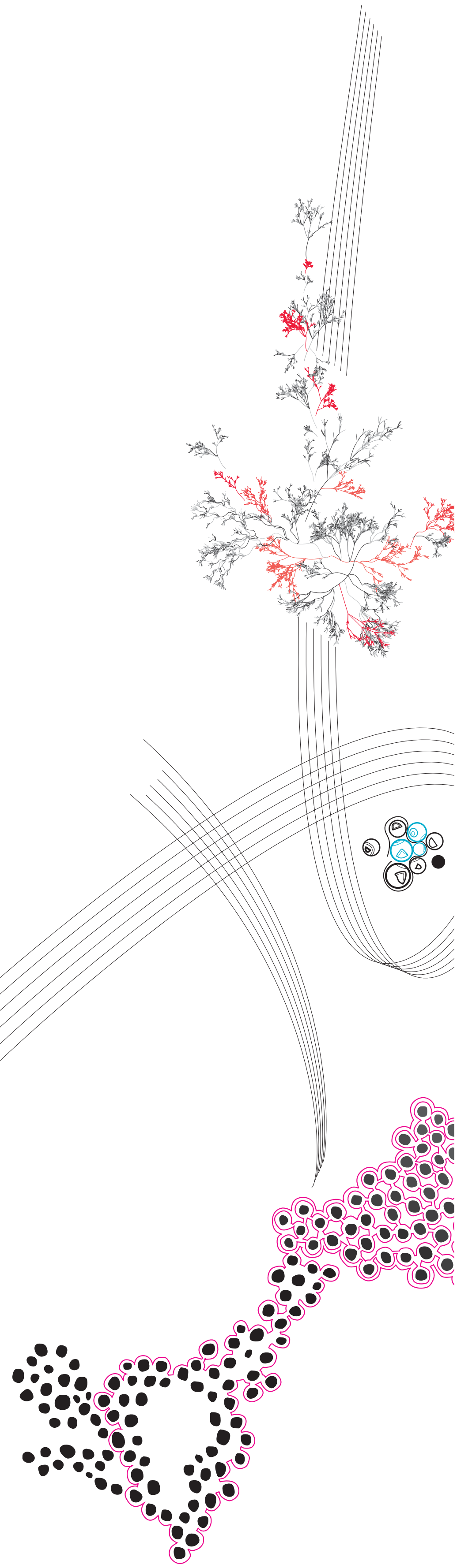
BSc Thesis Applied Mathematics



Probabilistic analysis of distinctive features for discovering growth mechanism in complex networks

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June 25, 2021

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Preface

I want to thank my parents for supporting me during my bachelor study.

I want to thank my supervisors Ms.Litvak and Ms.Bucur for their instructions during my working on my bachelor assignment.

I want to thank my friend Weiting Cai for his companionship during these three years.

I want to thank Ziggy, my neighbourhood's cat, for his visit and stay. He is cute.

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Abstract

Scale-free networks are networks that have power-law degree distribution, at least asymptotically. They are important in complex network theory because many real-world networks are found to be scale-free. In complex network theory, preferential attachment (PA) and fitness (F) are two hypothetical mechanisms that drive the evolution of scale-free networks. Although both of them are able to generate scale-free networks, they are different with respect to the temporal changes they produce during the development of the networks, which might have implications for the future structure of the networks. Therefore, how to discover the growth mechanisms behind a network becomes important. The goal of this work is to do mathematical analysis on distinctive features for discovering growth mechanisms in complex networks. We propose a F-based model with exponentially distributed fitness value and show empirically that it is able to generate scale-free networks for certain parameter values. In addition, we analyze a PA-based model and the F-based model, and show that they are different under certain conditions. In particular, we show that the expected value of the distinctive feature - the average number of new links that a group of nodes receive during a certain time interval after normalization - of the PA-based model is strictly greater than that of the F-based model. Note that this article is part of a larger project that aims to develop a classifier that, given a synthetic network, is able to tell which mechanism from PA and F fits the network the best. The analytical results in this work will be compared to the empirical results obtained in Weiting Cai's work [3] of developing the machine-learning classifier.

Keywords: complex network, scale-free network, power-law, preferential attachment, fitness

1 Introduction

Network is a powerful tool for modelling real-world complex systems, including social networks, biological networks and technological networks. Among these real-life networks, many are found to be scale-free, which means that the asymptotic degree distribution of the networks are power-law. So far, various mechanisms have been proposed to explain the existence of scale-free networks in real life, such as preferential attachment, fitness, aging etc. In addition, remarkable progress has been made in random graph theory on describing these mechanisms by formulating models based on these mechanisms and analyzing statistical measures that characterize these mechanisms. However, current attempts fail at discovering the mechanisms behind the growth of a given network, which might have implications for predicting the future structure of the network and solving related problems such as predicting the spreading of fake news on the internet.

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1.1 Related Work

This article is part of a larger assignment. In this section, only work that is specific to the scope of this article is discussed.

Preferential attachment (PA) and fitness (F) are two hypothetical mechanisms that are able to generate scale-free networks. As for the PA mechanism, it is assumed that, in a network, those vertices that already had many connections are more likely to receive new connections in comparison to those that had fewer connections, which is called "the rich-get-richer" effect. As for the F mechanism, it is assumed that every vertex has an intrinsic value that characterizes its initial attractiveness, and those vertices that have higher intrinsic values are more likely to receive new connections than those that have lower intrinsic values do.

Various models have been proposed for these two mechanisms. As for the PA mechanism, Albert-László Barabási and Réka Albert proposed the Albert-Barabási model [1]. This model combines growth, the concept that the number of vertices in the network increases over time, with the PA mechanism, trying to explain the existence of vertices with extremely high degree in real-life networks. However, this model fails at formulating the growing process of the network rigorously. It did not describe the status of the network at the very beginning, neither did it specify how the edges were attached to the vertices, which resulted in much confusion between mathematicians and theoretical physicists [6]. To solve this problem, Bollobás, Riordan, Spencer and Tusnády specified the initial network and the attachment rule of edges in [2], making self-loops and multi-edges possible, which is the PA-based model investigated in this article.

As for the F mechanism, Ghadge, Shilpa proposed a lognormal fitness attachment model in [5]. Motivated by the fact that the initial attractiveness of a paper in citation networks depends on multiple factors, this model assumes that the overall attractiveness of a paper depends multiplicatively on these factors. Furthermore, this model assumes that each factor that contributes to the overall attractiveness of a paper is of normal distribution. Therefore, in this model, the fitness value of each vertex is a random variable of lognormal distribution, which is a product of a number of normally distributed random variables. However, although it has been shown empirically that this model is able to generate scale-free networks, no formal proof was given. Furthermore, according to the attachment rule proposed in [5], this model does not allow self-loop or multi-edge, which makes it incomparable to the PA-based model proposed by Bollobás, Riordan, Spencer and Tusnády in [2]. Therefore, we formulate our own F-based model in Section 2.4, which allows self-loops and multi-edges, showe empirically that it is able to generate scale-free networks and did analysis based on it.

Although considerable progress on analyzing these two mechanisms has been achieved in random graph theory, they are not useful for identifying the mechanisms in a given network. In particular, current mathematical analysis is often limited to aggregated statistics such as degree distribution and clustering coefficient [7][9], which does not provide information about the temporal changes of the network. As both the PA mechanism and the F mechanism could lead to scale-free networks, given a network generated by the F mechanism and a network generated by the PA mechanism, it is likely that both networks have similar values for the aggregated statistics, making it difficult to determine which mechanism is the one that drives the development of the network. Therefore, to identify the mechanisms behind the growth of the networks, we need measures that incorporate the temporal changes of the networks.

1.2 Research Question

This work is part of a larger project that aims to develop a machine-learning classifier that, given a synthetic network, is able to tell which mechanism from PA and F fits the network the best. The larger project consists of two bachelor assignments, this one and the one of Weiting Cai [3]. In particular, in this work, by doing mathematical analysis on the distinctive feature used in [3] to train the machine-learning classifier, the result of this work explains the performance of the trained classifier in [3].

The overall research question of the larger project is as follows: *What features of a network can enable a machine learning classifier to identify the PA or F mechanism behind the evolution of a network in a mathematically interpretable way?*

Specifically, to answer the research question, the following tasks need to be completed:

- Task 1: Analyze a PA-based model and an F-based model and explore which statistical characteristics are different for the two models.
- Task 2: Design features for the machine-learning classifier.
- Task 3: Train the machine-learning classifier using synthetic data.
- Task 4: Evaluate feature importance based on the training data and the performance of the trained classifier, and compare the result with that of the mathematical analysis.
- Task 5: Interpret the results produced by the classifier based on the results of Task 1.

In this article, we will focus on Task 1 and Task 5, while Weiting Cai will focus on Task 2, Task 3 and Task 4 in his article [3].

1.3 Present Work

At the end of Section 1.1, we argued that current mathematical analysis of aggregated statistics of networks is not sufficient for identifying the mechanisms behind the growth of networks because it is possible for two networks generated by different mechanisms to have similar values for the aggregated statistics. Motivated by this, we propose and analyze an incremental statistic, the average number of new links a vertex s could receive during time interval $[t_0, t_1]$ provided $s < t_0 < t_1$, that reflects the temporal changes of the networks, showing that this incremental statistic could lead to a distinctive feature for discovering growth mechanisms in complex networks under certain conditions.

In this article, we focus on two possible mechanisms behind the evolution of scale-free networks: the preferential attachment (PA) mechanism and the fitness (F) mechanism. In particular, we analyze, for the PA-based model defined by Remco van der Hofstad in his book [6] and the F-based model defined in Section 2.4, the incremental statistic: the average number of new links a vertex s could receive during time interval $[t_0, t_1]$ provided $s < t_0 < t_1$. Based on the analysis on the incremental statistic, we explore under which conditions do the incremental statistics of these two models lead to a distinctive feature such that the scale-free networks generated by the models are distinguishable, which could provide insights into the performance of the machine-learning classifier developed in Weiting Cai's work [3].

Overall our work makes the following contribution:

1. We propose an F-based model with exponentially distributed fitness value and show empirically that it is able to generate scale-free networks given certain values of λ, which is the parameter of the exponential distribution of the fitness values.

2. We propose an incremental statistic that reflects the temporal changes of networks during their development.

3. Based on mathematical analysis on the incremental statistic, we show mathematically that the distinctive feature used in Weiting Cai's work [3] for networks generated by the PA model is always greater than that for networks generated by the F model under certain conditions.

2 Theory

In this section, concepts and theories required for understanding the results are introduced.

