**MASTER THESIS**

# COMPUTING THE SEQUENTIAL PRICE OF ANARCHY OF AFFINE CONGESTION GAMES USING LINEAR PROGRAMMING TECHNIQUES


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**Abstract**

For the class of Congestion Games with affine cost functions the Sequential Price of Anarchy (SPoA) has been determined exactly when two or three players are involved [15]. For more than three players, the exact value was unknown. There existed a Linear Program (LP) that could be used to determine the SPoA for any finite number of players [15]. However, this Linear Program has too many variables to be solved in reasonable time when the number of players is 4 or higher.

In this thesis column generation has been used to reduce the number of variables in that LP, in order to compute the SPoA for the 4 player case. Besides that, additional constraints have been added to analyse the worst case instance that the Linear Program describes for the 3 player case. Moreover, a variant of the LP has been presented that can determine the SPoA for any class of congestion games for which the number of resources and the action sets have been fixed. Finally, the LP has been modified in order to compute the exact SPoA for two classes of weighted congestion games, with the use of LP duality.

# Preface

During the past months I have been working on this thesis. This concludes my study in Applied Mathematics at the University of Twente. Within the specialisation of Operations Research I mostly enjoyed the courses related the the chair of Discrete Mathematics and Mathematical Programming. Therefore I am grateful that Marc Uetz offered me this graduation project at that chair.

During the project I have had regular meetings with my supervisors Marc Uetz and Matthias Walter. I would like to thank you for your support throughout the project. Although our meetings were held online due to the Covid-19 pandemic, your enthusiasm during the meetings motivated me to keep working. In particular, I would like to thank Marc for answering all my game theory related questions and suggesting new research directions whenever I got stuck. I would like to thank Matthias for helping me understand all the necessary theory on linear programming and suggestion possible solution methods, as well as helping me implement everything correctly.

I would like to thank the graduation committee for taking the time and making the effort to read my work. In particular, I would like to thank Christóbal as the third member for joining the graduation committee.

Because of the Covid-19 pandemic I have mostly been working on this project when the country was in lockdown. This meant that I have mostly worked from home. Moreover, most of my physical leisure activities came to a halt. Therefore I am grateful for all the online activities that have been organised and for the game nights that my friends have invited me to. While I missed the regular real life chats, the games during lunch breaks and the drinks in the evenings, I am thankful for my friends providing social distractions in online world instead.

Lastly I would like to thank my family for always keeping their faith in me and cheering me up whenever I was in a bad mood. I love you and I hope I can make you proud with this work.

Joran

# Contents

# Chapter 1

# Introduction

Non-cooperative Game Theory is a branch of mathematics that studies situations where several parties compete with each other [23]. Depending on the setting the competing parties either intend to maximise their utility or minimise their costs. The parties are not interested in working together to maximise their combined utility or minimise their total costs and reach a *social optimum*. Instead they individually make rational decisions such that they cannot be better off themselves, regardless of the implications for the other parties. When no player can improve their strategy given the strategies of other players, the strategies form a *Nash Equilibrium* [20]. In such an equilibrium the group as a whole may be worse off than in a social optimum. The quality of an equilibrium can be measured with the so-called *Price of Anarchy* [18].

The class of *Atomic Congestion Games* was introduced by Rosenthal [24]. In this class of games a set of resources is available for a number of players. Each player selects a subset of the resources, which causes those resources to be congested. Multiple players are allowed to buy the same resource. The price players pay for a resource is determined by the number of players that opted to congest it. Players select resources in such a way that they minimise the total costs over all resources they select, while they take the strategies of other players into account.

This class of games has been well studied. A lot of the results in this field assume that all players simultaneously decide which resources they opt to congest. First of all Rosenthal showed that each instance of a congestion game has a pure Nash Equilibrium [24]. Moreover, the quality of this equilibrium is known. Christodoulou and Koutsoupias [7] and Awerbuch et al. [2] have independently established that the Price of Anarchy of this class of games equals 2 for the case with 2 players and that the Price of Anarchy equals 2.5 for any arbitrary number of players that is 3 or higher.

This class of games has also been investigated as a *sequential game*. In that setting players take turns in selecting their subset of resources. For each player the decisions of previous players are known. This allows them to follow a strategy that allows them to make different decisions depending on the decisions of previous players and that takes decisions of future players into account. When no player can improve their strategy in a sequential game a so-called *subgame perfect equilibrium* has been reached. For this type of games the *Sequential Price of Anarchy* has been introduced in [21] to measure the quality of such an equilibrium.

Whereas the Price of Anarchy is known for affine congestion games for any number of players, the Sequential Price of Anarchy is not yet known in general. De Jong and Uetz [15] have computed the value when the number of players is at most 3. For the 4 player case a lower bound has been established by Kolev [17]. But when the number of players is 4 or larger, no exact value is known. Correa et al. [8] showed that computing the Sequential Price of Anarchy for congestion games is NP-hard when the number of players is fixed. In general, when the number of players is arbitrary, computing the Sequential Price of Anarchy for congestion games has been shown to be PSPACE-hard. [21]

However, De Jong and Uetz [15] have been able to derive a linear program that computes the Sequential Price of Anarchy for affine congestion games with 3 players. That program can be adapted to find the value for any fixed number of players. So despite the problem being NP-hard for an arbitrary fixed number of players, there does exist a concrete method to solve the problem. In [15] it was stated that the original version of this program becomes too large to solve for more than three players. In this thesis we reduce the size of that program with a linear programming technique called *column generation.* With this method we compute the Sequential Price of Anarchy of affine congestion games with 4 players. Moreover, we present adaptations of the program in order to compute the Sequential Price of Anarchy of some similar problems, specifically for instances with fixed sets of resources and actions and of two classes of weighted congestion games.

## 1.1 The Sequential Price of Anarchy

Congestion Games have been introduced by Rosenthal back in 1973 [24]. However, research into the Price of Anarchy is quite recent. After all Koutsoupias and Papadimitriou only introduced the concept in 1999 [18]. The Sequential Price of Anarchy has only been introduced in 2012, by Paes Leme et al. [21]. Consequently, research that applies the concept of the (Sequential) Price of Anarchy to the class of congestion games is recent too. In 2005 Christodoulou and Koutsoupias [7] and Awerbuch et al. [2] independently discovered that the Price of Anarchy of the class of affine congestion games equals 2 for the case with 2 players and 2.5 for cases with 3 or more players. The Sequential Price of Anarchy for 2 and 3 players was presented in 2014, as well as some general bounds [15], [16]. In particular, the Sequential Price of Anarchy for 2 and 3 players equal 1.5 and $1039/488 \approx 2.13$ respectively, which is lower than the Price of Anarchy. Biló further investigated the use of linear programs and their duals to compute the (Sequential) Price of Anarchy of several games [5]. There has also been done some research into specific instances of congestion games. For example Correa et al. [8], [9] established a lower bound of $\Omega(\sqrt{n})$ for the Sequential Price of Anarchy in network routing games. This means that the Sequential Price of Anarchy of congestion games with few players is lower than the Price of Anarchy, but higher when the number of players is large. In fact, it diverges to infinity when the number of players goes to infinity. However, for a large number of players no exact value has yet been determined. Groenland and Schafer introduced a framework to investigate Sequential Games where players only have a limited lookahead [11]. An example of research into the Sequential

Price of Anarchy of a different class of games is the article by Angelucci et al. [1], where they investigated Isolation Games. We can conclude that research into the Sequential Price of Anarchy is recent, but so far it already has been established that the Sequential Price of Anarchy behaves differently than the Price of Anarchy.

## 1.2 Outline

In Chapter 2 a formal problem definition is presented. In Chapter 3 we use the technique of column generation in order to compute the Sequential Price of Anarchy of affine congestion games with 4 players. In Chapter 4 some restricted instances of affine congestion games are investigated using adapted versions of the program presented in [15]. Finally, in Chapter 5 another version of that program is used to compute the Sequential Price of Anarchy for weighted affine congestion games, a generalisations of affine congestion games.

# Chapter 2

# Problem Description

In this chapter we formally define the problem that is discussed in this thesis. We also introduce the notation that is used throughout the thesis. After we have presented the definitions necessary for this thesis we present an example to clarify them.

## 2.1 Games

Each game in non-cooperative game theory can be represented as a *strategic form games*. These are defined as follows, as stated in [23].

**Definition 2.1** (Strategic Form Game)**.** [23] A *Stratgic Form Game* is a tuple $(N, \mathcal{S}_1, \ldots, \mathcal{S}_n, C_1, C_n)$ which represent the following. The set $N$ represents $n$ *players* of the game. Each player $i \in N$ is given a set of *pure strategies* $\mathcal{S}_i$ that they can select in the game. A tuple that contains the selected strategies of all players is called a *strategy profile*. The set of strategy profiles is denoted $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Finally each player $i$ is given a *cost function* $C_i : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathbb{R}$, which represents the cost the player pays when all players decide to play a specific strategy profile $S = (S_1, \ldots, S_n)$.

*Remark* 2.1. According to the definition stated in [23] the functions $C_i(S)$ of a strategic form game can either represent a payoff that can either be a utility, which players intend to maximise, or a cost, which they intend to minimise. In this thesis we only consider games where $C_i(S)$ represents a cost.

In general each player determines the strategy they are going to play independently. It can be seen as a plan that is made on beforehand. It dictates how the player is going to act in certain states of the game. For this plan it is relevant when a player has to commit to her action. If all players act simultaneously without knowledge of other player's actions, it may be beneficial for a player to stick to a different plan than when players play sequentially. After all, if one player already has committed an action when another player still has to decide, that latter player may want to create a flexible plan that bases their action on the first player's decision. Games where players act sequentially can be represented in *extensive form* [23].

**Definition 2.2** (Extensive Form Game)**.** [23] A game in *extensive form* is described by a *game tree*. This is a directed tree that describes the order in which players make decisions. The top vertex of the tree is called the *root* of the game tree and the the tree ends in several *end nodes* or *leaves*. Each vertex other than the leaves represent *decision states*. The arcs of the graph represent *actions* that a player can choose between in a decision state. We say that an action $A_i$ represented by arc $(v, v')$ is *prescribed* by strategy $S_i$ if $S_i$ maps decision vertex $v$ to arc $(v, v')$. The root represents the decision state of the first player. All arcs from a decision state for player $i$ lead to decision states for player $i + 1$, except for the decision states for the final player $n$. The arcs from a decision node for player $n$ lead to leaves. Each leaf $l$ is given a label $c_l$ that represents the costs of that leave. If all players play actions prescribed by a strategy profile $S$, then a unique path from the root to a leaf $l$ is prescribed by $S$. Then for all players $i$ it holds that $C_i(S)$ equals the $i$-th entry of leaf label $c_l$.

*Remark* 2.2. According to [23] an extensive form game can also contain so-called chance nodes besides the decision states and the leaves. Moreover, several decision states can be grouped into so-called information sets. In this thesis we only consider games without chance nodes and with perfect information. Therefore Definition 2.2 suffices.

In Definition 2.2 we make a distinction between a *strategy* and an *action* of a player in a game. This distinction is made formal in the following definition.

**Definition 2.3** (Action and Strategy)**.** Let player $i$ be a player of an extensive form game. An *action* $A_i$ denotes an available arc in the game tree. The set $\mathcal{A}_i$ denotes the set of actions available for player $i$ in the game tree. A *strategy* in an extensive form game represents a function that maps decision states to actions. If we denote by $V_i$ the set of all decision states for player $i$, then all strategies $S_i \in \mathcal{S}_i$ are functions of the form $S_i : V_i \to \mathcal{A}_i$. A tuple $A = (A_i)_{i \in N}$ that denotes one action for each player is called an *action profile*. When all players play actions that are prescribed by a strategy profile $S$, then the resulting action profile $A$ is called the *outcome* of the game. The set of all action profiles is denoted $\mathcal{A}$. By $(A_{-i}, A'_i)$ we denote the action profile where player $i$ chooses action $A'_i$ and all other players act according to action profile $A$. With $A_{<i}$ we denote the ordered set of actions $(A_1, \ldots, A_{i-1})$.

In an extensive form game the player costs are determined by the leaf of the action outcome that a strategy profile describes. This corresponds with a unique path in the game tree from the root to a specific leaf. In general this means that there exist decision states in the game tree that this path do not occur in the path corresponding to this outcome. However, a strategy $S_i$ maps each decision state $i \in V_i$ to an action $A_i \in \mathcal{A}_i$. If a state $v_i$ does not appear in the path of the outcome of strategy profile $S$, then any strategy $S'_i$ that maps $v_i$ to an arbitrary action yields the same outcome as $S_i$. That means that the player costs $C_i(S)$ and $C_i((S_{-i}, S'_i))$ are equal for all players.

**Observation 2.1.** *Let $S$ and $S'$ be two strategy profiles of an extensive form game that lead to the same outcome $A$. Then for all players it holds that*

$$C_i(S) = C_i(S'). \tag{2.1}$$

In Definition 2.1 it was stated that player costs are functions that map a strategy profile to a real number. Because of Observation 2.1 we also use the notation $C_i(A)$ to denote the player costs for player $i$ in action profile $A$. For any strategy profile $S$ for which action profile $A$ is the induced outcome, it holds that $C_i(S) = C_i(A)$. We will use the notations $C_i(S)$ and $C_i(A)$ interchangeably throughout this thesis.

In general a decision state for player $i$ only has outgoing arcs for a subset of $\mathcal{A}_i$. After all, the actions of previous players can influence the available actions a player can choose. However, in this thesis we only consider a special case of extensive form games where all actions are available in every decision state. Such games are called *sequential games* [21].

**Definition 2.4** (Sequential Game)**.** A *sequential game* is a special case of an extensive form game for which the game tree has the following structure. Each player $i \in N$ is given a set of actions $\mathcal{A}_i$. The root of the game tree has $|\mathcal{A}_1|$ outgoing arcs to decision states for player 2, one for each action that player 1 can choose. Likewise, each decision state $v_i$ for player $i$ has $|\mathcal{A}_i|$ outgoing arcs to decision states for player $i+1$. Player $n$ has $|\mathcal{A}_n|$ outgoing arcs to leaves.

## 2.2 Quality of Equilibria

In the previous section we defined a sequential game as a special case of an extensive form game. In general players are allowed to choose any action in any decision state of the game tree. However, players intend to play a strategy that lets the outcome of the game be such that their cost is as low as possible. If all players play optimally with respect to minimising their own costs, the resulting strategy profile is called an equilibrium. However, the outcome that follows from such a strategy profile may not be an outcome that minimises the sum of the costs for all players combined. In this section we define the *Sequential Price of Anarchy* as a measure for the quality of an equilibrium.

Firstly, we give the definition for an equilibrium of a strategic form game, as stated by Nash [20].

**Definition 2.5** (Nash Equilibrium)**.** [20] Let $I$ be an instance of a strategic form game. Then a *pure Nash equilibrium* $S^{\mathsf{NE}}$ is a strategy profile for which it holds that no player can play a different strategy $S_i$ in order to strictly decrease their cost. That means that for all players $i$ and all strategies $S_i \in \mathcal{S}_i$ the following *Nash inequality* holds:

$$C_i\left(S^{\mathsf{NE}}\right) \leq C_i\left(S^{\mathsf{NE}}_{-i}, S'_i\right) \tag{2.2}$$

In games where players choose their actions simultaneously, a Nash equilibrium is a strategy profile where no player can decrease their cost by deviating to a different strategy. However, in extensive form games a Nash equilibrium may not be a strategy in which both players play optimally, as will be illustrated in Example 2.1. After all, Observation 2.1 states that the player costs do not change when a player changes their strategy for a decision state that does not occur in the path of the outcome. Therefore a second definition of an equilibrium is required for extensive form games. To this end we firstly define a *subgame*, as defined in [23].

**Definition 2.6** (Subgame)**.** [23] A *subgame* of an extensive form game $I$ induced by decision state $v$ is a game $I'$ for which the game tree is a copy of the game tree for $I$, except that all paths that do not involve $v$ have been removed.

Definition 2.6 implies that in the game tree for $I'$ all decision states closer to the root than $v$ only have one outgoing arc. Decision state $v$ and states after $v$ still have the same outgoing arcs as the game tree of $I$.

We now use Definition 2.6 to define the so-called *subgame perfect equilibrium*, as stated in [23]. This equilibrium is suitable for extensive form games.

**Definition 2.7** (Subgame Perfect Equilibrium)**.** [23] A *subgame perfect equilibrium* $S^{\mathsf{SPE}}$ is a strategy profile for extensive form game $I$ that induces a Nash equilibrium in the subgame induced by $v$ for all decision states $v$ of the game tree of $I$. An action profile $A^{\mathsf{SPE}}$ is called a *subgame perfect outcome* if it is the outcome of a subgame perfect equilibrium $S^{\mathsf{SPE}}$. We call an action $A_i$ of player $i$ is subgame perfect in the subgame induced by state $v$ if there exists a subgame perfect equilibrium in the subgame induced by $v$ where player $i$ chooses $A_i$.

Note that a subgame perfect equilibrium can be computed using backward induction [23]. Player $n$ knows for each decision state what leaves her actions lead to. She can directly compute in which leaf her cost is minimised and choose to play the strategy that exactly prescribes those actions. Likewise any player $i$ can then use backward induction to compute the strategies for players $i+1$ until player $n$. Player $i$ then knows what her cost will be for all her strategies use that information to determine her own strategy. After every player has determined their strategy with this method, a subgame perfect equilibrium $S^{\mathsf{SPE}}$ has been computed.

The subgame perfect equilibrium is used to describe action profiles where players play optimally in such a way that they minimise their own cost. Now we define *social cost* as the price for all players combined.

**Definition 2.8** (Social Cost)**.** [21] The *social cost* $C(A)$ is a function that maps action profiles to $\mathbb{R}$, which is defined as follows:

$$C(A) = \sum_{i=1}^{n} C_i(A). \tag{2.3}$$

*Remark* 2.3. The cost function in (2.3) is called a *utilitarian* cost function. Social cost can also be defined with a so-called *egalitarian* cost function. In this thesis we always use the utilitarian social cost.

**Definition 2.9** (Social Optimum)**.** For any game $I$ the *social optimum* $A^{\mathsf{OPT}}$ is an action profile for which the social cost $C(A)$ is minimised. That is:

$$A^{\mathsf{OPT}} = \arg\min_{A \in \mathcal{A}} C(A). \tag{2.4}$$

The quality of a subgame perfect equilibrium with respect to the social optimum is then given by the *Sequential Price of Anarchy.* This is only defined for sequential games, not for extensive form games in general.

**Definition 2.10** (Sequential Price of Anarchy)**.** [21] Let $I$ be an instance of a sequential game and let $A^{\mathsf{SPE}}(I)$ denote the set of all subgame perfect equilibria of $I$. Then the Sequential Price of Anarchy for instance $I$ is defined as followed:

$$\mathsf{SPoA}(I) = \max_{A \in A^{\mathsf{SPE}}(I)} \frac{C\left(A^{\mathsf{SPE}}\right)}{C\left(A^{\mathsf{OPT}}\right)}. \tag{2.5}$$

Let $\mathcal{I}$ be a class of sequential games. Then the Sequential Price of Anarchy of class $\mathcal{I}$ is defined as followed:

$$\mathsf{SPoA}(\mathcal{I}) = \sup_{I \in \mathcal{I}} \mathsf{SPoA}(I). \tag{2.6}$$

When the class $\mathcal{I}$ is clear from context, we also write $\mathsf{SPoA}$.

## 2.3 Congestion Games

In this thesis we discuss the class of affine congestion games. In this section that class is defined. We also present some examples of congestion games to illustrate the definitions of the previous sections.

**Definition 2.11** (Affine Congestion Game)**.** [24] An *(atomic) congestion game* is defined as follows. The set $R$ denotes a set of resources. For each player $i \in N$ the set of actions $\mathcal{A}_i$ consists of subsets of resources: $\mathcal{A}_i \subseteq 2^R$. We say that player $i$ *chooses* resource $r \in R$ if she chooses an action $A_i \in \mathcal{A}_i$ for which it holds that $r \in A_i$. Each resource $r$ has a nondecreasing cost function $c_r : \mathbb{R}_+ \to \mathbb{R}_+$. Given an action profile $A = (A_1, \dots, A_n)$, each resource $r$ has a cost of $c_r(x_r)$. Here $x_r$ denotes the number of players who chose r, that is: $x_r = |\{i \in N | r \in A_i\}|$. For each player the player cost is defined as follows:

$$C_i(A) = \sum_{r \in R} c_r(x_r). \tag{2.7}$$

An *affine congestion game* is a congestion game where for each resource $r \in R$ the cost function $c_r(x_r)$ is defined as follows:

$$c_r(x_r) = \alpha_r + \beta_r x_r. \tag{2.8}$$

Here we have $\alpha_r \geq 0$ and $\beta_r \geq 0$. The constant term $\alpha_r$ of the cost function is called the *activation cost* of resource $r$ and the linear term $\beta_r$ is called the *weight* of resource $r$.

