# Stability of Time-delayed Systems

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# 1 Introduction

A time-delayed system (TDS) is a dynamical system where the output of the system depends delayed inputs. An example would be the voting population in a country which depends on the general population dynamics and a delay of 18 years. The study of TDSs has attracted a great amount of research, for example [1], [2] and [3].

An important aspect of a TDS is stability. Stability of dynamical systems is usually analysed via the characteristic function and the roots of this function [4]. A system is then said to be stable if for each root, the real part of that root is negative. This stability assessment is complicated in a TDS since the characteristic function of such a system has infinitely many roots and it is not straightforward to assess for each of these roots that its real part is negative.

Lev Pontryagin has published a standard work on precisely this problem in 1942 [5]. Since then his work is often cited, for example in [6] and [7]. The mathematical writing standard back in 1942 differs from today's standard and his work often lacks the details necessary to fully understand the methods he introduced. This report hopes to fill in the gaps in this piece of standard literature.

Pontryagin considers only exponential polynomials and their properties. In chapter 2 we discuss the theory behind TDSs and how their characteristic functions relate to such exponential polynomials. In particular we shall see that an exponential polynomial can have a *principal term*. In chapter 3 we relate the lack of a principal term in an exponential polynomial to the existence of infinite roots with positive real part. In chapter 4 we discuss a technical theorem on the number of roots of an exponential polynomial in a strip in the complex plane and in chapter 5 we tie everything together to prove Pontryagin's main theorem.

This theorem gives straightforward and easily verifiable conditions on the stability of a TDS. As Pontryagin's theorems are often cited, a critical assessment is warranted. It is our aim to provide the reader with an understanding of Pontryagin's work, his methods and their limitations.

# Contents




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## 2 TDSs and Exponential Polynomials

In this section we lay out the groundwork for the context of Pontryagin's theorems. We will relate time-delayed systems (TDS) and *exponential polynomials* which we define below:

**Definition 2.1 (Exponential Polynomial)** *Let $H(z)$ be a function of the complex variable $z$. We call $H(z)$ an exponential polynomial if there exists a bivariate polynomial $h(s,t)$ such that:*

$$H(z) = h(z, e^z). \tag{1}$$

Now let us consider the following retarded TDS:

$$\begin{aligned} \dot{x}(t) &= a_0 x(t) + a_1 x(t-\tau_1) + \cdots + a_n x(t-\tau_n), \\ x(\theta) &= \psi(\theta), \quad \theta \in [-\tau_n, 0]. \end{aligned} \tag{2}$$

Here $x(t) \in \mathbb{R}$ denotes the state of the system, $a_0, \cdots, a_n$ are real constants and $\tau_n > \cdots > \tau_1 > 0$. With $\psi$ we denote a function that serves as an initial condition for the system such that the system has a unique solution. We will look at the characteristic equation of system (2) to analyse it:

$$H(s) = -s + a_0 + a_1 \cdot e^{-\tau_1 s} + ... + a_n \cdot e^{-\tau_n s} = 0. \tag{3}$$

It is well known that system (2) is stable if and only if $\Re(s) < 0$ for all $s$ that solve equation (3). We can immediately see that $H(s)$ in (3) is not an exponential polynomial, writing:

$$H(s) = h(s, e^s) \quad \text{with} \quad h(s,t) = -s + a_0 + a_1 \cdot t^{-\tau_1} + ... + a_n \cdot t^{-\tau_n}.$$

A function with negative exponents cannot be a polynomial. In Lemma 2.3 we will prove that the zeros of this characteristic equation are via a scaling equal to the zeros of a corresponding exponential polynomial. Let us take a look at a multidimensional TDS which has the following general form:

$$\sum_{i=0}^{M} A_i y'(t-\tau_i) + \sum_{i=0}^{M} B_i y(t-\tau_i) = 0, \quad y(\theta) = \psi(\theta), \quad \theta \in [-\tau_M, 0]. \tag{4}$$

Here in similar fashion to (2) $y(t)$ denotes the $N$-dimensional state of the system, $(y_1(t), \cdots, y_N(t)), y_i : \mathbb{R} \to \mathbb{C}$. Again $\psi$ serves as an $N$-dimensional initial condition, $\tau_n > ... > \tau_1 > \tau_0 = 0$ and $A_i, B_i \in \mathbb{R}^{N \times N}$. Following the work of [7], it can be determined that zeros of the characteristic function (denoted as $H(s)$) of the $N$-dimensional TDS in (4) are given by:

$$H(s) = \det \left( \sum_{i=0}^{M} (A_i s + B_i) \, e^{-\tau_i s} \right) = 0. \tag{5}$$

**Example 2.2** *An often studied example [8] [9] is the following realisation of system* (4)*:*

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -2 & 0 \\ 0 & \frac{-9}{10} \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad (6)$$

*with* $\tau_0 = 0$ *and* $\tau_1 > 0$*. The characteristic function of this system is:*

$$\det\left(\begin{bmatrix} s - 2 - e^{-\tau_1 s} & 0 \\ -e^{-\tau_1 s} & s - \frac{9}{10} - e^{-\tau_1 s} \end{bmatrix}\right) = \\ s^2 - \frac{29s}{10} - 2se^{-\tau_1 s} + \frac{9}{5} + \frac{29}{10}e^{-\tau_1 s} + e^{-2\tau_1 s}.$$

Now we will prove the lemma stating that, under assumptions that the delays are rational numbers, the zeros of the characteristic equation of an $N$-dimensional TDS as in (5) are a scaled version of the zeros of a corresponding exponential polynomial.

**Lemma 2.3** *Let* $H(s)$ *be given in* (5)*. If all* $\tau_i \in \mathbb{Q}$*, then there exists an exponential polynomial* $h(s, e^s)$ *and a positive* $c \in \mathbb{R}$ *such that* $e^{N\tau_M s}H(s) = h(cs, e^{cs})$*. In particular the zeros of the characteristic function are equal to the zeros of an exponential polynomial, up to the scaling factor* $c$*.*

**Proof:** Let

$$G(s) := e^{\tau_M s}\left(\sum_{i=0}^{M} (A_i s + B_i)\, e^{-\tau_i s}\right). \quad (7)$$

Furthermore recall that $\tau_M > \tau_i$ for all $0 \leq i < M$. Bellman and Cooke on page 394-395 [7] find that $\det(G(s))$ has the following form:

$$g(s) = \det(G(s)) = e^{N\tau_M} H(s) = \sum_{j=0}^{K} p_j(s) e^{\beta_j s}, \quad (8)$$

where each number $\beta_j > 0$ is a linear combination of $\tau_0, ..., \tau_M$. Furthermore $K \in \mathbb{N}$ depending on $N$ and $M$ is the number of different $\beta_j$ with $\beta_K = N\tau_M$. In addition each function $p_j : \mathbb{C} \to \mathbb{C}$ is a polynomial in $s$ of degree at most $N$.

Now since $\tau_i \in \mathbb{Q}$, $0 \leq i \leq M$, the linear combination $\beta_j \in \mathbb{Q}$ as well. We write:

$$\beta_j = \frac{d_j}{e_j}, \quad (9)$$

with $d_j, e_j \in \mathbb{N}$, since we know that $\beta_j > 0$. Now we simply multiply each $\beta_j$ with a factor such that they all have the same denominator:

$$\beta_j = \frac{d_j \prod_{i \neq j}^{K} e_i}{\prod_{i=0}^{K} e_i}. \tag{10}$$

We define:

$$\epsilon := \prod_{i=0}^{K} e_i, \quad z := \frac{s}{\epsilon}, \quad c_j := d_j \prod_{i \neq j}^{K} e_j. \tag{11}$$

With this notation we have the following function:

$$\begin{aligned} g(s) &= \sum_{j=0}^{K} p_j(s) e^{\beta_j s} = \sum_{j=0}^{K} p_j(\epsilon z) e^{c_j z} \\ &= \sum_{j=0}^{K} \tilde{p}_j(z) e^{c_j z} := \tilde{g}(z). \end{aligned} \tag{12}$$

We see that the components $\tilde{p}_j(z)$ are still polynomials, since for a polynomial $f(x)$ we know that $f(cx)$ is still a polynomial for any $c \in \mathbb{R}$. Moreover $c_j \in \mathbb{N}$. So we see that $\tilde{g}(z)$ is an exponential polynomial with zeros denoted $z_0$. Furthermore using (12),(8) and (7) each zero $s_0$ of $H(s)$ is related to a $z_0$ as:

$$s_0 = z_0 \epsilon. \tag{13}$$

We put $c = \frac{1}{\epsilon}$ our scaling factor. This proves the lemma.

**Example 2.4** *Let us revisit system* (6) *with* $\tau_1 = \frac{3}{7}$. *We multiply the characteristic equation by* $e^{2\tau_1 s}$ *and substitute* $z = \frac{s}{7}$ *to obtain the following exponential polynomial:*

$$g(z) = \frac{1}{49} z^2 e^{6z} - \frac{29}{10} z e^{6z} - \frac{2}{7} z e^{3z} + \frac{9}{5} e^{6z} + \frac{29}{10} e^{3z} + 1. \tag{14}$$

We will now try to find conditions on an exponential polynomial such that all zeros are on the left side of the complex plane. For this we need the important notion of a *principal term*:

**Definition 2.5 (Principle Term)** *Let* $h(s,t)$ *denote a bivariate polynomial in the variables* $s$ *and* $t$ *of bidegree* $(D_1, D_2)$ *[see A.1]. We write the polynomial* $h(s,t)$ *in the following form:*

$$h(s,t) = \sum_{n,m=0}^{D_1, D_2} a_{n,m} s^n t^m. \tag{15}$$

*The principle term of the polynomial $h(s,t)$ is defined as the term $a_{D_1,D_2}s^{D_1}t^{D_2}$, provided $a_{D_1,D_2} \neq 0$. If the latter does not hold, then $h(s,t)$ is said to have no principal term.*

**Example 2.6** *The polynomial*

$$h_1(s,t) = s^2t^2 + 2$$

*has the principle term: $s^2t^2$, whereas the polynomial*

$$h_2(s,t) = s^2 + t^2 + 2$$

*does not have a principal term. The required term $a_{2,2}s^2t^2$ has $a_{2,2} = 0$.*

Note that we used a bivariate polynomial to define the *principal term*, but the definition can be extended to any multivariate polynomial:

**Definition 2.7 (Multivariate Principal Term)** *Let $f(x_1, \cdots, x_n)$ be a polynomial of multidegree $(D_1, \cdots, D_n)$ with $n > 2$. The principal term is given by $a_{(D_1,\cdots,D_N)}x_1^{D_1} \cdots x_N^{D_N}$ provided $a_{(D_1,\cdots,D_N)} \neq 0$.*

In the text we sometimes consider a principal term on polynomials of more than 2 variables.