2.1 Scale-free Networks

A network is scale-free if it has power-law degree distribution. Specifically, in a network, if the proportion of vertices having degree k, denoted by $P(k)$, goes to $P(k) \sim k^{-\gamma}$ as k goes large, then the network is a scale-free network. As an example, a typical scale-free network looks like this:

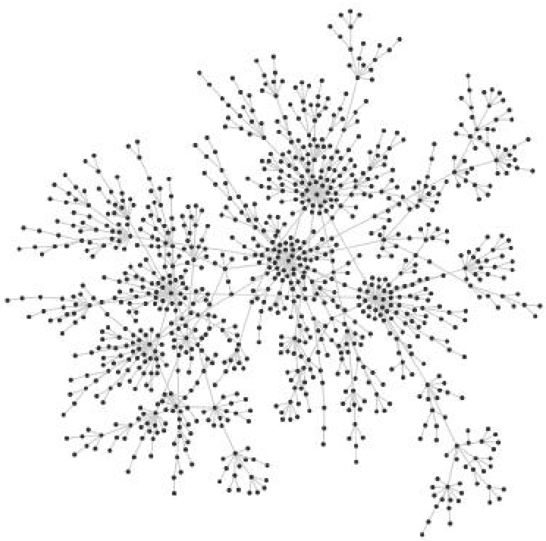

FIGURE 1: scale-free network generated by the graph randomizer of Cytoscape 3.8.2, Barabasi-Albert model with $N = 1000$ and $m = 1$

2.2 Log-log Plot

A log-log plot is a 2-dimensional plot that describes numerical data using logarithmic scales on both horizontal and vertical axes. Given equation

$$y = cx^a, \tag{1}$$

where c and a are constant real numbers, taking the logarithm of both sides gives:

$$\log y = a \log x + \log c, \tag{2}$$

which is a straight line with slope a and intercept $\log c$ on the vertical axis in the log-log plot. Therefore, given that the horizontal axis describes the logarithm of vertex degree k and the vertical axis describes the logarithm of the fraction $P(k)$ of vertices with degree k, if the log-log plot is close to a straight line, such as the plot shown in Figure 2, then it is likely that $P(k) \sim k^a$, where a is constant real number. That is, the degree distribution is power-law.

2.3 Growing Model with Preferential Attachment

In this work, for the PA mechanism, we analyze the PA-based model defined by Remco van der Hofstad in his book [6]. We introduce the model and a related theorem in this

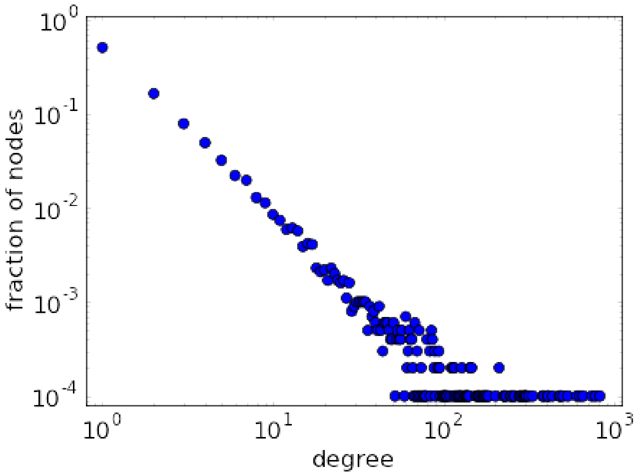

FIGURE 2: [8] log-log plot of a scale-free network

section. Note that the content in the quote blocks is directly quoted from the book [6].

In [6], the model is described as follows:

> The model produces a graph sequence denoted by $(\mathrm{PA}_t^{(m,\delta)})_{t\geq 1}$. At each time step, the model generates a graph with t nodes and mt edges. As $(\mathrm{PA}_t^{(m,\delta)})_{t\geq 1}$ is defined in terms of $(\mathrm{PA}_{mt}^{(1,\delta/m)})_{t\geq 1}$, we first introduce the special case $(\mathrm{PA}_t^{(1,\delta)})_{t\geq 1}$, and then introduce the general case $(\mathrm{PA}_t^{(m,\delta)})_{t\geq 1}$.
>
> First, we describe the case $m = 1$. In this case, $\mathrm{PA}_1^{(1,t)}$ contains a single vertex with a self-loop. We denote the vertices of $\mathrm{PA}_1^{(1,\delta)}$ by $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, .., v_t^{(1)}$. We denote the degree of vertex $v_i^{(1)}$ in $\mathrm{PA}_t^{(1,\delta)}$ by $D_i(t)$. By convention, a self-loop increases the degree by 2. At each time step t, a vertex $v_t^{(1)}$ arrives with an edge incident to it. The other end point of the edge is connected to $v_t^{(1)}$ with probability $(1+\delta)/(t(2+\delta)+(1+\delta))$, and to $v_i^{(1)}$ $i \in \{1, 2, .., t-1\}$ with probability $(D_i(t)+\delta)/(t(2+\delta)+(1+\delta))$.
>
> Next, we describe the model with $m > 1$ in terms of the model with $m = 1$. Fix $\delta \geq -m$, we start with $\mathrm{PA}_{mt}^{(1,\delta/m)}$ and denote the vertices in $\mathrm{PA}_{mt}^{(1,\delta/m)}$ by $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, .., v_{mt}^{(1)}$. Then we identify the vertices $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, .., v_m^{(1)}$ to be the vertex $v_1^{(m)}$ in $\mathrm{PA}_t^{(m,\delta)}$. In general, we collapse the vertices $v_{(j-1)m+1}^{(1)}, v_{(j-1)m+2}^{(1)}, v_{(j-1)m+3}^{(1)}, .., v_{jm}^{(1)}$ to be the vertex $v_j^{(m)}$ in $\mathrm{PA}_t^{(m,\delta)}$. The resulting graph would be a multigraph with t vertices and mt edges.

The PA mechanisms assumes that, in a network, those vertices that already had many connections are more likely to receive new connections in comparison to those that had fewer connections, which is called "the rich-get-richer" effect. This can be seen from the formula $(D_i(t)+\delta)/(t(2+\delta)+(1+\delta))$ in the model. Given δ, the value of $(D_i(t)+\delta)/(t(2+\delta)+(1+\delta))$ increases as the value of $D_i(t)$ increases, corresponding to the "rich-get-richer" effect. Note that in this model, the "rich-get-richer" effect is equivalent to the "old-get-richer" effect, meaning that vertices that arrived earlier are more likely to receive new links in comparison

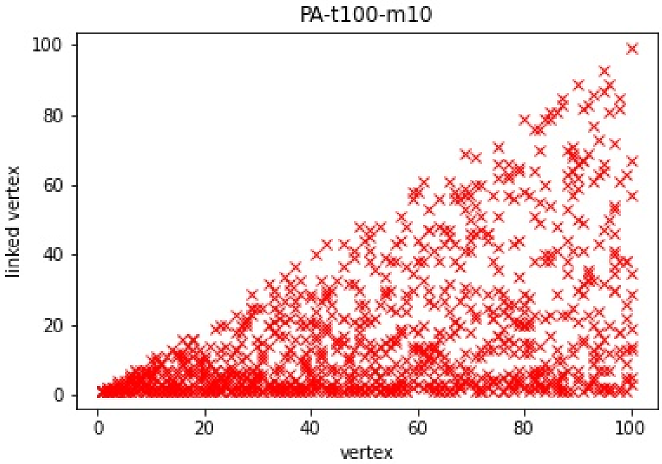

FIGURE 3: [3] visualization of the adjacency matrix of a PA-based synthetic network with number of vertices equal to 100 and $m = 10$

to those that arrived later. There is an intuitive explanation for this equivalence: vertices that arrived earlier have less competitors, thus are more likely to receive new connections, which resulted in their becoming vertices with many connections later, attracting even more new connections. Furthermore, to illustrate the idea of the "old-get-richer" effect, visualization of the adjacency matrix of a synthetic network generated by the model might help (Figure 3). The x-axis denotes the arriving vertices and the y-axis denotes the vertices to be linked by the arriving vertices [3]. From this figure, it can be seen that most of the new connections are linked to vertices $\{v_t | 0 < t < 10\}$, which are the vertices that arrived early.

In addition to the model, we introduce a related theorem that will be used in results:

Theorem 2.1. *[6] Fix $m = 1$ and $\delta > -1$. Then, $D_i(t)/t^{1/(2+\delta)}$ converges almost surely to a random variable ξ_i as $t \to \infty$ and*

$$\mathbf{E}[D_i(t) + \delta] = \frac{\Gamma(t+1)\Gamma(i - 1/(2+\delta))}{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i)} \tag{3}$$

2.4 Growing Model with Fitness

In this assignment, for the F mechanism, we introduce and analyze the F-based model defined by ourselves. In this model, the fitness value of each vertex is an exponentially distributed random variable with parameter λ. We introduce the model in this section and show empirically that this model is able to generate scale-free networks for certain values of λ. By analogy with PA, we consider $m = 1$ and $m > 1$ separately.

The model produces a graph sequence denoted by $(\mathrm{F}_t^{(m,\lambda)})_{t \geq 1}$. At each time step, the model generates a graph with t nodes and mt edges. We denote the vertices in $\mathrm{F}_t^{(m,\lambda)}$ by $v_1, v_2, v_3, .., v_t$. For each vertex v_i, we denote the fitness value of the vertex, which is a random variable of $exp(\lambda)$ distribution, by Φ_i.

Given $m \geq 1$, $\mathrm{F}_1^{(m,\delta)}$ contains m isolated vertices and no edges. At each time step $t \geq 1$, a vertex v_t with fitness value ϕ_t arrives with m edges incident to it. For each edge incident to

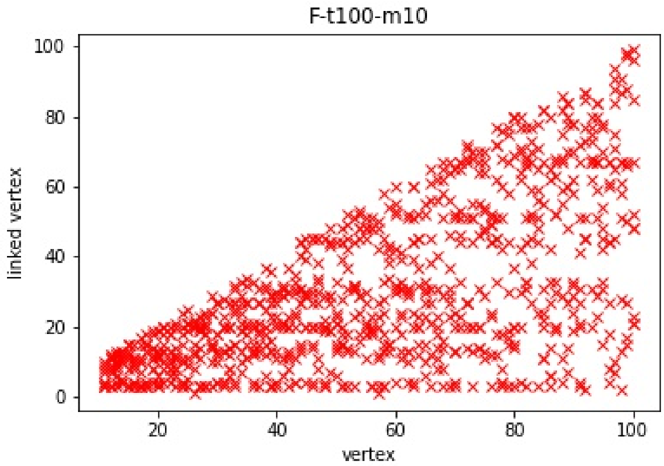

Figure 4: [3] visualization of the adjacency matrix of a F-based synthetic network with number of vertices equal to 100 and $m = 10$

v_t, the other end of the edge is connected to v_i with probability $\phi_i / \sum_j \phi_j$. The resulting graph would be a multigraph with t vertices and mt edges.

To illustrate the difference between the F-based model and the PA-based model, we present a visualization of the adjacency matrix of a synthetic network generated by the F-based model. In Figure 4, it can be seen that, unlike Figure 3, new connections no longer concentrate at the bottom of the figure, meaning that the "rich-get-richer" or the "old-get-richer" effect that exists in the PA networks disappears. That is, even a vertex arrived late, it may still attract large number of new connections due to its high fitness value.

Below in Figure 5 and Figure 6 are two log-log plots of the eventual degree distribution of the networks generated by the above F-based model. It can be seen from the plots that the above F-based model is able to generate scale-free networks for certain parameter values. Using these log-log plots, we have shown empirically that the F-based model we proposed is able to generate scale-free networks for $1 \leq \lambda < 10$. Since most of the networks used in Weiting Cai's work are of $1000 \leq T \leq 2000$ and $10 \leq m \leq 15$ [3], for consistency, we only show plots for networks with $T \in \{1000, 2000\}$ and $m \in \{10, 15\}$. For more plots, see Appendix A.

