UNIVERSITY OF TWENTE.

**Faculty of Electrical Engineering, Mathematics & Computer Science**

# Non-linear dynamic control for positive-real systems

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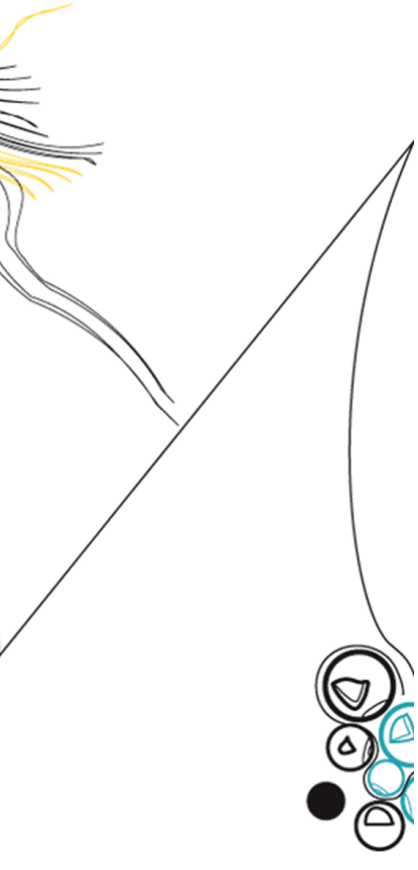

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**Abstract**

In this research a non-linear controller called the Hybrid Integrator-Gain System has been designed and analysed. The controller switches between integrator and gain control and it generates a control signal of the same sign as its input signal. It has been applied to the positive real system of the vibrating string. That system is an example of a linear Port-Hamiltonian system. In such systems the change in energy in the system is equal to the product of the input and output of the system. The model of the vibrating string has been approximated and is connected to the Hybrid Integrator-Gain System and integrator and gain control separately. The energy and stability of the resulting closed loop feedback system has been analysed. Although the integrator control does not stabilise the vibrating string on its own, the Hybrid Integrator-Gain System stabilises the system and the energy in the system decreases monotonically.

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# 1. Introduction

Many phenomena that evolve with time are modelled using dynamical systems. Often these phenomena are natural existing occurrences, such as changes in weather, the streaming of water or the moving of planets or man-made structures and machines, such as moving vehicles. We want to understand all these phenomena better or even influence them. In a dynamical system the behaviour of a such a phenomenon is mimicked by relating its underlying variables, such as temperature, density or position. When a dynamical system represents a real world occurrence correctly we can experiment with the way it will react to external influences and predict its behaviour in the future. For example, forces as wind, vibrations of an earthquake, or a push can influence the behaviour of a system. To get a system to behave in a certain way we can search for the correct external influences such that we get desired results, this is called controlling a system. The found control can then be applied in the real world.

An important variable in dynamical systems is the energy stored in the system. Some dynamical systems, without external influences, have the property that the energy stored remains unchanged or leaves the system. Such systems are called dissipative systems. When the change of the energy in the system is equal tot the product of the input and output then the system is called Port-Hamiltonian. The energy changes via the boundaries.

Port-Hamiltonian systems which are linear and time-invariant have a positive real transfer function and are called Positive Real. It is useful to control such a system in a way that the energy always decreases. This is possible using non-linear control. In this project we will apply non-linear dynamic control to Positive Real systems.

An more general example of non-linear dynamic control is reset control. It will induce less phase lag into a system by keeping the sign of the control input and output equal. This method does introduce discontinuous control signals which can cause disturbances in the system..

A method which keeps the control signal continuous and also keeps the integrator input and output sign equal is the Hybrid Integrator-Gain System (HIGS). These kind of systems use an adjusted integrator part which switches to a proportional controller while keeping the control signal continuous.

In this thesis the method of Hybrid-Integrator Gain Systems has been researched for positive real systems. How such systems work and what influence HIGS has on them is studied.

# 2. Recent Research

Over the years non-linear control methods have been the subject of many research projects. The notion of a Hybrid Integrator-Gain System based on reset control has been introduced and formulated in [4]. The Clegg method was used to reduce low frequency disturbance in a system while the HIGS would also handle high frequency disturbances, since it does not hard-reset the state of the controller. In [4] stability conditions for a Hybrid Integrator-Gain System are provided and it has been experimentally tested on a wafer stage system resulting in a performance improvement.

The paper [3] presents a mathematical framework to formally describe the HIGS based on projected dynamical systems. This way the well-posedness and global existence of solutions can be shown. It introduces two approached to show closed-loop stability of a HIGS-controlled system. Well-Posedness for non-linear controlled systems have been further researched in [1].

In this project the dynamical system model of the vibrating string will be used. It is a common system in physics and theory on Partial Differential Equations. The vibrating string is a well-known example of a Port-Hamiltonian system and the Hybrid-Integrator Gain System will be applied to the model.

# 3. Preliminaries

In this chapter some general definitions around dynamical systems and system and control theory are outlined which will be referred to during the report. An abstract definition of a dynamical system is given and some practical models are defined. Next to that some important theorems and properties are stated which will help in the analysis of the systems.

## 3.1 Models for Dynamical Systems

Dynamical systems can be modelled in several different ways. Properties of a dynamical system such as the notion of linearity and time-invariance are introduced and a well known way to model dynamical systems, called state-space representations, is introduced. The definitions of these properties and more background can be read in the source [10]. Some of the definitions have been put in this chapter for easy reference. An abstract definition of a dynamical system is given next.

**Definition 3.1.1** (Dynamical System)**.** *A dynamical system $\Sigma$ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subset \mathbb{R}$, called the time axis, $\mathbb{W}$ a set called the signal space, and $\mathfrak{B} \subseteq \{w : \mathbb{T} \to \mathbb{W}\}$ called the behaviour.*

An important and useful property of a dynamical system is linearity.

**Definition 3.1.2** (Linearity)**.** *A Dynamical System $\Sigma$ is linear if signal space $\mathbb{W}$ is a vector space and behaviour $\mathfrak{B}$ is a linear subspace of all maps $\{w : \mathbb{T} \to \mathbb{W}\}$, i.e.*

1. $0 \in \mathfrak{B}$
2. *if $w_1$, $w_2 \in \mathfrak{B}$ then $w_1 + w_2 \in \mathfrak{B}$*
3. *if $w \in \mathfrak{B}$ and $\lambda \in \mathbb{R}$ then $\lambda w \in \mathfrak{B}$, $\forall \lambda$.*

Next to linearity the property of time-invariance makes the analysis of dynamical systems simpler.

**Definition 3.1.3** (Time-Invariance)**.** *A system $\Sigma$ is time-invariant if time axis $\mathbb{T} = \mathbb{R} \vee \mathbb{Z}$ and for all $\tau \in \mathbb{T}$ and signal $w \in \mathfrak{B}$ the delay $\sigma^\tau w \in \mathfrak{B}$, where $\sigma$ is the delay operator $\sigma^\tau w(t) = w(t + \tau)$.*

Many dynamical systems can be represented as a so called input/output system. These systems can be influenced via the input and are very common

in control theory since the input can be determined in such a way that the output behaves as desired.

**Definition 3.1.4** (LTI input-output system)**.** *A system with behaviour* $\mathfrak{B} = \{(u, y) : \mathbb{T} \to (\mathbb{W}_u, \mathbb{W}_y) \mid y = \mathscr{H}(u)\}$*, where input* $u$ *determines output* $y$ *is called an input-output system defined by the map* $\mathscr{H} : u \to y$*.*
*An input-output system is Linear Time-Invariant (LTI) if the map* $\mathscr{H}$ *is linear, i.e. for all* $u_1$*,* $u_2$*,* $u \in \mathfrak{B}$*,* $\mathscr{H}(u_1+u_2) = \mathscr{H}(u_1)+\mathscr{H}(u_2)$ *and* $\mathscr{H}(\lambda u) = \lambda\mathscr{H}(u)$ *for all* $\lambda \in \mathbb{R}$*, and time-invariant, i.e.* $\mathscr{H}(\sigma^\tau u) = \sigma^\tau \mathscr{H}(u)$ *for all* $\tau \in \mathbb{T}$*.*

Next to an input and output a dynamical system can involve a state vector. The definition for linear time-invariant state-space systems is given in Definition 3.1.5 and will be used extensively.

**Definition 3.1.5** (State-Space Representation)**.** *A system of the form*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.1}$$

*is called a State-Space representation where* $u(t) \in \mathbb{R}^p$ *the input vector,* $y(t) \in \mathbb{R}^n$ *the output vector,* $x(t) \in \mathbb{R}^q$ *the state vector and* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times p}$*,* $C \in \mathbb{R}^{q\times n}$ *and* $D \in \mathbb{R}^{q\times p}$*.*

These are the types of models used in this report to apply control methods to.

### Existence of solutions for State-Space Representations

To control a dynamical system modeled by a state-space representation it is always good to know if there actually exists an output for any given input signal and if the state of the system exist in the case of state-space representations. This section will show that there exist solutions as long as some behavioural properties hold for the input signal of state-space systems.

A solution to a state-space representation is based on the input signal $u(t)$, which evolves with time $t$, and an initial condition of the state of the system, $x(t_0) = x_0$. This initial condition represents the situation of the system at the starting time $t_0$. A solution of the state will be determined by the state equation $\dot{x}(t) = Ax(t)+Bu(t)$ in Definition 3.1. Starting with an exponential candidate solution $x(t) = e^{At}z(t)$ for a general state-space representation, where $z(t) \in \mathbb{R}^n$. The candidate solution is differentiated,

$$\dot{x}(t) = Ae^{At}z(t) + e^{At}\dot{z}(t). \tag{3.2}$$

By the given state equation we have that $Bu(t) = \dot{x}(t) - Ax(t)$, which, substituting in our the expression for the candidate solution, results in an expression for the derivative of the candidate solution $\dot{z}(t)$,

$$Bu(t) = e^{At}\dot{z}(t) \implies \dot{z}(t) = e^{-At}Bu(t). \tag{3.3}$$

Now integration and substitution of the candidate solution results in an expression for the state of the representation,

$$\begin{aligned} z(t) &= z(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau \\ \implies x(t) &= e^{At}\left(e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau\right), \end{aligned} \tag{3.4}$$

which gives the expression,

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{3.5}$$

Clearly the solution for the state is completely determined by the initial condition $x_0$ and input signal $u(t)$. The output is then also determined by the initial state and input signal using the output equation $y(t) = Cx(t) + Du(t)$ in Definition (3.1), further described as

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{3.6}$$

These expressions for the state $x(t)$ and output $y(t)$ exist for all input signals $u(t)$ which are locally integrable functions. This confirms that the integral in (3.5) and (3.6) exist. Locally integrable function are functions for which the following property holds,

$$\int_{b}^{a} |u(\tau)|d\tau < \infty \quad \forall (a,b) \in \mathbb{R}. \tag{3.7}$$

This gives a condition on the input of a state-space representation. Another definition involving the behaviour of functions is the notion of Lipschitz continuity, [9], [5].

**Definition 3.1.6** (Lipschitz continuity)**.** *A function $f(x) \in \mathbb{R}$ is globally Lipschitz continuous if there exists a real constant $C > 0$ such that for all $x_1$,$x_2 \in \mathbb{R}$*

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|. \tag{3.8}$$

Intuitively a Lipschitz continuous function is a continuous function which does not change abruptly. It is an important property for the existence of solutions of differential equations.

## 3.2 Lyapunov stability

An important function in the analysis of dynamical systems is the Lyapunov function. Such a is function is useful in determining if a system is stable. A definition for a Lyapunov function is given in Definition (3.2.1).

**Definition 3.2.1** (Lyapunov function)**.** *Given a dynamical system represented by $\dot{x}(t) = f(x(t))$, where $x(t) \in \mathbb{R}^n$ and $f(0) = 0$. A Lyapunov function for such a system is a function $V : \mathbb{R}^n \to \mathbb{R}$ for which the property holds that*

$$V(x(t)) > 0 \quad \forall x(t) \neq 0 \tag{3.9}$$

*and the derivative $\dot{V}(x(t)) \leq 0$ along all solutions $x(t)$ of the system. When the derivative of $V(x(t))$ is equal to zero for all solutions $x(t)$ then it is called a weak Lyapunov function.*

When a Lyapunov function exists for a dynamical system of the form $\dot{x}(t) = f(x(t))$ then the dynamical system is asymptotically stable. This basically means that the system will converge towards an equilibrium. In the case of the existence of a weak Lyapunov function it means that such a system will neither converge nor diverge, but instead will stay in a certain bounded area.

If, for a Lyapunov function $V(x(t))$, the implication holds that if $\|x(t)\| \to \infty \implies V(x(t)) \to \infty$ then the Lyapunov function is called radially unbounded.

## 3.3 Transfer functions

Another way of writing a dynamical system modeled by a state-space representation is by the use of the its Transfer Function, [6]. Transfer Functions are functions in the frequency domain, $s \in \mathbb{C}$, that show the response of the a system on certain inputs. The Transfer Function of a state-space representation is defined as $H(s) = C(sI - A)^{-1}B + D$. The Transfer Function is the Laplace transform to the frequency response of a system. The relation between the input signal $U(s)$ and output signal $Y(s)$ in frequency domain is given by the equation $Y(s) = H(s)U(s)$. When the system described by a state-space representation is minimal then the eigenvalues of the matrix $A$ are also the poles of the function $H(s)$.

A useful theorem for analysing Transfer functions is the Final Value Theorem, stated in Theorem (3.3.1).

**Theorem 3.3.1** (Final Value Theorem)**.** *Given a function $f(t)$ where its final value $t \to \infty$ exists and it's Laplace transform $F(s)$ is defined for all $Re(s) > 0$, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{3.10}$$

The Final Value Theorem is used to see if the output of a dynamical system will converge or diverge.

## 3.4 Feedback systems

In this project we will control a dynamical system, which will be called the plant. Controllers will be connected to the plant via a general negative feedback loop, [6]. Such a feedback loop is visualised in the diagram below. The input of the system is a desired reference signal $r(t)$ of which the output

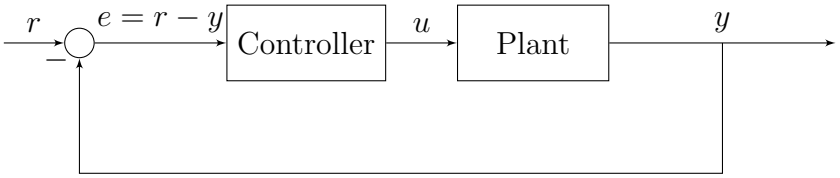



Figure 3.1: Feedback diagram

$y(t)$ must mimic its behaviour. The error signal $e(t) = r(t) - y(t)$ into the controller represents how far off of the desired behaviour the output signal is and determines via the controller what control signal $u(t)$ is used as input to the plant to achieve a better result. If the controller and plant both have known transfer functions a combined transfer function to describe the feedback loop as a whole can be derived. Given the transfer functions $K(s)$ of the controller and $P(s)$ of the plant and using the relations $Y(s) = P(s)U(s)$ and $U(s) = K(s)E(s) = K(s)(R(s)-Y(s))$ a transfer function for the system as a whole can be derived.

$$\begin{aligned} Y(s) &= P(s)K(s)(R(s) - Y(s)) \\ \implies Y(s)(1 + P(s)K(s)) &= P(s)K(s)R(s) \end{aligned} \tag{3.11}$$

The input output relation of the system is $Y(s) = \frac{P(s)K(s)}{1+P(s)K(s)}R(s)$, and the transfer function for a general negative feedback loop is,

$$H(s) = \frac{P(s)K(s)}{1 + P(s)K(s)}. \tag{3.12}$$

This relation is very useful for simulating dynamical feedback systems and controlling the output.

## 3.5 Port-Hamiltonian system

Port-Hamiltonian systems are dynamical systems in which the change in energy in the system is equal to the product of the input and output of the system. The energy only changes via the boundary of the system. A linear Port-Hamiltonian system can be described as a state-space representation. It will be defined in this section but first we need the notion of a Positive Definite matrix, [10].

**Definition 3.5.1** (Positive Definite matrix)**.** *A symmetric matrix $A \in \mathbb{C}^{n\times n}$, i.e. $A = A^*$, is called Positive Definite if, for all vectors $x \in \mathbb{C}^n$ and $x \neq 0$,*

$$x^*Ax > 0. \tag{3.13}$$

*We write $A \succ 0$.*

Now we are ready to introduce a Port-Hamiltonian state-space representations. [2].

**Definition 3.5.2** (Linear Port-Hamiltonian system)**.** *Let $\mathcal{H}$ be a positive-definite matrix, and let $J$ be a skew-adjoint matrix, i.e. $\mathcal{J}^* = -\mathcal{J}$. Then the system*

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}\mathcal{H}x(t) + Bu(t) \\ y(t) &= B\mathcal{H}x(t) \end{aligned} \tag{3.14}$$

*is called a linear Port-Hamiltonian system associated to $\mathcal{H}$ and $\mathcal{J}$. $\mathcal{J}$ is called the structure matrix and $\mathcal{H}$ is the Hamiltonian density. The Hamiltonian associated to $\mathcal{H}$ is $\frac{1}{2}x^*\mathcal{H}x$.*

# 4. Hybrid Integrator-Gain Systems

In this chapter we design and analyse a Hybrid Integrator-Gain System (HIGS). The inner workings of the control method will be described in detail. A HIGS application results in a switched dynamical system and is a form of reset control.

## 4.1 Reset control

Reset control is itself a switching system. This means the model of a dynamical system switches from one representation to another. These switches can be random but are often based on the behaviour of the signals in the system. When certain criteria are met the system switches representation. This results in a non-linear model and may result in a complex control problem. An example of a switching system is the Clegg controller. [4]. It is an integrator which resets its state right-hand limit, denoted by $x_r(t^+)$, to zero whenever the input and output are not of equal sign. The representation of the system thus only changes in the state vector, a reset to zero. Still, this is seen as a change in model representation. The Clegg controller has input signal $e(t) \in \mathbb{R}$, output signal $u_r(t) \in \mathbb{R}$ and integrator frequency coefficient $\omega_r \geq 0 \in \mathbb{R}$. For simplicity we consider the input signal $e(t)$ to be continuous. The integrator frequency $\omega_r$ is used as a tune-able parameter of the controller and can be adjusted when needed. To ensure the output signal will start at zero the state of the controller is initially in rest by initial condition $x_r(t_0) = 0$. The controller is described in the state-space form of (4.1),

$$Clegg(e(t), u_r(t)) = \begin{cases} \dot{x}_r(t) & = \omega_r e(t) \qquad \text{if } e(t)u_r(t) \geq 0 \\ x_r(t^+) & = 0 \qquad\qquad \text{if } e(t)u_r(t) \leq 0\,. \\ u_r(t) & = x_r(t). \end{cases} \tag{4.1}$$

By the second equation in the definition of the Clegg controller in (4.1) the right-hand limit is reset to zero when $e(t)u_r(t) \leq 0$. This happens when the input and output signal of the controller is of unequal sign. One of them may be positive while the other is negative and it does not matter which of the signals is positive or negative. When the right-hand limit of the state is set to zero it causes the state to be discontinuous. This results in the output signal becoming discontinuous also. When applying reset control, such as the Clegg controller, to a plant it is important to know how the plant will deal with a

discontinuous input signal, since the control input will be discontinuous. In certain dynamical systems this will be undesired and might cause problems like breaking of a machine or amplifying small disturbances.

