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DISCRETE MATHEMATICS AND MATHEMATICAL PROGRAMMING

The Price of Anarchy of Symmetric and Semi-Symmetric Uniform Congestion Games

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Abstract

The price of anarchy of a system indicates how bad the system may perform if it is not regulated and actors act selfishly. In this research, we analyse the price of anarchy of two subclasses of atomic congestion games: symmetric uniform congestion games, in which all players pick the same number of resources from a set, and semi-symmetric uniform congestion games, in which the number of resources that players pick may differ. For the symmetric games, we prove that the price of anarchy lies between 1.34 and 2.02 if there are affine cost functions. For such games in which every player picks exactly 2 resources, the price of anarchy lies between 4/3 and 1.81. The results are generalised for games with cost functions of maximum degree d and for small d improve upon the known upper bound for the price of anarchy of general atomic congestion games. For semi-symmetric games, we prove that the price of anarchy is at least 5/3 if there are affine cost functions. For such games in which every player picks at most 2 resources, the price of anarchy lies between 1.4 and 2. Again, the results are generalised for games with cost functions of maximum degree d and for small d improve upon the known upper bound for the price of anarchy of general atomic congestion games.

Keywords: atomic congestion games, price of anarchy, uniform matroid

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Through this research, I got introduced to mathematical theory that I had not worked with before. Game theory turns out to be a very interesting subject that appeals to the imagination. It is also a subject that one can relatively easily explain to friends and family, which is quite unique for a mathematical topic.

Contents

Chapter 1

Introduction

1.1 Motivation

In some systems with multiple actors, the actions of one actor may impact others. As an example, consider the citizens of Berlin that need to get back home after work by car. If one of the citizens, let us call her Alice, decides to drive on the Friedrichstraße, then this will increase the travel time of all other car drivers that are using this road. The travel time on this road is thus dependent on the number of car drivers that use it, which we call *load-dependent*. The load-dependency for each road can be described using a *latency function*. Depending on the system, the term *cost function* can be used as well.

Such a system can be considered a game, and more specifically a *congestion game*. In game theory, a congestion game is a system with resources that have load-dependent latencies or costs, a set of players and their valid strategies, as introduced by Rosenthal [1973a]. The individual latency for some player equals the sum of the latencies of the resources that the player picks. In the previous example, the roads form the set of resources, the car drivers are the players and their valid strategies are all routes from their work to their home. For each player, the individual latency is the travel time from work to home.

For the purpose of fewer cars on the streets of Berlin, it may be desirable to regulate the traffic and minimise the total or average travel time of the car drivers. However, it is difficult to get actors in a big system to play according to some strategy if it does not directly benefit themselves, as shown by Koller [2021], and adapting the system to force actors into some optimal strategy, for example using tolls or taxes, is difficult and time consuming, as shown by Bilò and Vinci [2019a] and Nickerl [2021]. It is therefore useful to know whether such regulations can significantly improve the situation. To that end, it makes sense to analyse what happens to the system if there are no regulations and all players play selfishly, i.e., all car drivers pick a route that gets them home quickest, given the routes chosen by the other car drivers. This is called a *Nash equilibrium*. It was proved by Rosenthal [1973a] that at least one (pure) Nash equilibrium exists for each congestion game. However, a Nash equilibrium does not necessarily lead to an optimal outcome of a game, as showed by Dubey [1986]. For example, the quickest way home leads Alice down the Friedrichstraße, but in order to minimise the total travel time of all car drivers of Berlin, it may be better for her to avoid this street.

The total latency of the worst-case Nash equilibrium can be related to the total latency of the optimal play using the concept of the *price of anarchy*, which was formulated by Koutsoupias and Papadimitriou [1999]. The higher the price of anarchy, the worse the system may function if it is not regulated. Note that the optimal play of a game can be defined in different ways. In the example, we defined the optimal play as the routing that minimises the total travel time, in order to decrease the number of cars on the streets. However, we could have defined the optimal play as the routing that minimises the longest travel time, in order to decrease the number of traffic incidents caused by fatigue of car drivers.

The example with Alice in Berlin is called a *network routing* congestion game. For games of this subclass of congestion games with linear latency functions and unsplittable players, the exact price of anarchy is known to be 5/2, as proved by Christodoulou and Koutsoupias [2005]. In this research, we consider another subclass of congestion games, namely *uniform* congestion games, for which the exact price of anarchy is still unknown. In these games, each player may choose a certain, fixed number of resources out of the set of resources. In contrast to the routing game example, any subset of resources that contains the correct number of resources is a valid strategy. We consider both games in which every player picks the same number of resources and games in which this number may differ. For this first class of games, the strategy spaces for the players are the bases of the uniform matroid of the resources. The effect of such a simple matroid structure of the strategy spaces on the price of anarchy can be very insightful. The second class of games does not have this property, but what is interesting is that it is a generalisation of the first class.

An example of a uniform congestion game is a group of mathematicians that each need to perform certain computations. In order to verify the results, they all want their computations to be performed on several independent machines. However, there is only a limited number of machines available and the more mathematicians use a machine, the higher its latency. Each mathematician picks a set of machines to perform their computations such that they experience as little latency as possible, which results in some Nash equilibrium play of the game. To maximise the average happiness of the mathematicians, an optimal play could be to minimise the average latency of the machines.

For uniform congestion games with linear latency functions in which each player picks exactly one resource (also called *singleton* congestion games), Fotakis [2007] and Lücking et al. [2008] proved that the price of anarchy is exactly 4/3. However, for uniform congestion games in which each player can pick more than one resource, only non-matching upper and lower bounds of the price of anarchy are known. In this research, we improve upon these bounds, both for games with affine latency functions and for games with more general polynomial latency functions.

1.2 Overview of our results

We study the price of anarchy (PoA) of uniform congestion games with cost functions that are polynomials with nonnegative coefficients and real exponents $\in [0, d]$ for some $d \in \mathbb{R}$ and with unweighted players. As social cost, we consider the total cost.

First, we consider symmetric k-uniform congestion games, in which every player picks exactly k resources. Our results of the upper and lower bounds of the PoA are summarised in Table 1.1, both for affine cost functions and for general cost functions of maximum degree d. Our result for affine cost functions improves upon the upper bound of 2.15 that was proved by de Jong et al. [2016]. For the special case $k = 2$, we provide an even better upper bound.

	affine	
	lower bound	upper bound
k-uniform	1.34	2.02
2-uniform	$4/3 \approx 1.33$	1.81

	maximum degree d	
	lower bound	upper bound
k-uniform	$\frac{4+\sqrt{2}(1+2^{d+1})}{4+3\sqrt{2}}$	$\begin{cases} (\rho_d - d(d+1)^{-(d+1)/d})^{-1} & 0 < d \leq 1.6 \\ (\rho_d)^{-d-1} & d > 1.6 \end{cases}$ with $\rho_d = \frac{2^{d/2}+1}{1+2^{d/2}-2^d+2^{3d/2}}$
2-uniform	$\frac{2+2^d}{3}$	$\begin{cases} (\tilde{\rho}_d - d(d+1)^{-(d+1)/d})^{-1} & 0 < d \leq 2.0 \\ (\tilde{\rho}_d)^{-d-1} & d > 2.0 \end{cases}$ with $\tilde{\rho}_d = \frac{4}{2^d+3}$

Table 1.1: Bounds for the PoA of symmetric k-uniform congestion games. Results for affine cost functions and cost functions of maximum degree d. Improved results for the case $k = 2$.

Second, we consider *semi-symmetric* k-uniform congestion games, in which every player picks a fixed number of resources that is *at most* k. As far as we are aware, this subclass of congestion games has not been studied before. Our results of the upper and lower bounds of the PoA are summarised in Table 1.2, both for affine cost functions and for general cost functions of maximum degree d, and both for general k and for $k = 2$.

	affine	
	lower bound	upper bound
k-uniform	$5/3 \approx 1.67$	-
2-uniform	$7/5 = 1.4$	2

	maximum degree d	
	lower bound	upper bound
k-uniform	$\frac{1+2^{d+1}}{3}$	-
2-uniform	$\frac{3+2^{d+1}}{5}$	$\begin{cases} (\hat{\rho}_d - d(d+1)^{-(d+1)/d})^{-1} & 0 < d \leq 1.7 \\ (\hat{\rho}_d)^{-d-1} & d > 1.7 \end{cases}$ with $\hat{\rho} = \frac{3}{2^d+2}$

Table 1.2: Bounds for the PoA of semi-symmetric k-uniform congestion games. Results for affine cost functions and cost functions of maximum degree d. Improved results for the case $k = 2$.

1.3 Related work

The most important subclasses of congestion games for which the price of anarchy has been studied can be categorised as atomic/non-atomic and symmetric/asymmetric. Atomic games have infinitely many players that each have a negligible demand, or players with a splittable demand. Non-atomic games have players with a non-negligible, unsplittable demand. In symmetric games, a valid strategy for one player is always a valid strategy for all other players too. In asymmetric games, this may not be the case. In addition, the price of anarchy may be defined with the social cost being the total cost or the maximum private cost.

In our research, we analyse atomic games and use the total cost as the social cost. The first subclass of games that we analyse is symmetric, and the second subclass is asymmetric. However, these is a certain symmetry in the way the second subclass of games is defined, which is why we call it 'semi-symmetric'. In this section, we present an overview of bounds for the price of anarchy of different subclasses of congestion games. Note that games can be weighted/unweighted, and we restrict our attention to unweighted games.

Atomic, symmetric, maximum cost

The price of anarchy (PoA) was first studied by Koutsoupias and Papadimitriou [1999], for a simple network routing congestion game with m parallel links from a source to a target node and linear latency functions. The PoA of these games is $\Theta(\log m / \log \log m)$ - Koutsoupias and Papadimitriou [1999] proved the lower bound and Czumaj and Vöcking [2001] proved the upper bound. Czumaj and Vöcking [2001] extended the analysis to networks with m parallel *paths* and proved that the price of anarchy is then $\Theta(\log m / \log \log \log m)$. For series-parallel network routing congestion games with affine latency functions, Hao and Michini [2020] proved an upper bound of 2 for the PoA.

Non-atomic, asymmetric, total cost

Non-atomic network routing congestion games were studied by Roughgarden and Tardos [2002]. They showed that the PoA is exactly 4/3 when there are linear latency functions. For general non-atomic congestion games with polynomial cost functions of maximum degree d, Roughgarden and Tardos [2004] proved the exact value of the PoA, which asymptotically grows as $\Theta(d/\log d)$. For affine cost functions, this PoA equals 4/3, too.

Atomic, asymmetric, total cost

The PoA of general atomic congestion games was studied by Christodoulou and Koutsoupias [2005]. They proved that the PoA for asymmetric games is at most $5/2$ when there are linear latency functions. When there are at least 3 players and there are no restrictions on the strategy spaces, they showed that this bound is tight. They also showed that this bound is tight for atomic network routing congestion games. The tightness of this result for singleton congestion games was proved by Caragiannis et al. [2011]. For singleton congestion games in which all resources have identical cost functions, the result is not tight: Suri et al. [2004] proved that the PoA is at most 2.15 for those games. Bilò and Vinci [2019c] studied asymmetric congestion games in which the strategy spaces of the players are very similar, both with atomic and with non-atomic players. For these games, the PoA is strictly smaller than $5/2$ too. Congestion games in which players may not have all information about the other players' choices were studied by Bilò et al. [2020]. The exact PoA of congestion games with certain Stackelberg strategies was proven by Bilò and Vinci [2019b].

For general atomic congestion games with polynomial latency functions of maximum degree d, Christodoulou and Koutsoupias [2005] proved upper and lower bounds for the PoA with a gap of $o(1)$. These results were improved by Aland et al. [2011], such that the upper and the lower bound match for general atomic congestion games with polynomial cost functions of any bounded degree. This exact PoA asymptotically grows as $\Theta(d/\log d)^{d+1}$.

Atomic, symmetric, total cost

For symmetric congestion games with atomic players, Christodoulou and Koutsoupias [2005] proved that the PoA is exactly $\frac{5N-2}{2N+1}$, where N is the number of players, when there are linear latency functions and the only restriction on the strategy spaces is the symmetry. The tightness of this result for atomic network routing congestion games was proved by Correa et al. [2019]. Fotakis [2007] and Lücking et al. [2008] proved that this result is not tight for singleton congestion games: for that subclass of congestion games, the PoA is exactly $4/3$ when there are affine latency functions. For singleton congestion games with cost functions of maximum degree d, Fotakis [2007] proved that the PoA is equals that of general non-atomic congestion games with polynomial cost functions of maximum degree d.

For symmetric uniform congestion games, the exact PoA remains unknown. For games with affine cost functions, de Jong et al. [2016] proved an upper bound for the PoA of 2.15, so we know that the general upper bound of $\frac{5N-2}{2N+1}$ is not tight for $N > 6$. To the best of our knowledge, symmetric uniform congestion games with polynomial cost functions of maximum degree d have not been studied before. The work of de Jong et al. [2016] is the main inspiration for our research, and most of our proof techniques are generalisations of theirs. Note that uniform congestion games are a generalisation of singleton congestion games, for which we already mentioned that the exact PoA is known.

Other work that relates to uniform congestion games is the work of Rosenthal [1973b] and Meyers [2007], which are analyses of integer-splittable congestion games and (integer) k-splittable congestion games, respectively. The main difference between their work and ours is that players in our games are not allowed to pick the same resource twice, whereas they are allowed to do so in their games.

1.4 Outline of the report

In Chapter 2, the model that we use for symmetric and semi-symmetric k-uniform congestion games is explained in detail. Chapter 3 is about the properties of instances of the games that are analysed in order to bound the PoA. In Chapter 4, the upper and lower bound for the PoA of symmetric k-uniform games is given. We give bounds for general k and improved bounds for the special case $k = 2$. In Chapter 5, the upper and lower bound for the PoA of semi-symmetric k-uniform games is given. Again, we give bounds for general k and improved bounds for the special case $k = 2$. The conclusion and an outlook on future work is given in Chapter 6.

Chapter 2

Symmetric and semi-symmetric k-uniform congestion games

In this section, the model used for the symmetric and semi-symmetric uniform congestion games is explained. Afterwards, an example instance of a semi-symmetric uniform congestion game is shown.

2.1 Preliminaries

An instance $I = (N, R, (c_r)_{r\in R}, X)$ of a congestion game is characterised by a finite set $N := \{1, \ldots, n\}$ of players, a finite set R of resources and their cost functions $(c_r)_{r\in R}$, and a finite set $X := X_1 \times \ldots \times X_n$ containing the possible strategies for each player. The set of possible strategies for some player $i \in N$ is denoted by $X_i \subseteq 2^R$. A strategy for a player is to pick a certain subset of resources, so $X_i = \{R_{i,1}, \ldots, R_{i,i'}\}$ with $R_{i,j} \subseteq R$, for some $i' \in \mathbb{N}$. In general, the available strategies may differ between players: players might not have access to the same resources in R, and some subset $R' \subseteq R$ might be a valid strategy for one player but not for another. Moreover, valid strategies may contain a different number of resources. Congestion games in which the available strategies are the same for all players, i.e., $X_i = X_j$ for all $i, j \in N$, are called *symmetric*. Congestion games without this property are called *asymmetric*.

We study a two subclasses of congestion games, which are both related to k-uniform congestion games. These games are characterised by a specific strategy space X: a valid strategy for each player $i \in N$ is to pick exactly k resources from a set $R_i \subseteq R$, i.e., for every player $i \in N$ there exists a subset $R_i \subseteq R$ s.t. $X_i = \{R' \subseteq R_i : |R'| = k\}$. An instance of a k-uniform congestion game is symmetric if $R_i = R_j$ for all $i, j \in N$. The first subclass of congestion games that we analyse is the class of symmetric k-uniform congestion games. The second subclass of congestion games that we analyse is a generalisation of symmetric k-uniform congestion games, which we call *semi-symmetric* k-uniform congestion games. This class has not been studied before, to the best of our knowledge. In this class of games, a valid strategy for each player $i \in N$ is to pick exactly $k_i \leq k$ resources from a set $R_i \subseteq R$, i.e., for every player $i \in N$ there exists a subset $R_i \subseteq R$ and a value $k_i \in \{1, \ldots, k\}$ such that $X_i = \{R' \subseteq R_i : |R'| = k_i\}$. In both classes of games that we analyse, all resources in R are available to all players, so $\forall i \in N : R_i = R$. We denote these two subclasses of congestion games by $\mathcal{G}_k$ and $\mathcal{G}_{\leq k}$, respectively. For convenience, we next present a summary of these two subclasses of congestion games.

- **Symmetric k-uniform congestion games ($\mathcal{G}_k$):**
  For each player, a valid strategy is to pick exactly k resources from R, so
  $\forall i \in N : X_i = \{R' \subseteq R : |R'| = k\}$.
- **Semi-symmetric k-uniform congestion games ($\mathcal{G}_{\leq k}$):**
  For each player, a valid strategy is to pick some fixed number of resources from R that is at most k, so
  $\forall i \in N : X_i = \{R' \subseteq R : |R'| = k_i\}$, with $k_i \in \{1, \ldots, k\}$.

Note that $\mathcal{G}_{\leq k}$ can be reduced to $\mathcal{G}_k$ by defining $k_i = k$ for all $i \in N$.

Each strategy vector $x \in X$ denotes a possible realisation of an instance, in which each player $i \in N$ plays according to a particular strategy $x_i \in X_i$. We denote by $x_r := |\{i \in N : r \in x_i\}|$ the number of players picking resource $r \in R$ according to the strategy vector x. A realisation of the instance comes with a cost $c_r(x_r)$ of each resource $r \in R$, a private cost $C_i(x)$ for each player $i \in N$ and the social cost $C(x)$. These costs are explained in detail next.

The cost $c_r(x_r)$ of a resource $r \in R$ depends on the number of players x_r picking this resource according to the strategy vector x. We assume for all resources $r \in R$ that the cost function is a polynomial function in x_r with nonnegative coefficients and real exponents. Note that these exponents need not be integers. We say that a game has cost functions of maximum degree d if the largest exponent that appears in some cost function $c_r(x_r), r \in R$, is d. This maximum degree is made explicit in the notation of the two subclasses of congestion games that we analyse: we use the notation $\mathcal{G}_k^d$ and $\mathcal{G}_{\leq k}^d$ for the two subclasses, respectively.

The private cost $C_i(x)$ for a player $i \in N$ is defined as the sum of the costs of each resource that this player picked according to the strategy vector x: $C_i(x) := \sum_{r \in x_i} c_r(x_r)$.

The social cost $C(x)$ is defined as the sum of all private costs: $C(x) := \sum_{i \in N} C_i(x)$. We call a strategy vector *optimal* for some instance if it minimises the social cost of this instance, and denote by (I) the set of all optimal strategy vectors in the instance I:

$$\text{OPT}(I) := \{x \in X : C(x) = \min_{x \in X} C(x)\}.$$

In this research, we are interested in the relation between optimal strategy vectors and suboptimal strategy vectors called *Nash equilibria*, which are explained in more detail. As in standard game theory notation, let $(y_i, x_{-i}) \in X$ be the strategy vector in which all players play as in $x \in X$, except for player $i \in N$ who plays $y_i \in X_i$. A Nash equilibrium is a strategy vector $x \in X$ with

$$C_i(x) \leq C_i(y_i, x_{-i}) \tag{2.1}$$

for all $y_i \in X_i$, $i \in N$. So in a Nash equilibrium, no player can decrease their private cost by playing a different strategy if the other players keep their strategies fixed. We denote by $\text{NE}(I)$ the set of Nash equilibria in the instance I:

$$\text{NE}(I) := \{x \in X : C_i(x) \leq C_i(y_i, x_{-i}) \text{ for all } y_i \in X_i, i \in N\}.$$

For convenience, we denote by $\text{NE}^*(I)$ the set of Nash equilibria in the instance I with maximum social cost:

$$\text{NE}^*(I) := \{x^{\text{NE}} \in \text{NE}(I) : C(x^{\text{NE}}) = \max_{x \in \text{NE}(I)} C(x)\}.$$

We want to know how the cost of Nash equilibria relates to the cost of the optimal strategy vectors in the two subclasses of congestion games that we explained before. To that end, we analyse the price of anarchy (PoA) of instances of these subclasses (Koutsoupias and Papadimitriou [1999]):

$$\begin{aligned}\text{PoA}(I) &:= \max_{x \in \text{NE}(I)} \frac{C(x)}{C(x^{\text{OPT}})} \\ &= \frac{C(x^{\text{NE}})}{C(x^{\text{OPT}})}\end{aligned}$$

for some $x^{\text{NE}} \in \text{NE}^*(I), x^{\text{OPT}} \in \text{OPT}(I)$. In particular, we are interested in the largest price of anarchy possible within these two subclasses of congestion games:

$$\text{PoA}(\mathcal{G}) := \max_{I \in \mathcal{G}} \text{PoA}(I),$$

where $\mathcal{G} \in \{\mathcal{G}_k^d, \mathcal{G}_{\leq k}^d\}$ denotes the subclass of congestion games.

An important note is that we are not interested in the price of anarchy for some specific k, but rather in the price of anarchy if k can be any value. Therefore, for the first class of games, the correct notation of the price of anarchy that we seek to find is

$$\begin{aligned}\mathrm{PoA}^* &:= \max_{k \in N} \max_{I \in \mathcal{G}_k^d} \mathrm{PoA}(I) \\ &= \max_{I \in G^*} \mathrm{PoA}(I),\end{aligned}$$

where $G^* := \bigcup_{k \in \mathbb{N}} \mathcal{G}_k^d$. In the remainder of the report, we abuse notation and use $\mathcal{G}_k^d$ to denote G^*. Similarly, we use $\mathcal{G}_{\leq k}^d$ to denote the union over all k of semi-symmetric k-uniform congestion games. An exception occurs whenever k is given an specific value, e.g. in $\mathcal{G}_3^d$ or $\mathcal{G}_{\leq 2}^d$. Then we do not refer to the union over all k, but rather to the class of games with this specific value of k.

2.2 Example instance

Figure 2.1 shows an instance $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq 2}^1$ of a semi-symmetric 2-uniform congestion game, with players $N = \{1, 2\}$, resources $R = \{1, 2, 3\}$ with cost functions $c_1(x) = x, c_2(x) = 1, c_3(x) = 2$, and valid strategies $X = X_1 \times X_2$ in which player 1 picks one resource and player 2 picks two resources. The strategy spaces for the players are

$$\begin{aligned}X_1 &= \{R' \in R : |R'| = 1\} = \{\{1\}, \{2\}, \{3\}\} \\ X_2 &= \{R' \in R : |R'| = 2\} = \{\{1, 2\}, \{1, 2\}, \{2, 3\}\}.\end{aligned}$$

Each column in Figure 2.1 represents a resource, whose cost function is shown at the top. The rows represent two realisations of the game: the first row shows an optimal strategy vector x^{OPT} and the second row shows a Nash equilibrium strategy vector x^{NE}. In these two realisations, player 1 and player 2 are represented as 1 and 2, respectively.

	Resources		
	$c_1(x) = x$	$c_2(x) = 1$	$c_3(x) = 2$
x^{OPT}	2	1 2	
x^{NE}	1	2	2

Figure 2.1: An instance of a semi-symmetric 2-uniform congestion game.

In the optimal strategy vector x^{OPT}, player 1 picks resource 2, and player 2 picks resources 1 and 2. We obtain as costs:

$$\begin{aligned}&\text{resource costs: } c_1(x_1^{\mathrm{OPT}}) = c_1(1) = 1, c_2(x_2^{\mathrm{OPT}}) = c_2(2) = 1, c_3(x_3^{\mathrm{OPT}}) = c_3(0) = 2 \\ &\text{private costs: } C_1(x^{\mathrm{OPT}}) = c_2(x_2^{\mathrm{OPT}}) = 1, C_2(x^{\mathrm{OPT}}) = c_1(x_1^{\mathrm{OPT}}) + c_2(x_2^{\mathrm{OPT}}) = 2 \\ &\text{social cost: } C(x^{\mathrm{OPT}}) = C_1(x^{\mathrm{OPT}}) + C_2(x^{\mathrm{OPT}}) = 3.\end{aligned}$$

In the Nash equilibrium strategy vector x^{NE}, player 1 picks resource 1, and player 2 picks resources 2 and 3. We obtain as costs:

$$\begin{aligned}&\text{resource costs: } c_1(x_1^{\mathrm{NE}}) = c_1(1) = 1, c_2(x_2^{\mathrm{NE}}) = c_2(1) = 1, c_3(x_3^{\mathrm{NE}}) = c_3(1) = 2 \\ &\text{private costs: } C_1(x^{\mathrm{NE}}) = c_1(x_1^{\mathrm{NE}}) = 1, C_2(x^{\mathrm{NE}}) = c_2(x_2^{\mathrm{NE}}) + c_3(x_3^{\mathrm{NE}}) = 3 \\ &\text{social cost: } C(x^{\mathrm{NE}}) = C_1(x^{\mathrm{NE}}) + C_2(x^{\mathrm{NE}}) = 4.\end{aligned}$$

In x^{NE}, no player can decrease their private cost by picking other resources. We will show this explicitly for player 2. In x^{NE}, player 2 plays the strategy $\{2, 3\}$, so we will show that if player 2 switches to strategy $\{1, 2\}$ or $\{1, 3\}$, their private cost does not decrease. Let $x'^{\mathrm{NE}} = (\{1, 2\}, x_{-2}^{\mathrm{NE}})$ be the strategy vector in which player 1 plays as in x^{NE}, and player 2 plays the strategy $\{1, 2\}$. We obtain as costs:

$$\begin{aligned}&\text{resource costs: } c_1(x_1'^{\mathrm{NE}}) = c_1(2) = 2, c_2(x_2'^{\mathrm{NE}}) = c_2(1) = 1, c_3(x_3'^{\mathrm{NE}}) = c_3(0) = 2 \\ &\text{private cost player 2: } C_2(x'^{\mathrm{NE}}) = c_1(x_1'^{\mathrm{NE}}) + c_2(x_2'^{\mathrm{NE}}) = 3,\end{aligned}$$

so $C_2(x'^{\mathrm{NE}}) = C_2(x^{\mathrm{NE}})$. Let $x''^{\mathrm{NE}} = (\{1,3\}, x^{\mathrm{NE}}_{-2})$ be the strategy vector in which player 1 plays as in x^{NE}, and player 2 plays the strategy $\{1,3\}$. We obtain as cost:

resource costs: $c_1(x''^{\mathrm{NE}}_1) = c_1(2) = 2, c_2(x''^{\mathrm{NE}}_2) = c_2(0) = 1, c_3(x''^{\mathrm{NE}}_3) = c_3(1) = 2$

private cost player 2: $C_2(x'^{\mathrm{NE}}) = c_1(x'^{\mathrm{NE}}_2) + c_3(x'^{\mathrm{NE}}_3) = 4,$

so $C_2(x''^{\mathrm{NE}}) > C_2(x^{\mathrm{NE}})$. A similar argument can be made for player 1. This proves that x^{NE} is indeed a Nash equilibrium strategy vector.

The price of anarchy of this instance is

$$\mathrm{PoA}(I) = \max_{x^{\mathrm{NE}} \in \mathrm{NE}(I)} \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} = \frac{4}{3}.$$

This proves a lower bound for the price of anarchy of semi-symmetric 2-uniform congestion games with cost functions of maximum degree 1:

$$\mathrm{PoA}(\mathcal{G}^1_{\leq 2}) = \max_{I \in \mathcal{G}^1_{\leq 2}} \mathrm{PoA}(I) \geq \frac{4}{3}.$$

Chapter 3

Properties of symmetric and semi-symmetric k-uniform congestion games

In this section, useful properties of symmetric and semi-symmetric uniform congestion games are discussed. First, properties that hold for all instances of these games are considered. Second, we elaborate on properties that only hold for worst-case instances.

3.1 Properties of all instances

The games that we described in Chapter 2 have some useful properties that are covered in this section. In Chapters 4 and 5, these properties are used to perform crucial steps in the proofs to bound the price of anarchy or to simplify those proofs. Some properties only hold for symmetric k-uniform congestion games; we elaborate on the properties of semi-symmetric k-uniform congestion games in Chapter 5.

We start with translating the general Nash inequality (2.1) to an easier to use Nash inequality that applies to instances of symmetric and semi-symmetric k-uniform congestion games.

Lemma 3.1 (Nash Inequality). *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$ be an instance of a (semi-)symmetric k-uniform congestion game and let $x^{\text{NE}} \in X$ be a strategy vector. Then, $x^{\text{NE}} \in \text{NE}(I)$ if and only if for all resources $r_1, r_2 \in R$ holds: if some player uses resource r_1 and not resource r_2 in x^{NE} then*

$$c_{r_1}(x^{\text{NE}}_{r_1}) \leq c_{r_2}(x^{\text{NE}}_{r_2} + 1).$$

Proof. First we prove the "only if" part by contradiction. Assume $x^{\text{NE}} \in \text{NE}(I)$. Let player i use resource r_1 and not resource r_2 in x^{NE} and assume $c_{r_1}(x^{\text{NE}}_{r_1}) > c_{r_2}(x^{\text{NE}}_{r_2} + 1)$. Now, let $y^{\text{NE}}_i \in X_i$ be the strategy vector for player i that is identical to x^{NE}_i, but with resource r_2 instead of resource r_1:

$$y^{\text{NE}}_i := (x^{\text{NE}}_i \backslash \{r_1\}) \cup \{r_2\}.$$

Then we obtain as private cost for player i

$$C_i(y^{\text{NE}}_i, x^{\text{NE}}_{-i}) = C_i(x^{\text{NE}}_i) - c_{r_1}(x^{\text{NE}}_{r_1}) + c_{r_2}(x^{\text{NE}}_{r_2} + 1) < C_i(x^{\text{NE}}_i).$$

This contradicts the Nash inequality (2.1).