In the remainder of this section two examples of affine congestion games are presented in order to illustrate the definitions presented in this chapter.

*Example* 2.1. In a network congestion game players travel within a network. This example, displayed in Figure 2.1, consists of two players who both start at vertex $s$ of the network. Player 1 intends to travel to vertex $t_1$, player 2 wants to travel to $t_2$. Both players have two possible routes to their destination available:

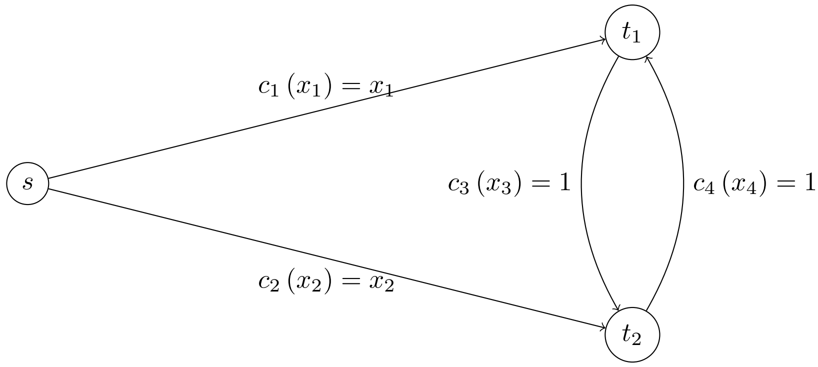



Figure 2.1: Network congestion game of Example 2.1

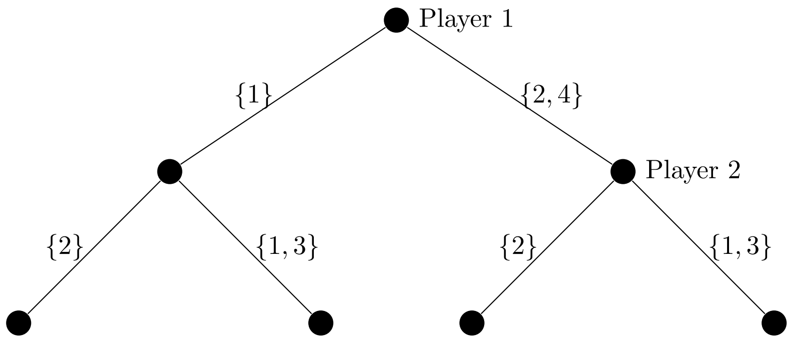



Figure 2.2: Game Tree of the network congestion game in Figure 2.1

| | | $\mathcal{S}_2$ | | | |
|---|---|---|---|---|---|
| | | $\{2\}, \{2\}$ | $\{2\}, \{1, 3\}$ | $\{1, 3\}, \{2\}$ | $\{1, 3\}, \{1, 3\}$ |
| $\mathcal{S}_1$ | $\{1\}$ | ***1,1*** | ***1,1*** | $2, 3$ | $2, 3$ |
| | $\{2, 4\}$ | $3, 2$ | $2, 2$ | $3, 2$ | *2,2* |

Table 2.1: Strategic form representation of the game tree in Figure 2.2

they can either choose a direct path or a detour. In order to travel over an edge in the graph they have to spend travel time, indicated by the functions $c_r(x_r)$. The players intend to arrive at their destination while spending as little travel time as possible.

This game can be seen as a congestion game. Here the edges of the graph represent the set of resources and the available routes for the players represent their action sets. That is, $R = \{1, 2, 3, 4\}$, $\mathcal{A}_1 = \{\{1\}, \{2, 3\}\}$ and $\mathcal{A}_2 = \{\{2\}, \{1, 3\}\}$. In this example we assume that player 1 leaves $s$ slightly before player 2, which means that this is a sequential game. The game tree of this game is displayed in Figure 2.2. The strategic form representation of the game is displayed in Table 2.1. Observe that player 1 has one decision state in the game tree and player 2 has two. Therefore each strategy for player 1 only consists of one action, while the strategies for player 2 consist of two actions. For each strategy profile the player costs are displayed.

This game has three Nash equilibria, for which the player costs have been displayed in italics in Table 2.1. For all those strategies it holds that neither player can decrease their cost by deviating to a different strategy if the other player sticks to the strategy of the Nash equilibrium. However, the strategy profile $S^{\mathsf{NE}} = ((\{2,4\}), (\{1,3\}, \{1,3]\}))$ is not a subgame perfect equilibrium. After all, in this game player 1 acts first. Therefore by choosing the strategy $S_1 = (\{1\})$ she gives an incentive to player 2 to pick the short route instead of the detour.

In this example the action profile with the lowest cost equals $A^{\mathsf{OPT}} = (\{1\}, \{2\})$. This turns out to also be the only outcome of the subgame perfect equilibria of the game. Therefore, in this example it holds that $\mathsf{SPoA} = 1$.

The congestion game of Example 2.1 has a Sequential Price of Anarchy of 1. But this is not true for all sequential games. The next example has a higher Sequential Price of Anarchy.

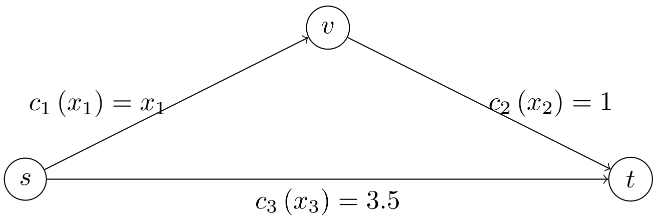



Figure 2.3: Network congestion game of Example 2.2

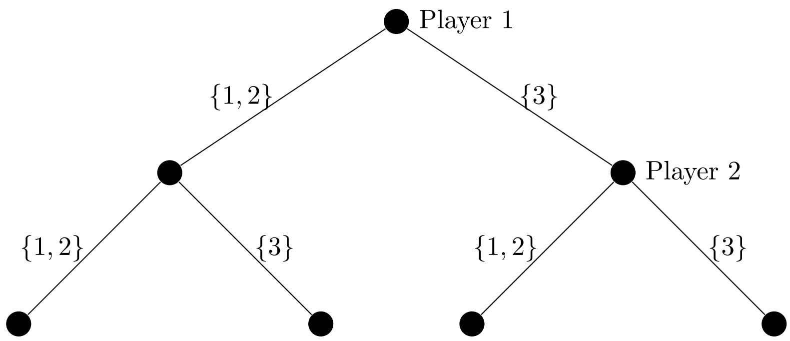



Figure 2.4: Game Tree of the network congestion game in Figure 2.3

| | | $\mathcal{A}_2$ | |
|---|---|---|---|
| | | $\{1,2\}$ | $\{3\}$ |
| $\mathcal{A}_1$ | $\{1,2\}$ | $3,3$ | $2,3.5$ |
| | $\{3\}$ | $3.5,2$ | $3.5,3.5$ |

Table 2.2: Outcomes of the game tree in Figure 2.4

*Example* 2.2. Again consider a network congestion game with two players. In this instance, which is displayed in Figure 2.3, both players want to travel from vertex $s$ to $t$. They have two routes available: they can choose the direct edge or the path via vertex $v$. The corresponding game tree is displayed in Figure 2.4. The outcomes of the game are displayed in Table 2.2.

In both decision states for player 2 it is benificial for her to choose the direct edge over the path via $v$. Her travel time then is respectively 3 and 2 instead of 3.5. Therefore, the only subgame perfect strategy for player 2 is $S_2^{\mathsf{SPE}} = (\{1,2\},\{1,2\})$. Player 1 then can deduce that her cost for the path via $v$ is the fastest. It follows that both player select the path via vertex $v$ in the subgame perfect equilibrium: $A^{\mathsf{SPE}} = (\{1,2\},\{1,2\})$. From Table 2.2 it follows that the social cost in this outcome is $C\left(A^{\mathsf{SPE}}\right) = 6$.

However, it can also be seen that the subgame perfect outcome is not the social optimum. After all, the social optimum is obtained when one player takes the direct edge and the other player the detour. In that scenario it holds that $C\left(A^{\mathsf{OPT}}\right) = 5.5$. We conclude that for this instance we have that $\mathsf{SPoA} = 1.2$.

# Chapter 3

# Congestion Games with 4 Players

The Sequential Price of Anarchy for affine congestion games is not yet known in general. In [15] de Jong and Uetz derive the values for 2 and 3 player games, which are 1.5 and 1039/488 respectively. For more players the Sequential Price of Anarchy is still unknown. However, in [15] the same authors presented a Linear Program (LP) which can generate a worst case instance of a sequential congestion game for an arbitrary number of players. So in theory the Sequential Price of Anarchy for any number of players can be found using that LP. But with more than 3 players the number of variables and constraints of the LP becomes too large to be practically computable in its original form. In [17] it is stated that the Sequential Price of Anarchy for the case with 4 players is bounded below by 2.5509150067, but no exact value was yet presented. In this chapter we apply column generation which allows us to use the LP from [15] to compute the Sequential Price of Anarchy for the case with 4 players.

## 3.1 LP Formulation

In this section a lemma from [15] is presented which ensures that a worst case instance of bounded size exists. This allowed the authors to formulate the LP that generates such an instance.

**Lemma 3.1.** [15] *For any instance $I$ of a congestion game there exists a congestion game $I'$ for which the following properties hold.*

1. *For all players $i \in N$ it holds that $|\mathcal{A}_i| \leq z_i$, where all $z_i$ are defined as follows:*

$$z_1 := 2 \text{ and } z_i := 1 + \prod_{j=1}^{i-1} z_j \text{ for all } i \geq 2. \tag{3.1}$$

2. *The amount of resources $|R|$ is at most $2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$.*
3. *For all resources $r \in R$ it holds that $\alpha_r + \beta_r \leq nC(A^{\mathsf{OPT}})$.*

*4.* $\mathsf{SPoA}(I') = \mathsf{SPoA}(I)$.

In [15] the authors defined an LP to compute the Sequential Price of Anarchy for 3 players. Here we formulate a similar LP for the case with 4 players. Because of Lemma 3.1 the LP is constrained such that it only considers instances of congestion games with $|\mathcal{A}_i| \leq z_i$ for all players $i \in N$ and with at most $2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$ resources.

The parameters and variables of the LP are introduced in Table 3.1 and 3.2 respectively. The subscripts of the parameters and variables refer to different sets that will be introduced here. The subscripts $a$ and $a'$ denote actions of players, so they are elements of the set $\mathcal{A}_1$. Similarly we have that $b, b' \in \mathcal{A}_2$, $c, c' \in \mathcal{A}_3$ and $d, d' \in \mathcal{A}_4$. When the subscript $\mu$ is used, we refer to the union of the actions of all players: $\mu \in \bigcup_{i=1}^4 \mathcal{A}_i$. The subscripts $p$ and $q$ in variable $o_{pq}$ also denote actions, but those actions belong to different players $i$ and $j$, that is $p \in \mathcal{A}_i$, $q \in \mathcal{A}_j$ such that $j > i$. When specific actions are mentioned, they are denoted in the form of $i.k$. Here the $i$ refers to the player and the $k$ to the action within $\mathcal{A}_i$. For instance, action profile $A = (1.1, 2.1, 3.1, 4.1)$ consists of the first action of the action sets for each player. Finally, the subscript $r$ denotes a resource from the set of resources $R$.

In this LP because of Lemma 3.1 we enforce that for each unique subset of the actions in $\bigcup_{i=1}^4 \mathcal{A}_i$ there exists exactly one resource that is selected in that exact subset of actions. This is encoded in parameter $\delta_{\mu r}$. These parameters are set such that for any two resources $r, r' \in R$ there exists an action $\mu \in \bigcup_{i=1}^4 \mathcal{A}_i$ such that $\delta_{\mu r} \neq \delta_{\mu r'}$. Moreover, it holds that $\sum_{\mu \in \bigcup_{i=1}^4} \delta_{\mu r} \geq 1$ for all resources $r \in R$. Because of Lemma 3.1 we set the number of actions for each player as follows:

$$|\mathcal{A}_1| = 2, \tag{3.2a}$$
$$|\mathcal{A}_2| = |\mathcal{A}_1| + 1 = 2 + 1 = 3, \tag{3.2b}$$
$$|\mathcal{A}_3| = |\mathcal{A}_1| \cdot |\mathcal{A}_2| + 1 = 2 \cdot 3 + 1 = 7, \tag{3.2c}$$
$$|\mathcal{A}_4| = |\mathcal{A}_1| \cdot |\mathcal{A}_2| \cdot |\mathcal{A}_3| + 1 = 2 \cdot 3 \cdot 7 + 1 = 43. \tag{3.2d}$$

As a result, the following holds for the number of resources:

$$|R| = 2^{\sum_{i=1}^4 |\mathcal{A}_i|} - 1 = 2^{2+3+7+43} - 1 = 2^{55} - 1. \tag{3.3}$$

In the LP we enforce which action profile corresponds to the social optimum and which corresponds to $S^{\mathsf{SPE}}$, namely as follows: $A^{\mathsf{OPT}} = (1.1, 2.1, 3.1, 4.1)$ and $A^{\mathsf{SPE}} = (1.2, 2.3, 3.7, 4.43)$. The parameters $z_a^1$, $z_{ab}^2$, $z_{abc}^3$ and $z_{abcd}^4$ are set equal to 1 whenever the action profile belongs to $S^{\mathsf{SPE}}$ and 0 otherwise.

Below the LP is stated. We intend to find the instance for which $\sup \frac{C(A^{\mathsf{SPE}})}{C(A^{\mathsf{OPT}})}$ is obtained. To this end we enforce that $C(A^{\mathsf{OPT}}) = 1$ in constraint (3.4h) and we maximise $C(A^{\mathsf{SPE}})$ in the objective (3.4a). After all, if we fix all resource costs and then multiply all variables $\alpha_r$ and $\beta_r$ with the same nonnegative constant $M$, then we it holds for each player cost that

$$C_i(A) = \sum_{r \in A_i} M\alpha_r + M\beta_r x_r = MC_i(A).$$

This means that normalising the social cost in the social optimum does not have an impact on the Sequential Price of Anarchy of the instance.

Constraints (3.4b) and (3.4c) define variables $v_\mu$ and $o_{pq}$, called the base cost and overlap, in terms of activation cost and weight of the resources. The variables $v_\mu$ and $o_{pq}$ are used to translate the affine cost functions of the resources into linear constraints. After all, for each resource in an action a player pays the activation cost and the linear term at least once. For each other player she has to pay the linear term of their overlapping resources another time. Therefore we can define the costs for the players in each action profile as the sum of the base cost and the overlap with each other player. This is done in constraint (3.4d), (3.4e), (3.4f) and (3.4g). Constraint (3.4i) ensures that the total costs for each action profile are at least the total costs of the social optimum $A^{\mathrm{OPT}}$. Constraints (3.4j), (3.4k), (3.4l) and (3.4m) make sure that the costs of the individual players are minimised in the action profiles that correspond to $S^{\mathrm{SPE}}$, using the auxiliary variables $C_1(a)$, $C_2(ab)$ and $C_3(abc)$. After all, by Definition 2.7 all decision states have to induce a Nash equilibrium in a subgame perfect equilibrium. This means that in each decision state of the game tree the Nash inequality from Definition 2.5 must satisfy for the action that belongs to $S^{\mathsf{SPE}}$. In constraints (3.4n), (3.4o) and (3.4p) the player costs in the corresponding action profiles are set equal to the auxiliary variables. Finally, constraint (3.4q) ensures that the LP maximises $C(A^{\mathrm{SPE}})$. Since the activation cost and the weight of resources in a congestion game are defined to be nonnegative, this has been added with (3.4r).

## Parameters

| | | |
|---|---|---|
| $\delta_{\mu r}$ | $\forall \mu, r$ | $\begin{cases} 1 & \text{if } r \in \mu \\ 0 & \text{otherwise} \end{cases}$ |
| $z_a^1$ | $\forall a$ | $\begin{cases} 1 & \text{if } a \text{ is prescribed by } S_1^{\mathsf{SPE}} \\ 0 & \text{otherwise} \end{cases}$ |
| $z_{ab}^2$ | $\forall a, b$ | $\begin{cases} 1 & \text{if } b \text{ is prescribed by } S_2^{\mathsf{SPE}} \text{ in state } a \\ 0 & \text{otherwise} \end{cases}$ |
| $z_{abc}^3$ | $\forall a, b, c$ | $\begin{cases} 1 & \text{if } c \text{ is prescribed by } S_3^{\mathsf{SPE}} \text{ in state } ab \\ 0 & \text{otherwise} \end{cases}$ |
| $z_{abcd}^4$ | $\forall a, b, c, d$ | $\begin{cases} 1 & \text{if } d \text{ is prescribed by } S_4^{\mathsf{SPE}} \text{ in state } abc \\ 0 & \text{otherwise} \end{cases}$ |

Table 3.1: Parameters of the LP

## Variables

| | | |
|---|---|---|
| $\alpha_r$ | $\forall r$ | activation cost of $r$ |
| $\beta_r$ | $\forall r$ | weight of $r$ |
| $v_\mu$ | $\forall \mu$ | total activation cost plus total weight of resources in $\mu$ |
| $o_{pq}$ | $\forall p, q$ | total weight of resources in $p \cap q$ |
| $C_i(abcd)$ | $\forall a, b, c, d, i$ | cost of player $i$ when players 1, 2, 3 and 4 choose $a$, $b$, $c$ and $d$ respectively |
| $C(A^{\mathsf{SPE}})$ | | costs in subgame perfect equilibrium $A^{\mathsf{SPE}}$ |
| $C_1(a)$ | $\forall a$ | cost of player 1 when she chooses $a$ and players 2, 3 and 4 choose according to $S^{\mathsf{SPE}}$ |
| $C_2(ab)$ | $\forall a, b$ | cost of player 2 when players 1, 2 choose $a$, $b$ and players 3 and 4 choose according to $S^{\mathsf{SPE}}$ |
| $C_3(abc)$ | $\forall a, b, c$ | cost of player 3 when players 1, 2, 3 choose $a$, $b$, $c$ and player 4 chooses according to $S^{\mathsf{SPE}}$ |

Table 3.2: Variables of the LP

### Linear Program

$$\max \quad C\left(A^{\mathsf{SPE}}\right) \tag{3.4a}$$

$$\text{s.t.} \quad v_\mu = \sum_{r \in R} \delta_{\mu r}\left(\beta_r + \alpha_r\right) \qquad \forall \mu \in \bigcup_{i=1}^{4} \mathcal{A}_i \tag{3.4b}$$

$$o_{pq} = \sum_{r \in R} \delta_{pr}\delta_{qr}\beta_r \qquad \forall p \in \mathcal{A}_i, \forall q \in \mathcal{A}_j, j > i \tag{3.4c}$$

$$C_1(abcd) = v_a + o_{ab} + o_{ac} + o_{ad} \qquad \forall a, b, c, d \tag{3.4d}$$

$$C_2(abcd) = v_b + o_{ab} + o_{bc} + o_{bd} \qquad \forall a, b, c, d \tag{3.4e}$$

$$C_3(abcd) = v_c + o_{ac} + o_{bc} + o_{cd} \qquad \forall a, b, c, d \tag{3.4f}$$

$$C_4(abcd) = v_d + o_{ad} + o_{bd} + o_{cd} \qquad \forall a, b, c, d \tag{3.4g}$$

$$\sum_{i=1}^{4} C_i(1.1, 2.1, 3.1, 4.1) = 1 \tag{3.4h}$$

$$\sum_{i=1}^{4} C_i(abcd) \geq 1 \qquad \forall a, b, c, d \tag{3.4i}$$

$$C_4(abcd) \leq C_4(abcd') \qquad \forall a, b, c, d | z^4_{abcd} = 1, \forall d' \tag{3.4j}$$

$$C_3(abc) \leq C_3(abc') \qquad \forall a, b, c | z^3_{a,b,c} = 1, \forall c' \tag{3.4k}$$

$$C_2(ab) \leq C_2(ab') \qquad \forall a, b | z^2_{ab} = 1, \forall b' \tag{3.4l}$$

$$C_1(a) \leq C_1(a') \qquad \forall a | z^1_a = 1, \forall a' \tag{3.4m}$$

$$C_1(a) = C_1(abcd) \qquad \forall a, b | z^2_{ab} = 1, c | z^3_{abc} = 1, d | z^4_{abcd} = 1 \tag{3.4n}$$

$$C_2(ab) = C_2(abcd) \qquad \forall a, b, c | z^3_{abc} = 1, d | z^4_{abcd} = 1 \tag{3.4o}$$

$$C_3(abc) = C_3(abcd) \qquad \forall a, b, c, d | z^4_{abcd} = 1 \tag{3.4p}$$

$$C\left(A^{\mathsf{SPE}}\right) = \sum_{i=1}^{4} C_i(1.2, 2.3, 3.7, 4.43) \tag{3.4q}$$

$$\alpha_r, \beta_r \geq 0 \tag{3.4r}$$

## 3.2 Column Generation

The LP as described above is an extension of the LP in [15] to suit congestion games with 4 players. De Jong and Uetz stated in [15] that this LP is too large to be practically solvable. In this section we apply column generation in order to find the solution of the LP. This allows us to determine the Sequential Price of Anarchy for congestion games with 4 players.