Next follows the main theorem of this section that gives a condition when the equivalent exponential polynomial of a multidimensional TDS has a principal term:

**Theorem 2.8** *Consider the TDS in (4) with characteristic function $g(s)$. Let $\tilde{g}(z)$ be the exponential polynomial from Lemma 2.3 which satisfies $e^{N\tau_M z}H(z) = \tilde{g}(z)$. We write $\tilde{g}(z)$ as:*

$$\tilde{g}(z) = \sum_{j=0}^{K} \tilde{p}_j(z)e^{c_j z}, \tag{16}$$

*where $\tilde{p}_j : \mathbb{C} \to \mathbb{C}$ are polynomials and $c_j \in \mathbb{N}$ with $c_0 < c_1 < \cdots < c_K$ If*

$$\det(A_0) \neq 0, \tag{17}$$

*then the exponential polynomial $\tilde{g}(z)$ has a principal term. Moreover if*

$$\det(A_0) = 0 \quad \text{and} \quad \det(A_M) \neq 0, \tag{18}$$

*then the exponential polynomial $\tilde{g}(z)$ has no principal term.*

**Proof:** By (5) and the fact that the matrices are $N \times N$ we have:

$$e^{N\tau_M s} H(s) = \det\left(\sum_{i=0}^{N} (A_i s + B_i)\, e^{(\tau_M - \tau_i)s)}\right). \tag{19}$$

Since $0 = \tau_0 < \cdots < \tau_M$ Bellman and Cooke [7] conclude that, $c_0 = 0$ and

$$\tilde{p}_0(z) = \det(A_M z + B_M), \quad \tilde{p}_K(z) = \det(A_0 z + B_0). \tag{20}$$

Subsequently they find that $\tilde{p}_K(z)$ is of degree $N$ when condition (17) holds. This can be proven by noticing:

$$\det\left(A_0 z + B_0\right) = z^N \det\left(A_0 + \frac{B_0}{z}\right), \tag{21}$$

since $A_0, B_0 \in \mathbb{R}^{N\times N}$. Multiplying by $z^{-N}$ and letting $z \to \infty$ we see that:

$$\lim_{z\to\infty} z^{-N}\tilde{p}_K(z) = \lim_{z\to\infty} \det\left(A_0 + \frac{B_0}{z}\right) = \det(A_0). \tag{22}$$

Therefore condition (17) holds. As we have seen $\tilde{p}_0(z) = \det\left(A_M z + B_M\right)$. A similar argument as in (22) shows that this polynomial is of degree $N$ if and only if $\det\left(A_M\right) \neq 0$. So under condition (18) $\tilde{p}_0(z)$ is of degree $N$ whereas $\tilde{p}_K(z)$ is of lower degree. Note that since $A_0$ and $B_0$ are not both zero, the exponential term is given by $e^{c_K z}$. Hence there is no principal term.

**Example 2.9** *Let us once again revisit system* (6)*. Since* $\det(A_0) \neq 0$*, we have by Theorem 2.8 that the exponential polynomial has a principal term. If* $\tau_1 = \frac{3}{7}$*, we see from* (14) *that the principal term is given by:* $\frac{1}{49}z^2 e^{6z}$.

Of course we have to explain why we need the notion of a *principal term* at all. The first theorem of Pontryagin in this report will discuss what happens when an exponential polynomial does not have a principal term.

# 3 Exponential Polynomials with Infinite Zeros

In this chapter we will see that if there is no principle term, there is an unbounded number of zeros with arbitrarily large real part. In both Pontryagin's paper [5] and in Bellman and Cooke [7] the following example is provided to give guidance for the general proof for this claim:

**Example 3.1** *We consider the following exponential polynomial and try to find the zeros for it:*

$$e^z - z = 0. \tag{23}$$

*Choosing* $z = x + iy$*, we can split this equation into two:*

$$e^x \cos y = x \tag{24}$$

*and*

$$e^x \sin y = y. \tag{25}$$

*Since we are trying to prove that there are infinitely many zeros with large real part, let's look for cases where* $x, y$ *are very large and positive. The first of these two equations reduces to* $\cos y = xe^{-x} \approx 0$ *for such* $x$ *and therefore we look at* $y$*'s close to the zeros of the cosine:* $y = k\pi + \frac{\pi}{2} + \delta_1, k \in \mathbb{N}$*, where* $\delta_1$ *is a small number. However since* $e^x > 0, y > 0$*, by* (25) *we see that* $\sin(y) > 0$ *and so we restrict our* $'y's$ *to* $y = 2k\pi + \frac{\pi}{2} + \delta_1$*, otherwise* $\sin(y)$ *would be approximately* $-1$*. Substituting this* $y$ *in* (25) *we get:*

$$e^x \sin(2k\pi + \frac{\pi}{2} + \delta_1) = 2k\pi + \frac{\pi}{2} + \delta_1.$$

*Since* $\sin(2k\pi + \frac{\pi}{2} + \delta_1) = \mu \approx 1$ *we have:*

$$\begin{aligned} e^x &= \mu^{-1}(2k\pi + \frac{\pi}{2} + \delta_1) \iff \\ x &= \ln(\mu^{-1}(2k\pi + \frac{\pi}{2} + \delta_1)) \iff \\ x &= \ln(2k\pi + \frac{\pi}{2} + \delta_1) - \ln(\mu) \iff \\ x &= \ln(2k\pi + \frac{\pi}{2} + \delta_1) - \delta_2, \end{aligned}$$

*where* $\delta_1, \delta_2 \approx 0$ *for large* $x, y$ *where* $k \gg 1$*. This example shows a direction to find the zeros of* $e^z - z$ *with large positive real and imaginary part. These expressions for* $x$ *and* $y$ *are the general form of the zeros where Pontryagin starts searching.*

To prove the existence of zeros with large real part for any exponential polynomial without principal term, we will look for zeros of similar form to $x, y$ that we found in this example. We will start by introducing the theorem that generalizes the claim that $e^z - z$ has roots with large real parts to any exponential polynomial without principal term. To prove this theorem we shall prove three necessary lemmas.

**Theorem 3.2 (Theorem 1 in Pontryagin [5])** *Let the bivariate polynomial $h(s,t)$ of bidegree of $(D_1, D_2)$ be written in the form of* (15)*. If $h(s,t)$ has no principal term, then the function $H(z) := h(z, e^z)$ has an unbounded number of zeros with an arbitrarily large real part.*

**Proof:** As we have seen in Example 3.1, the zeros might be of the form:

$$z_k = \alpha \ln(2k\pi) + 2k\pi i + \ln(\theta) + \zeta, \tag{26}$$

with $\alpha \in \mathbb{R}^+$, $\theta \in \mathbb{C}$ and with $\zeta \in \mathbb{C}$.
Note that $\zeta$ represents the $\delta$-quantities from example 3.1, This proof will consist of proving three lemmas. The first lemma proves the existence of $\alpha$ in (26) for exponential polynomials without a principal term, the second lemma shows the existence of $\theta \neq 0$ in (26) and the third lemma states an existence result of the zeros such that

$$\tilde{z}_k = \alpha \ln(2k\pi) + 2k\pi i + \ln(\theta) \tag{27}$$

only lies a small $\zeta$ from the true zeros $z_k$ (26).

Before we state the first lemma we perform some algebraic manipulation on $H(z)$: From this point we use $(D_1, D_2) = (N, M)$ for reasons of readability. If we substitute (27) in the exponential polynomial $H(z) = h(z, e^z)$, we get:

$$\begin{aligned}
H(\tilde{z}_k) &= \sum_{n,m=0}^{N,M} a_{n,m} \tilde{z}_k^m e^{n\tilde{z}_k}, \qquad (28)\\
&= \sum_{n,m=0}^{N,M} a_{n,m} (\alpha \ln(2k\pi) + 2k\pi i + \ln(\theta))^m e^{n\cdot(\alpha \ln(2k\pi) + 2k\pi i + \ln(\theta))},\\
&= \sum_{n,m=0}^{N,M} a_{n,m} (\alpha \ln(2k\pi) + 2k\pi i + \ln(\theta))^m (2k\pi)^{n\alpha} \theta^n,\\
&= \sum_{n,m=0}^{N,M} a_{n,m} (2k\pi)^{m+\alpha n} i^m \theta^n \left( \frac{\alpha}{2\pi k i} \ln(2k\pi) + 1 + \frac{\ln(\theta)}{2\pi k i} \right)^m. \qquad (29)
\end{aligned}$$

Next we define:

$$\delta_1(k) := \frac{\alpha}{2\pi ki}\ln(2k\pi) + \frac{\ln(\theta)}{2\pi ki}. \tag{30}$$

We see that $\lim_{k\to\infty}\delta_1(k) = 0$ and that the convergence rate of $\delta_1$ is $\ln(k)/k$. Using (29) and (30) $H(\tilde{z}_k)$ becomes:

$$H(\tilde{z}_k) = \sum_{n,m=0}^{N,M}(2k\pi)^{m+n\alpha}a_{n,m}i^m\theta^n(1+\delta_1(k))^m. \tag{31}$$

Next we combine several terms:

$$a_{k,n,m} = a_{n,m}i^m(1+\delta_1(k))^m, \tag{32}$$

where we see that $a_{k,n,m} \neq 0 \iff a_{n,m} \neq 0$. We obtain the following expression:

$$H(\tilde{z}_k) = \sum_{n,m=0}^{N,M}(2k\pi)^{m+n\alpha}a_{k,n,m}\theta^n := H(\alpha,\theta,k). \tag{33}$$

We will now find a suitable $\alpha$ and $\theta$. We first define:

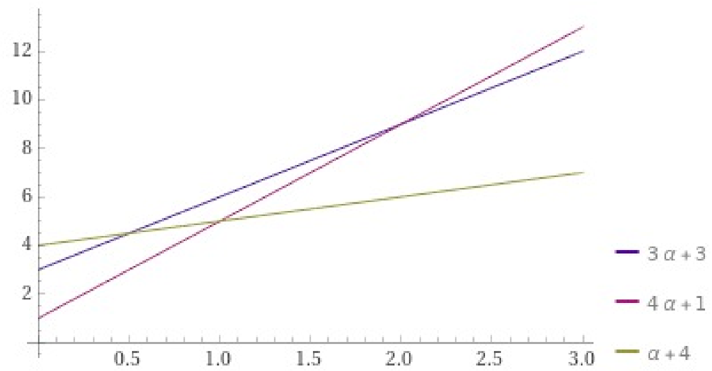
$$\beta(\alpha) := \max_{(n,m)\in\mathfrak{b}}(m+n\alpha), \tag{34}$$

where the set $\mathfrak{b}$ is defined as:

$$\mathfrak{b} = \left\{n,m\in\mathbb{N}^2 \mid 0\leq n\leq N, 0\leq m\leq M, a_{n,m}\neq 0\right\}. \tag{35}$$

Here $a_{n,m}$ are the coefficients of $H$. The number $\beta$ is easy to find for given $\alpha$, as is shown in the following example.

**Example 3.3** *Let* $H(z) = z^3e^{3z} + ze^{4z} + z^4e^z$*. From this we see that* $\mathfrak{b} = \{(3,3),(4,1),(1,4)\}$*. Let us sketch the lines* $m+n\alpha$ *for the pairs (3,3),(1,4) and (4,1):*




$\alpha \rightarrow$

*The maximum $\beta(\alpha)$ is the ceiling drawn by the three lines. It follows the green line in the interval $(0,\frac{1}{2})$, the blue line in $(\frac{1}{2},2)$ and the red line onwards.*

Now we are ready to state the following lemma:

**Lemma 3.4** *Given a function $H$ in the form (33) and $\mathfrak{b}$ as in (35) there exists an $\alpha \in \mathbb{R}^+$ such that equation (33) can be written in the following form:*

$$H(\alpha,\theta,k) = (2k\pi)^{\beta(\alpha)} \sum_{n=0}^{N} \theta^n \left( \sum_{m:m+\alpha n=\beta(\alpha)} a_{k,n,m} + \delta_2(k,n,\alpha) \right), \qquad (36)$$

*where $a_{k,n,m} \neq 0$ (32) for at least two different pairs $(n,m)$ satisfying $m + \alpha n = \beta(\alpha)$. Furthermore, $\lim_{k\to\infty} \delta_2(k,n,\alpha) = 0$. This $\alpha$ will be used in (26).*

**Proof:** Consider the function $H(\alpha,\theta,k)$ from equation (33).

$$\begin{aligned} H(\alpha,\theta,k) &= \sum_{m,n}^{N,M} (2k\pi)^{m+n\alpha} a_{k,n,m} \theta^n \\ &= \sum_{n=0}^{N} \theta^n \left( \sum_{m=0}^{M} (2k\pi)^{m+n\alpha} a_{k,n,m} \right). \end{aligned}$$