Now we prove the "if" part. Consider a strategy vector $x^{\text{NE}} \in X$. Assume that for all resources $r_1, r_2 \in R$ holds: if some player uses resource r_1 and not resource r_2 in x^{NE} then $c_{r_1}(x^{\text{NE}}_{r_1}) \leq c_{r_2}(x^{\text{NE}}_{r_2} + 1)$. Let $i \in N$ be some player and let $y^{\text{NE}}_i \in X_i$ be some strategy vector for player

i. Let $R' = x_i^{\mathrm{NE}} \backslash y_i^{\mathrm{NE}}$ be the set of resources picked by player i in x_i^{NE} but not in y_i^{NE} and let $R'' = y_i^{\mathrm{NE}} \backslash x_i^{\mathrm{NE}}$ be the set of resources picked by player i in y_i^{NE} but not in x_i^{NE}. Then,

$$c_{r'}(x_{r'}^{\mathrm{NE}}) \leq c_{r''}(x_{r''}^{\mathrm{NE}} + 1)$$

for all $r' \in R', r'' \in R''$, by the assumption. Moreover, we know $|R'| = |R''|$, since $|x_i^{\mathrm{NE}}| = |y_i^{\mathrm{NE}}|$. Therefore we obtain for the private cost of player i

$$C_i(y_i^{\mathrm{NE}}, x_{-i}^{\mathrm{NE}}) = C_i(x_i^{\mathrm{NE}}) - \sum_{r \in R'} c_r(x_r^{\mathrm{NE}}) + \sum_{r \in R''} c_r(x_r^{\mathrm{NE}} + 1) \geq C_i(x_i^{\mathrm{NE}}).$$

The Nash inequality (2.1) holds, therefore x^{NE} must be a Nash equilibrium. □

The next two lemmas show useful properties of the cost functions of symmetric and semi-symmetric k-uniform congestion games. The first lemma can be used to relate the cost of a resource to the cost of this resource if it would be used by one more player. It is a result of the way the cost functions are defined.

Lemma 3.2. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq k}^d$ be an instance of a symmetric or semi-symmetric k-uniform congestion game with cost functions of maximum degree d. Let $x \in X$ be any strategy vector and let resource $r \in R$ be used by some player according to x. Then*

$$c_r(x_r + 1) \leq 2^d c_r(x_r).$$

Proof. By definition, the cost function is of the form

$$c_r(x_r) = \alpha_1 x_r^{d_1} + \ldots + \alpha_p x_r^{d_p},$$

for some $p \in \mathbb{Z}_{\geq 0}$, $\alpha_i \in \mathbb{R}_{\geq 0}$ and $d_i \in [0, d]$ for $i = 1, \ldots, p$. We then obtain

$$\begin{aligned} c_r(x_r + 1) - 2^d c_r(x_r) &= \alpha_1 \Big((x_r + 1)^{d_1} - 2^d x_r^{d_1} \Big) + \ldots + \alpha_p \Big((x_r + 1)^{d_p} - 2^d x_r^{d_p} \Big) \\ &\leq \alpha_1 \Big((x_r + 1)^{d_1} - 2^{d_1} x_r^{d_1} \Big) + \ldots + \alpha_p \Big((x_r + 1)^{d_p} - 2^{d_p} x_r^{d_p} \Big) \\ &\leq \alpha_1 \Big((2x_r)^{d_1} - 2^{d_1} x_r^{d_1} \Big) + \ldots + \alpha_p \Big((2x_r)^{d_p} - 2^{d_p} x_r^{d_p} \Big) \\ &= 0, \end{aligned}$$

where the last inequality follows from $x_r \geq 1$. □

The following lemma provides an important insight into the cost of resources in a Nash equilibrium. It states that the costs of two resources in a Nash equilibrium cannot be too far apart, unless one of the resources is used by every player or by no player at all.

Lemma 3.3. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_k^d$ be an instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}(I)$. Then*

$$c_r(x_r^{\mathrm{NE}}) \leq 2^d c_s(x_s^{\mathrm{NE}} + 1)$$

for any $r, s \in R$ with $x_r^{\mathrm{NE}} \geq 1$ and $x_s^{\mathrm{NE}} < n$.

Proof. Let p be a player using resource r in x^{NE}. Let q be a player that does not use resource s in x^{NE}. If player q uses resource r then

$$c_r(x_r^{\mathrm{NE}}) \leq c_s(x_s^{\mathrm{NE}} + 1),$$

by the Nash inequality (Lemma 3.1), and we are done. If player q does not use resource r, then player q uses some resource $t \in R$ that is not used by player p (since both players pick the same number of resources). We then obtain

$$c_t(x_t^{\mathrm{NE}}) \leq c_s(x_s^{\mathrm{NE}} + 1),$$

by the Nash inequality (Lemma 3.1). Since player p uses resources r, and not resource t, we obtain

$$c_r(x_r^{\mathrm{NE}}) \leq c_t(x_t^{\mathrm{NE}} + 1),$$

again by the Nash inequality (Lemma 3.1). We can conclude

$$c_r(x_r^{\mathrm{NE}}) \leq c_t(x_t^{\mathrm{NE}} + 1) \leq 2^d c_t(x_t^{\mathrm{NE}}) \leq 2^d c_s(x_s^{\mathrm{NE}} + 1),$$

where the second inequality follows from Lemma 3.2. □

3.2 Properties of worst-case instances

We introduce a subset of the instances of both symmetric and semi-symmetric k-uniform congestion games, which we call *critical* instances. These instances have some additional useful properties, which are covered in this section. The critical instances are defined as follows.

Definition. (critical instance) *An instance of a (semi-)symmetric k-uniform congestion game is called* critical *if no instance of a (semi-)symmetric k-uniform congestion game has a larger PoA and if all instances of (semi-)symmetric k'-uniform congestion games with $k' < k$ have a smaller PoA.*

Clearly, we only need to analyse critical instances in order to bound the PoA of both symmetric k-uniform congestion games and semi-symmetric k-uniform congestion games.

The cost functions of resources in critical instances can be simplified. To do this, first consider some critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$ of a (semi-)symmetric k-uniform congestion game and strategy vectors $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Now the set of resources is split into *underloaded (U)*, *balanced (B)* and *overloaded (O)* resources, with respect to the two strategy vectors:

$$\begin{aligned}
U &= \{r \in R : x_r^{\text{NE}} < x_r^{\text{OPT}}\}\\
B &= \{r \in R : x_r^{\text{NE}} = x_r^{\text{OPT}}\}\\
O &= \{r \in R : x_r^{\text{NE}} > x_r^{\text{OPT}}\}.
\end{aligned}$$

For the instance in Figure 2.1, we obtain $U = \{2\}, B = \{1\}, O = \{3\}$. We will talk about underloaded, balanced and overloaded resources without adding "with respect to strategy vectors x and y" whenever the strategy vectors are clear from the context. Lemmas 3.4 and 3.5 present the simplified cost functions. Lemma 3.4 states that all underloaded resources in critical instances have a constant cost. This is an important insight, which is used in crucial steps in proofs in Sections 4 and 5. Lemma 3.5 states that we can assume that all balanced and overloaded resources in critical instances have a linear cost. This lemma is only used to simplify proofs. Both lemmas are based on similar lemmas for games with affine cost functions by de Jong et al. [2020].

Lemma 3.4. *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ be a critical instance of a symmetric k-uniform congestion game, and let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Then, all underloaded resources $u \in U$ that are used by some player according to strategy x^{NE} have constant cost $c_u(x) = \beta_u$.*

Proof. The proof is by contradiction. We assume the lemma is false for some critical instance I of a symmetric k-uniform congestion game. Then we construct a new instance $\hat{I}$ of a symmetric k-uniform congestion game to prove the existence of a resource r whose cost function satisfies certain inequalities. Using what we know about resource r, we construct another instance $\tilde{I}$ of a symmetric k-uniform congestion game and we show $\text{PoA}(\tilde{I}) > \text{PoA}(I)$. This is the contradiction.

Consider a critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ of a symmetric k-uniform congestion game and two strategy vectors $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Let $u \in U$ be an underloaded resource used in x^{NE} with non constant cost

$$c_u(x) = \alpha_1 x^{d_1} + \ldots + \alpha_p x^{d_p}$$

for some $p \in \mathbb{Z}_{\geq 0}$, $\alpha_i \in \mathbb{R}_{\geq 0}$ and $d_i \in [0, p]$ for $i = 1, \ldots, p$. Since $c_u(x)$ is non constant, we know $c_u(x_u^{\text{OPT}}) \geq c_u(x_u^{\text{NE}} + 1) > c_u(x_u^{\text{NE}})$ and we know that $\alpha_i > 0, d_i > 0$ for some $1 \leq i \leq p$. For small $\varepsilon > 0$ we now define a new cost function for u:

$$\begin{aligned}
\hat{c}_u(x) &:= p\varepsilon + (\alpha_1 - \frac{\varepsilon}{(x_u^{\text{NE}} + 1)^{d_1}})x^{d_1} + \ldots + (\alpha_p - \frac{\varepsilon}{(x_u^{\text{NE}} + 1)^{d_p}})x^{d_p}\\
&= c_u(x) + p\varepsilon - \frac{x^{d_1}}{(x_u^{\text{NE}} + 1)^{d_1}}\varepsilon - \ldots - \frac{x^{d_p}}{(x_u^{\text{NE}} + 1)^{d_p}}\varepsilon.
\end{aligned}$$

This new cost function has the following properties:

$$\begin{aligned}
\hat{c}_u(x_u^{\mathrm{NE}}+1) &= c_u(x_u^{\mathrm{NE}}+1)\\
\hat{c}_u(x_u^{\mathrm{NE}}) &= c_u(x_u^{\mathrm{NE}}) + p\varepsilon - \frac{(x_u^{\mathrm{NE}})^{d_1}}{(x_u^{\mathrm{NE}}+1)^{d_1}}\varepsilon - \ldots - \frac{(x_u^{\mathrm{NE}})^{d_p}}{(x_u^{\mathrm{NE}}+1)^{d_p}}\varepsilon\\
&> c_u(x_u^{\mathrm{NE}})\\
\hat{c}_u(x_u^{\mathrm{OPT}}) &= c_u(x_u^{\mathrm{OPT}}) + p\varepsilon - \frac{(x_u^{\mathrm{OPT}})^{d_1}}{(x_u^{\mathrm{NE}}+1)^{d_1}}\varepsilon - \ldots - \frac{(x_u^{\mathrm{OPT}})^{d_p}}{(x_u^{\mathrm{NE}}+1)^{d_p}}\varepsilon\\
&\leq c_u(x_u^{\mathrm{OPT}}).
\end{aligned}$$

For all other resources $r \in R\backslash\{u\}$, we define new cost functions $\hat{c}_r(x) = c_r(x)$ that are the same as the old cost functions. Let $\hat{I} = (N, R, (\hat{c}_r)_{r\in R}, X) \in \mathcal{G}_k^d$ be the same instance as I, except the new cost functions are used. If there is a small $\varepsilon > 0$ such that $x^{\mathrm{NE}} \in \mathrm{NE}(\hat{I})$ then we obtain $\mathrm{PoA}(\hat{I}) > \mathrm{PoA}(I)$, which is shown next.

We assume $x^{\mathrm{NE}} \in \mathrm{NE}(\hat{I})$. Let $y^{\mathrm{NE}} \in \mathrm{NE}^*(\hat{I})$ be a Nash equilibrium with the highest social cost. Then $\hat{C}(y^{\mathrm{NE}}) \geq \hat{C}(x^{\mathrm{NE}}) > C(x^{\mathrm{NE}})$, since resource u is used in x^{NE}. Similarly, let $y^{\mathrm{OPT}} \in \mathrm{OPT}(\hat{I})$ be an optimal strategy vector. Then $\hat{C}(y^{\mathrm{OPT}}) \leq \hat{C}(x^{\mathrm{OPT}}) \leq C(x^{\mathrm{OPT}})$. Regarding the PoA of $\hat{I}$, we obtain

$$\mathrm{PoA}(\hat{I}) = \frac{\hat{C}(y^{\mathrm{NE}})}{\hat{C}(y^{\mathrm{OPT}})} > \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} = \mathrm{PoA}(I).$$

This is not possible, since both I and $\hat{I}$ are instances of a symmetric k-uniform congestion game and I is critical. Therefore, we know that no $\varepsilon > 0$ exists such that $x^{\mathrm{NE}} \in \mathrm{NE}(\hat{I})$. By Lemma 3.1 we know that there exist some player $t \in N$ using some resource r_1 and not some other resource r in x^{NE}, with

$$\hat{c}_{r_1}(x_{r_1}^{\mathrm{NE}}) > \hat{c}_r(x_r^{\mathrm{NE}}+1)$$

for all $\varepsilon > 0$. In addition, we know that $c_{t,\max}^{\mathrm{NE}} \leq c_r(x_r^{\mathrm{NE}}+1)$, where

$$c_{t,\max}^{\mathrm{NE}} := \max_{r\in x_t} c_r(x_r^{\mathrm{NE}}),$$

else player t would already use resource r in x^{NE}, by Lemma 3.1. Next, we will prove that $r_1 = u$.

Firstly, we cannot have $r_1 \neq u$ and $r \neq u$, because then

$$\hat{c}_{r_1}(x_{r_1}^{\mathrm{NE}}) = c_{r_1}(x_{r_1}^{\mathrm{NE}}) \leq c_r(x_r^{\mathrm{NE}}+1) = \hat{c}_r(x_r^{\mathrm{NE}}+1),$$

by Lemma 3.1 since $x^{\mathrm{NE}} \in \mathrm{NE}(I)$. Secondly, we cannot have $r = u$, since then

$$\hat{c}_{r_1}(x_{r_1}^{\mathrm{NE}}) = c_{r_1}(x_{r_1}^{\mathrm{NE}}) \leq c_u(x_u^{\mathrm{NE}}+1) = \hat{c}_u(x_u^{\mathrm{NE}}+1),$$

again by Lemma 3.1 since $x^{\mathrm{NE}} \in \mathrm{NE}(I)$. Therefore, we must have $r_1 = u$ and

$$\hat{c}_u(x_u^{\mathrm{NE}}) > \hat{c}_r(x_r^{\mathrm{NE}}+1)$$

for some $r \in R$. Next, we prove that $c_u(x_u^{\mathrm{NE}}) = c_r(x_r^{\mathrm{NE}}+1)$. The first thing to note is that $\hat{c}_u(x) = c_u(x)$ whenever $\varepsilon = 0$. Secondly, the function $\hat{c}_u(x)$ is continuous in ε. Thirdly, $c_u(x_u^{\mathrm{NE}}) \leq c_r(x_r^{\mathrm{NE}}+1)$, by the Nash inequality (Lemma 3.1). Now we can conclude that $c_u(x_u^{\mathrm{NE}}) = c_r(x_r^{\mathrm{NE}}+1)$. Using this information about resource r and player t, we will construct another instance $\tilde{I}$ of a symmetric k-uniform congestion game with $\mathrm{PoA}(\tilde{I}) > \mathrm{PoA}(I)$, to form the final contradiction.

Let $\tilde{I} = (N, \tilde{R}, (\tilde{c}_r)_{r\in\tilde{R}}, \tilde{X}) \in \mathcal{G}_k^d$ be the same instance as I, but with resources $\tilde{R} = R \cup \{b\}$ and cost functions $\tilde{c}_b(x) = \frac{1}{2^d} c_{\max}^{\mathrm{NE}} x^d$, where

$$c_{\max}^{\mathrm{NE}} := \max_{r\in R:\, x_r^{\mathrm{NE}}\geq 1} c_r(x_r^{\mathrm{NE}}),$$

$\tilde{c}_u(x) = c_u(x+1)$ and $\tilde{c}_r(x) = c_r(x)$ for $r \in \tilde{R}\backslash\{b,u\}$. Let $\tilde{x}^{\text{NE}}$ be the same strategy vector as x^{NE}, except that player t uses resource b instead of resource u. Let $q \in N$ be a player that uses resource u in x^{OPT} but not in x^{NE} and let $\tilde{x}^{\text{OPT}}$ be the same strategy vector as x^{OPT}, except that player q uses resource b instead of resource u. We obtain the following costs:

$$
\begin{aligned}
\tilde{c}_b(\tilde{x}_b^{\text{NE}}) &= \frac{1}{2^d} c_{\max}^{\text{NE}} \\
\tilde{c}_b(\tilde{x}_b^{\text{NE}} + 1) &= c_{\max}^{\text{NE}} \\
\tilde{c}_b(\tilde{x}_b^{\text{NE}}) &= \frac{1}{2^d} c_{\max}^{\text{NE}} \\
\tilde{c}_r(\tilde{x}_r^{\text{NE}}) &= c_r(x_r^{\text{NE}}) \\
\tilde{c}_r(\tilde{x}_r^{\text{NE}} + 1) &= c_r(x_r^{\text{NE}} + 1) \\
\tilde{c}_r(\tilde{x}_r^{\text{NE}}) &= c_r(x_r^{\text{OPT}}),
\end{aligned}
$$

for all $r \in \tilde{R}\backslash\{b\}$. We will now prove $\tilde{x}^{\text{NE}} \in \text{NE}(\tilde{I})$ using Lemma 3.1.

Let some player p use resource r_1 and not resource r_2 in $\tilde{x}^{\text{NE}}$. We need to prove $\tilde{c}_{r_1}(\tilde{x}_{r_1}^{\text{NE}}) \leq \tilde{c}_{r_2}(\tilde{x}_{r_2}^{\text{NE}} + 1)$. To that end, we distinguish the following cases:

- $p \neq t$ and $r_1 \neq b, r_2 \neq b$
- $p \neq t$ and $r_1 \neq b, r_2 = b$
- $p = t$ and $r_1, r_2 \notin \{b, u\}$
- $p = t$ and $r_1 \notin \{b, u\}, r_2 = u$
- $p = t$ and $r_1 = b, r_2 \neq b$.

These cases cover all possibilities, since a player $p \neq t$ does not use resource b in $\tilde{x}^{\text{NE}}$ (so then $r_1 \neq b$) and player t does not use resource u in $\tilde{x}^{\text{NE}}$ (so then $r_1 \neq u$.).

Case 1: $p \neq t$ and $r_1 \neq b, r_2 \neq b$. Then,

$$\tilde{c}_{r_1}(\tilde{x}_{r_1}^{\text{NE}}) = c_{r_1}(x_{r_1}^{\text{NE}}) \leq c_{r_2}(x_{r_2}^{\text{NE}} + 1) = \tilde{c}_{r_2}(\tilde{x}_{r_2}^{\text{NE}} + 1),$$

by the Nash inequality (Lemma 3.1), since player p used resource r_1 and not resource r_2 in x^{NE}.
Case 2: $p \neq t$ and $r_1 \neq b, r_2 = b$. Then we know

$$\tilde{c}_{r_1}(\tilde{x}_{r_1}^{\text{NE}}) = c_{r_1}(x_{r_1}^{\text{NE}}) \leq c_{\max}^{\text{NE}} = \tilde{c}_b(\tilde{x}_b^{\text{NE}} + 1).$$

Case 3: $p = t$ and $r_1, r_2 \notin \{b, u\}$. Then,

$$\tilde{c}_{r_1}(\tilde{x}_{r_1}^{\text{NE}}) = c_{r_1}(x_{r_1}^{\text{NE}}) \leq c_{r_2}(x_{r_2}^{\text{NE}} + 1) = \tilde{c}_{r_2}(\tilde{x}_{r_2}^{\text{NE}} + 1),$$

by the Nash inequality (Lemma 3.1), since player t used resource r_1 and not resource r_2 in x^{NE}.
Case 4: $p = t$ and $r_1 \notin \{b, u\}, r_2 = u$. Then,

$$\tilde{c}_{r_1}(\tilde{x}_{r_1}^{\text{NE}}) = c_{r_1}(x_{r_1}^{\text{NE}}) \leq c_{t,\max}^{\text{NE}} = c_u(x_u^{\text{NE}}) \leq c_u(x_u^{\text{NE}} + 1) = \tilde{c}_u(\tilde{x}_u^{\text{NE}} + 1).$$

Case 5: $p = t$ and $r_1 = b, r_2 \neq b$. Then,

$$\tilde{c}_b(\tilde{x}_b^{\text{NE}}) = \frac{1}{2^d} c_{\max}^{\text{NE}} \leq c_{r_2}(x_{r_2}^{\text{NE}} + 1) = \tilde{c}_{r_2}(x_{r_2}^{\text{NE}} + 1),$$

where the inequality follows from Lemma 3.3. We covered all cases of player p and resources r_1 and r_2, so by the Nash inequality (Lemma 3.1) we know $\tilde{x}^{\text{NE}} \in \text{NE}(\tilde{I})$.

We proved that $\tilde{x}^{\text{NE}} \in \text{NE}(\tilde{I})$, but there may be another Nash equilibrium with a higher social cost. Let $y^{\text{NE}} \in \text{NE}^*(\tilde{I})$ be a Nash equilibrium with the highest social cost. Then we obtain $\tilde{C}(y^{\text{NE}}) \geq \tilde{C}(\tilde{x}^{\text{NE}}) = C(x^{\text{NE}}) - (c_u(x_u^{\text{NE}}) - \frac{1}{2^d} c_{\max}^{\text{NE}})$. Similarly, there may be a strategy vector with a lower social cost than $\tilde{x}^{\text{OPT}}$. Let $y^{\text{OPT}} \in \text{OPT}(\tilde{I})$ be an optimal strategy vector. Then we obtain $\tilde{C}(y^{\text{OPT}}) \leq \tilde{C}(\tilde{x}^{\text{OPT}}) = C(x^{\text{OPT}}) - (c_u(x_u^{\text{OPT}}) - \frac{1}{2^d} c_{\max}^{\text{NE}})$.

Regarding the PoA of $\tilde{I}$, we obtain

$$\begin{aligned}
\mathrm{PoA}(\tilde{I}) &= \frac{C(y^{\mathrm{NE}})}{C(y^{\mathrm{OPT}})} \geq \frac{C(\tilde{x}^{\mathrm{NE}})}{C(\tilde{x}^{\mathrm{OPT}})} = \frac{C(x^{\mathrm{NE}}) - (c_u(x_u^{\mathrm{NE}}) - \frac{1}{2^d} c_{\max}^{\mathrm{NE}})}{C(x^{\mathrm{OPT}}) - (c_u(x_u^{\mathrm{OPT}}) - \frac{1}{2^d} c_{\max}^{\mathrm{NE}})} \\
&> \frac{C(x^{\mathrm{NE}}) - (c_u(x_u^{\mathrm{NE}}) - \frac{1}{2^d} c_{\max}^{\mathrm{NE}})}{C(x^{\mathrm{OPT}}) - (c_u(x_u^{\mathrm{NE}}) - \frac{1}{2^d} c_{\max}^{\mathrm{NE}})} \geq \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} = \mathrm{PoA}(I),
\end{aligned}$$

where the second inequality follows from $c_u(x_u^{\mathrm{OPT}}) > c_u(x_u^{\mathrm{NE}})$ (as argued at the start of the proof) and the last inequality follows from $c_u(x_u^{\mathrm{NE}}) = c_r(x_r^{\mathrm{NE}} + 1) \geq \frac{1}{2^d} c_{\max}^{\mathrm{NE}}$ (by Lemma 3.3).

Since both I and $\tilde{I}$ are instances of a symmetric k-uniform congestion game, and I is critical, we must have $\mathrm{PoA}(I) \geq \mathrm{PoA}(\tilde{I})$. This is a contradiction. We can therefore conclude that a resource like u cannot exist, so all underloaded resources must have constant cost. □

Lemma 3.5. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ be a critical instance of a (semi-)symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. We may assume that all resources $r \in B \cup O$ have a cost function of the form $c_r(x) = \alpha_r x^d$, with $\alpha_r \in \mathbb{R}_{\geq 0}$.*

Proof. Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ be some critical instance of a (semi-)symmetric k-uniform congestion game and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$, $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. We will construct an instance $\tilde{I} = (N, R, (\tilde{c}_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ of a (semi-)symmetric k-uniform game with $\forall r \in B \cup O : \tilde{c}_r(x) = \alpha_r x^d$ for some $\alpha_r \in \mathbb{R}_{\geq 0}$ and $\mathrm{PoA}(\tilde{I}) = \mathrm{PoA}(I)$. This means that $\tilde{I}$ is also a critical instance, so we can focus our attention on instances like $\tilde{I}$.

By definition, the cost function of any resource $r \in R$ is of the form

$$c_r(x) = \alpha_1 x^{d_1} + \ldots + \alpha_p x^{d_p},$$

for some $p \in \mathbb{Z}_{\geq 0}$ and $d_1, \ldots, d_p \in [0, d]$. Now we define a new cost function for every resource $r \in B \cup O$ of the form $\tilde{c}_r(x) = \alpha_r x^d$, namely

$$\tilde{c}_r(x) := \frac{c_r(x_r^{\mathrm{NE}})}{(x_r^{\mathrm{NE}})^d} x^d.$$

These cost functions are well defined, since $x_r^{\mathrm{NE}} \geq 1$. If $x_r^{\mathrm{NE}} = 0$, then also $x_r^{\mathrm{OPT}} = 0$ (since $r \in B \cup O$), so then we can remove the resource from the game. The new cost functions have the following properties:

$$\begin{aligned}
\tilde{c}_r(x_r^{\mathrm{NE}}) &= c_r(x_r^{\mathrm{NE}}) \\
\tilde{c}_r(x_r^{\mathrm{OPT}}) &= \alpha_1 \frac{(x_r^{\mathrm{OPT}})^d}{(x_r^{\mathrm{NE}})^{d-d_1}} + \ldots + \alpha_p \frac{(x_r^{\mathrm{OPT}})^d}{(x_r^{\mathrm{NE}})^{d-d_p}} \\
&= \alpha_1 (x_r^{\mathrm{OPT}})^{d_1} \frac{(x_r^{\mathrm{OPT}})^{d-d_1}}{(x_r^{\mathrm{NE}})^{d-d_1}} + \ldots + \alpha_p (x_r^{\mathrm{OPT}})^{d_p} \frac{(x_r^{\mathrm{OPT}})^{d-d_p}}{(x_r^{\mathrm{NE}})^{d-d_p}} \\
&\leq \alpha_1 (x_r^{\mathrm{OPT}})^{d_1} + \ldots + \alpha_p (x_r^{\mathrm{OPT}})^{d_p} \\
&= c_r(x_r^{\mathrm{OPT}}). \\
\tilde{c}_r(x_r^{\mathrm{NE}} + 1) &= \alpha_1 (x_r^{\mathrm{NE}} + 1)^{d_1} \frac{(x_r^{\mathrm{NE}} + 1)^{d-d_1}}{(x_r^{\mathrm{NE}})^{d-d_1}} + \ldots + \alpha_p (x_r^{\mathrm{NE}} + 1)^{d_p} \frac{(x_r^{\mathrm{NE}} + 1)^{d-d_p}}{(x_r^{\mathrm{NE}})^{d-d_p}} \\
&\geq \alpha_1 (x_r^{\mathrm{NE}} + 1)^{d_1} + \ldots + \alpha_p (x_r^{\mathrm{NE}} + 1)^{d_p} \\
&= c_r(x_r^{\mathrm{NE}} + 1).
\end{aligned}$$

For resources $u \in U$ we use cost functions that are the same as the old ones: $\tilde{c}_u(x) = c_u(x)$. Let $\tilde{I} = (N, R, (\tilde{c}_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ be the same instance as I, except it has the new cost functions. We will now show that $x^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$ using Lemma 3.1.

Take any two resources $r_1, r_2 \in R$ with a player that uses resource r_1 and not resource r_2 in x^{NE}. Then $c_{r_2}(x^{\mathrm{NE}}_{r_2}+1) \geq c_{r_1}(x^{\mathrm{NE}}_{r_1})$ by the Nash inequality (Lemma 3.1), since $x^{\mathrm{NE}} \in \mathrm{NE}(I)$. We obtain

$$\tilde{c}_{r_2}(x^{\mathrm{NE}}_{r_2}+1) \geq c_{r_2}(x^{\mathrm{NE}}_{r_2}+1) \geq c_{r_1}(x^{\mathrm{NE}}_{r_1}) = \tilde{c}_{r_1}(x^{\mathrm{NE}}_{r_1}).$$

So also in I, no player has an incentive to switch to another resource when they play according to x^{NE}. By the Nash inequality (Lemma 3.1) we know $x^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$.

