod_{p \in M} T_p^*M \, ,$$

together with the canonical projection $\pi : T^*M \to M$ that projects T_p^*M down to p.

With respect to a chart (U, χ) of $(M, \mathcal{O}_M, \mathcal{A}_M)$, such that $\pi(p) \in U$ for some $p \in T^*M$, we can construct a chart induced homeomorphism $\widetilde{\chi} : T^*M \to \mathbb{R}^{2n}$ around p, defined by

$$\widetilde{\chi}(p) := (\chi^1(\pi(p)), \ldots, \chi^n(\pi(p)), \omega_1, \ldots, \omega_n) \in \mathbb{R}^{2n} \, ,$$

where ω is the covector at $\pi(p) \in M$, given by

$$\omega = \sum_{i=1}^{n} \omega_i (d\chi^i)_p \, .$$

The pair

$$\left(\coprod_{\pi(p) \in U} \pi(p), \widetilde{\chi} \right)$$

can be defined for any chart χ of $(M, \mathcal{O}_M, \mathcal{A}_M)$ and they are all smoothly compatible since the change of coordinates between $(d\chi^i)_p$-type objects is smooth. Hence we can turn the cotangent bundle into a smooth $2n$-dimensional manifold by picking the maximal atlas containing the above pairs corresponding charts of $(M, \mathcal{O}_M, \mathcal{A}_M)$.

With this smooth structure, one can show that (c.f. [7]) for a smooth map $\omega : M \to T^*M$ satisfying $\pi \circ \omega = \mathrm{id}_M$ holds that, for any chart χ of M, the component functions of $\widetilde{\chi} \circ \omega$ are smooth as a map $M \to \mathbb{R}$. This means that a one form can locally be expressed as

$$\omega_p = \sum_{i=1}^{n} \omega_i(p)(d\chi^i)_p \,,$$

where the ω_i are smooth functions from $\mathbb{R} \to \mathbb{R}$. If X is a vector field and ω is a one form, we can therefore, locally express the function $\omega(X) \in C^\infty(M)$ as the sum

$$\omega(X)(p) = \sum_{i=1}^{n} \omega_i(p) X^i(p) \,.$$

Where ω_i are the component functions of ω under a chart around p, and X^i are the component functions of X under the same chart around p.

We could even describe smooth (p, q)-type tensor fields in terms of smooth local coordinates (c.f. [2]), such that $T(\omega^{(1)} \dots \omega^{(p)}, X^{(1)}, \dots, X^{(q)}) \in C^\infty(M)$ becomes a sum

$$\sum_{a=1}^{n} \cdots \sum_{b=1}^{n} \sum_{c=1}^{n} \cdots \sum_{d=1}^{n} T^{a,\dots,b}{}_{c,\dots,d}(p) \omega_a^{(1)}(p) \dots \omega_b^{(p)}(p) X^{(1)\,c}(p) \dots X^{(q)\,d}(p) \,.$$

One can see that this becomes very messy very quick due to a large amount of summation symbols. In *Einstein summation convention*, sums are implied when the same index is repeated twice, the indices are then said to be *contracted*. E.g. the above sum would locally be denoted as

$$T^{a,\dots,b}{}_{c,\dots,d} \omega_a^{(1)} \dots \omega_b^{(p)} X^{(1)\,c} \dots X^{(q)\,d} \,.$$

One should note that we use superindices for components of objects that work on covector fields (i.e. vector fields or indices of a tensor) and subindices for components of objects that work on vector fields (i.e. covector fields or indices of a tensor). It should come as no surprise that, when a subindex and a superindex of a (p, q)-tensor are contracted, one is left with a $(p-1, q-1)$-tensor.

C. General relativity

General relativity is a theory that is described on a 4-dimensional smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$, which is equipped with a *metric*, that is a smooth $(0,2)$-type tensor field g that satiefies the following two conditions: for all vector fields $X, Y \in \mathfrak{X}(M)$, $g(X,Y) = g(Y,X)$, (in coordinates, $g_{ab} = g_{ba}$), we say g is *symmetric*, and for every point $p \in M$, there should exist vector fields $e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}$ such that