As an example the test signal $e(t) = \sin(2t) + \frac{1}{2}\sin(3t)$, with integrator frequency $\omega_h = 1$, is put into the Clegg controller.

In Figure 4.1 the output is clearly reset to zero where the input signal crosses the horizontal axes and becomes negative. After the reset the signs stay equal again until a reset occurs.

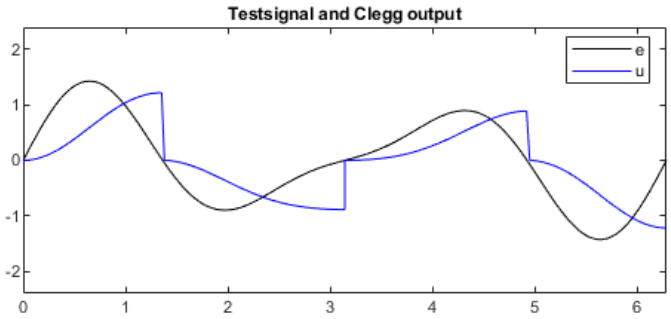



Figure 4.1: Clegg output with test input

The design of a HIGS will have a similar structure to the Clegg controller.

## 4.2 The HIGS design

The goal of a Hybrid Integrator-Gain System is to create a control signal of equal sign as the input signal. It is designed as a controller which switches between two control methods, [4]. These two methods are integrator control and gain control. Both well known methods in control theory and often used in controllers, [6]. When to switch is based on the relation between the input, the input derivative and the output of the controller. The vector $x_h(t)$ represents the state of the controller, $e(t)$ the input signal and $u(t)$ the output signal. Connected to a plant in a closed-loop system $u(t)$ will be the control signal into the plant. The HIGS will always be initially in rest, so $x_h(t_0) = 0$ for the starting time $t_0$. The coefficient $\omega_h$ is called the integrator frequency and $k_h$ the gain constant. Both $\omega_h$, $k_h \in (0, \infty)$ and must be seen as tune-able parameters to provide the desired output of the feedback system.

The HIGS block is designed to not reset the state completely to zero, as does the Clegg controller (4.1), but instead switch control mode and continue the state. This way the state will be continuous, which will result in a continuous output signal as well. The Hybrid Integrator-Gain System is defined in (4.2),

$$H(e,\dot{e},u) = \begin{cases} \dot{x}_h &= \omega_h e \quad \text{if } (e,\dot{e},u) \in \mathcal{F}_1 \\ x_h &= k_h e \quad \text{if } (e,\dot{e},u) \in \mathcal{F}_2 \\ u &= x_h. \end{cases} \tag{4.2}$$

The output signal of a HIGS is bounded by an area denoted as $\mathcal{F}$. This area consists of two subregions, one region for each control method, noted as the union $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. When the output signal is in $\mathcal{F}_1$ the HIGS is in integrator mode and in $\mathcal{F}_2$ it is in gain mode. The specific areas are described by constraints on the input, input derivative and output signals.

$$\mathcal{F} = \{(e,\dot{e},u) \in \mathbb{R}^3 \mid k_h e u \geq u^2\}. \tag{4.3}$$

The constraint for area $\mathcal{F}$ ensures that the input and output signal always have the same sign. The constraint holds if $e(t)$ and $u(t)$ are both positive or both negative. When $u(t) = 0$ the boundary of $\mathcal{F}$ is reached.

For the second area $\mathcal{F}_2$ the HIGS is in gain mode.

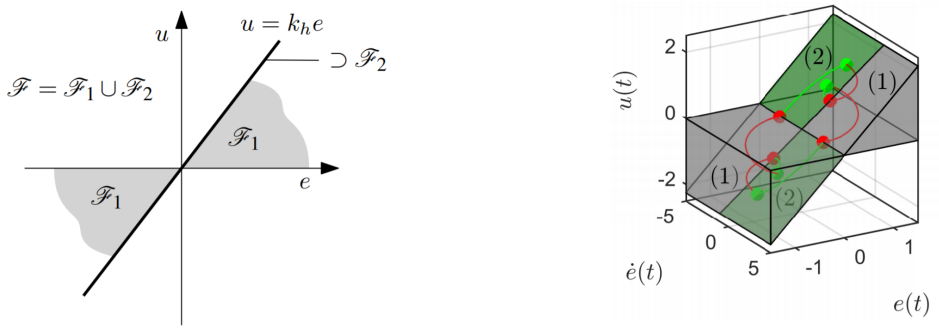
$$\mathcal{F}_2 = \{(e,\dot{e},u) \in \mathcal{F} \mid u = k_h e \wedge \omega_h e^2 > k_h \dot{e} e\}. \tag{4.4}$$

The area is part of the line $u(t) = k_h e(t)$ on the boundary of $\mathcal{F}$. There equality of the constraint for being in $\mathcal{F}$ holds, $k_h e(t) u(t) = u(t)^2$. In gain mode the state $x_h(t) = k_h e(t)$ results in the output signal $u(t) = k_h e(t)$, hence the output $u(t)$ follows the behaviour the input signal. The second constraint of area $\mathcal{F}_2$ compares the derivative of $u(t)$ in gain mode to what the derivative of $u(t)$ would be in integrator mode. The output derivative in gain mode is $\dot{u}(t) = k_h \dot{e}(t)$ and the derivative in integrator mode would be $\dot{u}(t) = \omega_h e(t)$.

The interior of $\mathcal{F}$ is the other subregion $\mathcal{F}_1$, here the HIGS is in integrator mode,

$$\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_2. \tag{4.5}$$

In Figure 4.2a a plot of the area $\mathcal{F}$ is given. The area $\mathcal{F}_1$ is the interior of a cone bounded by the line $u(t) = k_h e(t)$, which is area $\mathcal{F}_2$, and the horizontal axis. For more insight the visualisation of $\mathcal{F}$ is also given in three dimensional space with the derivative of $e(t)$ as third dimension, Figure 4.2b. In the areas (1) the HIGS block is in integrator mode and in (2) it is in gain mode.




(a) Two dimensional

(b) Three dimensional

Figure 4.2: Visualisation of $\mathcal{F}$, [4]

When a switch occurs the derivative of the state is abruptly changed resulting in a piecewise differentiable state and thus a piecewise differentiable output signal.

### Initialisation

The HIGS will always start in integrator mode and its state at the starting time $t_0$ will always be in rest, $x_h(t_0) = 0$. The output signal at that time will be zero, $u(t_0) = 0$ and will start to evolve according to the input signal $e(t)$. The constraint $k_h e(t_0)u(t_0) \geq u(t_0)^2$ will be equal at the starting time and so we start inside of $\mathcal{F}$.

### Crossing the horizontal axis

When both $e(t)$ and $u(t)$ are zero we are still inside of $\mathcal{F}$. This will be shown using simple Taylor series approximations of the input and output signals $e(t)$ and $u(t)$ around a point in time denoted $\tau$, assuming both $e(\tau)$ and $u(\tau)$ are zero at this time. To make the Taylor approximations we must have an appropriate differentiability for the signals in the system. The input signal must be at least once differentiable such that its series can be of first order and the output signal must be at least twice differentiable for a second order approximation. For $t$ around $\tau$ the two Taylor series are,

$$\begin{aligned} e(t) &\approx e(\tau) + \dot{e}(\tau)(t-\tau) \\ u(t) &\approx u(\tau) + \dot{u}(\tau)(t-\tau) + \frac{1}{2}\ddot{u}(\tau)(t-\tau)^2. \end{aligned} \tag{4.6}$$

Since $\tau$ is the point on the horizontal axis where the two signals touch, $u(\tau) = 0$ and $e(\tau) = 0$. Assume $\dot{e}(\tau)$ arbitrary, then, in integrator mode,

$\dot{u}(\tau) = \omega_h e(\tau)$ implies $\dot{u}(\tau) = 0$ as well. The Taylor approximations (4.6) become $e(t) \approx \dot{e}(\tau)(t-\tau)$ and $u(t) \approx \frac{1}{2}\ddot{u}(\tau)(t-\tau)^2$ for $t$ close to $\tau$. The series are substituted in both sides of the constraint for $\mathcal{F}$,

$$\begin{aligned} k_h e(t)u(t) &\approx k_h \dot{e}(\tau)(t-\tau)\frac{1}{2}\ddot{u}(\tau)(t-\tau)^2 \\ u(t)^2 &\approx (\frac{1}{2}\ddot{u}(\tau)(t-\tau)^2)^2 \end{aligned} \tag{4.7}$$

This results in $k_h \dot{e}(\tau)\frac{1}{2}\ddot{u}(\tau)(t-\tau)^3 \geq \frac{1}{4}u(\tau)^2(t-\tau)^4$, or simply,

$$k_h \dot{e}(\tau)\ddot{u}(\tau) \geq \frac{1}{4}\ddot{u}(\tau)^2(t-\tau). \tag{4.8}$$

The left-hand side is constant and the right-hand side is linear. The constraint holds for all $t$ close to $\tau$, where the right-hand side is close to zero. So the constraint for $\mathcal{F}$ is satisfied at $\tau$ and we stay in $\mathcal{F}$ when crossing the horizontal axis.

This method holds even if $\dot{e}(\tau) = 0$. In that situation the order of the Taylor series is increased and the same result will occur. The signals need to have one order of differentiability higher for those higher order Taylor series to exist.

The input signal is allowed to be piecewise differentiable, in this case the derivatives in the Taylor expansions must be seen as the right-hand limit derivatives at $\tau^+$. These limit derivatives do exist for piecewise differentiable signals.

## 4.3 Switching control modes

An overview will give insight in the switching of control modes of the Hybrid Integrator-Gain System.

**When in integrator mode at time $\tau$ in the interior of $\mathcal{F}$,** we are in $\mathcal{F}_1$ and the derivative of the output signal is $\dot{u}(\tau) = \omega_h e(\tau)$. If $e(\tau) > 0$ the signal $u(\tau)$ increases as long as we are inside the interior the inequality $u(\tau) < k_h e(\tau)$ holds, so we stay in integrator mode. When $e(\tau) < 0$ the signal decreases and as long as we are inside the interior the inequality $u(\tau) > k_h e(\tau)$ holds. Again, we stay in integrator mode. Combining the positive and negative case of $e(\tau)$ we only check if we stay in the interior of $\mathcal{F}$, the area $\mathcal{F}_1$.

- $u(\tau) \neq k_h e(\tau) \implies$ stay in integrator mode

**When in integrator mode at time $\tau$ on the boundary of $\mathcal{F}$** we are on the boundary $\mathcal{F}_1$ and the output signal touches the line $\mathcal{F}_2$. Here $u(\tau) = k_h e(\tau)$ and the second constraint of $\mathcal{F}_2$ is checked.

If $e(\tau) > 0$ the inequality $\omega_h e(\tau) > k_h \dot{e}(\tau)$ would imply that, by staying in integrator mode, we leave $\mathcal{F}$ since the derivative in integrator mode $\dot{u}(\tau) = \omega_h e(\tau)$ would increase faster than the derivative $\dot{u}(\tau) = k_h \dot{e}(\tau)$ in gain mode. The constraint $k_h e(\tau) u(\tau) \geq u(\tau)^2$ is violated if the HIGS would stay in integrator mode so it makes a switch to gain mode.

The same argument holds if $e(\tau) < 0$, in this case the inequality $\omega_h e(\tau_s) < k_h \dot{e}(\tau)$ is checked instead, and a switch to gain mode is made. The positive and negative cases of $e(\tau)$ are captured in the combined constraint $\omega_h e(t)^2 > k_h \dot{e}(t) e(t)$.

- $\omega_h e(\tau)^2 > k_h \dot{e}(\tau) e(\tau) \implies$ switch to gain mode

**When in gain mode at time $\tau$ on the boundary of $\mathcal{F}$** we are in the area $\mathcal{F}_2$ and the derivative of the output is $\dot{u}(\tau) = k_h \dot{e}(\tau)$. When $e(\tau) > 0$ then, if $\omega_h e(\tau) > k_h \dot{e}(\tau)$, a switch to integrator mode would result in leaving $\mathcal{F}$, so we stay in gain mode. The output in integrator mode would increase faster than in gain mode, which would cause the constraint $k_h e(\tau) u(\tau) \geq u(\tau)^2$ to be violated. When $e(\tau)$ is negative the same argument holds for $\omega_h e(\tau) < k_h \dot{e}(\tau)$ and we stay in gain mode. The output moves along the boundary, $\mathcal{F}_2$. Both cases are combined in the constraint $\omega_h e(t)^2 > k_h \dot{e}(t) e(t)$.

- $\omega_h e(\tau)^2 > k_h \dot{e}(\tau) e(\tau) \implies$ stay in gain mode

If instead, for $e(\tau) > 0$, the inequality $\omega_h e(\tau) \leq k_h \dot{e}(\tau)$ holds the output in integrator mode would increase slower than the output in gain mode. A switch to integrator mode would cause the output go into the interior of $\mathcal{F}$, so a switch to integrator mode is made. The same switch occurs for $e(\tau) < 0$ and the inequality $\omega_h e(\tau) \geq k_h \dot{e}(\tau)$. Both cases are combined in the constraint $\omega_h e(t)^2 \leq k_h \dot{e}(t) e(t)$.

- $\omega_h e(\tau)^2 \leq k_h \dot{e}(\tau) e(\tau) \implies$ switch to integrator mode

### Discontinuities

Imagine the following situation. When $e(t)$ is positive, in integrator mode, $u(t)$ is increasing. If $e(t)$ would jump from positive to negative then the output $u(t)$ will start to decrease but the signal remains positive just after the moment of the jump. In such a situation they are not of equal sign and $k_h e(t) u(t) < 0$. The constraint for $\mathcal{F}$ would be violated and the signal is

outside of $\mathcal{F}$, specifically in the upper left quadrant in Figure 4.2a. $u(t)$ is decreasing and going into the cone again, but it has been outside $\mathcal{F}$ for a while. Such situations we want to avoid by restricting the signal $e(t)$ to not have a discontinuities. In the next sections some example signals give more insight.

## 4.4 An example signal

To understand when the HIGS switches a test signal $e(t) = \sin(t) + \frac{1}{2}\sin(3t)$ is put into it, where the parameters are kept simple, $\omega_h = 1$ and $k_h = 1.5$. In Figure 4.3 the test signal and the output $u(t)$ is plotted. The signals in the governing constraints are plotted below it.

A switch to gain mode is applied when we hit $u = k_h e$ and the constraint $\omega_h e^2 > k_h \dot{e} e$ holds at that time. A switch back to integrator mode is applied when the constraint $\omega_h e^2 > k_h \dot{e} e$ does not hold anymore. The switching points can be compared in the plots of Figure 4.3.

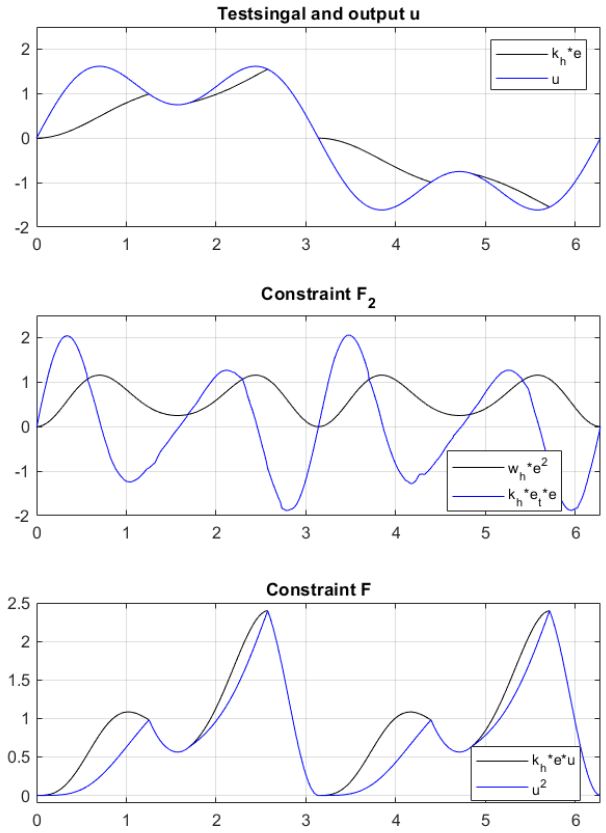



Figure 4.3: Test signal, $u(t)$ and the subregion constraints

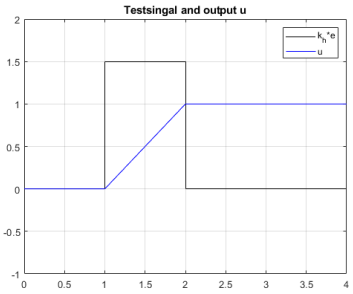
At $t = 0$ the test signal is zero and its derivative is positive. Moving slightly such that $t = \epsilon$, see that the derivative $\dot{e}$ still is positive, Figure 4.3. The constraint $\omega_h e(\epsilon)^2 > k_h \dot{e}(\epsilon) e(\epsilon)$ does not hold since $\dot{e}(\epsilon)$ is a lot larger than $e(\epsilon) \approx 0$. The HIGS starts in integrator mode until $u$ hits $k_h e$.

Clearly, in gain mode the signals are on the boundary of $\mathcal{F}$ and a switch to integrator mode makes the signal go back into the interior. The switch back to integrator mode is made when the signal $\omega_h e^2$ dives below $k_h \dot{e} e$ in the second plot. When the output signal crosses $k_h e(t)$ in the first plot a switch to gain mode is made, correctly according to the constraint in the second plot.

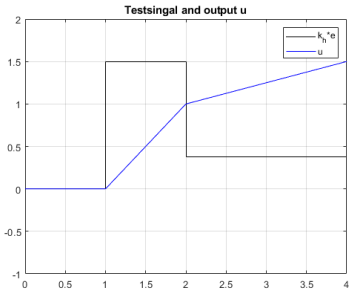
## 4.5 Dealing with discontinuities

In certain situations the HIGS can deal with a discontinuous input signal $e(t)$. When a jump in the signal $e(t)$ does not change sign it is possible to stay in $\mathcal{F}$. The jump will change the derivative of the output $u(t)$ and causes it to be non-differentiable. When $e(t) > 0$ the output signal $u(t)$ must still be positive after a jump. A jump up is no problem but a jump down must be smaller than $|k_h e(t^{-1})| - |u(t^{-1})|$.

In Figure 4.4a a step signal has been put in which jumps back to zero, this causes the output derivative to become zero and we leave $\mathcal{F}$ since the constraint $k_h eu \geq u^2$ does not hold anymore.




(a) Jump to zero




(b) Large jump

Figure 4.4: Leaving $\mathcal{F}$ after negative jumps

In Figure 4.4b a jump larger than $|k_h e(t)| - |u(t)|$ is made and we leave $\mathcal{F}$. The derivative of the output signal is still positive so $u(t)$ keeps increasing above $k_h e(t)$.