We proved that $x^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$, but there may be another Nash equilibrium with a higher social cost. Let $y^{\mathrm{NE}} \in \mathrm{NE}^*(\tilde{I})$ be a Nash equilibrium with the highest social cost. Then we obtain $\tilde{C}(y^{\mathrm{NE}}) \geq \tilde{C}(x^{\mathrm{NE}}) = C(x^{\mathrm{NE}})$. Similarly, there may be a strategy vector with a lower social cost than x^{OPT}. Let $y^{\mathrm{OPT}} \in \mathrm{OPT}(\tilde{I})$ be an optimal strategy vector. Then we obtain $\tilde{C}(y^{\mathrm{OPT}}) \leq \tilde{C}(x^{\mathrm{OPT}}) \leq C(x^{\mathrm{OPT}})$.

Regarding the PoA of $\tilde{I}$, we obtain

$$\mathrm{PoA}(\tilde{I}) = \frac{\tilde{C}(y^{\mathrm{NE}})}{\tilde{C}(y^{\mathrm{OPT}})} \geq \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} = \mathrm{PoA}(I).$$

Given that I is a critical instance and both I and $\tilde{I}$ are instances of a (semi-)symmetric k-uniform game, there must hold $\mathrm{PoA}(I) \geq \mathrm{PoA}(\tilde{I})$. We can conclude $\mathrm{PoA}(I) = \mathrm{PoA}(\tilde{I})$, which makes the instance $\tilde{I}$ also critical. We can thus focus our attention on instances like $\tilde{I}$. □

Lastly, we show that no resource is picked by all players in a Nash equilibrium with maximum social cost. This result is more general than a similar result by de Jong et al. [2016].

Lemma 3.6. *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$. Then we may assume that no resource $r \in R$ is picked by all players in x^{NE}, unless $\mathrm{PoA}(I) < \frac{1}{1-d(d+1)^{-(d+1)/d}}$.*

Proof. The proof is by contradiction. We consider a critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ of a symmetric k-uniform congestion game with $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$ and strategy vector $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ with $x^{\mathrm{NE}}_r = |N| = n$ for some resource $r \in R$. Then we will construct an instance $\tilde{I}$ of a symmetric $(k-1)$-uniform congestion game with $\mathrm{PoA}(\tilde{I}) \geq \mathrm{PoA}(I)$, which contradicts with I being critical.

Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ be a critical instance of a symmetric k-uniform congestion game with $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$ and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ with $x^{\mathrm{NE}}_r = |N| = n$ for some resource $r \in R$. Let $\tilde{I} = (N, \tilde{R}, (c_r)_{r\in\tilde{R}}, \tilde{X}) \in \mathcal{G}^d_{k-1}$ be an instance of a symmetric $(k-1)$-uniform congestion game that has the same set of players as I, and a set of resources $\tilde{R} = R\backslash\{r\}$. Now, let $\tilde{x}^{\mathrm{NE}}$ be the strategy vector in which every player plays as in x^{NE}, except they do not pick resource r. We will prove that $\tilde{x}^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$ using Lemma 3.1.

Let some player p use resource $r_1 \in \tilde{R}$ and not resource $r_2 \in \tilde{R}$ in $\tilde{x}^{\mathrm{NE}}$. Then

$$c_{r_1}(\tilde{x}^{\mathrm{NE}}_{r_1}) = c_{r_1}(x^{\mathrm{NE}}_{r_1}) \leq c_{r_2}(x^{\mathrm{NE}}_{r_2}+1) = c_{r_2}(\tilde{x}^{\mathrm{NE}}_{r_2}+1),$$

where the inequality follows from the Nash inequality (Lemma 3.1), since player p used resource r_1 and not resource r_2 in $x^{\mathrm{NE}} \in \mathrm{NE}(I)$. By Lemma 3.1 we know $\tilde{x}^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$.

We proved that $\tilde{x}^{\mathrm{NE}} \in \mathrm{NE}(\tilde{I})$, but there may be another Nash equilibrium with a higher social cost. Let $\tilde{y}^{\mathrm{NE}} \in \mathrm{NE}^*(\tilde{I})$ be a Nash equilibrium with the highest social cost. Then we obtain $C(\tilde{y}^{\mathrm{NE}}) \geq C(\tilde{x}^{\mathrm{NE}}) = C(x^{\mathrm{NE}}) - nc_r(n)$.

Let $\tilde{x}^{\mathrm{OPT}}$ be the same strategy vector as x^{OPT}, except the x^{OPT}_r players that pick resource r in x^{OPT} do not do so in $\tilde{x}^{\mathrm{NE}}$ and the other $n - x^{\mathrm{OPT}}_r$ players all pick one resource $u \in U$ less in $\tilde{x}^{\mathrm{OPT}}$ than in x^{OPT}. We will now prove that this is always possible or that we can rearrange which player pick which resources in the strategy vector x^{NE} to make it possible.

If $r \in B$ and thus $x^{\mathrm{OPT}}_r = x^{\mathrm{NE}}_r = n$, then the strategy vector $\tilde{x}^{\mathrm{OPT}}$ as described before is well defined. Otherwise, if $r \in O$ and thus $x^{\mathrm{OPT}}_r < n$, the strategy vector $\tilde{x}^{\mathrm{OPT}}$ may not be well defined if there exists some player $p \in N$ that does not use resource r nor any resource $u \in U$ in x^{NE}. We will now show how to rearrange which player pick which resources in the strategy vector x^{NE} to make sure player p picks an underloaded resource $u \in U$. To that end, first we observe

$$\sum_{u \in U} x_u^{\text{OPT}} \geq \sum_{u \in U} (x_u^{\text{OPT}} - x_u^{\text{NE}}) = \sum_{o \in O} (x_o^{\text{NE}} - x_o^{\text{OPT}}) \geq x_r^{\text{NE}} - x_r^{\text{OPT}} = n - x_r^{\text{OPT}}. \tag{3.1}$$

Inequality (3.1) tells us that if player p does not use resource r nor any underloaded resource, then there must be some other player $q \in N$ using both resource r (which is not underloaded) and at least one underloaded resource $u \in U$, or at least two underloaded resources $u \in U$. In addition, player p must use some resource $r' \in B \cup O$ that is not used by player q in x^{OPT}, otherwise player q would use at least 2 more resources than player p, which is impossible. Now we can rearrange x^{OPT} such that player p picks one of the underloaded resources that q originally picked instead of resource r', and player q picks resource r' instead of the underloaded resource that player p now picks. This rearrangement can be done for all players that do not use resource r nor any underloaded resource, and it does not change the social cost of the strategy vector. In the resulting strategy vector, every player picks either resource r or some underloaded resource (or both), so then $\tilde{x}^{\text{OPT}}$ can be formed using this strategy vector.

There may be another strategy vector that has a lower social cost than $\tilde{x}^{\text{OPT}}$. Let $\tilde{y}^{\text{OPT}} \in \text{OPT}(\tilde{I})$ be an optimal strategy vector. Then we obtain $C(\tilde{y}^{\text{OPT}}) \leq C(\tilde{x}^{\text{OPT}})$. To bound the social cost of $\tilde{x}^{\text{OPT}}$, we use an insight into the cost of underloaded resources:

$$c_u(x_u^{\text{OPT}}) \geq c_r(n) \text{ for all } u \in U.$$

We now prove this inequality. If for some $u \in U$ holds $c_u(x_u^{\text{NE}} + 1) < c_r(x_r^{\text{NE}})$, then all players picking resource r in x^{NE} must also pick resource u, by the Nash inequality (Lemma 3.1). Then we obtain $x_u^{\text{NE}} = x_r^{\text{NE}} = n$, which is not possible for an underloaded resource. Therefore,

$$c_u(x_u^{\text{OPT}}) = c_u(x_u^{\text{NE}}) = c_u(x_u^{\text{NE}} + 1) \geq c_r(x_r^{\text{NE}}) = c_r(n),$$

where the first two equalities follow from Lemma 3.4.

Now we obtain for the social cost of $\tilde{y}^{\text{OPT}}$:

$$\begin{aligned} C(\tilde{y}^{\text{OPT}}) \leq C(\tilde{x}^{\text{OPT}}) &\leq C(x^{\text{NE}}) - x_r^{\text{OPT}} c_r(x_r^{\text{OPT}}) - (n - x_r^{\text{OPT}}) c_r(n) \\ &\leq C(x^{\text{OPT}}) - (1 - d(d+1)^{-(d+1)/d}) n c_r(n). \end{aligned}$$

A proof of the final inequality is shown in Appendix A.1. Regarding the PoA of $\tilde{I}$, we obtain

$$\begin{aligned} \text{PoA}(\tilde{I}) &= \frac{C(\tilde{y}^{\text{NE}})}{C(\tilde{y}^{\text{OPT}})} \geq \frac{C(\tilde{x}^{\text{NE}})}{C(\tilde{x}^{\text{OPT}})} \\ &\geq \frac{C(x^{\text{NE}}) - n c_r(n)}{C(x^{\text{OPT}}) - (1 - d(d+1)^{-(d+1)/d}) n c_r(n)} = \frac{C(x^{\text{NE}}) - n c_r(n)}{\frac{1}{\text{PoA}(I)} C(x^{\text{NE}}) - (1 - d(d+1)^{-(d+1)/d}) n c_r(n)} \\ &\geq \frac{C(x^{\text{NE}}) - n c_r(n)}{\frac{C(x^{\text{NE}}) - n c_r(n)}{\text{PoA}(I)}} = \text{PoA}(I), \end{aligned}$$

where the last inequality follows from $\text{PoA}(I) \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}$. Since I is a critical instance of a symmetric k-uniform congestion game, and $\tilde{I}$ is an instance of a symmetric $(k-1)$-uniform congestion game, it must hold that $\text{PoA}(I) > \text{PoA}(\tilde{I})$. This contradicts our finding of $\text{PoA}(\tilde{I}) \geq \text{PoA}(I)$, so an instance like I cannot exist. □

Chapter 4

The PoA of symmetric k-uniform congestion games

In this section, the price of anarchy of symmetric k-uniform congestion games with cost functions of maximum degree d ($\mathcal{G}_k^d$) is studied. Before diving into technical details, an upper and lower bound can already be shown. As to the upper bound: $\mathrm{PoA}(\mathcal{G}_k^d)$ cannot exceed the price of anarchy of general atomic congestion games with cost functions of maximum degree d. The latter asymptotically grows as $\Theta(d/\log d)^{d+1}$ (Aland et al. [2011]), so $\mathrm{PoA}(\mathcal{G}_k^d)$ asymptotically grows as $O(d/\log d)^{d+1}$. As to the lower bound: $\mathrm{PoA}(\mathcal{G}_k^d)$ is larger than the price of anarchy of singleton congestion games with cost functions of maximum degree d, since $\mathcal{G}_1^d \subset \mathcal{G}_k^d$. The price of anarchy of singleton congestion games asymptotically grows as $\Theta(d/\log d)$ (Fotakis [2007]), so $\mathrm{PoA}(\mathcal{G}_k^d)$ asymptotically grows as $\Omega(d/\log d)$.

First, symmetric k-uniform congestion games with general k are considered. The upper and lower bound for the price of anarchy that are proven asymptotically grow as $\Theta(2^{d(d+1)})$ and $\Theta(2^d)$, respectively. This upper bound leaves room for improvement, since $2^{d(d+1)} \neq O(d/\log d)^{d+1}$. However, for small d it still improves upon the general bound and for $d = 1$ it improves upon the bound that was proved by de Jong et al. [2016]. We also present a conjecture that results in an improved upper bound that asymptotically grows as $\Theta(2^{d(d-1)})$.

Second, symmetric 2-uniform congestion games are considered. The upper and lower bound that are proven asymptotically grow as $\Theta(2^{d(d-1)})$ and $\Theta(2^d)$, respectively. Again, the upper bound cannot be tight for large d, since $2^{d(d-1)} \neq O(d/\log d)^{d+1}$.

4.1 Games with general k

4.1.1 Upper bound for the PoA

To prove an upper bound for the PoA of symmetric k-uniform congestion games, four lemmas are needed. Some of these lemmas hold for semi-symmetric k-uniform games as well.

The first lemma shows an insight into a factor that influences the PoA: the difference between the excess cost of resources in U in x^{OPT} and the excess cost resources in O in x^{NE}. The larger this difference, the larger the PoA. This difference appears explicitly in the lemma. The lemma is a generalisation of a similar lemma by Fotakis [2007].

Lemma 4.1. *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}_{\leq k}^d$ be an instance of a (semi-)symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then we obtain*

$$(1-\mu_d)C(x^{\mathrm{NE}}) \leq \nu_d\, C(x^{\mathrm{OPT}}) + \sum_{o\in O}\left(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}\right)c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}\left(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}\right)c_u(x_u^{\mathrm{NE}}+1)$$

where

$$\mu_d := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & d \le d^* \\ \frac{d}{d+1}\rho_d & d > d^* \end{cases}$$

$$\nu_d := \begin{cases} 1 & d \le d^* \\ \frac{1}{d+1}(\rho_d)^{-d} & d > d^* \end{cases}$$

$$\rho_d := \frac{2^{d/2}+1}{1+2^{d/2}-2^d+2^{3d/2}}$$

and $d^* \approx 1.6$ *is the unique solution to*

$$(d^*+1)^{-1/d^*} = \rho_{d^*}.$$

Proof. Let I, x^{NE} and x^{OPT} be as stated in the lemma. We will bound the cost of the underloaded resources, balanced resources and overloaded resources separately. Using those bounds, we can bound the social cost of x^{NE}.

For any $u \in U$, we will use

$$\begin{aligned} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) &= x_u^{\mathrm{OPT}} c_u(x_u^{\mathrm{OPT}}) - x_u^{\mathrm{OPT}} c_u(x_u^{\mathrm{NE}}) + x_u^{\mathrm{NE}} c_a(x_a^{\mathrm{NE}}) \\ &= x_u^{\mathrm{OPT}} c_u(x_u^{\mathrm{OPT}}) - x_u^{\mathrm{OPT}} c_u(x_u^{\mathrm{NE}}+1) + x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}+1) \\ &= x_u^{\mathrm{OPT}} c_u(x_u^{\mathrm{OPT}}) - (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) c_u(x_u^{\mathrm{NE}}+1). \end{aligned} \tag{4.1}$$

For any $b \in B$, we will use

$$x_b^{\mathrm{NE}} c_b(x_b^{\mathrm{NE}}) = x_b^{\mathrm{OPT}} c_b(x_b^{\mathrm{OPT}}). \tag{4.2}$$

For any $o \in O$, we use Lemma 3.5 to obtain

$$\begin{aligned} x_o^{\mathrm{OPT}} c_o(x_o^{\mathrm{NE}}) &= \alpha_o x_o^{\mathrm{OPT}} (x_o^{\mathrm{NE}})^d \\ &\le \alpha_o \lambda_d (x_o^{\mathrm{OPT}})^{d+1} + \alpha_o \mu_d (x_o^{\mathrm{NE}})^{d+1} \\ &= \lambda_d x_o^{\mathrm{OPT}} c_o(x_o^{\mathrm{OPT}}) + \mu_d x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}), \end{aligned}$$

where μ_d is as defined in the lemma and

$$\lambda_d := \frac{1}{(\mu_d)^d (d+1)} \Big(\frac{d}{d+1}\Big)^d.$$

The validity of the inequality for any $\mu_d > 0$ is proved in Appendix A.2. The particular value for μ_d is picked for optimisation purposes: it minimises the upper bound for the PoA. Now we obtain for any $o \in O$

$$x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) \le \lambda_d x_o^{\mathrm{OPT}} c_o(x_o^{\mathrm{OPT}}) + \mu_d x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) + (x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) c_o(x_o^{\mathrm{NE}}). \tag{4.3}$$

By combining inequalities (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned} C(x^{\mathrm{NE}}) &= \sum_{u\in U} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) + \sum_{b\in B} x_b^{\mathrm{NE}} c_b(x_b^{\mathrm{NE}}) + \sum_{o\in O} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) \\ &\le \sum_{r\in U\cup B} x_r^{\mathrm{OPT}} c_r(x_r^{\mathrm{OPT}}) + \lambda_d \sum_{o\in O} x_o^{\mathrm{OPT}} c_o(x_o^{\mathrm{OPT}}) + \mu_d \sum_{o\in O} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) \\ &\quad + \sum_{o\in O} (x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) c_u(x_u^{\mathrm{NE}}+1) \\ &\le \max\{1, \lambda_d\} C(x^{\mathrm{OPT}}) + \mu_d C(x^{\mathrm{NE}}) \\ &\quad + \sum_{o\in O} (x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) c_u(x_u^{\mathrm{NE}}+1). \end{aligned}$$

Rearranging this gives

$$\begin{aligned} (1-\mu_d) C(x^{\mathrm{NE}}) &\le \max\{1, \lambda_d\} C(x^{\mathrm{OPT}}) \\ &\quad + \sum_{o\in O} (x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) c_u(x_u^{\mathrm{NE}}+1). \end{aligned}$$

For $\mu_d = \frac{d}{d+1}(d+1)^{-1/d}$, we obtain $\lambda_d = 1$. For $\mu_d = \frac{d}{d+1}\rho_d$, we obtain $\lambda_d = (d+1)^{-1}(\rho_d)^{-d} \geq 1$ for $d \geq d^*$. In conclusion:

$$\max\{1, \lambda_d\} = \begin{cases} 1 & \text{if } \mu_d = \frac{d}{d+1}(d+1)^{-1/d} \\ \frac{1}{d+1}(\rho_d)^{-d} & \text{if } \mu_d = \frac{d}{d+1}\rho_d. \end{cases}$$

The uniqueness of d^* is proven in Appendix B. □

The next three lemmas are needed to bound

$$\sum_{o \in O} \left(x_o^{\text{NE}} - x_o^{\text{OPT}}\right) c_o(x_o^{\text{NE}}) - \sum_{u \in U} \left(x_u^{\text{OPT}} - x_u^{\text{NE}}\right) c_u(x_u^{\text{NE}} + 1)$$

from above. To that end, we consider an instance I of a (semi-)symmetric k-uniform congestion game, with $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Note that there may be resources $u \in U, o \in O$ with $c_o(x_o^{\text{NE}}) \leq c_u(x_u^{\text{NE}} + 1)$. If this holds for all such resources, then we obtain $\sum_{o \in O} \left(x_o^{\text{NE}} - x_o^{\text{OPT}}\right) c_o(x_o^{\text{NE}}) - \sum_{u \in U} \left(x_u^{\text{OPT}} - x_u^{\text{NE}}\right) c_u(x_u^{\text{NE}} + 1) \leq 0$ and we are done. For singleton games, this is indeed true, by the Nash inequality (Lemma 3.1). However, for (semi-)symmetric k-uniform games, this may not be the case. An example is the instance in Table 2.1, where $2 = c_3(x_3^{\text{NE}}) > c_2(x_2^{\text{NE}} + 1) = 1$.

In order to still use this information, we introduce variables $x'_r \in \mathbb{Z}_{\geq 0}$ for all $r \in U \cup O$ such that

$$\sum_{o \in O} x'_o c_o(x_o^{\text{NE}}) - \sum_{u \in U} x'_u c_u(x_u^{\text{NE}} + 1) \leq 0, \tag{4.4}$$

where $x'_u \leq x_u^{\text{OPT}} - x_u^{\text{NE}}$ for $u \in U$ and $x'_o \leq x_o^{\text{NE}} - x_o^{\text{OPT}}$ for $o \in O$. In particular, we want the values x'_r to satisfy

$$\sum_{o \in O} x'_o = \sum_{u \in U} x'_u \text{ , and}$$
$$c_o(x_o^{\text{NE}}) - c_u(x_u^{\text{NE}} + 1) > 0 \text{ for all } u \in U', o \in O'$$

where

$$U' := \{u \in U : x'_u < x_u^{\text{OPT}} - x_u^{\text{NE}}\},$$
$$O' := \{o \in O : x'_o < x_o^{\text{NE}} - x_o^{\text{OPT}}\}.$$

Such values x'_r can always be found, for example by using Algorithm 1.

Algorithm 1 Obtaining x'_r for $r \in U \cup O$

```
for r ∈ U ∪ O do
    x'_r = 0
end for
while ∃u ∈ U', o ∈ O' : c_o(x_o^NE) − c_u(x_u^NE + 1) ≤ 0 do
    x'_u = x'_u + 1
    x'_o = x'_o + 1
end while
```

For convenience, we introduce the variables $z_r := |x_r^{\text{OPT}} - x_r^{\text{NE}}| - x'_r$ for $r \in U \cup O$. If we interpret x_r as the part of $|x_r^{\text{NE}} - x_r^{\text{OPT}}|$ that can be bound using inequality (4.4), then we can interpret z_r as the 'leftover' part of $|x_r^{\text{NE}} - x_r^{\text{OPT}}|$ that we still need to deal with. This is shown in the next lemma.

Lemma 4.2. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ be an instance of a (semi-)symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Then*

$$\sum_{o \in O} \left(x_o^{\text{NE}} - x_o^{\text{OPT}}\right) c_o(x_o^{\text{NE}}) - \sum_{u \in U'} \left(x_u^{\text{OPT}} - x_u^{\text{NE}}\right) c_u(x_u^{\text{NE}} + 1) \leq \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}) - \sum_{u \in U'} z_u c_u(x_u^{\text{NE}} + 1).$$

Proof. By construction of x'_r, z_r, $r \in U \cup O$ and U', O', we obtain

$$
\begin{aligned}
&\sum_{o\in O}\Big(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}\Big)c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}\Big(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}\Big)c_u(x_u^{\mathrm{NE}}+1)\\
&- \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) + \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&= \sum_{o\in O}\Big(x'_o + z_o\Big)c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}\Big(x'_u + z_u\Big)c_u(x_u^{\mathrm{NE}}+1)\\
&- \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) + \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&= \sum_{o\in O} x'_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} x'_u c_u(x_u^{\mathrm{NE}}+1) + \sum_{o\in O} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&- \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) + \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&\le \sum_{o\in O} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U} z_u c_u(x_u^{\mathrm{NE}}+1) - \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) + \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&= \sum_{o\in O\setminus O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U\setminus U'} z_u c_u(x_u^{\mathrm{NE}}+1)\\
&= 0.
\end{aligned}
$$

□

Since $x_u^{\mathrm{NE}} \le n$ and $x_o^{\mathrm{NE}} \ge 1$ for all $u \in U, o \in O$, we can use Lemma 3.3 to bound $\sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1)$ from below with respect to $\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}})$. This results in the following lemma.

Lemma 4.3. *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}_k^d$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1) \le (1-2^{-d}) \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

Proof. Firstly, we know $\sum_{o\in O'} z_o = \sum_{u\in U'} z_u$, since

$$
\begin{aligned}
\sum_{o\in O'} z_o - \sum_{u\in U'} z_u &= \sum_{o\in O} z_o - \sum_{u\in U} z_u - \Big(\sum_{o\in O\setminus O'} z_o - \sum_{u\in U\setminus U'} z_u\Big)\\
&= \sum_{o\in O} z_o - \sum_{u\in U} z_u\\
&= \sum_{o\in O}(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}} - x'_o) - \sum_{u\in U}(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}} - x'_u)\\
&= \sum_{o\in O}(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) - \sum_{u\in U}(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) + \sum_{u\in U} x'_u - \sum_{o\in O} x'_o\\
&= 0,
\end{aligned}
$$

since $\sum_{o\in O}(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) = \sum_{u\in U}(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}})$ and $\sum_{u\in U} x'_u = \sum_{o\in O} x'_o$. Secondly, we know that the cost of each resource in U' can be bounded by the cost of a resource in O': we know that for all $o \in O'$ holds $x_o^{\mathrm{NE}} \ge x_o^{\mathrm{OPT}} + 1 \ge 1$ and for all $u \in U'$ holds $x_o^{\mathrm{NE}} \le x_o^{\mathrm{OPT}} - 1 < n$. Therefore, by Lemma 3.3 we obtain $c_u(x_u^{\mathrm{NE}}+1) \ge 2^{-d} c_o(x_o^{\mathrm{NE}})$ for all $o \in O'$, $u \in U'$. We can conclude

$$\sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1) \ge 2^{-d} \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

Finally, we obtain

$$\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1) \le \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - 2^{-d}\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) = (1-2^{-d})\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

□

Next, $\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}})$ can be bounded from above by some constant times the social cost $C(x^{\mathrm{NE}})$. To do this, an elaborate argument is needed. The social cost is split into the cost resulting from resources in U', resources in O' and resources in $R\backslash(U'\cup O')$. Then, these costs are all bounded from below by a constant times $\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}})$, using insights from the Nash inequality (Lemma 3.1) and from Lemma 3.3. The resulting lemma improves upon a similar lemma for games with affine cost functions by de Jong et al. [2016].

Lemma 4.4. *Let $I=(N,R,(c_r)_{r\in R},X)\in\mathcal{G}_k^d$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and $\mathrm{PoA}(I)\geq\frac{1}{1-d(d+1)^{-(d+1)/d}}$. Let $x^{\mathrm{NE}}\in\mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}}\in\mathrm{OPT}(I)$. Then*

$$\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}})\leq\frac{1-\rho_d}{1-2^{-d}}C(x^{\mathrm{NE}}).$$

Proof. Let I,x^{NE} and x^{OPT} be as in the lemma and let $\mathrm{PoA}(I)\geq\frac{1}{1-d(d+1)^{-(d+1)/d}}$. We start by observing

$$\begin{aligned}C(x^{\mathrm{NE}})&=\sum_{r\in R\backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}})+\sum_{o\in O'} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}})\\&=\sum_{r\in R\backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}})+\sum_{o\in O'}(z_o+x_o^{\mathrm{OPT}}+x_o')c_o(x_o^{\mathrm{NE}})\\&\geq\sum_{r\in R\backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}})+\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).\end{aligned}$$

Next, we will bound $\sum_{r\in R\backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}})$ in terms of $\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}})$. To this end, we use

$$\sum_{r\in R\backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}})=\sum_{u\in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}})+\sum_{r\in R\backslash(U'\cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}).\tag{4.5}$$

First, we bound the sum over $u\in U'$ in (4.5). We introduce p as the maximum number of resources in O' picked by a single player in x^{NE}, so

$$p:=\max_{i\in N}|\{r\in x_i^{\mathrm{NE}}:r\in O'\}|.$$

We now know that there are at least $\frac{1}{p}\sum_{o\in O'} x_o^{\mathrm{NE}}$ different players picking resources in O' in x^{NE}. By definition, $c_o(x_o^{\mathrm{NE}})>c_u(x_u^{\mathrm{NE}}+1)$ for all $o\in O',u\in U'$, so for the Nash inequality (Lemma 3.1) to hold, all players that pick a resource in O' must also use all resources in U' in x^{NE}. This gives

$$\begin{aligned}\sum_{u\in U'} x_u^{\mathrm{NE}}&\geq\frac{|U'|}{p}\sum_{o\in O'} x_o^{\mathrm{NE}}\\&=\frac{|U'|}{p}\sum_{o\in O'}(z_o+x_o^{\mathrm{OPT}}+x_o')\\&\geq\frac{|U'|}{p}\sum_{o\in O'} z_o.\end{aligned}$$

Note that this argumentation also implies $k\geq|U'|+p$. By Lemmas 3.4 and 3.3 we obtain $c_u(x_u^{\mathrm{NE}})=c_u(x_u^{\mathrm{NE}}+1)\geq 2^{-d}c_o(x_o^{\mathrm{NE}})$ for all $u\in U',o\in O'$. We can conclude

$$\sum_{u\in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}})\geq\frac{|U'|}{2^d p}\sum_{o\in O'} z_o c_o(x_u^{\mathrm{NE}}).\tag{4.6}$$

Next, we bound the sum over $r\in R\backslash(U'\cup O')$ in (4.5). We do this by arguing that there are players that cannot pick all their k resources in U' and O' in x^{NE}, so they must pick at least some resources in $R\backslash(U'\cup O')$ in x^{NE}.