$$\begin{aligned}
&g_{ab}(p)e^a_{(0)}(p)e^b_{(0)}(p) = 1\\
&g_{ab}(p)e^a_{(1)}(p)e^b_{(1)}(p) = g_{ab}(p)e^a_{(2)}(p)e^b_{(2)}(p) = g_{ab}(p)e^a_{(3)}(p)e^b_{(3)}(p) = -1\\
&g_{ab}(p)e^a_{(i)}(p)e^b_{(j)}(p) = 0 \text{ when } i \neq j\,,
\end{aligned}$$

we say g has *signature* $(+,-,-,-)$, the pair (M,g) is then called a *spacetime.* The *inverse metric* g^{-1} (or g^{ab} in coordinates) is a smooth $(2,0)$-type tensor field that satisfies at every point $p \in M$

$$g^{ab}(p)g_{bc}(p) = \delta^a_c\,.$$

A curve $\gamma : (\tau_I, \tau_F) \to M$ is a *timelike worldline* if for every $\tau \in (\tau_I, \tau_F)$, holds that $g_{ab}(\gamma(\tau))\dot{\gamma}(\tau)^a\dot{\gamma}(\tau)^b > 0$, a *non-spacelike* worldline is a curve $\gamma : (\tau_I, \tau_F) \to M$ if for every $\tau \in (\tau_I, \tau_F)$, holds that $g_{ab}(\gamma(\tau))\dot{\gamma}(\tau)^a\dot{\gamma}(\tau)^b \geq 0$. The *length*, or *proper time*, of a timelike worldline between parameters τ_0 and τ_1 is given by

$$L[\gamma] := \int_{\tau_0}^{\tau_1} d\tau\, \sqrt{g_{ab}(\gamma(\tau))\dot{\gamma}(\tau)^a\dot{\gamma}(\tau)^b}\,.$$

Stationary curves of this length functional are called *geodesics.* If the geodesics are parametrized such that $g_{ab}\dot{\gamma}^a\dot{\gamma}^b = 1$, they are solutions to the *geodesic equation*

$$\ddot{\gamma}^a + \Gamma^a_{bc}\dot{\gamma}^b\dot{\gamma}^c = 0\,,$$

where the Γ^a_{bc} are the *Christoffel symbols*, given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{am}\left(\frac{\partial g_{mc}}{\partial x^b} + \frac{\partial g_{bm}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^m}\right)\,.$$

Note that these Christoffel symbols are not $C^\infty(M)$-multilinear and are therefore not tensors.

In the expression for the Christoffel symbols, the coordinate functions of the metric are differentiated with respect to a coordinate, we can abbreviate this by defining an operator ∂_a in coordinates as:

$$\partial_a f := \frac{\partial f}{\partial x^a}.$$

Note that this is not a one-form, if we let it work on a vector v^a

$$\partial_a v^a := \frac{\partial v^a}{\partial x^a},$$

we can see that it is not $C^\infty(M)$-linear.

We describe the curvature on the spacetime M using the *Riemann tensor* $R^a{}_{bcd}$ in terms of coordinates

$$R^a{}_{bcd} := \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.$$

It can be shown that this is, indeed, a tensor (c.f. [1]). The *Einstein tensor* is given by

$$G_{ab} := R^m{}_{amb} - \frac{1}{2} R^m{}_{cmd} g^{cd} g_{ab}.$$

The *stess-energy tensor* of a matter field is related to the stationary points of the action of the matter field with respect to the metric, by

$$T_{ab} = \frac{2}{\sqrt{-\det g}} \frac{\delta S_{\text{matter}}}{\delta g^{ab}}.$$

In coordinates where all arising constant prefactors are equated to 1, the Einstein tensor is related to the stress energy tensor of the matter field by

$$G_{ab} = T_{ab}.$$

If we have a massive point particle on the spacetime that is affected by no field other than gravity, we shall call it a *free particle.* By the properties of the Einstein field equations, free massive particles follow timelike geodesics.

For more insight into general relativity, differential geometry and field theory, we refer the reader to [1, 2, 8].

III. HISTORIES WITH CONSTANT PARTICLE NUMBER

Let (M, g) be a spacetime and assume that g is a solution to Einstein's field equations with respect to some particular matter field content of the thus described universe. The associated cotangent bundle be denoted by $\pi_1 : T^*M \to M$.