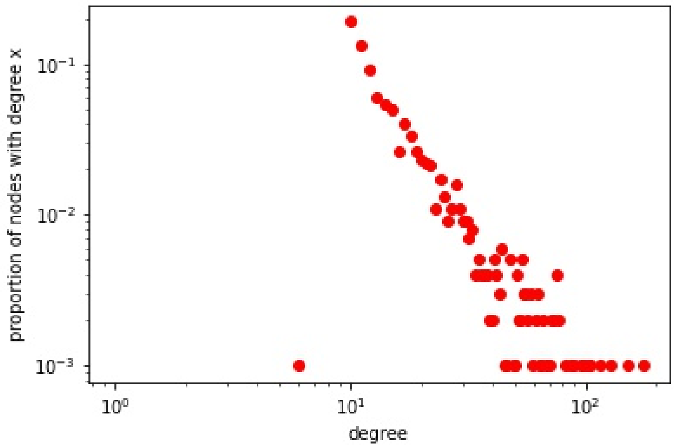

FIGURE 5: $m = 10$, $T = 1000$, $\lambda = 1$

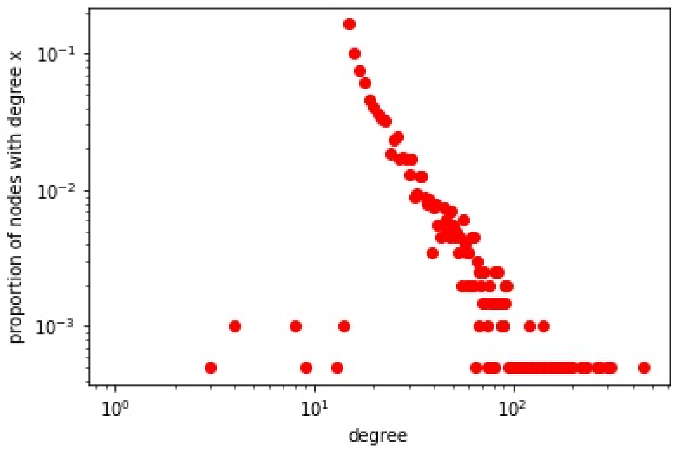

FIGURE 6: $m = 15$, $T = 2000$, $\lambda = 1$

2.5 The Distinctive Feature

As explained in Section 1.2, this work is part of a larger project, which consists of this bachelor assignment and the bachelor assignment of Weiting Cai. To connect our results to the results of Weiting Cai's work [3], in this section, we introduce the definition of the feature matrix used in Weiting Cai's work for training the machine-learning model.

In Weiting Cai's work [3], for feature engineering, he proposed a flexible and scalable feature design that organizes features in a matrix $M_{a\times b}$ as follows:

> Rows of this matrix correspond to time, columns correspond to groups of nodes, and the cells contain the network's incremental statistics.
>
> Formally, learned from the work of our supervisors, the general $a \times b$ feature matrix is defined as follows:
>
> Let $G(T)$ be the set of vertices generated on the interval $[0,T]$ and $d_k(t)$ be the degree of each vertex v_k at time step t. Define
>
> $$F_T(x) = \frac{1}{T}\sum_{v_k\in G(T)} 1\{d_k(T) \leq x\}, x \in [0, 1, .., \max_{v_k\in G(T)} d_k(T)],$$
>
> to be the empirical distribution of the vertex degrees in the final network. For each $d_k(T)$, let q_k be such that $d_k(T)$ is the q_k-quantile of $F_T(x)$. Then we divide $G(T)$ into b groups as follows:
>
> $$G_j = \{v_k \in G(T) : \frac{j-1}{b} < q_k \leq \frac{j}{b}\}, j \in \{1, 2, .., b\}.$$
>
> In particular, G_1 contains the vertices, of which degrees end up in the highest $\frac{1}{b}$%, while G_b contains the vertices with degrees ending up in the lowest $\frac{1}{b}$%.
>
> Then we divide the time into a consecutive non-overlapping intervals with equal length, which are $T_1 = [0,\frac{T}{a}]$, $T_2 = [\frac{T}{a},\frac{2T}{a}]$,.., $T_a = [\frac{(a-1)T}{a},T]$.
>
> Finally, we compute the incremental statistics in the cells - M_{ij} = the average number of new links received per vertex in vertex group G_j during time interval T_i) - as follows:
>
> $$M_{ij} = \frac{1}{|G_j|}\sum_{v_k\in G_j}\left(d_k\left(\frac{i}{b}T\right) - d_k\left(\frac{i-1}{b}T\right)\right), i \in \{1, 2, .., a\}, j \in \{1, 2, .., b\}.$$

Next, to reduce overfitting, he normalized M so that the sum of all entries of M is equal to 1. Let M' denote the normalized matrix, we have:

$$M'_{ij} = \frac{M_{ij}}{\sum_{k\in\{1,2,..,a\},l\in\{1,2,..,b\}} M_{kl}}, i \in \{1, 2, .., a\}, j \in \{1, 2, .., b\}.$$

In Weiting's work, he randomly picks $a = 3$ and $b = 4$. In this work, for consistency, we keep these values. Denote the feature in the upper left corner of the normalized matrix by (T_1, G_1), $T_1 = [0, T/3]$ and $G_1 = \{v_k \in G(T) : \frac{1}{4} < q_k \leq 1\}$.

3 Results

In this section, the results of analyzing the incremental statistic, the average number of new links a vertex s could receive during time interval $[t_0, t_1]$ provided $s < t_0 < t_1$, for both the PA-based model introduced in Section 2.3 and the F-based model defined in Section 2.4 are presented. Specifically, for both models, the expected value of the incremental statistic is computed. For the PA-based model, a lower bound is derived for the expected value. For the F-based model, an upper bound is derived for the expected value. Finally, we discuss a special case for $\delta = 0$ and $\lambda = 1$, by connecting the analysis on the incremental statistic to the distinctive feature defined in Section 2.5, we show that the distinctive feature introduced in Section 2.5 can be used to distinguish the synthetic networks generated by the two models under this condition.

3.1 Analysis of the PA-based model

In this section, the incremental statistic is analyzed for the PA-based model introduced in Section 2.3. First, we derive a formula for the expectation of the incremental statistic. Next, we derive a lower bound for the expectation.

In Theorem 3.1, the expectation of the incremental statistic for the PA-based model is derived for $m = 1$.

Theorem 3.1. *Let $(PA_t^{(1,\delta)})_{t\geq 1}$ be a graph sequence defined in Section 2.3. Suppose $X_s^{(1)}(t_0, t_1)$ is the number of new links that vertex $v_s^{(1)}$ could receive during time interval $[t_0, t_1]$ $(s < t_0 < t_1)$, then* $\mathbb{E}[X_s^{(1)}(t_0, t_1)] = \sum_{t=t_0}^{t_1} \frac{1+\delta}{t(2+\delta)+(1+\delta)} \frac{\Gamma(t+1)\Gamma(s-1/(2+\delta))}{\Gamma(t+(1+\delta)/(2+\delta))\Gamma(s)}$.

Proof. Let $v_s \leftarrow v_t$ denote the fact that v_t is connected to v_s.

$$\begin{aligned}
\mathbb{E}[X_s^{(1)}(t_0, t_1)|D_s(t_0), D_s(t_0+1), .., D_s(t_1)] &= \mathbb{E}[\sum_{t=t_0}^{t_1} \mathbf{1}[v_s \leftarrow v_t|D_s(t)]] \\
&= \sum_{t=t_0}^{t_1} \mathbb{E}[\mathbf{1}[v_s \leftarrow v_t|D_s(t)]] \\
&= \sum_{t=t_0}^{t_1} P(v_s \leftarrow v_t|D_s(t)) \\
&= \sum_{t=t_0}^{t_1} \frac{D_s(t)+\delta}{t(2+\delta)+(1+\delta)},
\end{aligned} \tag{4}$$

where $D_s(t)$ is a random variable. By Theorem 2.1, the expected value of $D_s(t) + \delta$ is:

$$\mathbb{E}[D_s(t) + \delta] = (1+\delta)\frac{\Gamma(t+1)\Gamma(s-1/(2+\delta))}{\Gamma(t+(1+\delta)/(2+\delta))\Gamma(s)} \tag{5}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X_s^{(1)}(t_0, t_1)] &= \mathbb{E}[\mathbb{E}[X_s^{(1)}(t_0, t_1)|D_s(t_0), D_s(t_0+1), .., D_s(t_1)]] \\
&= \sum_{t=t_0}^{t_1} \frac{1+\delta}{t(2+\delta)+(1+\delta)} \frac{\Gamma(t+1)\Gamma(s-1/(2+\delta))}{\Gamma(t+(1+\delta)/(2+\delta))\Gamma(s)}
\end{aligned} \tag{6}$$

□

In Theorem 3.2, the result of Theorem 3.1 is extended to $m > 1$.

Theorem 3.2. *Let $(PA_t^{(m,\delta)})_{t\geq 1}$ be a graph sequence defined in 2.3. Note that $m > 1$. Suppose $X_s^{(m)}(t_0, t_1)$ is the number of new links that vertex $v_s^{(m)}$ could receive during time interval $[t_0, t_1]$ $(s < t_0 < t_1)$. We have*

$$\mathbb{E}[X_s^{(m)}(t_0, t_1)] = \frac{1+\delta/m}{t(2+\delta/m)+(1+\delta/m)} \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{\Gamma(t+1)\Gamma(k-1/(2+\delta/m))}{\Gamma(t+(1+\delta/m)/(2+\delta/m))\Gamma(k)}. \tag{7}$$

Proof. By definition of $(\mathrm{PA}_t^{(m,\delta)})_{t\geq 1}$ $(m \geq 1)$, it is obtained by collapsing $(\mathrm{PA}_{mt}^{(1,\delta/m)})_{t\geq 1}$. Therefore, the number of new links that node $v_s^{(m)}$ could receive during time interval $[t_0, t_1]$ is equal to the number of new links that nodes $v_{m(s-1)+1}^{(1)}, v_{m(s-1)+2}^{(1)}, .., v_{sm}^{(1)}$ could receive during time interval $[mt_0, mt_1]$.

By Theorem 3.1, for a graph sequence $\mathrm{PA}_{mt}^{(1,\delta/m)})_{t\geq 1}$, the expected number of new links that $v_k^{(1)}$ could receive during time interval $[mt_0, mt_1]$, denoted by $X_k^{(1)}(mt_0, mt_1)$ is

$$\mathbb{E}[X_k^{(1)}(mt_0, mt_1)] = \sum_{t=mt_0}^{mt_1} \frac{1+\delta/m}{t(2+\delta/m)+(1+\delta/m)} \frac{\Gamma(t+1)\Gamma(k-1/(2+\delta/m))}{\Gamma(t+(1+\delta/m)/(2+\delta/m))\Gamma(k)} \tag{8}$$

Therefore,

$$\begin{aligned}\mathbb{E}[X_s^{(m)}(t_0, t_1)] &= \sum_{k=m(s-1)+1}^{ms} \mathbb{E}[Z_k^{(1)}(mt_0, mt_1)] \\ &= \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{1+\delta/m}{t(2+\delta/m)+(1+\delta/m)} \frac{\Gamma(t+1)\Gamma(k-1/(2+\delta/m))}{\Gamma(t+(1+\delta/m)/(2+\delta/m))\Gamma(k)}\end{aligned} \tag{9}$$

□

In Theorem 3.3, a lower bound is derived for the expectation obtained in Theorem 3.2.