A smaller jump than $|k_h e(t)| - |u(t)|$ is seen in Figure 4.5. $u(t)$ is still increasing but slower. Such a small jump does not violate $k_h eu \geq u^2$ and we stay in $\mathcal{F}$.

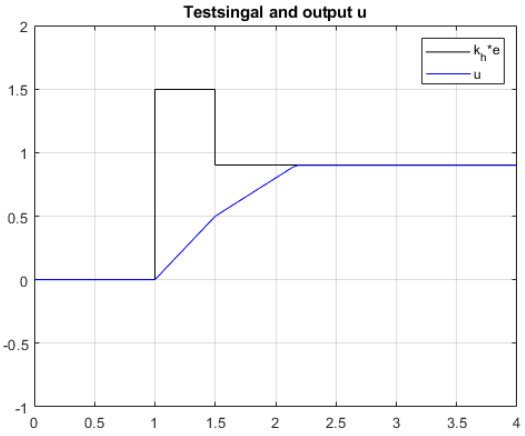
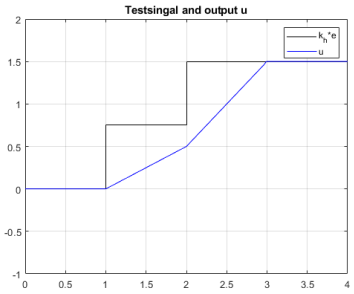
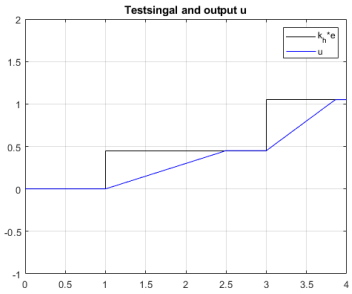



Figure 4.5: Staying in $\mathcal{F}$ after a small negative jump

In Figure 4.7 and Figure 4.6b a positive jump is made. The derivative of $u(t)$ in the first plot becomes larger and the signal increases faster but the constraint for $\mathcal{F}$ will stay satisfied. In the second plot $u(t)$ is determined by Gain mode of the HIGS when the jump happens and it causes no problems, at the jump it switches to Integrator mode because the constraint $\omega_h e^2 > k_h \dot{e} e$ is violated by the jump where the derivative of $e(t)$ is infinite and after the jump $u(t)$ does not equal $k_g e(t)$ anymore.




(a) A positive jump




(b) Another positive jump

Figure 4.6: Staying in $\mathcal{F}$ with positive jumps

When the input signal jumps to across the horizontal axis it changes sign and we will briefly leave $\mathcal{F}$. This is seen in Figure 4.7. After the jump the output signal changes direction since the derivative switches sign. The moment where $u(t)$ and $k_h e(t)$ are of opposite sign the constraint $k_h e u \geq u^2$ is violated.

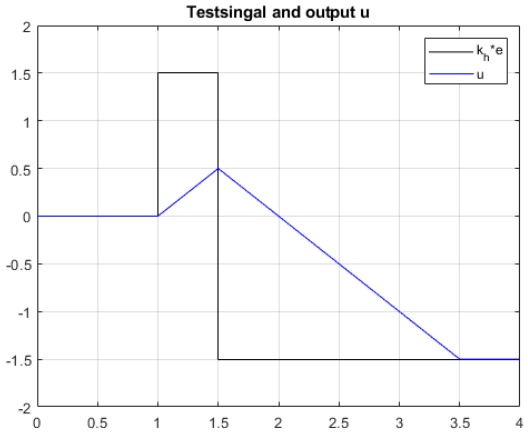



Figure 4.7: Jumping past the horizontal axis

Similar arguments can be made for negative step functions as input $e(t)$.

These are all examples of step functions but the situations do extend to any discontinuous piecewise differentiable functions $e(t)$.

In practice it is a good idea to avoid these discontinuous input signals entirely. Signals that do have jumps like this can best be approximated by a continuous function that increases or decreases very fast at the original jumping points. This way the output signal will be crossing $k_h e(t)$ and the HIGS will switch to gain mode at the jumping moment.

## 4.6 Existence of closed loop HIGS solutions

In this section a Hybrid Integrator-Gain System is applied to the general model of a plant. We will use a closed loop feedback system as in the diagram in Figure 4.8.

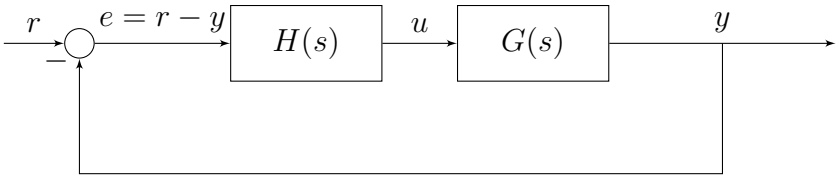



Figure 4.8: HIGS $H(s)$ applied to a plant $G(s)$

The goal is to proof the existence of solutions for such closed-loop systems.

The plant considered will be a general Port-Hamiltonian state-space representation.

The Port-Hamiltonian plant is of the form (3.5.2), restated here,

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}\mathcal{H}x(t) + Bu(t) \\ y(t) &= B^*\mathcal{H}x(t). \end{aligned} \tag{4.9}$$

The control signal $u(t)$ is the output of the HIGS and the input of the HIGS is the error signal $e(t) = r(t) - y(t)$.

Consider the plant to be in motion, the state is non-zero at some starting time $t_0$, then the HIGS should steer the plant to rest. In this case the reference signal is set to zero, $r(t) = 0$, and the only focus of the HIGS is to stabilise the plant. This makes the error signal to just be $e(t) = -y(t)$.

The output of the plant is stated as $y(t) = B^*\mathcal{H}x(t)$, resulting in the error signal $e(t) = -B^*\mathcal{H}x(t)$.

Assume the HIGS is in integrator mode for a certain time interval, the state of the HIGS in that interval is then described as $\dot{x}_h = -\omega_h B^*\mathcal{H}x$ and the control signal output is $u(t) = x_h(t)$. Combining the Port-Hamiltonian plant and integrator form of the HIGS the closed loop system in the integrator mode time interval can be written into one state-space representation with a new state, [10],

$$\begin{aligned} \dot{x}_i(t) &= \begin{pmatrix} \mathcal{J}\mathcal{H} & -B \\ \omega_h B^*\mathcal{H} & 0 \end{pmatrix} x_i(t) \\ y(t) &= \begin{pmatrix} B^*\mathcal{H} & 0 \end{pmatrix} x_i(t). \end{aligned} \tag{4.10}$$

A solution exists for this system, just as in (3.6), when the initial condition of the new state $x_i(t_0)$ is given. The input is the reference signal $r(t) = 0$, which is just a constant.

Now an interval in which the HIGS is in gain mode gives a control signal $u(t) = -k_h y(t)$ and results in a combined state-space representation, with state $x_p(t)$

$$\begin{aligned} \dot{x}_p(t) &= \mathcal{J}\mathcal{H}x_p(t) - (k_h B B^*\mathcal{H})x_p(t) \\ y(t) &= B^*\mathcal{H}x_p(t), \end{aligned} \tag{4.11}$$

which simplifies to the system

$$\begin{aligned} \dot{x}_p(t) &= (\mathcal{J} - k_h B B^*)\mathcal{H}x_p(t) \\ y(t) &= B^*\mathcal{H}x_p(t). \end{aligned} \tag{4.12}$$

This is a proportional controlled system with damping matrix $R = k_h BB^*$. For a time interval in which the system is in gain mode the closed loop system is described by (4.12). A unique solution exists when an initial condition for $x_p(t_0)$ is given.

In non-overlapping time intervals the HIGS controlled closed loop system is either described by (4.12) or (4.10). For both control modes there exist solutions of the system. Together the separate solutions for all intervals combine to one solution over the whole time axis of the HIGS controlled system.

The whole solution can be written in one term. Say for a number $n \in \mathbb{N}$ there exist solutions $y_i(t)$ to the closed-loop integrator controlled system (4.10) on the respective continuous time intervals $(t_i, t_{i_{end}}]$ where $t_i < t_{i_{end}}$ and $i = 1, 2, \ldots n$. Also, for any number $m \in \mathbb{N}$, $y_j(t)$ are $m - n$ solutions to the closed loop gain controlled system (4.12) on the respective continuous time intervals $(t_j, t_{j_{end}}]$ where $t_j < t_{j_{end}}$ and $j = n+1, n+2 \ldots m$. The time intervals do not overlap such that $(t_i, t_{i_{end}}] \cap (t_j, t_{j_{end}}] = \emptyset$ for all $i$ and $j$. The solution for the closed loop HIGS controlled system is written as the summation

$$y(t) = \sum_i y_i(t)\mathbb{1}((t_i, t_{i_{end}}]) + \sum_j y_j(t)\mathbb{1}((t_j, t_{j_{end}}]). \tag{4.13}$$

It is important that the time intervals, in which the system is either in integrator or gain mode, are non-empty and continuous. If the system would switch between control modes at one point in time infinitely a problem called a sliding mode would occur, [11]. The HIGS is defined such that this problem does not happen, at any time the control mode of the HIGS is clear.

The representation of a general solution assumes that the states of the two closed loop systems are of equal dimension. For the two closed loop state-space representations we have that the states can be different dimensions. A switch will change the dimension of the state of the whole closed loop system and the analysis of the closed loop system becomes more complex.

### State continuity

Lets look at the control block as a general state-space representation where $A = 0$ in both domains $\mathcal{F}_1$ and $\mathcal{F}_2$. $B_{\mathcal{F}_1} = \omega_h$ and $B_{\mathcal{F}_2} = k_h$ in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Where $e(t)$ is the input in $\mathcal{F}_1$ and derivative $\dot{e}(t)$ is the input in $\mathcal{F}_2$.

Using a candidate solution $x_h(t) = e^{At}z(t)$ where, with any function $z(t) \in \mathbb{R}$, we get $x_h(t) = z(t)$.

Since $\omega_h, k_h \in \mathbb{R}$ the solution for the HIGS state $x_h(t)$ can be written, depending on the control mode it is in, as

$$\begin{aligned} x_h(t) &= x_h(t_0) + \int_{t_0}^{t} \omega_h e(\tau) d\tau \text{ in } \mathcal{F}_1, \\ x_h(t) &= x_h(t_0) + \int_{t_0}^{t} k_h \dot{e}(\tau) d\tau \text{ in } \mathcal{F}_2. \end{aligned} \tag{4.14}$$

These expressions exist for locally integrable function $e(t)$ in $\mathcal{F}_2$ and locally integrable derivative $\dot{e}(t)$ in $\mathcal{F}_1$ and are determined by the initial condition of the state.

The output function $u_h(t)$ is then given by the state $x_h(t)$ itself.

Now consider a switch to the other control mode. When this happens the state it left off with will be used as new initial condition. Lets say $x_{switch}(t_{switch})$ is the state at switching time $t_{switch}$. When going from integrator to gain mode at time $t_{switch}$ set $x_h(t_0) = x_{switch}(t_{switch})$ and see,

$$x_h(t) = x_{switch}(t_{switch}) + \int_{t_{switch}}^{t} k_h \dot{e}(\tau) d\tau \text{ in } \mathcal{F}_1. \tag{4.15}$$

Going back from gain to integrator mode at time $t_{switch}$, see

$$x_h(t) = x_{switch}(t_{switch}) + \int_{t_{switch}}^{t} \omega_h e(\tau) d\tau \text{ in } \mathcal{F}_2. \tag{4.16}$$

This way the state will be continuous. The state is Lipschitz continuous if $e(t)$ is integrable and, in case of gain mode in $\mathcal{F}_1$, also $\dot{e}(t)$ must be integrable. The definition of Lipschitz continuity is given in (3.1.6). The result is put in Theorem (4.6.1).

**Theorem 4.6.1.** *Given the real integrable function $e(t)$ and it's integrable derivative $\dot{e}(t)$, then the state and output of the Hybrid Integrator-Gain System as defined in* 4.2 *are Lipschitz continuous.*

*Proof.* Given that $e(t)$ and it's derivative are locally integrable then there exist real constants $C_1$,$C_2 \in \mathbb{R}$ such that for any time $t$ and initial time $t_0$ the following integrals are bounded,

$$\int_{t_0}^{t} |e(\tau)| d\tau \leq C_1 \text{ and } \int_{t_0}^{t} |\dot{e}(\tau)| d\tau \leq C_2. \tag{4.17}$$

Then, in section $\mathcal{F}_2$, for all two points in time $t_1$, $t_2 \in \mathbb{R}$ and initial condition $x_h(t_0)$

$$\begin{aligned} |x_h(t_1) - x_h(t_2)| &= \left| x_h(t_0) + \int_{t_0}^{t_1} \omega_h e(\tau) d\tau - x_h(t_0) - \int_{t_0}^{t_2} \omega_h e(\tau) d\tau \right| \\ &= \left| \int_{t_1}^{t_2} \omega_h e(\tau) d\tau \right| \leq \int_{t_1}^{t_2} |\omega_h e(\tau)| d\tau \leq |\omega_h| C_1 \\ &= K|t_1 - t_2| \end{aligned} \tag{4.18}$$

for some real constant $K = \frac{|\omega_h| C_1}{|t_1 - t_2|}$, which proves Lipschitz continuity of $x_h(t)$ in $\mathcal{F}_2$.

A similar proof shows the Lipschitz continuity of the state in $\mathcal{F}_1$ using $C_2$ and the integrability of $\dot{e}(t)$.

Since the current state is used as initial condition when a switch happens there are no discontinuities in the trajectory of the state. □

# 5. Vibrating string system

An example of a positive real system is the model of the vibrating string. It is described by the well known partial differential equation of the wave equation. In this chapter we will rewrite the model of the vibrating string into a fitting state-space representation in order to apply control. While the model is derived it is also shown why it is a positive real system.

## 5.1 The Partial Differential Equation

The model of the vibrating string is a version of the wave equation. [8]. It models the movement of a string, which can up and down by applying a force on the string. The motion of the string is governed by the following partial differential equation (5.1),

$$\frac{\partial^2 \omega}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left( T(\zeta) \frac{\partial \omega}{\partial \zeta}(\zeta, t) \right). \tag{5.1}$$

Here $T$ and $\rho$ represent the physical properties of the string at the horizontal location $\zeta$. The parameter $T$ represents Young's modulus, which is an indication of the tension on the string. The parameter $\rho$ is the mass density of the string and $\omega(\zeta, t)$ is a variable which represents the vertical displacement of the string at location $\zeta$ and time $t$. The string begins and ends at the points $a$ and $b$ and there exists an initial condition for the begin position of the string at time $t_0 = 0$, stated as a function of $\zeta$,

$$\omega(\zeta, 0) = \omega_0(\zeta). \tag{5.2}$$

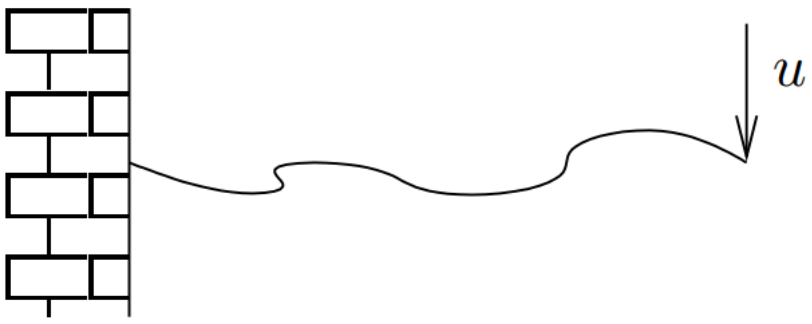



Figure 5.1: The vibrating string [2]

The string is fixed on the left and can move freely on the right where a force can be applied. Lets take a look at the units of the arguments of the model. An overview is given in the Table 5.1.

| Description: | Variable: | Units: |
|---|---|---|
| Vertical coordinate | $\omega$ | meter $(m)$ |
| Horizontal displacement | $\zeta$ | meter $(m)$ |
| Time | $t$ | Seconds $(s)$ |
| Density | $\rho$ | Kg per meter $(\frac{Kg}{m})$ |
| Young's modulus (elasticity) | $T$ | Newton per meter $(N)$ |
| Begin & end points | $a$, $b$ | meter $(m)$ |

Table 5.1: Unit overview

The second derivative term on the left of the equal sign in (5.1) has unit 'meter per seconds squared' $\left(\frac{m}{s^2}\right)$. For the model to make sense the unit of the term on the right-hand side should be the same as on the left-hand side, which will be checked. The derivative of $\omega$ to $\zeta$ is dimensionless since it is the derivative of a variable in 'meter' to 'meter'. This unit multiplied by the unit of $T$ gives the term in the brackets of the right-hand side in (5.1) the unit 'Newton'. The derivative to $\zeta$ has the unit 'Newton per meter' $\left(\frac{N}{m}\right)$. A force can be expressed as mass times acceleration. So unit 'Newton' is expressed as 'kilogram times meter per second squared' $\left(Kg\frac{m}{s^2}\right)$. So $\frac{N}{m}$ is equal to $\frac{Kg}{s^2}$. Because $\frac{1}{\rho}$ has unit 'meter per kilogram' $\left(\frac{m}{Kg}\right)$ the term on the right will become $\frac{m}{Kg}\frac{Kg}{s^2}$, which simplifies to $\frac{m}{s^2}$. Indeed this is the same unit as that of the term on the left-hand side so it is concluded that the model makes sense in terms of units.

The goal is to write the PDE for the string model into a state-space representation.

## 5.2 The Port-Hamiltonian representation

That the model of the vibrating string is a positive real system and equation (5.1) is a Port-Hamiltonian partial differential equation will be shown in this section. A linear state-space representation of a Port-Hamiltonian system has already been defined in Definition 3.5.2 but there also exists a class of Port-Hamiltonian PDE's. A system being Port-Hamiltonian means that the

energy can only enter or leave the system via its boundaries, also called its 'Ports'. The 'Hamiltonian' is a specific equation which describes the energy in the system. The class is described in Definition 5.2.1.

**Definition 5.2.1** (Linear, First-order Port-Hamiltonian PDE)**.** *Let $P_1 \in \mathbb{C}^{n\times n}$ be invertible and self-adjoint, let $P_0 \in \mathbb{C}^{n\times n}$ be skew-adjoint, i.e. $P_0^* = -P_0$, and let $\mathcal{H} \in L_\infty([a,b];\mathbb{C}^{n\times n})$ be symmetric, i.e. $\mathcal{H}(\zeta)^* = \mathcal{H}(\zeta)$ and positive then the differential equation*

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)) \tag{5.3}$$

*is called a linear, first order Port-Hamiltonian system with the associated Hamiltonian $E : [0,\infty) \to [0,\infty)$ given by*

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta,t)x(\zeta,t)d\zeta. \tag{5.4}$$

The model of the vibrating string in (5.1) has the structure of a Port-Hamiltonian system,

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho(\zeta)\frac{\partial \omega}{\partial t}\\ \frac{\partial \omega}{\partial \zeta}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial \zeta}\left(\begin{pmatrix}\frac{1}{\rho(\zeta)} & 0\\ 0 & T(\zeta)\end{pmatrix}\begin{pmatrix}\rho(\zeta)\frac{\partial \omega}{\partial t}\\ \frac{\partial \omega}{\partial \zeta}\end{pmatrix}\right), \tag{5.5}$$

where the state is $x(\zeta,t) = \begin{pmatrix}\rho(\zeta)\frac{\partial \omega}{\partial t}\\ \frac{\partial \omega}{\partial \zeta}\end{pmatrix}$. The variables in the state represent the energy in the system, $\rho(\zeta)\frac{\partial \omega}{\partial t}$ the momentum and $\frac{\partial \omega}{\partial \zeta}$ the strain. The matrices of the model in equation (5.3) are specified as

$$P_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \mathcal{H}(\zeta) = \begin{pmatrix}\frac{1}{\rho(\zeta)} & 0\\ 0 & T(\zeta)\end{pmatrix} \text{ and } P_0 = 0.$$

For solutions of the model there need to be boundary conditions, which will be determined in next section.