A. Histories of one particle

We will focus first only on massive particle histories. A massive one-particle history (p, m) is a continuous curve

$$p : (0, 1) \to T^*M$$

and some positive real m such that for all $\lambda \in (0, 1)$ it holds that

$$p(\lambda) = m \frac{g(\dot{\gamma}(\lambda), \cdot)}{\sqrt{g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))}}\,, \tag{1}$$

where $\gamma := \pi_1 \circ p$ denotes the projection of the one-particle history to M. Note that these conditions are invariant under strictly increasing smooth reparametrizations of p and that they imply that $\gamma : (0, 1) \to M$ is a timelike continuously differentiable curve on M.

B. Histories of several particles

Rather than looking at b independent one-particle histories, we find it advantageous to capture the same information in one single object. For every positive integer b, we accomplish this by denoting $T^*M^{\times b}$ as the b-fold Cartesian product of T^*M, which we will equip with the product topology and product smooth manifold structure canonically inherited from T^*M. There are now b different smooth projection maps $\pi_\beta : T^*M^{\times b} \to M$ for $\beta = 1, \dots b$, with each of them corresponding to the projection map of the corresponding Cartesian factor T^*M of $T^*M^{\times b}$.

We then define a b-particle history (p, m) as a continuous curve

$$p : (0, 1) \to T^*M^{\times b}$$

together with an $m \in \mathbb{R}^b$ such that for $\beta = 1, \dots, b$ the *one-particle maps* $p_\beta : (0, 1) \to T^*M$ and the components m_β of m give rise to one-particle histories (p_β, m_β). Likewise, we will refer, somewhat loosely, to $T^*M^{\times b}$ as the b-particle phase space.

C. Histories of no particles

Defining the set $T^*M^{\times 0}$ as the one-element set $\{\emptyset\}$, we define a 0-particle history as a curve

$$p : (0,1) \to T^*M^{\times 0}\,,$$

of which there is thus only one instance, namely the constant curve $p(\lambda) = \emptyset$. By itself this is quite trivial, but serves as an important building block for histories with a variable number of particles.

IV. HISTORIES WITH VARIABLE PARTICLE NUMBER

A. Classical Fock phase space (Idea at set-theoretic level)

We now wish to construct a space such that a history with a variable number of particles can be captured in terms of one single curve. At the mere set-theoretic level — later we will have reason to choose a particular topology on that space — this space is given by the disjoint union of all b-particle phase spaces

$$FM := \coprod_{b=0}^{\infty} T^*M^{\times b}\,,$$

which provides a classical version of the Fock space construction employed in quantum mechanics. The rough idea is now to define a history with a variable number of particles as a curve

$$p : (0,1) \to FM$$

such that the restriction of p to any maximal parameter interval whereon $p(\lambda) \in T^*M^{\times b}$ for some fixed b gives rise to a b-particle history $(p, \{m_1, \dots, m_b\})$. Without having decided yet on the topology of the Fock space FM, however, we cannot even require such curves to be continuous. Which topology serves our purpose, is discussed in the following subsections.

Before going into this crucial detail, however, it is useful to clarify the difference between our classical Fock space construction and the one that is used in quantum mechanics. While a quantum mechanical Fock space (c.f. [4])

$$F\mathcal{H} = \bigoplus_{b=0}^{\infty} \mathcal{H}^{\otimes b}$$

is a direct sum of tensor products of the Hilbert space for one quantum particle, our classical version here is a disjoint union of mere Cartesian products of the classical phase space for one particle. Due to the disjoint union, any element in FM corresponds to the state of some definite number of particles, whereas the direct sum in quantum mechanics allows superposition of states with different particle numbers. Likewise, due to the simple product manifold structure of the individual constant particle spaces, any b-particle phase space $T^*M^{\times b}$ can be broken down into b one-particle phase spaces, whereas the tensor product in quantum mechanics encodes the fact that a two-particle quantum system can be in an entangled state that can no longer be broken down into two quantum systems of one particle.

B. Histories

Endow each $T^*M^{\times b}$ with the product topology and the product smooth atlas inherited from the smooth manifold T^*M. For some $N \in \mathbb{N}$ consider a curve

$$p : (0,1)\backslash\{\lambda_1, \dots, \lambda_{N-1}\} \to FM$$

with the *singularity parameters* $0 < \lambda_1 < \dots < \lambda_{N-1} < 1$ and the *masses*

$$m \equiv (m^{(1)}, \dots, m^{(N)}) \in \mathbb{R}^{b_1+\dots+b_N}$$

for some non-negative particle numbers $b_1, \dots b_N$. The pair (p, m) is called a *history* if the following conditions hold:

(a) For every $n = 1, \dots, N$ the map

$$p^{(n)} : (0,1) \to T^*M^{\times b_n}\,, \qquad \lambda \mapsto p(\lambda_{n-1} + \lambda(\lambda_n - \lambda_{n-1}))$$

gives rise to a b_n-particle history $(p^{(n)}, m^{(n)})$, which we refer to a the *n-branch* of the history (p, m).