Now we split the sum over $m$ into two parts. For fixed $n$, a part with those $m$'s such that $\beta(\alpha) = \max_{\mathfrak{b}} (m + n\alpha)$ and a part with $m + \alpha n < \beta(\alpha)$:

$$\begin{aligned} H(\alpha,\theta,k) = \sum_{n=0}^{N} \theta^n \Bigg( & \sum_{m:m+\alpha n=\beta(\alpha)} (2k\pi)^{\beta(\alpha)} a_{k,n,m} \\ & + \sum_{m:m+\alpha n<\beta(\alpha)} (2k\pi)^{m+n\alpha} a_{k,n,m} \Bigg). \end{aligned} \qquad (37)$$

To find the zeros of this function we take out the term $(2k\pi)^{\beta(\alpha)}$:

$$\begin{aligned} H(\alpha,\theta,k) = (2k\pi)^{\beta(\alpha)} \sum_{n=0}^{N} \theta^n \Bigg( & \sum_{m:m+\alpha n=\beta(\alpha)} a_{k,n,m} \\ & + \sum_{m:m+\alpha n<\beta(\alpha)} (2k\pi)^{m+n\alpha-\beta(\alpha)} a_{k,n,m} \Bigg). \end{aligned} \qquad (38)$$

Now when $m + n\alpha < \beta(\alpha)$, there holds:

$$\lim_{k\to\infty} (2k\pi)^{m+n\alpha-\beta(\alpha)} = 0. \tag{39}$$

Therefore we write:

$$\delta_2(k, n, \alpha) = \sum_{m:m+\alpha n<\beta(\alpha)} (2k\pi)^{m+n\alpha-\beta(\alpha)} a_{n,m} i^m, \tag{40}$$

and we conclude from (39), (32) and (30) that:

$$\lim_{k\to\infty} \delta_2(k, n, \alpha) = 0. \tag{41}$$

We still need to prove that $a_{k,n,m} \neq 0$ for at least 2 different pairs $(n, m)$ when we choose $\alpha$ properly, by (32) this is equivalent to proving that $a_{n,m} \neq 0$ for those $(n, m)$ for certain $\alpha$. Looking back at Example 3.3 we see that this happens for $\alpha = \frac{1}{2}$ or $\alpha = 2$.

Assume $h(z, t)$ has a principal term, where $h(z, e^z) = H(z)$. Then for every $\alpha \in \mathbb{R}$, $\beta(\alpha) = M + N\alpha$. Here $(N, M)$ is the only pair such that $a_{N,M} \neq 0$. Thus if we want that $\beta(\alpha)$ is attained for more one pair $h(z, t)$ needs to have no principal term. Consider the finite set $\mathfrak{b} = \{(n_1, m_1), (n_2, m_2), \cdots\}$ where $\max_i n_i = n_1 = N$. Since $h(z, t)$ has no principal term, there exists a term $(n_2, m_2)$ without loss of generality such that $N = n_1 > n_2$ for all $i \neq 1, 2$ but $m_2 > m_1$. Assume that $n_2 > n_i$ for all $i \neq 1, 2$ where $m_2 > m_i$. Then for large $\alpha$ the term $\beta(\alpha)$ is given by $m_1 + N\alpha$. If we decrease $\alpha$ however, there comes a point where:

$$\beta(\alpha) = m_1 + N\alpha = m_2 + n_2\alpha. \tag{42}$$

This $\alpha$ is positive as $N > n_1$, $m_2 > m_1$ and:

$$\alpha = \frac{m_2 - m_1}{N - n_2}. \tag{43}$$

As both pairs $(N, m_1), (n_2, m_2) \in \mathfrak{b}$ we have $a_{N,m_1} \neq 0$ and $a_{n_2,m_2} \neq 0$. This proves the lemma.

**Remark:** Note that the construction in the proof may not be the only suitable $\alpha$ as we could have alternatively used the minima for the pairs $(n, m)$. The $\alpha$ derived suffices however. Also note that there might be more than two pairs that reach the same $\beta(\alpha)$ for some $\alpha$. Let the exponential polynomial $G(z) = e^{2z} + ze^z + z^2$. Then we have three pairs $(2, 0), (1, 1), (0, 2)$ which all reach $\beta(\alpha) = 2$ for $\alpha = 1$.

Next we will determine $\theta$ in (26).

**Lemma 3.5** *Let $\alpha$ be determined in Lemma 3.4. Let $b_n := a_{n,m}$ for any fixed pair $(n, m)$ such that $\beta(\alpha) = m + n\alpha$ with $a_{n,m}$ the coefficients of the function $H(z)$ (28). Then there exists a $\Theta \neq 0 \in \mathbb{C}$ where $\Theta$ is a zero of the polynomial $K$:*

$$K(\theta) := \sum_{n=0}^{N} b_n \theta^n. \tag{44}$$

*Subsequently we use $\alpha$ and $\Theta = \theta$ in (26).*

**Proof:** Let us restate the following form (36) for $H(\alpha, \theta, k)$ from Lemma 3.4:

$$H(\alpha, \theta, k) = (2k\pi)^{\beta(\alpha)} \sum_{n=0}^{N} \theta^n \left( \sum_{m:m+\alpha n=\beta(\alpha)} a_{k,n,m} + \delta_2(k, n, \alpha) \right). \tag{45}$$

We define:

$$\tilde{b}_{k,n,\alpha} := a_{k,n,m} + \delta_2(k, n, \alpha). \tag{46}$$

We substitute this in (36), resulting in the polynomial:

$$\tilde{K}(\theta) := (2k\pi)^{\beta(\alpha)} \sum_{n=0}^{N} \tilde{b}_{k,n,\alpha} \theta^n. \tag{47}$$

By (41) and (32) we see that $\delta_2(k, n, \alpha) \to 0$ and $a_{k,n,m} \to a_{n,m}$. So we see that the polynomial $\sum_{n=0}^{N} \tilde{b}_{k,n,\alpha} \theta^n$ converges to a different polynomial $K(\theta)$:

$$K(\theta) := \sum_{n=0}^{N} b_n \theta^n. \tag{48}$$

For the given $\alpha, \beta(\alpha)$ and $n$ only one $m$ exists such that $m + \alpha n = \beta(\alpha)$ so we have $b_n = a_{n,m}$ and $b_{k,n,\alpha} \to b_n$. This polynomial $K$ has at most $N$ roots by the Fundamental Theorem of Algebra A.2, let us denote one such root with $\Theta$. To prove that $\Theta \neq 0$ we note that by Lemma 3.4 for our choice of $\alpha$ there are at least two different pairs $(n, m)$ satisfying $m + \alpha n = \beta(\alpha)$. This means there are at least two non-zero coefficients $b_n = a_{n,m}$, which proves we always have at least one root $\Theta \neq 0$. By construction we see that $H(\alpha, \Theta, k) \to 0$ for sufficiently large $k$. Therefore we use $\Theta = \theta$ in (26). This proves Lemma 3.5.

**Remark:** Recall that we are looking for zeros of the form (26). So far we have proven the existence of a suitable $\alpha$ and $\Theta$. However the existence of $\zeta$ in (26) poses a problem in the original paper as Pontryagin states that $\zeta = \zeta \to 0$ as $\frac{1}{k}$ but does not provide a proof. In fact this rate $\frac{1}{k}$ seems unfounded. Due to the different approximations it is hard to put a maximally allowed convergence rate on $\zeta$. In fact we will have to show a weaker version: that there exists a boundary $\zeta$ such that the real zero (26) lies within the bound $\zeta$ of (27).

The question of the existence of $\zeta$ can phrased differently: given a suitable $\alpha$ and $\Theta$ that solve (48), we need to prove that the roots of $H(z)$ lie within a bound of $\|\zeta\|$ from $\tilde{z}_k$ (27).

**Lemma 3.6** *Let $\alpha$ be determined from Lemma 3.4 and $\theta$ be determined from Lemma 3.5. There is a constant number $\zeta$ such that some zeros with large real part of $H$ are given by* (26) *for sufficiently high $k$.*

**Proof:** We are going to use the Theorem of Rouché A.4. We define a family of curves $\mathfrak{C}_k$ with radius $\|\zeta\|$ centering around $\tilde{z}_k$. To consider $H$ on the curves $\mathfrak{C}_k$, let us evaluate $H(z_k)$ instead of $H(\tilde{z}_k)$. We define $\delta_3(k, \alpha, \Theta, \zeta)$:

$$\begin{aligned} \delta_3(k, \alpha, \Theta, \zeta) &:= \sum_{n=0}^{N} \Theta^n e^{\zeta} \delta_2(k, n, \alpha) \\ &= \sum_{n=0}^{N} \Theta^n e^{\zeta} \left( \sum_{m: m+\alpha n < \beta(\alpha)} (2k\pi)^{m+n\alpha-\beta(\alpha)} a_{n,m} i^m \right), \end{aligned} \tag{49}$$

where we used for $\delta_2$ the definition in (40). We substitute $\Theta$ from Lemma 3.5, add the contribution of $\zeta$ and this expression for $\delta_3$ in (36) to obtain:

$$H(\alpha, \Theta, k, \zeta) = (2k\pi)^{\beta(\alpha)} \left( \sum_{n=0}^{N} \Theta^n e^{\zeta} \left( \sum_{m: m+\alpha n = \beta(\alpha)} a_{k,n,m} \right) + \delta_3(k, \alpha, \Theta) \right). \tag{50}$$

We now have two analytic functions on $\mathfrak{C}_k$:

$$\begin{aligned} N(\alpha, \Theta, k, \zeta) &:= (2k\pi)^{\beta(\alpha)} \left( \sum_{n=0}^{N} \Theta^n e^{\zeta} \left( \sum_{m: m+\alpha n = \beta(\alpha)} a_{k,n,m} \right) \right), \\ M(\alpha, \Theta, k, \zeta) &:= (2k\pi)^{\beta(\alpha)} \delta_3(k, \alpha, \Theta, \zeta) \end{aligned} \tag{51}$$

To invoke Rouché's Theorem we have to find out if the following is true:

$$|M| < |N| \quad \text{along } \mathfrak{C}_k. \tag{52}$$

Since $\delta_3 \to 0$ as $k \to \infty$, there is always some $K \in \mathbb{N}$ such that for all $k \geq K$ condition (52) holds!

Note that $\|\zeta\|$ can be arbitrarily small but not 0 as for $\zeta = 0$ we have that $N(\alpha, \Theta, k, 0) = 0$ for very large $k$.

Therefore for an exponential polynomial $H(z)$ where $h(z, t)$ does not have a principal term, there exist zeros in the form:

$$z_k = \alpha \ln(2k\pi) + 2k\pi i + \ln(\Theta) + \zeta, \tag{53}$$

for high enough $k$. Since $\alpha$ is positive, $\Theta$ is constant and $\zeta \in \mathbb{C}$, these zeros have positive real part. This proves Theorem 3.2.

Note that this theorem does not allow us to precisely calculate the zeros of arbitrary exponential polynomials with no principal term. It is limited to showing the existence of infinite roots with positive real term for these exponential polynomials. A direct corollary to this theorem can also be given:

**Corollary 3.7** *Consider the TDS in* (4)*. Let the realisation of this system be such that both conditions in* (18) *hold. Then the system is unstable.*

**Proof:** That system has no principal term and by Theorem 3.2 the corresponding exponential polynomial to that system's characteristic equation has infinite zeros $z_0$ with $\Re(z_0) > 0$. Therefore it is unstable.