## 5.3 Boundary conditions

The boundary conditions of a Port-Hamiltonian PDE are of the special form (5.6), where the matrix $M = [M_1, M_2]$ must be of full rank $n$, [2].

$$M_1(\mathcal{H}x)(b,t) + M_2(\mathcal{H}x)(a,t) = 0, \tag{5.6}$$

The left side of the string will be stationary and the right side will have no force applied to it at the starting time. This results in a homogeneous boundary condition on the right side. Together this gives the two boundary conditions,

$$T(b)\frac{\partial \omega}{\partial \zeta}(b,t) = 0 \text{ and } \frac{\partial \omega}{\partial t}(a,t) = 0. \tag{5.7}$$

This makes the matrices in (5.6) to be chosen as $M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ such that the desired boundary conditions are written in the correct form. Clearly these boundary conditions are a sufficient choice since the chosen matrix $M = [M_1, M_2]$ has rank 2.

## 5.4 Hamiltonian

The Hamiltonian of a Port-Hamilton system is defined in Definition 5.2.1. It represents the energy in the system, which can be seen as a function of time,

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta,t)x(\zeta,t)d\zeta. \tag{5.8}$$

Filling in the term inside the integral for the specific string model results in,

$$\begin{aligned} x(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t) &= \begin{pmatrix}\rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t) & \frac{\partial\omega}{\partial\zeta}(\zeta,t)\end{pmatrix}\begin{pmatrix}\frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{pmatrix}\begin{pmatrix}\rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t) \\ \frac{\partial\omega}{\partial\zeta}(\zeta,t)\end{pmatrix} \\ &= \begin{pmatrix}\frac{\partial\omega}{\partial t}(\zeta,t) & T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t)\end{pmatrix}\begin{pmatrix}\rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t) \\ \frac{\partial\omega}{\partial\zeta}(\zeta,t)\end{pmatrix} \\ &= \rho(\zeta)\left(\frac{\partial\omega}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial\omega}{\partial\zeta}(\zeta,t)\right)^2, \end{aligned} \tag{5.9}$$

and the Hamiltonian of the vibrating string system is determined as

$$E(t) = \frac{1}{2}\int_a^b \rho(\zeta)\left(\frac{\partial\omega}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial\omega}{\partial\zeta}(\zeta,t)\right)^2 d\zeta. \tag{5.10}$$

It is known that kinetic energy is expressed as $E = \frac{1}{2}mv^2$, where $m$ is mass and $v$ velocity. The unit of energy is $Kg(\frac{m}{s})^2$. To be certain if the Hamiltonian is correct the unit will be checked. Since $\rho$ has $\frac{Kg}{m}$ and $(\frac{\partial\omega}{\partial t})^2$ is $(\frac{m}{s})^2$, $T$ is in 'Newton' and $(\frac{\partial\omega}{\partial\zeta})^2$ is unit-less. The term inside the integral in (5.10)

is than in 'Newton' $\left(N = Kg\frac{m}{s^2}\right)$. Because the integral is over $\zeta$ in 'meters' we get a unit of $Kg(\frac{m}{s})^2$ for the integral as a whole. This is indeed the unit of energy. From a unit standpoint the Hamiltonian makes sense.

The change in energy is the derivative of the Hamiltonian, so differentiating the energy, while making use of the PDE in (5.1), the power balance equation is derived,

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_a^b 2\rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t)\left(\frac{\partial^2\omega}{\partial t^2}(\zeta,t)\right) \\
&\quad + 2T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t)\frac{\partial}{\partial t}\left(\frac{\partial\omega}{\partial\zeta}(\zeta,t)\right)d\zeta \\
&= \int_a^b \rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t)\frac{1}{\rho(\zeta)}\frac{\partial}{\partial\zeta}\left(T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t)\right) \\
&\quad + T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t)\frac{\partial}{\partial\zeta}\left(\frac{\partial\omega}{\partial t}(\zeta,t)\right)d\zeta \\
&= \int_a^b \frac{\partial}{\partial\zeta}\left(\frac{\partial\omega}{\partial t}(\zeta,t)T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t)\right)d\zeta \\
&= \frac{\partial\omega}{\partial t}(b,t)T(b)\frac{\partial\omega}{\partial\zeta}(b,t) - \frac{\partial\omega}{\partial t}(a,t)T(a)\frac{\partial\omega}{\partial\zeta}(a,t)
\end{aligned} \tag{5.11}$$

As can be seen from power balance, the change of energy is only possible via the boundaries of the string. The right side of the string will be used to influence the system and change the energy in the string.

## 5.5 Input/Output formulation

When adding a controller to the system it must have an input, which will be the control signal, as well as an output to measure the results. In this section the input and output of the vibrating string model are defined. The input is denoted as $u(t)$ and the output as $y(t)$. A Port-Hamiltonian system combined with homogeneous boundary conditions and an input and output is summarised as the system (5.12), [13]

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) \\
0 &= M_{11}(\mathcal{H}x)(b,t) + M_{12}(\mathcal{H}x)(a,t) \\
u(t) &= M_{21}(\mathcal{H}x)(b,t) + M_{22}(\mathcal{H}x)(a,t) \\
y(t) &= C_1(\mathcal{H}x)(b,t) + C_2(\mathcal{H}x)(a,t),
\end{aligned} \tag{5.12}$$

where $rank \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ C_1 & C_2 \end{bmatrix} = n + rank\begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

The input will be a force applied to the right of the string, so $u(t) = T(b)\frac{\partial \omega}{\partial \zeta}(b,t)$. The output will represent the velocity of the string on the right, so $y(t) = \frac{\partial \omega}{\partial t}(b)$. The following matrices are chosen to describe input, output and boundary conditions of the vibrating string model in one input/output system,

$$\begin{aligned} M_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ M_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, M_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{aligned} \tag{5.13}$$

With this input/output description the transfer function of the vibrating string model can now be found.

## 5.6 Transfer function

In this section the transfer function for the vibrating string model is derived using the matrices (5.13) and the useful term

$$\begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix} = \begin{pmatrix} \frac{\partial \omega}{\partial t}(b,t) \\ T(b)\frac{\partial \omega}{\partial \zeta}(b,t) \\ \frac{\partial \omega}{\partial t}(a,t) \\ T(a)\frac{\partial \omega}{\partial \zeta}(a,t) \end{pmatrix}. \tag{5.14}$$

The input and output of 5.12 are written in a new form, [2]

$$u(t) = W_B \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix} \tag{5.15}$$

$$y(t) = W_C \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix}. \tag{5.16}$$

The desired boundary conditions are achieved with the matrices $W_B = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ and $W_C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, since $u(t) = T(b)\frac{\partial \omega}{\partial \zeta}(b,t)$ and $y(t) = \frac{\partial \omega}{\partial t}(b,t)$. The transfer function is found by solving a differential equation with inhomogeneous boundary conditions following Theorem 5.2.1 in [13].

The theorem states that, if for a given $s \in \mathbb{C}$ and $u_0 \in \mathbb{R}$, we can solve

$$\begin{aligned} sx_0(\zeta) &= P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x_0(\zeta)) + P_0(\mathcal{H}(\zeta)x_0(\zeta)) \\ u_0 &= W_B \begin{pmatrix} \mathcal{H}(b)x_0(b) \\ \mathcal{H}(a)x_0(a) \end{pmatrix} \\ G(s)u_0 &= W_C \begin{pmatrix} \mathcal{H}(b)x_0(b) \\ \mathcal{H}(a)x_0(a) \end{pmatrix}, \end{aligned} \tag{5.17}$$

then $G(s)$ is the transfer function of the system. For simplicity the parameters $\rho$ and $T$ are considered to be constant, which makes the differential equation solvable. The differential equation of the string results in the system

$$\begin{aligned} sx_0(\zeta) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{d\zeta} \left( \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix} x_0(\zeta) \right) \\ u_0 &= T\frac{\partial \omega}{\partial \zeta}(b) \qquad \frac{\partial \omega}{\partial t}(a) = 0 \\ G(s)u_0 &= \frac{\partial \omega}{\partial t}(b). \end{aligned} \tag{5.18}$$

Now two terms are defined, $Q_0(\zeta) = \rho\frac{\partial \omega}{\partial t}$ and $\phi_0(\zeta) = \frac{\partial \omega}{\partial \zeta}$. This way the state and boundary conditions at the starting time are $x_0(\zeta) = \begin{pmatrix} Q_0(\zeta) \\ \phi_0(\zeta) \end{pmatrix}$, $u_0 = T\phi_0(b)$ and $y_0 = \frac{1}{\rho}Q_0(b)$. This gives the following two useful equations,

$$\begin{aligned} sQ_0(\zeta) &= T\frac{\partial \phi_0}{\partial \zeta}(\zeta) \\ s\phi_0(\zeta) &= \frac{1}{\rho}\frac{\partial Q_0}{\partial \zeta}(\zeta). \end{aligned} \tag{5.19}$$

Now a candidate solution for $Q_0(\zeta)$ is defined as $Q_0(\zeta) = e^{sB\zeta}$. The derivative of the candidate solution is $\frac{\partial Q_0}{\partial \zeta}(\zeta) = sBe^{sB\zeta}$. From the second equation in (5.19) a term for $\phi_0(\zeta)$ can now be derived,

$$s\phi_0(\zeta) = \frac{1}{\rho}\frac{\partial Q_0}{\partial \zeta}(\zeta) = \frac{1}{\rho}sBe^{sB\zeta} \implies \phi_0(\zeta) = \frac{1}{\rho}Be^{sB\zeta}. \tag{5.20}$$

Now that term is differentiated as well, $\frac{\partial \phi_0}{\partial \zeta}(\zeta) = \frac{1}{\rho}sB^2e^{sB\zeta}$, and a new expression for $Q_0(\zeta)$ is found, using the first equation of (5.19), see

$$sQ_0(\zeta) = T\frac{\partial \phi_0}{\partial \zeta}(\zeta) = \frac{T}{\rho}sB^2e^{sB\zeta} \implies Q_0(\zeta) = \frac{T}{\rho}B^2e^{sB\zeta}. \tag{5.21}$$

With the chosen candidate solution for $Q_0(\zeta)$ an equation for the constant $B$ is derived,

$$\frac{T}{\rho}B^2 e^{sB\zeta} = e^{sB\zeta} \implies \frac{T}{\rho}B^2 = 1. \tag{5.22}$$

Concluding a term for the constant $B = \pm\sqrt{\frac{\rho}{T}}$ and resulting in the two general solutions to equations (5.19)

$$\begin{aligned} Q_0(\zeta) &= \alpha e^{s\sqrt{\frac{\rho}{T}}\zeta} + \beta e^{-s\sqrt{\frac{\rho}{T}}\zeta} \\ \phi_0(\zeta) &= \alpha\frac{1}{\rho}\sqrt{\frac{\rho}{T}}e^{s\sqrt{\frac{\rho}{T}}\zeta} - \beta\frac{1}{\rho}\sqrt{\frac{\rho}{T}}e^{-s\sqrt{\frac{\rho}{T}}\zeta}. \end{aligned} \tag{5.23}$$

With these two solutions the differential equation of (5.18) can be solved for the transfer function $G(s)$. Combining the general solutions (5.23) with the given input and output conditions of (5.18) expressions for $\alpha$ and $\beta$ are found, see

$$\begin{aligned} \begin{pmatrix} u_0 \\ 0 \end{pmatrix} &= \begin{pmatrix} T\phi_0(b) \\ Q_0(a) \end{pmatrix} = \begin{pmatrix} \frac{T}{\rho}\sqrt{\frac{\rho}{T}}e^{s\sqrt{\frac{\rho}{T}}b} & -\frac{T}{\rho}\sqrt{\frac{\rho}{T}}e^{-s\sqrt{\frac{\rho}{T}}b} \\ e^{s\sqrt{\frac{\rho}{T}}a} & e^{-s\sqrt{\frac{\rho}{T}}a} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \frac{1}{det}\begin{pmatrix} e^{-s\sqrt{\frac{\rho}{T}}a} & \frac{T}{\rho}\sqrt{\frac{\rho}{T}}e^{-s\sqrt{\frac{\rho}{T}}b} \\ -e^{s\sqrt{\frac{\rho}{T}}a} & \frac{T}{\rho}\sqrt{\frac{\rho}{T}}e^{s\sqrt{\frac{\rho}{T}}b} \end{pmatrix}\begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \end{aligned} \tag{5.24}$$

where the determinant is expressed as

$$det = \frac{T}{\rho}\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{s\sqrt{\frac{\rho}{T}}(a-b)}). \tag{5.25}$$

The full expression for $\alpha$ and $\beta$ are stated

$$\begin{aligned} \alpha &= \frac{1}{det}e^{-s\sqrt{\frac{\rho}{T}}a}u_0 = \frac{e^{-s\sqrt{\frac{\rho}{T}}a}}{\frac{T}{\rho}\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{s\sqrt{\frac{\rho}{T}}(a-b)})}u_0 \\ \beta &= -\frac{1}{det}e^{s\sqrt{\frac{\rho}{T}}a}u_0 = \frac{-e^{s\sqrt{\frac{\rho}{T}}a}}{\frac{T}{\rho}\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{s\sqrt{\frac{\rho}{T}}(a-b)})}u_0. \end{aligned} \tag{5.26}$$

The output equation of (5.18) is used to derive the transfer function itself,

$$\begin{aligned}
y_0 &= \frac{1}{\rho}Q_0(b) = \begin{pmatrix} \frac{1}{\rho}e^{s\sqrt{\frac{\rho}{T}}b} & \frac{1}{\rho}e^{-s\sqrt{\frac{\rho}{T}}b} \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
&= (\frac{1}{\rho}e^{s\sqrt{\frac{\rho}{T}}b})(\frac{1}{det}e^{-s\sqrt{\frac{\rho}{T}}a}u_0) + (\frac{1}{\rho}e^{-s\sqrt{\frac{\rho}{T}}b})(-\frac{1}{det}e^{s\sqrt{\frac{\rho}{T}}a}u_0) \\
&= \frac{(\frac{1}{\rho}e^{s\sqrt{\frac{\rho}{T}}b})(e^{-s\sqrt{\frac{\rho}{T}}a}u_0)}{\frac{T}{\rho}\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{s\sqrt{\frac{\rho}{T}}(a-b)})} - \frac{(\frac{1}{\rho}e^{-s\sqrt{\frac{\rho}{T}}b})(e^{s\sqrt{\frac{\rho}{T}}a}u_0)}{\frac{T}{\rho}\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{s\sqrt{\frac{\rho}{T}}(a-b)})} \\
&= \frac{(e^{s\sqrt{\frac{\rho}{T}}(b-a)} - e^{-s\sqrt{\frac{\rho}{T}}(b-a)})u_0}{T\sqrt{\frac{\rho}{T}}(e^{s\sqrt{\frac{\rho}{T}}(b-a)} + e^{-s\sqrt{\frac{\rho}{T}}(b-a)})} = \frac{1}{T\sqrt{\frac{\rho}{T}}}\tanh\left(s\sqrt{\frac{\rho}{T}}(b-a)\right)u_0 \\
\implies G(s)u_0 &= \frac{1}{T\sqrt{\frac{\rho}{T}}}\tanh\left(s\sqrt{\frac{\rho}{T}}(b-a)\right)u_0.
\end{aligned} \tag{5.27}$$

Resulting in the transfer function for the vibrating string model expressed by

$$G(s) = \frac{1}{T\sqrt{\frac{\rho}{T}}}\tanh\left(s\sqrt{\frac{\rho}{T}}(b-a)\right). \tag{5.28}$$

In the next sections we analyse this transfer function and approximate it numerically using a Laurent Series, making use of the Residue Theorem, [5].

## 5.7 Approximation by Residue Theorem

The transfer function (5.28) involves a hyperbolic term. In order to run simulations that term must be approximated of a polynomial. One way to achieve this is by using the Residue Theorem, [5].

**Theorem 5.7.1** (Residue theorem)**.** *Given a complex function $f(z)$ and some contour $C$, then*

$$\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty}\frac{a_n}{(z-p_n)} \\
a_n &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-p_n)}dz
\end{aligned} \tag{5.29}$$

*where $p_n$ represent the poles of the complex function $f(z)$ and the coefficients $a_n$ are the called the 'residues' of $f(z)$ at the corresponding pole $p_n$.*

The poles of the transfer function $G(s)$ of the vibrating string model are where $\cosh\left(s\sqrt{\frac{\rho}{T}}(b-a)\right) = 0$, which occurs at the frequencies $s\sqrt{\frac{\rho}{T}}(b-a) =$

$\frac{i\pi}{2} + i\pi n$, for $n \in \mathbb{Z}$. This implies that the poles of the transfer function are at $s = \frac{\frac{i\pi}{2}+i\pi n}{\sqrt{\frac{\rho}{T}}(b-a)}$ and they are simple poles, all of them have a multiplicity of one, [10]. The residues of a function $G(s) = \frac{P(s)}{Q(s)}$ are found by the formula, [5]

$$Res(G(s), p_n) = \frac{P(p_n)}{Q'(p_n)}.$$

In case of the vibrating string transfer function the definition of the hyperbolic $\tanh(s) = \frac{\sinh(s)}{\cosh(s)}$ and the poles $s = \frac{\frac{i\pi}{2}+i\pi n}{\sqrt{\frac{\rho}{T}}(b-a)}$ result in a constant residue for every pole,

$$Res(G(s), \frac{\frac{i\pi}{2} + i\pi n}{\sqrt{\frac{\rho}{T}}(b-a)}) = \frac{1}{\rho(b-a)} \frac{\sinh\left(\frac{i\pi}{2} + i\pi n\right)}{\sinh\left(\frac{i\pi}{2} + i\pi n\right)} = \frac{1}{\rho(b-a)}.$$

By the Residue Theorem 5.29 the transfer function of the string can be written as the infinite summation,

$$G(s) = \sum_{n=-\infty}^{\infty} \frac{1}{\rho(b-a)(s - \frac{\frac{i\pi}{2}+i\pi n}{\sqrt{\frac{\rho}{T}}(b-a)})}.$$

In practice, using an infinite amount of poles in a simulation is not possible. Instead, the transfer function is approximated by the use of a finite amount of poles in the summation. When one pole is used the complex conjugate of that pole is also used, so when pole $p_n$ is used then pole $p_{-n}$ is used in the approximation as well. This way the poles are chosen symmetric and they will appear in pairs. The result is a finite approximation of the hyperbolic term in the transfer function. The approximation is a rational transfer function where its order is determined by the amount of poles used. The more poles are used, the better the approximation of the original transfer function becomes, but simultaneously, the more complex the numerical simulations will be. Say $2N + 1$ poles are used, then

$$G(s) = \sum_{n=-N}^{N} \frac{1}{\rho(b-a)(s - \frac{\frac{i\pi}{2}+i\pi n}{\sqrt{\frac{\rho}{T}}(b-a)})}. \tag{5.30}$$

Looking at the units of the approximation of the transfer function and comparing the result to the units of the original transfer function (5.28) will indicate if the approximation makes sense. They should both have the same unit.