Theorem 3.3. *Let $(PA_t^{(m,\delta)})_{t\geq 1}$ be a graph sequence defined in Section 2.3. Note that $m \geq 1$. Suppose $X_s^{(m)}(t_0, t_1)$ is the number of new links that vertex s could receive during time interval $[t_0, t_1]$ $(s < t_0 < t_1)$. Then we have*

$$\mathbb{E}[X_s^{(m)}(t_0, t_1)] < \frac{1+\delta}{2+\delta} \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{\Gamma(k-1/(2+\delta))}{\Gamma(k)} \frac{t^{1/(2+\delta)}}{t+(1+\delta)/(2+\delta)} \tag{10}$$

.

Proof. By Theorem 3.2,

$$\mathbb{E}[X_s^{(m)}(t_0, t_1)] = \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{1+\delta/m}{t(2+\delta/m)+(1+\delta/m)} \frac{\Gamma(t+1)\Gamma(k-1/(2+\delta/m))}{\Gamma(t+(1+\delta/m)/(2+\delta/m))\Gamma(k)} \tag{11}$$

By Gautschi's inequality [4], $x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$ for positive and real number x and $s \in (0,1)$. Therefore,

$$\frac{\Gamma(t+1)}{\Gamma(t+(1+\delta/m)/(2+\delta/m))} > t^{1/(2+\delta/m)} \tag{12}$$

, which implies:

$$\begin{aligned}
\mathbb{E}[X_s] &= \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{1+\delta/m}{t(2+\delta/m)+(1+\delta/m)} \frac{\Gamma(t+1)\Gamma(s-1/(2+\delta/m))}{\Gamma(t+(1+\delta/m)/(2+\delta/m))\Gamma(s)} \\
&> \frac{1+\delta}{2+\delta} \sum_{k=m(s-1)+1}^{ms} \sum_{t=mt_0}^{mt_1} \frac{\Gamma(k-1/(2+\delta))}{\Gamma(k)} \frac{t^{1/(2+\delta)}}{t+(1+\delta)/(2+\delta)}
\end{aligned} \tag{13}$$

□

3.2 Analysis of the F-based model

In this section, the incremental statistic is analyzed for the F-based model defined in Section 2.4. First, we derive a formula for the expectation of the incremental statistic. Next, we derive an upper bound for the expectation.

In Theorem 3.4, for the F-based model, the expectation of the incremental statistic conditioned on the fitness value Φ_s of vertex v_s is derived.

Theorem 3.4. *Let $(F_t^{(m,\lambda)})_{t\geq 1}$ be a graph sequence defined in Section 2.4. Suppose $X_s(t_0,t_1)$ is the number of new links that vertex v_s could receive during time interval $[t_0,t_1]$ ($s < t_0 < t_1$). Given that the fitness value of vertex s is ϕ_s, $\mathbb{E}[X_s(t_0,t_1)] = \sum_{t=t_0}^{t_1} \mathbf{E}[\frac{\phi_s}{Y_t+\phi_s}]$, where Y_t is a random variable of distribution Gamma$(t-1,\lambda)$.*

Proof. By definition, in a graph sequence $(\mathrm{F}_t^{(m,\lambda)})_{t\geq 1}$, each vertex v_i arrives with m edges incident to it. We denote the edges that are incident to v_i when v_i arrives by $e_1^i, .., e_m^i$. Let $v_s \leftarrow e_i^t$ denote the fact that the other end of e_i^t is connected to v_s. By definition,

$$\begin{aligned}
\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] &= \mathbb{E}[\sum_{t=t_0}^{t_1}\sum_{i=1}^{m} \mathbf{1}[v_s \leftarrow e_i^t|\Phi_s=\phi_s]] \\
&= \mathbb{E}[\sum_{t=t_0}^{t_1}\sum_{i=1}^{m} P(v_s \leftarrow e_i^t|\Phi_s=\phi_s)] \\
&= \sum_{t=t_0}^{t_1}\sum_{i=1}^{m} \mathbb{E}[P(v_s \leftarrow e_i^t|\Phi_s=\phi_s)] \\
&= \sum_{t=t_0}^{t_1}\sum_{i=1}^{m} \mathbb{E}[P(v_s \leftarrow e_i^t|\Phi_s=\phi_s)] \\
&= \sum_{t=t_0}^{t_1}\sum_{i=1}^{m} \mathbb{E}[\frac{\phi_s}{\Phi_1+..+\phi_s+\Phi_{s+1}+..+\Phi_t}] \\
&= m\sum_{t=t_0}^{t_1} \mathbb{E}[\frac{\phi_s}{\Phi_1+..+\phi_s+\Phi_{s+1}+..+\Phi_t}]
\end{aligned} \tag{14}$$

Since sum of n exponentially distributed random variables with parameter λ is a Gamma distribution with parameters $(n-1, \lambda)$, the above expression is equal to:

$$\begin{aligned}
\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] &= m\sum_{t=t_0}^{t_1}\mathbb{E}[\frac{\phi_s}{\Phi_1+..+\phi_s+\Phi_{s+1}+..+\Phi_t}] \\
&= m\sum_{t=t_0}^{t_1}\mathbb{E}[\frac{\phi_s}{Y_t+\phi_s}]
\end{aligned} \tag{15}$$

, where Y_t is a random variable of distribution $Gamma(t-1, \lambda)$. □

In Theorem 3.5, for the F-based model, an upper bound is derived for the expectation of the incremental statistic derived in Theorem 3.4.

Theorem 3.5. *Let $(F_t^{(m,\lambda)})_{t\geq 1}$ be a graph sequence defined in Section 2.4. Suppose $X_s(t_0,t_1)$ is the number of new links that vertex v_s could receive during time interval $[t_0,t_1]$ provided $s < t_0 < t_1$ and $t_0 > 2$. Given that the fitness value of vertex s is ϕ_s, then $\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] < \phi_s\sum_{t=t_0}^{t_1}\lambda^{t-1}\frac{1}{t-2}$.*

Proof. By Theorem 3.4, the expectation of $X_s(t_0,t_1)$ is equal to $m\sum_{t=t_0}^{t_1}\mathbb{E}[\frac{\phi_s}{Y+\phi_s}]$. Assume $\lambda \geq 1$, given $Y \sim Gamma(t-1,\lambda)$, substitute the probability density function of Y, $p(y) = \frac{\lambda^{t-1}y^{t-2}e^{-\lambda y}}{\Gamma(t-1)}$, into the expectation, we obtain:

$$\begin{aligned}
\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] &= m\sum_{t=t_0}^{t_1}\mathbb{E}[\frac{\phi_s}{Y+\phi_s}] \\
&= m\sum_{t=t_0}^{t_1}\int_0^\infty \frac{\phi_s}{y+\phi_s}p(y)dy \\
&= m\sum_{t=t_0}^{t_1}\int_0^\infty \frac{\phi_s}{y+\phi_s}\frac{\lambda^{t-1}y^{t-2}e^{-\lambda y}}{\Gamma(t-1)}dy \\
&\leq m\sum_{t=t_0}^{t_1}\lambda^{t-1}\int_0^\infty \frac{\phi_s}{y+\phi_s}\frac{y^{t-2}e^{-y}}{\Gamma(t-1)}dy \\
&= m\sum_{t=t_0}^{t_1}\frac{\phi_s\lambda^{t-1}}{\Gamma(t-1)}\int_0^\infty \frac{y^{t-2}e^{-y}}{y+\phi_s}dy \\
&< m\sum_{t=t_0}^{t_1}\frac{\phi_s\lambda^{t-1}}{\Gamma(t-1)}\int_0^\infty \frac{y^{t-2}e^{-y}}{y}dy \\
&= m\sum_{t=t_0}^{t_1}\frac{\phi_s\lambda^{t-1}}{\Gamma(t-1)}\int_0^\infty y^{t-3}e^{-y}dy
\end{aligned} \tag{16}$$

By definition of Gamma function,$\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, $\Gamma(y) = \int_0^\infty x^{y-1}e^{-y}dy$ for

$y \in \mathbb{R}$. Therefore, the inequality in (16) is equivalent to:

$$\begin{aligned}
\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] &= m\sum_{t=t_0}^{t_1} E[\frac{\phi_s}{Y+\phi_s}] \\
&< m\phi_s \sum_{t=t_0}^{t_1} \lambda^{t-1}\frac{\Gamma(t-2)}{\Gamma(t-1)} \\
&= m\phi_s \sum_{t=t_0}^{t_1} \lambda^{t-1}\frac{1}{t-2}
\end{aligned} \tag{17}$$

□

In Theorem 3.6, for the F-based model, an upper bound is derived for the expectation of the incremental statistic without conditioning.

Theorem 3.6. *Let $(F_t^{(m,\lambda)})_{t\geq 1}$ be a graph sequence defined in Section 2.4. Suppose $X_s(t_0,t_1)$ is the number of new links that vertex v_s could receive during time interval $[t_0,t_1]$ provided $s<t_0<t_1$ and $t_0>2$. Then we have $\mathbb{E}[X_s(t_0,t_1)] < m\sum_{t=t_0}^{t_1}\lambda^{t-2}\frac{1}{t-2}$.*

Proof. By Theorem 3.5,

$$\mathbb{E}[X_s(t_0,t_1)|\Phi_s=\phi_s] < m\phi_s \sum_{t=t_0}^{t_1} \lambda^{t-1}\frac{1}{t-2} \tag{18}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X_s(t_0,t_1)] &= \mathbb{E}[\mathbb{E}[X_s(t_0,t_1)|\Phi_s]] \\
&< \mathbb{E}[m\Phi_s \sum_{t=t_0}^{t_1} \lambda^{t-1}\frac{1}{t-2}] \\
&= m\mathbb{E}[\Phi_s] \sum_{t=t_0}^{t_1} \lambda^{t-1}\frac{1}{t-2} \\
&= m \sum_{t=t_0}^{t_1} \lambda^{t-2}\frac{1}{t-2}
\end{aligned} \tag{19}$$

□

3.3 Special Case

In this section, we discuss a special case of the above results, when $\delta = 0$ and $\lambda = 1$. We combine our results and the results obtained in Weiting Cai's work [3], trying to explain the performance of the machine-learning classifier in [3].

According to the feature importance analysis in Weiting Cai's work, (T_1, G_1) alone is sufficient to distinguish the networks generated by the two models with $\delta = 0$ and $\lambda = 1$. Therefore, in this section, we will only analyze (T_1, G_1).

First, using the expected values of the incremental statistic we obtained in Section 3.1, we give an estimate of the expected value of the feature (T_1, G_1) for the networks generated by the PA-based model.

Let $(\mathrm{PA}_t^{(m,\delta)})_{1\le t\le T}$ be a graph sequence generated by the PA-based model during time interval $[0,T]$. Let T_1 and G_1 be the same as in Section 2.5. That is, $T_1 = [0, \frac{T}{3}]$ and $G_1 = \{v_k \in G(T) : \frac{1}{4} < q_k \le 1\}$. As for the PA-based model, due to the "old-get-richer" effect, we may use vertices that arrive during time interval $[0, \frac{T}{4}]$ to approximate the vertices that end up in G_1. Since vertices that arrive earlier are more likely to attract new connections than those that arrive later, it is likely that the vertices that arrive during time interval $[0, \frac{T}{4}]$ are also vertices that end up with degrees in the first quantile of the overall degree distribution at time step T (vertices in G_1). Given this approximation, we formulate an estimate for the expected value of the feature $(T_1,\ G_1)$ for the PA-based model:

$$\mathbb{E}[(T_1, G_1)] \approx \sum_{s\in[0,\frac{T}{4}]} \mathbb{E}[X_s(s, \frac{T}{3})], \tag{20}$$

where the definition of $X_s(s, \frac{T}{3})$ is the same as in Section 3.1 with $t_0 = s$ and $t_1 = \frac{T}{3}$. Note that $t_0 = s$ because vertex v_s will start receiving new connections only after it has arrived at time step s.