# 4 Exponential Polynomials with Finite Zeros

In the previous chapter we found that exponential polynomials without a principal term have infinite many zeros with positive real part. The corresponding time-delayed systems (TDS) are therefore unstable. In this section we will look at exponential polynomials with a principal term. Theorem 4.4 asserts a finite amounts of zeros for a polynomial with a principal term within a certain strip in the complex plane. We will need the following definition of a homogeneous function:

**Definition 4.1 (Homogeneous Function of degree $D$)** *A function $f : \mathbb{C}^n \to \mathbb{C}$ is called homogeneous of degree $D \in \mathbb{N}$ if:*

$$f(\lambda x_1, ..., \lambda x_n) = \lambda^D f(x_1, ..., x_n), \tag{54}$$

*for all $\lambda \in \mathbb{C}$.*

**Example 4.2** *A polynomial $h(s,t) = s^2t^3$ is homogeneous of degree 5 as $h(\lambda s, \lambda t) = \lambda^5 h(s,t)$. A polynomial $g(s,t) = s^2t^3 + s^2$ is not homogeneous of degree 5. In fact it is not homogeneous of any degree. We can see that by assuming there exist some $D \in \mathbb{R}$ such that $g(\lambda s, \lambda t) = \lambda^D g(s,t)$ for all $\lambda \in \mathbb{C}$. Thus we have to solve for $D$:*

$$\lambda^5 s^2 t^3 + \lambda^2 s^2 = \lambda^D \left(s^2t^3 + s^2\right). \tag{55}$$

*We see that no such $D$ exists for all $\lambda \in \mathbb{C}$.*

**Lemma 4.3** *A multivariate monomial with degree $(D_1, \cdots, D_n)$ is defined as:*

$$M(x_1, ..., x_n) := x_1^{D_1} \cdot \cdots \cdot x_n^{D_n}. \tag{56}$$

*If a multivariate polynomial $p(x_1, \cdots, x_n)$ of degree $(D_1, \cdots, D_n)$ is homogeneous of degree $D$, then it is a sum of monomials such that the sum of the degrees of each monomial is $D$.*

**Proof:** A polynomial is a sum of monomials where a monomial is a polynomial with a single term. Since $p(\lambda x_1, \cdots, \lambda x_n) = \lambda^D p(x_1, \cdots, x_n)$ we can see that the polynomial is a sum of monomials where the sum of the degrees of each variable adds up the same number:

$$p(x_1, \cdots, x_n) = \sum_i^N a_i M_i(x_1, \cdots, x_n), \tag{57}$$

where $N \in \mathbb{N}, a_i \in \mathbb{C}$ and where for each $M_i$ we have $\sum_j^n D_j = \tilde{D} \in \mathbb{N}$. Now we immediately see that we have $\tilde{D} = D$. In particular for a bivariate polynomial $f(x, y)$ homogeneous of degree $D$, it means that we have the following general form:

$$f(x,y) = \sum_{i=0}^{D} a_i x^i y^{D-i}, \tag{58}$$

with $a_i \in \mathbb{C}$.
The notion of homogeneous polynomials will be important in the statement of the following theorem and its proof.

**Theorem 4.4 (Theorem 3 in Pontryagin [5])** *Let the polynomial $f(z, u, v)$ be written as:*

$$f(z,u,v) = \sum_{m,n=0}^{r,s} z^m \phi_m^n(u,v), \tag{59}$$

*where $r, s \in \mathbb{N}$ and $\phi_m^n$ is a polynomial of bidegree $(N, M)$ such that $n = N + M$ with real coefficients. Let us denote the $\phi$-coefficient of $z^r$ by*

$$\phi_*^s(u,v) = \sum_{n=0}^{s} \phi_r^n(u,v). \tag{60}$$

*Now we let*

$$\Phi_*^s(z) := \phi_*^s(\sin z, \cos z). \tag{61}$$

*Assume $\phi_m^n$ is homogeneous of degree $n$. Assume that $\phi_m^n(u, v)$ is not divisible by $u^2 + v^2$ for all $n, m$ and that $\phi_r^s(u, v) \neq 0$ for some $u, v$. If $\epsilon \in \mathbb{R}$ is such that $\Phi_*^s(\epsilon + iy)$ does not take the value 0 for a real $y$, then in the strip $-2k\pi + \epsilon \leq x < 2k\pi + \epsilon, z = x + iy$, the function $f(z, \sin z, \cos z)$ will have, for all sufficiently large values of $k \in \mathbb{N}$, exactly $4ks+r$ zeros. Thus, in order for $f(z, \sin z, \cos z)$ to only have real roots, it is necessary and sufficient that in the strip $-2k\pi + \epsilon \leq x < 2k\pi + \epsilon$ it has exactly $4ks + r$ real roots starting with sufficiently large $k$.*

**Proof:** We are going to find the number of zeros of $z^r \Phi_*^s(z)$ and compare that number to the zeros of $f(z, \sin(z), \cos(z))$. We use a Rouché-like argument to show the number of zeros is $4ks + r$ in a large rectangle depending on $k$. We will prove that, if the rectangle is large enough, that $f$ has exactly $4ks + r$ zeros inside this rectangle.

We are going to prove four lemmas. The proof of the theorem follows naturally as a result of combining the four lemmas.

Since in what follows it is assumed that $u = \cos(z), v = \sin(z)$, it is natural to assume that the polynomial $\phi_m^n(u, v)$ is not divisible by $u^2+v^2$. If $\phi_m^n(u, v)$ were divisible by $u^2 + v^2$, we could write it as:

$$\phi_m^n(u, v) = \hat{\phi}_m^n(u, v) \cdot \left(u^2 + v^2\right), \tag{62}$$

which becomes after substitution of $u = \cos(z), v = \sin(z)$:

$$\phi_m^n(\cos(z), \sin(z)) = \hat{\phi}_m^n(\cos(z), \sin(z)). \tag{63}$$

This assumption directly leads to the following lemma:

**Lemma 4.5** *Let the polynomial $\phi_m^n(u, v)$ be homogeneous of degree $n$ and of bidegree $(N, M)$. If and only if $\phi_m^n$ has no factor $u^2 + v^2$:*

$$\phi_m^n(1, \pm i) \neq 0. \tag{64}$$

**Proof:** The introduction to this lemma suffices as the proof that, given a factor $u^2 + v^2$, we have $\phi_m^n(1, \pm i) \neq 0$.
To prove the other direction we note that as $\phi_m^n(u, v)$ is of bidegree $(N, M)$ it is not constantly 0 for all $u, v$. Using the homogeneity of $\phi_m^n$ we write for $u \neq 0$:

$$\phi_m^n(u, v) = u^n \phi_m^n \left(1, \frac{v}{u}\right).$$

Next we assume that $\phi_m^n(1, \pm i) = 0$ and try to find a contradiction. Let $w = \frac{v}{u}$, then:

$$\begin{aligned}\phi_m^n(u, v) = u^n \phi_m^n \left(1, \frac{v}{u}\right) \\ = & u^n \phi_m^n (1, w) = u^n (w - i)(w + i) \tilde{\phi}_m^{n-2} (1, w),\end{aligned}$$

where we have used the assumption that $\phi_m^n(1, \pm i) = 0$. Here $\tilde{\phi}_m^{n-2}$ is a polynomial of degree $n - 2$ as we have factorized out the two roots in $i = 1, i = -1$. Since $w = \frac{v}{u}$, we get:

$$u^n (w - i)(w + i) \tilde{\phi}_m^{n-2} (1, w) = u^n \left(\frac{v^2}{u^2} + 1\right) \tilde{\phi}_m^{n-2} \left(1, \frac{v}{u}\right).$$

It follows that:

$$u^n \left(\frac{v^2}{u^2} + 1\right) \tilde{\phi}_m^{n-2} \left(1, \frac{v}{u}\right) = u^{n-2} \left(v^2 + u^2\right) \tilde{\phi}_m^{n-2} \left(1, \frac{v}{u}\right).$$

To show that $\phi_m^n(u,v)$ has a factor $(u^2+v^2)$ we need to show that the term $u^{n-2}\tilde{\phi}_m^{n-2}\left(1,\frac{v}{u}\right)$ is a polynomial. It is clear that $\tilde{\phi}_m^{n-2}$ is a polynomial in $w=\frac{v}{u}$ of degree $n-2$. After multiplication with $u^{n-2}$ we see that all exponents of the variables $u,v$ are a positive natural number. Therefore $\phi_m^n(u,v)$ has a factor $(v^2+u^2)$ assuming that $\phi_m^n(1,\pm i)=0$. This concludes the proof.

Theorem 4.4 assumes that $\phi_m^n(u,v)$ has no factor $(u^2+v^2)$ for all $n,m$. By Lemma 4.5 we conclude then that $\phi_m^n(1,\pm i)\neq 0$.

We will investigate the function $\Phi_*^s(z)$. We will use Lemma 4.5 to show that this function has exactly $4ks$ zeros in a strip $-2k\pi+a\leq x\leq 2k\pi+a$ for any $a\in\mathbb{R}$ and $k\in\mathbb{N}$.

**Lemma 4.6** *Let a polynomial $f(z,u,v)$ be represented in the form* (59)*. Denote the $\phi$-coefficient of $z^r$ by $\phi_*^s(u,v)$ as in* (60) *in similar fashion as Theorem 4.4. Then $\Phi_*^s(z)=\phi_*^s(\sin(z),\cos(z))$ has $4ks$ simple zeros on the strip $-2k\pi+a\leq x\leq 2k\pi+a$ for any $a\in\mathbb{R}$ and $k\in\mathbb{N}$.*

**Proof:** First we prove that on the strip $a\leq x\leq 2\pi+a, a\in\mathbb{R}$ the $\phi$-coefficient of $z^r$ (60) has exactly $2s$ zeros. To go from the polynomial $\phi_*^s(u,v)$ in two variables to $\Phi_*^s(z)$ in one variable, we first substitute for $u,v$:

$$u=\frac{1}{2}\left(t+\frac{1}{t}\right),\quad v=\frac{1}{2i}\left(t-\frac{1}{t}\right). \tag{65}$$

Then later we will use that $t=e^{iz}$ results in we have $u=\cos(z), v=\sin(z)$. Substituting (65) in the polynomial (60) we get a finite series $\tilde{\phi}_*^s(t)$ for positive and negative powers of $t$:

$$\begin{aligned}\tilde{\phi}_*^s(t)&=\phi_*^s\left(\frac{1}{2}\left(t+\frac{1}{t}\right),\frac{1}{2i}\left(t-\frac{1}{t}\right)\right)\\&=\sum_{n=0}^{n=s}\phi_r^n\left(\frac{1}{2}\left(t+\frac{1}{t}\right),\frac{1}{2i}\left(t-\frac{1}{t}\right)\right).\end{aligned}\tag{66}$$

We first make use of the property of homogeneity of all polynomials $\phi_r^n$:

$$\tilde{\phi}_*^s(t)=t^s\sum_{n=0}^{s}t^{n-s}\phi_r^n\left(\frac{1}{2}\left(1+\frac{1}{t^2}\right),\frac{1}{2i}\left(1-\frac{1}{t^2}\right)\right). \tag{67}$$

Similarly:

$$\tilde{\phi}_*^s(t)=t^{-s}\sum_{n=0}^{s}t^{s-n}\phi_r^n\left(\frac{1}{2}\left(t^2+1\right),\frac{1}{2i}\left(t^2-1\right)\right). \tag{68}$$

We now look to find the number of zeros of the finite series $\tilde{\phi}_*^s(t)$. To find the value of the coefficient of the term $t^s$, denoted by $b_s$, we proceed by a limit argument using equation (67):

$$b_s = t^{-s} \lim_{t\to\infty} \tilde{\phi}_*^s(t) = \lim_{t\to\infty} \sum_{n=0}^{s} t^{n-s} \phi_r^n \left( \frac{1}{2}\left(1+\frac{1}{t^2}\right), \frac{1}{2i}\left(1-\frac{1}{t^2}\right)\right). \quad (69)$$

Here for $n \neq s$ all terms $t^{n-s}$ tend to 0 for $t \to \infty$. Therefore we get:

$$b_s = \phi_r^s \left(\frac{1}{2}, \frac{-i}{2}\right). \quad (70)$$

We use a similar argument using (68) to find for $b_{-s}$:

$$b_{-s} = \phi_r^s \left(\frac{1}{2}, \frac{i}{2}\right). \quad (71)$$

So by looking at the extreme cases of $t$ we have determined that:

$$\tilde{\phi}_*^s(t) = \phi_r^s \left(\frac{1}{2}, \frac{-i}{2}\right) t^s + \cdots + \phi_r^s \left(\frac{1}{2}, \frac{i}{2}\right) t^{-s}. \quad (72)$$

As we know $\phi_r^s(u,v)$ is homogeneous we can state with the help of lemma (4.5):

$$\phi_r^s \left(\frac{1}{2}, \frac{\pm i}{2}\right) = \left(\frac{1}{2}\right)^s \phi_r^s(1, \pm i) \neq 0.$$

This guarantees the the polynomial $t^s\tilde{\phi}_*^s(t)$ has exactly 2s zeros. We can also see that $\phi_*^s(0) \neq 0$.