First the original transfer function (5.28) is checked. From the unit table (5.1) it is known that the term $\left(\frac{\rho}{T}\right)$ has unit $\frac{Kg}{mN}$. Using $N = Kg\frac{m}{s^2}$, which has unit $\frac{Kg}{mKg\frac{m}{s^2}}$, gives the unit $\frac{s^2}{m^2}$. Resulting in the term $\sqrt{\frac{\rho}{T}}$ to have the unit $\frac{s}{m}$.

Inside the hyperbolic term the length of the string $(b-a)$ in 'meter' and the frequency $s$ has the unit $\frac{1}{s}$. The unit $\frac{s}{m}\frac{1}{s}m$ is dimensionless, which makes sense as argument in the hyperbolic function.

Now $\frac{1}{T\sqrt{\frac{\rho}{T}}}$ has unit $\frac{m}{Ns}$, which becomes $\frac{m}{Kg\frac{m}{s^2}s}$. Resulting in the complete transfer function to have unit $\frac{s}{Kg}$.

Now for the unit of the approximated transfer function. It has a term $\rho(b-a)$ with unit $Kg$. Again, frequency is in $\frac{1}{s}$ and the poles are in terms of $\frac{1}{s}$ as well. The approximation has unit $\frac{s}{Kg}$ which is indeed the same unit as the original transfer function.

Another way of checking if the approximation could be correct is by looking at the relation $Y(s) = G(s)U(s)$ and comparing the unit of the input and output functions. $U(s)$ has unit 'Newton' and $Y(s)$ has unit 'meter per second' $\left(\frac{m}{s}\right)$. The right term $\frac{s}{Kg}N$ written in full is $\frac{s}{Kg}Kg\frac{m}{s^2}$, which results in unit $\frac{m}{s}$. Indeed, this is the same unit as the unit of $Y(s)$.

The approximated transfer function will be used to write the vibrating string model in a suitable state-space representation.

## 5.8 Some realisations

To give an insight in the vibrating string system the approximation to the transfer function has been generated in MATLAB. To keep the model simple the parameter values $T = 1$ Newton per meter and $\rho = 1$ kilogram per meter are used and the string has a length of 1 meter, $[a, b] = [0, 1]$. In the approximation 20 poles will be used.

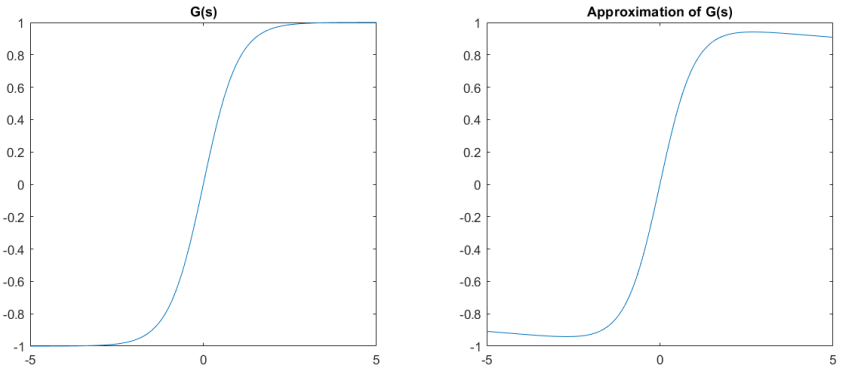
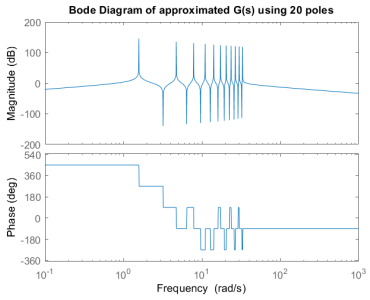
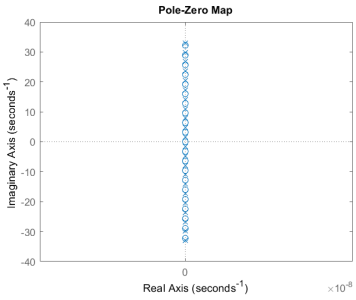



Figure 5.2: $G(s)$ and it's approximation, $s \in \mathbb{R}$

In this comparison, Figure 5.2, it is seen that the approximation is close to the original function around the origin. Since we have used a limited amount of poles, far away from the origin, in either the positive or negative direction, the approximation will go back to zero.



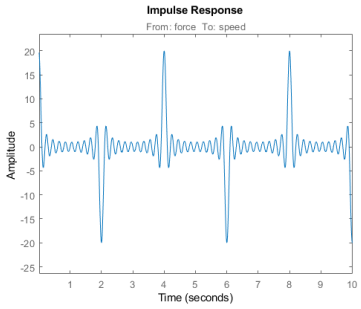

(a) Bode Diagram of $G(s)$



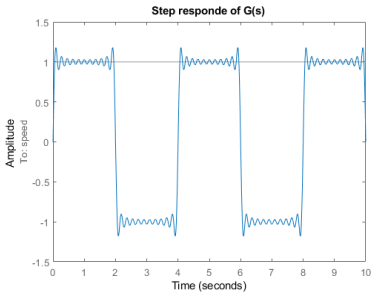

(b) Pole-Zero map of G(s)

Because all the poles are on the imaginary axis the Bode-plot in Figure 5.3a does not give clear information. The frequencies where the Bode diagram shoots up are the locations of the poles and where it shoots down the zeros are located. Because of the logarithmic scale of the plots the poles and zeroes after a certain frequency are not visible in the plot since they lay too close to each other.

The poles in Figure 5.3b all lay on the imaginary axis which indicates that the system is marginally stable. To stabilise the system one should shift these poles to the left of the imaginary axis. This is possible by connecting a controller which will be done in later chapters.




(a) Impulse response of G(s) using 20 poles




(b) Step response of G(s) using 20 poles

In Figure 5.4a the impulse response is plotted. The impulse causes the system to oscillate, which is common behaviour for marginally stable systems. In Figure 5.4b the output, the speed at the right-hand side of the string, is plotted for a constant input signal of 1. Since the output is the speed of the string the plot can be interpreted as moving in one direction for two seconds and in the opposite direction for another two seconds.

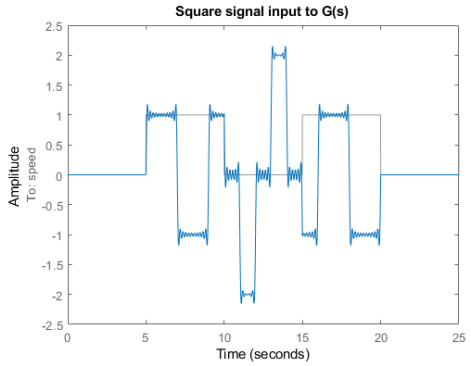



Figure 5.5: Response to a square signal input of G(s)

For an input of a square wave, a step to 1 for ten seconds, the output is plotted. The behaviour of the string becomes a bit more chaotic already.

For clarity and simplicity just two poles will be used in the approximation of the transfer function. Namely poles $p_0 = \frac{i\pi}{2}$ and $p_{-1} = -\frac{i\pi}{2}$. The transfer function will be a simple rational function,

$$G(s) = \frac{2s}{s^2 + \frac{\pi^2}{4}}. \tag{5.31}$$

Clearly it has a zero at $s = 0$ and a pair of complex conjugate poles at $s \approx \pm\sqrt{\frac{\pi^2}{4}}i$.

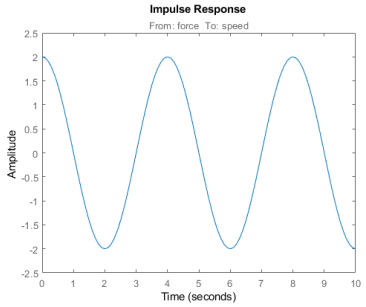
The inverse Laplace Transform of (5.31) equals the impulse response $h(t)$ of the approximated vibrating string system. [7]. Using the Laplace transform of cosine stated,

$$\frac{s}{s^2 + b^2} \longleftrightarrow \mathbb{L}\{\cos(bt)\}, \quad \forall\, \mathrm{Re}(s) > 0, \tag{5.32}$$

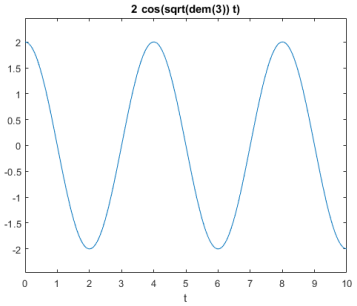
the function (5.31) results in an expression for the impulse response,

$$h(t) \approx 2\cos(\pm\sqrt{\frac{\pi^2}{4}}t), \quad \forall t \geq 0. \tag{5.33}$$

Indeed in Figure (5.6a) and Figure (5.6b) the two impulse responses are the equal.




(a) Impulse response approximation with two poles




(b) Inverse Laplacian of approximated G(s)

## 5.9 Positive realness

In this section the property of positive realness of the transfer function for the vibrating string system will be shown. The positive realness is the reason the vibrating string model is a Port-Hamiltonian system. First a definition of a Positive Real complex function is given in Definition 5.9.1.

**Definition 5.9.1** (Positive Real function)**.** *A complex function $Z(s)$, $s \in \mathbb{C}$ is called Positive Real if for all $s$ with $Re(s) > 0 \implies Re(Z(s)) \geq 0$.*

When a transfer function is positive real the system it describes is a positive real system.

To show positive realness the Hamiltonian of the vibrating string model will be used. The Hamiltonian matrix $\mathcal{H}$ is positive definite 3.5.1 and is restated here,

$$\mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}. \tag{5.34}$$

The parameters $\rho$ and $T$ are positive scalars or real and positive functions of $\zeta$. This makes the Hamiltonian matrix self-adjoint, i.e. $\mathcal{H} = \mathcal{H}^*$ and its property of being positive definite follows from the quadratic terms below,

$$\begin{aligned} x^*\mathcal{H}(\zeta)x &= \begin{pmatrix} \rho(\zeta)\frac{\partial\omega}{\partial t}^* & \frac{\partial\omega}{\partial\zeta}^* \end{pmatrix} \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix} \begin{pmatrix} \rho(\zeta)\frac{\partial\omega}{\partial t} \\ \frac{\partial\omega}{\partial\zeta} \end{pmatrix} \\ &= \begin{pmatrix} \rho(\zeta)\frac{\partial\omega}{\partial t}^* & \frac{\partial\omega}{\partial\zeta}^* \end{pmatrix} \begin{pmatrix} \frac{\partial\omega}{\partial t} \\ T(\zeta)\frac{\partial\omega}{\partial\zeta} \end{pmatrix} \\ &= \rho(\zeta)\left|\frac{\partial\omega}{\partial t}\right|^2 + T(\zeta)\left|\frac{\partial\omega}{\partial\zeta}\right|^2 > 0. \end{aligned} \tag{5.35}$$

Indeed $x^*\mathcal{H}x > 0$ for any $x \neq 0$.

In Port-Hamiltonian partial differential equation systems the change in energy of the system obeys the following equation, [13],

$$\frac{d}{dt}E(t) = \frac{1}{2}[(\mathcal{H}(\zeta)x(\zeta,t))^* P_1(\mathcal{H}(\zeta)x(\zeta,t))]_a^b. \tag{5.36}$$

In previous section the derivative of the energy in the system of the vibrating string has already been derived, (5.11), that result is restated here,

$$\frac{d}{dt}E(t) = \frac{\partial\omega}{\partial t}(b,t)T(b)\frac{\partial\omega}{\partial\zeta}(b,t) - \frac{\partial\omega}{\partial t}(a,t)T(a)\frac{\partial\omega}{\partial\zeta}(a,t). \tag{5.37}$$

Now the right-hand term of (5.36) is derived to check if the result is similar,

$$\begin{aligned} &\frac{1}{2}[(\mathcal{H}(\zeta)x(\zeta,t))^* P_1(\mathcal{H}(\zeta)x(\zeta,t))]_a^b \\ &\quad= \frac{1}{2}[\begin{pmatrix} \frac{\partial\omega}{\partial t}^*(\zeta,t) & T(\zeta)\frac{\partial\omega}{\partial\zeta}^*(\zeta,t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial\omega}{\partial t}(\zeta,t) \\ T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t) \end{pmatrix}]_a^b \\ &\quad= \frac{1}{2}[\begin{pmatrix} \frac{\partial\omega}{\partial t}^*(\zeta,t) & T(\zeta)\frac{\partial\omega}{\partial\zeta}^*(\zeta,t) \end{pmatrix} \begin{pmatrix} T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t) \\ \frac{\partial\omega}{\partial t}(\zeta,t) \end{pmatrix}]_a^b \\ &\quad= \frac{1}{2}[\frac{\partial\omega}{\partial t}^*(\zeta,t)T(\zeta)\frac{\partial\omega}{\partial\zeta}(\zeta,t) + T(\zeta)\frac{\partial\omega}{\partial\zeta}^*(\zeta,t)\frac{\partial\omega}{\partial t}(\zeta,t)]_a^b \\ &\quad= \frac{1}{2}\left(\frac{\partial\omega}{\partial t}^*(b,t)T(b)\frac{\partial\omega}{\partial\zeta}(b,t) + T(b)\frac{\partial\omega}{\partial\zeta}^*(b,t)\frac{\partial\omega}{\partial t}(b,t)\right. \\ &\quad\left. -\frac{\partial\omega}{\partial t}^*(a,t)T(a)\frac{\partial\omega}{\partial\zeta}(a,t) - T(a)\frac{\partial\omega}{\partial\zeta}^*(t,a)\frac{\partial\omega}{\partial t}(a,t)\right). \end{aligned} \tag{5.38}$$

When the arguments are considered to be real it simplifies to equation (5.37),

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial\omega}{\partial t}^*(b,t)T(b)\frac{\partial\omega}{\partial\zeta}(b,t)+T(b)\frac{\partial\omega}{\partial\zeta}^*(b,t)\frac{\partial\omega}{\partial t}(b,t)\right.\\&\left.-\frac{\partial\omega}{\partial t}^*(a,t)T(a)\frac{\partial\omega}{\partial\zeta}(a,t)-T(a)\frac{\partial\omega}{\partial\zeta}^*(t,a)\frac{\partial\omega}{\partial t}(a,t)\right)\\&=\frac{\partial\omega}{\partial t}(b,t)T(b)\frac{\partial\omega}{\partial\zeta}(b,t)-\frac{\partial\omega}{\partial t}(a,t)T(a)\frac{\partial\omega}{\partial\zeta}(a,t).\end{aligned}\tag{5.39}$$

So indeed equation (5.36) holds. A result of the vibrating string being a Port-Hamiltonian system.

The chosen input, output and boundary conditions are substituted in the equation to get an expression for the change in energy in the system using the input and output arguments,

$$\frac{d}{dt}E(t)=\frac{1}{2}\left(y(t)^*u(t)+u(t)^*y(t)\right).\tag{5.40}$$

Now for the positive realness of the transfer function, consider general solutions for the input, output and state, $u(t)=u_0e^{st}$, $y(t)=y_0e^{st}$ and $x(t)=x_0e^{st}$ respectively, [13]. The boundary condition $y_0=G(s)u_0$ from (5.18) is applied as well. For generality the initial input is $u_0\in\mathbb{C}$. The change in energy in the system with these general solutions results in a positive expression,

$$\begin{aligned}\frac{1}{2}\left(y(t)^*u(t)+u(t)^*y(t)\right)&=\frac{1}{2}\left((y_0e^{st})^*u_0e^{st}+(u_0e^{st})^*y_0e^{st}\right)\\&=\frac{1}{2}\left((G(s)u_0e^{st})^*u_0e^{st}+(u_0e^{st})^*G(s)u_0e^{st}\right)\\&=\frac{1}{2}\left|u_0e^{st}\right|^2\left(G(s)^*+G(s)\right)\\&=\frac{1}{2}\left|u_0e^{st}\right|^2 2Re(G(s))=\left|u_0e^{st}\right|^2Re(G(s)).\end{aligned}\tag{5.41}$$

Similarly the Hamiltonian equation is differentiated using the general solution for the state,

$$\begin{aligned}E(t)&=\frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta\\&=\frac{1}{2}\int_a^b (x_0(\zeta)e^{st})^*\mathcal{H}(\zeta)x_0(\zeta)e^{st}d\zeta\\&=\frac{1}{2}\int_a^b \left|e^{st}\right|^2x_0(\zeta)^*\mathcal{H}(\zeta)x_0(\zeta)d\zeta\\&=\frac{1}{2}\int_a^b e^{2Re(s)t}x_0(\zeta)^*\mathcal{H}(\zeta)x_0(\zeta)d\zeta,\end{aligned}\tag{5.42}$$

which results in the following expression,

$$\frac{d}{dt}E(t) = Re(s)e^{2Re(s)t}\int_a^b x_0(\zeta)^*\mathcal{H}(\zeta)x_0(\zeta)d\zeta. \tag{5.43}$$

Since $\mathcal{H}$ is positive definite, which was shown in (5.35), the integral is positive and the whole expression is positive.

The two results for the derivative of the energy function combine to form the equation,

$$Re(s)e^{2Re(s)t}\int_a^b x_0(\zeta)^*\mathcal{H}(\zeta)x_0(\zeta)d\zeta = \left|u_0e^{st}\right|^2 Re(G(s)). \tag{5.44}$$

Since $\left|u_0e^{st}\right|^2 > 0$ for all $s \in \mathbb{C}$, for $Re(s) > 0$ the transfer function must be positive real, i.e. $Re(G(s)) > 0$. The change in energy is equal to the product of the input and output which is a typical property of positive-real systems. Another property of such systems is the Positive Real Lemma, which is explained in next section.

## 5.10 The Positive Real lemma

A positive real transfer function can be written into a state-space representation of Hamiltonian form 3.5.2 by use of the Positive Real lemma. The lemma itself states that there exists a positive real transfer function for a linear Port-Hamiltonian system, (3.5.2). It is used the other way around, to form a state-space representation from an existing positive real transfer function. The lemma is stated next.

**Lemma 5.10.1** (Positive Real lemma)**.** *Consider the state-space system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(0) &= x_0, \end{aligned} \tag{5.45}$$

*where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ and $(A, B)$ controllable and $(A, C)$ observable. It's transfer function, $H(s) \in \mathbb{R}^{m\times m}$, $s \in \mathbb{C}$ is $H(s) = C(sI - A)^{-1}B + D$, exists and is Positive Real if and only if there exist matrices $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ such that*

$$\begin{aligned} A^TP + PA &= -L^TL \\ PB - C^T &= -LW \\ D + D^T &= W^TW \end{aligned} \tag{5.46}$$

The equation $A^TP + PA = -L^TL$ is known as the Lyapunov equation for matrices, [10].