Since the resources in U' are underloaded, there must be some set of players $N'\subseteq N$ and $q_i\in\{0,\ldots,|U'|-1\},q_i'\in\{1,\ldots,|U'|\}$ for all $i\in N'$ such that each player $i\in N'$ uses q_i resources in U' in x^{NE} and $q_i + q_i'$ resources in U' in x^{OPT}. In x^{NE}, each of these players uses k resources, and at most $|U'| - 1$ of those in U'. If such a player would use a resource in O', then they would have to use all resources in U', by the definition of U' and O' and by the Nash inequality (Lemma 3.1). Therefore, in x^{NE}, none of the players in N' use a resource in O'. We then obtain that all remaining resources are picked in $R \backslash (U' \cup O')$. Thus

$$\begin{aligned}\sum_{r \in R \backslash (U' \cup O')} x_r^{\mathrm{NE}} &\geq \sum_{i \in N'} (k - q_i) \\ &\geq \sum_{i \in N'} (k + q_i' - |U'|) \\ &= |N'|(k - |U'|) + \sum_{i \in N'} q_i',\end{aligned}$$

since $q_i + q_i' \leq |U'|$. Moreover, we can bound the number of players in N' by observing that each such player cannot pick a resource in U' multiple times. We then obtain

$$\begin{aligned}|N'| &\geq \frac{1}{|U'|} \sum_{u \in U'} (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) \\ &= \frac{1}{|U'|} \sum_{u \in U'} (z_u + x_u') \\ &\geq \frac{1}{|U'|} \sum_{u \in U'} z_u \\ &= \frac{1}{|U'|} \sum_{o \in O'} z_o,\end{aligned}$$

where the last equality was proved within the proof of Lemma 4.3. We thus obtain

$$\sum_{r \in R \backslash (U' \cup O')} x_r^{\mathrm{NE}} \geq \frac{k - |U'|}{|U'|} \sum_{o \in O'} z_o + \sum_{i \in N'} q_i'.$$

Next, we observe

$$\sum_{i \in N'} q_i' = \sum_{u \in U'} (x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}) = \sum_{u \in U'} (z_u + x_u') \geq \sum_{u \in U'} z_u = \sum_{o \in O'} z_o.$$

This gives us

$$\begin{aligned}\sum_{r \in R \backslash (U' \cup O')} x_r^{\mathrm{NE}} &\geq \frac{k - |U'|}{|U'|} \sum_{o \in O'} z_o + \sum_{o \in O'} z_o \\ &= \frac{k}{|U'|} \sum_{o \in O'} z_o \\ &\geq \frac{|U'| + p}{|U'|} \sum_{o \in O'} z_o \\ &= (1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o. \end{aligned} \tag{4.7}$$

By Lemmas 3.2 and 3.3 we obtain $c_r(x_r^{\mathrm{NE}}) \geq 2^{-d} c_r(x_r^{\mathrm{NE}} + 1) \geq 2^{-2d} c_o(x_o^{\mathrm{NE}})$ for all $r \in R, o \in O' :$ $x_r^{\mathrm{NE}} < n$. Since $\mathrm{PoA}(I) \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}$, we obtain by Lemma 3.6 that $x_r^{\mathrm{NE}} < n$ for all $r \in R$, so we can conclude

$$\sum_{r \in R \backslash (U' \cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) \geq 2^{-2d} (1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}). \tag{4.8}$$

By inequalities (4.5), (4.6) and (4.8), we obtain

$$\begin{aligned}\sum_{r \in R \backslash O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) &\geq \frac{|U'|}{2^d p} \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}) + 2^{-2d}(1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}) \\ &= \Big(\frac{|U'|}{2^d p} + \frac{1}{2^{2d}} + \frac{p}{2^{2d} |U'|} \Big) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}).\end{aligned}$$

This inequality could hold for any $p, |U'| \in \mathbf{Z}_{\geq 1}$, so we obtain

$$\begin{aligned}
\sum_{r \in R \setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) &\geq \min_{p, |U'| \in \mathbf{Z}_{\geq 1}} \left(\frac{|U'|}{2^d p} + \frac{1}{2^{2d}} + \frac{p}{2^{2d}|U'|} \right) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}) \\
&\geq \min_{z \in \mathbf{R}_{>0}} \left(\frac{z}{2^d} + \frac{1}{2^{2d}} + \frac{1}{2^{2d} z} \right) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}) \\
&= (2^{1-3d/2} + 2^{-2d}) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}),
\end{aligned}$$

where the minimum is attained at $z = 2^{-d/2}$. Finally we obtain

$$\begin{aligned}
C(x^{\mathrm{NE}}) &\geq \sum_{r \in R \setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) + \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \\
&\geq (1 + 2^{1-3d/2} + 2^{-2d}) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}),
\end{aligned}$$

or

$$\begin{aligned}
\sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}) &\leq \frac{1}{1 + 2^{1-3d/2} + 2^{-2d}} C(x^{\mathrm{NE}}) \\
&= \frac{1 - \rho_d}{1 - 2^{-d}} C(x^{\mathrm{NE}}),
\end{aligned}$$

where ρ_d is defined as in Lemma 4.1. □

We conclude with the main theorem.

Theorem 1. *For the class $\mathcal{G}_k^d$ of symmetric k-uniform congestion games with cost functions of maximum degree d holds*

$$\mathrm{PoA}(\mathcal{G}_k^d) \leq \frac{\nu_d}{\rho_d - \mu_d},$$

where ν_d, ρ_d and μ_d are defined as in Lemma 4.1.

Proof. Let $\mathcal{G}_k^d$ be the class of symmetric k-uniform congestion games with cost functions of maximum degree d and $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_k^d$. Now two cases are distinguished: $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$ and $\mathrm{PoA}(I) < \frac{1}{1-d(d+1)^{-(d+1)/d}}$. In the first case, we obtain an upper bound for $\mathrm{PoA}(I)$ that is larger than $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. This upper bound thus also holds in the second case. Therefore, it can be presented as the upper bound in all cases.

We assume $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Then

$$\begin{aligned}
(1 - \mu_d) C(x^{\mathrm{NE}}) &\leq \nu_d \, C(x^{\mathrm{OPT}}) + \sum_{o \in O} \left(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}} \right) c_o(x_o^{\mathrm{NE}}) - \sum_{u \in U} \left(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}} \right) c_u(x_u^{\mathrm{NE}} + 1) \\
&\leq \nu_d \, C(x^{\mathrm{OPT}}) + \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u \in U'} z_u c_u(x_u^{\mathrm{NE}} + 1) \\
&\leq \nu_d \, C(x^{\mathrm{OPT}}) + (1 - 2^{-d}) \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \\
&\leq \nu_d \, C(x^{\mathrm{OPT}}) + (1 - \rho_d) C(x^{\mathrm{NE}}),
\end{aligned}$$

where the first inequality follows from Lemma 4.1, the second from Lemma 4.2, the third from Lemma 4.3 and the fourth from Lemma 4.4. We obtain

$$\mathrm{PoA}(I) \leq \frac{\nu_d}{\rho_d - \mu_d}.$$

The proof of

$$\frac{\nu_d}{\rho_d - \mu_d} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}$$

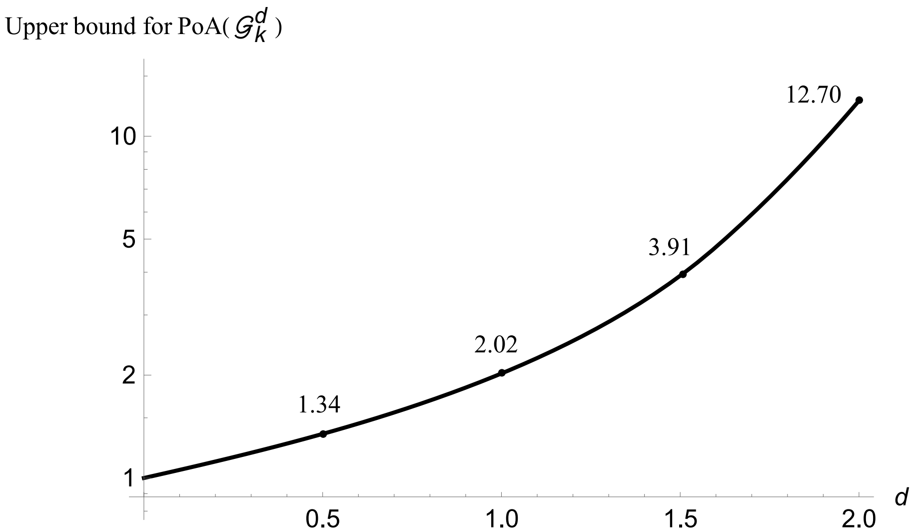

Figure 4.1: Upper bound for $\text{PoA}(\mathcal{G}_k^d)$

is given in Appendix A.4. Finally we obtain

$$\begin{aligned}\text{PoA}(\mathcal{G}_k^d) &= \max_{I \in \mathcal{G}_k^d} \text{PoA}(I) \\ &= \max_{I \in \mathcal{G}_k^d} \max_{x^{\text{NE}} \in \text{NE}(I)} \frac{C(x^{\text{NE}})}{C(x^{\text{OPT}})} \\ &\leq \frac{\nu_d}{1 - \mu_d - (1 - \rho_d)} \\ &= \frac{\nu_d}{\rho_d - \mu_d}.\end{aligned}$$

□

The upper bound presented in Theorem 1 asymptotically grows as $\Theta(2^{d(1+d)})$, which is proven in Appendix C. The bound is shown in Figure 4.1 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly.

4.1.2 Lower bound for the PoA

In this section, we present a lower bound for the PoA of symmetric k-uniform congestion games with cost functions of maximum degree d.

Theorem 2. *For the class $\mathcal{G}_k^d$ of symmetric k-uniform congestion games with cost functions of maximum degree d holds*

$$\text{PoA}(\mathcal{G}_k^d) \geq \frac{4 + \sqrt{2}(1 + 2^{d+1})}{4 + 3\sqrt{2}}.$$

Proof. For some $p, q \in \mathbb{N}$, we will construct an instance $I \in \mathcal{G}_{p+q}^d$ of a symmetric $(p+q)$-uniform congestion game with cost functions of maximum degree d and $\text{PoA}(I) \to \frac{4+\sqrt{2}(1+2^{d+1})}{4+3\sqrt{2}}$ if $p/q \to \sqrt{2}$.

Let the set of players be $M \cup N$, with $|M| = p$ and $|N| = 2q$. Let the set of resources be $R = U \cup V \cup W$, with $|U| = q$, $|V| = pq$ and $|W| = p(p+q)$. Resources $u \in U$ have cost $c_u(x) = 1$ and resources $r \in V \cup W$ have cost $c_r(x) = x^d$.

There is a Nash equilibrium x^{NE} that looks as follows. All players in N pick all q resources in U, and moreover pick p resources in V, each of those shared with one other player from N. All players in M pick $p+q$ resources in W, none of those shared. This results in $c_u(x_u^{\text{NE}}) = 1$ for all $u \in U$, $c_v(x_v^{\text{NE}}) = 2^d$ for all $v \in V$ and $c_w(x_w^{\text{NE}}) = 1$ for all $w \in W$. No player can improve their costs by deviating; the players in M already chose the cheapest resources (with cost 1) and the players in N

cannot choose more resources in U nor pick a better resource in $V \cup W$, since choosing another resource $v \in V$ would cost $c_v(x_u^{\text{NE}} + 1) = 3^d$ and choosing another resource $w \in W$ would cost $c_w(x_w^{\text{NE}} + 1) = 2^d$. The cost of this equilibrium is $2^d p + q$ for players in N and $p + q$ for players in M. This results in

$$C(x^{\text{NE}}) = p^2 + (2^{d+1} + 1)pq + 2q^2.$$

A system optimum solution x^{OPT} looks as follows. All players pick all q resources in U and moreover pick p resources in $V \cup W$, none of those shared. The cost of this optimum solution is $p + q$ for all players. This results in

$$C(x^{\text{OPT}}) = p^2 + 3pq + 2q^2.$$

This implies for the price of anarchy of symmetric k-uniform congestion games with cost functions of maximum degree d that

$$\begin{aligned}
\text{PoA}(\mathcal{G}_k^d) &\geq \sup_{p,q \in \mathbb{N}} \frac{p^2 + (2^{d+1} + 1)pq + 2q^2}{p^2 + 3pq + 2q^2} = \sup_{p,q \in \mathbb{N}} \frac{(p/q)^2 + (2^{d+1} + 1)p/q + 2}{(p/q)^2 + 3p/q + 2} \\
&= \max_{z \in \mathbb{R}_{\geq 0}} \frac{z^2 + (2^{d+1} + 1)z + 2}{z^2 + 3z + 2} = \frac{4 + \sqrt{2}(1 + 2^{d+1})}{4 + 3\sqrt{2}},
\end{aligned}$$

where the maximum is attained at $z = \sqrt{2}$. For $p, q \in \mathbb{N}$ we can let $\frac{p}{q}$ get as close to $\sqrt{2}$ as we want. □

A small example of the lower bound presented in Theorem 2, with $p = 2, q = 1$ and $d = 1$, is shown in Figure 4.2.

	Resources			
	$c_1(x) = 1$	$c_2(x) = x$	$c_3(x) = x$	$c_4(x) = x$
x^{OPT}	1 2 3 4	1	1	2
x^{NE}	1 2	1 2	1 2	3

	Resources				
	$c_5(x) = x$	$c_6(x) = x$	$c_7(x) = x$	$c_8(x) = x$	$c_9(x) = x$
x^{OPT}	2	3	3	4	4
x^{NE}	3	3	4	4	4

Figure 4.2: An instance as described in the proof of Theorem 2 with $p = 2, q = 1$ and $d = 1$. We have $U = \{1\}$, $V = \{2, 3\}$, $W = \{4, \ldots, 9\}$ and $N = \{1, 2\}$, $M = \{3, 4\}$.

The lower bound presented in Theorem 2 asymptotically grows as $\Theta(2^d)$, which is proven in Appendix C. The bound is shown in Figure 4.3 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly. For comparison, the upper bound as presented in Theorem 1 is shown in the plot as well.

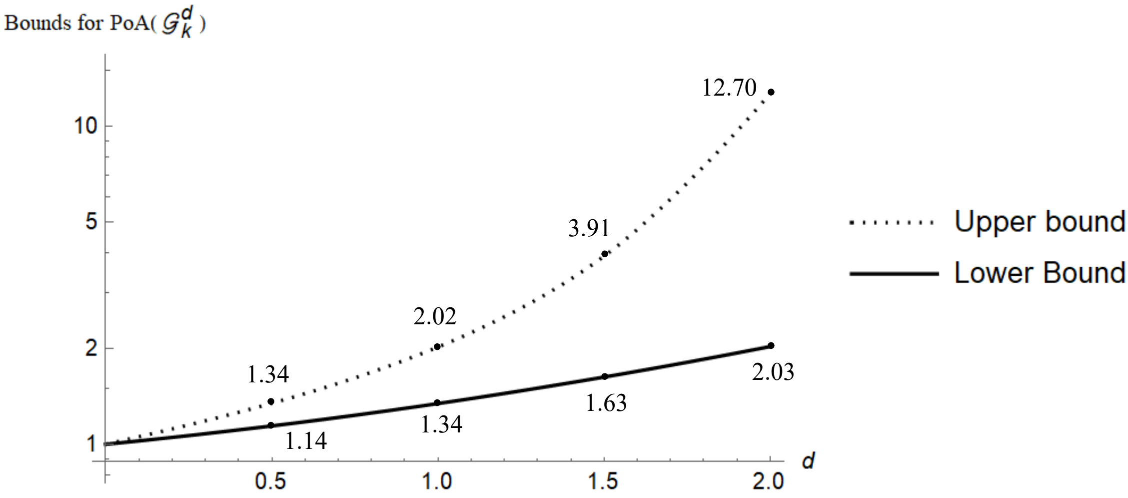

Figure 4.3: Lower bound for $\text{PoA}(\mathcal{G}_k^d)$

4.1.3 Conjecture

In this section, we present a conjecture for the cost of resources in a Nash equilibrium of a critical instance of a symmetric k-uniform congestion game. The conjecture is similar to Lemma 3.3, but only considers critical instances and is more powerful for these instances. If the conjecture is correct, then we obtain a better upper bound for $\text{PoA}(\mathcal{G}_k^d)$. For $d = 1$, we would obtain $\text{PoA}(\mathcal{G}_k^1) \leq 1.81$. First, we present the conjecture. Second, we present two intuitive arguments as to why the conjecture may hold. Third, we suggest a proof structure and fourth we present the consequences if the conjecture is proven to be correct.

Conjecture 4.1. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_k^d$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Then*

$$c_r(x_r^{\text{NE}}) \geq 2^{-d} c_o(x_o^{\text{NE}})$$

for all resources $r \in R, o \in O'$.

Recall that $O' \subseteq O$ contains the resources $o \in O$ with $x'_o < x_o^{\text{NE}} - x_o^{\text{OPT}}$.

We present two intuitive arguments as to why the conjecture may hold. Firstly, the price of anarchy of instances for which the conjecture does not hold seems to be relatively small, which suggests that such instances are never critical. To illustrate this, an instance I of a symmetric 3-uniform congestion game for which the conjecture does not hold is shown in Figure 4.4. In particular, resources $1 = r$ and $2 = o$ do not meet the conjecture. The price of anarchy of this instance is $\text{PoA}(I) = 13/11 \approx 1.18$. A different instance $\tilde{I}$ of a symmetric 3-uniform congestion game with the same number of resources and players for which the conjecture does hold is shown in Figure 4.5. The price of anarchy of this instance is $\text{PoA}(\tilde{I}) = 4/3 \approx 1.33$. Since $\text{PoA}(\tilde{I}) > \text{PoA}(I)$, instance I cannot be critical.

	Resources					
	$c_1(x) = x$	$c_2(x) = 4x$	$c_3(x) = 2$	$c_4(x) = 2x$	$c_5(x) = 2x$	$c_6(x) = 2x$
x^{OPT}	1		1 2	1	2	2
x^{NE}	1	1	1	2	2	2

Figure 4.4: An instance of a non-critical symmetric 3-uniform congestion game for which Conjecture 4.1 does not hold.

	Resources					
	$c_1(x) = x$	$c_2(x) = x$	$c_3(x) = x$	$c_4(x) = 2$	$c_5(x) = 2$	$c_6(x) = 2$
x^{OPT}	1	1	1	2	2	2
x^{NE}	1 2	1 2	1 2			

Figure 4.5: An instance of a symmetric 3-uniform congestion game for which Conjecture 4.1 holds.

Secondly, we already proved that the conjecture holds for symmetric 2-uniform congestion games in Lemma 4.5. The properties of symmetric 2-uniform congestion games may of course differ from the properties of symmetric k-uniform congestion games, but the fact that the conjecture holds for at least a subclass of symmetric k-uniform congestion games does support the conjecture.

Now we present a proof structure that may be used to prove the conjecture.

Suggested proof structure. By contradiction. Consider a critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}_k^d$ of a symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Assume there are resources $r \in R, o \in O'$ with $c_r(x^{\text{NE}}) < 2^{-d} c_o(x_o^{\text{NE}})$.

• Observe that $r \in B \cup O$ (where B is the set of balanced resources), because if $r \in U$ then

$$c_r(x_r^{\text{NE}}) = c_r(x_r^{\text{NE}} + 1) \geq 2^{-d} c_o(x_o^{\text{NE}}),$$

where the equality follows from Lemma 3.4 and the inequality follows from Lemma 3.3.

• It seems likely that $r \in B$, since this resource is relatively cheap. For the instance in Figure 4.4, this is indeed the case. Then one can make the argument that balanced resources decrease the price of anarchy of instances. Therefore, it is likely that there is an instance without such a balanced resource that has a larger price of anarchy. Note that the instance in Figure 4.5 indeed has no balanced resources. Alternatively, one can look at the following arguments.

• Since $o \in O'$, we know $x_o^{\text{NE}} \geq 1$. Let player $p \in N$ use resource o in x^{NE}, then this player must also use resource r, by the Nash inequality (Lemma 3.1):

$$c_r(x_r^{\text{NE}} + 1) \leq 2^d c_r(x_r^{\text{NE}}) < c_r(x_o^{\text{NE}}),$$

where the first inequality follows from Lemma 3.2.

• By Lemma 3.6, we know that there exists a player $q \in N$ not using resource r. Let this player use some resource $s \in R$. Then we obtain

$$c_s(x_s^{\text{NE}}) \leq c_r(x_r^{\text{NE}} + 1) \leq 2^d c_r(x_r^{\text{NE}}) < c_o(x_o^{\text{NE}}),$$

where the first inequality follows from the Nash inequality (Lemma 3.1) and the second inequality follows from Lemma 3.2. At least two such resources s_1, s_2 are not used by player p, since player p and player q use the same number of resources. For these resources must hold $s_i \in B \cup O, i = 1, 2$, since if $s_i \in U$ then

$$c_{s_i}(x_{s_i}^{\text{NE}} + 1) = c_{s_i}(x_{s_i}^{\text{NE}}) < c_o(x_o^{\text{NE}}),$$

where the equality follows from Lemma 3.4. By the Nash inequality (Lemma 3.1), player q would then also use resources s_1 and s_2.

• It seems likely that for all underloaded resources $u \in U$ holds

$$c_u(x_u^{\text{NE}}) < c_o(x_o^{\text{NE}}),$$

since it would not make sense that the resources that are used more often in the optimal strategy than in the Nash equilibrium are relatively expensive. It follows that player p uses all underloaded resources, by the Nash inequality (Lemma 3.1):

$$c_u(x_u^{\text{NE}} + 1) = c_u(x_u^{\text{NE}}) < c_o(x_o^{\text{NE}}),$$

where the equality follows from Lemma 3.4. Since there must be at least one underloaded resource, it follows that there must be at least one other resource $s_3 \in B \cup O$ used by player q and not by player p. If all these resources s_1, s_2, s_3 are overloaded, then it seems likely that there exists more than one underloaded resource. This would again result in more resources $s_i \in B \cup O$ used by player p and not by player q. Therefore, some resources $s_i \in B \cup O$ are probably balanced. This is indeed true for the instance in Figure 4.4, for the resources $4 = s_1, 5 = s_2, 6 = s_3$. Again, one can make the argument that balanced resources decrease the price of anarchy of instances. ◀

Lastly, we elaborate on the consequences of Conjecture 4.1, if it were true. The first consequence is an improvement of Lemma 4.4.

Consequence 4.1.1 (Conjecture 4.1)**.** *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_k^d$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \leq \frac{1}{1 + 3/2^d} C(x^{\mathrm{NE}}).$$

The proof is identical to the proof of Lemma 4.6, which states the same property but for the case $k = 2$, for which we are able to prove the validity of Conjecture 4.1. The second consequence is an upper bound for the PoA that improves upon the upper bound as stated in Theorem 1.

Consequence 4.1.2 (Conjecture 4.1)**.** *For the class $\mathcal{G}_k^d$ of symmetric k-uniform congestion games with cost functions of maximum degree d, we obtain*

$$\mathrm{PoA}(\mathcal{G}_k^d) \leq \frac{\hat{\nu}_d}{\hat{\rho}_d - \hat{\mu}_d}$$

where $\hat{\mu}_d$, $\hat{\nu}_d$ and $\hat{\rho}_d$ are as defined in Theorem 3.

Again, the proof is almost identical to the proof of Theorem 3, which states the same property but for the case $k = 2$. A visualisation of this result can therefore be found in Figure 4.6. For $d = 1$ we obtain $\mathrm{PoA}(\mathcal{G}_k^1) \leq 1.81$.

4.2 Games with $k = 2$

4.2.1 Upper bound for the PoA

For symmetric 2-uniform congestion games with cost functions of maximum degree d ($\mathcal{G}_2^d$), we can improve upon the upper bound for the PoA that was shown for general k. The improved upper bound is a result of the use of Lemma 4.5. This lemma states that only a limited number of resources $r \in R$ can have a cost $c_r(x_r^{\mathrm{NE}}) < 2^{-d} c_o(x_o^{\mathrm{NE}})$ for some resource $o \in O'$, otherwise players using o must use more than k resources.

Lemma 4.5. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq k}^d$ be an instance of a (semi-)symmetric k-uniform congestion game and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then there are at most $k-2$ resources $r \in R$ for which there exists a resource $o \in O'$ with $c_r(x_r^{\mathrm{NE}}) < 2^{-d} c_o(x_o^{\mathrm{NE}})$.*

Proof. The proof is by contradiction. Let I, x^{NE} and x^{OPT} be as in the lemma and assume there are $k-1$ distinct resources $r_1, \ldots, r_{k-1} \in R$ with $c_{r_i}(x_{r_i}^{\mathrm{NE}}) < 2^{-d} c_{o_i}(x_{o_i}^{\mathrm{NE}}), o_i \in O', 1 \leq i \leq k-1$. Let $O_{\max} = \{o \in \{o_1, \ldots, o_{k-1}\} : c_o(x_o^{\mathrm{NE}}) = \max_i c_{o_i}(x_{o_i}^{\mathrm{NE}})\}$ and let $o \in O_{\max}$. Then

$$c_{r_i}(x_{r_i}^{\mathrm{NE}} + 1) \leq 2^d c_{r_i}(x_{r_i}^{\mathrm{NE}}) < c_{o_i}(x_{o_i}^{\mathrm{NE}}) \leq c_o(x_o^{\mathrm{NE}}),$$

where the first inequality follows from Lemma 3.3. By the Nash inequality (Lemma 3.1), players using resource o must also use all resources $r_1, \ldots, r_{k-1}$ in x^{NE}.

Furthermore, since $o \in O'$ we know $c_u(x_u^{\mathrm{NE}} + 1) < c_o(x_o^{\mathrm{NE}})$ for all $u \in U'$. By the Nash inequality (Lemma 3.1), players using resource o must also use all resources $u \in U'$ in x^{NE}. We can assume that there exists at least one such resource $u \in U'$, because if $|U'| = 0$ then $|O'| = 0$ and the lemma would be trivial. Moreover,

$$c_u(x_u^{\mathrm{NE}}) = c_u(x_u^{\mathrm{NE}} + 1) \geq 2^{-d} c_o(x_o^{\mathrm{NE}}),$$

where the equality follows from Lemma 3.4 and the inequality follows from Lemma 3.3. Therefore, $u \neq r_i$ for $i = 1, \dots, k-1$.

In conclusion, players using o must also use resources $r_1, \dots, r_{k-1}$ and all resources in U', which results in the usage of at least $k+1$ resources. This is a contradiction. $\square$

For 2-uniform congestion games, Lemma 4.5 simplifies to

$$c_r(x_r^{\mathrm{NE}}) \geq 2^{-d} c_o(x_o^{\mathrm{NE}})$$

for all resources $r \in R, o \in O'$. This can be used to obtain a lemma that improves upon Lemma 4.4.

Lemma 4.6. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_2^d$ be a critical instance of a symmetric 2-uniform congestion game with cost functions of maximum degree d and $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \leq \frac{1}{1 + 3/2^d} C(x^{\mathrm{NE}}).$$

Proof. The proof is similar to the proof of Lemma 4.4. Again, we use

$$\begin{aligned}
C(x^{\mathrm{NE}}) &\geq \sum_{r \in R \setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) + \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}), \\
\sum_{r \in R \setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) &= \sum_{u \in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) + \sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}), \\
\sum_{u \in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) &\geq \frac{|U'|}{2^d p} \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}}), \text{ and} \\
\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} &\geq (1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o.
\end{aligned}$$

Different than in the proof of Lemma 4.4, we now can use Lemma 4.5 to obtain $c_r(x_r^{\mathrm{NE}}) \geq 2^{-d} c_o(x_o^{\mathrm{NE}})$ for all $r \in R \setminus (U' \cup O'), o \in O'$. This gives

$$\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) \geq (1 + \frac{p}{|U'|}) \frac{1}{2^d} \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

Finally, we can conclude

$$\begin{aligned}
C(x^{\mathrm{NE}}) &\geq \left(1 + \frac{|U'|}{2^d p} + (1 + \frac{p}{|U'|}) \frac{1}{2^d}\right) \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \\
&= \left(1 + \frac{1}{2^d}(1 + \frac{|U'|}{p} + \frac{p}{|U'|})\right) \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \\
&\geq \left(1 + \frac{3}{2^d}\right) \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}).
\end{aligned}$$

Rearranging this proves the lemma. $\square$

Next, Lemma 4.6 can be used to obtain an upper bound for the PoA of symmetric 2-uniform congestion games that is better than the upper bound for the PoA of symmetric k-uniform congestion games for general k.

Theorem 3. *For the class $\mathcal{G}_2^d$ of symmetric 2-uniform congestion games with cost functions of maximum degree d holds*

$$\mathrm{PoA}(\mathcal{G}_2^d) \leq \frac{\hat{\nu}_d}{\hat{\rho}_d - \hat{\mu}_d}$$

where

$$\hat{\mu}_d := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & d \leq \hat{d} \\ \frac{d}{d+1}\hat{\rho}_d & d > \hat{d} \end{cases}$$

$$\hat{\nu}_d := \begin{cases} 1 & d \leq \hat{d} \\ \frac{1}{d+1}(\hat{\rho}_d)^{-d} & d > \hat{d} \end{cases}$$

$$\hat{\rho}_d := \frac{4}{2^d + 3}$$

and $\hat{d} \approx 2.0$ *is the unique solution to*

$$(\hat{d}+1)^{-1/\hat{d}} = \hat{\rho}_{\hat{d}}.$$

Proof. Different than in the analysis of symmetric k-uniform congestion games with general k, the existence of a critical instance of a symmetric 2-uniform congestion game is not guaranteed. Therefore, two cases are distinguished. Case 1: there exists some critical instance $I \in \mathcal{G}_2^d$, so $\mathrm{PoA}(\mathcal{G}_2^d) > \mathrm{PoA}(\mathcal{G}_1^d)$. Case 2: there exists no critical instance $I \in \mathcal{G}_2^d$, so $\mathrm{PoA}(\mathcal{G}_2^d) \leq \mathrm{PoA}(\mathcal{G}_1^d)$. For case 1, we prove an upper bound for $\mathrm{PoA}(\mathcal{G}_2^d)$ that is larger than an upper bound for $\mathrm{PoA}(\mathcal{G}_1^d)$. Since this upper bound is also valid in case 2, it can be presented as the upper bound for $\mathrm{PoA}(\mathcal{G}_2^d)$ in all cases.