(b) For each $n = 1, \dots, N-1$ and every spacetime point $f \in M$ we define the set

$$B^{\uparrow}_{n,f} := \{\beta \in \{1, \dots, b_n\} \mid \lim_{\lambda\uparrow 1}(\pi_\beta \circ p^{(n)}_\beta)(\lambda) = f\}\,,$$

which contains the labels β of all one-particle maps $p^{(n)}_\beta$ of the n-branch of (p, m) that end at f, and, similarly, the set

$$B^{\downarrow}_{n,f} := \{\beta \in \{1, \dots, b_{n+1}\} \mid \lim_{\lambda\downarrow 0}(\pi_\beta \circ p^{(n+1)}_\beta)(\lambda) = f\}\,,$$

which contains the labels of all one-particle maps $p_\beta^{(n+1)}$ of the $(n+1)$-branch of (p, m) that begin at f, cf. figure 1, and require that the history satisfies

$$\sum_{\beta \in B_{f,n}^{\uparrow}} \lim_{\lambda \uparrow 1} p_\beta^{(n)}(\lambda) = \sum_{\beta \in B_{f,n}^{\downarrow}} \lim_{\lambda \downarrow 0} p_\beta^{(n+1)}(\lambda)$$

for all $f \in M$ and $n = 1, \ldots, N-1$.

Physically, the first condition means that each history can only accommodate a finite number of points where all incoming one-particle histories are annihilated and some outgoing one-particle histories are created. In this process, the total number of particles may change. Technically, these points must be excluded from the history, which corresponds to the removal of the singularity parameters $\lambda_1, \ldots, \lambda_{N-1}$ from the domain. Note that the continuity of any one-particle history, which is a curve on T^*M, implies that its projection to a worldline on M is at least a C^1-curve (once continuously differentiable), and analogously for each of the b_n one-particle histories $p_\beta^{(n)} : (0,1) \to T^*M$ that constitute the history in the parameter range $(\lambda_{n-1}, \lambda_n)$ of p. The physical meaning of the second condition is that energy-momentum is conserved where some one-particle histories come to and end and some others start. Note that for the case of only one incoming and one outcoming one-particle history, such as at f_2 in figure 1, this implies that the momentum of the one-particle history remains the same, so that the phase space point that is missing from the history can be uniquely inserted, such that the one-particle histories before and after project to an overall C^1-worldline.

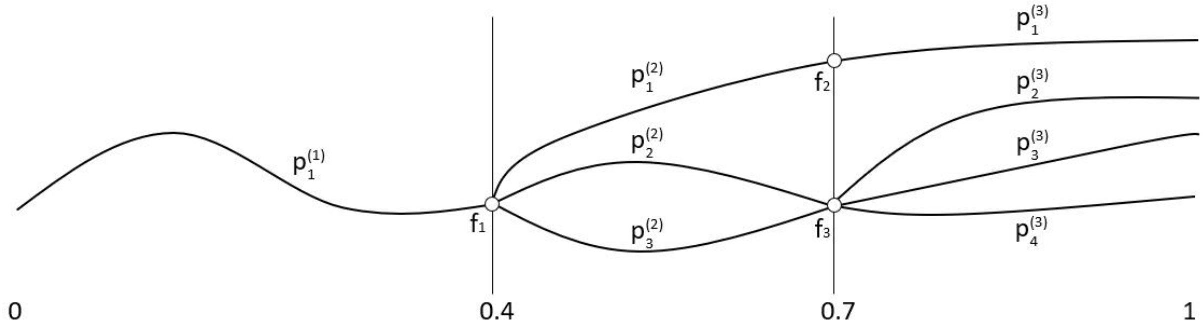

FIGURE 1: Particle history projected down on M, particle number changes happen at $\lambda_1 = 0.4$ and $\lambda_2 = 0.7$. $B_{1,f_1}^{\uparrow} = \{1\}$, $B_{1,f_1}^{\downarrow} = \{1,2,3\}$, $B_{2,f_2}^{\uparrow} = \{1\}$, $B_{2,f_2}^{\downarrow} = \{1\}$, $B_{2,f_3}^{\uparrow} = \{2,3\}$ and $B_{2,f_3}^{\downarrow} = \{2,3,4\}$.