The transfer function of the vibrating string is used to rewrite it to an ordinary differential equation. From there a state-space representation can be derived and the matrices $P, L$ and $W$ will be found.

The three equations in the lemma can be written as one Linear Matrix Inequality, [12], which is feasible for a positive symmetric matrix $P > 0$, defined as

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D^T - D \end{bmatrix} = -\begin{bmatrix} L \\ W^T \end{bmatrix} \begin{bmatrix} L^T & W \end{bmatrix} \leq 0. \tag{5.47}$$

When $D + D^T > 0$ it can even be written as a Quadratic Matrix Inequality related to the Algebraic Riccati Equality, [12]

$$A^TP + PA + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T \leq 0. \tag{5.48}$$

When $D = 0$ it follows that that $W = 0$ and $PB = C^T$. This is the case for the vibrating string model, just the Lyapunov inequality is left. The matrix $P$ is the Hamiltonian matrix of the linear Port-Hamiltonian system.

## 5.11 State-space representation

The approximate transfer function will be written into a state-space representation. This is possible by use of the Positive Real Lemma 5.10.1 from previous section.

Let us restate the transfer function involving two poles for the uncontrolled system as in (5.31),

$$P(s) = \frac{2s}{s^2 + \frac{\pi^2}{4}}. \tag{5.49}$$

This will be called the plant. Based on the corresponding ordinary differential equation of the transfer function state variables will be chosen for the state-space representation. The ODE is derived by use of the relation $Y(s) = G(s)U(s)$.

$$s^2Y + \frac{\pi^2}{4}Y = 2sU \implies \ddot{y}(t) + \frac{\pi^2}{4}y(t) = 2\dot{u}(t). \tag{5.50}$$

To define the states variables $x_1$ and $x_2$ the differential equation is rewritten and integrated twice,

$$\ddot{y} = 2\dot{u} - \frac{\pi^2}{4}y \implies y = \int\int [2\dot{u} - \frac{\pi^2}{4}y] \implies y = \int [2u + \int [-\frac{\pi^2}{4}y]]. \tag{5.51}$$

Choosing $x_1 = \int[-\frac{\pi^2}{4}y]$ and $x_2 = \int[2u + x_1]$, where $y = x_2$, their derivatives are $\dot{x}_1 = -\frac{\pi^2}{4}y$ and $\dot{x}_2 = 2u + x_1$. Together they form the state-space representation of the plant as a whole,

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x. \end{aligned} \tag{5.52}$$

It can be checked that this system is correct by use of the definition of a transfer function for general state-pace representations. As stated in the section 3.3, a transfer function of certain state-space representations is defined as $H(s) = C(sI - A)^{-1}B + D$, [12]. In the case of the derived system (5.52) see

$$H(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \left( s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{2s}{s^2 + \frac{\pi^2}{4}}. \tag{5.53}$$

The state-space representation is correct. Note that the matrix $D = 0$, which, by the equations in the Positive Real lemma (5.10.1), implies that $W = 0$ and $PB = C^T$.

The eigenvalues of $A$ are imaginary, namely $\lambda == \pm\sqrt{\frac{\pi^2}{4}}i = \pm\frac{\pi}{2}i$, which indicates that the system is marginally stable. This property was already concluded from the pole-zero map of the approximated transfer function ion Figure 5.3b, where twenty poles were used. In the coming sections the system is stabilised by controllers which will cause the real parts of the eigenvalues to become negative.

When the system is controlled it is very useful to know how the energy in the system behaves. To get an insight in the energy behaviour the values in the Hamiltonian matrix for the approximated model will be determined. The Lyapunov equation for the plant is solved for a symmetric function $P$. Since the system is marginally stable we choose $L = 0$ and solve for $P$,

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ -\frac{\pi^2}{4} & 0 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & 0 \end{pmatrix} = 0 \\ \implies &\begin{pmatrix} p_3 & p_4 \\ -\frac{\pi^2}{4}p_1 & -2.467p_2 \end{pmatrix} + \begin{pmatrix} p_2 & -\frac{\pi^2}{4}p_1 \\ p_4 & -\frac{\pi^2}{4}p_3 \end{pmatrix} = 0. \end{aligned} \tag{5.54}$$

The four equations $p_3 + p_2 = 0$, $p_4 - \frac{\pi^2}{4}p_1 = 0$, $-\frac{\pi^2}{4}p_1 + p_4 = 0$ and $-\frac{\pi^2}{4}p_2 - \frac{\pi^2}{4}p_3 = 0$ result in a unique solution matrix, where the elements $p_1$ and $p_2$

are variables, giving the solution matrix,

$$P = \begin{pmatrix} p_1 & p_2 \\ -p_2 & \frac{\pi^2}{4}p_1 \end{pmatrix}. \tag{5.55}$$

This solution matrix is the corresponding Hamiltonian density $\mathcal{H}$ for the uncontrolled approximated model of the vibrating string.

Using the standard Port-Hamiltonian state-space representation form of 3.5.2 and the equations in the Positive Real lemma 5.10.1, namely $C = B^T\mathcal{H}$ and $B^TP = C$, the elements in the matrix can be specifically determined,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ -p_2 & \frac{\pi^2}{4}p_1 \end{pmatrix}, \tag{5.56}$$

concluding in the values $p_1 = \frac{1}{2(\frac{\pi^2}{4})}$ and $p_2 = 0$. The matrix is $\mathcal{H} = \begin{pmatrix} \frac{2}{\pi^2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. If this matrix is correct the equation $A = \mathcal{J}^*\mathcal{H}$, in a Port-Hamiltonian representation, should reveal a skew-adjoint matrix $\mathcal{J}$. Indeed,

$$\begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & 0 \end{pmatrix} = \mathcal{J}^*\begin{pmatrix} \frac{2}{\pi^2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \implies \mathcal{J} = \begin{pmatrix} 0 & -\frac{\pi^2}{2} \\ \frac{\pi^2}{2} & 0 \end{pmatrix} \tag{5.57}$$

gives a matrix $\mathcal{J}$ which is skew-adjoint. The found Hamiltonian matrix used in the function $\frac{1}{2}x^T\mathcal{H}x$ correctly describes the energy in the approximated model and the approximated vibrating string system is a linear Port-Hamiltonian system.

A weak Lyapunov function is derived for the approximated uncontrolled vibrating string following the definition 3.2.1 in section 3.2. It is the function $V(x) = x^T\mathcal{H}x$, where indeed $V(0) = 0$ and

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T\mathcal{H}x + x^T\mathcal{H}\dot{x} = (Ax)^T\mathcal{H}x + x^T\mathcal{H}Ax \\ &= x^T(A^T\mathcal{H} + \mathcal{H}A)x = 0. \end{aligned} \tag{5.58}$$

## 5.12 Energy behaviour

The Hamiltonian function $E = \frac{1}{2}x^*\mathcal{H}x$, which represents the energy in the system, is plotted in Figure 5.7.

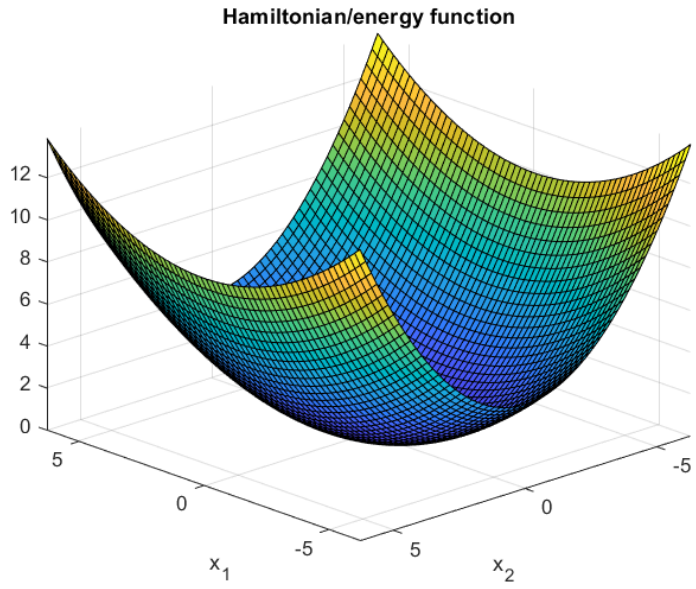
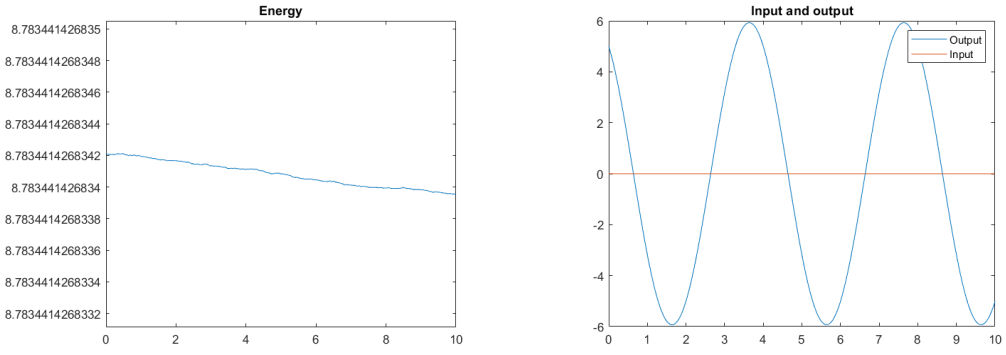



Figure 5.7: Energy function of uncontrolled system

A point on the energy function in Figure 5.7 is set as initial condition to see how the energy will evolve when the system is simulated. As a clear initial condition point $x_0 = \big(-5, 5\big)$ is chosen since, as seen in the energy function, the energy would start in the front corner at about a height of 9. See how the energy evolves in the open-loop system of the plant using an input signal $u(t) = 0$ and the chosen initial condition for the state of the plant.




(a) Energy (b) Input and output signals

Figure 5.8: Simulation of the uncontrolled vibrating string

The energy in Figure 5.8a is seen to decay very slowly, which is caused by small numerical errors in the simulation. In practice the energy is constant at the value where the simulation started, the chosen initial condition.

The output and input of the uncontrolled system are plotted in Figure 5.8b. The input is zero and the output is oscillating. This result makes sense because the uncontrolled system is marginally stable and therefore it will neither diverge nor converge.

Remember that the system is Port-Hamiltonian. The change in energy is equal to the product of input and output. For this simulation of the open-loop system the input signal is $u(t) = 0$, which causes the change in energy to be zero everywhere, regardless of the output of the system. The energy will be constant for any initial condition of the plant, as long as the signal input stays at zero.

# 6. PID-control for the vibrating string

In this chapter two variations of PID-control [6] are applied to the vibrating string model. Namely proportional gain control and integrator control. These two methods are chosen since they form the basis of the Hybrid Integrator-Gain system (4.2). The controllers will be connected in a negative feedback loop as in Figure 6.1. In this figure the signal $r(t)$ represents the reference speed that the vibrating string should achieve. The behaviour of the energy of the plant will be analysed.

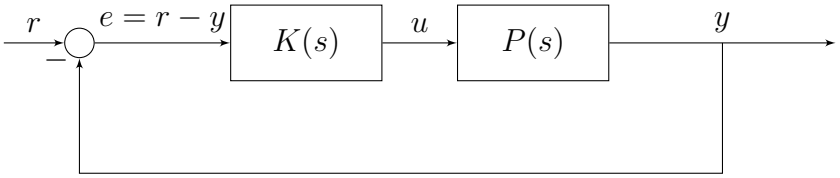



Figure 6.1: Vibrating string $P(s)$ with PID controller $K(s)$ in negative feedback

A transfer function of a general negative feedback loop has already been derived in equation (3.12) and is restated here,

$$H(s) = \frac{P(s)K(s)}{1 + P(s)K(s)}. \tag{6.1}$$

A PID-controller always has a transfer function of the form $K(s) = k_p + \frac{1}{s}k_i + k_d s$, where the PID coefficients are constant. This causes a general closed-loop PID controlled feedback system of the vibrating string (5.49) to have the general transfer function, including a pole-zero cancellation,

$$\mathcal{F}_{CL}(s) = \frac{2k_d s^2 + 2k_p s + 2k_i}{(2k_d + 1)s^2 + 2k_p s + 2k_i + \frac{\pi^2}{4}}. \tag{6.2}$$

Note that if we use differentiator control, i.e. $k_d \neq 0$, the transfer function will just be proper and not strictly proper [10]. Looking at the limit

$$\lim_{s \to 0} s\mathcal{F}_{CL}(s) = 0, \tag{6.3}$$

then, by the Final Value Theorem 3.3.1, every form of PID-control will result in an output that converges to zero. This makes PID-control only useful

to bring a system into rest. Just like the transfer function of the plant the general closed-loop transfer function can be written into a state-space representation. Equation (6.2) corresponds to the following differential equation,

$$(2k_d+1)\ddot{y}(t)+2k_p\dot{y}(t)+(2k_i+\frac{\pi^2}{4})y(t)=2k_d\ddot{r}(t)+2k_p\dot{r}(t)+2k_ir(t). \quad (6.4)$$

Rewriting equation (6.4),

$$\ddot{y}=\frac{1}{2k_d+1}\left(-2k_p\dot{y}-(2k_i+\frac{\pi^2}{4})y+2k_d\ddot{r}+2k_p\dot{r}+2k_ir\right) \quad (6.5)$$

integrating twice,

$$\begin{aligned} y &= \frac{1}{2k_d+1}\int\int\left[-2k_p\dot{y}-(2k_i+\frac{\pi^2}{4})y+2k_d\ddot{r}+2k_p\dot{r}+2k_ir\right] \\ &= \frac{1}{2k_d+1}\left(2k_dr+\int\left[-2k_py+2k_pr+\int\left[-(2k_i+\frac{\pi^2}{4})y+2k_ir\right]\right]\right). \end{aligned} \quad (6.6)$$

The state variables are defined as $x_1=\int\left[-(2k_i+\frac{\pi^2}{4})y+2k_ir\right]$ and $x_2=\int[-2k_py+2k_pr+x_1]$ while the output is $y=\frac{1}{2k_d+1}(2k_dr+x_2)$. Using this expression for the output the derivatives of the state variables are derived

$$\begin{aligned} \dot{x}_1 &= \frac{-(2k_i+\frac{\pi^2}{4})}{2k_d+1}x_2+\frac{-(2k_i+\frac{\pi^2}{4})2k_d}{2k_d+1}r+2k_ir \\ \dot{x}_2 &= \frac{-2k_p}{2k_d+1}x_2+\frac{-4k_pk_d}{2k_d+1}r+2k_pr+x_1. \end{aligned} \quad (6.7)$$

The general closed-loop state-space representation from $r(t)$ to $y(t)$ using PID-control is the system,

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & \frac{-(2k_i+\frac{\pi^2}{4})}{2k_d+1} \\ 1 & \frac{-2k_p}{2k_d+1} \end{pmatrix}x(t)+\begin{pmatrix} \frac{-(2k_i+\frac{\pi^2}{4})2k_d}{2k_d+1}+2k_i \\ \frac{-4k_pk_d}{2k_d+1}+2k_p \end{pmatrix}r(t) \\ y(t) &= \begin{pmatrix}0 & 1\end{pmatrix}x(t)+\frac{2k_d}{2k_d+1}r(t). \end{aligned} \quad (6.8)$$

The definition for the transfer function of this state-space representation is exactly the transfer function of equation (6.2), so we know that it describes

the same system,

$$\begin{aligned} H(s) &= C(sI-A)^{-1}B + D \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & \frac{-(2k_i+\frac{\pi^2}{4})}{2k_d+1} \\ 1 & \frac{-2k_p}{2k_d+1} \end{pmatrix}\right)^{-1}\begin{pmatrix} \frac{-(2k_i+\frac{\pi^2}{4})2k_d}{2k_d+1} + 2k_i \\ \frac{-4k_pk_d}{2k_d+1} + 2k_p \end{pmatrix} + \frac{2k_d}{2k_d+1} \\ &= \frac{2k_ds^2 + 2k_ps + 2k_i}{(2k_d+1)s^2 + 2k_ps + 2k_i + \frac{\pi^2}{4}}. \end{aligned} \tag{6.9}$$

## 6.1 Proportional control

To apply proportional control only a negative feedback law for the input signal, $u(t) = -k_p y(t)$ is used. Proportional control is also called gain control. The proportional transfer function is connected to the approximated transfer function of the vibrating string (5.31) to find the transfer function of the whole feedback loop, as in equation (6.2), where are $k_i = k_d = 0$,

$$F_P(s) = \frac{2k_ps}{s^2 + 2k_ps + \frac{\pi^2}{4}}.$$

The state-space representation is derived from (6.8) and becomes,

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & -2k_p \end{pmatrix}x(t) + \begin{pmatrix} 0 \\ 2k_p \end{pmatrix}r(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix}x(t). \end{aligned} \tag{6.10}$$

The proportional system is a damped version of the uncontrolled system (5.52), implying that there should be a matrix $R$ which represents the damping of the string in the model. The standard Port-Hamiltonian state-space form of (3.5.2) is extended with a damping matrix $R$, [2]. The extended system is still Port-Hamiltonian and is stated here,

$$\begin{aligned} \dot{x}(t) &= (\mathcal{J} - R)\mathcal{H}x(t) + Br(t) \\ y(t) &= B\mathcal{H}x(t) \end{aligned} \tag{6.11}$$

Equation $A = (\mathcal{J} - R)\mathcal{H}$ is used to derive the matrix $R$,

$$\begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & -2k_p \end{pmatrix} = \left(\begin{pmatrix} 0 & -\frac{\pi^2}{2} \\ \frac{\pi^2}{2} & 0 \end{pmatrix} - R\right)\begin{pmatrix} \frac{1}{\frac{\pi^2}{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tag{6.12}$$

resulting in damping matrix $R \approx \begin{pmatrix} 0 & 0 \\ 0 & 4k_p \end{pmatrix}$. This matrix is positive semi-definite.