Let us denote these roots by $t_1, t_2, ..., t_{2s}$. To return to $\Phi_*^s(z) = 0$ we have to solve $e^{iz} = t_j$. Then the zeros of $e^{siz}\tilde{\phi}_*^s(e^{siz})$ has the same zeros as $\tilde{\phi}_*^s(e^{siz}) = \Phi_*^s(z)$. With fixed $j$, $e^{iz} = t_j$ has exactly one root $z_j$ in the strip $a \leq x < 2\pi + a, a \in \mathbb{R}$. The multiplicity of the zeros is preserved. If all $t_j$ are different, all $z_j$ are different too. Then in a larger strip $-2k\pi + a \leq x < 2k\pi + a, k \in \mathbb{N}, \Phi_*^s(z)$ has $4ks$ zeros. This proves the lemma.

We will now work to rewrite $F(z) = f(z, \sin(z), \cos(z))$ into:

$$F(z) = z^r \Phi_*^s(z) \left(1 + \delta_4(z)\right). \quad (73)$$

for a function $\delta_4$ with $\delta_4(z) \to 0$ as $\|z\| \to \infty$. More precisely:

**Lemma 4.7** *If* $\epsilon \in \mathbb{R}$ *is such that* $\Phi_*^s(\epsilon + iy) \neq 0, y \in \mathbb{R}$, *then there exists* $\delta_4 : \mathbb{C} \to \mathbb{C}$ *on the boundaries of the rectangle* $P_{kb} := \{z = x+iy \mid -2k\pi+\epsilon \leq x \leq 2k\pi + \epsilon,\ -b \leq y \leq b\}$ *such that the function* $F(z) = f(z, \sin(z), \cos(z))$ *(59) can be represented in the following form:*

$$F(z) = z^r \Phi_*^s(z) \left(1 + \delta_4(z)\right). \tag{74}$$

*Here* $\delta_4(z) \to 0$ *for* $z$ *on the boundaries of rectangle* $P_{kb}$ *as* $k, b \to \infty$.

**Proof:** We have to find an analytical form for $\delta_4(z)$ and show that it is defined properly on the boundaries of $P_{kb}$. First we find out about the behaviour of the equation

$$\Phi_*^s(z) = \phi_*^s(\cos z, \sin z) \tag{75}$$

for large values of $y$ in $z = x+iy$, both positive and negative. We substitute:

$$\begin{aligned} \Phi_m^n(x+iy) &= \Phi_m^n(z) \\ &= \phi_m^n(\cos(z), \sin(z)) \\ &= \phi_m^n\left(\frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right), \frac{1}{2i}\left(e^{ix-y} - e^{-ix+y}\right)\right). \end{aligned}$$

Let us first consider the case $y \to -\infty$. We use the homogeneity of $\phi_m^n$ to obtain:

$$\Phi_m^n(x+iy) = e^{nix-ny}\phi_m^n\left(\frac{1}{2}\left(1 + e^{-2ix+2y}\right), \frac{1}{2i}\left(1 - e^{-2ix+2y}\right)\right).$$

We shall use a binomial expansion on $\phi_m^n(x+iy)$. Let the homogeneous polynomial $\phi_m^n(u,v)$ be given in the form (58).

$$\phi_m^n(u,v) = \sum_{i=0}^{n} c_i M_i(u,v) = \sum_{i=0}^{n} c_i u^i v^{n-i}. \tag{76}$$

Here $c_i \in \mathbb{C}$ for all $i$ are constants. Then for $u = u_1 + u_2$ and $v = v_1 + v_2$ and using the binomial expansion, we get:

$$\begin{aligned}
&\phi_m^n(u_1+u_2, v_1+v_2)\\
&= \sum_{i=0}^{n} c_i \left( \left( \sum_{k=0}^{i} \binom{i}{k} u_1^k u_2^{i-k} \right) \left( \sum_{j=0}^{n-i} \binom{n-i}{j} v_1^j v_2^{i-j} \right) \right)\\
&= \sum_{i=0}^{n} c_i \left( \left( u_1^i + \sum_{k=1}^{i} \binom{i}{k} u_1^k u_2^{i-k} \right) \left( v_2^{n-i} + \sum_{j=1}^{n-i} \binom{n-i}{j} v_1^j v_2^{n-i-j} \right) \right)\\
&= \sum_{i=0}^{n} c_i \left( \left( u_1^i + f_{i,1}(u_1,u_2) \right) \left( v_2^{n-i} + f_{i,2}(v_1,v_2) \right) \right)\\
&= \sum_{i=0}^{n} c_i \left( u_1^i v_2^{n-i} + f_{i,3}(u_1,u_2,v_1,v_2) \right).
\end{aligned} \tag{77}$$

The functions $f_{i,1}, f_{i,2} : \mathbb{C}^2 \to \mathbb{C}$ are bivariate polynomials for all $i$. In addition $\lim_{u_i \to 0} f_{i,1}(u_1,u_2) = 0$, $\lim_{v_i \to 0} f_{i,2}(v_1,v_2) = 0$ for all $i = 1,2$.

Then $f_{i,3} : \mathbb{C}^4 \to \mathbb{C}$ is a polynomial in 4 variables. For $f_{i,3}$ we have that: $\lim_{u_k,v_j \to 0} f_3(u_1,u_2,v_1,v_2) = 0$ for all $k,j = 1,2$.

Let us now set:

$$u_1 = \frac{1}{2}, \quad u_2 = \frac{e^{-2ix+2y}}{2}, \quad v_2 = \frac{-i}{2}, \quad v_1 = \frac{ie^{-2ix+2y}}{2}. \tag{78}$$

As $y \to -\infty$ we see that $u_2, v_1 \to 0$ uniformly; in $x$. Therefore:

$$\delta_{5,n}(x,y) = \sum_{i=0}^{n} c_i f_{i,3}\left( \frac{1}{2}, \frac{e^{-2ix+2y}}{2}, \frac{ie^{-2ix+2y}}{2}, \frac{-i}{2} \right) \tag{79}$$

we have the original results of Pontryagin:

$$\Phi_m^n(x+iy) \;=\; e^{nix-ny} \left( \phi_m^n\left( \frac{1}{2}, \frac{-i}{2} \right) + \delta_{5,n}(x,y) \right). \tag{80}$$

Here $\delta_{5,n} : \mathbb{C} \to \mathbb{C}$ go to zero as $y \to -\infty$.

By noticing that we could also use the homogeneity of $\phi_m^n$ as follows:

$$\Phi_m^n(x+iy) = e^{-nix+ny} \phi_m^n \left( \frac{1}{2}\left(e^{2ix-2y}+1\right), \frac{1}{2i}\left(e^{2ix-2y}-1\right) \right),$$

We can then follow similar steps and prove:

$$\Phi_m^n(x+iy) \quad = \quad e^{-nix+ny}\left(\phi_m^n\left(\frac{1}{2},\frac{i}{2}\right)+\delta_{6,n}(x,y)\right). \tag{81}$$

Here $\delta_{6,n}:\mathbb{C}\to\mathbb{C}$ go to zero as $y\to\infty$. We will extend this derivation for the inhomogeneous function $\Phi_*^s(z)$ to obtain:

$$\Phi_*^s(x+iy) \quad = \quad e^{six-sy}\left(\phi_r^s\left(\frac{1}{2},\frac{i}{2}\right)+\delta_7(x,y)\right), \tag{82}$$

$$\Phi_*^s(x+iy) \quad = \quad e^{-six+sy}\left(\phi_r^s\left(\frac{1}{2},\frac{-i}{2}\right)+\delta_8(x,y)\right). \tag{83}$$

Here both $\delta_7,\delta_8:\mathbb{C}\to\mathbb{C}$ go to zero as either $y\to\infty$ or $-\infty$. To prove this we can see by (80):

$$\begin{aligned}\Phi_*^s(z) &= \Phi_*^s(x+iy)\\ &= \sum_{n=0}^{s}\phi_r^n(x+iy)\\ &= \phi_r^s(x+iy)+\sum_{n=0}^{s-1}\phi_r^s(x+iy)\\ &= e^{six-sy}\left(\phi_r^s\left(\frac{1}{2},\frac{i}{2}\right)+\delta_{5,s}(x,y)+\sum_{n=0}^{s-1}\frac{e^{nix-ny}}{e^{six-sy}}\left(\phi_r^n\left(\frac{1}{2},\frac{i}{2}\right)+\delta_{5,n}(x,y)\right)\right).\end{aligned} \tag{84}$$

So for:

$$\delta_7(x,y)=\delta_{5,s}(x,y)+\sum_{n=0}^{s-1}\frac{e^{nix-ny}}{e^{six-sy}}\left(\phi_r^n\left(\frac{1}{2},\frac{i}{2}\right)+\delta_{5,n}(x,y)\right), \tag{85}$$

we get (82). We can see that:

$$\frac{e^{nix-ny}}{e^{six-sy}}\to 0 \quad \text{as } y\to-\infty. \tag{86}$$

Therefore we have indeed that $\delta_7(x,y)\to 0$ as $y\to-\infty$ since $s>n$ for all $n$ in the sum. Again for $y\to\infty$ the proof is similar with:

$$\delta_8(x,y)=\delta_{6,s}(x,y)+\sum_{n=0}^{s-1}\frac{e^{-nix+ny}}{e^{-six+sy}}\left(\phi_r^n\left(\frac{1}{2},\frac{i}{2}\right)+\delta_{6,n}(x,y)\right). \tag{87}$$

We now choose some large $b'>0\in\mathbb{R}$ independent of $x$ such that $\Phi_*^s(x+iy)\neq$

0 for all $\|y\| > b'$. This can be done since $\Phi_r^s(\frac{1}{2}, \frac{i}{2}) \neq 0$ and by using (82).

Using equations (80) and (82) we obtain:

$$\left|\frac{\Phi_m^n(x+iy)}{\Phi_*^s(x+iy)}\right| < c_1, \tag{88}$$

for $|y| > b'$ and $c_1 > 0$ a constant that depends on our original polynomial $f(z,u,v)$ (59) and $b'$. Most importantly this quantity is well-defined, that is, $\Phi_*^s(x+iy) \neq 0$ for large $|y|$.