In the LP variables $\alpha_r$ and $\beta_r$ are the only variables that are defined for each resource. The other variables in the LP are defined for actions. The only constraints in which $\alpha_r$ and $\beta_r$ appear are (3.4b) and (3.4c). Those are the constraints where the base cost and overlap variables $v_\mu$ and $o_{pq}$ are defined. However, recall that the amount of resources is exponentially larger than the amount of actions: $|R| = 2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$. This suggests that we may be able to reduce the number of variables representing the resources by an exponential factor using column generation.

In order to apply column generation we initialise the LP with a polynomial number of resources. We only add the variables $\alpha_r$ and $\beta_r$ for which there exist only one action $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$ such that $\delta_{\mu r} = 1$. So initially the reduced LP only has 55 resources instead of $2^{55} - 1$. If we solve this LP, some of the constraints of the dual problem corresponding to the complete LP can be violated. If there exists a violated dual constraint, we add the corresponding constraint to the LP and solve it again. We continue this process until no dual constraints of the complete LP are violated anymore. At that point the obtained solution to the LP is optimal to the complete LP.

If resources are removed from the LP, the only constraints of the dual LP that potentially will be violated, correspond to variables $\alpha_r$ and $\beta_r$ in the primal LP for those resources $r \in R$ that are removed. Let us therefore derive those dual constraints.

Since $\alpha_r$ and $\beta_r$ appear in constraints (3.4b) and (3.4c) we create dual variables that correspond to those primal constraints. Let $\tau_\mu$ be the dual variable that corresponds to primal constraint (3.4b) where variable $v_\mu$ is defined and let $\sigma_{pq}$ be the dual variable that corresponds to primal constraint (3.4c) where $o_{pq}$ is defined. Since those primal constraints are equality constraints, the dual variables $\tau_\mu$ and $\sigma_{pq}$ are unrestricted in sign. Since primal variables $\alpha_r$ and $\beta_r$ are nonnegative and the primal LP is a maximisation problem, the dual constraints will be $\geq$-constraints. Since neither $\alpha_r$ nor $\beta_r$ appears in the objective function of the primal LP, the right hand side of the dual constraints will be 0. The dual constraints are as follows, where (3.5) corresponds to primal variable $\alpha_r$ and (3.6) to $\beta_r$.

$$\sum_{\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i} \delta_{\mu r} \tau_\mu \geq 0 \tag{3.5}$$

$$\sum_{\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i} \delta_{\mu r} \tau_\mu + \sum_{x \in \mathcal{A}_i} \sum_{y \in \mathcal{A}_j | j > i} \delta_{pr} \delta_{qr} \sigma_{pq} \geq 0 \tag{3.6}$$

Consider an optimal solution to the dual of the reduced LP and let $\hat{\tau}_\mu$ and $\hat{\sigma}_{pq}$ be the optimal values for dual variables $\tau_\mu$ and $\sigma_{pq}$ respectively. In order to find a resource $r \in R$ for which a corresponding constraint is violated in the dual of the complete LP, we look for a vector $\mathbf{x}$ that represents all values $\delta_{\mu r}$ such that (3.5) or (3.6) is violated if $\tau_\mu = \hat{\tau}_\mu$ for all $\mu$ and $\sigma_{pq} = \hat{\sigma}_{pq}$ for all $p, q$. In order to figure out if either constraint is violated we can solve the following two pricing problems.

$$\begin{aligned} \min \quad & \sum_{\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i} \hat{\tau}_\mu x_\mu \\ \text{s.t.} \quad & x_\mu \in \{0, 1\} \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
\min \quad & \sum_{\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i} \hat{\tau}_\mu x_\mu + \sum_{p \in \mathcal{A}_i} \sum_{q \in \mathcal{A}_j | j > i} \hat{\sigma}_{pq} y_{pq} \\
\text{s.t.} \quad & y_{pq} \leq x_p \\
& y_{pq} \leq x_q \\
& y_{pq} \geq x_p + x_q - 1 \\
& y_{pq} \geq 0 \\
& x_p, x_q \in \{0, 1\}
\end{aligned}
\tag{3.8}
$$

**Proposition 3.2.** *The optimal objective value of pricing problem* (3.7) *is strictly negative if and only if there exists a constraint* (3.5) *in the dual problem of the complete LP* (3.4) *which is violated. Moreover, if the objective value of* (3.7) *is strictly negative, then the values of $x_\mu$ in the optimal solution describe the values of LP parameter $\delta_{\mu r}$ for a resource $r \in R$ for which* (3.5) *is violated.*

*Proof.* For all $2^{55} - 1$ resources $r \in R$ there exists a variable $\alpha_r$ in the complete LP (3.4). The dual problem of the complete LP hence have $2^{55} - 1$ constraints of the form in (3.5). Each resource is represented by binary parameters $\delta_{\mu r}$ for all actions $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$, as defined in Section 3.1. By the pigeonhole principle either $x_\mu = 0$ for all $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$, or the binary variables $x_\mu$ in the optimal solution of pricing problem (3.7) are equal to the parameters $\delta_{\mu r}$ for exactly one of the resources $r \in R$. The values of the parameters $\hat{\tau}_\mu$ represent the optimal solution of variables $\tau_\mu$ in the dual problem of the restricted version of (3.4). So the objective value of (3.7) represents the left hand side of dual constraint (3.5) for some resource $r \in R$, given that $x_\mu = 1$ for some $\mu \in \bigcup_{i=4}^{4} \mathcal{A}_i$.

If the objective value of pricing problem (3.7) is strictly negative, then there exists a $\mu \in \bigcup_{i=4}^{4} \mathcal{A}_i$ such that $x_\mu = 1$. After all, if $x_\mu = 0$ for all $\mu \in \bigcup_{i=4}^{4} \mathcal{A}_i$ then the objective value equals 0. So the variables $x_\mu$ in the optimal solution of (3.7) represent the parameters $\delta_{\mu r}$ for some $r \in R$. The left hand side of dual constraint (3.5) is strictly negative, so the constraint is violated.

Conversely, let the optimal objective value of (3.7) be nonnegative. Since (3.7) is a minimisation problem, it holds that the objective value is nonnegative for all values $x_\mu$, with $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$. It follows that for all resources $r \in R$ the left hand side of (3.5) is nonnegative, so none of those constraints are violated. ■

**Proposition 3.3.** *The optimal objective value of pricing problem* (3.8) *is strictly negative if and only if there exists a constraint* (3.6) *in the dual problem of the complete LP* (3.4) *which is violated. Moreover, if the objective value of* (3.8) *is strictly negative, then the values of $x_\mu$ in the optimal solution describe the values of LP parameter $\delta_{\mu r}$ for a resource $r \in R$ for which* (3.6) *is violated.*

*Proof.* The constraints of (3.8) ensure that $y_{pq} = 1$ for some $p, q$ if both $x_p = 1$ and $x_q$ and that $y_{pq} = 0$ otherwise. Since all $x_p$ and $x_q$ are binary it holds that $y_{pq} = x_p x_q$ for all $p, q$.

By similar reasoning as in the proof of Proposition 3.2 the objective of (3.8) represents the left hand side of dual constraint (3.6), where $x_\mu = \delta_{\mu r}$ and $y_{pq} = \delta_{pr}\delta_{qr}$, for some resource $r \in R$. Likewise the result of Proposition 3.3 follows. ■

As stated before, we initially start the column generation process by solving the LP with only those resources $r \in R$ for which there exists exactly one $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$ such that $\delta_{\mu r} = 1$. Consequently, it suffices to only solve pricing problem (3.8) in order to find a resource with a violated dual constraint.

**Lemma 3.4.** *Let $m \in \mathbb{N}$ be arbitrary. Consider the following inequality:*

$$\sum_{j=1}^{m} t_j x_j \geq 0. \tag{3.9}$$

*Suppose that $\hat{\mathbf{t}}$ satisfies (3.9) for all unit vectors $\boldsymbol{x} = \mathbf{e}_i$, where $1 \leq i \leq m$. Then $\hat{\mathbf{t}}$ satisfies (3.9) for all $\boldsymbol{x} \in \{0,1\}^m$.*

*Proof.* Let $i$ be arbitrary such that $1 \leq i \leq m$ and choose $\mathbf{x} = \mathbf{e}_i$. If this unit vector $\mathbf{x}$ satisfies (3.9) then the following holds:

$$0 \leq \sum_{j=1}^{m} t_j x_j = t_i.$$

So if $\mathbf{e}_i$ satisfies (3.9) it follows that $t_i \geq 0$. Hence if $\hat{\mathbf{t}}$ satisfies (3.9) for all unit vectors $\mathbf{x} = \mathbf{e}_i$, then it holds that $\hat{t}_i \geq 0$ for all $i$ where $1 \leq i \leq m$. Consequently, (3.9) holds for all $\mathbf{x} \geq \mathbf{0}$, in particular for all $\mathbf{x} \in \{0,1\}^m$. ■

It follows from Lemma 3.4 that the optimum of pricing problem (3.7) is nonnegative with the current initialisation. Therefore it suffices to only solve pricing problem (3.8) within each iteration of our column generation process. If on the one hand the optimum is strictly negative, we add variable $\beta_r$ and all parameters $\delta_{\mu r}$ for the resource $r \in R$ that has a violated dual constraint. If on the other hand the optimum of (3.8) is nonnegative, then all dual constraints are satisfied and the optimum to the primal LP has been found. The column generation process is summarised in Algorithm 3.1.

**Algorithm 3.1** Column Generation for SPoA of 4 player affine congestion games

**Require:** Restricted Master LP (3.4) with only $\alpha_r$, $\beta_r$ and $\delta_{\mu r}$ for those resources $r \in R$ that are only used by one action $\mu \in \bigcup_{i=1}^{4} \mathcal{A}_i$, pricing problem (3.8).
**Ensure:** SPoA for 4 player affine congestion games.
  **repeat**
    Solve Restricted Master LP (3.4)
    Solve pricing problem (3.8).
    **if** Objective value of (3.8) $< 0$ **then**
      Add $\beta_r$ and all $\delta_{\mu r}$ to Restricted Master LP (3.4) for the resource $r \in R$ for which pricing problem (3.8) found a violated constraint in the dual problem of the Complete Master LP (3.4).
    **end if**
  **until** Objective value of (3.8) $\geq 0$
  **return** Objective Value of (3.4).

## 3.3 Sequential Price of Anarchy

As stated in the beginning of the chapter the Sequential Price of Anarchy for affine congestion games with 2 or 3 players equals 1.5 and 1039/488 respectively [15]. A lower bound for the Sequential Price of Anarchy for the case with 4 players is 2.5509150067 [17]. By applying the column generation process as described in Algorithm 3.1, we could determine the exact Sequential Price of Anarchy for 4 player affine congestion games.

**Theorem 3.5.** *For affine congestion games with 4 players the following holds:*

$$\mathsf{SPoA} = 28679925/10823887 \approx 2.65. \tag{3.10}$$

*Proof.* We implemented Algorithm 3.1 in Python using the Gurobi extension [12]. The objective value of the final Master LP equals 28679925/10823887. ∎

It can be seen that the numerator and denominator of the Sequential Price of Anarchy cannot be reduced to a small value. As a matter of fact the worst case instance for which the SPoA is obtained has cost functions with weights where the denominator is large, too. The instance found by the LP consists of more than a hundred resources. The number of resources is significantly lower than $2^{55} - 1$, which is the number of resources that the complete version of LP (3.4) considers. But the obtained instance is too large to display in the report.

# Chapter 4

# Restricted Instances of Congestion Games

In the cases with 3 or 4 players the Sequential Price of Anarchy of affine congestion games are fractions with large denominators that cannot be simplified. In [15] it is stated that with three players this value equals 1039/488 and Theorem 3.5 states that the value for the case with 4 players equals 28679925/10823887. It seems unlikely that such values will occur in any practical instances of congestion games. In this chapter it is investigated how restricted instances can be handled in case the input parameters of the cost functions are bounded. This would correspond to more realistic scenarios from a practical perspective. We do this in order to understand if this would have meaningful consequences on the Sequential Price of Anarchy. In order to do this it is investigated whether adaptations to the LP from [15] can be made in order to evaluate the Sequential Price of Anarchy with such restrictions. In Section 4.1 we impose restrictions on the worst case instance that the original LP describes. In Section 4.2 we discuss a variant of the LP that describes any arbitrary instance with a fixed number of actions and resources.

## 4.1 Bounded Cost Parameters

The LP that computes the Sequential Price of Anarchy for the class of affine congestion games essentially determines the parameters $\alpha_r$ and $\beta_r$ for all resources in the worst case instance, which satisfies the properties described by Lemma 3.1. These instances have resources with weights that vary a lot across resources. For instance, in [15] the worst case instance for the case with three players has some resources with weight 3 and 4 but other resources have weights of 82, 93 and 374. In order to restrict instances to represent realistic congestion games it is intuitive to restrict the activation costs and weights of resources to only vary a little.

However, even if we consider the class of congestion games where the activation costs and weights of all resources to be either 0 or the same positive value, there still exists an instance for which $\mathsf{SPoA}(I)$ equals the Sequential Price of Anarchy of the class of all congestion games. In order to prove this, we use the following lemma.

**Lemma 4.1.** *There exists a solution to LP* (3.4) *such that all variables are rational.*

*Proof.* By constraints (3.4h) and (3.4i) it holds that for any solution we have that $C\left(A^{\mathsf{OPT}}\right) = 1$. By the third property of Lemma 3.1 it holds that $\alpha_r$ and $\beta_r$ are bounded between 0 and $nC\left(A^{\mathsf{OPT}}\right) = n$. Hence we know that there exists an optimal solution to the LP. It follows that there exists an optimal solution which is a basic feasible solution [3].

Furthermore it is possible to write the constraints in the following form: [3]

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

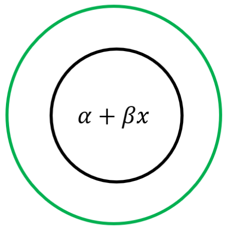
The optimal solution $\mathbf{x}$ can be described in the form $B\mathbf{x} = \mathbf{b}$, where $B$ is a matrix with the coefficients of the basis variables and $\mathbf{b}$ represents the right hand side of the constraints [3]. We can compute the solution of $x$ using Cramer's rule. [10] Since all coefficients of $B$ and $\mathbf{b}$ are integer, so are the determinants of the matrices used in Cramer's rule. We conclude that the optimal solution $x$ is rational. ■

Using Lemma 4.1 we prove the following theorem.

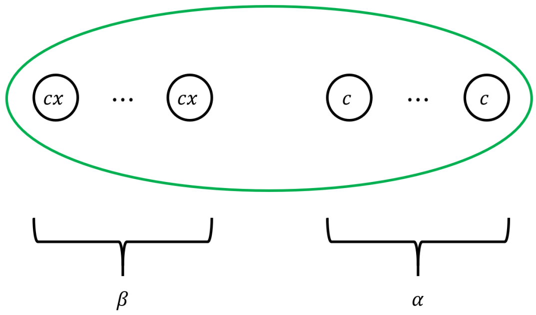
**Theorem 4.2.** *Let $\mathcal{I}$ be the class of affine congestion games with $n$ players. Let $c > 0$ be an arbitrary number. Let $\mathcal{J}$ be the class of affine congestion games with $n$ players such that the cost functions of all resources can only have a activation term of 0 or $c$ and a linear term of 0 or $c$. Then $\mathsf{SPoA}(\mathcal{I}) = \mathsf{SPoA}(\mathcal{J})$.*

*Proof.* Since all instances of class $\mathcal{J}$ also are instances of class $\mathcal{I}$ it suffices to show that there exists an instance $J \in \mathcal{J}$ such that $\mathsf{SPoA}(J) = \mathsf{SPoA}(\mathcal{I})$ Let $I$ be the worst case instance of class $\mathcal{I}$ given by LP (3.4). By Lemma 4.1 we know that all $\alpha_r$ and $\beta_r$ of this instance are rational. If this solution has any resources that have a noninteger activation cost or weight, then multiply all $\alpha_r$ and $\beta_r$ by the product of the denominators of all noninteger activation costs and linear terms. This new instance then has the same Sequential Price of Anarchy as the original instance. Now, replace each resource $r$ in this instance with $\alpha_r$ resources that have a activation cost of $c$ and no linear cost, and with $\beta_r$ resources that have a linear cost of $c$ and no activation cost. Include these new resources in exactly those actions that $r$ appeared in. This has been illustrated in Figure 4.1 Then we have constructed an instance $J \in \mathcal{J}$ where all player costs $C_i(A)$ have been multiplied by the same constant, namely the product of $c$ and the denominators of all noninteger activation costs and weights of the resources of $I$. This means that that $\mathsf{SPoA}(J) = \mathsf{SPoA}(\mathcal{I})$. ■

Theorem 4.2 states that there is no point in creating a program that only bounds the parameters of the cost functions of resources. After all, such a program would yield the same objective value as the program in [15]. However, we can investigate restricted versions of the worst case instance described by LP (3.4). In order to understand the effect of bounding the cost parameters of this particular instance we focus on the case with 3 players. In [15] a worst case instance for the 3 player case was presented for which that all resource weights were integer. This solution was a scaled version of an optimal solution




(a) Resource before replacement




(b) Replacing resources

Figure 4.1: Resource replacement in the proof of Theorem 4.2

to the original 3 player version of the LP as stated in [15]. In this solution all resource weights were between 0 and 374. Let us now restrict the worst case instance for 3 players such that for all rescource it holds that $\alpha_r$ and $\beta_r$ are integer and bounded between 0 and some upper bound $U$ lower than 374. It is to be expected that the Sequential Price of Anarchy of this instance is lower than 1039/488, the value for the unrestricted instance [15]. However, there may be instances with a Sequential Price of Anarchy close to that value for which the maximum of the variables $\alpha_r$ and $\beta_r$ is much lower than 374. This is investigated in this section.