Furthermore, by Theorem 3.3,

$$\mathbb{E}[X_s(s, \frac{T}{3})] > \frac{1}{2} \sum_{k=m(s-1)+1}^{ms} \sum_{t=ms}^{m\frac{T}{3}} \frac{\Gamma(k-1/2)}{\Gamma(k)} \frac{t^{1/2}}{t+1/2} \tag{21}$$

Therefore, this estimate has a lower bound for $\delta = 0$:

$$\begin{aligned} \mathbb{E}[(T_1, G_1)] &\approx \sum_{s\in[0,\frac{T}{4}]} \mathbb{E}[X_s(s, \frac{T}{3})] \\ &> \frac{1}{2} \sum_{s\in[0,\frac{T}{4}]} \sum_{k=m(s-1)+1}^{ms} \sum_{t=ms}^{m\frac{T}{3}} \frac{\Gamma(k-1/2)}{\Gamma(k)} \frac{t^{1/2}}{t+1/2} := L_{pa} \end{aligned} \tag{22}$$

Let $(\mathrm{F}_t^{(m,\lambda)})_{1\le t\le T}$ be a graph sequence generated by the F-based model during time interval $[0,T]$. Let T_1 and G_1 be the same as in Section 2.5. That is, $T_1 = [0, \frac{T}{3}]$ and $G_1 = \{v_k \in G(T) : \frac{1}{4} < q_k \le 1\}$. As for the F-based model, due to the effect of the fitness value, vertices that arrive during time interval $[0, \frac{T}{4}]$ are not good approximate of the vertices that end up in G_1. However, for those vertices that arrive after $\frac{T}{4}$ but end up in G_1, the number of new connections they could receive during time interval $[0, \frac{T}{4}]$ is 0, which does not contribute to (T_1, G_1), while those vertices that arrive during $[0, \frac{T}{4}]$ but did not end up in G_1 could receive more connections during $[0, \frac{T}{4}]$. Therefore, although replacing the vertices in G_1 by the vertices that arrive $[0, \frac{T}{4}]$ could not lead to a good estimate of the expected value of (T_1, G_1), it could result in an upper bound for (T_1, G_1). Therefore, by the above analysis and Theorem 3.6, for $\lambda = 1$,

$$\begin{aligned} \mathbb{E}[(T_1, G_1)] &\le \sum_{s\in[0,m]} \mathbb{E}[X_s(m+1, \frac{T}{3})] + \sum_{s\in[m+1,\frac{T}{4}]} \mathbb{E}[X_s(s, \frac{T}{3})] \\ &< m \sum_{s\in[0,m]} \sum_{t=m+1}^{\frac{T}{3}} \frac{1}{t-2} + m \sum_{s\in[m+1,\frac{T}{4}]} \sum_{t=s}^{\frac{T}{3}} \frac{1}{t-2} := U_f \end{aligned} \tag{23}$$

T	m	U_f	L_pa
1000	10	0.3187957712	0.3260765191
1000	15	0.312871218	0.3263613002
1500	10	0.3205477326	0.3266499592
1500	15	0.3165980304	0.3268518749
1500	30	0.3056019807	0.3270704295
1500	45	0.2950382106	0.3271499061
2000	10	0.3209287661	0.3265115227
2000	15	0.3179664895	0.3266696533
2500	10	0.3214063876	0.3267784607
2500	15	0.3190365663	0.3269093004
3000	15	0.3197224979	0.3270704295
5000	15	0.3207061959	0.3270509992

FIGURE 7: values of U_f and L_{pa} for various T and m

Note that the addition in the inequality (23) is due to the initial status of the F-based model: at time step 1, there are m isolated vertices in the graph.

Computing the lower bound L_{pa} in the inequality (22) and the upper bound U_f in the inequality (23) for various T and m, we obtain the results in Figure 7. It can be seen that L_{pa} varies around 0.326 and U_f varies between 0.295 and 0.321, indicating that the expected value of the feature (T_1, G_1) for the PA-based model is always greater than that for the F-based model and that the threshold value to distinguish the networks generated by the two models is around 0.321. Furthermore, according to the results obtained by Weiting Cai, the threshold value that the decision tree uses to classify the synthetic networks generated by the two models is around 0.315 [3], which is very close to the theoretical approximation 0.321 we obtained. Note that the choices of T and m are made based on the training and testing data used in Weiting Cai's work [3].

4 Discussion

In this work, only the case with $\delta = 0$ and $\lambda = 1$ is analysed, which is not sufficient to provide insights into the performance of the machine learning classifier in general. In addition, while the expected values of the distinctive features of the two models are compared, the variances of the distinctive features are not analyzed.

Furthermore, it should be noted that the results are based on approximation of the lower bound and the upper bound of the distinctive features.

5 Conclusions

In conclusion, in this article, we propose an F-based model and show empirically that it is able to generate scale-free networks for certain values of λ. We define an incremental statistic for complex networks and derive a lower bound for the expected value of this incremental statistic in PA networks and an upper bound for that in F networks. Using the lower bound and the upper bound of the incremental statistic, we give an estimate on a lower bound of the expected value of the distinctive feature (defined in Section 2.5) for the PA networks and an upper bound of that for the F networks.

Finally, by computing the estimated lower bound and upper bound of the distinctive feature for the two models separately and for different T and m, we show that under certain conditions, the estimated lower bound is always greater than the estimated upper bound, implying that these two types of networks are distinguishable under these conditions. Furthermore, according to the computation result, the estimated threshold value that distinguishes the two types of networks is around 0.321, which is close to the empirical value 0.315 obtained in Weiting Cai's work [3]. This, in combination with the results obtained by Weiting Cai [3], answers the general research question.

6 Recommendations

In this assignment, only two separate mechanisms are analyzed, namely the preferential attachment mechanism and the fitness mechanism. If time allows, it would be interesting to analyze the combination of the two mechanisms and other mechanisms such as aging (A), which assumes that a vertex's attractiveness decline over time. Intuitively, it is expected that the combination of PA and A could exhibit similar characteristics as F only because both the A mechanism and the F mechanism could counteract the "old-get-richer" effect as network grows. In addition, it would also be exciting to prove that the F-based model we proposed indeed generates scale-free networks, as the plots show exactly this. Furthermore, our current analysis does not investigate how varying values δ and λ will influence the performance of the classifier, while Weiting Cai has found that these two types of networks are still distinguishable with varying values of parameters, just with a different feature and a different threshold value [3]. Therefore, it would also be interesting to investigate the general relationship of the two derived bounds.

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A Log-log plots of degree distribution for F networks