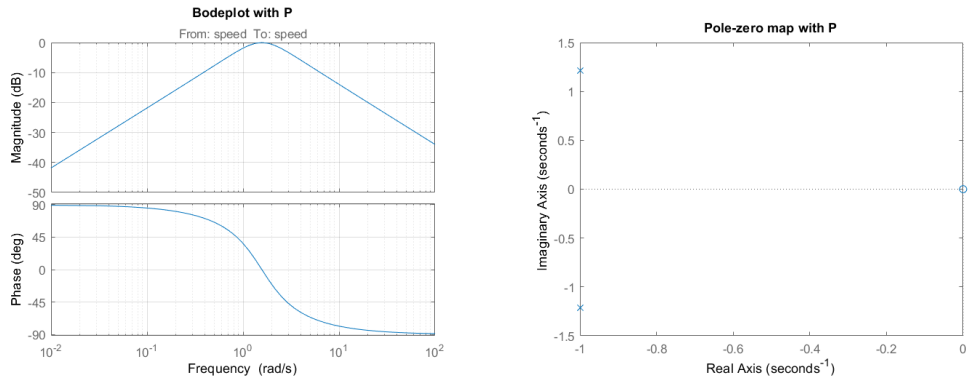
### Closing the loop

Another approach to get to the state-space representation of the proportional controlled system is by adding a control rule directly to the state-space representation of the uncontrolled system (5.52). The negative feedback rule $u(t) = -k_p y(t)$ is added to the plant and the loop is closed with the reference signal $r(t)$ set to zero. The feedback rule is added by substituting $-k_p y(t) = -\begin{pmatrix}0 & k_p\end{pmatrix} x(t)$ as input signal in (5.52), which gives

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix}0 & -2.467\\ 1 & -2k_p\end{pmatrix} x(t) \\ y(t) &= \begin{pmatrix}0 & 1\end{pmatrix} x(t) \end{aligned} \tag{6.13}$$

This system is the exact state-space representation found before (6.10), where the reference signal $r(t)$ is zero. Applying the feedback rule directly to the state-space representation removes the need to use the transfer function and its corresponding differential equation.

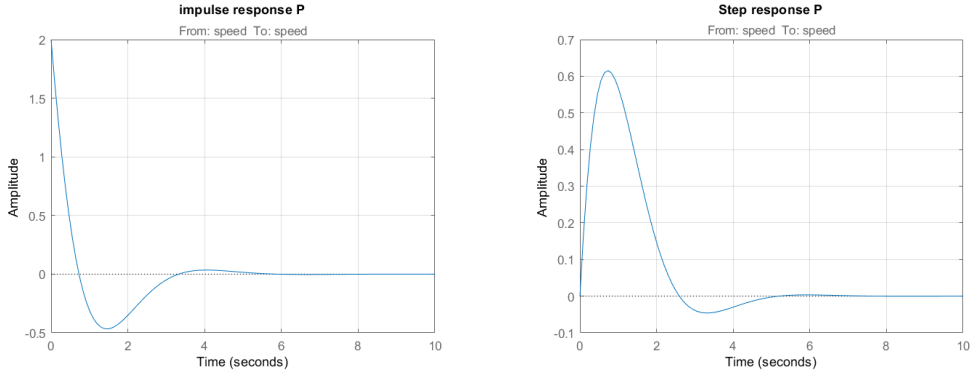
The system is generated with gain coefficient $k_p = 1$. The magnitude plot in Figure (6.2a) shows that signals with low and high frequencies get reduced. In the phase plot the signals with low frequencies get advanced and high frequencies get delayed.




(a) Bode Diagram (b) Pole-Zero map

Figure 6.2: Proportional controlled system, $k_p = 1$

The poles of the closed-loop system in Figure 6.2b lay in the left half plane and at $-1 \pm 1.2114i$. They both have negative real part, which indicate that the system is asymptotically stable but there still is a zero at the origin.




(a) Impulse response (b) Step response

Figure 6.3: Input responses

Since the system is stable the impulse response in Figure 6.3a goes to zero, i.e. the system goes into rest. The step response in Figure 6.3b goes to zero as well. When the gain constant is increased to $k_p = 100$ the step response touches 1 and goes back to zero after, see Figure 6.4.

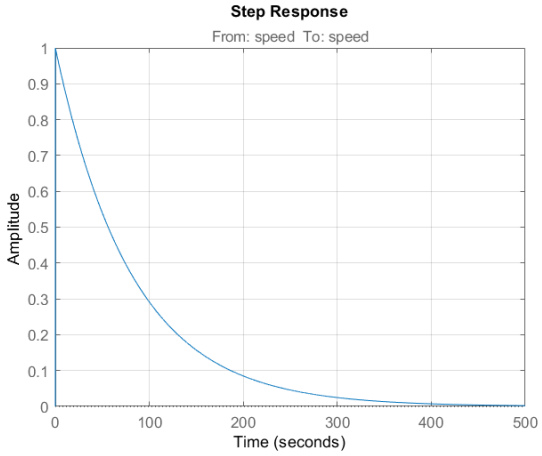



Figure 6.4: Step response, $k_p = 100$

## 6.2 Proportional energy behaviour

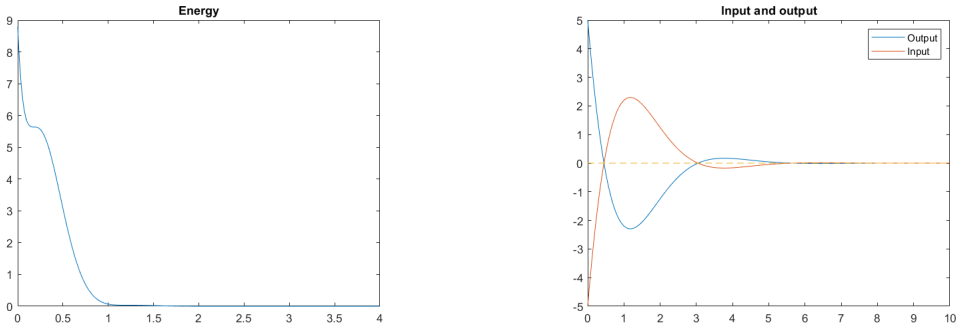
The energy in the proportional controlled vibrating string system will be analysed in this section. The proportional feedback rule $u(t) = -k_p y(t)$ is substituted into the input/output expression of the derivative of the power

balance from (5.40),

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{1}{2}\left(y(t)^* u(t) + u(t)^* y(t)\right) \\ &= \frac{1}{2}\left(y(t)^*(-k_p y(t)) + (-k_p y(t))^* y(t)\right) \\ &= \frac{1}{2} - k_p\left(|y(t)|^2 + |y(t)|^2\right) \\ &= -k_p |y(t)|^2.\end{aligned} \tag{6.14}$$

When $k_p > 0$ the change of energy in the system is negative (6.14), implying that the energy decreases. The system will go into rest once the energy hits zero.

Now the initial condition of the plant is set to $x_0 = \big(-5, 5\big)$ and the reference signal is zero, $r(t) = 0$. The energy in Figure 6.5a is seen to converge to zero after a hurdle between 0 and 0.5 seconds. The energy is monotonically decreasing.




(a) Energy curve

(b) Control signal and output

Figure 6.5: Proportional controlled model

Since the plant is connected via a negative feedback loop the control signal $u(t)$ is equal to the negative of the output signal $y(t)$, see Figure 6.5b. The input and the output of the plant are of opposite sign at any time. Knowing that the plant is Port-Hamiltonian it makes sense that the energy is monotonically decreasing, the change in energy is equal to the input times the output of the plant. At crossings of the horizontal axis the change in energy is zero, there the energy is constant. These are exactly at the points where the energy curve in Figure 6.5a is constant.

### Tracking a reference signal

When the output $y(t)$ of the plant should behave in a desired way a reference signal $r(t)$ can be put into the system. This section will show that gain control alone will not result in good signal tracking. Suppose the output must be constant $y(t) = 1$, then the reference signal $r(t) = 1$ is set as input. The initial condition of the plant is set to $x(0) = (0, 0)$, i.e. the plant is initially at rest.

Since the gain constant $k_p = 1$ is low, the output fails to track the reference signal, see Figure 6.6.

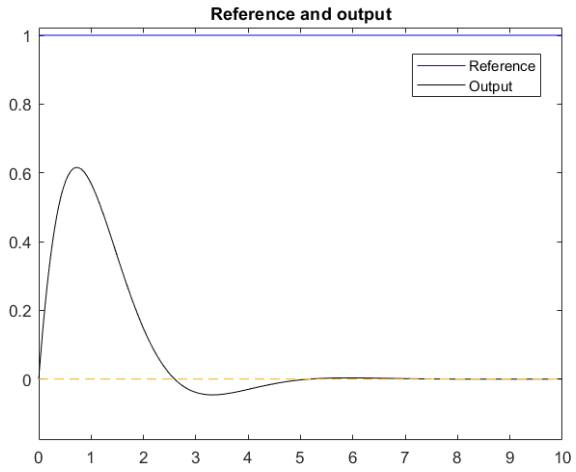
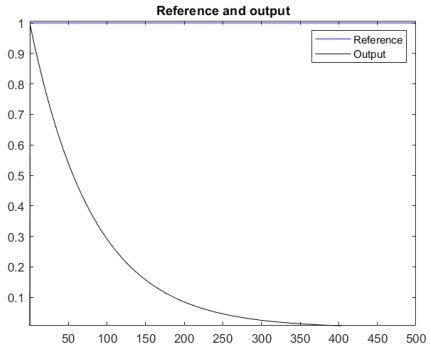
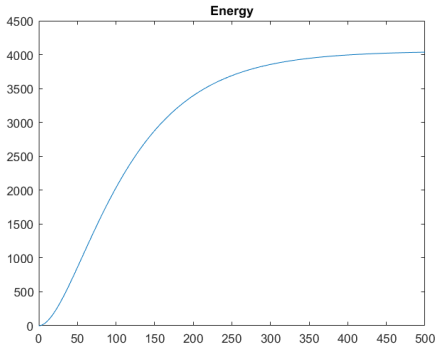



Figure 6.6: Reference and output signal

When the gain is increased to $k_p = 100$, the controller will be more aggressive and the reference signal is reached. The output diverges from the desired signal immediately and will become zero again, see Figure 6.7a. The energy of the system converges to a high value and stays there, since the output becomes zero the energy becomes constant, Figure 6.7b.




(a) Reference and output




(b) Energy

Figure 6.7: Aggressive gain control

This is in line with the Final Value Theorem, 3.3.1. The transfer function of the closed-loop proportional controlled vibrating string is defined for all $Re(s) > 0$ and the limit of the function is

$$\lim_{s \to 0} s \frac{2s}{s^2 + 2s + \frac{\pi^2}{4}} = 0. \tag{6.15}$$

Reference tracking will not work with just a proportional controller.

## 6.3 Integrator control

In this section an integrator part is added to the proportional controller from previous section. To keep the model simple an integrator coefficient of $k_i = 1$ is used and the proportional coefficient is $k_p = 1$ again. These control coefficients are to be tuned based on the results. By the transfer function derived as (6.2) the transfer function of the whole negative feedback closed-loop system here is

$$F_{PI}(s) = \frac{2s + 2}{s^2 + 2s + \frac{8+\pi^2}{4}}. \tag{6.16}$$

The respective state-space representation, derived as (6.8), will be

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & -\frac{8+\pi^2}{4} \\ 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} r(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t). \end{aligned} \tag{6.17}$$

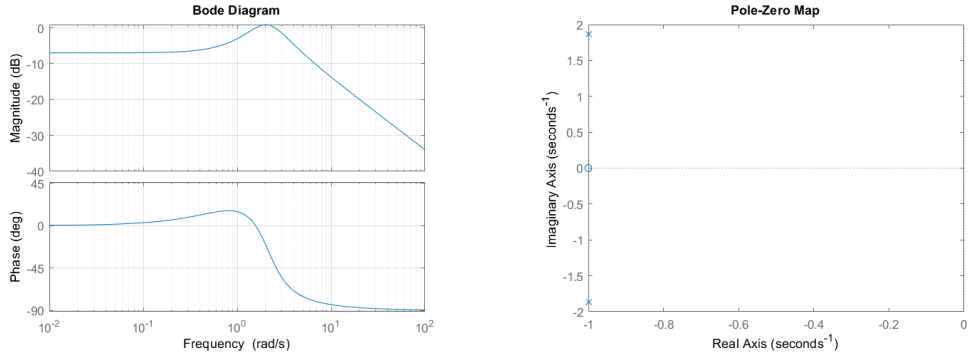
This system is effected by the zero in the approximated transfer function (5.31). The pole-zero cancellation results in the state of the closed-loop system being two dimensional. If the system were to be written into state-space representation without use of the pole-zero cancellation it would result in a three dimensional state and the system might not be Port-Hamiltonian anymore.

There is a damping matrix $R$ again and using the matrix $\mathcal{J}$ and the derived Hamiltonian matrix of the energy in the plant $R$ is derived by equation $A = (\mathcal{J} - R)\mathcal{H}$,

$$\begin{pmatrix} 0 & -\frac{\pi^2}{2} \\ 1 & -2 \end{pmatrix} = \left( \begin{pmatrix} 0 & -\frac{\pi^2}{2} \\ \frac{\pi^2}{2} & 0 \end{pmatrix} - R \right) \begin{pmatrix} \frac{1}{\frac{\pi^2}{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{6.18}$$

By $A\mathcal{H}^{-1} = \mathcal{J} - R$ the damping matrix is $R = \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix}$, which is positive semi-definite.

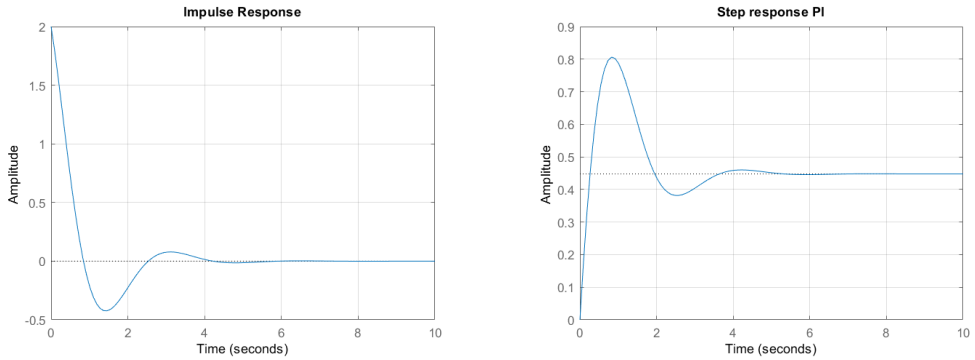
The magnitude plot of the integrator controlled system in Figure 6.8a shows that high frequencies get filtered out. The Figure 6.8b shows the poles with negative real part, which indicate asymptotic stability. The zero has been moved from the origin into the left half-plane.




(a) Bode diagram (b) Pole-Zero map

Figure 6.8: PI controlled system, $k_p = 1$, $k_i = 1$

In Figure 6.9a the system goes to zero after an impulse at starting time zero. Next to it, in Figure 6.9b, the output to a step response converges to a positive value.

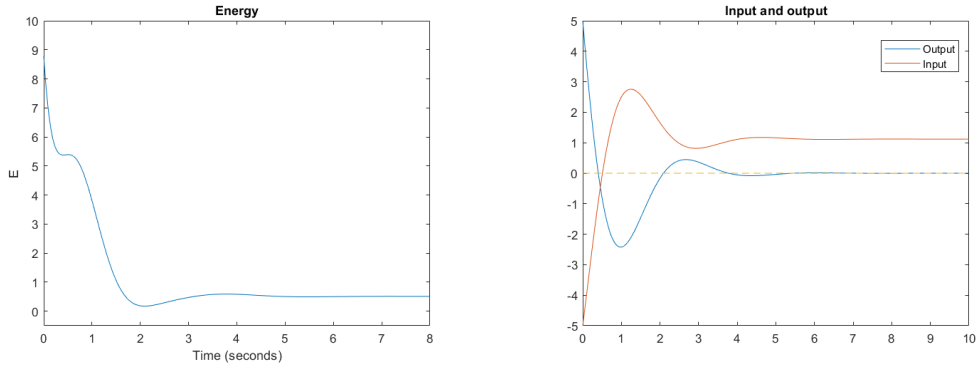



(a) Impulse response (b) Step response

Figure 6.9: Input responses

## 6.4 Integrator energy behaviour

The closed-loop system of (6.17) with initial condition of the plant at $x_0 = [-5, 5]$ has been simulated. Figure 6.10b shows the control signal into plant and its resulting output. The output $y(t)$ goes to zero which drives the system into rest. However, the control signal converges to a positive value just higher than one. To keep the system in rest a force needs to constantly be applied to the system and so the control signal will not go to zero.




(a) Energy curve (b) Control and output signal

Figure 6.10: PI controlled system

With the result of the Final Value Theorem 3.3.1, again, the following limit,

$$\lim_{s\to 0} s\frac{2s+2}{s^2+2s+\frac{8+\pi^2}{4}} = 0, \tag{6.19}$$

implies that the output of the impulse response to the system will go to zero regardless of the chosen PI-coefficients.

## 6.5 A pure integrator

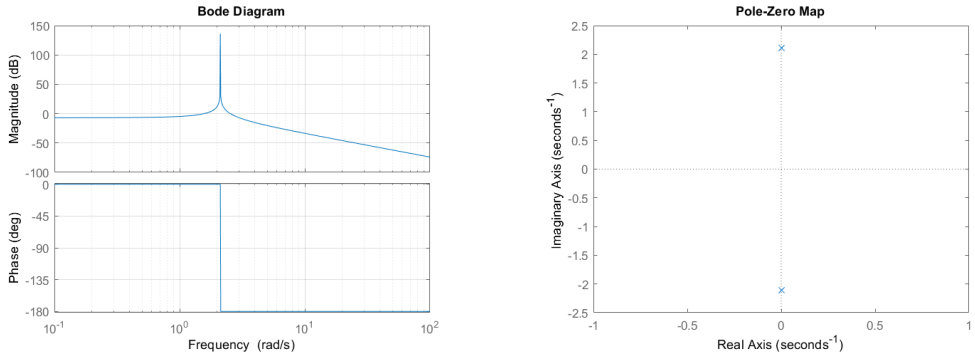
The Hybrid Integrator-Gain System makes use of pure integrator control and proportional control. Just a pure integrator does not steer the system of the vibrating string into rest. This is because of the zero at $s = 0$ in the transfer function of the plant. The initial condition for the plant at $x_0 = (-5, 5)$ is used here as well. The closed-loop transfer function with $k_i = 1$ and $k_p = k_d = 0$ gives,

$$F_I(s) = \frac{2}{s^2 + \frac{8+\pi^2}{4}}. \tag{6.20}$$

The corresponding pure integrator controlled state-space representation including the pole-zero cancellation is,

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & -\frac{8+\pi^2}{4} \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t). \end{aligned} \tag{6.21}$$

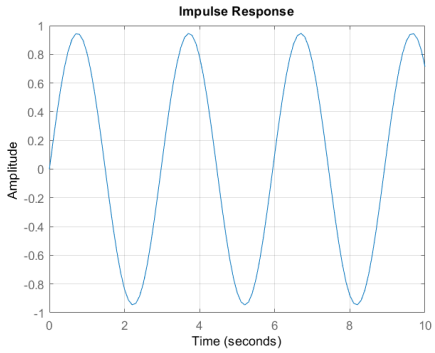
The matrix $A$ has eigenvalues with zero real part, indicating that this system is just marginally stable. A Bode-plot does not give sufficient information and the pole-zero map shows the poles being on the imaginary axis.



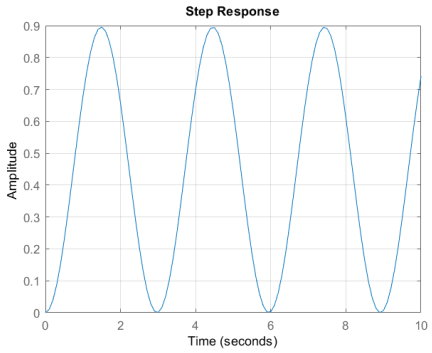

(a) Bode diagram (b) Pole-Zero map

Figure 6.11: Pure integrator controlled system, $k_i = 1$

The impulse and step responses for this system give similar oscillating results for the output signal, see Figure 6.12a and Figure 6.12b.