C. Topology of Fock phase space

Let $\mathcal{O}_{T^*M^{\times b}}$ denote the product topology on $T^*M^{\times b}$. Consider any subset $V \subseteq FM$, which uniquely decomposes into a disjoint unions of subsets $V_b \subseteq T^*M^{\times b}$ such that

$$V = \coprod_{b=0}^{\infty} V_b \,. \tag{2}$$

We define a topology $\mathcal{O}_{FM}$ by declaring precisely those subsets U as elements $\mathcal{O}_{FM}$ for which each V_b is an element of $\mathcal{O}_{T^*M^{\times b}}$. This is the disjoint union topology. It is easy to see that this is indeed a topology. The empty set is disjoint union of empty sets, and FM is a disjoint union of all $T^*M^{\times b}$, so both certainly are in $\mathcal{O}_{FM}$. For two arbitrary sets $U, V \in \mathcal{O}_{FM}$, we have

$$U \cap V = \left(\coprod_{b=0}^{\infty} U_b\right) \cap \left(\coprod_{\tilde{b}=0}^{\infty} V_{\tilde{b}}\right) \,.$$

Since U_b and $V_{\tilde{b}}$ only lie in the same space when $b = \tilde{b}$, the intersection distributes over the disjoint union and the cross terms drop out, so that we have

$$U \cap V = \coprod_{b=0}^{\infty} (U_b \cap V_b) \,.$$

Now, since U_b and V_b lie in $\mathcal{O}_{T^*M^{\times b}}$, their intersection does too. Hence $U \cap V$ is the disjoint union of open sets in $T^*M^{\times b}$ for every b, so by the definition of $\mathcal{O}_{FM}$, it must also be open. Lastly, consider an arbitrary subset $\mathcal{U} \subseteq \mathcal{O}_{FM}$, whence

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} \coprod_{b=0}^{\infty} U_b \,.$$

Once again, U_b and $U_{\tilde{b}}$ lie in the same connected component of FM if and only if $b = \tilde{b}$, so we can swap the union and the disjoint union to obtain

$$\bigcup_{U \in \mathcal{U}} U = \coprod_{b=0}^{\infty} \bigcup_{U \in \mathcal{U}} U_b \,.$$

Since $U_b \in \mathcal{O}_{T^*M^{\times b}}$ for every $U \in \mathcal{U}$, the union on the right hand side of the above equation must also lie in $\mathcal{O}_{T^*M^{\times b}}$. So by the definition of $\mathcal{O}_{FM}$, the left hand side of that equation must lie in $\mathcal{O}_{FM}$. Hence $\mathcal{O}_{FM}$ is a topology.

This is indeed the appropriate topology to be established on the Fock space FM, since it renders our histories continuous. To see this, suppose we have a history

$p : U \to FM$ with $U := (0,1)\backslash\{\lambda_1, \ldots, \lambda_{N-1}\}$. Decomposing any open set $V \in \mathcal{O}_{FM}$ as in (2), we have

$$\mathrm{preim}_p(V) = \coprod_{b=0}^{\infty} \mathrm{preim}_p(V_b)\,.$$

But since $V_b \in \mathcal{O}_{T^*M^{\times b}}$, continuity of p in each of the intervals $(\lambda_{n-1}, \lambda_n)$ for $n = 1, \ldots, N$, we have that $\mathrm{preim}_p(V_b)$ lies in the subset topology on $U \subset \mathbb{R}$ induced from the standard topology on $\mathbb{R}$. Hence also $\mathrm{preim}_p(V)$, as a union of such subsets, is open. Hence p is continuous.

V. FREE HISTORIES

A. Lifting free dynamics for a massive particle worldline to a one-particle history

The equations of motion for a worldline $\gamma : (0,1) \to M$ representing a massive particle in spacetime (M, g), which is only under the influence of gravity, are the stationary points of the action functional

$$S^{\text{free}}_{\text{worldline}}[\gamma] = \int_0^1 d\tau\, m\sqrt{g_{ab}(\gamma)\dot{\gamma}^a\dot{\gamma}^b}\,, \tag{3}$$

where the integral kernel is the Lagrangian L of the system. In order to lift these dynamics to phase space, we need to rewrite these equations in terms of the one-particle history that corresponds to the same equations of motion.