We thus assume that there exists some critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}_2^d$. In addition, as in the proof of Theorem 1, we assume that $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Then we obtain an upper bound for $\mathrm{PoA}(I)$ that is larger than $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. This upper bound thus also holds if $\mathrm{PoA}(I) < \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Therefore, it can be presented as the upper bound in all cases. As shown in the proof of Lemma 4.1, for any $\mu > 0$ holds

$$(1-\mu)C(x^{\mathrm{NE}}) \leq \max\{1,\lambda\}C(x^{\mathrm{OPT}}) + \sum_{o\in O}(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}})c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}})c_u(x_u^{\mathrm{NE}}+1),$$

where

$$\lambda_d := \frac{1}{(\mu_d)^d(d+1)}\Big(\frac{d}{d+1}\Big)^d.$$

Since the result of Theorem 1 is improved using the result of Lemma 4.6, there is a new value $\hat{\mu_d} \neq \mu_d$ that minimises the PoA.

We now obtain for all $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}_2^d$:

$$\begin{aligned}(1-\hat{\mu_d})C(x^{\mathrm{NE}}) &\leq \hat{\nu_d}C(x^{\mathrm{OPT}}) + \sum_{o\in O}(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}})c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}})c_u(x_u^{\mathrm{NE}}+1) \\ &\leq \hat{\nu_d}\,C(x^{\mathrm{OPT}}) + \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}) \\ &\leq \hat{\nu_d}\,C(x^{\mathrm{OPT}}) + (1-2^{-d})\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) \\ &\leq \hat{\nu_d}\,C(x^{\mathrm{OPT}}) + \frac{2^d-1}{2^d+3}C(x^{\mathrm{NE}}),\end{aligned}$$

where the first inequality follows from Lemma 4.1, the second from Lemma 4.2, the third from Lemma 4.3 and the fourth from Lemma 4.6. Finally we obtain

$$\begin{aligned}\mathrm{PoA}(\mathcal{G}_2^d) &= \max_{I\in\mathcal{G}_2^d}\mathrm{PoA}(I) \\ &= \max_{I\in\mathcal{G}_2^d}\max_{x^{\mathrm{NE}}\in\mathrm{NE}(I)}\frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} \\ &\leq \frac{\hat{\nu_d}}{\hat{\rho_d} - \hat{\mu_d}},\end{aligned}$$

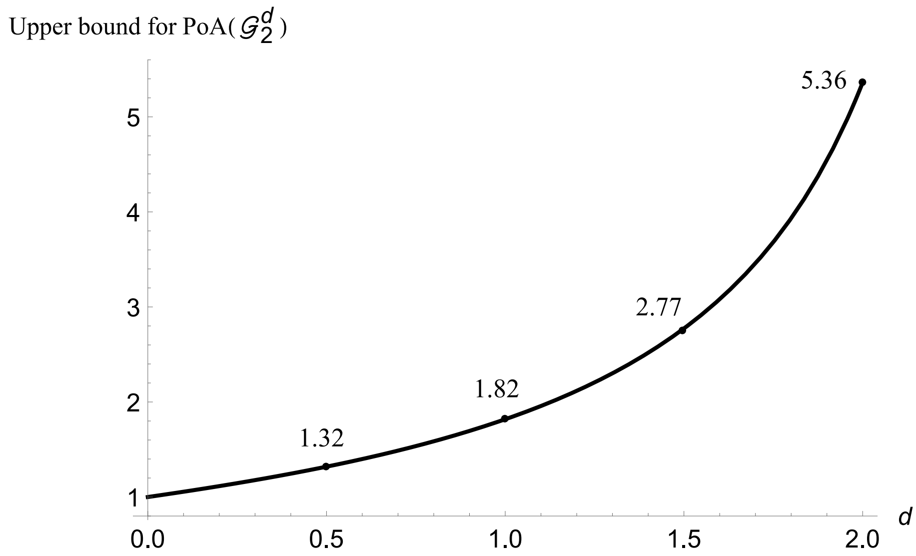

Figure 4.6: Upper bound for PoA($\mathcal{G}_2^d$).

where $\hat{\mu_d}, \hat{\nu_d}$ and $\hat{\rho_d}$ are defined as in the theorem.

Now, we prove that this upper bound exceeds an upper bound for PoA($\mathcal{G}_1^d$). To that end, we consider the upper bound that was proven by Fotakis [2007]:

$$\text{PoA}(\mathcal{G}_1^d) \leq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

In Appendix A.5 is proven that the general upper bound proven by Fotakis [2007] reduces to this form for the games that we consider. In Appendix A.4 is proven that

$$\frac{\hat{\nu_d}}{\hat{\rho_d} - \hat{\mu_d}} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Note that this is both a proof that the upper bound exceeds an upper bound for PoA($\mathcal{G}_1^d$) and that the upper bound exceeds $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. The uniqueness of $\hat{d}$ is proven in Appendix B. □

The upper bound presented in Theorem 3 asymptotically grows as $\Theta(2^{d(d-1)})$, which is proven in Appendix C. The bound is shown in Figure 4.6 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly.

4.2.2 Lower bound for the PoA

In this section, we present a lower bound for the PoA of symmetric 2-uniform congestion games with cost functions of maximum degree d.

Theorem 4. *For the class $\mathcal{G}_2^d$ of symmetric 2-uniform congestion games with cost functions of maximum degree d holds*

$$\text{PoA}(\mathcal{G}_2^d) \geq \frac{2 + 2^d}{3}.$$

Proof. We will construct an instance $I \in \mathcal{G}_2^d$ of a symmetric 2-uniform congestion game with cost functions of maximum degree d and $\text{PoA}(I) = \frac{2+2^d}{3}$. The game is explained in more detail next and is shown in Figure 4.7 for $d = 1$. Let the set of players be $N = \{1, 2, 3\}$ and the set of resources $R = \{1, 2, 3, 4\}$. Resource 1 has cost 1 and resources 2, 3 and 4 have cost $c_r(x) = x^d$, $r = 2, 3, 4$.

There is a Nash equilibrium x^{NE} that looks as follows: players 1 and 2 pick resources 1 and 2. Player 3 picks resources 3 and 4. This results in $c_1(x_1^{\text{NE}}) = c_3(x_3^{\text{NE}}) = c_4(x_4^{\text{NE}}) = 1$ and $c_2(x_2^{\text{NE}}) = 2^d$. No player can improve their costs by deviating, since $c_1(x_1^{\text{NE}}+1) = 1$ and $c_3(x_3^{\text{NE}}+1) = c_4(x_4^{\text{NE}}+1) = 2^d$. This results in

$$C(x^{\text{NE}}) = 4 + 2^{d+1}.$$

A system optimum solution x^{OPT} looks as follows: all players pick resource 1, player 1 picks resource 2, player 2 picks resource 3 and player 3 picks resource 4. This results in

$$C(x^{\text{OPT}}) = 6.$$

This implies for the price of anarchy of symmetric 2-uniform congestion games with cost functions of degree d that

$$\text{PoA}(\mathcal{G}_2^d) \geq \frac{4 + 2^{d+1}}{6} = \frac{2 + 2^d}{3}.$$

□

	Resources			
	$c_1(x) = 1$	$c_2(x) = x$	$c_3(x) = x$	$c_4(x) = x$
x^{OPT}	1 2 3	1	2	3
x^{NE}	1 2	1 2	3	3

Figure 4.7: The instance as described in the proof of Lemma 4 with $d = 1$.

The lower bound presented in Theorem 4 asymptotically grows as $\Theta(2^d)$, which is proven in Appendix C. The bound is shown in Figure 4.8 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly. For comparison, the upper bound as presented in Theorem 3 is shown in the plot as well.

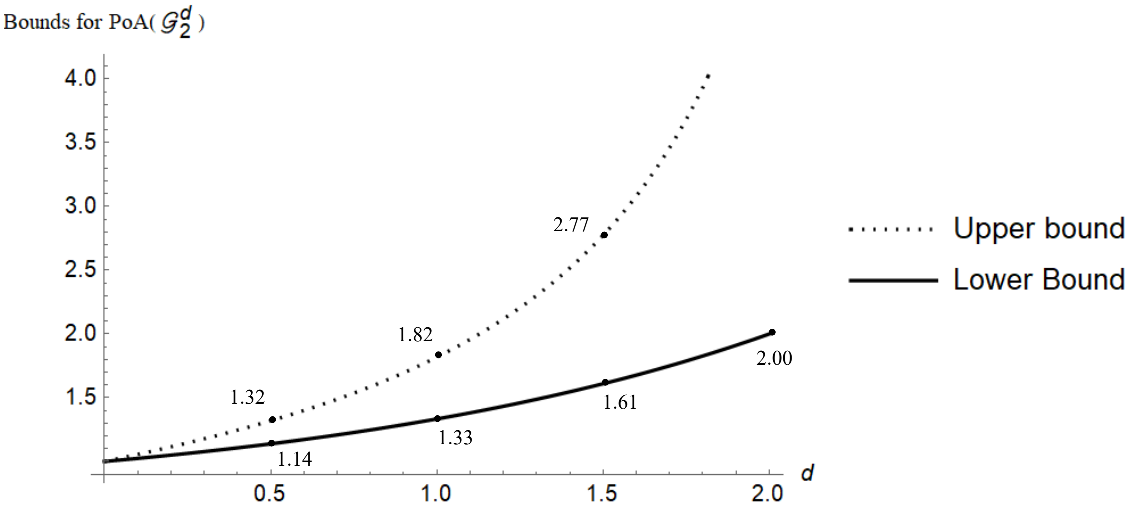

Figure 4.8: Lower bound for $\text{PoA}(\mathcal{G}_2^d)$.

It it worth mentioning that the lower bound presented in Theorem 4 with $d = 1$ is $4/3$, which equals the price of anarchy of singleton congestion games with $d = 1$. It is therefore tempting to suspect that the price of anarchy of symmetric 2-uniform congestion games equals the price of anarchy of singleton congestion games. However, the price of anarchy of singleton congestion asymptotically grows as $\Theta(d/\log d)$ and because of Theorem 4 we know that the price of anarchy of symmetric 2-uniform congestion games asymptotically grows as $\Omega(2^d)$. Since 2^d outgrows $d/\log d$, $\text{PoA}(\mathcal{G}_2^d) \neq \text{PoA}(\mathcal{G}_1^d)$ for large d. In particular, for $d > 1$, we obtain $\text{PoA}(\mathcal{G}_2^d) > \text{PoA}(\mathcal{G}_1^d)$. For the case $d = 2$ holds $\text{PoA}(\mathcal{G}_2^2) \geq 2$ and $\text{PoA}(\mathcal{G}_1^2) \leq 1.63$ (Fotakis [2007]).

Chapter 5

The PoA of semi-symmetric k-uniform congestion games

In this section, the price of anarchy of semi-symmetric k-uniform congestion games with cost functions of maximum degree d ($\mathcal{G}^d_{\leq k}$) is studied. To the best of our knowledge, this has not been studied before. The same trivial upper and lower bound that hold for the price of anarchy of symmetric k-uniform congestion games also hold for the price of anarchy of semi-symmetric k-uniform congestion games. As to the upper bound: $\text{PoA}(\mathcal{G}^d_{\leq k})$ cannot exceed the price of anarchy of general atomic congestion games with cost functions of maximum degree d. The latter asymptotically grows as $\Theta(d/\log d)^{d+1}$ (Aland et al. [2011]), so $\text{PoA}(\mathcal{G}^d_{\leq k})$ asymptotically grows as $O(d/\log d)^{d+1}$. As to the lower bound: $\text{PoA}(\mathcal{G}^d_{\leq k})$ is larger than the price of anarchy of singleton congestion games with cost functions of maximum degree d, since $\mathcal{G}^d_1 \subset \mathcal{G}^d_{\leq k}$. The price of anarchy of singleton congestion games asymptotically grows as $\Theta(d/\log d)$ (Fotakis [2007]), so $\text{PoA}(\mathcal{G}^d_{\leq k})$ asymptotically grows as $\Omega(d/\log d)$.

First, semi-symmetric k-uniform congestion games with general k are considered. We elaborate on the properties of these games and prove a lower bound for the price of anarchy that asymptotically grows as $\Theta(2^d)$. We present a conjecture that results in an upper bound that asymptotically grows as $\Theta(3^{d(d+1)})$. Moreover, we present a second conjecture that results in an upper bound that asymptotically grows as $\Theta(2^{d^2})$. Both upper bounds leave room for improvement, since $3^{d(d+1)} \neq O(d/\log d)^{d+1}$ and $2^{d^2} \neq O(d/\log d)^{d+1}$. However, for small enough d, both upper bounds improve upon the general bound.

Second, semi-symmetric 2-uniform congestion games are considered. The upper and lower bound that are proven asymptotically grow as $\Theta((2/3)^d 2^{d^2})$ and $\Theta(2^d)$, respectively. The upper bound is not tight, since $(2/3)^d 2^{d^2} \neq O(d/\log d)^{d+1}$. However, for small d it still improves upon the general bound.

5.1 Games with general k

5.1.1 Properties

There are some lemmas that we were not able to prove for semi-symmetric k-uniform congestion games. The most important among these are Lemmas 3.3 and 4.4: all other lemmas that do not hold for semi-symmetric k-uniform congestion games use these. In this section, we present a lemma that is similar to a part of Lemma 4.4. We have not proven an upper bound for the price of anarchy of semi-symmetric k-uniform congestion games with general k, since we have not proven a lemma similar to Lemma 3.3.

There are two reasons why we cannot prove that Lemma 4.4 holds for semi-symmetric k-uniform congestion games. Firstly, Lemma 3.3 is used. Secondly, inequality (4.7) used in the proof of Lemma 4.4 does not hold for semi-symmetric k-uniform congestion games. This inequality states for a critical instance $I \in \mathcal{G}^d_k$ of a symmetric k-uniform congestion game with $x^{\text{NE}} \in \text{NE}^*(I)$ and

$x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$ that

$$\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} \geq (1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o,$$

where

$$p := \max_{i \in N} |\{r \in x_i^{\mathrm{NE}} : r \in O'\}|.$$

A counterexample of this inequality for semi-symmetric k-uniform games is shown in Figure 5.1. For this instance holds $U' = \{1\}, O' = \{3\}, p = 1$ and thus $\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} = 1 < 2 = (1 + p/|U'|) \sum_{o \in O'} z_o$.

	Resources		
	$c_1(x) = 1$	$c_2(x) = x$	$c_3(x) = 2x$
x^{OPT}	1 2	2	
x^{NE}	2	1	2

Figure 5.1: An instance of a semi-symmetric 2-uniform congestion game for which inequality (4.7) does not hold.

We obtain an inequality similar to inequality (4.7) that does hold for semi-symmetric k-uniform congestion games.

Lemma 5.1. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}^d_{\leq k}$ be a a critical instance of a semi-symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} \geq \sum_{o \in O'} z_o.$$

Proof. Similarly as in the proof of Lemma 4.4, we use

$$\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} \geq \sum_{i \in N'} (k_i - q_i).$$

However, now that k_i depends on i, we can only conclude

$$\begin{aligned} \sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} &\geq \sum_{i \in N'} (k_i - q_i) \\ &\geq \sum_{i \in N'} q_i' \\ &\geq \sum_{o \in O'} z_o, \end{aligned}$$

where we used $q_i + q_i' \leq k_i$. For the proof of $\sum_{i \in N'} q_i' \geq \sum_{o \in O'} z_o$ we refer to the proof of Lemma 4.4. □

It is worth mentioning that we have not found an instance of a semi-symmetric k-uniform congestion game that does not meet the property of Lemma 4.4, even though we found a counterexample of an inequality in the proof. In Section 5.1.3 we present a conjecture that is related to this observation.

5.1.2 Lower bound for the PoA

In this section, we present a lower bound for the PoA of semi-symmetric k-uniform congestion games with cost functions of maximum degree d.

Theorem 5. *For the class $\mathcal{G}^d_{\leq k}$ of semi-symmetric k-uniform congestion game with cost functions of maximum degree d holds*

$$\mathrm{PoA}(\mathcal{G}^d_{\leq k}) \geq \frac{1+2^{d+1}}{3}.$$

Proof. For some $p, q \in \mathbb{N}$, we will construct an instance $I \in \mathcal{G}^d_{\leq p+q}$ of a semi-symmetric $(p+q)$-uniform congestion game with cost functions of maximum degree d and $\mathrm{PoA}(I) \to \frac{1+2^{d+1}}{3}$ if $p/q \to \infty$.

Let the set of players be $M \cup N$, with $|M| = p$ and $|N| = 2q$. The players in M pick q resources and the players in N pick $p+q$ resources. Let the set of resources be $R = U \cup V \cup W$, with $|U| = q$, $|V| = pq$ and $|W| = pq$. Resources $u \in U$ have cost $c_u(x) = 1$ and resources $r \in V \cup W$ have cost $c_r(x) = x^d$.

There is a Nash equilibrium x^{NE} that looks as follows. All players in N pick all q resources in U, and moreover pick p resources in V, each of those shared with one other player from N. All players in M pick q resources in W, none of those shared. No player can improve their costs by deviating; the players in M already chose the cheapest resources (with cost 1) and the players in N cannot choose more resources in U nor pick a better resource $r \in V \cup W$, since then $c_r(x_r^{\mathrm{NE}} + 1) \geq 2^d$. The cost of this equilibrium is $q + 2^{d+1}p$ for players in N and q for players in M. This results in

$$C(x^{\mathrm{NE}}) = 2q^2 + (2^{d+1} + 1)pq.$$

A system optimum solution x^{OPT} looks as follows. All players pick all q resources in U and all players in N pick p resources in $V \cup W$, none of those shared. The cost of this optimum solution is $p + q$ for players in N and q for players in M. This results in

$$C(x^{\mathrm{OPT}}) = 2q^2 + 3pq.$$

This implies for the price of anarchy of semi-symmetric k-uniform congestion games with cost functions of maximum degree d that

$$\mathrm{PoA}(\mathcal{G}^d_{\leq k}) \geq \sup_{p,q\in\mathbb{N}} \frac{2q^2 + (2^{d+1}+1)pq}{2q^2+3pq} = \sup_{p,q\in\mathbb{N}} \frac{2 + (2^{d+1}+1)p/q}{2+3p/q} = \frac{1+2^{d+1}}{3},$$

for $p/q \to \infty$. □

A small example of the lower bound presented in Theorem 5, with $p = 2, q = 1$ and $d = 1$, is shown in Figure 5.2.

	Resources				
	$c_1(x) = 1$	$c_2(x) = x$	$c_3(x) = x$	$c_4(x) = x$	$c_5(x) = x$
x^{OPT}	1 2 3 4	1	1	2	2
x^{NE}	1 2	1 2	1 2	3	4

Figure 5.2: An instance as described in the proof of Theorem 5 with $p = 2, q = 1$ and $d = 1$. We have $U = \{1\}$, $V = \{2, 3\}$, $W = \{4, 5\}$ and $N = \{1, 2\}$, $M = \{3, 4\}$.

The lower bound presented in Theorem 5 asymptotically grows as $\Theta(2^d)$, which is proven in Appendix C. The bound is shown in Figure 5.3 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly.

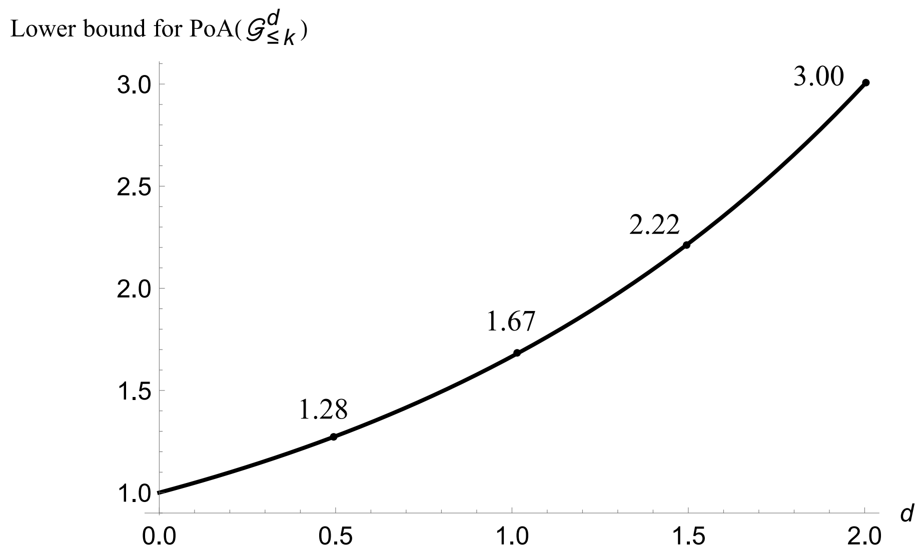

Figure 5.3: Lower bound for PoA($\mathcal{G}^d_{\leq k}$).

5.1.3 Conjectures

In this section, we present two conjectures for semi-symmetric k-uniform congestion games. The first is a conjecture for the cost of resources in a Nash equilibrium a semi-symmetric k-uniform congestion game. This conjecture is similar to Lemma 3.3. If the conjecture is correct, then we obtain a lemma similar to Lemma 4.3 for semi-symmetric k-uniform congestion games and an upper bound for PoA($\mathcal{G}^d_{\leq k}$) that is larger than the upper bound 5/2 for general congestion games as proved by Christodoulou and Koutsoupias [2005]. The second is a conjecture on the ratio of the cost of the overloaded resources and the social cost of a Nash equilibrium in a semi-symmetric k-uniform congestion game. This conjecture is similar to Lemma 4.4. It is a conjecture for both symmetric and semi-symmetric k-uniform congestion games, and we present it in this section because it improves more on the results that we have for semi-symmetric k-uniform congestion games than on the results that we have for symmetric k-uniform congestion games. If both conjectures were correct, then we could obtain a new upper bound for PoA($\mathcal{G}^d_{\leq k}$). For both conjectures, we suggest a proof structure and elaborate on the consequences if they were correct.

The first conjecture is similar to Lemma 3.3, which states that

$$c_r(x_r^{\mathrm{NE}}) \leq 2^d c_s(x_s^{\mathrm{NE}} + 1)$$

for every instance $I = (N, R, (c_r)_{r\in R}, X)$ of a symmetric k-uniform congestion game with cost functions of maximum degree d, $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x_r^{\mathrm{NE}} \geq 1, x_s^{\mathrm{NE}} < n$. This lemma does not hold for semi-symmetric k-uniform congestion games with $k \geq 3$: see the instance in Figure 5.4. In this instance we obtain $c_2(x_2^{\mathrm{NE}}) = 6 > 4 = 2c_3(x_3^{\mathrm{NE}} + 1)$. In Section 5.2.1, we prove that Lemma 3.3 does hold for instances of semi-symmetric 2-uniform congestion games.

	Resources			
	$c_1(x) = x$	$c_2(x) = 6x$	$c_3(x) = 2$	$c_4(x) = 3x$
x^{OPT}	2 3		1 2 3	2
x^{NE}	1 2	2	2 3	3

Figure 5.4: An instance of a semi-symmetric 3-uniform congestion game for which Lemma 3.3 does not hold.

Conjecture 5.1. *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$ be a critical instance of a semi-symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\text{NE}} \in \text{NE}^*(I)$. Then*

$$c_r(x_r^{\text{NE}}) \leq 3^d c_s(x_s^{\text{NE}} + 1)$$

for any $r, s \in R$ with $x_r^{\text{NE}} \geq 1$ and $x_s^{\text{NE}} < n$.

Now we present a proof structure that may be used to prove the conjecture.

Suggested proof structure. Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$ be an instance of a semi-symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Let $r^{\max}$ be the set of most expensive resources in x^{NE} that are used by at least one player:

$$r^{\max} := \{r \in R^{\geq 1} : c_r(x_r^{\text{NE}}) \geq c_{r'}(x_{r'}^{\text{NE}})\ \forall r' \in R^{\geq 1}\},$$

where $R^{\geq 1} := \{r \in R : x_r^{\text{NE}} \geq 1\}$. Similarly, let $r^{\min}$ be the set of least expensive resources in x^{NE} that are *not* used by at least one player:

$$r^{\min} := \{r \in R^{<n} : c_r(x_r^{\text{NE}}) \leq c_{r'}(x_{r'}^{\text{NE}})\ \forall r' \in R^{<n}\},$$

where $R^{<n} := \{r \in R : x_r^{\text{NE}} < n\}$. Now, let $r \in r^{\max}$ and let $s \in r^{\min}$. To prove the conjecture, one needs to prove that $c_r(x_r^{\text{NE}}) \leq 3^d c_s(x_s^{\text{NE}} + 1)$ for these resources r and s.

• Let $p \in N$ be a player that uses resource r and let $q \in N$ be a player that does not use resource s. If player q uses resource r, then we obtain

$$c_r(x_r^{\text{NE}}) \leq c_s(x_s^{\text{NE}} + 1),$$

by the Nash inequality (Lemma 3.1), and we are done. Otherwise, if player q uses some resource $v \in R$ not used by player p then we obtain

$$c_r(x_r^{\text{NE}}) \leq c_v(x_v^{\text{NE}} + 1) \leq 2^d c_v(x_v^{\text{NE}}) \leq 2^d c_s(x_s^{\text{NE}} + 1),$$

where the first inequality follows from the Nash inequality (Lemma 3.1) since player p uses resource r and not resource v, the second inequality follows from Lemma 3.2 and the third inequality follows from the Nash inequality (Lemma 3.1) since player q uses resource v and not resource s. In this case, the conjecture is proven as well.

• If player q does not use resource r and all resources that player q uses are also used by player p then the previous proof does not hold. In symmetric k-uniform congestion games this was not possible, since players p and q must use the same number of resources there. In semi-symmetric k-uniform congestion games, it may be possible for player q to pick a subset of the resources of player p: $x_q^{\text{NE}} \subseteq x_p^{\text{NE}} \backslash \{r\}$. This implies $x_v^{\text{NE}} \geq 2$.

• It seems likely that player p does not use all resources. Then there must be a resource $t \in R$ not used by player p. We obtain

$$c_r(x_r^{\text{NE}}) \leq c_t(x_t^{\text{NE}} + 1),$$

by the Nash inequality (Lemma 3.1). If resource t is not used by any player in x^{NE} then we obtain $t \in U$ and $c_t(x_t^{\text{OPT}}) \geq c_t(x_t^{\text{NE}} + 1) \geq c_r(x_r^{\text{NE}})$, which seems unlikely since resource r is relatively expensive. Therefore, resource t is probably used by some player $a \in N$ in x^{NE}.

• If player a uses resource v then this resource is used by at least 3 players. In all instances with the largest PoA that we have found, no resource is ever picked by more than 2 players in x^{NE}, both in symmetric and in semi-symmetric k-uniform congestion games. Therefore, we assume that player a does not use resource v. We obtain

$$c_r(x_r^{\text{NE}}) \leq c_t(x_t^{\text{NE}} + 1) \leq 2^d c_t(x_t^{\text{NE}}) \leq 2^d c_v(x_v^{\text{NE}} + 1) \leq 3^d c_v(x_v^{\text{NE}}) \leq 3^d c_s(x_s^{\text{NE}} + 1)$$

and we are done. The first inequality follows from the Nash inequality (Lemma 3.1), since player p uses resource r and not resource t. The second inequality follows from Lemma 3.2. The third inequality follows from the Nash inequality (Lemma 3.1), since player a uses resource t and not resource v. The fourth inequality follows from a property of resources used by at least two players that is similar to Lemma 3.2. We prove this property in Appendix A.3. The fifth inequality follows from the Nash inequality (Lemma 3.1), since player q uses resource v and not resource s. Note that an important cause of the appearance of the factor 3^d is that at least two players are using resource v. ◀

Lastly, we elaborate on three consequences of Conjecture 5.1, if it were true. The first consequence is a lemma similar to Lemma 4.3 for semi-symmetric k-uniform congestion games.

Consequence 5.1.1 (Conjecture 5.1). *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$ be a critical instance of a semi-symmetric k-uniform game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}} + 1) \leq (1 - 3^{-d}) \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

The proof this consequence is very similar to the proof of Lemma 4.3. The only difference is within the use of Conjecture 5.1 instead of Lemma 3.3. The second consequence is similar to Lemma 4.4 for semi-symmetric k-uniform congestion games.

Consequence 5.1.2 (Conjecture 5.1). *Let $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_k$ be a critical instance of a symmetric k-uniform congestion game with cost functions of maximum degree d and $\mathrm{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) \leq \frac{1}{1 + 3^{-2d}} C(x^{\mathrm{NE}}).$$

Proof. The proof is similar to the proof of Lemma 4.4. Again, we use

$$C(x^{\mathrm{NE}}) \geq \sum_{r\in R\setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) + \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}),$$

$$\sum_{r\in R\setminus O'} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) = \sum_{u\in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) + \sum_{r\in R\setminus(U'\cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}), \text{ and } \sum_{u\in U'} x_u^{\mathrm{NE}} \geq \frac{|U'|}{p} \sum_{o\in O'} z_o.$$

Different than in the proof of Lemma 4.4, we now use Conjecture 5.1 to obtain $c_u(x_u^{\mathrm{NE}}) \geq 3^{-d} c_u(x_u^{\mathrm{NE}} + 1) \geq 3^{-2d} c_o(x_o^{\mathrm{NE}})$ for all $u \in U', o \in O'$. This gives

$$\sum_{u\in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) \geq \frac{|U'|}{3^{3d} p} \sum_{o\in O'} z_o.$$

Next, Conjecture 5.1 is used to obtain $c_r(x_r^{\mathrm{NE}}) \geq 3^{-d} c_r(x_r^{\mathrm{NE}} + 1) \geq 3^{-2d} c_o(x_o^{\mathrm{NE}})$ for all $r \in R\setminus(U' \cup O'), o \in O'$. Combined with Lemma 5.1, we obtain

$$\sum_{r\in R\setminus(U'\cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) \geq \frac{1}{3^{2d}} \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}).$$

Finally, we can conclude

$$\begin{aligned} C(x^{\mathrm{NE}}) &\geq \Big(1 + \frac{|U'|}{3^{2d} p} + \frac{1}{3^{2d}}\Big) \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) \\ &= \Big(1 + \frac{1}{3^{2d}}\Big) \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}). \end{aligned}$$

Rearranging this proves the lemma. □

The third consequence is an upper bound for the price of anarchy.