## Mixed Integer Program

$$\max \quad C\left(A^{\mathsf{SPE}}\right) \tag{4.1a}$$

$$\text{s.t.} \quad v_\mu = \sum_{r\in R} \delta_{\mu r}\left(\beta_r + \alpha_r\right) \qquad \forall \mu \in \bigcup_{i=1}^{3} \mathcal{A}_i \tag{4.1b}$$

$$o_{pq} = \sum_{r\in R} \delta_{pr}\delta_{qr}\beta_r \qquad \forall p \in \mathcal{A}_i, \forall q \in \mathcal{A}_j, j > i \tag{4.1c}$$

$$C_1(abc) = v_a + o_{ab} + o_{ac} \qquad \forall a,b,c \tag{4.1d}$$

$$C_2(abc) = v_b + o_{ab} + o_{bc} \qquad \forall a,b,c \tag{4.1e}$$

$$C_3(abc) = v_c + o_{ac} + o_{bc} \qquad \forall a,b,c \tag{4.1f}$$

$$\sum_{i=1}^{3} C_i(1.1, 2.1, 3.1) = x \tag{4.1g}$$

$$\sum_{i=1}^{3} C_i(abc) \geq x \qquad \forall a,b,c \tag{4.1h}$$

$$C_3(abc) \leq C_3(abc') \qquad \forall a,b,c|z^3_{a,b,c} = 1, \forall c' \tag{4.1i}$$

$$C_2(ab) \leq C_2(ab') \qquad \forall a,b|z^2_{ab} = 1, \forall b' \tag{4.1j}$$

$$C_1(a) \leq C_1(a') \qquad \forall a|z^1_a = 1, \forall a' \tag{4.1k}$$

$$C_1(a) = C_1(abc) \qquad \forall a,b|z^2_{ab} = 1, c|z^3_{abc} = 1 \tag{4.1l}$$

$$C_2(ab) = C_2(abc) \qquad \forall a,b,c|z^3_{abc} = 1 \tag{4.1m}$$

$$C\left(A^{\mathsf{SPE}}\right) = \frac{1}{x}\sum_{i=1}^{4} C_i(1.2, 2.3, 3.7) \tag{4.1n}$$

$$0 \leq \alpha_r, \beta_r \leq U \qquad \forall r \tag{4.1o}$$

$$\alpha_r, \beta_r \in \mathbb{Z}_+ \tag{4.1p}$$

In order to investigate this restricted instance we use program (4.1), which is an adapted variant of the LP in [15]. Observe that this is now a Mixed Integer Program (MIP), because we set $\alpha_r, \beta_r \in \mathbb{Z}_+$. Besides the fact that $\alpha_r$ and $\beta_r$ have been bounded, which is stated in constraint (4.1o), there has been made another change. In the original LP it can be assumed that $C\left(A^{\mathsf{OPT}}\right) = 1$. After all, any instance can by scaled by multiplying $\alpha_r$ and $\beta_r$ with any positive number for all resources $r \in R$ without changing the Sequential Price of Anarchy of the instance. However, if $\alpha_r$ and $\beta_r$ are restricted by some upper bound $U$ such a scaling is not always possible. So in order to find the Sequential Price of Anarchy for all instances bounded by $U$ all different possible values for $C\left(A^{\mathsf{OPT}}\right)$ have to be considered.

Constraints (4.1g), (4.1h) and (4.1n) have been set up such that the total cost in the social optimum equals some integer value $x$. If one solves the program for a given upper bound $U$ for all possible social optimum values, the Sequential Price of Anarchy of all instances with that upper bound then is the maximum of all those solutions. In order to apply this approach we need to determine all possible values the social optimum can have for which the Sequential Price of Anarchy can be maximal. According to Lemma 3.4 there exists an optimal solution for which $\alpha_r = 0$ for all resources that are selected by at least two different actions. In [5] it is even stated that there exists a worst case instance with only linear cost functions. Therefore the highest possible social optimum for which the Sequential Price of Anarchy could be maximal is reached in the solution where $\beta_r = U$ and $\alpha_r = 0$ for all $r \in R$. Observe that the number of actions in (4.1) is 12. It follows that for any action $\mu$ there are $2^{11}$ resources for which $\delta_{\mu,r} = 1$. Moreover, for any action pair $(p, q)$ there exist $2^{10}$ resources such that $\delta_{p,r} = \delta_{q,r} = 1$. If we set $\beta_r = U$ and $\alpha_r = 0$ for all resources, it follows from constraints (4.1b) and (4.1c) that $v_\mu = 2^{11}U$ and $o_{pq} = 2^{10}U$ for all actions $\mu, p, q$. Hence in this solution it holds that

$$\begin{aligned} C\left(A^{\mathsf{OPT}}\right) &= \sum_{i=1}^{3} C_i(1.1, 2.1, 3.1) \\ &= \sum_{i=1}^{3} 2^{11}U + 2^{10}U + 2^{10}U \\ &= 12288U. \end{aligned}$$

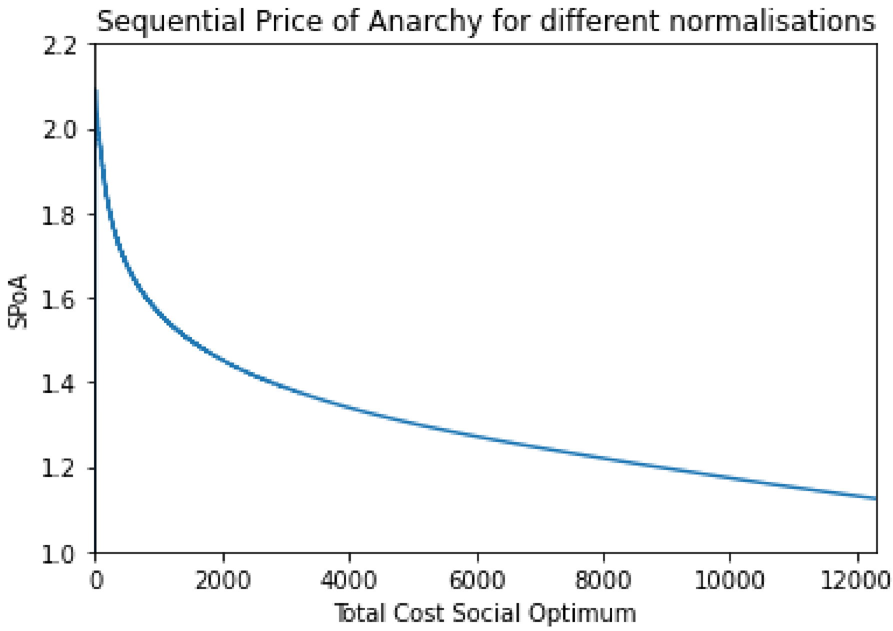



Figure 4.2: SPoA when $U = 1$

In Figure 4.2 the Sequential Price of Anarchy is shown for $U = 1$ and the social optimum between 1 and 12288. It holds that over all normalisations the Sequential Price of Anarchy is approximately 2.09, a value that is obtained when $C\left(A^{\mathsf{OPT}}\right) = 11$. It can also be observed that the Sequential Price of Anarchy is reached when the social optimum is still low. After all, all instances with a Sequential Price of Anarchy of at least 2 have a social cost in the social optimum of at most 62. On the other hand, already when $C\left(A^{\mathsf{OPT}}\right) = 3$ there exists an instance for which the Sequential Price of Anarchy is 2.

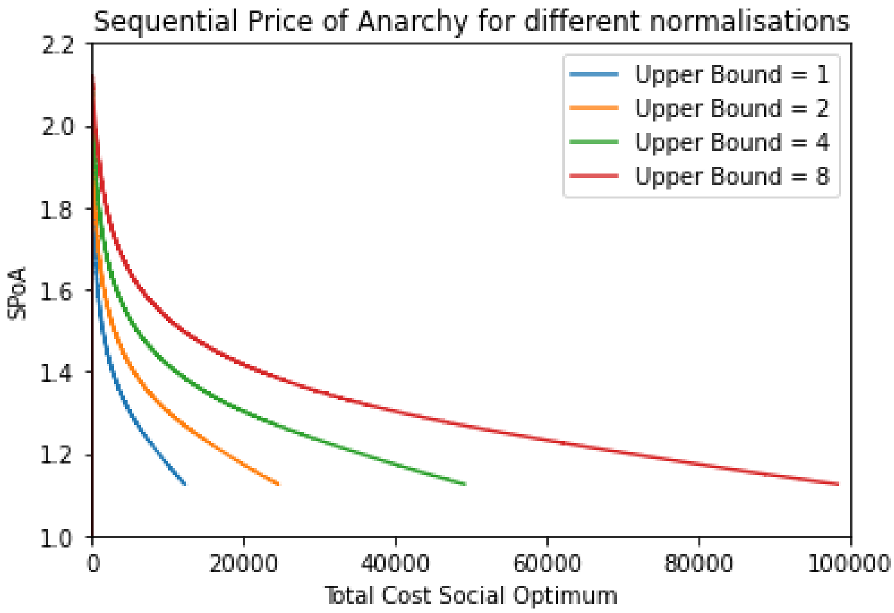



Figure 4.3: SPoA for different upper bounds $U$

It can be conjectured that also for higher upper bounds the maximal Sequential Price of Anarchy is obtained when the social optimum is relatively small in comparison to $12288U$. Figure 4.3 depicts the Sequential Price of Anarchy for all normalisations when the upper bound is 1, 2, 4 or 8. It can be seen that for all these upper bounds the same pattern occurs.

We now use program (4.1) to show how for increasing upper bound $U$ on the cost parameters of the worst case instance the Sequential Price of Anarchy converges to 1039/488, the Sequential Price of Anarchy for the class of all affine congestion games with 3 players [15]. Figure 4.4 shows this convergence. In this report we do not prove that the Sequential Price of Anarchy is obtained when the social cost of the social optimum is low. Instead the figure shows the Sequential Price of Anarchy over all instances with a social cost in the social optimum of at most 1000. When we set the upper bound $U$ equal to 127, the instance with bounded cost parameters has a Sequential Price of Anarchy of 1039/488. It can be seen that the Sequential Price of Anarchy converges to this value rather quickly. However, it could be the case that for some upper bounds the value is even closer to 1039/488 than depicted in the figure. After all we have not formally proven that for any upper bound $U$ the highest Sequential Price of Anarchy is obtained in an instance where the social cost in the social optimum is at most 1000.

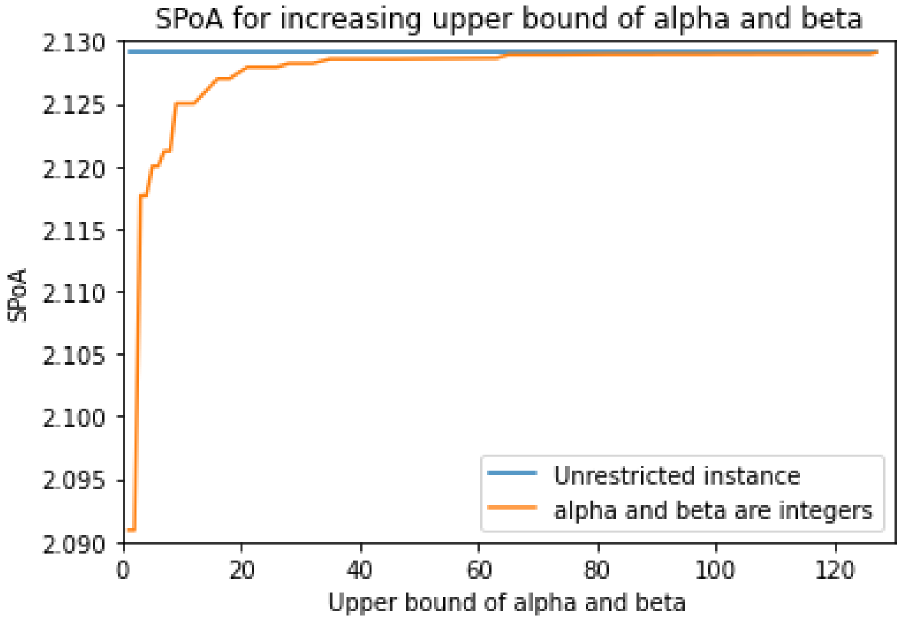



Figure 4.4: SPoA for increasing upper bound $U$

## 4.2 Program for Specified Number of Actions and Resources

The LP presented in [15] describes a congestion game instance that has been proven to be the worst case instance for the entire class of congestion games. It finds the resource cost functions for all resources of that instance such that the Sequential Price of Anarchy of the worst case instance is maximised.

However, within the general class of affine congestion games there exist subclasses with a more specified structure. An example of such a subclass is the class of network congestion games, which has been introduced in Example 2.1 and 2.2. The Sequential Price of Anarchy of the general class serves as an upper bound for the Sequential Price of Anarchy of such subclasses. Moreover for the class of network congestion games the lower bound of $\Omega(\sqrt{n})$ has been presented, where $n$ represents the number of players [8]. But in general no exact Sequential Price of Anarchy for such subclasses is known. Even for a specific instance of an affine congestion game where the set of resources and the sets of actions have been fixed, it is not clear how the cost parameters of the resources of the instance can be set such that the Sequential Price of Anarchy of the instance is maximised.

In this section we investigate to what extent the program in [**?**] can be altered in order to compute the Sequential Price of Anarchy of subclasses or specific instances of affine congestion games. It turns out that it is not possible to derive a variant of the program to compute the Sequential Price of Anarchy for subclasses of affine congestion games. However, we do derive a generalised variant of that program which describes an arbitrary instance of a congestion game for which the set of resources and the sets of actions for all players have been fixed.

For such instances this variant of the program finds the cost parameters of the resource cost functions for which the Sequential Price of Anarchy of the instance is maximised. The program displayed in this section holds for games with three players.

In the instance of the original program all actions are symmetrical. That is, each action contains the same number of resources and for each unique subset of the actions the instance has exactly one resource that is contained in exactly those actions. However, in general an arbitrary instance do not have to be symmetrical. Therefore we do not know on beforehand which actions turn out to be part of the social optimum or the worst Subgame Perfect Equilibrium of the game. In order to maximise the Sequential Price of Anarchy of the instance the Nash inequalities have to be enforced for those actions that form the Subgame Perfect Equilibrium in the optimal solution. Therefore we have introduced the binary variables $z_1(a)$, $z_2(ab)$, $z_3(abc)$ and $s(abc)$ to denote the actions of the Subgame Perfect Equilibrium and the social optimum, as is denoted in Table 4.3. Note that the variables $z_1(a)$, $z_2(ab)$ and $z_3(abc)$ were parameters in the original LP.

The program is displayed in (4.2). Here constraints (4.2g), (4.2j), (4.2k) and (4.2l) ensure that exactly one action pattern is selected to be the social optimum and that in each node of the game tree exactly one action is selected to be included in $S^{\mathsf{SPE}}$. Then the normalisation constraint (4.2h) and the Nash inequalities in (4.2m), (4.2n) and (4.2o) are only enforced for those actions that belong to the social optimum and $S^{\mathsf{SPE}}$ respectively. Since the total costs in the social optimum have been set to 1, the objective value of the program is the total costs in the Subgame Perfect Outcome that follows from $z_1(a)$, $z_2(ab)$ and $z_3(abc)$, as denoted in (4.2r).

## Sets

| | | |
|---|---|---|
| $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ | $\mu$, $\nu$, $a$, $a'$, $b$, $b'$, $c$, $c'$ | Actions |
| $R$ | $r$ | Resources |

Table 4.1: Sets

## Parameters

| | | |
|---|---|---|
| $\delta_{r\mu}$ | $\forall r, \mu$ | $\begin{cases} 1 & \text{if edge } r \text{ in action } \mu \\ 0 & \text{otherwise} \end{cases}$ |

Table 4.2: Parameters of the LP

## Variables

| | | |
|---|---|---|
| $\alpha_r$ | $\forall r$ | activation cost of $e$ |
| $\beta_r$ | $\forall r$ | weight of $e$ |
| $z_1(a)$ | $\forall a$ | $\begin{cases} 1 & \text{if action } a \text{ is played in SPE by player 1} \\ 0 & \text{otherwise} \end{cases}$ |
| $z_2(ab)$ | $\forall a, b$ | $\begin{cases} 1 & \text{if action } b \text{ is played in SPE in state } a \text{ by player 2} \\ 0 & \text{otherwise} \end{cases}$ |
| $z_3(abc)$ | $\forall a, b, c$ | $\begin{cases} 1 & \text{if action } c \text{ is played in SPE in state } ab \text{ by player 3} \\ 0 & \text{otherwise} \end{cases}$ |
| $s(abc)$ | $\forall a, b, c$ | $\begin{cases} 1 & \text{if actions } a, b, c \text{ are played in social optimum} \\ 0 & \text{otherwise} \end{cases}$ |
| $v_\mu$ | $\forall \mu$ | total activation cost plus total weight of edges in $\mu$ |
| $o_{\mu\nu}$ | $\forall \mu, \nu$ | total weight of resources in $\mu \cap \nu$ |
| $C_i(abc)$ | $\forall a, b, c, i$ | cost of player $i$ when players 1, 2 and 3 choose $a$, $b$ and $c$ respectively |
| $C_2(ab)$ | $\forall a, b$ | cost of player 2 when players 1 and 2 choose $a$ and $b$ and player 3 plays according to $S^{\mathsf{SPE}}$. |
| $C_1(a)$ | $\forall a$ | cost of player 1 when player 1 chooses $a$ and players 2 and 3 play according to $S^{\mathsf{SPE}}$. |
| $C(A^{\mathsf{SPE}})$ | | costs in subgame perfect equilibrium $A^{\mathsf{SPE}}$ |

Table 4.3: Variables of the LP

### Non-linear Mixed Integer Program

$$\max \quad C\left(A^{\mathsf{SPE}}\right) \tag{4.2a}$$

$$\text{s.t.} \quad v_\mu = \sum_{r \in R} \delta_{r\mu}\left(\alpha_r + \beta_r\right) \qquad \forall \mu \in \bigcup_{i=1}^{3} \mathcal{A}_i \tag{4.2b}$$

$$o_{pq} = \sum_{r \in R} \delta_{rp}\delta_{rq}\beta_r \qquad \forall p \in \mathcal{A}_i, \forall q \in \mathcal{A}_j, j > i \tag{4.2c}$$

$$C_1(abc) = v_a + o_{ab} + o_{ac} \qquad \forall a, b, c \tag{4.2d}$$

$$C_2(abc) = v_b + o_{ab} + o_{bc} \qquad \forall a, b, c \tag{4.2e}$$

$$C_3(abc) = v_c + o_{ac} + o_{bc} \qquad \forall a, b, c \tag{4.2f}$$

$$\sum_{a \in \mathcal{A}_1} \sum_{b \in \mathcal{A}_2} \sum_{c \in \mathcal{A}_3} s(abc) = 1 \tag{4.2g}$$

$$\sum_{a \in \mathcal{A}_1} \sum_{b \in \mathcal{A}_2} \sum_{c \in \mathcal{A}_3} s(abc) \sum_{i=1}^{3} C_i(abc) = 1 \tag{4.2h}$$

$$\sum_{i=1}^{3} C_i(abc) \geq 1 \qquad \forall a, b, c \tag{4.2i}$$

$$\sum_{a \in \mathcal{A}_1} z_1(a) = 1 \tag{4.2j}$$

$$\sum_{b \in \mathcal{A}_2} z_2(ab) = 1 \qquad \forall a \tag{4.2k}$$

$$\sum_{c \in \mathcal{A}_3} z_3(abc) = 1 \qquad \forall a, b \tag{4.2l}$$

$$z_3(abc)C_3(abc) \leq C_3(abc') \qquad \forall a, b, c, c' \tag{4.2m}$$

$$z_2(ab)z_3(abc)C_2(abc) \leq C_2(ab') \qquad \forall a, b, b', c \tag{4.2n}$$

$$z_1(a)z_2(ab)z_3(abc)C_1(abc) \leq C_1(a') \qquad \forall a, a', b, c \tag{4.2o}$$

$$C_2(ab) = \sum_{c \in \mathcal{A}_3} z_3(abc)C_2(abc) \qquad \forall a, b \tag{4.2p}$$

$$C_1(a) = \sum_{b \in \mathcal{A}_\in} z_2(ab) \sum_{c \in \mathcal{A}_3} z_3(abc)C_1(abc) \qquad \forall a \tag{4.2q}$$

$$C\left(A^{\mathsf{SPE}}\right) = \sum_{a \in \mathcal{A}_1} z_1(a) \sum_{b \in \mathcal{A}_2} z_2(ab) \sum_{c \in \mathcal{A}_3} z_3(abc) \sum_{i=1}^{3} C_i(abc) \tag{4.2r}$$

$$\alpha_r, \beta_r \geq 0 \qquad \forall r, \mu, \nu, p_i, p_j \tag{4.2s}$$

$$z_1(a), z_2(ab), z_3(abc), s(abc) \in \{0, 1\} \qquad \forall a, b, c \tag{4.2t}$$

It can be observed that program (4.2) is non-linear. The reason for this is the fact that multiplications of binary variables and at most one continuous variable occur in the constraints. This means that the program can be linearised by adding auxilary variables that replace those multiplications [19]. Any multiplication of two binary variable $x_1$ and $x_2$ can be replaced by variable $y$ by adding the constraints displayed in (4.3). If $x_1$ is binary and $x_2$ a continuous variable such that $0 \leq x_2 \leq u$, then the multiplication can be replaced by variable $y$ that satisfies the constraints displayed in (4.4).

$$\begin{aligned} y &\leq x_1 \\ y &\leq x_2 \\ y &\geq x_1 + x_2 - 1 \end{aligned} \tag{4.3}$$

$$\begin{aligned} y &\leq u x_1 \\ y &\leq x_2 \\ y &\geq x_2 - u(1 - x_1) \\ y &\geq 0 \end{aligned} \tag{4.4}$$

**Proposition 4.3.** *Program* (4.2) *can be transformed in a Mixed Integer Program with only linear constraints.*

*Proof.* Let us construct a MIP with only linear constraints by replacing multiplications with auxiliary variables. Firstly, replace all multiplications of the binary variables $z_1(a)$, $z_2(ab)$ and $z_3(abc)$ in constraints (4.2n), (4.2o), (4.2q) and (4.2r). When a multiplication of three binary variables appears, then we can firstly replace the multiplication of the first two variables and then the multiplication of the auxiliary variable with the third binary variable. This new program only has multiplications of binary variables and cost variables left. Since the social optimum has been normalised to 1, it follows from Lemma 3.1 that $\alpha_r + \beta_r \leq 1$ for all resources $r \in R$. Hence the following is true for all cost variables.

$$C_i(abc) \leq 3|R|.$$

It also follows from the constraints that the cost variables are nonnegative. Consequently, we can replace all multiplications of a binary variable with a continuous variable using the auxiliary variable and constraints displayed in (4.4). ■

The reason that program (4.2) can describe any instance of a congestion game, is the fact that it has not been predetermined how many resources and actions are contained by the sets $R$ and $\mathcal{A}_i$. Hence this program can be seen as a generalisation of the LP presented in [15]. After all, if we define the sets of resources and actions as in the LP in [15], then the objective value of program (4.2) equals the objective value of the LP in [15]. In that instance all actions are symmetric, because for each subset of the actions there is exactly one resource that is included in exactly those actions. Therefore the values of the binary variables does not have an impact on the objective value. As a result, they could be replaced by the original parameters.