Calculating the canonical momentum

$$p_a = \frac{\partial L}{\partial \dot{\gamma}^a} = m\frac{g_{as}\dot{\gamma}^s}{\sqrt{g_{mn}\dot{\gamma}^m\dot{\gamma}^n}}$$

of the above dynamics, one finds that not all four components are independent, since direct calculation shows that they are bound by the constraint

$$g^{ab}p_a p_b - m^2 = 0\,.$$

This makes it impossible to express, conversely, the velocities $\dot{\gamma}$ in terms of the canonical momenta, which requires to perform the needed transition to the Hamiltonian through the Dirac procedure for constrained systems rather than the standard Legendre formation (c.f. [9]). Doing so one finds that there are no further constraints (which in principle could come up by systematically investigating the dynamical consistency of the initial constraint above) and that the dynamics is a gauge theory. The total Hamiltonian at the end of the Dirac algorithm turns out to be the pure constraint Hamiltonian

$$H(\gamma,p)=\lambda(g^{ab}p_a p_b-m^2)\,, \tag{4}$$

where the Lagrange multiplier $\lambda:(0,1)\to\mathbb{R}$ is an a priori arbitrary smooth curve in $\mathbb{R}$.

We will now show that variation of the action functional

$$S^{\text{free}}_{\text{1-particle history}}[(p,m),\lambda]=\int_0^1 d\tau\ \left[\dot\gamma^a p_a-\lambda(g^{ab}(\gamma)p_a p_b-m^2)\right]\qquad\text{where } \gamma:=\pi\circ p \tag{5}$$

with respect to the map p of the one-particle history (p,m) and a Lagrange multiplier function $\lambda:(0,1)\to\mathbb{R}$ yield the same equations of motion as variation of the action (3) with respect to the curve γ on M. Note that γ is now only a shorthand for the projection of the phase space curve $p:(0,1)\to T^*M$ down to M, by virtue of the canonical bundle projection $\pi:T^*M\to M$ and that we do not assume any a priori relation between the the values $p(\tau)$ and the tangent vector $\dot\gamma(\tau)$ to γ. In fact, we will see that, apart from the time-orientation, the above dynamics automatically yield the conditions (1) for a one-particle history.

Variation of the action (5) with respect to the curve p yields the stationarity condition

$$\dot\gamma^m-2\lambda g^{mb}p_b=0\,, \tag{6}$$

while variation with respect to the projected curve γ^m yields

$$\dot p_m+\lambda(\partial_m g^{ab})p_a p_b=0\,. \tag{7}$$

Solving (6) for p and inserting this into (7), one obtains the equation of motion

$$-\ln(\lambda)^{\cdot}g_{mn}\dot\gamma^n+(\partial_s g_{mn})\dot\gamma^s\dot\gamma^n+g_{mn}\ddot\gamma^n+\frac{1}{2}(\partial_m g^{ab})g_{as}g_{bt}\dot\gamma^s\dot\gamma^t=0\,. \tag{8}$$

In order to eliminate λ, invoke the variation of (5) with respect to λ, yielding the stationarity condition

$$g^{ab}p_a p_b-m^2=0\,,$$

or, plugging in the solution of (6),

$$g_{mn}\dot{\gamma}^m\dot{\gamma}^n = 4\lambda^2 m^2\,. \tag{9}$$

One way to proceed is to solve this equation for λ and then to eliminate λ from equation (8), which recovers precisely the Euler-Lagrange equations corresponding to the action (3),

$$-\ln\left(\frac{\sqrt{g_{ab}\dot{\gamma}^a\dot{\gamma}^b}}{2|m|}\right)^{\cdot} g_{mn}\dot{\gamma}^n + g_{mn}\ddot{\gamma}^n + \frac{1}{2}\left(\partial_u g_{mv} + \partial_v g_{mu} - \partial_m g_{uv}\right)\dot{\gamma}^u\dot{\gamma}^v = 0\,. \tag{10}$$