Consequence 5.1.3 (Conjecture 5.1)**.** *For the class $\mathcal{G}^d_{\leq k}$ of semi-symmetric k-uniform congestion games with cost functions of maximum degree d holds*

$$\text{PoA}(\mathcal{G}^d_{\leq k}) \leq \frac{\nu''_d}{\rho''_d - \mu''_d},$$

where

$$\mu''_d = \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & d \leq d'' \\ \frac{d}{d+1}\rho''_d & d > d'' \end{cases}$$

$$\nu''_d = \begin{cases} 1 & d \leq d'' \\ \frac{1}{d+1}(\rho''_d)^{-d} & d > d'' \end{cases}$$

$$\rho''_d = \frac{1+3^d}{1+9^d}$$

and $d'' \approx 0.9$ is the unique solution to

$$(d''+1)^{-1/d''} = \rho'_{d''}.$$

Proof. Let $\mathcal{G}^d_{\leq k}$ be the class of semi-symmetric k-uniform congestion games with cost functions of maximum degree d and $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$. Now two cases are distinguished: $\text{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$ and $\text{PoA}(I) < \frac{1}{1-d(d+1)^{-(d+1)/d}}$. In the first case, we obtain an upper bound for $\text{PoA}(I)$ that is larger than $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. This upper bound thus also holds in the second case. Therefore, it can be presented as the upper bound in all cases.

We thus assume $\text{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Then,

$$\begin{aligned}(1-\mu''_d)C(x^{\text{NE}}) &\leq \nu''_d\, C(x^{\text{OPT}}) + \sum_{o\in O}\left(x^{\text{NE}}_o - x^{\text{OPT}}_o\right)c_o(x^{\text{NE}}_o) - \sum_{u\in U}\left(x^{\text{OPT}}_u - x^{\text{NE}}_u\right)c_u(x^{\text{NE}}_u+1) \\ &\leq \nu''_d\, C(x^{\text{OPT}}) + \sum_{o\in O'} z_o c_o(x^{\text{NE}}_o) - \sum_{u\in U'} z_u c_u(x^{\text{NE}}_u+1) \\ &\leq \nu''_d\, C(x^{\text{OPT}}) + (1-3^{-d})\sum_{o\in O'} z_o c_o(x^{\text{NE}}_o) \\ &\leq \nu''_d\, C(x^{\text{OPT}}) + \frac{1-3^{-d}}{1+3^{-2d}}C(x^{\text{NE}}) \\ &= \nu''_d\, C(x^{\text{OPT}}) + (1-\rho''_d)C(x^{\text{NE}}),\end{aligned}$$

where the first inequality follows from Lemma 4.1, the second from Lemma 4.2, the third from the first consequence of Conjecture 5.1 and the fourth from the second consequence of Conjecture 5.1. We obtain

$$\text{PoA}(I) \leq \frac{\nu''_d}{\rho''_d - \mu''_d}.$$

The proof of

$$\frac{\nu''_d}{\rho''_d - \mu''_d} \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$$

is given in Appendix A.4. Finally we obtain

$$\begin{aligned}\text{PoA}(\mathcal{G}^d_{\leq k}) &= \max_{I\in\mathcal{G}^d_{\leq k}} \text{PoA}(I) \\ &= \max_{I\in\mathcal{G}^d_{\leq k}} \max_{x^{\text{NE}}\in\text{NE}(I)} \frac{C(x^{\text{NE}})}{C(x^{\text{OPT}})} \\ &\leq \frac{\nu''_d}{1-\mu''_d-(1-\rho''_d)} \\ &= \frac{\nu''_d}{\rho''_d - \mu''_d}.\end{aligned}$$

The uniqueness of d'' is proven in Appendix B. □

The upper bound presented in the Consequence 5.1.3 asymptotically grows as $\Theta(3^{d(d+1)})$, which is proven in Appendix C.

The second conjecture improves upon Consequence 5.1.2 and is thus also similar to Lemma 4.4, which states that

$$\sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \leq \frac{1-\rho_d}{1-2^{-d}} C(x^{\mathrm{NE}})$$

for every instance $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_k^d$ of a symmetric k-uniform congestion game with cost functions of maximum degree d, $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. As mentioned in Section 5.1.1, we have not found an instance of a semi-symmetric k-uniform congestion game that does not meet the property of Lemma 4.4. In fact, all (semi-)symmetric k-uniform congestion games seem to meet an even stronger inequality.

Conjecture 5.2. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq k}^d$ be an instance of a (semi-)symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Then*

$$\sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) \leq \frac{1}{2^{1-d}+1} C(x^{\mathrm{NE}}).$$

We have observed that the property of the conjecture holds for all semi-symmetric k-uniform congestion games that we could find. An instance with $d = 1$ for which the inequality is tight is shown in Figure 5.5. For this instance holds $\sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}}) = 2 = \frac{1}{2} C(x^{\mathrm{NE}})$.

	Resources		
	$c_1(x) = 1$	$c_2(x) = x$	$c_3(x) = 2x$
x^{OPT}	1 2	2	
x^{NE}	2	1	2

Figure 5.5: An instance of a semi-symmetric 2-uniform congestion game for which Conjecture 5.2 is tight.

Now we present a proof structure that may be used to prove the conjecture.

Suggested proof structure. Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq k}^d$ be an instance of a semi-symmetric k-uniform congestion game with cost functions of maximum degree d and let $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$ and $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$.

• Let resource $o \in O'$ be used by player $p \in N$. By definition of O' and U', player p must also use all resources in U'. Let $u \in U'$. Since this resource is underloaded, there must be a player $q \in N$ not using resource u. Let this player use resource r_1. Then $r_1 \in R \backslash O'$. Moreover,

$$c_{r_1}(x_{r_1}^{\mathrm{NE}}) \leq c_u(x_u^{\mathrm{NE}} + 1).$$

If player p uses resource r_1 and there is no such other player q using some resource r_i that is not used by player p then player p uses relatively many resources in $R \backslash O'$, which makes the conjecture more likely to hold. Therefore, we assume that player p does not use resource r_1. (For symmetric k-uniform congestion games, this assumption is w.l.o.g.) We obtain

$$c_o(x_o^{\mathrm{NE}}) \leq c_{r_1}(x_{r_1}^{\mathrm{NE}} + 1) \leq 2^d c_{r_1}(x_{r_1}^{\mathrm{NE}}),$$

where the last inequality follows from Lemma 3.2.

• If we assume that resource u has constant cost, as is the case in symmetric k-uniform congestion games (Lemma 3.4), then we obtain

$$\begin{aligned}C(x^{\text{NE}}) &\geq c_u(x_u^{\text{NE}}) + c_{r_1}(x_{r_1}^{\text{NE}}) + c_o(x_o^{\text{NE}}) \\ &\geq 2c_{r_1}(x_{r_1}^{\text{NE}}) + c_o(x_o^{\text{NE}}) \\ &\geq \frac{2}{2^d} c_o(x_o^{\text{NE}}) + c_o(x_o^{\text{NE}}) \\ &= (1 + 2^{1-d}) c_o(x_o^{\text{NE}})\end{aligned}$$

and thus

$$c_o(x_o^{\text{NE}}) \leq \frac{1}{1 + 2^{1-d}} C(x^{\text{NE}}).$$

If $z_o = 1$ and $O' = \{o\}$ then we obtain

$$\sum_{o \in O'} z_o c_o(x_o^{\text{NE}}) = c_o(x_o^{\text{NE}}) \leq \frac{1}{1 + 2^{1-d}} c_o(x_o^{\text{NE}})$$

and the conjecture holds.

• It remains to analyse what happens if $z_o \geq 2$ or if $O' \supset \{o\}$. We start with analysing the first case. Assume $z_o = 2$, then $x_o^{\text{NE}} - x_o^{\text{OPT}} = z_o + x'_o \geq z_o$. Let players p_1 and p_2 pick resource o in x^{NE} and not in x^{OPT}. Then in x^{OPT}, these players must pick a resource different from resource o. If they pick the same resource r_1 and in x^{NE} this resource is still only picked by player q, then this results in this resource r_1 being underloaded. We again assume its cost is then constant, and we obtain

$$c_o(x_o^{\text{NE}}) \leq c_{r_1}(x_{r_1}^{\text{NE}} + 1) = c_{r_1}(x_{r_1}^{\text{NE}}) \leq c_u(x_u^{\text{NE}} + 1) = c_u(x_u^{\text{NE}}).$$

This is impossible, since for $o \in O'$ and $u \in U'$ must hold $c_o(x_o^{\text{NE}}) > c_u(x_u^{\text{NE}})$. Therefore, we know that either more players are picking resource r_1 in x^{NE} or players p_1 and p_2 are using different resources r_1 and r_2 in x^{OPT}. The first option seems to make the conjecture more likely to hold, so we assume the latter. Resource r_2 also cannot be underloaded, by the same argument, so there must be some player using this resource in x^{NE}. W.l.o.g. let $c_{r_1}(x_{r_1}^{\text{NE}}) \geq c_{r_2}(x_{r_2}^{\text{NE}})$. We obtain

$$\begin{aligned}C(x^{\text{NE}}) &\geq 2c_u(x_u^{\text{NE}}) + c_{r_1}(x_{r_1}^{\text{NE}}) + c_{r_2}(x_{r_2}^{\text{NE}}) + 2c_o(x_o^{\text{NE}}) \\ &\geq 2c_u(x_u^{\text{NE}}) + 2c_{r_1}(x_{r_1}^{\text{NE}}) + 2c_o(x_o^{\text{NE}}) \\ &\geq 4c_{r_1}(x_{r_1}^{\text{NE}}) + 2c_o(x_o^{\text{NE}}) \\ &= \frac{4}{2^d} c_o(x_o^{\text{NE}}) + 2c_o(x_o^{\text{NE}}) \\ &= 2(1 + 2^{1-d}) c_o(x_o^{\text{NE}})\end{aligned}$$

and thus

$$\sum_{o \in O'} z_o c_o(x_o^{\text{NE}}) = 2c_o(x_o^{\text{NE}}) \leq \frac{1}{1 + 2^{1-d}} c_o(x_o^{\text{NE}}).$$

• Now we analyse the second case. Assume $O' = \{o, o_2\}$. If another player $p_2 \neq p$ uses resource o_2 then similar arguments as in the case $z_o = 2$ apply. If player p uses resource o_2 then these arguments apply too. In this case it is trivial that resource r_2 must exist. ◀

Lastly, we elaborate on the consequence of Conjectures 5.1 and 5.2, if they were both true.

Consequence 5.2.1 (Conjectures 5.1 and 5.2)**.** *For the class $\mathcal{G}^d_{\leq k}$ of semi-symmetric k-uniform congestion games with cost functions of maximum degree d holds*

$$\text{PoA}(\mathcal{G}^d_{\leq k}) \leq \frac{\nu'_d}{\rho'_d - \mu'_d},$$

where

$$\mu'_d = \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & d \leq d' \\ \frac{d}{d+1}\rho'_d & d > d' \end{cases}$$

$$\nu'_d = \begin{cases} 1 & d \leq d' \\ \frac{1}{d+1}(\rho'_d)^{-d} & d > d' \end{cases}$$

$$\rho'_d = \frac{2^{1-d} + 3^{-d}}{1 + 2^{1-d}}$$

and $d' \approx 1.5$ *is the unique solution to*

$$(d'+1)^{-1/d'} = \rho'_{d'}.$$

Proof. Let $\mathcal{G}^d_{\leq k}$ be the class of semi-symmetric k-uniform congestion games with cost functions of degree d. Then we obtain for all instances $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq k}$:

$$\begin{aligned}(1-\mu'_d)C(x^{\mathrm{NE}}) &\leq \nu'_d\, C(x^{\mathrm{OPT}}) + \sum_{o\in O}\left(x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}\right)c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U}\left(x_u^{\mathrm{OPT}} - x_u^{\mathrm{NE}}\right)c_u(x_u^{\mathrm{NE}}+1) \\ &\leq \nu'_d\, C(x^{\mathrm{OPT}}) + \sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) - \sum_{u\in U'} z_u c_u(x_u^{\mathrm{NE}}+1) \\ &\leq \nu'_d\, C(x^{\mathrm{OPT}}) + (1-3^{-d})\sum_{o\in O'} z_o c_o(x_o^{\mathrm{NE}}) \\ &\leq \nu'_d\, C(x^{\mathrm{OPT}}) + \frac{1-3^{-d}}{1+2^{1-d}}C(x^{\mathrm{NE}}),\end{aligned}$$

where the first inequality follows from Lemma 4.1, the second from Lemma 4.2, the third from Conjecture 5.1 and the fourth from Conjecture 5.2. Finally we obtain

$$\begin{aligned}\mathrm{PoA}(\mathcal{G}^d_{\leq k}) &= \max_{I\in\mathcal{G}^d_{\leq k}} \mathrm{PoA}(I) \\ &= \max_{I\in\mathcal{G}^d_{\leq k}} \max_{x^{\mathrm{NE}}\in\mathrm{NE}(I)} \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} \\ &\leq \frac{\nu'_d}{1-\mu'_d-(1-\rho'_d)} \\ &= \frac{\nu'_d}{\rho'_d - \mu'_d}.\end{aligned}$$

The uniqueness of d' is proven in Appendix B. □

The upper bound presented in Consequence 5.2.1 asymptotically grows as $\Theta(2^{d^2})$, which is proven in Appendix C.

5.2 Games with $k = 2$

5.2.1 Upper bound for the PoA

In this section, we present an upper bound for the PoA of semi-symmetric 2-uniform congestion games. We start with proving the general Lemma 3.3 for these games. It follows that Lemmas 3.4, 3.6 and 4.3 also hold for these games. Then, we use Lemma 5.1 to formulate a lemma similar to Lemma 4.4. Finally, we present the upper bound.

First, we prove that Lemma 3.3 holds for semi-symmetric 2-uniform congestion games.

Lemma 5.2. *Let* $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq 2}$ *be an instance of a semi-symmetric* 2*-uniform congestion game with cost functions of maximum degree* d *and let* $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$*. Then,*

$$c_r(x_r^{\mathrm{NE}}) \leq 2^d c_s(x_s^{\mathrm{NE}}+1)$$

for any $r, s \in R$ *with* $x_r^{\mathrm{NE}} \geq 1$ *and* $x_s^{\mathrm{NE}} < n$.

Proof. By contradiction. Let I and x^{NE} be as in the lemma and assume there are some resources $r, s \in R$ with $x_r^{\text{NE}} \geq 1$ and $x_s^{\text{NE}} < n$ and $c_r(x_r^{\text{NE}}) > 2^d c_s(x_s^{\text{NE}} + 1)$. Let player p use resource r in x^{NE}. By the Nash inequality (Lemma 3.1), player p must also use resource s in x^{NE}.

There must be another player q not using resource s. Let player q use some resource $v \in R \backslash \{r, s\}$. Then $c_v(x_v^{\text{NE}}) \leq c_s(x_s^{\text{NE}} + 1)$, by the Nash inequality (Lemma 3.1). We obtain

$$c_v(x_v^{\text{NE}} + 1) \leq 2^d c_v(x_v^{\text{NE}}) \leq 2^d c_s(x_s^{\text{NE}} + 1) < c_r(x_r^{\text{NE}}),$$

where the first inequality follows from Lemma 3.2. By the Nash inequality (Lemma 3.1), players using resource r must also use resource v. Therefore, player p uses at least 3 resources, which is not possible. □

Combining Lemmas 5.1 and 5.2 allows for a lemma similar to Lemma 4.4.

Lemma 5.3. *Let $I = (N, R, (c_r)_{r \in R}, X) \in \mathcal{G}_{\leq 2}^d$ be an instance of a semi-symmetric 2-uniform congestion game with cost functions of maximum degree d and $\text{PoA}(I) \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}$. Let $x^{\text{NE}} \in \text{NE}^*(I)$ and $x^{\text{OPT}} \in \text{OPT}(I)$. Then*

$$\sum_{o \in O'} z_o c_o(x_o^{\text{NE}}) \leq \frac{1}{1 + 2^{1-d}} C(x^{\text{NE}}).$$

Proof. The proof is similar to the proof of Lemma 4.4. Again, we use

$$C(x^{\text{NE}}) \geq \sum_{r \in R \backslash O'} x_r^{\text{NE}} c_r(x_r^{\text{NE}}) + \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}),$$

$$\sum_{r \in R \backslash O'} x_r^{\text{NE}} c_r(x_r^{\text{NE}}) \geq \sum_{u \in U'} x_u^{\text{NE}} c_u(x_u^{\text{NE}}) + \sum_{r \in R \backslash (U' \cup O')} x_r^{\text{NE}} c_r(x_r^{\text{NE}}), \text{ and}$$

$$\sum_{u \in U'} x_u^{\text{NE}} c_u(x_u^{\text{NE}}) \geq \frac{|U'|}{2^d p} \sum_{o \in O'} z_o c_o(x_u^{\text{NE}}).$$

Different than in the proof of Lemma 4.4, we now use Lemmas 4.5 and 5.1 to obtain

$$\sum_{r \in R \backslash (U' \cup O')} x_r^{\text{NE}} c_r(x_r^{\text{NE}}) \geq \frac{1}{2^d} \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}).$$

We then obtain

$$C(x^{\text{NE}}) \geq \left(1 + \frac{|U'|}{2^d p} + \frac{1}{2^d}\right) \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}).$$

Next, we look into the possible values of $|U'|$ and p in a semi-symmetric 2-uniform congestion game. We start with the possible values of $|U'|$. By the definition of U' and O' and by the Nash inequality (Lemma 3.1), players using a resource $o \in O'$ must use all resources $u \in U'$. We can assume $|O'| \geq 1$ and thus $|U'| \geq 1$, otherwise the lemma is trivial. If $|U'| \geq 2$ then players using a resource $o \in O'$ must use at least 3 resources, which is impossible. Therefore, $|U'| = 1$. Now we look into possible values of p. Recall that

$$p := \max_{i \in N} |\{r \in x_i^{\text{NE}} : r \in O'\}|.$$

Since we can assume $|O'| \geq 1$, we obtain $p \geq 1$. If $p \geq 2$ then the player using p resources in O' must use at least 3 resources, since they also use all resources in $|U'|$. This is impossible, so $p = 1$. Finally, we can conclude

$$\begin{aligned} C(x^{\text{NE}}) &\geq \left(1 + \frac{2}{2^d}\right) \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}) \\ &= (1 + 2^{1-d}) \sum_{o \in O'} z_o c_o(x_o^{\text{NE}}). \end{aligned}$$

Rearranging this proves the lemma. □

Finally, we can present an upper bound for the PoA of semi-symmetric 2-uniform congestion games.

Theorem 6. *For the class $\mathcal{G}^d_{\leq 2}$ of semi-symmetric 2-uniform congestion games with cost functions of maximum degree d holds*

$$\text{PoA}(\mathcal{G}^d_{\leq 2}) \leq \frac{\tilde{\nu_d}}{\tilde{\rho_d} - \tilde{\mu_d}},$$

where

$$\tilde{\mu}_d := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & d \leq \tilde{d} \\ \frac{d}{d+1}\tilde{\rho}_d & d > \tilde{d} \end{cases}$$

$$\tilde{\nu}_d := \begin{cases} 1 & d \leq \tilde{d} \\ \frac{1}{d+1}(\tilde{\rho}_d)^{-d} & d > \tilde{d} \end{cases}$$

$$\tilde{\rho}_d := \frac{3}{2^d+2}$$

and $\tilde{d} \approx 1.7$ is the unique solution to

$$(\tilde{d}+1)^{-1/\tilde{d}} = \tilde{\rho}_{\tilde{d}}.$$

Proof. Different than in the analysis of semi-symmetric k-uniform congestion games with general k, the existence of a critical instance of a semi-symmetric 2-uniform congestion game is not guaranteed. For $d \geq 0.7$, the lower bound for $\text{PoA}(\mathcal{G}^d_{\leq 2})$ as presented in Lemma 7 exceeds an upper bound for $\text{PoA}(\mathcal{G}^d_{\leq 1})$, namely the bound presented by Fotakis [2007] for $\text{PoA}(\mathcal{G}^d_1)$. Note that $\mathcal{G}^d_{\leq 1} = \mathcal{G}^d_1$. Therefore, for $d \geq 0.7$, critical instances of semi-symmetric 2-uniform congestion games with cost functions of maximum degree d are proven to exist. Since such an argument cannot be made for $d < 0.7$, it is required to distinguish two cases. Case 1: there exists some critical instance $I \in \mathcal{G}^d_{\leq 2}$, so $\text{PoA}(\mathcal{G}^d_{\leq 2}) > \text{PoA}(\mathcal{G}^d_1)$. Case 2: there exists no critical instance $I \in \mathcal{G}^d_{\leq 2}$, so $\text{PoA}(\mathcal{G}^d_{\leq 2}) \leq \text{PoA}(\mathcal{G}^d_1)$. For case 1, we prove an upper bound for $\text{PoA}(\mathcal{G}^d_{\leq 2})$ that is larger than an upper bound for $\text{PoA}(\mathcal{G}^d_1)$. Since this upper bound is also valid in case 2, we can present it as the upper bound for $\text{PoA}(\mathcal{G}^d_{\leq 2})$ in all cases.

We thus assume that there exists some critical instance $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq 2}$. In addition, as in the proof of Theorem 1, we assume that $\text{PoA}(I) \geq \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Then we obtain an upper bound for $\text{PoA}(I)$ that is larger than $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. This upper bound thus also holds if $\text{PoA}(I) < \frac{1}{1-d(d+1)^{-(d+1)/d}}$. Therefore, it can be presented as the upper bound in all cases. As shown in the proof of Lemma 4.1, for any $\mu > 0$ holds

$$(1-\mu)C(x^{\text{NE}}) \leq \max\{1, \lambda\}C(x^{\text{OPT}}) + \sum_{o\in O}(x^{\text{NE}}_o - x^{\text{OPT}}_o)c_o(x^{\text{NE}}_o) - \sum_{u\in U}(x^{\text{OPT}}_u - x^{\text{NE}}_u)c_u(x^{\text{NE}}_u + 1),$$

for any $\mu > 0$, where

$$\lambda_d := \frac{1}{(\mu_d)^d(d+1)}\Big(\frac{d}{d+1}\Big)^d.$$

Since Lemma 5.1 has to be used instead of Lemma 4.4 and Lemma 4.5 is used, there is a new value $\tilde{\mu_d} \neq \mu_d$ that minimises the PoA.

We now obtain for all $I = (N, R, (c_r)_{r\in R}, X) \in \mathcal{G}^d_{\leq 2}$:

$$\begin{aligned}(1-\tilde{\mu_d})C(x^{\text{NE}}) &\leq \tilde{\nu_d}C(x^{\text{OPT}}) + \sum_{o\in O}(x^{\text{NE}}_o - x^{\text{OPT}}_o)c_o(x^{\text{NE}}_o) - \sum_{u\in U}(x^{\text{OPT}}_u - x^{\text{NE}}_u)c_u(x^{\text{NE}}_u + 1)\\
&\leq \tilde{\nu_d}\,C(x^{\text{OPT}}) + \sum_{o\in O'} z_o c_o(x^{\text{NE}}_o) - \sum_{u\in U'} z_u c_u(x^{\text{NE}}_u + 1)\\
&\leq \tilde{\nu_d}\,C(x^{\text{OPT}}) + (1-2^{-d})\sum_{o\in O'} z_o c_o(x^{\text{NE}}_o)\\
&\leq \tilde{\nu_d}\,C(x^{\text{OPT}}) + \frac{2^d-1}{2^d+2}C(x^{\text{NE}}),\end{aligned}$$

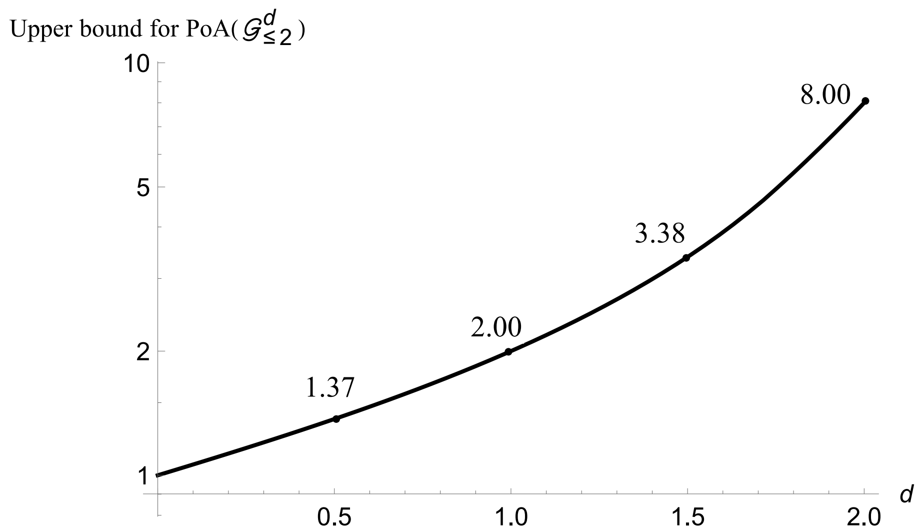

Figure 5.6: Upper bound for $\mathrm{PoA}(\mathcal{G}^d_{\leq 2})$

where the first inequality follows from Lemma 4.1, the second from Lemma 4.2, the third from Lemma 4.3 (which we can use because of Lemma 5.2) and the fourth from Lemma 5.3. Finally we obtain

$$\begin{aligned}\mathrm{PoA}(\mathcal{G}^d_{\leq 2}) &= \max_{I \in \mathcal{G}^d_2} \mathrm{PoA}(I) \\ &= \max_{I \in \mathcal{G}^d_{\leq k}} \max_{x^{\mathrm{NE}} \in \mathrm{NE}(I)} \frac{C(x^{\mathrm{NE}})}{C(x^{\mathrm{OPT}})} \\ &\leq \frac{\tilde{\nu_d}}{\tilde{\rho_d} - \tilde{\mu_d}},\end{aligned}$$

where $\tilde{\mu_d}, \tilde{\nu_d}$ and $\tilde{\rho_d}$ are defined as in the theorem.

Now, we prove that this upper bound exceeds an upper bound for $\mathrm{PoA}(\mathcal{G}^d_1)$. To that end, we consider the upper bound that was proven by Fotakis [2007]:

$$\mathrm{PoA}(\mathcal{G}^d_1) \leq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

In Appendix A.5 is proven that the general upper bound proven by Fotakis [2007] reduces to this form for the games that we consider. In Appendix A.4 is proven that

$$\frac{\tilde{\nu_d}}{\tilde{\rho_d} - \tilde{\mu_d}} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Note that this is both a proof that the upper bound exceeds an upper bound for $\mathrm{PoA}(\mathcal{G}^d_1)$ and that the upper bound exceeds $\frac{1}{1-d(d+1)^{-(d+1)/d}}$. The uniqueness of $\tilde{d}$ is proven in Appendix B. □

The upper bound presented in Theorem 3 asymptotically grows as $\Theta((2/3)^d 2^{d^2})$, which is proven in Appendix C. The bound is shown in Figure 5.6 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly.

5.2.2 Lower bound for the PoA

In this section, we present a lower bound for the PoA of semi-symmetric 2-uniform congestion games with cost functions of maximum degree d.

Theorem 7. *For the class $\mathcal{G}^d_{\leq 2}$ of semi-symmetric 2-uniform congestion games with cost functions of maximum degree d holds*

$$\mathrm{PoA}(\mathcal{G}^d_{\leq 2}) \geq \frac{3 + 2^{d+1}}{5}.$$

Proof. We will construct an instance $I \in \mathcal{G}^d_{\leq 2}$ of a semi-symmetric 2-uniform congestion game with cost functions of maximum degree d and $\text{PoA}(I) = \frac{3+2^{d+1}}{5}$. The instance is shown in Figure 5.7 and is explained in more detail next. Let the set of players be $N = \{1,2,3\}$ and the set of resources $R = \{1,2,3\}$. Resources 1 and 2 have cost $c_r(x) = x^d$, $r = 1,2$, and resource 3 has cost $c_3(x) = 1$.