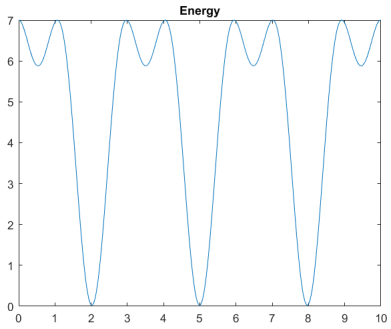

(a) Impulse response



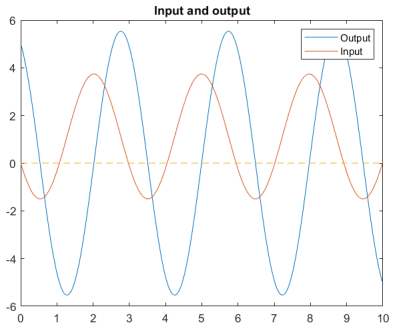

(b) Step response

Figure 6.12: Input responses

The energy in the system oscillates and does not converge. Looking at the control signal and output this makes sense because of the Port-Hamiltonian property of the plant. See Figures 6.13a and 6.13b.




(a) Energy curve




(b) Control and output signals

Figure 6.13: Pure integrator control

# 7. HIGS applied to the vibrating string

In this chapter the Hybrid-Integrator Gain System is applied the vibrating string model. It is connected via negative feedback. To keep the system simple the gain constant $k_h = 1$ and integrator frequency $w_h = 1$ are used. In practice these values can be tuned based on the output of the system. It is applied to the approximated model of the vibrating string using 2 poles and the initial condition of the plant is set to $x_0 = [-5, 5]$. The reference input $r(t)$ is set to zero because the goal is to steer the plant into rest.

## 7.1 HIGS as switched system

In integrator mode the closed-loop system is described by the already derived state-space representation (6.21) using the pole-zero cancellation, here the state of this system is called $x_i(t)$,

$$\begin{aligned} \dot{x}_i(t) &= \begin{pmatrix} 0 & -\frac{8+\pi^2}{4} \\ 1 & 0 \end{pmatrix} x_i(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_i(t). \end{aligned} \tag{7.1}$$

In gain mode another representation has been derived, (6.10), with another state $x_p(t)$,

$$\begin{aligned} \dot{x}_p(t) &= \begin{pmatrix} 0 & -\frac{\pi^2}{4} \\ 1 & -2 \end{pmatrix} x_p(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} r(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_p(t). \end{aligned} \tag{7.2}$$

Both these representations have been derived while including the pole-zero cancellation of the transfer function.

For a switched system the existence of a radially unbounded common Lyapunov function implies Global Uniform Asymptotic Stability (GUAS) of the system, as stated on page 23 of [11],

**Theorem 7.1.1.** *If all systems in the family of a switched system share a radially unbounded common Lyapunov function for the systems then the switched system is GUAS.*

In practice only the controller switches and the plant keeps the same representation. This results in 2 state values when the controller is in gain mode

and 3 state values in integrator mode. When the HIGS switches control mode the closed-loop system switches between a two dimensional and three dimensional state. The found energy function and Hamiltonian matrix of the plant is no common Lyapunov function of the switched system and Theorem 7.1.1 is not useful because of the dimension changes of the state.

In integrator mode the closed-loop system is marginally stable and in gain mode it is asymptotically stable. In next section we see that the HIGS does stabilise the vibrating string.

## 7.2 Simulations

The switched system is simulated in MATLAB using Simulink.

The input signal of the HIGS, $e(t) = -y(t)$, is plotted with the control signal $u(t)$ in Figure 7.1. The sign of the input and output of the block is the same everywhere. The control signal is seen to be continuous and piecewise-differentiable. At switching points the derivative of the control signal $u(t)$ is seen to change abruptly.

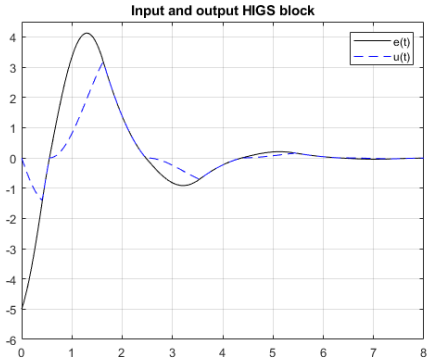
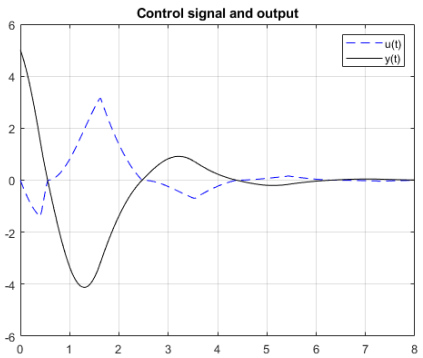


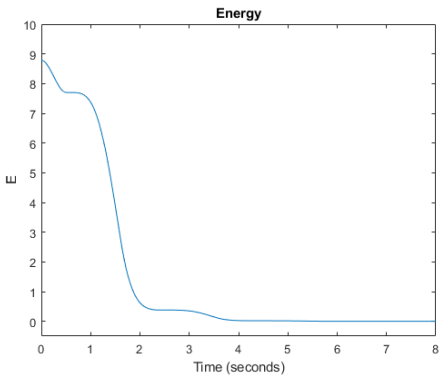

Figure 7.1: Input and output of HIGS

In Figure 7.2a the input and output of the plant is shown. The output of the plant goes to zero, which means the plant goes in rest, which is the desired result. The output $y(t)$ and input signal $u(t)$ are seen to keep switching sign, they are always the opposite sign of one another. Remember that the vibrating string is a Port-Hamiltonian system and so the change in energy equal to $\frac{d}{dt}E(t) = u(t)y(t)$. Since the signals are always of opposite sign the change in energy is always negative and the energy decreases. This is seen in the energy curve in Figure 7.2b as well. It decreases monotonically. The

energy never increases, what could cause problems in certain systems. In the vibrating string model increasing energy might not be a problem but in other applications an increase in energy can set problems in motion.




(a) Input and output signals




(b) Energy curve

Figure 7.2: HIGS control

Figure 7.3 shows the constraint when a switch in control mode is made. In the bottom plot it can be seen that the output stays in $\mathcal{F}$ and is on the boundary when in gain mode. The HIGS starts in gain mode, since the derivative $\dot{e}(0)$ is positive and $e(0)$ is negative and in the constraint $k_h\dot{e}(0)e(0)$ is negative while $\omega_h e(t)^2$ is positive. The constraint for $\mathcal{F}_2$ holds and $u(0) = k_h e(0)$.

The Figure 7.3 shows the moments the control mode switches. The second plot shows the constraint for $\mathcal{F}_2$ and the third shows the constraint for being in $\mathcal{F}$.

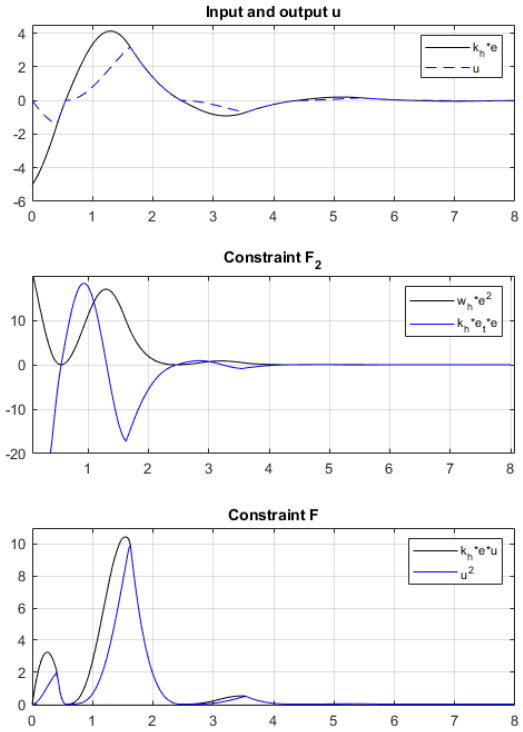



Figure 7.3: $k_h e$ and output $u$, constraint for $\mathcal{F}_2$ and constraint for $\mathcal{F}$ overall

Concluding, in practice the switched system does stabilise the vibrating string and the energy decreases monotonically. The HIGS is specifically designed to generate a control signal of opposite sign to the output of the plant resulting in the use of the Port-Hamiltonian nature of the plant.

# 8. Conclusion

In this report a switching non-linear control method has been designed. The method, called a Hybrid Integrator-Gain System, switches between pure integrator control and proportional control. It is designed such that the input and output signal are of opposite sign. The specific design details and area of solutions to the system have been described in detail.

A Partial Differential Equation for a model of a vibrating string has been approximated and its Port-Hamiltonian structure has been shown. The approximated model is written into a suitable state-space representation and has been simulated using various methods of PID-control connected in a closed-loop feedback system. Gain control has resulted in an asymptotically stable system and the energy in the string behaved nicely while integrator control did not.

The Hybrid Integrator Gain System is applied to the approximated model of the vibrating string. The combination of the Port-Hamiltonian property of the change in energy in the vibrating string system and the opposing signs of the input and control signals of the HIGS result in a monotonically decreasing energy in the vibrating string. Although pure integrator control does not stabilise the vibrating string the switching to gain control in the HIGS does result in a stabilising feedback loop.

## 8.1 Recommendation

The model of the vibrating string has been kept very minimal in this report. The string has not been modeled using realistic parameter settings so a real-life interpretation of the model is not clear. In its approximation the Residue Theorem has been applied to using only two poles. This approximation resulted in low dimensional state-space representations but does not approach the vibrating string system realistically.

To better see the impact of a Hybrid Integrator-Gain System it more interesting to use it on a more advanced Port-Hamiltonian system than that of the vibrating string. Preferably a system that benefits greatly from a monotonic decrease in energy.

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# A. Appendix

## A.1 Differential Equations for the HIGS in closed-loop

The closed-loop system connecting HIGS to the vibrating string model as plant can be described by a differential equation making use of indicator functions. The indicators are 1 when its arguments are in the subscripted set and 0 when the arguments are not, giving the rather long differential equation

$$\begin{aligned}\mathbb{1}_{\mathcal{F}_1}(e,\dot{e},u)(\ddot{y}(t)+(2\omega_h+2.467)y(t))+\mathbb{1}_{\mathcal{F}_2}(e,\dot{e},u)(\ddot{y}(t)+2k_h\dot{y}(t)+2.467y(t))\\ =\mathbb{1}_{\mathcal{F}_1}(e,\dot{e},u)(2\omega_h r(t))+\mathbb{1}_{\mathcal{F}_2}(e,\dot{e},u)(2k_h\dot{r}(t)).\end{aligned} \tag{A.1}$$

Expression (A.1) consists of two differential equations representing the two forms of control.

$$\begin{aligned}\ddot{y}(t)+(2\omega_h+2.467)y(t)&=2\omega_h r(t)\text{ in }\mathcal{F}_1\\ \ddot{y}(t)+2k_h\dot{y}(t)+2.467y(t)&=2k_h\dot{r}(t)\text{ in }\mathcal{F}_2\end{aligned} \tag{A.2}$$

Two terms in the indicator functions on the left side will always be in the equation so they can be moved outside the indicators functions, making the equation a bit more compact,

$$\begin{aligned}\ddot{y}(t)+2.467y(t)+\mathbb{1}_{\mathcal{F}_1}(e,\dot{e},u)(2\omega_h)+\mathbb{1}_{\mathcal{F}_2}(e,\dot{e},u)(2k_h\dot{y}(t))\\ =\mathbb{1}_{\mathcal{F}_1}(e,\dot{e},u)(2\omega_h r(t))+\mathbb{1}_{\mathcal{F}_2}(e,\dot{e},u)(2k_h\dot{r}(t)).\end{aligned} \tag{A.3}$$

A solution to the second order differential equations (A.2) exists only if there are real initial conditions $y(t_0)$ and $\dot{y}(t_0)$ for an initial time $t_0$. Combining the two differential equations as in (A.3) the solution to the close-loop system switches between the solutions of the two separate equations. When a switch in the HIGS block happens at $t_{switch}$ the initial condition of the next differential equation to which is being switched will be set to $y(t_0) = y(t_{switch})$. This way the solution to (A.3), after a switch, will continue at the same point before the switch.

## A.2 Variation of Parameters

The method of variation of parameters [9] is used to solve for solutions of both differential equations and will be combined for the closed-loop differential

equations as a whole. First we show the method of variation of parameters in a general case. Given the functions $p(t)$, $q(t)$ and $g(t)$ a solution $y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t)$ can be found to the second order inhomogeneous differential equation, stated

$$\ddot{y}(t) + q(t)\dot{y}(t) + p(t)y(t) = g(t). \tag{A.4}$$

A solution to such a second order homogeneous differential equation (A.4) exists on an interval $(a, b)$, $a, b \in \mathbb{R}$, if the functions $q(t)$, $p(t)$ and $g(t)$ are continuous on that interval and for a number $t_0 \in (a, b)$, there exist real initial conditions $y(t_0)$ and $\dot{y}(t_0)$. The resulting solution will be unique.

A particular solution $y_p(t)$ is of the form $y_p = v_1y_1 + v_2y_2$ where $y_1$ and $y_2$ are linear independent particular solutions to the homogeneous version of the differential equation. $v_1$ and $v_2$ will be found later and they will be used for $y_h(t) = C_1y_1(t) + C_2y_2(t)$. The inhomogenous version of equation one of the equations is

$$\ddot{y}(t) + q(t)\dot{y}(t) + p(t)y(t) = 0. \tag{A.5}$$

Th expression for $y_p$ is differentiated,

$$\dot{y}_p = \dot{v}_1y_1 + v_1\dot{y}_1 + \dot{v}_2y_2 + v_2\dot{y}_2, \tag{A.6}$$

and the equation $\dot{v}_1y_1 + \dot{v}_2y_2 = 0$ is set. Now Differentiating again,

$$\ddot{y}_p = \dot{v}_1\dot{y}_1 + v_1\ddot{y}_1 + \dot{v}_2\dot{y}_2 + v_2\ddot{y}_2 \tag{A.7}$$

and substituting the two derivatives into (A.4) gives

$$\begin{aligned}\ddot{y} + q\dot{y} + py &= (\dot{v}_1\dot{y}_1 + v_1\ddot{y}_1 + \dot{v}_2\dot{y}_2 + v_2\ddot{y}_2) + q(v_1\dot{y}_1 + v_2\dot{y}_2) + p(v_1y_1 + v_2y_2) \\ &= v_1(\ddot{y}_1 + q\dot{y}_1 + py_1) + v_2(\ddot{y}_2 + q\dot{y}_2 + py_2) + \dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2 = g.\end{aligned} \tag{A.8}$$

Since $y_1(t)$ and $y_2(t)$ are solutions to the homogenous differential equation (A.5), see that $\dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2 = g$. Now system of linear equations for $\dot{v}_1$ and $\dot{v}_2$ should be solved, it is made up of the homogenous and inhomogenous differential equations

$$\begin{aligned}\dot{v}_1y_1 + \dot{v}_2y_2 &= 0 \\ \dot{v}_1\dot{y}_1 + \dot{v}_2\dot{y}_2 &= g.\end{aligned} \tag{A.9}$$

Giving the terms, $\dot{v}_1 = \frac{-y_2g}{y_1\dot{y}_2 - \dot{y}_1y_2}$ and $\dot{v}_2 = \frac{y_1g}{y_1\dot{y}_2 - \dot{y}_1y_2}$. Integrating the solutions and substituting them into the particular solution $y_p = v_1y_1 + v_2y_2$ gives,

$$\begin{aligned}v_1 &= \int \frac{-y_2g}{y_1\dot{y}_2 - \dot{y}_1y_2}dt \\ v_2 &= \int \frac{y_1g}{y_1\dot{y}_2 - \dot{y}_1y_2}dt.\end{aligned} \tag{A.10}$$

This method is used for the vibrating string model in next section.

## A.3 Solutions for the vibrating string

The method of variation of parameters is applied to the model (A.2). In $\mathcal{F}_1$ we have $g(t) = 2\omega_h r(t)$. Two linear independent solutions to the inhomogenous version are $y_1(t) = \cos\big(\sqrt{2\omega_h + 2.467}t\big)$ and $y_2(t) = \sin\big(\sqrt{2\omega_h + 2.467}t\big)$. They can be checked by substituting them into the homogenous differential equation

$$\ddot{y}(t) + (2\omega_h + 2.467)y(t) = 0. \tag{A.11}$$

The expression $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$ is already found.

Using the formulas (A.10), the terms $v_1$ and $v_2$ simplify into

$$\begin{aligned} v_1 &= \int \frac{-\sin\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}(\cos\big(\sqrt{2\omega_h + 2.467}t\big)^2 + \sin\big(\sqrt{2\omega_h + 2.467}t\big)^2)}dt \\ v_2 &= \int \frac{\cos\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}(\cos\big(\sqrt{2\omega_h + 2.467}t\big)^2 + \sin\big(\sqrt{2\omega_h + 2.467}t\big)^2)}dt. \end{aligned} \tag{A.12}$$

In the denominator $\sin(x)^2 + \cos(x)^2 = 1$ appears, giving a bit more compact expressions,

$$\begin{aligned} v_1 &= \int \frac{-\sin\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}}dt \\ v_2 &= \int \frac{\cos\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}}dt. \end{aligned} \tag{A.13}$$

Substituting $v_1$ and $v_2$ into the particular form $y_p = v_1 y_1 + v_2 y_2$,

$$\begin{aligned} y_p(t) = &\int \frac{-\sin\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}}dt\cos\big(\sqrt{2\omega_h + 2.467}t\big) \\ &+ \int \frac{\cos\big(\sqrt{2\omega_h + 2.467}t\big)2\omega_h r(t)}{\sqrt{2\omega_h + 2.467}}dt\sin\big(\sqrt{2\omega_h + 2.467}t\big). \end{aligned} \tag{A.14}$$

Now we have a general solution,

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t). \tag{A.15}$$

Trigonometric functions are bounded and integrable, so these integrals exist for appropriate reference signal $r(t)$. The reference signal should be integrable.

Now for $\mathcal{F}_2$ the function $g(t) = 2k_h\dot{r}(t)$ and the second order inhomogenous differential equation is different. Particular solutions to the homogeneous version must be found first. The homogenous version of the second equation is

$$\ddot{y}(t) + 2k_h\dot{y}(t) + 2.467y(t) = 0, \tag{A.16}$$

which has a general exponential solution $y(t) = e^{\lambda t}$. The characteristic polynomial of (A.16) is

$$\lambda^2 + 2k_h\lambda + 2.467 = 0 \tag{A.17}$$

with $p = 2k_h$ and $q = 2.467$. If the roots of the characteristic polynomial are real or imaginary is determined by the term $p^2 - 4q = 4k_h^2 - 9.868$. The choice of $k_h$ determines the general form of a solution to (A.16). It will not be analysed any further.