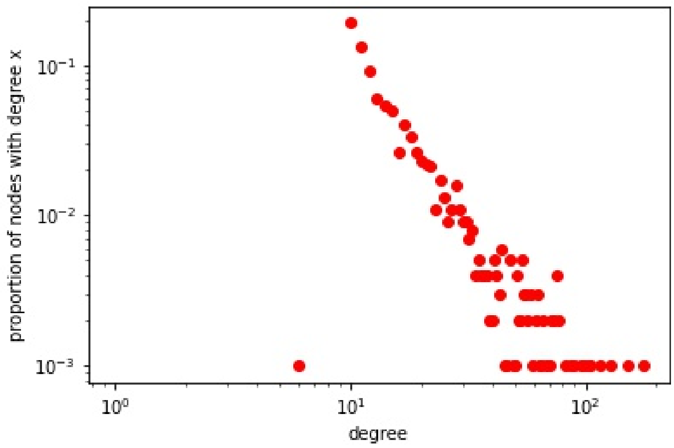

FIGURE 8: $m = 10$, $T = 1000$, $\lambda = 1.0$

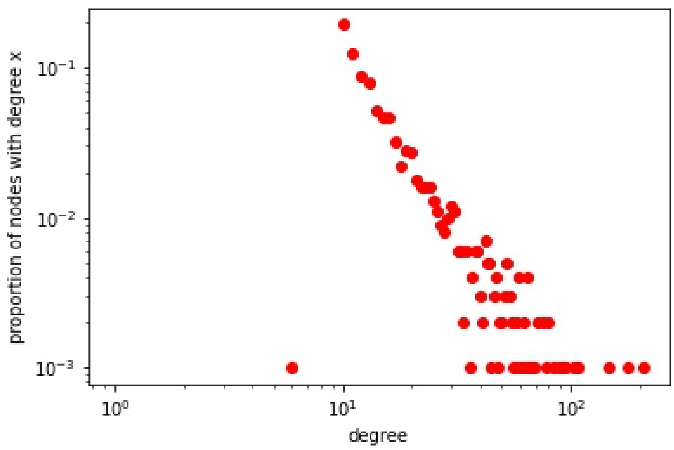

FIGURE 9: $m = 10$, $T = 1000$, $\lambda = 1.5$

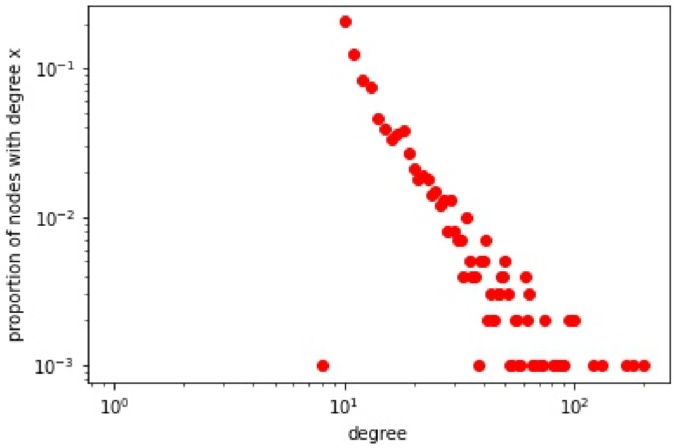

FIGURE 10: $m = 10$, $T = 1000$, $\lambda = 2.0$

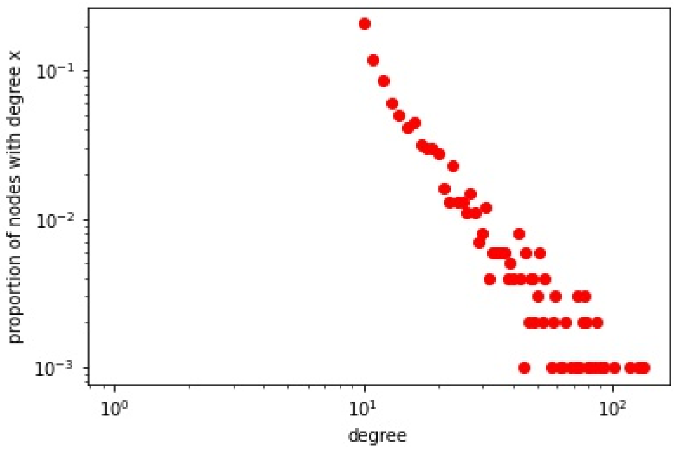

FIGURE 11: $m = 10$, $T = 1000$, $\lambda = 2.5$

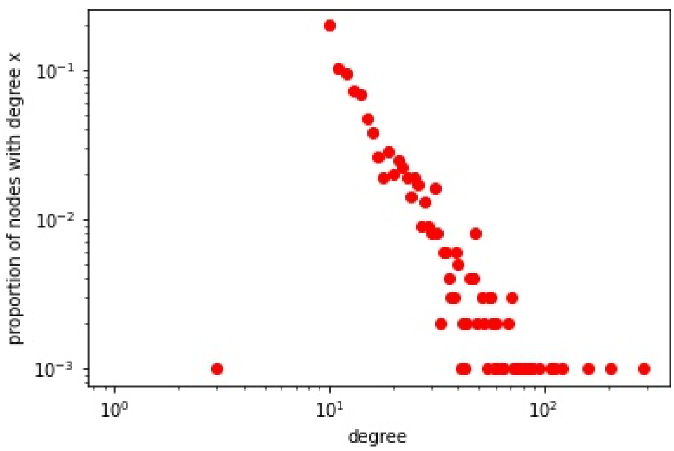

FIGURE 12: $m = 10$, $T = 1000$, $\lambda = 3.0$

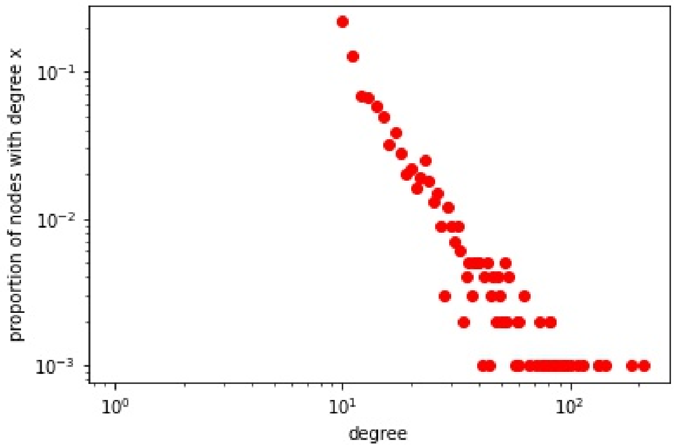

FIGURE 13: $m = 10$, $T = 1000$, $\lambda = 3.5$

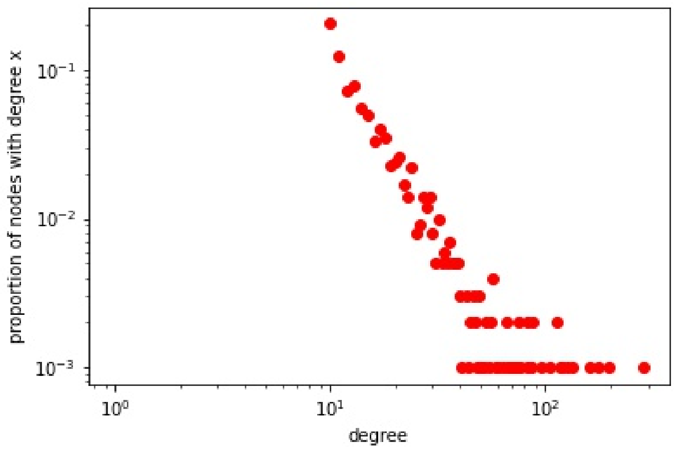

FIGURE 14: $m = 10$, $T = 1000$, $\lambda = 7$

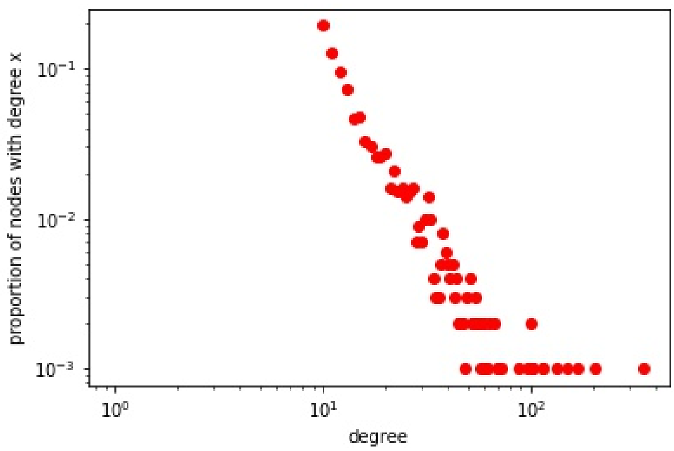

FIGURE 15: $m = 10$, $T = 1000$, $\lambda = 8$

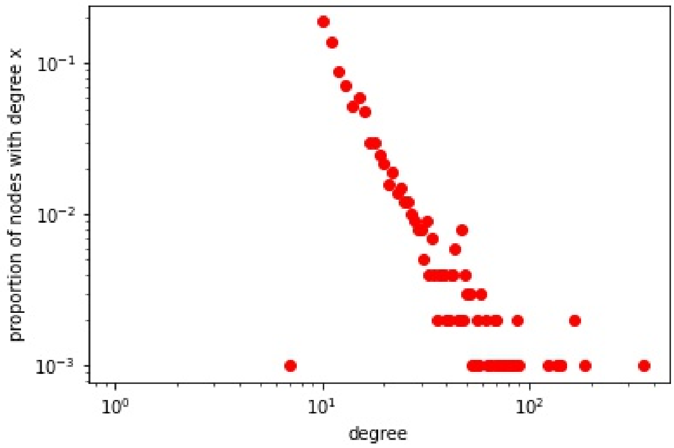

FIGURE 16: $m = 10$, $T = 1000$, $\lambda = 9$

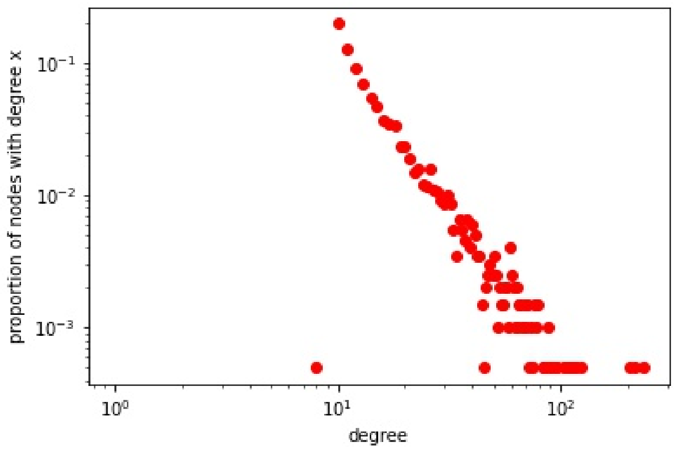

FIGURE 17: Caption

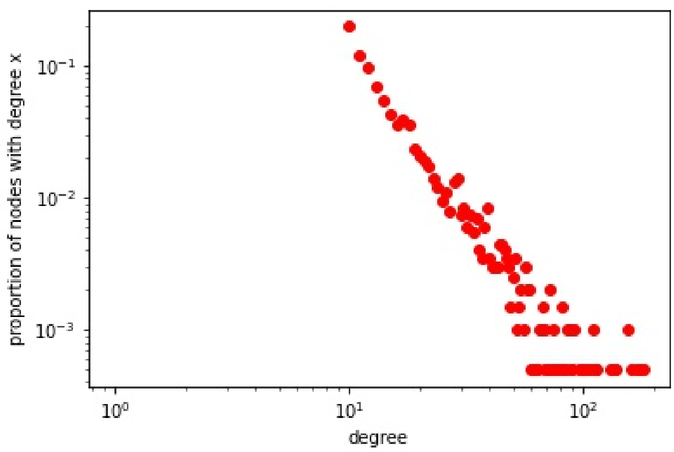

FIGURE 18: $m = 10$, $T = 2000$, $\lambda = 1.5$

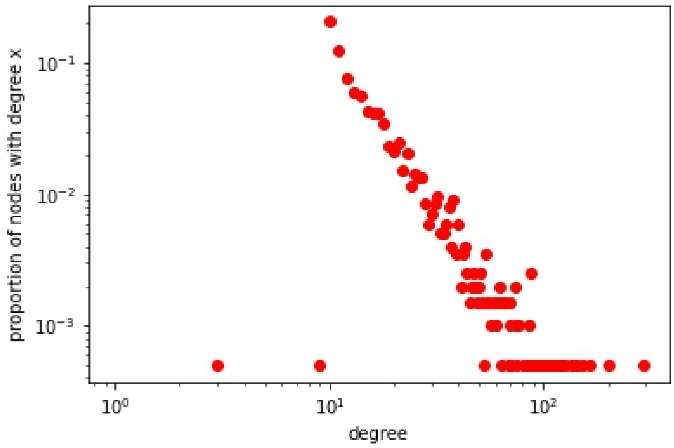

FIGURE 19: $m = 10$, $T = 2000$, $\lambda = 2.0$

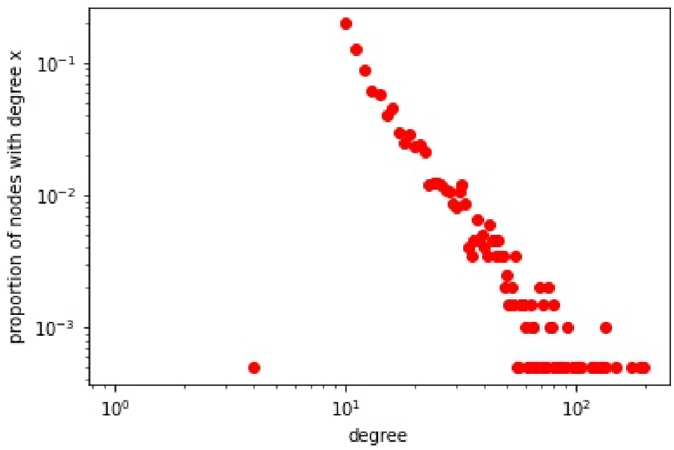