There is a Nash equilibrium x^{NE} that looks as follows: players 1 and 2 pick resources 1 and 3 and player 3 picks resource 2. This results in $c_1(x_1^{\text{NE}}) = 2^d, c_2(x_2^{\text{NE}}) = 1, c_3(x_3^{\text{NE}}) = 1$. No player can improve their costs by deviating, since $c_1(x_1^{\text{NE}}+1) = 3^d$ and $c_2(x_2^{\text{NE}}+1) = 2^d$. This results in

$$C(x^{\text{NE}}) = 3 + 2^{d+1}.$$

A system optimum solution x^{OPT} looks as follows: all players pick resource 3, player 1 picks resource 1 and player 2 picks resource 2. This results in

$$C(x^{\text{OPT}}) = 5.$$

This implies for the price of anarchy of semi-symmetric 2-uniform congestion games with cost functions of maximum degree d that

$$\text{PoA}(\mathcal{G}^d_{\leq 2}) \geq \frac{3+2^{d+1}}{5}.$$

□

	Resources		
	$c_1(x) = x$	$c_2(x) = x$	$c_3(x) = 1$
x^{OPT}	1	2	1 2 3
x^{NE}	1 2	3	1 2

Figure 5.7: The instance as described in the proof of Lemma 7 with $d = 1$.

The lower bound presented in Theorem 7 asymptotically grows as $\Theta(2^d)$, which is proven in Appendix C. The bound is shown in Figure 5.8 for $0 \leq d \leq 2$. For some values of d, the approximate result is given explicitly. For comparison, the upper bound as presented in Theorem 6 is shown in the plot as well.

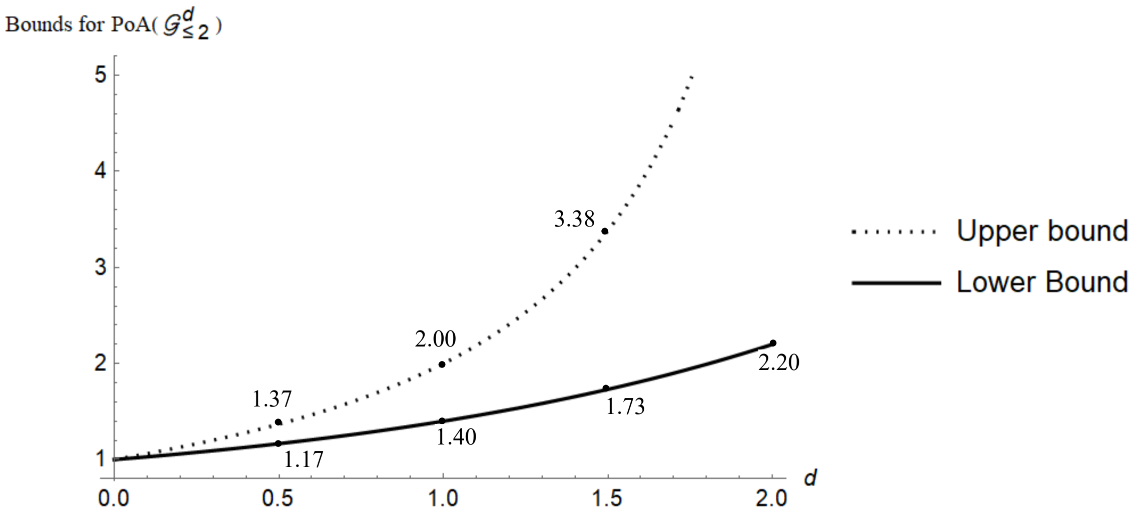

Figure 5.8: Lower bound for $\text{PoA}(\mathcal{G}^d_{\leq 2})$

Chapter 6

Conclusion and discussion

In this research, the price of anarchy of symmetric and semi-symmetric k-uniform congestion games was studied, both for games with general k and for games with $k = 2$. For the price of anarchy of symmetric k-uniform congestion games with cost functions of maximum degree d, we obtained an upper bound that improves upon the result for general congestion games by Aland et al. [2011] if d is not too large. For $d = 1$, this bound improves upon the result by de Jong et al. [2016] as well. Our results are improved for games with $k = 2$. For the price of anarchy of semi-symmetric k-uniform congestion games, we could not prove an upper bound. For games with $k = 2$, we did obtain an upper bound that improves upon the result for general congestion games by Aland et al. [2011] if d is not too large.

None of the upper bounds that we proved match the corresponding lower bounds, which means that the exact price of anarchy of any of the games that were analysed remain unknown. As mentioned at the start of Chapters 4 and 5, the upper bounds that we obtained asymptotically outgrow the result for general congestion games, so for large d the upper bounds cannot be tight. For all d, we can prove that no instance of a symmetric or semi-symmetric k-uniform congestion game with the price of anarchy presented as upper bounds in Theorems 1, 3 and 6 can exist (Lemma 6.1). This suggests that the upper bounds are not tight for small d either, as is also suggested in an unpublished paper by de Jong et al. [2017]. This paper contains a proof (which contains some bugs) of an upper bound of 1.41 for the price of anarchy of symmetric k-uniform congestion games with affine cost functions, which is significantly lower than our bound of 2.02. The conjectures that we present result in improved upper bounds, but they are not equal to the corresponding lower bounds.

The proof below can be modified for Theorems 3 and 6.

Lemma 6.1. *No instance $I \in \mathcal{G}_k^d$ of a symmetric k-uniform congestion games with a price of anarchy as presented in Theorem 1 can exist.*

Proof. The upper bound in Theorem 1 is proved using Lemmas 4.1, 4.3 and 4.4. We first show some requirements for these lemmas to be tight. Then, we show where these requirements contradict.

Let $I = (N, R, (c_r)_{r \in R}, X)$ and $x^{\mathrm{NE}} \in \mathrm{NE}^*(I)$, $x^{\mathrm{OPT}} \in \mathrm{OPT}(I)$. Lemma 4.1 can only be tight if the following holds:

- $\sum_{o \in O} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) = \lambda_d \sum_{o \in O} x_o^{\mathrm{OPT}} c_o(x_o^{\mathrm{NE}}) + \mu_d \sum_{o \in O} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) + \sum_{o \in O} (x_o^{\mathrm{NE}} - x_o^{\mathrm{OPT}}) c_o(x_o^{\mathrm{NE}})$
- $\sum_{o \in O} x_o^{\mathrm{NE}} c_o(x_o^{\mathrm{NE}}) = C(x^{\mathrm{NE}})$,

which leads to the following requirements:

1. $\forall o \in O : x_o^{\mathrm{NE}} = (d+1)^{1/d} x_o^{\mathrm{OPT}}$
2. $\forall r \in U \cup B : x_r^{\mathrm{NE}} = 0$.

Lemma 4.3 can only be tight if the following holds:

- $\sum_{u \in U'} z_u c_u(x_u^{\mathrm{NE}} + 1) = 2^{-d} \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}})$,

which leads to the following requirement:

1. $\forall u \in U', o \in O' : c_u(x_u^{\mathrm{NE}} + 1) = 2^{-d} c_o(x_o^{\mathrm{NE}})$.

Lemma 4.4 can only be tight if the following holds:

- $\sum_{o \in O'} (x_o^{\mathrm{OPT}} + x'_o) c_o(x_o^{\mathrm{NE}}) = 0$,
- $\sum_{u \in U'} x_u^{\mathrm{NE}} c_u(x_u^{\mathrm{NE}}) = \frac{|U'|}{2^d p} \sum_{o \in O'} z_o c_o(x_o^{\mathrm{NE}})$
- $\sum_{r \in R \setminus (U' \cup O')} x_r^{\mathrm{NE}} c_r(x_r^{\mathrm{NE}}) = 2^{-2d}(1 + \frac{p}{|U'|}) \sum_{o \in O'} z_o c_o(x_u^{\mathrm{NE}})$

which leads to the following requirements:

1. $\forall o \in O' : x_o^{\mathrm{OPT}} = x'_o = 0$,
2. $\forall u \in U', o \in O' : c_u(x_u^{\mathrm{NE}}) = 2^{-d} c_o(x_o^{\mathrm{NE}})$, and
3. $\forall r \in R \setminus (U' \cup O'), o \in O' : c_r(x_r^{\mathrm{NE}}) = 2^{-2d} c_o(x_o^{\mathrm{NE}})$.

The instance I cannot meet all these requirements. In particular, 3 contradictions occur.

Firstly, if $\forall o \in O : x_o^{\mathrm{NE}} = (d+1)^{1/d} x_o^{\mathrm{OPT}}$ holds, as is required for tightness of Lemma 4.1, then we obtain $x_o^{\mathrm{OPT}} \geq 1$ for all $o \in O$, so the first requirement for Lemma 4.4 does not hold.

Secondly, if $\forall r \in U \cup B : x_r^{\mathrm{NE}} = 0$ holds, as is required for tightness of Lemma 4.1, then we obtain $c_u(x_u^{\mathrm{NE}} + 1) \geq c_o(x_o^{\mathrm{NE}})$ for all $u \in U, o \in O$ by the Nash inequality (Lemma 3.1). This results in an upper bound for the price of anarchy of

$$\frac{1}{1 - d(d+1)^{-(d+1)/d}},$$

which is significantly lower than the upper bound presented in Theorem 1 (as proven in Appendix A.4).

Thirdly, if $\forall u \in U', o \in O'$ holds $c_u(x_u^{\mathrm{NE}} + 1) = 2^{-d} c_o(x_o^{\mathrm{NE}})$, as is required for tightness of Lemmas 4.3 and 4.4, and $\forall r \in R \setminus (U' \cup O')$ holds $c_r(x_r^{\mathrm{NE}}) = 2^{-2d} c_o(x_o^{\mathrm{NE}})$, as is required for tightness of Lemma 4.4, then all players using resources in O' must use all resources in U', by the Nash inequality (Lemma 3.1). In addition, these players must also use all resources $r \in R \setminus (U' \cup O')$, again by the Nash inequality (Lemma 3.1):

$$c_r(x_r^{\mathrm{NE}} + 1) \leq 2^d c_r(x_r^{\mathrm{NE}}) = 2^{-d} c_o(x_o^{\mathrm{NE}}),$$

where the first inequality follows from Lemma 3.2. Players using resources in O' thus use all resources in $U' \cup (R \setminus (U' \cup O')) = R \setminus O'$, so they use at least $1 + |R \setminus O'|$ resources. Therefore,

$$k \geq 1 + |R \setminus O'|.$$

Since every player picks k resources, every player must pick at least one resource in O'. In conclusion, every player picks all resources in $R \setminus O'$. This contradicts with Lemma 3.6. Note that this contradiction does not occur if $O' = \emptyset$, but in that case we obtain $c_u(x_u^{\mathrm{NE}} + 1) \geq c_o(x_o^{\mathrm{NE}})$ for all $u \in U, o \in O$, and the second contradiction occurs again. □

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Appendix A

Omitted proofs

A.1 Addition to the proof of Lemma 3.6

In this section, we prove the inequality

$$x_r^{\mathrm{OPT}} c_r(x_r^{\mathrm{OPT}}) + (n - x_r^{\mathrm{OPT}}) c_r(n) \geq (1 - d(d+1)^{-(d+1)/d}) n c_r(n).$$

Proof. Firstly, we know $c_r(x) = \alpha_r x^d$ for some $\alpha_r \in \mathbb{R}_{\geq 0}$ by Lemma 3.5. If $\alpha_r = 0$ then the inequality that we want to prove holds. Therefore, it remains to prove that the inequality holds for $\alpha_r > 0$. We know

$$\begin{aligned}
& x_r^{\mathrm{OPT}} c_r(x_r^{\mathrm{OPT}}) + (x_r^{\mathrm{NE}} - x_r^{\mathrm{OPT}}) c_r(x_r^{\mathrm{NE}}) - (1 - d(d+1)^{-(d+1)/d}) n c_r(n) \geq 0 \\
\iff & \alpha_r \Big((x_r^{\mathrm{OPT}})^{d+1} + n^{d+1} - x_r^{\mathrm{OPT}} n^d - (1 - d(d+1)^{-(d+1)/d}) n^{d+1} \Big) \geq 0 \\
\iff & (x_r^{\mathrm{OPT}})^{d+1} + n^{d+1} - x_r^{\mathrm{OPT}} n^d - (1 - d(d+1)^{-(d+1)/d}) n^{d+1} \geq 0
\end{aligned}$$

for all x_r^{OPT}. We will prove that the final inequality holds. To that end, let $z = x_r^{\mathrm{OPT}}/n$, which is well-defined, since $n > 0$. Then

$$\begin{aligned}
& (x_r^{\mathrm{OPT}})^{d+1} + n^{d+1} - x_r^{\mathrm{OPT}} n^d - (1 - d(d+1)^{-(d+1)/d}) n^{d+1} \geq 0 \\
\iff & z^{d+1} + 1 - z - (1 - d(d+1)^{-(d+1)/d}) \geq 0 \\
\iff & z^{d+1} - z + d(d+1)^{-(d+1)/d} \geq 0.
\end{aligned}$$

This final inequality is true, since

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^{d+1} - z + d(d+1)^{-(d+1)/d}] = 0 \iff z = \frac{1}{(d+1)^{1/d}}$$

and

$$\begin{aligned}
z^{d+1} - z + d(d+1)^{-(d+1)/d}|_{z=(d+1)^{-1/d}} &= 0 \\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}[z^{d+1} - z + d(d+1)^{-(d+1)/d}]|_{z=(d+1)^{-1/d}} &= d(d+1)^{1/d} > 0.
\end{aligned}$$

□

A.2 Addition to the proof of Lemma 4.1

In this section, the inequality

$$x_o^{\mathrm{OPT}} (x_o^{\mathrm{NE}})^d \leq \lambda_d (x_o^{\mathrm{OPT}})^{d+1} + \mu_d (x_o^{\mathrm{NE}})^{d+1}$$

is proven for any $\mu_d > 0$, where

$$\lambda_d := \frac{1}{(\mu_d)^d (d+1)} \Big(\frac{d}{d+1} \Big)^d.$$

Proof. Let $z = (x_o^{\mathrm{OPT}})/(x_o^{\mathrm{NE}})$, which is well-defined, since $x_o^{\mathrm{NE}} \geq x_o^{\mathrm{OPT}} + 1 \geq 1$. Then

$$\begin{aligned} & x_o^{\mathrm{OPT}}(x_o^{\mathrm{NE}})^d - \lambda_d (x_o^{\mathrm{OPT}})^{d+1} - \mu_d (x_o^{\mathrm{NE}})^{d+1} \leq 0 \\ \Longleftrightarrow\ & z - \lambda_d z^{d+1} - \mu_d \leq 0. \end{aligned}$$

The final inequality is true, since

$$\frac{\mathrm{d}}{\mathrm{d}z}[z - \lambda_d z^{d+1} - \mu_d] = 0 \Longleftrightarrow z = ((d+1)\lambda_d)^{-1/d}$$

and

$$\begin{aligned} z - \lambda_d z^{d+1} - \mu_d \,|_{z=((d+1)\lambda_d)^{-1/d}} &= 0 \\ \frac{\mathrm{d}^2}{\mathrm{d}z^2}[z - \lambda_d z^{d+1} - \mu_d] \,|_{z=((d+1)\lambda_d)^{-1/d}} &= \frac{-d}{\sqrt[d]{\lambda_d(d+1)}} < 0. \end{aligned}$$

□

A.3 Addition to the proof of Conjecture 5.1

In this section, we prove the inequality

$$c_v(x_v + 1) \leq \left(\frac{3}{2}\right)^d c_v(x_v)$$

for any cost function $c_v(x)$ of maximum degree d and $x_v \in \mathbb{N}_{\geq 2}$.

Proof. Let $c_v(x)$ and x_v be as presented before. By definition, the cost function is of the form

$$c_v(x_v) = \alpha_1 x_v^{d_1} + \ldots + \alpha_p x_v^{d_p},$$

for some $p \in \mathbb{Z}_{\geq 0}$, $\alpha_i \in \mathbb{R}_{\geq 0}$ and $d_i \in [0, d]$ for $i = 1, \ldots, p$. We then obtain

$$\begin{aligned} c_v(x_v+1) - \left(\frac{3}{2}\right)^d c_v(x_v) &= \alpha_1 \Big((x_v+1)^{d_1} - \left(\frac{3}{2}\right)^d x_v^{d_1} \Big) + \ldots + \alpha_p \Big((x_v+1)^{d_p} - \left(\frac{3}{2}\right)^d x_v^{d_p} \Big) \\ &\leq \alpha_1 \Big((x_v+1)^{d_1} - \left(\frac{3}{2}\right)^{d_1} x_v^{d_1} \Big) + \ldots + \alpha_p \Big((x_v+1)^{d_p} - \left(\frac{3}{2}\right)^{d_p} x_v^{d_p} \Big) \\ &\leq \alpha_1 \Big((\frac{3}{2}x_v)^{d_1} - \left(\frac{3}{2}\right)^{d_1} x_v^{d_1} \Big) + \ldots + \alpha_p \Big((\frac{3}{2}x_v)^{d_p} - \left(\frac{3}{2}\right)^{d_p} x_v^{d_p} \Big) \\ &= 0, \end{aligned}$$

where the last inequality follows from $x_v \geq 2$. □

A.4 Upper bounds exceed $\mathrm{PoA}(\mathcal{G}_1^d)$

In this section, we prove the inequalities

$$\begin{aligned} \frac{\nu_d}{\rho_d - \mu_d} &\geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}, \\ \frac{\hat{\nu_d}}{\hat{\rho_d} - \hat{\mu_d}} &\geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}, \\ \frac{\nu''_d}{\rho''_d - \mu''_d} &\geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}, \text{ and} \\ \frac{\tilde{\nu_d}}{\tilde{\rho_d} - \tilde{\mu_d}} &\geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}, \end{aligned}$$

for all $d \geq 0$. To that end, we prove that each side of the inequalities is a solution to a minimisation problem. Then, for each inequality we prove that the solution to the first minimisation problem must be larger than the solution to the second minimisation problem.

First, three general lemmas are presented. Then, each inequality is presented in a lemma. The first general lemma is a simple fact which will be used in the proof of the second general lemma.

Lemma A.1. *For any r_d with $0 < r_d \leq 1$ for all $d \geq 0$ holds*

$$(r_d)^{d+1} - r_d + (d+1)^{-1/d}\frac{d}{d+1} \geq 0.$$

Proof. Let $p := d+1$ and $q := \frac{d+1}{d}$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{aligned}(r_d)^{d+1} - r_d + (d+1)^{-1/d}\frac{d}{d+1} \geq 0 &\iff (r_d)^p - r_d + \frac{p^{-q+1}}{q} \geq 0 \\ &\iff \frac{(r_d)^p}{p} - \frac{r_d}{p} + \frac{1}{q}\Big(\frac{1}{p}\Big)^q \geq 0.\end{aligned}$$

Let $a := r_d$ and $b := \frac{1}{p}$. Then $a \geq 0$, $b \geq 0$ and

$$\frac{(r_d)^p}{p} - \frac{r_d}{p} + \frac{1}{q}\Big(\frac{1}{p}\Big)^q = \frac{a^p}{p} + \frac{b^q}{q} - ab.$$

By Young's inequality (Young [1912]), the latter is nonnegative. □

The second general lemma shows the solution to a minimisation problem.

Lemma A.2. *For any r_d with $0 < r_d \leq 1$ and $F_d = \frac{1}{d+1}\Big(\frac{d}{d+1}\Big)^d$, the minimisation problem*

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d}F_d\}}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

has the solution

$$\frac{n}{r_d - m},$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1}r_d \\ \frac{d}{d+1}r_d & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1}r_d. \end{cases}$$

$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1}r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1}r_d. \end{cases}$$

Proof. To solve the minimisation problem, two cases are distinguished: $m^{-d}F_d \leq 1$ and $m^{-d}F_d \geq 1$.

Case 1: $m^{-d}F_d \leq 1$. We obtain $m \geq (F_d)^{1/d}$ and

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d}F_d\}}{r_d - m} \ \text{s.t.} \ m < r_d \quad = \quad \min_{m \in \mathbf{R}_{>0}} \frac{1}{r_d - m} \ \text{s.t.} \ (F_d)^{1/d} \leq m < r_d.$$

For d with $(F_d)^{1/d} < r_d$, the minimisation problem may be solved by $m = (F_d)^{1/d}$. For d with $(F_d)^{1/d} \geq r_d$, there exists no m s.t. $m^{-d}F_d \geq 1$, so case 2 should be considered.

Case 2: $m^{-d}F_d \geq 1$. We obtain $m \leq (F_d)^{1/d}$ and

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d}F_d\}}{r_d - m} \ \text{s.t.} \ m < r_d \quad = \quad \min_{m \in \mathbf{R}_{>0}} \frac{m^{-d}F_d}{r_d - m} \ \text{s.t.} \ m < r_d,\ m \leq (F_d)^{1/d} \quad = \quad \min_{m \in \mathbf{R}_{>0}} \frac{F_d}{r_d m^d - m^{d+1}} \ \text{s.t.} \ m < r_d,\ m \leq (F_d)^{1/d}.$$

Basic calculus shows $r_d m^d - m^{d+1}$ is maximised for $m = \frac{d}{d+1}r_d$. Trivially, $\frac{d}{d+1}r_d < r_d$. Therefore, for d with $\frac{d}{d+1}r_d \leq (F_d)^{1/d}$, the minimisation problem may be solved by $m = \frac{d}{d+1}r_d$. For d with $\frac{d}{d+1}r_d > (F_d)^{1/d}$, the minimisation problem may be solved by $m = (F_d)^{1/d}$.

In conclusion,

- for d with $(F_d)^{1/d} < \frac{d}{d+1} r_d$, the problem is solved by $m = (F_d)^{1/d}$,
- for d with $\frac{d}{d+1} r_d \le (F_d)^{1/d} < r_d$, the problem is solved by $m = \frac{d}{d+1} r_d$ (or by $m = (F_d)^{1/d}$),
- for d with $r_d \le (F_d)^{1/d}$, the problem is solved by $m = \frac{d}{d+1} r_d$.

If $\frac{d}{d+1} r_d \le (F_d)^{1/d} < r_d$ then the solution with $m = \frac{d}{d+1} r_d$ (case 2) is smaller than the solution with $m = (F_d)^{1/d}$ (case 1), so the former should be used. We now prove that this solution is indeed the smallest of the two. To that end, we need to prove

$$\frac{\left(\frac{d}{d+1} r_d\right)^{-d} F_d}{r_d - \frac{d}{d+1} r_d} \le \frac{1}{r_d - (F_d)^{1/d}}.$$

First, observe

$$\frac{\left(\frac{d}{d+1} r_d\right)^{-d} F_d}{r_d - \frac{d}{d+1} r_d} = \frac{\left(\frac{d}{d+1} r_d\right)^{-d} \frac{1}{d+1}\left(\frac{d}{d+1}\right)^{d}}{\frac{r_d}{d+1}} = \frac{1}{(r_d)^{d+1}}.$$

Therefore,

$$\frac{\left(\frac{d}{d+1} r_d\right)^{-d} F_d}{r_d - \frac{d}{d+1} r_d} \le \frac{1}{r_d - (F_d)^{1/d}} \iff (r_d)^{d+1} \ge r_d - (F_d)^{1/d}.$$

The latter inequality is true by Lemma A.1. Substituting the correct values for m into the minimisation problem yields the result. $\square$

The third general lemma shows the minimisation problem that the right hand sides of the inequalities that need to be proven are the solution to.

Lemma A.3. *For $d \ge 0$ holds*

$$\frac{1}{1 - d(d+1)^{-(d+1)/d}} = \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{1 - m} \quad \text{s.t.} \quad m < 1,$$

where $F_d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d$.

Proof. Let $r_d = 1$. Then $0 < r_d \le 1$. By Lemma A.2,

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} = \frac{n}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{d}{d+1} r_d & \text{if } (F_d)^{1/d} \ge \frac{d}{d+1} r_d. \end{cases}$$

$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \ge \frac{d}{d+1} r_d. \end{cases}$$

Next, we prove $(F_d)^{1/d} < \frac{d}{d+1} r_d$ for all $d > 0$. We obtain

$$(F_d)^{1/d} < \frac{d}{d+1} r_d \iff (d+1)^{-1/d} < 1.$$

Note that there is no solution $d > 0$ to $(d+1)^{-1/d} = 1$, since

$$\begin{aligned}(d+1)^{-1/d} = 1 &\iff d+1 = 1 \\ &\iff d = 0.\end{aligned}$$

In addition, $(d+1)^{-1/d}$ is continuous for $d > 0$ and $(d+1)^{-1/d}|_{d=1} = \frac{1}{2} < 1$. We thus obtain $(d+1)^{-1/d} < 1$ for all $d > 0$. Therefore, the solution to the minimisation problem is

$$\frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

This concludes the proof. □

The next three lemmas conclude the proofs of the three inequalities.

Lemma A.4. *For $d \geq 0$ holds*

$$\frac{\nu_d}{\rho_d - \mu_d} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. To prove the inequality, first we show that the left hand side is the solution to a minimisation problem. Lemma A.3 gives that the right hand side is the solution to another minimisation problem. Then, we show that the solution to the first minimisation problem must be larger than the solution to the second minimisation problem.

First we prove

$$\frac{\nu_d}{\rho_d - m_d} = \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{\rho_d - m} \quad \text{s.t.} \quad m < \rho_d$$

where $F_d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d$. To that end, let $r_d = \rho_d = \frac{2^{d/2}+1}{1+2^{d/2}-2^d+2^{3d/2}}$. Then $0 < r_d \leq 1$, since $-2^d + 2^{3d/2} \geq 0$. By Lemma A.2,

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} = \frac{n}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{d}{d+1} r_d & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

It remains to prove that $n = \nu_d$ and $m = \mu_d$. To that end, we need to prove that

$$(F_d)^{1/d} \geq \frac{d}{d+1} r_d \iff d \geq d^*,$$

or

$$(d+1)^{-1/d} \geq r_d \iff d \geq d^*.$$

The uniqueness of d^* is proven in Appendix B. In addition, r_d and $(d+1)^{-1/d}$ are continuous for $d > 0$ and

$$\lim_{d \to 0} r_d - (d+1)^{-1/d} = 1 - \frac{1}{e} > 0$$
$$\lim_{d \to \infty} r_d - (d+1)^{-1/d} = 0 - 1 < 0.$$

Therefore, we obtain $(d+1)^{-1/d} \geq r_d \iff d \geq d^*$ and the proof is complete.

By Lemma A.3, it remains to prove that

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} \quad \text{s.t.} \quad m < r_d \qquad \geq \qquad \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{1 - m} \quad \text{s.t.} \quad m < 1.$$

Since $r_d \leq 1$, this is trivial. □

Lemma A.5. *For $d \geq 0$ holds*

$$\frac{\hat{\nu}_d}{\hat{\rho}_d - \hat{\mu}_d} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. To prove the inequality, first we show that the left hand side is the solution to a minimisation problem. Lemma A.3 gives that the right hand side is the solution to another minimisation problem. Then, we show that the solution to the first minimisation problem must be larger than the solution to the second minimisation problem.

First we prove

$$\frac{\hat{\nu}_d}{\hat{\rho}_d - \hat{\mu}_d} = \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{\hat{\rho}_d - m} \quad \text{s.t.} \quad m < \hat{\rho}_d$$

where $F_d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d$. To that end, let $r_d = \hat{\rho}_d = \frac{4}{2^d+3}$. Then $0 < r_d \leq 1$. By Lemma A.2,

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} = \frac{n}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{d}{d+1} r_d & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$
$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

It remains to prove that $n = \hat{\nu}_d$ and $m = \hat{\mu}_d$. To that end, we need to prove that

$$(F_d)^{1/d} \geq \frac{d}{d+1} r_d \iff d \geq \hat{d},$$

or

$$(d+1)^{-1/d} \geq r_d \iff d \geq \hat{d}.$$

The uniqueness of $\hat{d}$ is proven in Appendix B. In addition, r_d and $(d+1)^{-1/d}$ are continuous for $d > 0$ and

$$\lim_{d \to 0} r_d - (d+1)^{-1/d} = 1 - \frac{1}{e} > 0$$
$$\lim_{d \to \infty} r_d - (d+1)^{-1/d} = 0 - 1 < 0.$$

Therefore, we obtain $(d+1)^{-1/d} \geq r_d \iff d \geq \hat{d}$ and the proof is complete.

By Lemma A.3, it remains to prove that

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} \;\; \text{s.t.} \;\; m < r_d \quad \geq \quad \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{1 - m} \;\; \text{s.t.} \;\; m < 1.$$

Since $r_d \leq 1$, this is trivial. □

Lemma A.6. *For $d \geq 0$ holds*

$$\frac{\nu''_d}{\rho''_d - \mu''_d} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. To prove the inequality, first we show that the left hand side is the solution to a minimisation problem. Lemma A.3 gives that the right hand side is the solution to another minimisation problem. Then, we show that the solution to the first minimisation problem must be larger than the solution to the second minimisation problem.

First we prove

$$\frac{\nu''_d}{\rho''_d - m_d} = \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{\rho''_d - m} \quad \text{s.t.} \quad m < \rho''_d$$

where $F_d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d$. To that end, let $r_d = \rho''_d = \frac{1+3^d}{1+9^d}$. Then $0 < r_d \leq 1$. By Lemma A.2,

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} = \frac{n}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{d}{d+1} r_d & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

It remains to prove that $n = \nu_d$ and $m = \mu_d$. To that end, we need to prove that

$$(F_d)^{1/d} \geq \frac{d}{d+1} r_d \iff d \geq d'',$$

or

$$(d+1)^{-1/d} \geq r_d \iff d \geq d''.$$

The uniqueness of d'' is proven in Appendix B. In addition, r_d and $(d+1)^{-1/d}$ are continuous for $d > 0$ and

$$\lim_{d \to 0} r_d - (d+1)^{-1/d} = 1 - \frac{1}{e} > 0$$
$$\lim_{d \to \infty} r_d - (d+1)^{-1/d} = 0 - 1 < 0.$$

Therefore, we obtain $(d+1)^{-1/d} \geq r_d \iff d \geq d''$ and the proof is complete.