For any instance for which the number of players, resources and the actions have been fixed, program 4.2 can find the resource costs such that the Sequential Price of Anarchy is maximised. The LP in [15] finds the Sequential Price of Anarchy of the entire class of affine congestion players. However, there also exist some subclasses of congestion games for which the sets of actions have a specific structure. An example of this is the class of network congestion games. The presented programs cannot be used to find the Sequential Price of Anarchy of such subclasses. In this thesis it will not be proven that no such program exists. However, in Example 4.1 it is argued that the reasoning of Lemma 3.1 from [15] does not hold in general for such subclasses.

*Example* 4.1. Consider the network routing instance in Figure 4.5. In this instance all players choose a path from vertex $s$ to $t$ in the graph. The edges are the resources of this congestion game and the resource costs represent the travel time over the edges. For each player the set of actions consists of three paths: the top route, the bottom route and the zigzag route. According to Lemma 3.1 this instance can be transformed into a congestion game that satisfies all properties stated in the lemma while maintaining the same Sequential Price of Anarchy. However, that method constructs an instance where $|\mathcal{A}_1| = z_1 = 2$. So if we apply that method to this network routing instance, that would disallow player 1 to select one of the three routes, while all other players can still opt between all three routes. But a network congestion game has been defined such that any player is free to select any available path from the source to the target vertex. So while such an adapted instance is still a congestion game, it is not a network congestion game anymore.

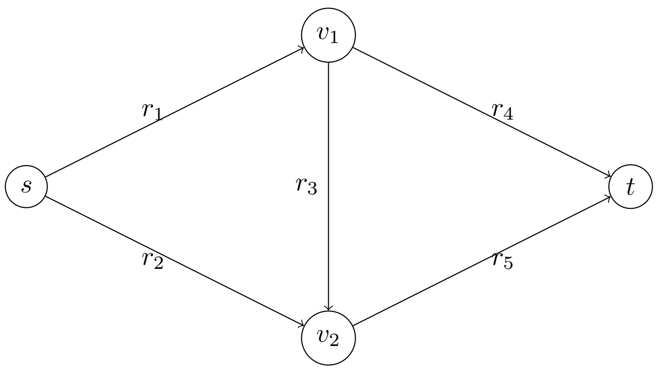



Figure 4.5: Instance of a network congestion game

Example 4.1 shows that one cannot construct a general worst case instance for a subclass of the affine congestion games such as the network routing. Lemma 3.1 allowed for the construction of a worst case instance for the general class of affine congestion games. This allowed for the creation of the LP in [15] as a tool to find the Sequential Price of Anarchy of the entire class of affine congestion games. But Example 4.1 shows that this lemma does not hold in general for a subclass. If one removes actions from an instance of a subclass such as the network congestion games in order to create an instance where $|\mathcal{A}_i| \leq z_i$ for all players $i \in N$, then one may construct an instance outside of the subclass. Therefore it is not possible to use Lemma 3.1 to provide an upper bound on the number of resources and actions for which a worst case instance exists. Hence we cannot create a program to find the Subgame Perfect Equilibrium of a subclass to the affine congestion games.

# Chapter 5

# Weighted Congestion Games

Chapter 3 and [15] focus on the Sequential Price of Anarchy for unweighted congestion games. That class of games models a scenario where any two players who congest the same resource contribute an equal amount to the total congestion of that resource. In this chapter we focus on the class of weighted congestion games, as described in [13] and [4]. Here each player is given a weight that represents the amount that they contribute to the total congestion of a resource. Recall from Section 2 that the cost of a resource is defined by the function $c_r(x_r)$. For unweighted congestion games the total congestion $x_r$ was defined as the number of players to whom resource $r$ was allocated: $x_r = |\{i \in N | r \in A_i\}|$. However, in *weighted congestion games* the total congestion of a resource $r$ equals to sum of the weights of the players to whom the resource is allocated: $x_r = \sum_{i \in N : r \in A_i} w_i$. In this chapter we will investigate the impact of player weights on the Sequential Price of Anarchy. Similarly to previous chapters we only allow for affine resource costs. We focus on games with $n = 2$.

We consider two variants of the weighted congestion game model: the model with *proportional costs* and the model with *uniform costs*. In a weighted congestion game with proportional costs the player costs are proportional to the player weights. This means that for all actions $A \in \mathcal{A}$ and all players $i \in N$ the player cost is defined as follows:

$$C_i(A) = w_i \sum_{r \in A_i} c_r(x_r). \tag{5.1}$$

In the model with uniform costs each player pays the same amount for allocated resources. In this model the weights of the players are only used to determine the total congestion of a resource. The player costs are then defined similarly to unweighted congestion games:

$$C_i(A) = \sum_{r \in A_i} c_r(x_r). \tag{5.2}$$

In this chapter we prove the following results, which we will discuss in detail in Sections 5.1 and 5.2 respectively.

**Theorem 5.1.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with proportional costs and 2 players it holds that the Sequential Price of Anarchy is given by the following formula:*

$$\mathsf{SPoA}(w_1, w_2) = 1 + \frac{w_1 w_2}{w_1^2 + w_2^2}. \tag{5.3}$$

**Corollary 5.1.1.** *For the class of weighted affine congestion games with proportional costs and 2 players it holds that* $\mathsf{SPoA} = 1.5$.

**Theorem 5.2.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with uniform costs and 2 players it holds that the Sequential Price of Anarchy is given by the following formula:*

$$\mathsf{SPoA}(w_1, w_2) = \begin{cases} 1 + \frac{w_1}{w_1 + w_2} & \text{if } w_2 \leq w_1 \\ 1 + \frac{2w_1 w_2}{2w_1^2 + w_1 w_2 + w_2^2} & \text{if } w_1 < w_2 \leq 2w_1 \\ 1 + \frac{w_2}{2w_1 + w_2} & \text{if } 2w_1 < w_2 \end{cases}. \tag{5.4}$$

**Corollary 5.2.1.** *For the class of weighted affine congestion games with uniform costs and 2 players it holds that* $\mathsf{SPoA} = 2$.

Recall that the Sequential Price of Anarchy for (unweighted) affine congestion games with 2 players it has been established that $\mathsf{SPoA} = 1.5$ [15]. On the one hand Corollary 5.1.1 states that for weighted congestion games with proportional costs the Sequential Price of Anarchy is the same as for the unweighted case. On the other hand Corollary 5.2.1 states that for weighted congestion games with uniform costs is strictly higher.

Theorems 5.1 and 5.2 are proven in two steps. Firstly, it is proven that the right hand sides of (5.12) and (5.4) serve as lower bounds for the Sequential Price of Anarchy. Secondly, it is certified that they also serve as upper bounds.

## 5.1 Proportional Costs

In this section we compute the Sequential Price of Anarchy for the class of games where players are assigned proportional resource costs. In order to do that we take the following approach. Firstly, we propose a potential worst case instance and compute the Sequential Price of Anarchy of that instance. Secondly, we certify that the proposed instance is indeed a worst case instance with a linear programming approach. This forms a proof for Theorem 5.1. Finally, we find the weights that maximise the Sequential Price of Anarchy of the worst case instance. That value is the Sequential Price of Anarchy of the entire class, which is the proof of Corollary 5.1.1.

### 5.1.1 Worst Case Instance

The worst case instance for this problem is similar to the worst case instance for unweighted congestion games. That instance is described in [15] and consists of three resources. The first player can either select resource 1 or resource 2, then player 2 either picks resource 2 or resource 3. We set the cost functions to be linear, so $c_r(x_r) = \beta_r x_r$ for all $r \in R$. We intend to define the resource costs in such a way that $A^{\mathsf{OPT}} = (\{1\}, \{2\})$ and $A^{\mathsf{SPE}} = (\{2\}, \{3\})$. This is displayed in Figure 5.1.

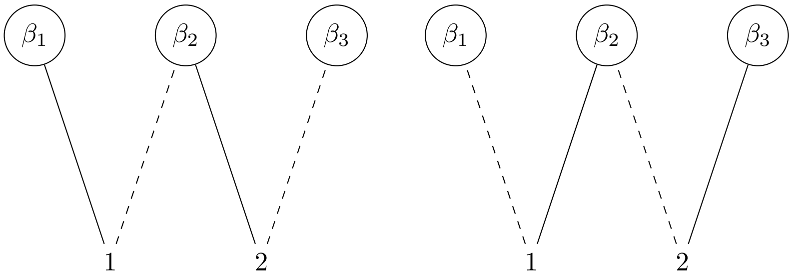



Figure 5.1: Optimal and subgame perfect action profiles for worst case instance for weighted congestion games with proportional costs

In order to determine the Sequential Price of Anarchy of this instance we need to find the values of $\beta_1$, $\beta_2$ and $\beta_3$. First of all we show that we only need to define two of the three cost functions.

**Lemma 5.3.** *Suppose we are given a specific instance $I$ of a weighted congestion game with three resources as described above. Let $\beta_1$, $\beta_2$ and $\beta_3$ be the linear terms of the resource cost functions. Let $I'$ be a similar instance with $a\beta_1$, $a\beta_2$ and $a\beta_3$ as linear terms, with $a > 0$. Then* $\mathsf{SPoA}(I) = \mathsf{SPoA}(I')$

*Proof.* Let $i \in N$ be an arbitrary player and let $(A_1, A_2) \in \mathcal{A}$ be an arbitrary action outcome of the games. Let us call the cost for player $i$ in this action outcome as $C_i^I(A_1, A_2)$ and $C_i^{I'}(A_1, A_2)$ in instances $I$ and $I'$ respectively. Then for instance $I$ the cost is defined as follows:

$$C_i^I(A_1, A_2) = w_i \beta_r x_r.$$

Now for instance $I'$ the cost for player $i$ at the same action outcome is as follows:

$$\begin{aligned} C_i^{I'}(A_1, A_2) &= w_i c \beta_r x_r \\ &= a C_i^I(A_1, A_2). \end{aligned}$$

As a result the Sequential Price of Anarchy is the following:

$$
\begin{aligned}
\mathsf{SPoA}(I') &= \frac{C_1^{I'}(A^{\mathsf{SPE}}) + C_2^{I'}(A^{\mathsf{SPE}})}{C_1^{I'}(A^{\mathsf{OPT}}) + C_2^{I'}(A^{\mathsf{OPT}})} \\
&= \frac{aC_1^{I}(A^{\mathsf{SPE}}) + aC_2^{I}(A^{\mathsf{SPE}})}{aC_1^{I}(A^{\mathsf{OPT}}) + aC_2^{I}(A^{\mathsf{OPT}})} \\
&= \frac{C_1^{I}(A^{\mathsf{SPE}}) + C_2^{I}(A^{\mathsf{SPE}})}{C_1^{I}(A^{\mathsf{OPT}}) + C_2^{I}(A^{\mathsf{OPT}})} \\
&= \mathsf{SPoA}(I).
\end{aligned}
$$

■

By Lemma 5.3 we can set $\beta_2 := 1$. In order to define $\beta_1$ and $\beta_3$ we have to ensure that the outcomes of the social optimum and the Subgame Perfect Equilibrium occur as described by Figure 5.1b. This is achieved by enforcing the Nash inequalities for each action of $S^{\mathsf{SPE}}$ in each decision state of the game tree. These can be formulated using the game tree, which is denoted in Figure 5.2.

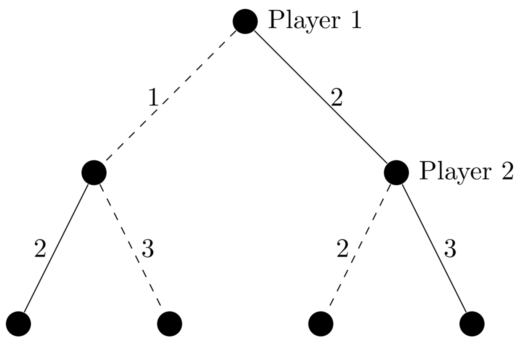



Figure 5.2: Game Tree of the worst case instance of Figure 5.1

The desired Subgame Perfect Outcome occurs when player 1 opts for the right path in the game tree and player 2 then also chooses the right path. This is enforced by two Nash inequalities. Firstly, in the subgame induced by the second state, when player 1 selected resource 2, it has to be ensured that picking resource 3 is a weakly dominant strategy over picking resource 2. This is described by constraint (5.5). Secondly, it has to be ensured that resource 2 is at least as cheap as resource 1 for player 1, under the assumption that player 2 picks resource 3 in the right state of the game tree. Since player 1 always pays the same price for resource 1 regardless of the strategy of player 2, we can formulate the Nash inequality for player 1 without formulating Nash inequalities for the subgame induced by the first state of player 2. So the inequality for player 1 is given by (5.6).

$$w_2^2\beta_3 \le w_2(w_1 + w_2)\beta_2 \tag{5.5}$$
$$w_1^2\beta_2 \le w_1^2\beta_1 \tag{5.6}$$

Now, let us define the Sequential Price of Anarchy of this game:

$$\mathsf{SPoA} = \frac{w_1^2\beta_2 + w_2^2\beta_3}{w_1^2\beta_1 + w_2^2\beta_2}. \tag{5.7}$$

From (5.7) the Sequential Price of Anarchy is maximised by setting $\beta_1$ as low as possible and $\beta_3$ as high as possible. This occurs when the Nash inequalities are tight. So $\beta_1$ and $\beta_3$ are as follows:

$$\begin{aligned} \beta_1 &= 1 \\ \beta_3 &= \frac{w_1 + w_2}{w_2}. \end{aligned}$$

Now all resource costs have been defined, so we can evaluate the Sequential Price of Anarchy of the game thus establish a lower bound for the Sequential Price of Anarchy for the entire class.

$$\begin{aligned} \mathsf{SPoA} &= \frac{w_1^2\beta_2 + w_2^2\beta_3}{w_1^2\beta_1 + w_2^2\beta_2} \\ &= \frac{w_1^2 + w_1w_2 + w_2^2}{w_1^2 + w_2^2} \\ &= 1 + \frac{w_1w_2}{w_1^2 + w_2^2}. \end{aligned}$$

**Lemma 5.4.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then the following formula is a lower bound for the Sequential Price of Anarchy for the class of weighted affine congestion games with proportional costs and 2 players:*

$$\mathsf{SPoA}(w_1, w_2) \geq 1 + \frac{w_1w_2}{w_1^2 + w_2^2}. \tag{5.8}$$

*Proof.* As demonstrated in this section, there exists an instance for which the Sequential Price of Anarchy is equal to this lower bound, namely the instance described by Figure 5.1 with $\beta_1 = \beta_2 = 1$ and $\beta_3 = \frac{w_1+w_2}{w_2}$. Hence this is a lower bound for the entire class. ■

### 5.1.2 A Certificate for the Upper Bound

In order to prove that the game presented above is indeed a worst-case instance, we use a linear program similar to the program presented in [15]. The program in [15] describes an optimal solution for the general worst case instance for affine congestion games. In this section we derive a general worst case instance for the class of weighted affine congestion games with proportional costs. This allows us to create a program that finds an optimal solution for that instance. While it is not possible to directly solve the program within reasonable time in order to find the Sequential Price of Anarchy, it can be used to provide an upper bound to the Sequential Price of Anarchy.

In order to prove the existence of a general worst case instance we prove the following lemma, which is a variant of Lemma 3.1 from [15]. The proof follows a similar structure to the proof of 3.1 presented in [15].

**Lemma 5.5.** *For any instance $I$ of a weighted affine congestion game with proportional costs there exists a weighted congestion game $I'$ for which the following properties hold.*

1. *For all players $i \in N$ it holds that $|\mathcal{A}_i| \leq z_i$, where all $z_i$ are defined as follows:*

$$z_1 := 2 \text{ and } z_i := 1 + \prod_{j=1}^{i-1} z_j \text{ for all } i \geq 2. \tag{5.9}$$

2. *The amount of resources $|R|$ is at most $2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$.*

3. $\mathsf{SPoA}(I') = \mathsf{SPoA}(I)$.

*Proof.* Suppose we are given an instance $I$ of a weighted affine congestion game with proportional costs. Let us construct an instance $I'$ for which the stated properties hold.

Firstly, note that for $I$ there exist both a social optimum and a Subgame Perfect Equilibrium. Let $A_1^{\mathsf{OPT}}$ and $A_1^{\mathsf{SPE}}$ be the actions that player 1 plays in the respective outcomes. If we now consider a game where all other actions for player 1 are removed, then in that game it holds that $|\mathcal{A}_1| \leq z_1$. Besides that, the same social optimum and Subgame Perfect Equilibrium still exist. Therefore the Sequential Price of Anarchy of the new instance has not changed. Now suppose that we have constructed a game where for the first $k$ players it holds that $|\mathcal{A}_j| \leq z_j$. Then the game tree of this new instance has at most $\prod_{j=1}^{k} z_j$ states in which player $k+1$ has to make a decision. In each state there exists one action that belongs to the subgame perfect strategy of player $k+1$. Besides that, there exists one action for player $k+1$ that is played in the social optimum. If we consider a new game in which all other actions for player $k+1$ are removed, then the same social optimum and Subgame Perfect Equilibrium as in $I$ still exist, and it holds that $|\mathcal{A}_{k+1}| \leq z_{k+1}$.

Secondly, suppose the newly constructed game has more than $2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$ resources. By the pigeonhole principle there exist two resources $r_1, r_2 \in R$ such that for all actions $A_i$ it holds that either both resources are picked in $A_i$ or none of them are. Then we can replace $r_1$ and $r_2$ by a new resource $r'$ without changing the total costs for any player regardless of the actions that are played. Let $i \in N$ be an arbitrary player to whom $r_1$ and $r_2$ are allocated when action profile $A$ is played. Since the total congestion $x_{r_1}$ of $r_1$ is equal to the total congestion $x_{r_2}$ of $r_2$ it holds that

$$\begin{aligned}
C_i(A) &= w_i c(r_1) + w_i c(r_2) + w_i \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r) \\
&= w_i(\alpha_{r_1} + \beta_{r_1} x_{r_1} + \alpha_{r_2} + \beta_{r_2} x_{r_2}) + w_i \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r) \\
&= w_i((\alpha_{r_1} + \alpha_{r_2}) + (\beta_{r_1} + \beta_{r_2}) x_{r_1}) + w_i \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r).
\end{aligned}$$

Now let us define the activation cost and the linear term of the resource cost for $r'$ as follows:

$$\alpha_{r'} = \alpha_{r_1} + \alpha_{r_2}$$
$$\beta_{r'} = \beta_{r_1} + \beta_{r_2}.$$

Then the Sequential Price of Anarchy of the game with $r'$ is equal to the game that involves both $r_1$ and $r_2$. This process can be repeated until we end up with the instance $I'$, for which $|R| \leq 2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$. ■

From Lemma 5.5 it follows that there exists a weighted affine congestion game with proportional costs for which $|\mathcal{A}_i| = z_i$ and $|R| = 2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$. This game can be described by a linear program similar to the LP in [15].