Another way is to study the direct relation between the choice of λ and the choice of parameterization of the curve γ that is imparted by equation (9). To this end, observe from comparing the first terms of (8) to (10) that choosing $\lambda = \frac{1}{2|m|}$ in (8) corresponds to choosing the parametrization to be given by the proper time along the curve γ in (10). This choice of parametrization yields the commonly known form of the geodesic equation in geometric (or, physically speaking, proper time) parametrization,

$$g_{mn}\ddot{\gamma}^n + \frac{1}{2}\left(\partial_u g_{mv} + \partial_v g_{mu} - \partial_m g_{uv}\right)\dot{\gamma}^u\dot{\gamma}^v = 0\,. \tag{11}$$

Choosing, instead, any strictly increasing reparametrization $\sigma : (0,1) \to (0,1)$ of the proper time parametrization of the curve p, and thus γ, then corresponds to the choice $\lambda = \dot{\sigma}/2m$ and so explicitly provides the claimed link between the choice of λ and the choice of parametrization of the curve.

In any case, the phase space dynamics encapsulated in the action (5) recovers the equation of motion for the projection $\pi \circ p : (0,1) \to M$ of the curve p. In order to see that the dynamics not only yield the desired projection $\pi \circ p$, but indeed a unique one-particle history $p : (0,1) \to T^*M$ with this property, note that, even without a special parameter choice, solving (9) for λ and inserting this into (6), one obtains

$$p_a = m\frac{g_{ab}\dot{\gamma}^b}{\sqrt{g_{mn}\dot{\gamma}^m\dot{\gamma}^n}}\,, \tag{12}$$

which is precisely the relation between the values of the curve p and the tangent vectors of the underlying worldline projection. Thus the action (5) is recognized as the lift of the free massive particle dynamics for worldlines on M to one-particle histories on T^*M.

B. Dynamics on histories with constant particle number

Now we wish to extend the dynamics from the case of a free one-particle history to a b-particle history. This is simply accomplished by stipulation of the action functional

$$S^{\text{free}}_{b\text{-particle history}}[(p,m),\lambda] = \sum_{\beta=1}^{b} S^{\text{free}}_{\text{1-particle history}}[(p_\beta, m_\beta), \lambda_\beta]\,, \tag{13}$$

where (p, m) now denotes a b-particle history and $p_\beta : (0, 1) \to T^*M$ are the one-particle maps of p and $\lambda_\beta : (0, 1) \to \mathbb{R}$ are the component maps of the Lagrange multiplier map $\lambda : (0, 1) \to \mathbb{R}^b$ on which the action functional also depends. It is clear that variation with respect the b-particle map p and the Lagrange multiplier map λ, which reduces to variation with respect to all one particle maps $p_1, \dots, p_b$ and Lagrange multiplier functions $\lambda_1, \dots, \lambda_b$, produces b independent geodesic equations for the underlying one-particle worldlines $\pi_\beta \circ p_\beta$ after elimination of the Lagrangian multipliers.

C. Dynamics on histories with variable particle number

For a history (p, m), which accommodates N branches with particle numbers $b_1, \dots, b_N$ the free action is now simply obtained from the constant particle history dynamics by summing over all branches,

$$S^{\text{free}}_{\text{history}}[(p,m),\lambda] = \sum_{n=1}^{N}\sum_{\beta=1}^{b_n} S^{\text{free}}_{\text{1-particle history}}[(p^{(n)}_\beta, m^{(n)}_\beta), \lambda^{(n)}_\beta]\,, \tag{14}$$

where $\lambda \equiv (\lambda^{(1)}, \dots, \lambda^{(N)})$ denotes a Lagrange multiplier map $\lambda : (0, 1) \to \mathbb{R}^{b_1+\dots+b_N}$.

D. Sketch of applications

We briefly sketch the ideas for three applications of histories in standard general relativity.

The first application concerns the interpretation of the spacetime curvature. When focusing on conventional worldlines, the entire Riemann curvature tensor at a point can be reconstructed from the relative acceleration of a family of neighbouring geodesics to one of them which runs right through the point p. This is afforded by the Jacobi equation [3], whose derivation rest in the use of geodesics and hence standard worldlines. The physical relevance of the spacetime curvature then of course arises through the identification of freely falling point particles with their geodesic worldlines. In the relativistically more consistent framework provided by histories, rather than worldlines, novel insights into the role of curvature seem possible. Particularly if the annihilation of some particles in favour of the creation of others depends on the local curvature, the interpretation of the latter through histories must be expected to differ from the one obtained through the relativistically unduly restricted focus on worldlines.