Additionally from equations (80) and (82) it follows that:

$$\left|\frac{\Phi_m^n(\pm 2k\pi + \epsilon + iy)}{\Phi_*^s(\pm 2k\pi + \epsilon + iy)}\right| < c_2, \tag{89}$$

where $c_2 > 0$ is a constant dependent on $f(z,u,v)$ and the number $\epsilon \in \mathbb{R}$. In addition $k \in \mathbb{N}$. Next we denote a rectangle with $P_{kb}$ and define it as:

$$P_{kb} := \{z = x + iy \mid -2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon, -b \leq y \leq b\} \tag{90}$$

Here $b \geq b'$. We write the function $F(z)$ in the form:

$$F(z) = z^r \Phi_*^s(z) \left(1 + \sum_{m,n=0}^{r,s-1} z^{m-r} \frac{\Phi_m^n(z)}{\Phi_*^s(z)}\right). \tag{91}$$

Here the exponent $m - r$ is negative and therefore, since the term $\frac{\Phi_m^n(z)}{\Phi_*^s(z)}$ is bounded from above via (88) and (89), we see that on the boundary of the rectangle $P_{kb}$ for sufficiently large $k$ and $b$ we can write:

$$F(z) = z^r \Phi_*^s(z)\left(1 + \delta_4(z)\right), \tag{92}$$

with $\delta_4 : \mathbb{C} \to \mathbb{C}$ and $\delta_4 \to 0$ uniformly as $k, b \to \infty$ and

$$\delta_4(z) = \sum_{m,n=0}^{r,s-1} z^{m-r} \frac{\Phi_m^n(z)}{\Phi_*^s(z)}. \tag{93}$$

This proves Lemma 4.7.

**Corollary 4.8** *Let $w = iz$. If $\epsilon_2 \in \mathbb{R}$ is such that $\Phi_*^s(x + i\epsilon_2) \neq 0$ for some $x \in \mathbb{R}$, then on the boundaries of the rectangle $P'_{kb} := \{z = x+iy \mid -2k\pi+\epsilon \leq y \leq 2k\pi + \epsilon,\ -b \leq x \leq b\}$ the function $f(w, \sin(w), \cos(w))$ (59) exists in the following form:*

$$F(w) = w^k \Phi_*^s(w)\left(1 + \delta'_4(w)\right). \tag{94}$$

*Here $\delta'_4 : \mathbb{C} \to \mathbb{C}$ and $\delta'_4(s) \to 0$ for $s$ on the boundaries of rectangle $P_{kb}$ as $k, b \to \infty$.*

**Proof:** We use Lemma 4.7 and take $w = iz$.

The last part needed to prove Theorem 4.4 is to confirm that the number of zeros inside the rectangle of $F(z)$ and $z^k\Phi_*^s(z)$ is the same. To prove this we need the principle of the argument A.3:

**Lemma 4.9** *Inside the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$, $F(z)$ has the same number of zeros as $z^k\Phi_*^s(z)$. Moreover this number of zeros is found to be $4ks + r$.*

**Proof:** Let $\mathfrak{C}$ be some closed contour in the plane of the complex variable $z$ and let $g(z)$ be an analytic function that does not have singularities both on and inside the contour $\mathfrak{C}$ and additionally it does not vanish on $\mathfrak{C}$. Then, by virtue of the Principle of the Argument A.3, the number of zeros of the function $g(z)$ inside the contour $\mathfrak{C}$ is equal to the total number of revolutions around the origin of the vector $w = g(z)$ when the variable $z$ describes the contour $\mathfrak{C}$. Let now $\tilde{g}(z)$ also be analytic on this contour and inside and let it be defined by $\tilde{g}(z) = g(z)(1 + \delta(z))$, where $|\delta(z)| < 1$. Consider now the function $g(z,\tau) = g(z)(1+\tau\delta(z))$ on the same contour $\mathfrak{C}$ and $\tau$ is a real number. For a fixed $\tau$ the vector $w = g(z,\tau)$ describes the amount of complete revolutions around origin while $z$ runs through the contour $\mathfrak{C}$. If now $\tau$ is continuously varied from 0 to 1 the term $(1+\tau\delta(z)) > 0$ and thus the vector $w$ never turns to 0 and therefore the number of complete revolutions cannot change. Thus the number of zeros of $\tilde{g}(z)$ and $g(z)$ is the same inside the contour $\mathfrak{C}$.

Now let $g(z) = z^r\Phi_*^s(z)$, $\mathfrak{C} = P_{kb}$ and Lemma 4.9 follows as the term $(1 + \delta_4(z)) > 0$ for sufficiently large $P_{kb}$.

We fix $k$ to a sufficiently large value and we let $b \to \infty$. Then we see that the number of zeros of $F(z)$ and $z^r\Phi_*^s(z)$ is the same in the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$. For the function $z^r\Phi_*^s(z)$ this number equals $4ks + r$: $r$ zeros from $z^r$; $2s$ zeros in the strip $a \leq x \leq 2\pi + a$ from Lemma 4.6 and subsequently $4ks$ zeros in a larger strip $-2k\pi + a \leq x \leq 2k\pi + a$. This completes the proof of Theorem 4.4.

We can state another lemma. If we impose more properties from the form of $H(iy) = f(y,u,v)+ig(y,u,v)$, we get that a linear combination $J_{\lambda,\mu}(y,u,v) = \lambda f(y,u,v)+\mu g(y,u,v)$ also fullfills the assumptions that are needed to apply Lemma 4.6 and thus Theorem 4.4. To be more precise:

**Lemma 4.10** *Let the exponential polynomial $H(z)$ (2.1) on the imaginary axis be written as:*

$$H(iy) = f(y, u, v) + ig(y, u, v). \tag{95}$$

*Let the polynomials $f(y, u, v), g(y, u, v)$ be given as:*

$$f(y, u, v) = \sum_{m,n=0}^{r,s} y^m \phi_m^n(u, v), \quad g(y, u, v) = \sum_{m,n=0}^{r,s} y^m \psi_m^n(u, v). \tag{96}$$

*Let $\mu, \lambda \in \mathbb{R}$ not be simultaneously 0. If:*

$$H(iy) = \sum_{m,n=0}^{r,s} a_{m,n}(iy)^m (u + iv)^n \tag{97}$$

*then the term*

$$\pi_r^s(u, v) = \lambda\phi_r^s(u, v) + \mu\psi_r^s(u, v) \tag{98}$$

*satisfies the following equation:*

$$\pi_r^s(1, \pm i) \neq 0. \tag{99}$$

**Proof:** We write:

$$\begin{aligned} J_{\lambda,\mu}(y, u, v) &= \lambda f(y, u, v) + \mu g(y, u, v) \\ &= \sum_{m,n=0}^{r,s} y^m \left(\lambda\phi_m^n(u, v) + \mu\psi_m^n(u, v)\right), \end{aligned} \tag{100}$$

for $\lambda, \mu$ not simultaneously zero. This gives for the term (98):

$$y^r \gamma^s(u, v) := y^r \left(\lambda\phi_r^s(u, v) + \mu\psi_r^s(u, v)\right). \tag{101}$$

To proceed we take a look at the function $e^{niy} = (\cos(y) + i\sin(y))^n = (u + iv)^n$ We put:

$$(u + iv)^n = \alpha^n(u, v) + i\beta^n(u, v), \tag{102}$$

where $\alpha^n, \beta^n : \mathbb{C}^2 \to \mathbb{C}$ are polynomials with real coefficients and bidegree $(n_1, n_2)$ where $n = n_1 + n_2$. Then we have:

$$\alpha^n(u, v) = \frac{1}{2}\left((u + iv)^n + (u - iv)^n\right), \tag{103}$$

$$\beta^n(u,v) = \frac{1}{2i}\left((u+iv)^n - (u-iv)^n\right). \tag{104}$$

Let $a$ and $b$ be real and not vanish simultaneously. Let us define the polynomial $\pi^n : \mathbb{C}^2 \to \mathbb{C}$:

$$\pi^n(u,v) := a\alpha^n(u,v) + b\beta^n(u,v). \tag{105}$$

For all $n$ we see immediately that:

$$\pi^n(1,\pm i) = 2^{n-1}(a \pm ib) \neq 0, \tag{106}$$

as $a$ and $b$ are real and non simultaneously 0. Before we can apply this, we have to look at our exponential polynomial, which we assume is of the following form:

$$H(z) = \sum_{n,m=0}^{s,r} a_{m,n} z^m e^{nz}. \tag{107}$$

Let us look at this polynomial on the imaginary axis and split it in its real and imaginary parts. Let us also set $u = \cos(y), v = \sin(y)$. Then:

$$H(iy) = f(y,u,v) + ig(y,u,v), \tag{108}$$

with polynomials $f, g$. Then we see that:

$$\begin{aligned} H(iy) = &\sum_{m,n}^{r,s} a_{m,n}(iy)^m(u+iv)^n \\ =&\sum_{m,n}(a'_{m,n} + ia''_{m,n})i^m y^m\left(\alpha^n(u,v) + i\beta^n(u,v)\right) \\ =&f(y,u,v) + ig(y,u,v). \end{aligned} \tag{109}$$

Here $a_{m,n} = a'_{m,n} + ia''_{m,n}, a'_{m,n}, a''_{m,n} \in \mathbb{R}$. Since we have:

$$f(y,u,v) = \sum_{m,n=0}^{r,s} y^m \phi_m^n(u,v), \quad g(y,u,v) = \sum_{m,n=0}^{r,s} y^m \psi_m^n(u,v), \tag{110}$$

then we obtain for the terms $\phi_r^s(u,v), \psi_r^s(u,v)$:

$$\phi_r^s(u,v) = \pm(a'_{r,s}\alpha^s(u,v) - a''_{r,s}\beta^s(u,v)); \tag{111}$$

$$\psi_r^s(u,v) = \pm(a''_{r,s}\alpha^s(u,v) + a'_{r,s}\beta^s(u,v)). \tag{112}$$

Where the signs depend on the term $i^r$ and thus on $r$. Let now $\lambda, \mu$ two real numbers that do not vanish simultaneously, then

$$\lambda f(y,u,v) + \mu g(y,u,v) = \sum_{m,n=0}^{r,s} y^m \left(\lambda \phi_m^n(u,v) + \mu \psi_m^n(u,v)\right). \tag{113}$$

By (105), (111) and (112) it follows that

$$\lambda \phi_r^s(u,v) + \mu \psi_r^s(u,v) = a\alpha^s(u,v) + b\beta^s(u,v) = \pi^s(u,v), \tag{114}$$

with

$$a = \pm(\lambda a'_{r,s} + \mu a''_{r,s}), \quad b = \pm(\lambda a'_{r,s} - \mu a''_{r,s}). \tag{115}$$

We see that $a$ and $b$ can not be simultaneously 0. Then by (106) we have that $\pi^s(1, \pm i) \neq 0$. This proves Lemma 4.10.

Now we can use the same strategy as in the proof of Theorem 4.4 for $J_{\lambda,\mu}(y,u,v)$ starting with lemma 4.6. This shows that the function $J_{\lambda,\mu}(y,u,v)$ has $4ks + r$ zeros in the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$, $-b \leq y \leq b$ for some $\epsilon \in \mathbb{R}$ and for $k \in \mathbb{N}, b \in \mathbb{R}$ large enough.

The difference is that in Theorem 4.4 we require the polynomials $\phi_m^n(u,v)$ not to have a factor $u^2 + v^2$ while for this Lemma 4.10 we require instead that $H(iy)$ is an exponential polynomial of the form (97). From the form (97) it is not immediately clear that the real and imaginary components $f(y,u,v), g(y,u,v)$ have no factors $u^2 + v^2$, therefore this extra corollary has to be included.

# 5 Pontryagin's Criterium

Recall that we denote our exponential polynomial with $H(z)$. In this chapter we will prove the main result of Pontryagin: Theorem 5.6. We will consider the total change of the argument of $H(z)$ as $z$ travels along the imaginary axis, thus $z = iy$. In this chapter therefore we use the following form for $H(iy)$ :

$$H(iy) = F(y) + iG(y), \tag{116}$$

where $f(y, u, v), g(y, u, v)$ are multivariate polynomials such that:

$$F(y) = f(y, \sin(y), \cos(y)), \quad G(y) = g(y, \sin(y), \cos(y)). \tag{117}$$

We will use the existence of a principal term of $H$ to show that this total rotation is related to the number of zeros in the plane $\Re(z) > 0$. Then we will use Theorem 4.4 to relate the number of zeros of $F(y)$ and $G(y)$ as $H(iy) = F(y) + iG(y)$ to the speed of the rotation. For this we will first define the total rotation and the speed of that rotation.