**Sets**

| | | |
|---|---|---|
| $P$ | $i, j$ | Players |
| $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ | $\mu$, $\nu$, $a$, $a'$, $b$, $b'$ | Actions |
| $R$ | $r$ | Resources |

Table 5.1: Sets

**Parameters**

| | | |
|---|---|---|
| $w_i$ | $\forall i$ | weight of player $i$ |
| $\delta_{\mu r}$ | $\forall \mu, r$ | $\begin{cases} 1 & \text{if } r \in \mu \\ 0 & \text{otherwise} \end{cases}$ |
| $z_a^1$ | $\forall a$ | $\begin{cases} 1 & \text{if } a \text{ is prescribed by } S_1^{\mathsf{SPE}} \\ 0 & \text{otherwise} \end{cases}$ |
| $z_{ab}^2$ | $\forall a, b$ | $\begin{cases} 1 & \text{if } b \text{ is prescribed by } S_2^{\mathsf{SPE}} \text{ in state } a \\ 0 & \text{otherwise} \end{cases}$ |

Table 5.2: Parameters

**Variables**

| | | |
|---|---|---|
| $\alpha_r$ | $\forall r$ | activation cost of resource $r$ |
| $\beta_r$ | $\forall r$ | weight of resource $r$ |
| $v_\mu$ | $\forall \mu$ | total activation cost plus total weight of resources in $\mu$ |
| $o_{ab}$ | $\forall a.b$ | total weight of resources in $a \cap b$ |
| $C_i(ab)$ | $\forall a, b, i$ | cost of player $i$ when players 1 and 2 choose $a$ and $b$ respectively |
| $C\left(A^{\mathsf{SPE}}\right)$ | | costs in subgame perfect equilibrium $A^{\mathsf{SPE}}$ |
| $C_1(a)$ | $\forall a$ | cost of player 1 when she chooses $a$ and player 2 chooses according to $S^{\mathsf{SPE}}$ |

Table 5.3: Variables

**Linear Program**

$$
\begin{aligned}
\max \quad & C\left(A^{\mathsf{SPE}}\right) && && (5.10a)\\
\text{s.t.} \quad & \sum_{r\in R}\delta_{\mu r}w_i\left(w_i\beta_r+\alpha_r\right)-v_\mu=0 && \forall \mu\in\mathcal{A}_i, i\in[2] && (5.10b)\\
& \sum_{r\in R}\delta_{ar}\delta_{br}w_1w_2\beta_r-o_{ab}=0 && \forall a\in\mathcal{A}_1,\forall b\in\mathcal{A}_2 && (5.10c)\\
& v_a+o_{ab}-C_1(ab)=0 && \forall a,b && (5.10d)\\
& v_b+o_{ab}-C_2(ab)=0 && \forall a,b && (5.10e)\\
& \sum_{i=1}^{2}C_i(1.1,2.1)=1 && && (5.10f)\\
& \sum_{i=1}^{2}C_i(ab)\geq 1 && \forall a,b && (5.10g)\\
& C_2(ab)-C_2(ab')\leq 0 && \forall a,b|z^2_{ab}=1,\forall b' && (5.10h)\\
& C_1(a)-C_1(a')\leq 0 && \forall a|z^1_a=1,\forall a' && (5.10i)\\
& C_1(a)-C_1(ab)=0 && \forall a,b|z^2_{ab}=1 && (5.10j)\\
& \sum_{i=1}^{2}C_i(1.2,2.3)-C\left(A^{\mathsf{SPE}}\right)=0 && && (5.10k)\\
& \alpha_r,\beta_r,v_\mu,o_{ab},C_i(ab),C\left(A^{\mathsf{SPE}}\right),C_1(a)\geq 0 && \forall a,b,\mu,r,i && (5.10l)
\end{aligned}
$$

Observe that the player weights appear as a parameter in this LP, namely in constraints (5.10b) and (5.10c). The LP aims to set the resource costs in such a way that given any two player weights the Sequential Price of Anarchy is maximised. However, in order to find the Sequential Price of Anarchy of the entire class we need to find those weights for which the objective value is maximised. If one would like to achieve that by directly solving the program, then the weights have to be defined as variables rather than weights. As a consequence constraints (5.10b) and (5.10c) would then contain multiplications of two nonnegative variables. In other words, one would need to solve a quadratic program in order to directly find the Sequential Price of Anarchy of the class.

Instead we intend to use this program to derive an upper bound for the Sequential Price of Anarchy. This upper bound serves as a certificate that the instance described in Section 5.1.1 is also a worst case instance. To that end, let us multiply each constraint with a constant described by Table 5.4. If we then add up all the constraints, we end up with the following constraint:

$$C\left(A^{\mathsf{SPE}}\right)+\zeta\leq 1+\frac{w_1w_2}{w_1^2+w_2^2}. \tag{5.11}$$

Here $\zeta$ is an expression consisting of LP variables other than $C\left(A^{\mathsf{SPE}}\right)$. If we can show that $\zeta\geq 0$, then we have found an upper bound for $C\left(A^{\mathsf{SPE}}\right)$. Observe that the right hand side of (5.11) is equal to the Sequential Price of Anarchy of the game discussed in Section 5.1.1. Consequently, if $\zeta\geq 0$ then the game in Section 5.1.1 is indeed a worst case instance.

| constraint | multiplier |
|---|---|
| (5.10b) when $\mu = 1.2$ | $x$ |
| (5.10b) when $\mu = 2.2$ | $x$ |
| (5.10c) when $a = 1.1$ and $b = 2.1$ | $x + 1$ |
| (5.10c) when $a = 1.2$ and $b = 2.2$ | $-1$ |
| (5.10d) when $a = 1.1$ and $b = 2.1$ | $x + 1$ |
| (5.10d) when $a = 1.1$ and $b = 2.2$ | $-x - 1$ |
| (5.10e) when $a = 1.1$ and $b = 2.2$ | $x + 1$ |
| (5.10e) when $a = 1.2$ and $b = 2.2$ | $-1$ |
| (5.10d) when $a = 1.2$ and $b = 2.3$ | $x$ |
| (5.10f) | $x + 1$ |
| (5.10h) when $a = 1.1$, $b = 2.2$ and $b' = 2.1$ | $x + 1$ |
| (5.10h) when $a = 1.2$, $b = 2.3$ and $b' = 2.2$ | $1$ |
| (5.10i) when $a = 1.2$ and $a' = 1.1$ | $x + 1$ |
| (5.10j) when $a = 1.1$ | $x + 1$ |
| (5.10j) when $a = 1.2$ | $-x - 1$ |
| (5.10k) | $-1$ |

Table 5.4: Multipliers for LP constraints, where $x = \frac{w_1 w_2}{w_1^2 + w_2^2}$. All other constraints have a multiplier of 0.

**Lemma 5.6.** *In* (5.11) *it holds that* $\zeta \geq 0$.

*Proof.* The term $\zeta$ is defined by the following expression:

$$\begin{aligned}
\zeta = {} & \frac{w_1 w_2}{w_1^2 + w_2^2} o_{1.2,2.3} \\
& + \frac{w_1 w_2}{w_1^2 + w_2^2} \left( \sum_{r \in R} \left( \delta_{1.2,r} w_1 (w_1 \beta_r + \alpha_r) + \delta_{2.2,r} w_2 (w_2 \beta_r + \alpha_r) \right) \right) \\
& + \left( 1 + \frac{w_1 w_2}{w_1^2 + w_2^2} \right) \left( \sum_{r \in R} \delta_{1.1,r} \delta_{2.1r} w_1 w_2 \beta_r \right) \\
& - \sum_{r \in R} \delta_{1.2,r} \delta_{2.2r} w_1 w_2 \beta_r .
\end{aligned}$$

Since all parameters and variables of the LP are nonnegative, it suffices to show that

$$\frac{w_1 w_2}{w_1^2 + w_2^2} \left( \sum_{r \in R} \left( \delta_{1.2,r} w_1^2 \beta_r + \delta_{2.2,r} w_2^2 \beta_r \right) \right) - \sum_{r \in R} \delta_{1.2,r} \delta_{2.2r} w_1 w_2 \beta_r \geq 0.$$

Observe that none of the resources $r \in R$ appear in the second summation for which $\delta_{1.2,r} = 0$ or $\delta_{2.2,r} = 0$. Therefore we only have to consider those resources $r \in R$ such that $\delta_{1.2,r} = 1$ and $\delta_{2.2,r} = 1$. For those resources the following holds:

$$\frac{w_1 w_2}{w_1^2 + w_2^2}\left(w_1^2\beta_r + w_2^2\beta_r\right) - w_1 w_2 \beta_r = \frac{w_1 w_2}{w_1^2 + w_2^2}\left(w_1^2 + w_2^2\right) - w_1 w_2$$
$$= w_1 w_2 - w_1 w_2$$
$$= 0.$$

■

From this lemma the upper bound for the Sequential Price of Anarchy follows.

**Lemma 5.7.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then the following formula is an upper bound for the Sequential Price of Anarchy for the class of weighted affine congestion games with proportional costs and 2 players:*

$$\mathsf{SPoA}(w_1, w_2) \leq 1 + \frac{w_1 w_2}{w_1^2 + w_2^2}. \tag{5.12}$$

*Proof.* If we multiply the LP constraints from (5.10) with the multiplier indicated in Table 5.4 it follows that for all feasible solutions to LP (5.10) have to satisfy constraint (5.11). From (5.11) and Lemma 5.6 it then follows that for all $w_1, w_2$ it holds that

$$C\left(A^{\text{SPE}}\right) \leq 1 + \frac{w_1 w_2}{w_1^2 + w_2^2}.$$

By Lemma 5.5 the objective value $C\left(A^{\mathsf{SPE}}\right)$ of LP (5.10) is equal to the Sequential Price of Anarchy. ■

### 5.1.3 Sequential Price of Anarchy

In the previous sections we have found a lower and an upper bound to the Sequential Price of Anarchy. This is sufficient to prove Theorem 5.1 and Corollary 5.1.1.

**Theorem 5.1.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with proportional costs and 2 players it holds that the Sequential Price of Anarchy is given by the following formula:*

$$\mathsf{SPoA}(w_1, w_2) = 1 + \frac{w_1 w_2}{w_1^2 + w_2^2}. \tag{5.3}$$

*Proof.* By Lemmas 5.4 and 5.7 the result follows. ■

**Corollary 5.1.1.** *For the class of weighted affine congestion games with proportional costs and 2 players it holds that $\mathsf{SPoA} = 1.5$.*

*Proof.* Let us choose $w_1 = w_2 = 1$. For those weights it holds that

$$\begin{aligned}
\mathsf{SPoA} &= \sup_{w_1,w_2} \mathsf{SPoA}(w_1,w_2) \\
&= \sup_{w_1,w_2} \left(1 + \frac{w_1 w_2}{w_1^2 + w_2^2}\right) \\
&\geq \mathrm{SPoA}(1,1) \\
&= 1 + \frac{1 \cdot 1}{1^2 + 1^2} \\
&= 1.5.
\end{aligned}$$

Furthermore for all $w_1$ and $w_2$ it holds that

$$(w_1 - w_2)^2 = w_1^2 - 2w_1w_2 + w_2^2 \geq 0.$$

It follows that

$$\begin{aligned}
\mathsf{SPoA} &= \sup_{w_1,w_2} \mathsf{SPoA}(w_1,w_2) \\
&= \sup_{w_1,w_2} \left(1 + \frac{2w_1 w_2}{2(w_1^2 + w_2^2)}\right) \\
&\leq \sup_{w_1,w_2} \left(1 + \frac{w_1^2 + w_2^2}{2(w_1^2 + w_2^2)}\right) \\
&= 1.5.
\end{aligned}$$

■

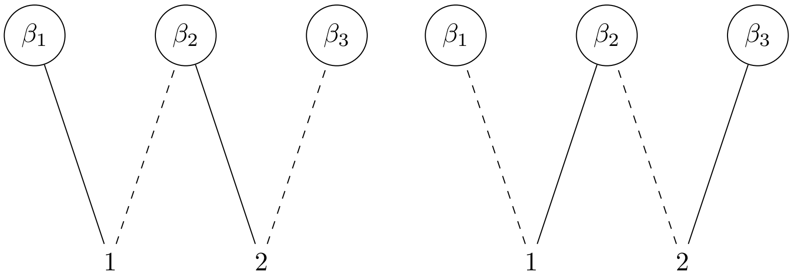
## 5.2 Uniform Costs

In this section we derive the Sequential Price of Anarchy for uniform costs, which is displayed in Theorem 5.2 and Corollary 5.2.1. We use a similar approach as in Section 5.1. Firstly, we present three instances in order to derive a lower bound. Then we use the LP to show that the lower bound is tight.

### 5.2.1 Worst Case Instances

In this section we present three instances of weighted congestion games with uniform costs. The Sequential Price of Anarchy of these instances serve as a lower bound for each interval indicated in the formula for $\mathsf{SPoA}(w_1, w_2)$ as shown in Theorem 5.2. For each instance we present the player costs for each action in the corresponding game tree. Using these values we derive the Subgame Perfect Equilibrium and the social optimum, which lead to the Sequential Price of Anarchy of the instance.

The first instance, which we will call $I_1$, is similar to the instance presented in Figure 5.3. This is the same instance as the wort case instance for the case with proportional costs, as displayed in figure 5.4. Only in this instance we set $\beta_1 = \beta_2 = 1$ and $\beta_3 = 1 + \frac{w_1}{w_2}$. The game tree of this instance is displayed in Figure 5.4. The player costs for each player are displayed in Table 5.5.




(a) $A^{\mathsf{OPT}}$ (b) $A^{\mathsf{SPE}}$

Figure 5.3: Optimal and subgame perfect action profiles for worst case instance for weighted congestion games with proportional costs

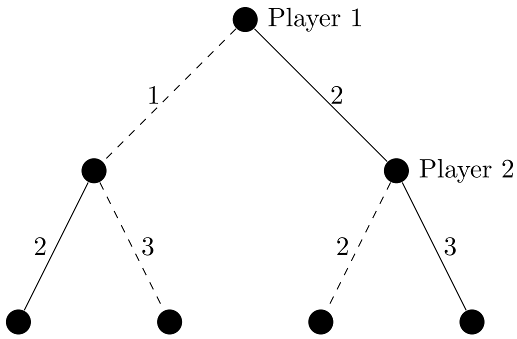



Figure 5.4: Game Tree of the worst case instance of Figure 5.3

| Action Outcome $(a, b)$ | $C_1(ab)$ | $C_2(ab)$ |
|---|---|---|
| $(\{1\}, \{2\})$ | $w_1$ | $w_2$ |
| $(\{1\}, \{3\})$ | $w_1$ | $w_1 + w_2$ |
| $(\{2\}, \{2\})$ | $w_1 + w_2$ | $w_1 + w_2$ |
| $(\{2\}, \{3\})$ | $w_1$ | $w_1 + w_2$ |

Table 5.5: Player costs for each outcome of the game tree displayed in Figure 5.2

It can be seen that the social optimum occurs when player 1 selects resource 1 and player 2 selects resource 2. Furthermore it can be seen that in any Subgame Perfect Equilibrium player 2 should select resource 2 if player 1 selects resource 1. But if player 1 selects resource 2, then it does not matter for player 2 which resource she selects. On the other hand player 1 should select resource 1 if she expects player 2 to pick resource 2 when she selects that resource herself too. Otherwise player 1 is indifferent.

It follows that the worst Subgame Perfect Equilibrium $S$ in this instance is as follows: $S = (\{2\}, (\{2\}, \{3\}))$. This leads to subgame perfect outcome $(\{2\}, \{3\})$. The Sequential Price of Anarchy for this instance then is as follows:

$$\mathsf{SPoA}(I_1) = \frac{w_1 + (w_1 + w_2)}{w_1 + w_2} = 1 + \frac{w_1}{w_1 + w_2}. \tag{5.13}$$

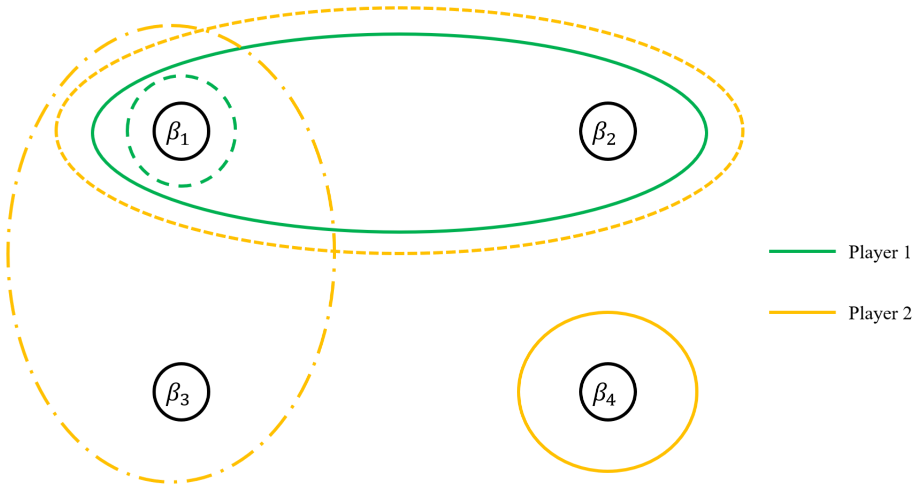



Figure 5.5: Worst case instance when $w_1 < w_2 \leq 2w_1$

Instance $I_2$ consists of four resources. It is displayed in Figure 5.5 and the corresponding game tree is presented in Figure 5.6. The cost functions of the resources do not have any activation costs. The linear terms are the following:

$$\begin{aligned}
\beta_1 &= 1 \\
\beta_2 &= \frac{w_2}{w_1} \\
\beta_3 &= 1 + \frac{w_1}{w_2} \\
\beta_4 &= 2 + \frac{w_1}{w_2} + \frac{w_2}{w_1}.
\end{aligned}$$

The action sets for the players are defined as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \{\{1\}, \{1, 2\}\} \\
\mathcal{A}_2 &= \{\{1, 2\}, \{2, 3\}, \{4\}\}.
\end{aligned}$$

In Table 5.6 the player costs for all outcomes of $I_2$ are displayed. First of all we can observe that $(\{1\}, \{2, 3\})$ is the social optimum. The worst Subgame Perfect Equilibrium will be presented in the following lemma:

**Lemma 5.8.** *Let strategy $S$ be the strategy defined as follows: $S = (\{1, 2\}, (\{1, 2\}, \{4\}))$. Then $S$ represents the worst Subgame Perfect Equilibrium of $I_2$.*

*Proof.* The outcome of strategy $S$ is action pattern $(\{1, 2\}, \{4\})$. In order to show that $S$ is a Subgame Perfect Equilibrium we verify that none of the players

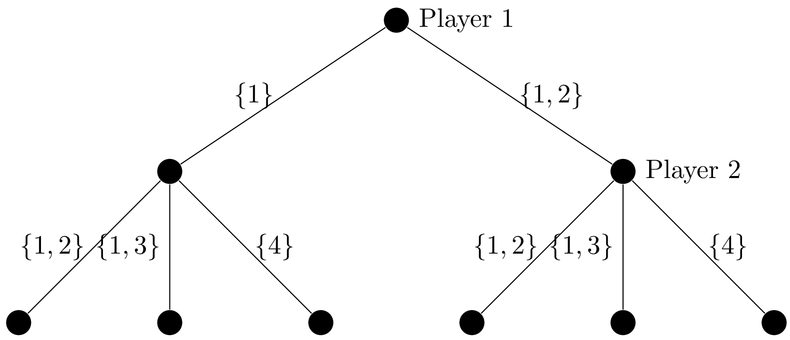



Figure 5.6: Game Tree of the worst case instance in Figure 5.7

| Action Outcome $(a, b)$ | $C_1(ab)$ | $C_2(ab)$ |
|---|---|---|
| $(\{1\}, \{1, 2\})$ | $w_1 + w_2$ | $w_1 + w_2 + \frac{w_2^2}{w_1}$ |
| $(\{1\}, \{2, 3\})$ | $w_1$ | $w_1 + w_2 + \frac{w_2^2}{w_1}$ |
| $(\{1\}, \{4\})$ | $w_1$ | $w_1 + 2w_2 + \frac{w_2^2}{w_1}$ |
| $(\{1, 2\}, \{1, 2\})$ | $w_1 + 2w_2 + \frac{w_2^2}{w_1}$ | $w_1 + 2w_2 + \frac{w_2^2}{w_1}$ |
| $(\{1, 2\}, \{2, 3\})$ | $w_1 + w_2 + \frac{w_2^2}{w_1}$ | $w_1 + 2w_2 + \frac{w_2^2}{w_1}$ |
| $(\{1, 2\}, \{4\})$ | $w_1 + w_2$ | $w_1 + 2w_2 + \frac{w_2^2}{w_1}$ |

Table 5.6: Player costs for each outcome of the game tree in Figure 5.6

can achieve a lower cost by selecting a different action. In Table 5.6 it can be seen that player 2 always has the same cost when player 1 selects resources 1 and 2. So player 2 cannot improve her cost. Likewise player 1 has the same cost for all outcomes that follow from the strategy of player 2. So $S$ is indeed a Subgame Perfect Equilibrium.