FIGURE 20: $m = 10$, $T = 2000$, $\lambda = 2.5$

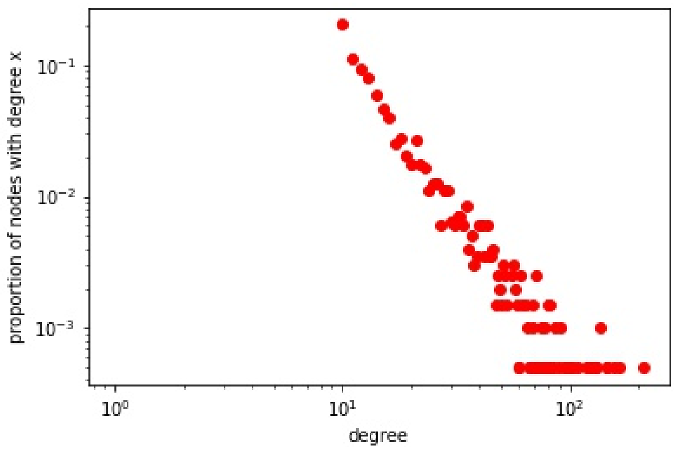

FIGURE 21: $m = 10$, $T = 2000$, $\lambda = 3.0$

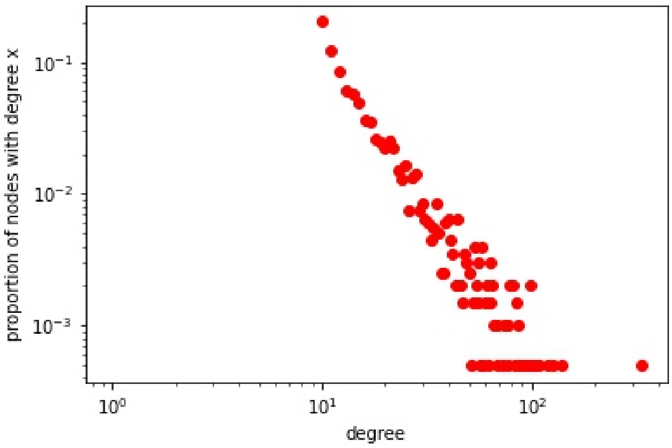

FIGURE 22: $m = 10$, $T = 2000$, $\lambda = 3.6$

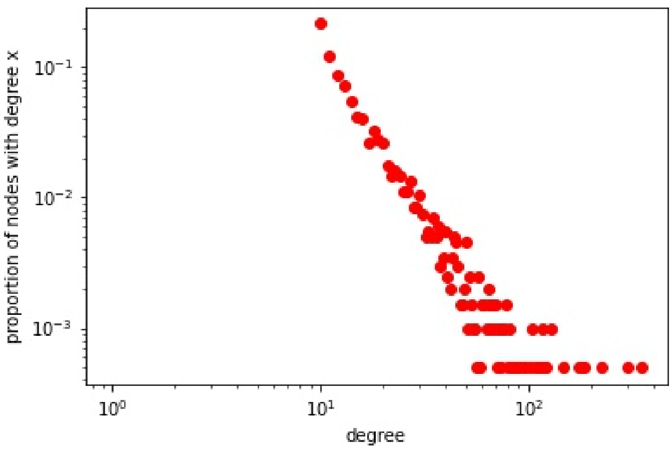

FIGURE 23: $m = 10$, $T = 2000$, $\lambda = 7$

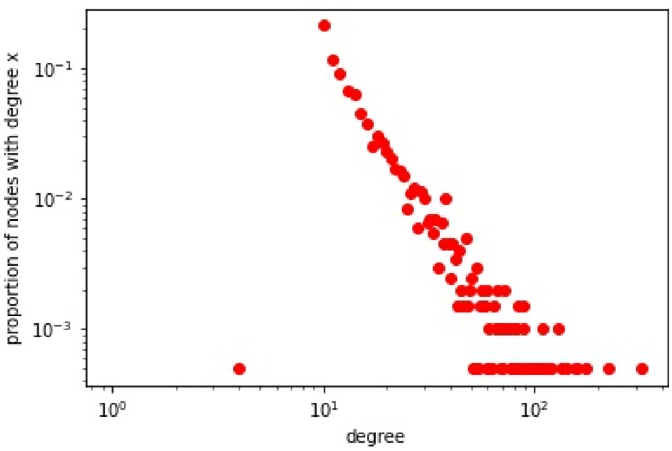

FIGURE 24: $m = 10$, $T = 2000$, $\lambda = 8$

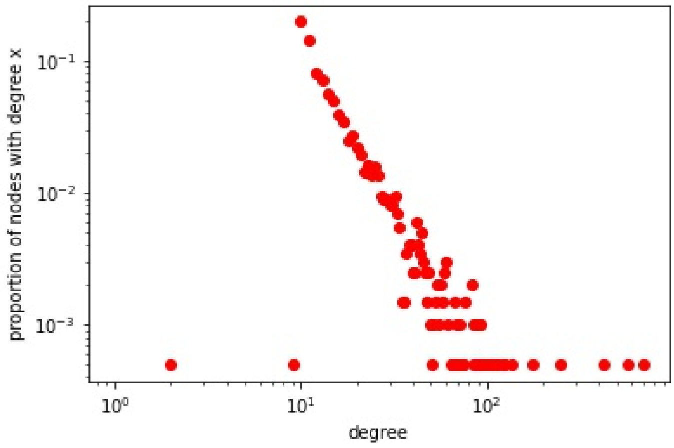

FIGURE 25: $m = 10$, $T = 2000$, $\lambda = 9$

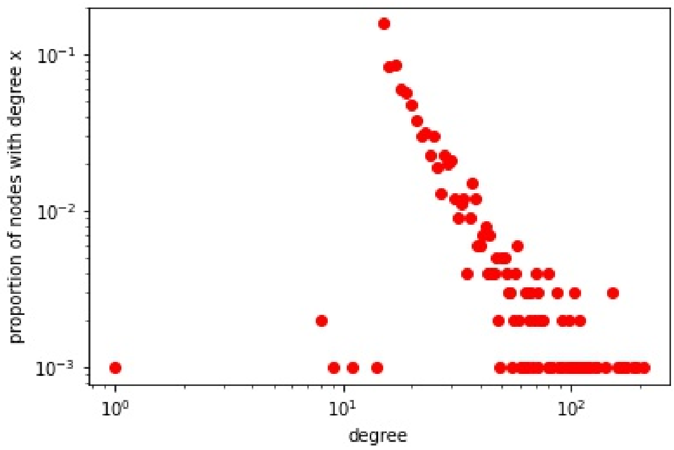

FIGURE 26: $m = 15$, $T = 1000$, $\lambda = 1.0$

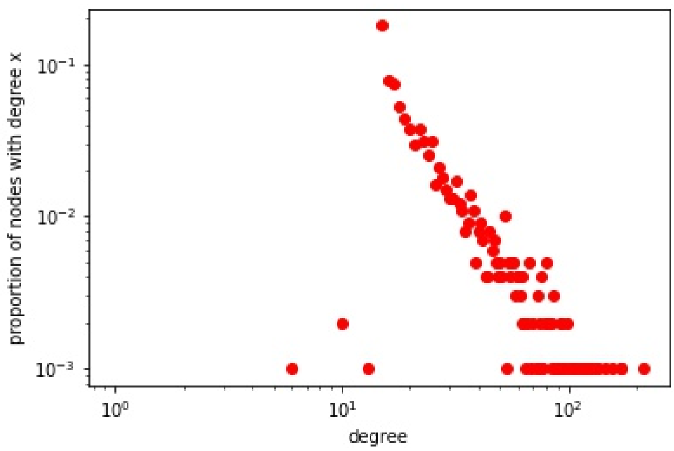

FIGURE 27: $m = 15$, $T = 1000$, $\lambda = 1.5$

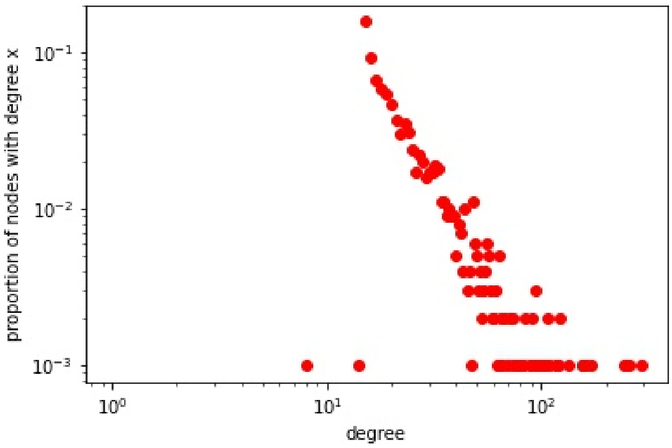

FIGURE 28: $m = 15$, $T = 1000$, $\lambda = 2.0$

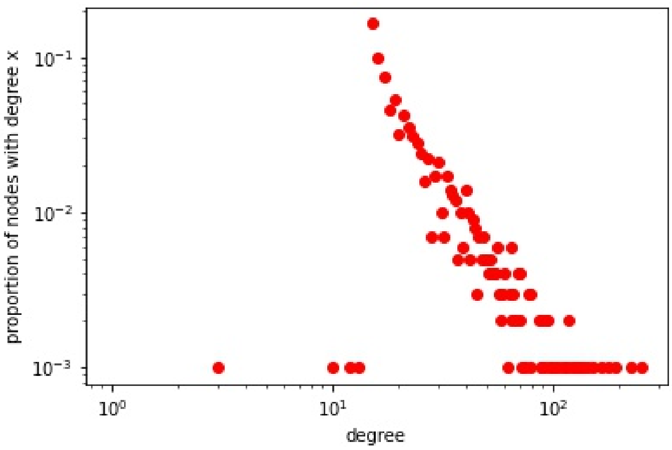

FIGURE 29: $m = 15$, $T = 1000$, $\lambda = 2.5$

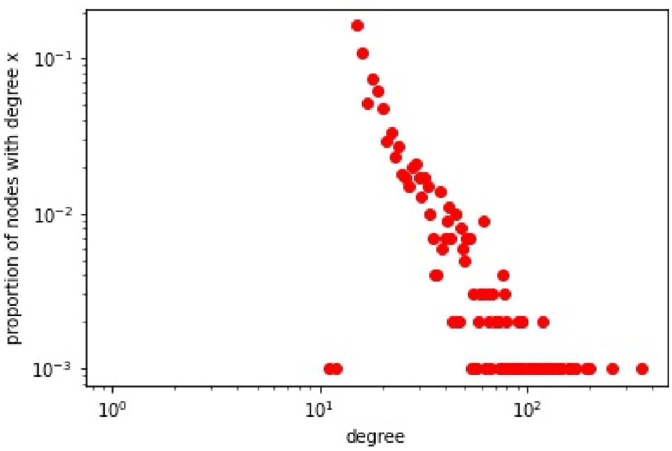

FIGURE 30: $m = 15$, $T = 1000$, $\lambda = 3.0$

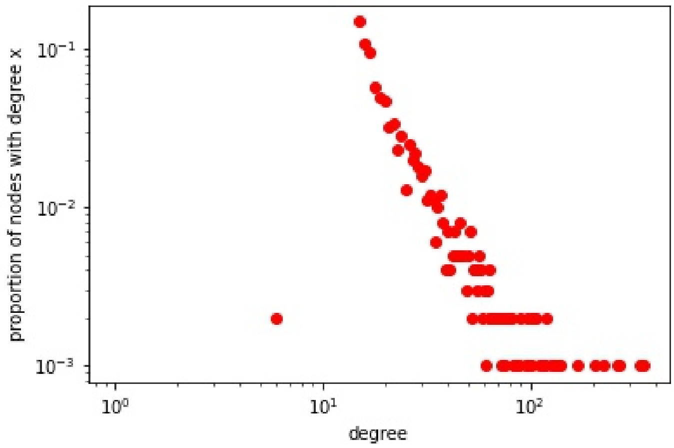

FIGURE 31: $m = 15$, $T = 1000$, $\lambda = 3.5$