The second application concerns causality. The causal future and past of a point p on a spacetime M are described by sets that consist of all points that can be reached by continuous non-spacelike worldlines [2, 3]. On some spacetimes, such worldlines are required to be non differentiable ([2] has a couple examples of this happening), which describes rather odd behaviour of the worldline. Therefore, it seems more elegant to describe causality using histories. It seems possible to define the future of a point $p \in M$ by the set of all points $q \in M$ such that there exists a free falling history that, when projected down onto M, starts at p and has a one particle map ending at q. For instance, if we remove a strip from a flat spacetime, we could still have free falling histories that "go around" the strip, c.f. figure 2.

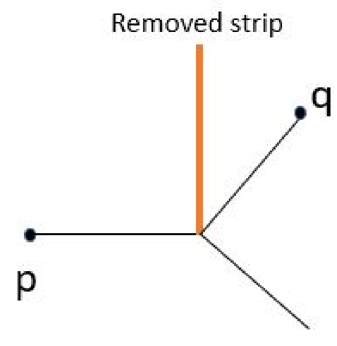

FIGURE 2: A free falling history on a flat spacetime, connecting the points p and q. A straight line between p and q would have to go through the removed strip, hence there is no geodesic connecting p and q.

The third application concerns singularities. Singularity theory uses worldlines on a curved spacetime to indicate singularites, which are points outside of the spacetime, that can be reached by a worldline of finite length [2, 3]. If such a worldline is timelike, a spaceship with enough fuel could reach the end of the universe in finite time, which is a rather strange phenomenon. A history, on the other hand is allowed to annihilate some particles and create different particles, this would allow a particle to be "ripped apart" by tidal forces near such a singularity, thus preventing the particle to reach the end of the universe. Thus approaching singularity theory from a history point of view could provide some new insights into singularities that properly account for very large tidal forces.

VI. CONCLUSIONS

A massive particle on a spacetime is an object with both a position and a momentum and is therefore viewed as a history, a curve on phase spacetime. If the particle is only affectd by gravity, such a massive one-particle history can be equipped with the Hamiltonian that arises from the constraint the mass puts on the momentum. This Hamiltonian can then be used to find the action functional for the particle and thus, ultimately, the dynamics corresponding to the history. These dynamics are identical to the dynamics derived from a Lagrangian description of a free falling massive particle, i.e. the geodesic equation.

Whenever there are several massive particles on the spacetime, we describe them by a single curve on several copies of the phase spacetime, such that every particle is described by a one-particle history on its own phase spacetime. If the particles are only affected by gravity, there is a Hamiltonian description for these particles by summing up the individual particle Hamiltonians. In this way we recover an action functional for the history as the sum over the individual one-particle actions.

Furthermore, if the particles on the spacetime are able to split, recombine or collide without particle number conservation, more structure is required to properly formulate dynamics. Thus we take a disjoint union over all multi particle phase spacetimes to obtain the Fock phase space. This space can be turned into a topological space by endowing it with the disjoint union topology, a topology that respects the topological structure on all multi particle phase spacetimes that constitute the Fock phase space.

A massive particle history that doesn't conserve particle number can then be viewed as a curve on the Fock phase space. These histories are rendered continuous by removing those parameters from the domain of the curve, where particle number is not conserved. When the history is only affected by gravity, the Hamiltonian is formulated as the sum over all individual one-particle Hamiltonians. Thus we can describe the dynamics for such histories using the action functional arising from this Hamiltonian description.

With this formulation of particle histories, one could look for possible applications in the field of general relativity. A first thought would lead one to the idea of describing curvature with the help of free falling histories. If a free falling particle could potentially split and recombine with other free falling particles, there should be a description of curvature that takes these events into account.

Secondly, the description of causal future of a point uses a description of non-spacelike worldlines that can potentially be not continuously differentiable. But such timelike worldlines wouldn't be physically significant for one-particle histories, therefore we could potentially define the chronological future of a point to be the set of all points that can be reached by a free falling history.

Thirdly, one could study singularities using histories. Since a singularity is a point that is outside the spacetime, but can be reached in finite proper time, a one-particle history could potentially reach the singularity and thus the end of the universe. Allowing variable particle number would mean the particle could be forced to split, thus avoiding the singularity.

Before these extensions can be made, however, a broader theory of histories needs to be formulated. Massless particles need to be introduced and dynamics for particles coupled to a field should be formulated. Furthermore, as of now only kinematical contraints are given for splitting/colliding particles, one could work on a formulation of dynamics of these events.

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