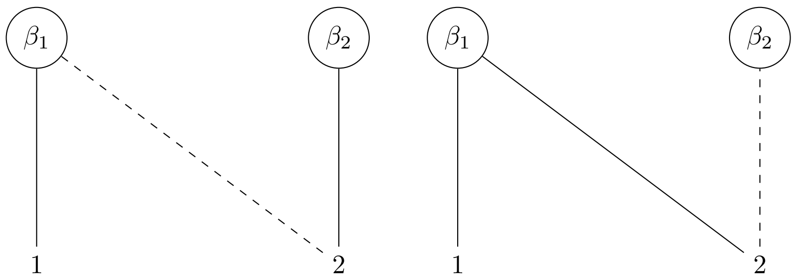
Now assume for the sake of contradiction that this is not the worst Subgame Perfect Equilibrium in this instance. Then there must exist a Subgame Perfect Equilibrium with either $(\{1, 2\}, \{1, 2\})$ or $(\{1, 2\}, \{2, 3\})$ as outcome. However, if player 2 selects $\{1, 2\}$ or $\{2, 3\}$ then player 1 has a strictly lower cost if she selects action $\{1\}$ instead of $\{1, 2\}$. It follows that $(\{1, 2\}, \{1, 2\})$ and $(\{1, 2\}, \{2, 3\})$ cannot be Subgame Perfect Outcomes, so $S$ is indeed the worst Subgame Perfect Equilibrium of $I_2$. ■

By Lemma 5.8 the Sequential Price of Anarchy of $I_2$ is as follows:

$$\begin{aligned} \mathsf{SPoA}(I_2) &= \frac{(w_1 + w_2) + \left(w_1 + 2w_2 + \frac{w_2^2}{w_1}\right)}{w_1 + \left(w_1 + w_2 + \frac{w_2^2}{w_1}\right)} \\ &= 1 + \frac{2w_1 w_2}{2w_1^2 + w_1 w_2 + w_2^2}. \end{aligned}$$

Finally, instance $I_3$ consists of two resources. The instance is displayed in Figure 5.7 and the corresponding game tree in Figure 5.8. The resource costs of this instance also only have linear terms, which are defined as follows:

$$\beta_1 = 1$$
$$\beta_2 = 1 + \frac{w_1}{w_2}.$$




(a) $A^{\mathsf{OPT}}$ (b) $A^{\mathsf{SPE}}$

Figure 5.7: Worst case instance when $2w_1 \leq w_2$

Table 5.7 shows the player costs for the only two outcomes of this game. It can be observed that $(\{1\}, \{2\})$ is the social optimum. Since player 1 can only select resource 1 and player 2 has the same cost in both outcomes, it follows that the worst Subgame Perfect Equilibrium is $S^{\mathsf{SPE}} = (\{1\}, (\{1\}, \{1\})$. The Sequential Price of Anarchy of $I_3$ is the following:

$$\mathsf{SPoA}(I_3) = \frac{(w_1 + w_2) + (w_1 + w_2)}{w_1 + (w_1 + w_2)} = 1 + \frac{w_2}{2w_1 + w_2}$$

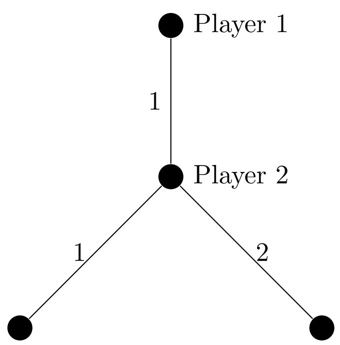



Figure 5.8: Game Tree of the worst case instance depicted in Figure 5.7

These three instances provide a lower bound for the formula depicted in Theorem 5.2. This result is presented in the following lemma.

**Lemma 5.9.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with uniform costs and 2 players it holds that a lower bound for the Sequential Price of Anarchy is given by the following formula:*

| Action Outcome $(a, b)$ | $C_1(ab)$ | $C_2(ab)$ |
|---|---|---|
| $(\{1\}, \{1\})$ | $w_1 + w_2$ | $w_1 + w_2$ |
| $(\{1\}, \{2\})$ | $w_1$ | $w_1 + w_2$ |

Table 5.7: Player costs for each outcome of the game tree displayed in Figure 5.2

$$\mathsf{SPoA}(w_1, w_2) \geq \begin{cases} 1 + \frac{w_1}{w_1+w_2} & \text{if } w_2 \geq w_1 \\ 1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2} & \text{if } w_1 < w_2 \leq 2w_1 \\ 1 + \frac{w_2}{2w_1+w_2} & \text{if } 2w_1 < w_2 \end{cases} . \tag{5.14}$$

*Proof.* It holds that $\mathsf{SPoA}(w_1, w_2) \geq \max\{\mathsf{SPoA}(I_1), \mathsf{SPoA}(I_2), \mathsf{SPoA}(I_3)\}$. So to complete the proof let us derive the intervals for which each of the three instances has the largest Sequential Price of Anarchy. Firstly we determine for which weights it holds that $\mathsf{SPoA}(I_1) \geq \mathsf{SPoA}(I_2)$.

$$\begin{aligned} 1 + \frac{w_1}{w_1 + w_2} &\geq 1 + \frac{w_1w_2}{2w_1^2 + w_1w_2 + w_2^2} \\ \iff w_1(2w_1^2 + w_1w_2 + w_2^2) &\geq w1(2w_1w_2 + 2w_2^2) \\ \iff w_1(2w_1 + w_2)(w_1 - w_2) &\geq 0 \\ \overset{w_1,w_2 \in \mathbb{R}_+}{\iff} w_1 &\geq w_2. \end{aligned}$$

Secondly we determine for which weights we have that $\mathsf{SPoA}(I_2) \geq \mathsf{SPoA}(I_3)$.

$$\begin{aligned} 1 + \frac{w_1w_2}{2w_1^2 + w_1w_2 + w_2^2} &\geq 1 + \frac{w_2}{2w_1 + w_2} \\ \iff w_2(4w_1^2 + 2w_1w_2) &\geq w_2(2w_1^2 + w_1w_2 + w_2^2) \\ \iff w_2(2w_1 - w_2)(w_1 + w_2) &\geq 0 \\ \overset{w_1,w_2 \in \mathbb{R}_+}{\iff} 2w_1 &\geq w_2. \end{aligned}$$

Finally we determine when $\mathsf{SPoA}(I_1) \geq \mathsf{SPoA}(I_3)$.

$$\begin{aligned} 1 + \frac{w_1}{w_1 + w_2} &\geq 1 + \frac{w_2}{2w_1 + w_2} \\ \iff 2w_1^2 + w_1w_2 &\geq w_1w_2 + w_2^2 \\ \overset{w_1,w_2 \in \mathbb{R}_+}{\iff} w_1\sqrt{2} &\geq w_2. \end{aligned}$$

We conclude that $\max\{\mathsf{SPoA}(I_1), \mathsf{SPoA}(I_2), \mathsf{SPoA}(I_3)\}$ is equal to the right hand side of (5.24). ■

### 5.2.2 A Certificate for the Upper Bound

In this section we provide an upper bound for the Sequential Price of Anarchy. Again we make use of an LP. In order to do that we first prove another version of Lemma 3.1 from [15].

**Lemma 5.10.** *For any instance $I$ of a weighted affine congestion game with uniform costs there exists a weighted congestion game $I'$ for which the following properties hold.*

1. *For all players $i \in N$ it holds that $|\mathcal{A}_i| \leq z_i$, where all $z_i$ are defined as follows:*

$$z_1 := 2 \text{ and } z_i := 1 + \prod_{j=1}^{i-1} z_j \text{ for all } i \geq 2. \tag{5.15}$$

2. *The amount of resources $|R|$ is at most $2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$.*
3. $\mathsf{SPoA}(I') = \mathsf{SPoA}(I)$.

*Proof.* Let $I$ be an arbitrary instance of a weighted affine congestion game with uniform costs. Let us construct an instance $I'$ that satisfies all properties stated above. This construction has the same approach as in the proof of Lemma 5.5: firstly we remove actions from the instance such that $|\mathcal{A}_i| \leq z_i$ for all players $i$. Secondly we remove all resources that are not used by any remaining actions and we merge resources that appear in exactly the same actions. The first step of this process does not have an impact on the Sequential Price of Anarchy of the instance regardless of how the resource costs are defined. Therefore the first part of the proof to Lemma 5.5 also applies. So we only have to prove that the second step of the construction preserves the Sequential Price of Anarchy.

So suppose that we constructed a new instance from $I$ for which $|\mathcal{A}_i| \leq z_i$ for all players $i \in N$ which has more than $2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$ resources. By the pigeonhole principle there exist two resources $r_1, r_2 \in R$ such that for all actions it holds that either both resources are picked in or none of them. Then we can replace $r_1$ and $r_2$ by a new resource $r'$ without changing the total costs for any player regardless of the actions that are played. Let $i \in N$ be an arbitrary player to whom $r_1$ and $r_2$ are allocated when action profile $A$ is played. Since the total congestion $x_{r_1}$ of $r_1$ is equal to the total congestion $x_{r_2}$ of $r_2$ it holds that

$$\begin{aligned} C_i(A) &= c(r_1) + w_i c(r_2) + \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r) \\ &= \alpha_{r_1} + \beta_{r_1} x_{r_1} + \alpha_{r_2} + \beta_{r_2} x_{r_2} + \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r) \\ &= (\alpha_{r_1} + \alpha_{r_2}) + (\beta_{r_1} + \beta_{r_2}) x_{r_1} + \sum_{r \in A_i \setminus \{r_1, r_2\}} c(r). \end{aligned}$$

Now let us define the activation cost and the linear term of the resource cost for $r'$ as follows:

$$\begin{aligned} \alpha_{r'} &= \alpha_{r_1} + \alpha_{r_2} \\ \beta_{r'} &= \beta_{r_1} + \beta_{r_2}. \end{aligned}$$

Then the Sequential Price of Anarchy of the game with $r'$ is equal to the game that involves $r_1$ and $r_2$. This process can be repeated until we end up with the instance $I'$, for which $|R| \leq 2^{\sum_{i \in N} |\mathcal{A}_i|} - 1$. ■

From Lemma 5.10 it follows that there exists an weighted affine congestion game with proportional costs for which $|\mathcal{A}_i| = z_i$ and $|R| = 2^{\sum_{i\in N}|\mathcal{A}_i|} - 1$. This game can be described by a variant of the LP in [15]. This variant has the same sets, parameters and variables as LP (5.10). Constraints (5.10b), (5.10c), (5.10d) and (5.10e) have been altered to represent uniform costs rather than proportional costs. The adapted constraints are stated in constraints (5.16b), (5.16c), (5.16d) and (5.16e) respectively.

**Linear Program**

$$\begin{aligned}
\max \quad & C\left(A^{\mathsf{SPE}}\right) && && (5.16a)\\
\text{s.t.} \quad & \sum_{r\in R}\delta_{\mu r}\left(w_i\beta_r+\alpha_r\right)-v_\mu=0 && \forall \mu\in\mathcal{A}_i, i\in[2] && (5.16b)\\
& \sum_{r\in R}\delta_{ar}\delta_{br}\beta_r-o_{ab}=0 && \forall a\in\mathcal{A}_1,\forall b\in\mathcal{A}_2 && (5.16c)\\
& v_a+w_2o_{ab}-C_1(ab)=0 && \forall a,b && (5.16d)\\
& v_b+w_1o_{ab}-C_2(ab)=0 && \forall a,b && (5.16e)\\
& \sum_{i=1}^{2}C_i(1.1,2.1)=1 && && (5.16f)\\
& \sum_{i=1}^{2}C_i(ab)\geq 1 && \forall a,b && (5.16g)\\
& C_2(ab)-C_2(ab')\leq 0 && \forall a,b|z_{ab}^2=1,\forall b' && (5.16h)\\
& C_1(a)-C_1(a')\leq 0 && \forall a|z_a^1=1,\forall a' && (5.16i)\\
& C_1(a)-C_1(ab)=0 && \forall a,b|z_{ab}^2=1 && (5.16j)\\
& \sum_{i=1}^{2}C_i(1.2,2.3)-C\left(A^{\mathsf{SPE}}\right)=0 && && (5.16k)\\
& \alpha_r,\beta_r,v_\mu,o_{ab},C_i(ab),C\left(A^{\mathsf{SPE}}\right),C_1(a)\geq 0 && \forall a,b,\mu,r,i && (5.16l)
\end{aligned}$$

Note that like the LP for the proportinal model this program is linear because we set the player weights to be parameters. That is why we will only use the program to derive an upper bound to SPoA$(w_1, w_2)$, rather than setting the weights as variables and solving it as a quadratic program. We present an upper bound for each interval of the weights as indicated in Theorem 5.2.

Firstly we consider the case where $w_1 \geq w_2$. Let us multiply all constraints of (5.16) with the multipliers stated in Table 5.8 and add them all up. It follows that each feasible solution to the LP has to satisfy the following:

$$C\left(A^{\mathsf{SPE}}\right)+x\leq 1+\frac{w_1}{w_1+w_2}. \tag{5.17}$$

Here $x$ is defined as follows:

| constraint | multiplier |
|---|---|
| (5.16b) when $\mu = 1.2$ | $\frac{w_1}{w_1+w_2}$ |
| (5.16b) when $\mu = 2.2$ | $\frac{w_1}{w_1+w_2}$ |
| (5.16c) when $a = 1.1$ and $b = 2.1$ | $w_2\left(1 + \frac{w_1}{w_1+w_2}\right)$ |
| (5.16c) when $a = 1.1$ and $b = 2.2$ | $\frac{2w_1^2 - w_1w_2 - w_2^2}{w_1+w_2}$ |
| (5.16c) when $a = 1.2$ and $b = 2.2$ | $-w_1$ |
| (5.16d) when $a = 1.1$ and $b = 2.1$ | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16d) when $a = 1.1$ and $b = 2.2$ | $-1 - \frac{w_1}{w_1+w_2}$ |
| (5.16e) when $a = 1.1$ and $b = 2.2$ | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16e) when $a = 1.2$ and $b = 2.2$ | $-1$ |
| (5.16d) when $a = 1.2$ and $b = 2.3$ | $\frac{w_1}{w_1+w_2}$ |
| (5.16f) | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16h) when $a = 1.1$, $b = 2.2$ and $b' = 2.1$ | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16h) when $a = 1.2$, $b = 2.3$ and $b' = 2.2$ | $1$ |
| (5.16i) when $a = 1.2$ and $a' = 1.1$ | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16j) when $a = 1.1$ | $1 + \frac{w_1}{w_1+w_2}$ |
| (5.16j) when $a = 1.2$ | $-1 - \frac{w_1}{w_1+w_2}$ |
| (5.16k) | $-1$ |

Table 5.8: Multipliers for LP constraints for the case that $w_1 \geq w_2$. All other constraints have a multiplier of 0.

$$\begin{aligned} x = {} & \frac{w_1 w_2}{w_1 + w_2} o_{1.2,2.3} + \frac{w_1}{w_1 + w_2} \sum_{r \in R} (\delta_{1.2,r}(\alpha_r + w_1\beta_r) + \delta_{2.2,r}(\alpha_r + w_2\beta_r)) \\ & + w_2 \left(1 + \frac{w_1}{w_1 + w_2}\right) \sum_{r \in R} \delta_{1.1,r}\delta_{2.1,r}\beta_r \\ & + \frac{2w_1^2 - w_1w_2 - w_2^2}{w_1 + w_2} \sum_{r \in R} \delta_{1.1,r}\delta_{2.2,r}\beta_r - w_1 \sum_{r \in R} \delta_{1.2,r}\delta_{2.2,r}\beta_r \end{aligned} \tag{5.18}$$

Similarly to the approach for the proportional model we now show that $x \geq 0$ in order to show that the right hand side of (5.17) is an upper bound to the Sequential Price of Anarchy when $w_1 \geq w_2$.

**Lemma 5.11.** *Let $x$ be defined as in* (5.18)*. Then it holds that $x \geq 0$ for all feasible solutions to LP* (5.16) *if for the player weights $w_1, w_2 \in \mathbb{R}_+$ it holds that that $w_1 \geq w_2$.*

*Proof.* Let us assume that $w_1 \geq w_2$. Since all parameters and variables are set to be nonnegative, it follows that all terms on the first two lines of (5.18) are nonnegative. So it is sufficient to show that the terms on the final line of (5.18) are either nonnegative or compensated for by the other terms.

Firstly, let us consider the first term on the last line of (5.18). We will show that the coefficient is nonnegative.

$$\frac{2w_1^2 - w_1 w_2 - w_2^2}{w_1 + w_2} \overset{w_1 \geq w_2}{\geq} \frac{2w_1^2 - w_1^2 - w_1^2}{w_1 + w_2}$$
$$= 0.$$

Secondly, consider the final term of (5.18). Let $r \in R$ be any recourse for which $\delta_{1.2,r} = \delta_{2.2,r} = 1$ and $\beta_r > 0$. Then this resource also appears in two terms of the first line of (5.18). Together they yield the following:

$$\frac{w_1}{w_1 + w_2}(w_1\beta_r + w_2\beta_r) - w_1\beta_r = \beta_r \left( \frac{w_1(w_1 + w_2)}{w_1 + w_2} - w_1 \right)$$
$$= 0.$$

It follows that $x \geq 0$ for all feasible solutions of LP (5.16). ■

| constraint | multiplier |
|---|---|
| (5.16b) when $\mu = 1.1$ | $\frac{-2w_1^2+w_1w_2+w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16b) when $\mu = 1.2$ | $\frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16b) when $\mu = 2.2$ | $\frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16c) when $a = 1.1$ and $b = 2.1$ | $w_2\left(1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}\right)$ |
| (5.16c) when $a = 1.1$ and $b = 2.2$ | $-w_1\left(\frac{-2w_1^2+w_1w_2+w_2^2}{2w_1^2+w_1w_2+w_2^2}\right)$ |
| (5.16c) when $a = 1.2$ and $b = 2.2$ | $-w_1$ |
| (5.16d) when $a = 1.1$ and $b = 2.1$ | $1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16d) when $a = 1.1$ and $b = 2.2$ | $-1 - \frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16e) when $a = 1.1$ and $b = 2.2$ | $1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16e) when $a = 1.2$ and $b = 2.2$ | $-1$ |
| (5.16d) when $a = 1.2$ and $b = 2.3$ | $\frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16f) | $1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16h) when $a = 1.1$, $b = 2.2$ and $b' = 2.1$ | $1 + \frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16h) when $a = 1.2$, $b = 2.3$ and $b' = 2.2$ | $1$ |
| (5.16i) when $a = 1.2$ and $a' = 1.1$ | $1 + \frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16j) when $a = 1.1$ | $1 + \frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16j) when $a = 1.2$ | $-1 - \frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}$ |
| (5.16k) | $-1$ |

Table 5.9: Multipliers for LP constraints for the case that $w_1 < w_2 \leq 2w_1$. All other constraints have a multiplier of 0.