**Definition 5.1 (Total Rotation)** *Let the vector $w = H(iy)$ be rotated around the origin as $y$ ranges from $a$ to $b$. The argument of $w$ is given by:*

$$\arg(w) = \arctan\frac{\Im(w)}{\Re(w)}. \tag{118}$$

*By the fundamental theorem of calculus we can then define the total change in $\arg(w)$ as the line integral from $a$ to $b$ of its derivative. Let us denote this total change with $v_w(a, b)$. Let us have $G(y) = \Im(w)$ and $F(y) = \Re(w)$ by 116. Now we define:*

$$v_w(a, b) = \int_a^b \left(\frac{F(y)}{G^2(y) + F^2(y)} dG(y) - \frac{G(y)}{G^2(y) + F^2(y)} dF(y)\right). \tag{119}$$

*The speed $\frac{\partial}{\partial y} v_w(0, y)$ of the rotation vector $w$ at $y$ is given by the formula:*

$$\frac{\partial}{\partial y} v_w(0, y) = \frac{G'(y)F(y) - G(y)F'(y)}{G^2(y) + F^2(y)}. \tag{120}$$

**Remark:** to evaluate equation (119), one notes that equation (118) is the primitive of equation (120) which means:

$$v_w(a, b) = \arctan\frac{G(a)}{F(a)} - \arctan\frac{G(b)}{F(b)}. \tag{121}$$

In addition also note that if $w$ makes a full rotation, for any path $\mathfrak{P}$ that begins at $p \in \mathbb{P}$ and ends at $p$, by the Argument Principle A.3 we see that:

$$v_w(\mathfrak{P}) = \arg\left(H(\mathfrak{P})\right) = \int_{\mathfrak{P}} \frac{H'(z)}{H(z)} dz = 2\pi i N, \tag{122}$$

where $N \in \mathbb{N}$ is the amount of zeros encircled by $\mathfrak{P}$. Let us now consider the total rotation of $w = H(iy)$ along the imaginary axis. The following theorem considers a large rectangle covering the right side of the complex plane and uses the argument principle A.3 to determine the rotation along the imaginary axis in relation to the number of zeros in the right side of the complex plane. By increasing length of the sides of the large rectangle, most of the contribution to the total rotation along those sides comes from the principal term of the bivariate polynomial $h(z,t) = h(z,e^z)$.

**Theorem 5.2 (Theorem 5 in [5])** *Let $h(z,t)$ be a bivariate polynomial with principal term $a_{r,s}z^r t^s$. Let $h(z,u,v)$ be $h(z,t)$ after the substitutions* (65). *Then furthermore denote the $\phi$-coefficient of $z^r$ by $\phi_*^s(u,v)$ as given in* (60). *Following Theorem 4.4 further we denote:*

$$\Phi_*^s(z) = \phi_*^s(\sin(z), \cos(z)).$$

*The number of zeros of the function $H(z) = h(z,e^z)$ in the strip $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon, x > 0, z = x + iy$ we denote by $N_k$ where $\epsilon$ a real number such that $\Phi_*^s(x + i\epsilon) \neq 0$ for arbitrary real $x$. We suppose further that the function $H(iy) \neq 0$. We denote by $v_w(-2k\pi + \epsilon, 2k\pi + \epsilon)$ the total rotation 5.1 of the vector $w = H(iy)$ around the origin when $y$ ranges through the interval $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$. Then:*

$$v_w(-2k\pi + \epsilon, 2k\pi + \epsilon) = 2\pi\left(2sk - N_k + \frac{r}{2}\right) + \delta_9(k), \tag{123}$$

*where $\delta_9(k) \to 0$ as $k \to \infty$.*

**Proof:** Consider the rectangle $P_{ka}$ defined by the conditions $0 \leq x \leq a$ and $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$ and estimate the total rotation of the vector $w = H(iy)$ when $z$ runs counterclockwise along three sides of rectangle $P_{ka}$ excluding side $x = 0$. So consider $w$ on the sides bottom, right and top. Since we have $\epsilon$ such that $\Phi_*^s(x + i\epsilon) \neq 0$ for arbitrary real $x$, by Corollary 4.8, we can write $H(z)$ in the form (73).

$$H(z) = z^r \Phi_*^s(z)\left(1 + \delta_4(z)\right). \tag{124}$$

As the total rotation of $H(z)$ is the sum of the rotation of $z^r$, $\Phi_*^s(z)$ and $(1+\delta_4(z))$, we determine these first: In this form $\delta_4(z)$ tends to zero uniformly on the three sides considered of the rectangle $P_{ka}$ as $k, a \to 0$. Thus for $k \to \infty, a \to \infty$ the total rotation of the vector $w' = (1+\delta_4(z))$ goes to 0. Therefore the total rotation of $w$ can be written as the total rotation of $w'' = z^r\Phi_*^s(z)$ from $-2k\pi + \epsilon$ to $2k\pi + \epsilon$:

$$v_w(-2k\pi + \epsilon, 2k\pi + \epsilon) = v_{w''}(-2k\pi + \epsilon, 2k\pi + \epsilon). \tag{125}$$

This is equal to the sum of rotations of $z^r$ and for $\Phi_*^s(z)$. The rotation along the three sides of $P_{ka}$ of $z$ is equal to

$$\arctan\frac{-2k\pi + \epsilon}{0} - \arctan\frac{2k\pi + \epsilon}{0} = \pi, \tag{126}$$

thus the rotation along those three sides of $P_{ka}$ of $z^r$ is $\pi r$.

Since the principal term of $H(z)$ is given by $a_{r,s}z^r e^{sz}$, the term with the largest exponent in $\Phi_*^s(z)$ is given by $a_{r,s}e^{sz}$. The total rotation of $\Phi_*^s(z)$ differs only little from $a_{r,s}e^{sz}$ which amounts to:

$$\arctan\frac{\sin(s(-2k\pi+\epsilon))}{\cos(s(-2k\pi+\epsilon)} - \arctan\frac{\sin(s(2k\pi+\epsilon))}{\cos(s(2k\pi+\epsilon)} = 4\pi ks. \tag{127}$$

To this we have to add the minor contributions of the other terms in $\Phi_*^s(z)$ and $1+\delta_4(z)$ to the total rotation. Let us call this contribution $\delta_9(k)$ where $\delta_9(k) \to 0$ as $k \to \infty$.

Therefore the total rotation $v_w(-2k\pi+\epsilon, 2k\pi+\epsilon)$ of $w = H(z)$ is $4\pi sk + \pi r + \delta_9(k)$. Since the number of zeros $N_k$ of the function $H(z)$ inside the rectangle $P_{ka}$ is equal to the number of complete rotations of the vector $w = H(z)$ when $z$ runs all sides of the rectangle $P_{ka}$, by equation (122). Doing the calculation we get:

$$v_w(-2k\pi + \epsilon, 2k\pi + \epsilon) = 2\pi\left(2sk - N_k + \frac{r}{2}\right) + \delta_9(k). \tag{128}$$

This proves Theorem 5.2.

**Remark:** We have our doubts that the term $a_{r,s}e^{sz}$ dominates on $P_{ka}$, which is especially dubious around the imaginary axis where $\Re(z)$ is small. As an example, let:

$$\Phi_*^s(z) = e^{sz} + e^{(s-1)z}, \tag{129}$$

for $s > 0 \in \mathbb{N}$. Then the total rotation of that part becomes:

$$\begin{aligned}(s-1/2)4k\pi = \; & \arctan \frac{\sin(s(-2k\pi+\epsilon)) + \sin((s-1)(-2k\pi+\epsilon))}{\cos s(-2k\pi+\epsilon) + \cos(s-1)(-2k\pi+\epsilon)} \\ & - \arctan \frac{\sin(s(2k\pi+\epsilon)) + \sin((s-1)(2k\pi+\epsilon))}{\cos(s(2k\pi+\epsilon)) + \cos((s-1)(2k\pi+\epsilon))}.\end{aligned} \tag{130}$$

If $s$ is relatively small, say $s = 2$, there is a significant difference. Therefore we cannot confirm with certainty that the proof of Theorem 5.2 is correct.

Theorem 5.2 checks the total rotation of $w$ along the interval $(-2k\pi+\epsilon, 2k\pi+\epsilon)$ on the imaginary axis. It turns out that we can also look at an easier interval: $(-2k\pi, 2k\pi)$ for large $k$.

**Lemma 5.3** *Let $w = H(iy)$. Then $v_w(a+\epsilon, b+\epsilon) = v_{w'''}(a,b) + \delta_{10}(a,b)$ for $a, b, \epsilon \in \mathbb{R}$. Here $\delta_{10}(a,b) \to 0$ for fixed $\epsilon$ as $a, b \to \infty$ and $w''' = a_{r,s} z^r e^{sz}$ for $s, r \in \mathbb{N}$ and $a_{r,s} \in \mathbb{C}$ a constant.*

**Proof:** It is clear that $v_w(a,c) = v_w(a,b) + v_w(b,c)$ for $a, b, c \in \mathbb{C}$ since equation (119) is path-independent. Therefore we can write: $v_w(a+\epsilon, b+\epsilon) = v_w(a,b) + v_w(b, b+\epsilon) - v_w(a, a+\epsilon)$. Letting $a, b \to \pm\infty$ we can use that $v_w(a,b) = v_{w'''}(a,b) + \delta_{10}(a,b)$ for $w''' = a_{r,s} z^r e^{sz}$. This we can in turn split up in the rotation of $z^r$ and of $e^{sz}$. Then we see:

$$\begin{aligned}v_{w'''}(a, a+\epsilon) = \; & \arctan\left(\frac{\sin(s(a+\epsilon))}{\cos(s(a+\epsilon))}\right) + r \arctan\left(\frac{a+\epsilon}{0}\right) \\ & - \arctan\left(\frac{\sin(sa)}{\cos(sa)}\right) - r\arctan\left(\frac{a}{0}\right) \\ = & s\epsilon.\end{aligned} \tag{131}$$

For $b \to \infty$ we do a similar calculation and find the same:

$$v_{w'''}(b, b+\epsilon) = s\epsilon. \tag{132}$$

Then we see: $v_w(a+\epsilon, b+\epsilon) = v_{w'''}(a,b) + v_{w'''}(b, b+\epsilon) - v_{w'''}(a, a+\epsilon) + \delta_{10}(a,b) = v_{w'''}(a,b)$. Thus we end up with:

$$v_w(a+\epsilon, b+\epsilon) = v_{w'''}(a,b) + \delta_{10}(a,b), \tag{133}$$

proving Lemma 5.3.

**Remark:** This proof has the same possible issue as the proof of Theorem 5.2, specifically the assertion that $v_w(a,b) = v_{w'''}(a,b) + \delta_{10}(a,b)$ for $w''' = a_{r,s}z^r e^{sz}$.

If we add the assumption on the exponential polynomial $H(z)$ that

$$\Phi_*^s(z) = \Phi_r^s(z) = a_{r,s}e^{sz} \tag{134}$$

thus only consists of a single term, we can be sure that the arguments in Theorem 5.2 and Lemma 5.3 are correct.

The next lemma contains the most important idea of Pontryagin as he relates the direction of rotation to the properties of zeros.