By Lemma A.3, it remains to prove that

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} \geq \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{1 - m}$$
$$\text{s.t.} \quad m < r_d \qquad\qquad \text{s.t.} \quad m < 1.$$

Since $r_d \leq 1$, this is trivial. □

Lemma A.7. *For $d \geq 0$ holds*

$$\frac{\tilde{\nu}_d}{\tilde{\rho}_d - \tilde{\mu}_d} \geq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. To prove the inequality, first we show that the left hand side is the solution to a minimisation problem. Lemma A.3 gives that the right hand side is the solution to another minimisation problem. Then, we show that the solution to the first minimisation problem must be larger than the solution to the second minimisation problem.

First we prove

$$\frac{\tilde{\nu}_d}{\tilde{\rho}_d - \tilde{\mu}_d} = \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{\tilde{\rho}_d - m} \quad \text{s.t.} \quad m < \tilde{\rho}_d$$

where $F_d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d$. To that end, let $r_d = \tilde{\rho}_d = \frac{3}{2^d+2}$. Then $0 < r_d \leq 1$. By Lemma A.2,

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} = \frac{n}{r_d - m} \quad \text{s.t.} \quad m < r_d,$$

where

$$m := \begin{cases} \frac{d}{d+1}(d+1)^{-1/d} & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{d}{d+1} r_d & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

$$n := \begin{cases} 1 & \text{if } (F_d)^{1/d} < \frac{d}{d+1} r_d \\ \frac{1}{d+1}(r_d)^{-d} & \text{if } (F_d)^{1/d} \geq \frac{d}{d+1} r_d. \end{cases}$$

It remains to prove that $n = \tilde{\nu}_d$ and $m = \tilde{\mu}_d$. To that end, we need to prove that

$$(F_d)^{1/d} \geq \frac{d}{d+1} r_d \iff d \geq \tilde{d},$$

or

$$(d+1)^{-1/d} \geq r_d \iff d \geq \tilde{d}.$$

The uniqueness of $\tilde{d}$ is proven in Appendix B. In addition, r_d and $(d+1)^{-1/d}$ are continuous for $d > 0$ and

$$\lim_{d \to 0} r_d - (d+1)^{-1/d} = 1 - \frac{1}{e} > 0$$
$$\lim_{d \to \infty} r_d - (d+1)^{-1/d} = 0 - 1 < 0.$$

Therefore, we obtain $(d+1)^{-1/d} \geq r_d \iff d \geq \tilde{d}$ and the proof is complete.

By Lemma A.3, it remains to prove that

$$\min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{r_d - m} \;\text{s.t.}\; m < r_d \quad \geq \quad \min_{m \in \mathbf{R}_{>0}} \frac{\max\{1, m^{-d} F_d\}}{1 - m} \;\text{s.t.}\; m < 1.$$

Since $r_d \leq 1$, this is trivial. □

A.5 Upper bound for $\mathrm{PoA}(\mathcal{G}_1^d)$

In this section, we show that the upper bound for $\mathrm{PoA}(\mathcal{G}_1^d)$ proved by Fotakis [2007] equals

$$\frac{1}{1 - d(d+1)^{-(d+1)/d}}$$

for the classes of games that we consider. The bound proved by Fotakis [2007] is

$$\mathrm{PoA}(\mathcal{G}_1^d) \leq \sup_{f \in F} \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{x f(x)}{y f(y) + (x-y) f(x)},$$

where F is the class of cost functions that appear in $\mathcal{G}_1^d$. By Lemmas 3.4 and 3.5, we know that there are two types of cost functions that appear in $\mathcal{G}_1^d$, namely

1. $c_r(x) = \beta_r$ for $r \in U$ and
2. $c_r(x) = \alpha_r^d$ for $r \in B \cup O$.

If we only consider the first type of cost functions, then we obtain

$$\begin{aligned}
\mathrm{PoA}(\mathcal{G}_1^d) &\leq \sup_{\beta_r \in \mathbb{R}_{\geq 0}} \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{\beta_r x}{\beta_r y + \beta_r (x-y)} \\
&= \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{x}{y + (x-y)} \\
&= \sup_{x \in \mathbb{R}:\, x \geq 0} \frac{x}{x} \\
&= 1.
\end{aligned}$$

If we only consider the second type of cost functions, then we obtain

$$\begin{aligned}
\mathrm{PoA}(\mathcal{G}_1^d) &\leq \sup_{\alpha_r \in \mathbb{R}_{\geq 0}} \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{\alpha_r x^{d+1}}{\alpha_r y^{d+1} + \alpha_r x^d (x-y)} \\
&= \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{x^{d+1}}{y^{d+1} + x^{d+1} - x^d y} \\
&= \sup_{x,y \in \mathbb{R}:\, x \geq y \geq 0} \frac{1}{(y/x)^{d+1} + 1 - y/x} \\
&= \max_{z \in [0,1]} \frac{1}{z^{d+1} + 1 - z} \\
&= \frac{1}{1 - d(d+1)^{-(d+1)/d}},
\end{aligned}$$

where the maximum is attained for $z = (d+1)^{-1/d}$. Note that we may divide by x^{d+1} because we may assume $x > 0$. If $x = 0$ then we obtain $y = 0$ and

$$\lim_{x,y \to 0} \frac{x^{d+1}}{y^{d+1} + x^{d+1} - x^d y} = \lim_{x \to 0} \frac{x^{d+1}}{x^{d+1} + x^{d+1} - x^{d+1}} = \lim_{x \to 0} \frac{x^{d+1}}{x^{d+1}} = 1.$$

Since this result is smaller than the result when we assume $x > 0$, we must use the latter.

Finally we can conclude that

$$\max\left\{1, \frac{1}{1 - d(d+1)^{-(d+1)/d}}\right\} = \frac{1}{1 - d(d+1)^{-(d+1)/d}},$$

so the bound by Fotakis [2007] is

$$\mathrm{PoA}(\mathcal{G}_1^d) \leq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Appendix B

Uniqueness of d^*, $\hat{d}$, d', d'' and $\tilde{d}$

In this section, the uniqueness of d^*, $\hat{d}$, d', d'' and $\tilde{d}$ is proven. To that end, first we prove that the function $(d+1)^{-1/d}$ is a strictly increasing function.

Lemma B.1. *The function* $(d+1)^{-1/d}$ *is strictly increasing for* $d > 0$.

Proof. To prove that $(d+1)^{-1/d}$ is a strictly increasing function, we use that the function is continuous for $d > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}d}(d+1)^{-1/d} > 0$$

for all $d > 0$. To prove the latter, observe that

$$\frac{\mathrm{d}}{\mathrm{d}d}(d+1)^{-1/d} = (d+1)^{-1/d}\Big(-\frac{1}{d(d+1)} + \frac{\log(d+1)}{d^2}\Big).$$

Then,

$$\begin{aligned}(d+1)^{-1/d}\Big(-\frac{1}{d(d+1)} + \frac{\log(d+1)}{d^2}\Big) > 0 &\iff -\frac{1}{d(d+1)} + \frac{\log(d+1)}{d^2} > 0\\ &\iff \frac{-d+(1+d)\log(d+1)}{d^2(d+1)} > 0\\ &\iff -d+(1+d)\log(d+1) > 0.\end{aligned}$$

Moreover, $-d+(1+d)\log(d+1)|_{d=0} = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}d}(-d+(1+d)\log(d+1)) = \log(d+1) > 0.$$

Since $-d+(1+d)\log(d+1)$ is continuous for $d > 0$, we obtain that $-d+(1+d)\log(d+1) > 0$ for all $d > 0$. Since $(d+1)^{-1/d}$ is continuous for $d > 0$, we obtain that $(d+1)^{-1/d}$ is a strictly increasing function. □

Lemma B.2. *There is a unique solution* $d = d^* \in \mathbf{R}_{>0}$ *to*

$$(d+1)^{-1/d} = \rho_d.$$

Proof. We first prove that ρ_d is a strictly decreasing function. Using Lemma B.1, we conclude that the functions can intersect at most once. Then we show that the functions intersect exactly once.

To prove that ρ_d is a strictly decreasing function, we use that the function is continuous for $d > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}d}\rho_d < 0$$

for all $d > 0$. To prove the latter, observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\rho_d &= \frac{\mathrm{d}}{\mathrm{d}d}\frac{1+2^{d/2}}{1+2^{d/2}-2^d+2^{3d/2}} \\
&= \frac{(1+2^{d/2}-2^d+2^{3d/2})(2^{d/2-1}\log(2)) - (1+2^{d/2})(2^{d/2-1}\log(2) - 2^d\log(2) + \frac{3}{2}2^{3d/2}\log(2))}{(1+2^{d/2}-2^d+2^{3d/2})^2} \\
&= \frac{-2^{3d/2-1}\log(2) + 2^{2d-1}\log(2) + 2^d\log(2) + 2^{3d/2}\log(2) - \frac{3}{2}2^{3d/2}\log(2) - \frac{3}{2}2^{2d}\log(2)}{(1+2^{d/2}-2^d+2^{3d/2})^2} \\
&= \frac{2^d\log(2)(-2^{d/2-1} + 2^{d-1} + 1 + 2^{d/2} - \frac{3}{2}2^{d/2} - \frac{3}{2}2^d)}{(1+2^{d/2}-2^d+2^{3d/2})^2} \\
&= \frac{2^d\log(2)(-2^{d/2} - 2^d + 1)}{(1+2^{d/2}-2^d+2^{3d/2})^2}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\rho_d < 0 &\iff \frac{2^d\log(2)(-2^{d/2}-2^d+1)}{(1+2^{d/2}-2^d+2^{3d/2})^2} < 0 \\
&\iff 2^d\log(2)(-2^{d/2}-2^d+1) < 0 \\
&\iff -2^{d/2}-2^d+1 < 0.
\end{aligned}$$

The final inequality is true, since $2^d \geq 2^0 = 1$. Since ρ_d is continuous for $d > 0$, we obtain that ρ_d is a strictly decreasing function. Combining this with Lemma B.1 yields that there is at most one solution to $(d+1)^{-1/d} = \rho_d$.

To prove that a solution exists, observe that

$$\begin{aligned}
\lim_{d\to 0}(d+1)^{-1/d} &= \frac{1}{e}, \\
\rho_0 &= 1, \\
\rho_4 &= \frac{5}{53}.
\end{aligned}$$

Since $1 > \frac{1}{e}$ and $\frac{5}{53} < \frac{1}{e}$, we know that $(d+1)^{-1/d}$ and ρ_d intersect at least once for $d > 0$. In conclusion, there is a unique solution $d = d^*$ to $(d+1)^{-1/d} = \rho_d$. □

Lemma B.3. *There is a unique solution $d = \hat{d} \in \mathbf{R}_{>0}$ to*

$$(d+1)^{-1/d} = \hat{\rho}_d.$$

Proof. We first prove that $\hat{\rho}_d$ is a strictly decreasing function. Using Lemma B.1, we conclude that the functions can intersect at most once.

To prove that $\hat{\rho}_d$ is a strictly decreasing function, we prove

$$\frac{\mathrm{d}}{\mathrm{d}d}\hat{\rho}_d < 0$$

for all $d > 0$. To that end, observe

$$\frac{\mathrm{d}}{\mathrm{d}d}\rho_d = \frac{-2^{d+2}\log(2)}{(2^d+3)^2} < 0.$$

Since $\hat{\rho}_d$ is continuous for $d > 0$, we obtain that $\hat{\rho}_d$ is a strictly decreasing function. Combining this with Lemma B.1 yields that there is at most one solution to $(d+1)^{-1/d} = \hat{\rho}_d$.

To prove that a solution exists, we observe

$$\begin{aligned}
\lim_{d\to 0}(d+1)^{-1/d} &= \frac{1}{e}, \\
\hat{\rho}_0 &= 1, \\
\hat{\rho}_3 &= \frac{4}{11}.
\end{aligned}$$

Since $1 > \frac{1}{e}$ and $\frac{4}{11} < \frac{1}{e}$, we know that $(d+1)^{-1/d}$ and $\hat{\rho}_d$ intersect at least once for $d > 0$. In conclusion, there is a unique solution $d = d^*$ to $(d+1)^{-1/d} = \hat{\rho}_d$. □

Lemma B.4. *There is a unique solution* $d = d' \in \mathbf{R}_{>0}$ *to*

$$(d+1)^{-1/d} = \rho'_d.$$

Proof. We first prove that ρ'_d is a strictly decreasing function. Using Lemma B.1, we conclude that the functions can intersect at most once. Then we show that the functions intersect exactly once.

To prove that ρ'_d is a strictly decreasing function, we use that the function is continuous for $d > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}d}\rho'_d < 0$$

for all $d > 0$. To prove the latter, observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\rho'_d &= \frac{\mathrm{d}}{\mathrm{d}d}\frac{2^{1-d}+3^{-d}}{1+2^{1-d}} \\
&= \frac{(1+2^{1-d})(-2^{1-d}\log(2) - 3^d\log(3)) - (2^{1-d}+3^{-d})(-2^{1-d}\log(2))}{(1+2^{1-d})^2} \\
&= \frac{-2^{2-2d}\log(2) - 3^d\log(3) - 2(3/2)^d\log(3) + 2/(6^d)\log(2)}{(1+2^{1-d})^2} \\
&= \frac{2^{d+1}\log(2)(-2\cdot 3^d + 2^d) - 3^d\log(3) - 2(3/2)^d\log(3)}{(1+2^{1-d})^2} \\
&< 0
\end{aligned}$$

Since ρ'_d is continuous for $d > 0$, we obtain that ρ'_d is a strictly decreasing function. Combining this with Lemma B.1 yields that there is at most one solution to $(d+1)^{-1/d} = \rho'_d$.

To prove that a solution exists, observe that

$$\begin{aligned}
\lim_{d\to 0}(d+1)^{-1/d} &= \frac{1}{e}, \\
\hat{\rho}_0 &= 1, \\
\hat{\rho}_3 &= \frac{31}{135}.
\end{aligned}$$

Since $1 > \frac{1}{e}$ and $\frac{31}{135} < \frac{1}{e}$, we know that $(d+1)^{-1/d}$ and ρ'_d intersect at least once for $d > 0$. In conclusion, there is a unique solution $d = d'$ to $(d+1)^{-1/d} = \rho'_d$. □

Lemma B.5. *There is a unique solution* $d = d'' \in \mathbf{R}_{>0}$ *to*

$$(d+1)^{-1/d} = \rho''_d.$$

Proof. We first prove that ρ''_d is a strictly decreasing function. Using Lemma B.1, we conclude that the functions can intersect at most once. Then we show that the functions intersect exactly once.

To prove that ρ''_d is a strictly decreasing function, we use that the function is continuous for $d > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}d}\rho''_d < 0$$

for all $d > 0$. To prove the latter, observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\rho''_d &= \frac{\mathrm{d}}{\mathrm{d}d}\frac{1+3^d}{1+9^d} \\
&= \frac{(1+9^d)(3^d\log(3)) - (1+3^d)(9^d\log(9)}{(1+9^d)^2} \\
&= \frac{3^d\log(3) + 27^d\log(3) - 9^d\log(9) - 27^d\log(9)}{(1+9^d)^2} \\
&< 0.
\end{aligned}$$

Since ρ''_d is continuous for $d > 0$, we obtain that ρ''_d is a strictly decreasing function. Combining this with Lemma B.1 yields that there is at most one solution to $(d+1)^{-1/d} = \rho''_d$.

To prove that a solution exists, we observe

$$\begin{aligned}\lim_{d\to 0}(d+1)^{-1/d} &= \frac{1}{e},\\ \hat{\rho}_0 &= 1,\\ \hat{\rho}_2 &= \frac{5}{41}.\end{aligned}$$

Since $1 > \frac{1}{e}$ and $\frac{5}{41} < \frac{1}{e}$, we know that $(d+1)^{-1/d}$ and ρ''_d intersect at least once for $d > 0$. In conclusion, there is a unique solution $d = d''$ to $(d+1)^{-1/d} = \rho''_d$. □

Lemma B.6. *There is a unique solution $d = \tilde{d} \in \mathbf{R}_{>0}$ to*

$$(d+1)^{-1/d} = \tilde{\rho}_d.$$

Proof. We first prove that $\tilde{\rho}_d$ is a strictly decreasing function. Using Lemma B.1, we conclude that the functions can intersect at most once. Then we show that the functions intersect exactly once.

To prove that $\tilde{\rho}_d$ is a strictly decreasing function, we use that the function is continuous for $d > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}d}\tilde{\rho}_d < 0$$

for all $d > 0$. To prove the latter, observe that

$$\frac{\mathrm{d}}{\mathrm{d}d}\tilde{\rho}_d = \frac{\mathrm{d}}{\mathrm{d}d}\frac{3}{2^d+2}\frac{-3\cdot 2^d\log(2)}{(2^d+2)^2} < 0.$$

Since $\tilde{\rho}_d$ is continuous for $d > 0$, we obtain that $\tilde{\rho}_d$ is a strictly decreasing function. Combining this with Lemma B.1 yields that there is at most one solution to $(d+1)^{-1/d} = \tilde{\rho}_d$.

To prove that a solution exists, observe that

$$\begin{aligned}\lim_{d\to 0}(d+1)^{-1/d} &= \frac{1}{e},\\ \hat{\rho}_0 &= 1,\\ \hat{\rho}_3 &= \frac{3}{10}.\end{aligned}$$

Since $1 > \frac{1}{e}$ and $\frac{3}{10} < \frac{1}{e}$, we know that $(d+1)^{-1/d}$ and $\tilde{\rho}_d$ intersect at least once for $d > 0$. In conclusion, there is a unique solution $d = d'$ to $(d+1)^{-1/d} = \tilde{\rho}_d$. □

Appendix C

Asymptotic growth of the price of anarchy

In this section we prove that

- the upper bound for the price of anarchy of symmetric k-uniform congestion games as presented in Theorem 1 asymptotically grows as $\Theta(2^{d(1+d)})$,
- the lower bound for the price of anarchy of symmetric k-uniform congestion games as presented in Theorem 2 asymptotically grows as $\Theta(2^d)$,
- the upper bound for the price of anarchy of symmetric 2-uniform congestion games as presented in Theorem 3 asymptotically grows as $\Theta(2^{d(d-1)})$,
- the lower bound for the price of anarchy of symmetric 2-uniform congestion games as presented in Theorem 4 asymptotically grows as $\Theta(2^d)$,
- the lower bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Theorem 5 asymptotically grows as $\Theta(2^d)$,
- the upper bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Consequence 5.1.3 asymptotically grows as $\Theta(3^{d(d+1)})$,
- the upper bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Consequence 5.2.1 asymptotically grows as $\Theta(2^{d^2})$,
- the upper bound for the price of anarchy of semi-symmetric 2-uniform congestion games as presented in Theorem 6 asymptotically grows as $\Theta((2/3)^d 2^{d^2})$,
- the lower bound for the price of anarchy of semi-symmetric 2-uniform congestion games as presented in Theorem 7 asymptotically grows as $\Theta(2^d)$.

Lemma C.1. *The upper bound for the price of anarchy of symmetric k-uniform congestion games as presented in Theorem 1 asymptotically grows as $\Theta(2^{d(1+d)})$.*

Proof. For $d > d^*$, the upper bound for $\mathrm{PoA}(\mathcal{G}_k^d)$ is

$$\mathrm{PoA}(\mathcal{G}_k^d) \leq (\rho_d)^{-1-d} = \Big(\frac{1 + 2^{d/2} - 2^d + 2^{3d/2}}{1 + 2^{d/2}}\Big)^{1+d}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \Big(\frac{1 + 2^{d/2} - 2^d + 2^{3d/2}}{1 + 2^{d/2}}\Big)^{1+d} \frac{1}{2^{d(1+d)}} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\begin{aligned}\lim_{d\to\infty}\Big(\frac{1+2^{d/2}-2^d+2^{3d/2}}{1+2^{d/2}}\Big)^{1+d}\frac{1}{2^{d(1+d)}} &= \lim_{d\to\infty}\Big(\frac{1+2^{d/2}-2^d+2^{3d/2}}{(1+2^{d/2})2^d}\Big)^{1+d}\\ &= \lim_{d\to\infty}\Big(\frac{1+2^{d/2}-2^d+2^{3d/2}}{2^d+2^{3d/2}}\Big)^{1+d}\\ &= \lim_{d\to\infty}\Big(\frac{2^{-3d/2}+2^{-d}-2^{-d/2}+1}{2^{-d/2}+1}\Big)^{1+d}\\ &= 1.\end{aligned}$$

□

Lemma C.2. *The lower bound for the price of anarchy of symmetric k-uniform congestion games as presented in Theorem 2 asymptotically grows as $\Theta(2^d)$.*

Proof. The lower bound for $\mathrm{PoA}(\mathcal{G}_k^d)$ is

$$\mathrm{PoA}(\mathcal{G}_k^d) \geq \frac{4+\sqrt{2}(1+2^{d+1})}{4+3\sqrt{2}}.$$

It is thus needed to prove that

$$\lim_{d\to\infty}\frac{4+\sqrt{2}(1+2^{d+1})}{4+3\sqrt{2}}\frac{1}{2^d} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\lim_{d\to\infty}\frac{4+\sqrt{2}(1+2^{d+1})}{4+3\sqrt{2}}\frac{1}{2^d} = \lim_{d\to\infty}\frac{2^{2-d}+\sqrt{2}(2^{-d}+2)}{4+3\sqrt{2}} = \frac{2\sqrt{2}}{4+3\sqrt{2}}.$$

□

Lemma C.3. *The upper bound for the price of anarchy of symmetric 2-uniform congestion games as presented in Theorem 3 asymptotically grows as $\Theta(2^{d(d-1)})$.*

Proof. For $d > \hat{d}$, the upper bound for $\mathrm{PoA}(\mathcal{G}_2^d)$ is

$$\mathrm{PoA}(\mathcal{G}_2^d) \leq (\hat{\rho}_d)^{-1-d} = \Big(\frac{2^d+3}{4}\Big)^{1+d}.$$

It is thus needed to prove that

$$\lim_{d\to\infty}\Big(\frac{2^d+3}{4}\Big)^{1+d}\frac{1}{2^{d(d-1)}} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\begin{aligned}\lim_{d\to\infty}\Big(\frac{2^d+3}{4}\Big)^{1+d}\frac{1}{2^{d(d-1)}} &= \lim_{d\to\infty}(2^d+3)^{1+d}2^{-d(d-1)}4^{-(d+1)}\\ &= \lim_{d\to\infty}(2^d)^{1+d}2^{-d(d-1)-2(d+1)}\\ &= \frac{1}{4}.\end{aligned}$$

□

Lemma C.4. *The lower bound for the price of anarchy of symmetric 2-uniform congestion games as presented in Theorem 4 asymptotically grows as $\Theta(2^d)$.*

Proof. The lower bound for $\mathrm{PoA}(\mathcal{G}_2^d)$ is

$$\mathrm{PoA}(\mathcal{G}_2^d) \geq \frac{2+2^d}{3}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \frac{2+2^d}{3}\frac{1}{2^d} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\lim_{d\to\infty} \frac{2+2^d}{3}\frac{1}{2^d} = \lim_{d\to\infty} \frac{2^{1-d}+1}{3} = \frac{1}{3}.$$

$\square$

Lemma C.5. *The lower bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Theorem 5 asymptotically grows as $\Theta(2^d)$.*

Proof. The lower bound for $\mathrm{PoA}(\mathcal{G}_{\leq k}^d)$ is

$$\mathrm{PoA}(\mathcal{G}_{\leq k}^d) \geq \frac{1+2^{d+1}}{3}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \frac{1+2^{d+1}}{3}\frac{1}{2^d} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\lim_{d\to\infty} \frac{1+2^{d+1}}{3}\frac{1}{2^d} = \lim_{d\to\infty} \frac{2^{-d}+2}{3} = \frac{2}{3}.$$

$\square$

Lemma C.6. *The upper bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Consequence 5.1.3 asymptotically grows as $\Theta(3^{d(d+1)})$.*

Proof. For $d > d''$, the conjecture gives

$$\mathrm{PoA}(\mathcal{G}_{\leq k}^d) \leq (\rho_d'')^{-1-d} = \left(\frac{1+9^d}{1+3^d}\right)^{1+d}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \left(\frac{1+9^d}{1+3^d}\right)^{1+d} \frac{1}{3^{d(d+1)}} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\begin{aligned}
\lim_{d\to\infty} \left(\frac{1+9^d}{1+3^d}\right)^{1+d} \frac{1}{3^{d(d+1)}} &= \lim_{d\to\infty} (1+9^d)^{1+d}(1+3^d)^{-1-d}3^{-d(d+1)} \\
&= \lim_{d\to\infty} (9^d)^{1+d}(1+3^d)^{-1-d}3^{-d(d+1)} \\
&= \lim_{d\to\infty} 3^{2d+2d^2}(3^d(3^{-d}+1))^{-1-d}3^{-d(d+1)} \\
&= \lim_{d\to\infty} 3^{d+d^2}3^{-d-d^2}(3^{-d}+1)^{-1-d} \\
&= 1.
\end{aligned}$$

$\square$

Lemma C.7. *The upper bound for the price of anarchy of semi-symmetric k-uniform congestion games as presented in Consequence 5.2.1 asymptotically grows as $\Theta(2^{d^2})$.*

Proof. For $d > d'$, the conjecture gives

$$\mathrm{PoA}(\mathcal{G}^d_{\leq k}) \leq (\rho'_d)^{-1-d} = \Big(\frac{1+2^{1-d}}{2^{1-d}+3^{-d}}\Big)^{1+d}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \Big(\frac{1+2^{1-d}}{2^{1-d}+3^{-d}}\Big)^{1+d} \frac{1}{2^{d^2}} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\begin{aligned}
\lim_{d\to\infty} \Big(\frac{1+2^{1-d}}{2^{1-d}+3^{-d}}\Big)^{1+d} \frac{1}{2^{d^2}} &= \lim_{d\to\infty} \Big(\frac{2^d+2}{2+(2/3)^d}\Big)^{1+d} \frac{1}{2^{d^2}} \\
&= \lim_{d\to\infty} (2^d+2)^{1+d}(2+(2/3)^d)^{-1-d}2^{-d^2} \\
&= \lim_{d\to\infty} (2^d)^{1+d}(2+(2/3)^d)^{-1-d}2^{-d^2} \\
&= \lim_{d\to\infty} 2^d(2+(2/3)^d)^{-1-d} \\
&= \lim_{d\to\infty} 2^d(2(1+\frac{1}{2}(2/3)^d))^{-1-d} \\
&= \lim_{d\to\infty} 2^d 2^{-1-d}(1+\frac{1}{2}(2/3)^d)^{-1-d} \\
&= \frac{1}{2}\lim_{d\to\infty} (1+\frac{1}{2}(2/3)^d)^{-1-d} \\
&= \frac{1}{2}.
\end{aligned}$$

□

Lemma C.8. *The upper bound for the price of anarchy of semi-symmetric 2-uniform congestion games as presented in Theorem 6 asymptotically grows as $\Theta((2/3)^d 2^{d^2})$.*

Proof. For $d > \tilde{d}$, the upper bound for $\mathrm{PoA}(\mathcal{G}^d_{\leq 2})$ is

$$\mathrm{PoA}(\mathcal{G}^d_{\leq 2}) \leq (\tilde{\rho}_d)^{-1-d} = \Big(\frac{2^d+2}{3}\Big)^{1+d}.$$

It is thus needed to prove that

$$\lim_{d\to\infty} \Big(\frac{2^d+2}{3}\Big)^{1+d} \frac{1}{(2/3)^d 2^{d^2}} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\begin{aligned}
\lim_{d\to\infty} \Big(\frac{2^d+2}{3}\Big)^{1+d} \frac{1}{(2/3)^d 2^{d^2}} &= \lim_{d\to\infty} (2^d+2)^{1+d}3^{-1-d}(2/3)^{-d}2^{-d^2} \\
&= \lim_{d\to\infty} (2^d)^{1+d}3^{-1-d}(2/3)^{-d}2^{-d^2} \\
&= \frac{1}{3}.
\end{aligned}$$

□

Lemma C.9. *The lower bound for the price of anarchy of semi-symmetric 2-uniform congestion games as presented in Theorem 7 asymptotically grows as $\Theta(2^d)$.*

Proof. The lower bound for $\mathrm{PoA}(\mathcal{G}^d_{\leq 2})$ is

$$\mathrm{PoA}(\mathcal{G}^d_{\leq 2}) \geq \frac{3 + 2^{d+1}}{5}.$$

It is thus needed to prove that

$$\lim_{d \to \infty} \frac{3 + 2^{d+1}}{5} \frac{1}{2^d} = c$$

for some finite $c \in \mathbf{R}_{>0}$. We obtain

$$\lim_{d \to \infty} \frac{3 + 2^{d+1}}{5} \frac{1}{2^d} = \lim_{d \to \infty} \frac{3 \cdot 2^{-d} + 2}{5} = \frac{2}{5}.$$

□