Now we have established the first upper bound we continue with the case where $w_1 < w_2 \leq 2w_1$. Let us now multiply the LP constraints with the multipliers stated in Table 5.9 and add them up. This yields a different upper bound:

$$C\left(A^{\mathsf{SPE}}\right) + y \leq 1 + \frac{2w_1w_2}{2w_1^2 + w_1w_2 + w_2^2}. \tag{5.19}$$

Here $y$ is defined as follows:

$$\begin{aligned}
y = &\frac{2w_1^2 + w_1w_2 - w_2^2}{2w_1^2 + w_1w_2 + w_2^2} o_{1.2,2.3} + \frac{-2w_1^2 + w_1w_2 + w_2^2}{2w_1^2 + w_1w_2 + w_2^2} \sum_{r \in R} \delta_{1.1,r}(\alpha_r + w_1\beta_r) \\
&+ \frac{2w_1^2 + w_1w_2 - w_2^2}{2w_1^2 + w_1w_2 + w_2^2} \sum_{r \in R} \delta_{1.2,r}(\alpha_r + w_1\beta_r) \\
&+ \frac{2w_1w_2}{2w_1^2 + w_1w_2 + w_2^2} \sum_{r \in R} \delta_{2.2,r}(\alpha_r + w_2\beta_r) \\
&+ w_2 \left(1 + \frac{2w_1w_2}{2w_1^2 + w_1w_2 + w_2^2}\right) \sum_{r \in R} \delta_{1.1,r}\delta_{2.1,r}\beta_r \\
&- w_1 \left(\frac{-2w_1^2 + w_1w_2 + w_2^2}{2w_1^2 + w_1w_2 + w_2^2}\right) \sum_{r \in R} \delta_{1.1,r}\delta_{2.2,r}\beta_r - w_1 \sum_{r \in R} \delta_{1.2,r}\delta_{2.2,r}\beta_r
\end{aligned} \tag{5.20}$$

Let us now show that $y \geq 0$, from which it follows that we have found the second upper bound.

**Lemma 5.12.** *Let $y$ be defined as in* (5.20)*. Then it holds that $y \geq 0$ for all feasible solutions to LP* (5.16) *if for the player weights $w_1, w_2 \in \mathbb{R}_+$ it holds that that $w_1 < w_2 \leq 2w_1$.*

*Proof.* All parameters and variables are nonnegative, so let us show for which weights all variable coefficients are nonnegative. Firstly, let us consider the coefficient of $o_{1.2,2.3}$.

$$\begin{aligned}
\frac{2w_1^2 + w_1w_2 - w_2^2}{2w_1^2 + w_1w_2 + w_2^2} o_{1.2,2.3} &= \frac{(2w_1 - w_2)(w_1 + w_2)}{2w_1^2 + w_1w_2 + w_2^2} o_{1.2,2.3} \geq 0 \\
&\overset{w_1, w_2 \in \mathbb{R}_+}{\iff} w_2 \leq 2w_1.
\end{aligned}$$

Moreover, if we set $w_2 \leq 2w_1$ then the coefficient on the second line of (5.20) is also nonnegative.

Secondly, let $r \in R$ be any resource for which $\delta_{1.1,r} = 1$ and $\alpha_r > 0$. In order to let all coefficients of $\alpha_r$ for such resources be nonzero the following must be satisfied:

$$\begin{aligned}
\frac{-2w_1^2 + w_1w_2 + w_2^2}{2w_1^2 + w_1w_2 + w_2^2} &= \frac{-(2w_1^2 + w_2)(w_1 - w_2)}{2w_1^2 + w_1w_2 + w_2^2} \geq 0 \\
&\overset{w_1, w_2 \in \mathbb{R}_+}{\iff} w_2 \geq w_1.
\end{aligned}$$

It remains to show that the terms on the final line of (5.20) are being compensated for. The first term is only nonzero for those resources $r \in R$ for which $\delta_{1.1,r} = \delta_{2.2,r} = 1$ and $\beta_r > 0$. For those resources on the first and third line of

(5.20) two other terms involving $\beta_r$ appear. If we add up those coefficients, we get:

$$w_1\left(\frac{-2w_1^2+w_1w_2+w_2^2}{2w_1^2+w_1w_2+w_2^2}\right)-w_1\left(\frac{-2w_1^2+w_1w_2+w_2^2}{2w_1^2+w_1w_2+w_2^2}\right)+\frac{2w_1w_2^2}{2w_1^2+w_1w_2+w_2^2}$$
$$=\frac{2w_1w_2^2}{2w_1^2+w_1w_2+w_2^2}$$
$$\geq 0.$$

Finally, let $r \in R$ be an arbitrary resource for which $\delta_{1.2,r} = \delta_{2.2,r} = 1$ and $\beta_r > 0$. By similar reasoning for such resources we have the following:

$$w_1\left(\frac{2w_1^2+w_1w_2-w_2^2}{2w_1^2+w_1w_2+w_2^2}\right)+\frac{2w_1w_2^2}{2w_1^2+w_1w_2+w_2^2}-w_1\left(\frac{2w_1^2+w_1w_2+w_2^2}{2w_1^2+w_1w_2+w_2^2}\right)$$
$$=\frac{2w_1^3+w_1^2w_2-w_1w_2^2+2w_1w_2^2-2w_1^3-w_1^2w_2-w_1w_2^2}{2w_1^2+w_1w_2+w_2^2}$$
$$=0.$$

The two terms on the last line of (5.20) are compensated for by different other terms of $y$. Therefore even for resources for which $\delta_{1.1,r} = \delta_{1.2,r} = \delta_{2.2,r}$ and $\beta_r > 0$ it holds that the final terms are compensated for. Therefore we conclude that $y \geq 0$ if $w_1 < w_2 \leq 2w_1$. ■

| constraint | multiplier |
|---|---|
| (5.16b) when $\mu = 1.1$ | $\frac{w_2}{2w_1+w_2}$ |
| (5.16b) when $\mu = 2.2$ | $\frac{w_2}{2w_1+w_2}$ |
| (5.16c) when $a = 1.1$ and $b = 2.1$ | $w_2\left(1+\frac{w_2}{2w_1+w_2}\right)$ |
| (5.16c) when $a = 1.1$ and $b = 2.2$ | $\frac{2w_1^2-w_2^2}{2w_1+w_2}$ |
| (5.16c) when $a = 1.2$ and $b = 2.2$ | $-w_1$ |
| (5.16d) when $a = 1.1$ and $b = 2.1$ | $1+\frac{w_2}{2w_1+w_2}$ |
| (5.16d) when $a = 1.1$ and $b = 2.2$ | $-1$ |
| (5.16e) when $a = 1.1$ and $b = 2.2$ | $1+\frac{w_2}{2w_1+w_2}$ |
| (5.16e) when $a = 1.2$ and $b = 2.2$ | $-1$ |
| (5.16f) | $1+\frac{w_2}{2w_1+w_2}$ |
| (5.16h) when $a = 1.1$, $b = 2.2$ and $b' = 2.1$ | $1+\frac{w_2}{2w_1+w_2}$ |
| (5.16h) when $a = 1.2$, $b = 2.3$ and $b' = 2.2$ | 1 |
| (5.16i) when $a = 1.2$ and $a' = 1.1$ | 1 |
| (5.16j) when $a = 1.1$ | 1 |
| (5.16j) when $a = 1.2$ | $-1$ |
| (5.16k) | $-1$ |

Table 5.10: Multipliers for LP constraints for the case that $2w_1 < w_2$. All other constraints have a multiplier of 0.

Finally we derive an upper bound for the case that $2w_1 < w_2$. Again we multiply the constraints of LP (5.16) with some multipliers and then add up the constraints. These multipliers are stated in Table 5.10. From this it follows that any feasible solution to (5.16) has to satisfy the following constraint:

$$C\left(A^{\mathsf{SPE}}\right) + z \leq 1 + \frac{w_2}{2w_1 + w_2}. \tag{5.21}$$

Here $z$ is defined as followed:

$$\begin{aligned} z = &\frac{w_2}{2w_1 + w_2} \sum_{r \in R} \left(\delta_{1.1,r}(\alpha_r + w_1\beta_r) + \delta_{2.2,r}(\alpha_r + w_2\beta_r)\right) \\ &+ w_2\left(1 + \frac{w_2}{2w_1 + w_2}\right) \sum_{r \in R} \delta_{1.1,r}\delta_{2.1,r}\beta_r \\ &+ \frac{2w_1^2 - w_2^2}{2w_1 + w_2} \sum_{r \in R} \delta_{1.1,r}\delta_{2.2,r}\beta_r - w_1 \sum_{r \in R} \delta_{1.2,r}\delta_{2.2,r}\beta_r \end{aligned} \tag{5.22}$$

Now we show that $z \geq 0$ in order to establish the third upper bound.

**Lemma 5.13.** *Let $z$ be defined as in* (5.22)*. Then it holds that $z \geq 0$ for all feasible solutions to LP* (5.16) *if for the player weights $w_1, w_2 \in \mathbb{R}_+$ it holds that that $2w_1 \leq w_2$.*

*Proof.* It suffices to show that the final two terms in (5.22) are compensated for. Firstly, let $r \in R$ be an arbitrary resource for which $\delta_{1.1,r} = \delta_{2.2,r} = 1$ and $\beta_r > 0$. If we add the coefficients of $\beta_r$ on the first and last line of (5.22) we get the following:

$$\frac{w_1w_2}{2w_1 + w_2} + \frac{w_2^2}{2w_1 + w_2} + \frac{2w_1^2 - w_2^2}{2w_1 + w_2} = \frac{2w_1^2 + w_1w_2}{2w_1 + w_2} \geq 0. \tag{5.23}$$

Secondly, consider the final term of (5.22). This is negative for those resources $r \in R$ that have $\delta_{1.2,r} = \delta_{2.2,r} = 1$ and $\beta_r > 0$. Let us add the coefficients of $\beta_r$ on the first and last line of (5.22). In order to let all these coefficients be nonnegative the following must be satisfied:

$$\begin{aligned} \frac{w_2^2}{2w_1 + w_2} + \frac{2w_1^2 - w_1w_2}{2w_1 + w_2} &= \frac{2w_1^2 - w_1w_2 - w_2^2}{2w_1 + w_2} \\ &= \frac{-(2w_1 - w_2)(w_1 + w_2)}{2w_1 + w_2} \\ &\geq 0. \\ \overset{w_1, w_2 \in \mathbb{R}_+}{\Longleftrightarrow} \; & 2w_1 \leq w_2. \end{aligned}$$

Finally, let $r \in R$ be any resource for which $\delta_{1.1,r} = \delta_{1.2,r} = \delta_{2.2,r}$ and $\beta_r$. Then all terms of (5.22) are involved. Using (5.23) the following then holds:

$$\begin{aligned}
&\frac{2w_1^2+w_1w_2}{2w_1+w_2}\beta_r + w_2\left(\frac{2w_1+2w_2}{2w_1+w_2}\right)\beta_r - \frac{w_1(2w_1+w_2)}{2w_1+w_2}\beta_r \\
=&\beta_r\left(\frac{2w_1^2+w_1w_2+2w_1w_2+2w_2^2-2w_1^2-w_1w_2}{2w_1+w_2}\right) \\
=&\beta_r\left(\frac{2w_2^2+2w_1w_2}{2w_1+w_2}\right) \\
\geq&0.
\end{aligned}$$

We conclude that $z \geq 0$ if $2w_1 < w_2$. ■

We are now ready to give an upper bound to the Sequential Price of Anarchy for all weights.

**Lemma 5.14.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with uniform costs and 2 players it holds that an upper bound for the Sequential Price of Anarchy is given by the following formula:*

$$\mathsf{SPoA}(w_1,w_2) \leq \begin{cases} 1+\frac{w_1}{w_1+w_2} & \text{if } w_2 \geq w_1 \\ 1+\frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2} & \text{if } w_1 < w_2 \leq 2w_1 \\ 1+\frac{w_2}{2w_1+w_2} & \text{if } 2w_1 < w_2 \end{cases}. \tag{5.24}$$

*Proof.* By Lemma 5.11 and LP constraint (5.17) the upper bound holds when $w_1 \geq w_2$. By Lemma 5.12 and LP constraint (5.19) the upper bound holds when $w_1 < w_2 \leq 2w_1$. By Lemma 5.13 and LP constraint (5.21) the upper bound holds when $2w_1 < w_2$. ■

### 5.2.3 Sequential Price of Anarchy

In this section we use the lower and upper bound from the previous section to prove Theorem 5.2 and Corollary 5.2.1.

**Theorem 5.2.** *Let $w_1, w_2 \geq 0$ be arbitrary player weights. Then for the class of weighted affine congestion games with uniform costs and 2 players it holds that the Sequential Price of Anarchy is given by the following formula:*

$$\mathsf{SPoA}(w_1,w_2) = \begin{cases} 1+\frac{w_1}{w_1+w_2} & \text{if } w_2 \leq w_1 \\ 1+\frac{2w_1w_2}{2w_1^2+w_1w_2+w_2^2} & \text{if } w_1 < w_2 \leq 2w_1 \\ 1+\frac{w_2}{2w_1+w_2} & \text{if } 2w_1 < w_2 \end{cases}. \tag{5.4}$$

*Proof.* By Lemmas 5.9 and 5.14 the result follows. ■

**Corollary 5.2.1.** *For the class of weighted affine congestion games with uniform costs and 2 players it holds that* $\mathsf{SPoA} = 2$.

*Proof.* Firstly we show that the Sequential Price of Anarchy is at least 2:

$$\mathsf{SPoA} = \sup_{w_1,w_2} \mathsf{SPoA}(w_1,w_2) \geq \mathsf{SPoA}(1,0) = 1 + \frac{1}{1+0} = 2.$$

Secondly we show that the Sequential Price of Anarchy is at most 2. We do this by giving an upper bound to $\mathsf{SPoA}(w_1, w_2)$ for all three cases. If $w_1 \geq w_2$ we have:

$$\mathsf{SPoA}(w_1,w_2) = 1 + \frac{w_1}{w_1+w_2} \leq 2.$$

In order to prove the upper bound for the case that $w_1 < w_2 \leq 2w_1$ we use that $w_1^2 + w_2^2 \geq 2w_1w_2$. This follows from the fact that

$$(w_1 - w_2)^2 = w_1^2 - 2w_1w_2 + w_2^2 \geq 0.$$

It follows that

$$\mathsf{SPoA}(w_1,w_2) = 1 + \frac{2w_1w_2}{2w_1^2 + w_1w_2 + w_2^2} \leq 1 + \frac{2w_1w_2}{w_1^2 + 3w_1w_2} \leq 2$$

Finally if $w_2 > 2w_1$ we have

$$\mathsf{SPoA}(w_1,w_2) = 1 + \frac{w_2}{2w_1+w_2} \leq 2.$$

We can conclude that

$$\mathsf{SPoA} = \sup_{w_1,w_2} \mathsf{SPoA}(w_1,w_2) = 2.$$

■

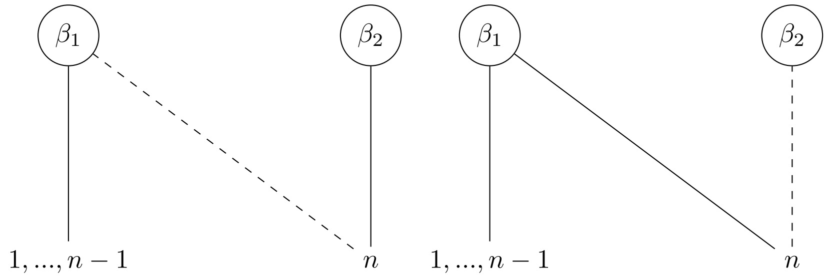
### 5.2.4 Lower Bound on SPoA for $n$ Players

The results presented in this chapter all concern games with 2 players. For games with more players we do not have the exact value of the Sequential Price of Anarchy. However, for games with uniform costs we can present a lower bound.

**Theorem 5.15.** *For the class of weighted affine congestion games with uniform costs and n players it holds that* $\mathsf{SPoA} \geq n$.

*Proof.* Let $I$ be the instance presented in Figure 5.9. This instance has two resources. The cost functions of these resources only have a linear term, which we set to be $\beta_1 = \beta_2 = 1$. The first $n-1$ players only have one available action, namely to select the first resource. Player $n$ can then select either the first or the second resource.

We now set $w_i = 0$ for all players $i$ such that $1 \leq i \leq n-1$ and we set $w_n = 1$. Then player $n$ always has a cost of 1, regardless of the resource she




(a) $A^{\mathsf{OPT}}$ (b) $A^{\mathsf{SPE}}$

Figure 5.9: Lower bound instance $I$ for $n$ players

selects. However, the other players have a cost of 1 if player 1 picks the first resource and a cost of 0 if player $n$ picks the second resource. So in the social optimum player $n$ chooses resource 2 and in the worst subgame perfect outcome she selects resource 1. We conclude that

$$\mathsf{SPoA} \geq \mathsf{SPoA}(I) = \frac{n}{1} = n.$$

■

# Chapter 6

# Conclusion

In this thesis we examined to what extent the linear program presented in [15] could be used to find the Sequential Price of Anarchy of affine congestion games. The resources and actions of a worst case instance for that class determined the size of that program. The number of resources of that instance exponentially larger than the number of actions. Since the number of constraints was of linear order in the number of actions, it was possible to apply column generation in order to solve the program for the case with 4 players. However, by Lemma 3.1 the number of actions already was of exponential size in the number of players. In the 5 player the number of actions also is so large that the program is not solvable in reasonable time, even when column generation is applied. This also means that this method is not sufficient to find the Sequential Price of Anarchy for any larger number of players.

Furthermore we presented variants of the program to analyse a restricted version of the worst case instance that the original LP describes. This gained some insight in the existence of instances with a Sequential Price of Anarchy that is close to the value of the whole class but with less varying resource cost functions. However, by restricting that worst case instance one cannot get insight in restricted classes of congestion games, since then Lemma 3.1 does not hold anymore.

We also presented a Mixed Integer Program that can set the resource costs of any instance such that the Sequential Price of Anarchy of that instance is maximised. An advantage of this problem is that the number of actions and resources remains the same for any number of players. This partly prevents the number of variables and constraints to become large, as is the case with the problem for which the original LP was created. But it is not possible to use this program to analyse subclasses of congestion games such as the network routing problem. After all, such restricted classes of games do not have a general worst case instance, as Lemma 3.1 does not hold anymore.

Finally we investigated two variants of weighted congestion games. We can only use a variant of the program to find the Sequential Price of Anarchy for fixed weights. We cannot use it to find the Sequential Price of Anarchy for all weights, as the weights make the program non-linear. By solving the program for several fixed weights, we were able to find a pattern in both the primal and dual solution, which allowed us to find the solution for arbitrary weights. A reason this method worked, was the fact that we effectively were searching for a linear function. After all, the Sequential Price of Anarchy did not alter when both player weights were multiplied by the same constant. When the number of players is more than 2, it may become harder to apply this method, since one must then find a multidimensional function. Moreover, solving the program with fixed weights has the same limitations as the program for unweighted congestion games has. Therefore it is not clear if this method can be applied for 3 or more players.

We conclude that it is possible to alter the original LP in [15] to gain further insights in the Sequential Price of Anarchy of affine congestion games. However, the applications of this approach are limited, because the size of the program gets too large when the number of players gets large, it is not suitable to describe subclasses of affine congestion games and it becomes non-linear when player weights are introduced.

However, we did gain the insight that for the case with 2 players it holds that the Sequential Price of Anarchy of weighted affine congestion games with uniform costs is larger than the Sequential Price of Anarchy for the unweighted case. In fact, in Section 5.2.4 we presented a lower bound of $n$ for the class of weighted affine congestion games with uniform costs, while [14] states that for the class of unweighted affine congestion games it holds that $\mathsf{SPoA} \leq n$. On the other hand, we have shown that for the case with 2 players the Sequential Price of Anarchy of weighted affine congestion game with proportional costs is equal to the Sequential Price of Anarchy for unweighted affine congestion games. We do not know if for a general number of players $n$ it holds if the class of weighted affine congestion games with uniform costs has a strictly higher Sequential Price of Anarchy than the class with proportional cost. Neither do we know if it holds for any number of players $n$ that weighted affine congestion games with proportional costs have the same Sequential Price of Anarchy as unweighted affine congestion games. Therefore we recommend further research into the relation between the Sequential Price of Anarchy of unweighted affine congestion games and of weighted affine congestion games with proportional or uniform costs.

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