**Lemma 5.4** *Let us assume that* $v_w(-2k\pi, 2k\pi) = \tau(4\pi ks + \pi r) + \delta_{11}(k)$ *where* $v_w(a,b)$ *is defined in* (5.1) *with* $\tau = \pm 1$ *and* $\delta_{11}(k) \to 0$ *as* $k \to \infty$. *Let* $w = H(iy)$. *Let* $\lambda, \mu \in \mathbb{R}$ *not simultaneously zero and define a line in the* $w$*-plane by the equation:*

$$\lambda\Re(w) + \mu\Im(w) = 0. \tag{135}$$

*Then all zeros of* $\lambda F(y) + \mu G(y)$ *are real and simple and do not exceed* $4ks + r$ *in number. In addition we have the inequality:*

$$\tau(G'(y)F(y) - G(y)F'(y)) > 0. \tag{136}$$

**Proof:** By Lemma 4.10 and Theorem 4.4 we see that there exists $\epsilon \in \mathbb{R}$ such that the function $J_{\lambda,\mu}(y,u,v) = \lambda f(y,u,v) + \mu g(y,u,v)$ has $4ks+r$ zeros in the strip $-2\pi k + \epsilon \leq y \leq 2k\pi + \epsilon$. Now we see that if $v(-2k\pi, 2k\pi) = \tau(4\pi ks + \pi r) + \delta_{11}$, for $k \to \infty$, the vector $w = H(iy)$ intersects every line of the form (135) exactly $4ks + r$ times. At every intersection we can write $w = \lambda f(y,u,v) + \mu g(y,u,v)$ and we know that this function has exactly $4ks + r$ zeros. Since we have as many intersections as zeros we can assert that all these zeros are simple. All the zeros are also real since each intersection happens for one value of $y \in \mathbb{R}$. Since this works for every line of the form (135) we see that the speed of the argument (120) never shifts signs and therefore we have the concluding inequality:

$$\tau(G'(y)F(y) - G(y)F'(y)) > 0, \tag{137}$$

for all $y$. Lemma 5.4 is proved.

We will prove one more lemma before we can turn our attention to Pontryagin's main theorem. This lemma asserts that, given the inequality (137), the zeros of $G(y)$ and $F(y)$ alternate.

**Lemma 5.5** *Let $F(y)$ and $G(y)$ be the real and imaginary part of $H(iy)$, our exponential polynomial of interest on the imaginary axis. Let $F(y)$ and $G(y)$ have only simple zeros. Let two functions have alternating zeros if and only if each of the functions has no multiple zero and between every two zeros of one of these functions there exists at least one zero of the other and the functions are never simultaneously zero. The zeros of $F(y)$ and $G(y)$ alternate if* (137) *holds.*

**Proof:** Let $\tau = 1$ in (137). Then the speed of the rotation vector $w = H(iy)$ is always positive and therefore

$$\arg(w) = \arctan\left(\frac{G(y)}{F(y)}\right) \tag{138}$$

always increases. Let $G(y) = 0$ for some $y$. Then $\arg(w) = 0$. Since the argument always increases it eventually becomes $\pi/2$ at which point $F(y) = 0$. Geometrically it then becomes $\pi$ for which $G(y) = 0$ again and so the loop continues as the argument increases. The same argument applies when $\tau = -1$ where $\arg(w)$ always decreases. None of the zeros of $G(y)$ or $F(y)$ can be multiple as we have assumed only simple zeros.

**Theorem 5.6 (Theorem 6 in Pontryagin [5])** *Let $H(z) = h(z, e^z)$ where $h(z,t)$ is a polynomial with a principal term. The function $H(iy)$ is now separated into real and imaginary parts; that is. we set $H(iy) = F(y) + iG(y)$. Let us add the assumption in* (134)*. If all the zeros of the function $H(z)$ lie to the left side of the imaginary axis, then the zeros of the functions $F(y)$ and $G(y)$ are:*

- *Real;*
- *Alternating: each of the functions have no multiple zero and between every two zeros of one of these functions there exists at least one zero of the other and the functions are never simultaneously zero;*
- *For each $y \in \mathbb{R}$:* $$G'(y)F(y) - G(y)F'(y) > 0 \tag{139}$$

*Moreover, in order that all the zeros of the function lie to the left of imaginary axis, it is sufficient that one of the following conditions be satisfied:*

- *All the zeros of the functions $F(y)$ and $G(y)$ are real and alternate and the inequality* (139) *is satisfied for at least one value of $y$.*
- *All the zeros of the function $F(y)$ are real and for each zero $y = y_0$ inequality* (139) *is satisfied, that is, $F'(y_0)G(y_o) < 0$.*

- *All the zeros of the function G(y) are real and for each zero* $y = y_0$ *inequality* (139) *is satisfied, that is,* $G'(y_0)F(y_o) > 0$.

**Proof (→):** If all the zeros of $H(z)$ lie to the left side of the imaginary axis, we have by Theorem 5.2 and Lemma 5.3 that $v_w(-2k\pi + \epsilon, 2k\pi + \epsilon) = v_w(-2k\pi, 2k\pi) = 4\pi ks + \pi r + \delta_9(k)$ for $k \to \infty$. Then by Lemma 5.4 we have $\tau = 1$ and thus inequality (139). In addition by Lemma 4.10 for special cases $\lambda = 1, \mu = 0$ and $\lambda = 0, \mu = 1$ we have that since $\lambda f(y, u, v) + \mu g(y, u, v)$ has only simple and real zeros, $F(y), G(y)$ also have only simple and real zeros. Finally by Lemma 5.5 we have that the zeros of $F(y)$ and $G(y)$ are alternating.

**Remark:** To prove the (←) direction, Pontryagin asserts that, because the functions $F(y)$ and $G(y)$ are alternating and real, the inequality (137) holds. If then the inequality (139) is found to be true for one value of $y$, we can establish that $\tau = 1$ and the argument of $w$ increases in counterclockwise direction. The rest of the proof is very similar to (→), involving intersections with lines and invoking Theorem 5.2 again.

It remains out of grasp to confirm Pontryagin's assertion that (137) holds under these circumstances. It is very imaginable that, even though the zeros of $F(y)$ and $G(y)$ alternate, for values of $y$ in between two such zeros the argument switching signs. There also seems no straightforward addition assumption to alter Theorem 5.6.

**Remark:** Unfortunately the practical side of Theorem 5.6 is to either check each zero of $F(y)$ or $G(y)$ and check inequality (139). To adequately review this part, more research is needed. It is possible to adjust the assumptions to:

1. All the zeros of the functions $F(y)$ and $G(y)$ are real and alternate and the inequality (139) is satisfied for all $y$.

This would allow us to prove the converse. Even then it is not clear that the second and third point lead to this new assumption 1.

# 6 Conclusion

In this report we have showed proofs of Pontryagin's theorems in modern mathematical language. We saw a link between characteristic functions of time-delayed systems (TDSs) and exponential polynomials. We saw that having a principal term was necessary for stability as without such a term the corresponding TDS is guaranteed instable. We then saw a theorem bounding the number of zeros in a strip in the complex plane of polynomials, which we used to prove a wonderful argument relating that number of zeros to the intersections of the image of the exponential polynomial on the imaginary axis with a collection of straight lines.

We also saw some gaps in the proof. We saw that it is not necessarily true that $\zeta$ converges to 0 with convergence rate $1/k$. We also saw that the dominating term on the rectangle in Theorem 5.2 seems to be a error that is not easily repaired since the proof of Theorem 5.6 relies heavily on the equality between the number of zeros and number of intersections. Furthermore we were unable to verify the proof of the sufficiency of the three conditions to have all zeros of $H(z)$ lie on the left of the imaginary axis. The main problem here was that alternating zeros do not necessary lead to a monotonous increase/decrease in the rotation. More research is definitely needed on this problem. Finally Theorem 2.8 has not been formulated to its maximal strength. That is, no criterium has been derived on the rank of the other polynomials $\tilde{p}_i$. We recommend further research into this topic.

The goal of this report was to clear up Pontryagin's theorems and formulate them in a more modern style. After reading this report the reader knows the mechanisms behind the theorems and the topics which have not been fully enlightened.

# A Compendium of Theorems and Definitions

This appendix contains well-known theorems and definitions used in the text for reference.

**Definition A.1 (Multivariate Polynomials)** *A multivariate polynomial $f$ in the variables $(x_1, \cdots, x_n)$ of multidegree $(D_1, \cdots, D_n)$ with $D_n, n \in \mathbb{N}$ is a function $f : \mathbb{C}^n \to \mathbb{C}$ such that $f$ has the following form:*

$$f(x_1, \cdots, x_n) = \sum_{k_1=0}^{D_1} \cdots \sum_{k_n=0}^{D_n} a_{k_1,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n}, \tag{140}$$

*with $a_{k_1,\cdots,k_n} \in \mathbb{C}$ and $k_i \in \mathbb{N}$ for all $i \in \mathbb{N}$.*

**Theorem A.2 (Fundamental Theorem of Algebra [10])** *A polynomial $P(z)$ of degree $n$ has $n$ values $z_i$ for which $P(z_i) = 0$.*

**Theorem A.3 (The Principle of the Argument [7])** *Let $f(z)$ by a meromorphic function on an open set $\Omega$ and $C$ is a closed curve on $\Omega$, contractible to a point in $\Omega$. Furthermore let $n(C, z)$ be the winding number of $C$ around $z$ and we have:*

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = \sum_{Z} n(C, z) - \sum_{P} n(C, z) \tag{141}$$

*Where the summation of the first term goes over all zeros of $f(z)$ inside $\mathfrak{C}$, multiplied by their multiplicity and the summation of the second term goes over all poles of $f(z)$ inside $\mathfrak{C}$ multiplied by their order. For an analytic function $f$, so no poles, in and on $\mathfrak{C}$ the interpretation of this theorem might also be seen as the amount of revolutions are the origin of the vector $w = f(z)$.*

**Theorem A.4 (Rouche [7])** *If two functions $f(z), g(z)$ $\mathbb{C} \to \mathbb{C}$ are analytic on and inside a closed contour $\mathfrak{C}$ and $|g(z)| < |f(z)|$ everywhere on $\mathfrak{C}$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $\mathfrak{C}$.*

# References

[1] S.-I. Niculescu and K. Gu, Eds., *Advances in time-delay systems*, ser. Lecture Notes in Computational Science and Engineering. Springer-Verlag, Berlin, 2004, vol. 38. [Online]. Available: https://doi.org/10.1007/978-3-642-18482-6

[2] V. L. Kharitonov, "Lyapunov functionals and matrices for neutral type time delay systems," in *Time delay systems: methods, applications and new trends*, ser. Lect. Notes Control Inf. Sci. Springer, Berlin, 2012, vol. 423, pp. 3–17. [Online]. Available: https://doi.org/10.1007/978-3-642-25221-1_1

[3] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional-differential equations*, ser. Applied Mathematical Sciences. Springer-Verlag, New York, 1993, vol. 99. [Online]. Available: https://doi.org/10.1007/978-1-4612-4342-7

[4] R. C. Dorf and R. H. Bishop, *Modern control systems.* Pearson, London, 2011.

[5] L. Pontrjagin, "On zeros of some transcendental functions," *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, vol. 6, no. 3, pp. 115–134, 1942.

[6] V. L. Kharitonov, "Robust stability analysis of time delay systems: A survey," *Annual Reviews in Control*, vol. 23, pp. 185–196, 1999.

[7] R. Bellman and K. L. Cooke, *Differential-difference equations.* Academic Press, New York-London, 1963.

[8] A. Seuret, F. Gouaisbaut, and L. Baudouin, "D1. 1-overview of lyapunov methods for time-delay systems," 2016.

[9] J. Sun, G. Liu, J. Chen, and D. Rees, "Improved delay-range-dependent stability criteria for linear systems with time-varying delays," *Automatica*, vol. 46, no. 2, pp. 466–470, 2010.

[10] S. G. Krantz, *Handbook of complex variables.* Birkhäuser Boston, Inc., Boston, MA, 1999. [Online]. Available: https://doi.org/10.1007/978-1-4612-1588-2