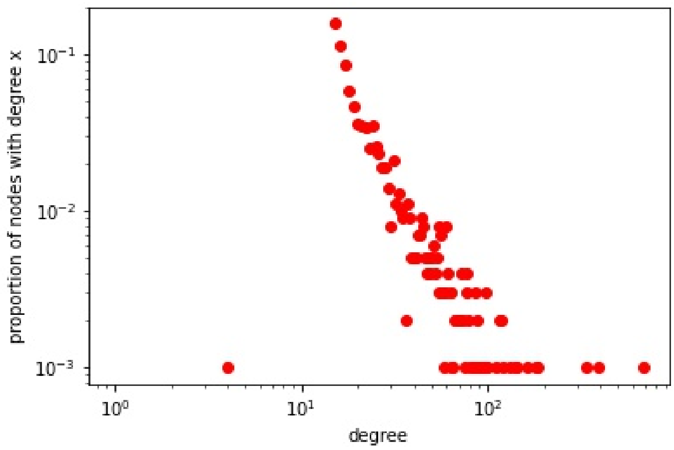

FIGURE 32: $m = 15$, $T = 1000$, $\lambda = 7$

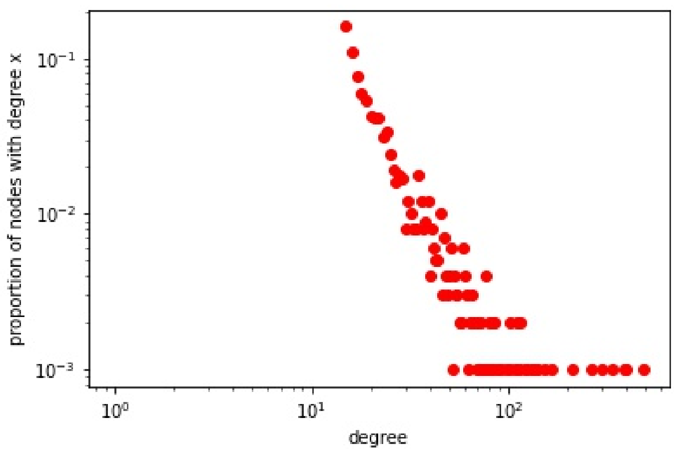

FIGURE 33: $m = 15$, $T = 1000$, $\lambda = 8$

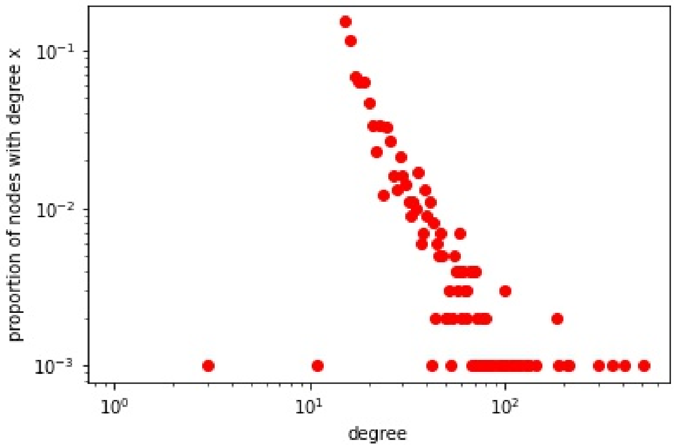

FIGURE 34: $m = 15$, $T = 1000$, $\lambda = 9$

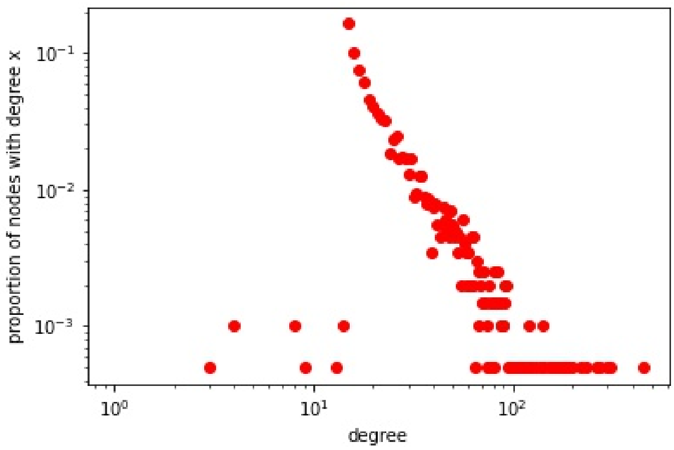

FIGURE 35: $m = 15$, $T = 2000$, $\lambda = 1.0$

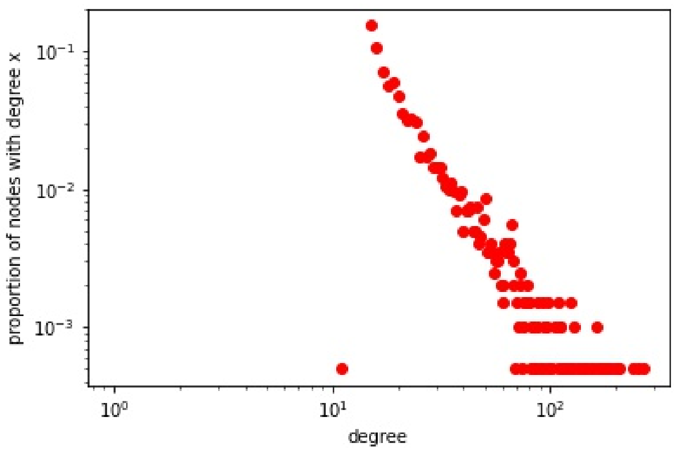

FIGURE 36: $m = 15$, $T = 2000$, $\lambda = 1.5$

BscThesisAM/f_scale_free_plots/m15_t2000_lambda2.0_F.jpg

FIGURE 37: $m = 15$, $T = 2000$, $\lambda = 2.0$

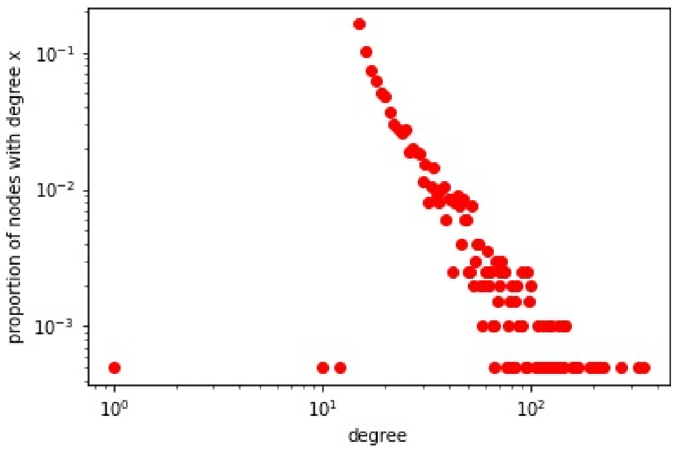

FIGURE 38: $m = 15$, $T = 2000$, $\lambda = 2.5$

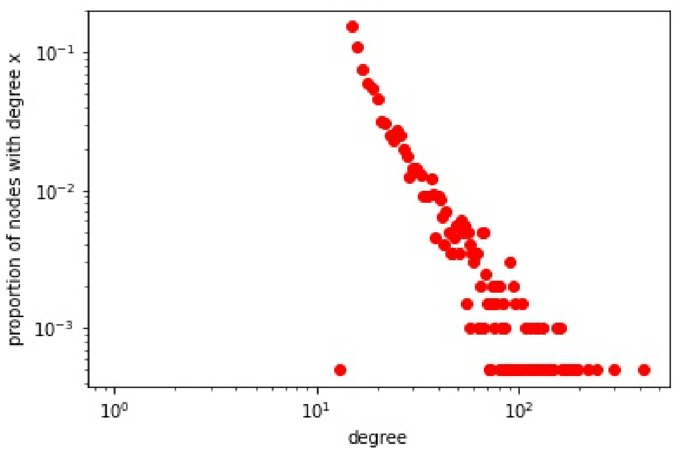

FIGURE 39: $m = 15$, $T = 2000$, $\lambda = 3.0$

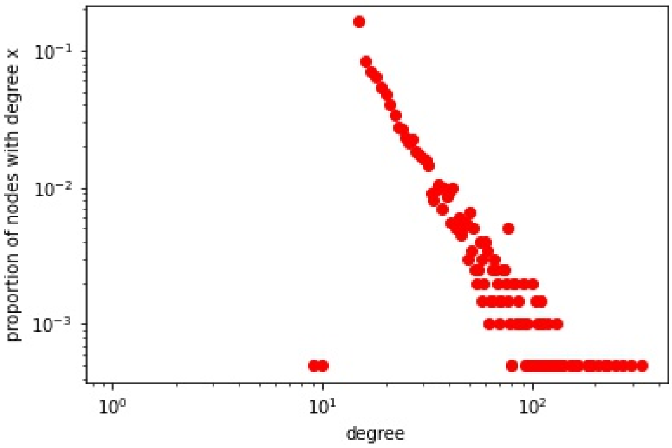

FIGURE 40: $m = 15$, $T = 2000$, $\lambda = 3.5$

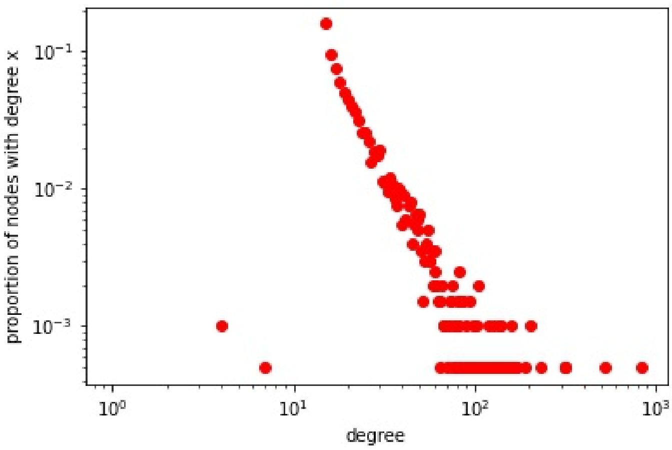

FIGURE 41: $m = 15$, $T = 2000$, $\lambda = 7$

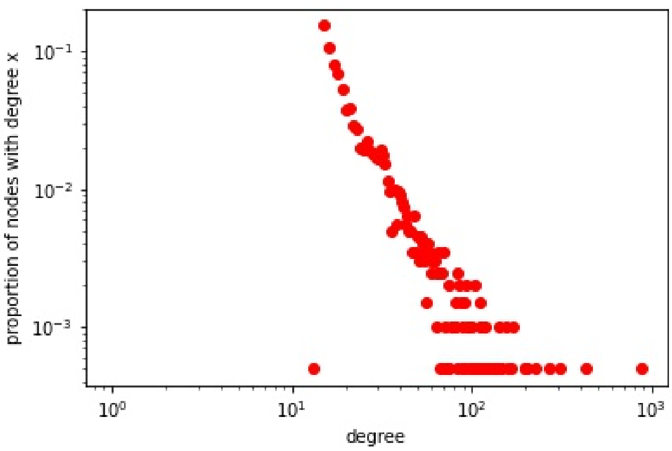

FIGURE 42: $m = 15$, $T = 2000$, $\lambda = 8$

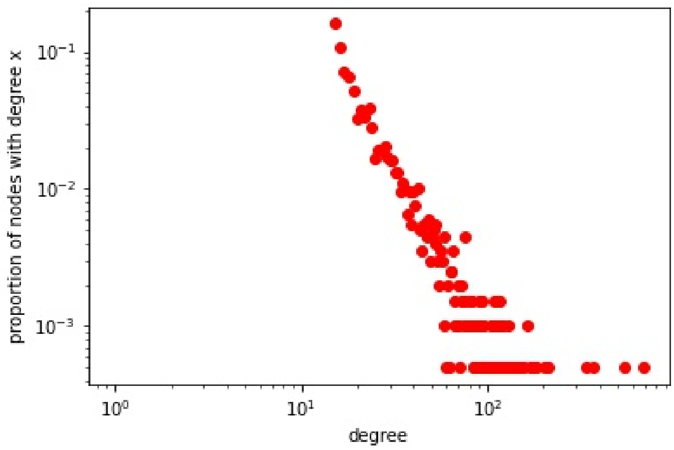

FIGURE 43: $m = 15$, $T = 2000$, $\lambda = 9$