University of Twente

Faculty of Behavioural, Management and Social sciences

Contest theory models of long-term advertising and short-term competitive behaviour

Master's Thesis in Financial Engineering and Management

Written by:
Thom Sparrius - s1744100

Supervised by:
Dr. R.A.M.G Joosten
Ir. R.L.A Harmelink

December 16, 2021

ABSTRACT

In this research I aim to establish a set of feasible limiting average rewards from a series of short-term competitive forms based on a long-term dyadic choice for investment in a duopolistic repeated simultaneous game. This is done through computing in Python on short-term market structures using an aggregated individual agent-based deterministic market distribution from contest theory and the law of large numbers. These computations result in a series of visualisations on which some effects are isolated to determine how eight short-term competition forms react to changes in variable advertisement cost (of the long-term variable) and changes in demand. The visualisations show that for the computations minor differences (e.g. limiting average rewards) occur when a baseline input is used on the eight models and that increase in variable advertisement cost and decrease in demand result in alterations to the limiting average reward structures.

Keywords: short-term competition, long-term decision making, fast and slow variables, contest theory, python, stochastic game theory, advertisement.

PREFACE

I present my Master's Thesis on the topic of "Contest theory models of long-term advertising and short-term competitive behaviour." The research is internal, meaning that no external company/firm commissioned it. This makes the academic fields in which the research problem is positioned, my supervisors and myself the problem owner.

The research includes computations of several short-term competition forms when exposed to dyadic long-term advertisement decisions. These computations are done in Python, with the intention of better visualising the differences or similarities between several short-term competitive models.

The paper is highly inspired and based upon the previous contributions by Reinoud Joosten and the modelling of Rogier Harmelink. It intends to combine previous research on the topics of stochastic game theory with theory on contest theory and long-term dyadic decision making. I attempt to determine limiting average rewards through the use and alterations on the model by Rogier Harmelink. With this it intends to fill a knowledge gap in the above-mentioned fields and contribute to the academic world.

In this preface I would also like to make some acknowledgements. Firstly I would like to thank Reinoud Joosten for his role as my main supervisor and sparring partner for this thesis. He introduced in the many lectures I attended of him the ideas of game theory and its fascinating aspects, and helped me find the subject of this thesis. He also guided me through the, what sometimes felt like unending, journey of writing a proper thesis, helped me understand the complex workings of the theoretical models and literature of stochastic game theory, and took the time to meet on many occasions, sometimes talking for hours about the thesis but also a plethora of other interesting subjects. Additionally, I would like to thank Rogier Harmelink for his much-needed assistance with regards to everything related to Python, his guidance through his work in Jupyter Notebook and assistance in thinking about potential applications of my model were essential. I also wish to thank to my girlfriend, family and close-friends for motivating me to find determination, for trusting me in my skills and helping me find the work-ethic to finish this thesis.

My intention is to have this thesis serve as the start to a publication in the field of game theory, advertising or contest theory. An intention which I hope to realize together with Reinoud and Rogier, parallel to my future career.

Thom Sparrius,
December 16, 2021

CONTENTS

1 INTRODUCTION

This thesis will be considering a series of infinitely repeated games, from which using different short-term competition models, the effects of long-term decision making in the form of advertisement costs is explained. Short-term competition in this case, is a series of games resulting in prices, quantities and profits, these short-term competitions are all duopolistic (i.e., two firms, or players, have control over the market). Combined with a dyadic long-term choice for two players in an advertisement battle (i.e., the choice to advertise at a certain points in time), this results in a stream of profits and payoffs per period which leads to a set of limiting average rewards.

The distinction between long- and short-term is made due to the fact that in the infinitely repeated stage games two distinct moments in time are important. The advertisement decisions are a choice made by both players to advertise. These decision are a set of infinite dyadic choices which are made before the short-term stage games take place (therefore long-term decision-making). The long-term decisions determine for the next period of time how the market shares are distributed through advertisement expenditure. The short-term competition is the entity influenced over a continuous period of play. In the short-term competition decisions are made with regards to short-term variables of the game such as prices and quantities from which profits emerge. The outcomes of the short-term competition are a stream of rewards emerging from the infinite sequence of stage games (several moments in time).

I attempt to elaborate how the distribution of market shares and short-term outcomes are influenced by the long-term decision making through the introduction of contest theory. Contest theory postulates that in any competition individual agents influence the outcomes of contests through separate stochastic processes. I employ this line of thinking to explain how in the long-term, historical advertisement behaviour of both players influences the payoffs of a collective of individual agents in the short-term. It is important to know that this influence is explicitly not on the short-term parameters such as price, quantity and cost (since these are intrinsic results of the regular competition) but rather on awareness of the brand or other external variables. The influence of advertisement then creates a preference within these individual agents, in turn I then employ the law of large numbers to aggregate this preference of the individual agents for either player in the short-term competition. This aggregated individual agent preference determines the market share distribution and size.

Python is used to model and visualise the outcomes of the short- and long-term interaction and elaborate on how certain parameters influence each other. The Python model contains the theoretical model computations and shows through several visualisations how changing demand factors and advertisement investment cost influences the sets of limiting average rewards for both players. This is done in eight different short-term competition models.

The main objective for the research is to understand, in a duopolistic market, the interaction of a long-term decision making variable with several types of short-term competition. Where contest theory principles are used to make the interaction between the long- and short-term explicit.

The general thesis outline is as follows. In Section 1, I shortly elaborate on the intention of my thesis. In Section 2, I explain the literary position of the thesis in the fields of game theory (classes of games), several modes of competition, contest theory, and short- and long-term variables. In Section 3, the elements of the model and the assumptions are explained. In Section 4, I show the theoretical model, explain the workings and elaborate on its intention. Section 5 contains the results of the models with which I will explain the effects of certain parameters. Sections 6 and 7 are the conclusions and the discussion and recommendations.

2 ACADEMIC POSITIONING OF THE THESIS

We position this thesis in the intersection of stochastic game theory, contest theory and industrial optimisation. To do so we address the literature in four sections, the first section covers some classes in stochastic game theory, and more specifically the use of two different categories of output (one being dyadic long-term decisions and the other short-term continuous limiting average rewards), the second section addresses the market type (e.g. Cournot and Bertrand competition) subdivided in several theoretical short-term competition forms in game theory, this section also contains the degree of collusion between players (i.e., secret cooperation), the third section is on the topic of contest theory and its applications, and the last section is the literature on the combination of long- and short-term variables and advertisement.

2.1 Classes in stochastic game theory

The field of stochastic game theory was pioneered by Shapley (1953). With the emergence of this challenging field a lot of work was made possible. In the field a few avenues emerge with regards to the criterion under which the stochastic games are assessed. The first is stochastic games evaluated under the discounted rewards criterion, these games are well-understood, hence solved. Value and optimal strategies, in case of zero-sum stochastic games, or a (Nash) equilibrium, in case of non zero-sum stochastic games, are known to exist. For the second avenue, that of limiting average reward criterion, results are not entirely as complete. The infinite stream of stage payoffs under this criterion is averaged over an infinite time horizon. This induces the problem that the long-term average payoffs need not converge. In order to solve this problem the mathematical operations of taking the limit inferior or limit superior of the average payoffs is taken. For infinitely repeated stochastic games under the limiting average reward criterion, in case of zero-sum stochastic games, values but not optimal strategies exist. ε-optimal strategies do exist. These are strategies for which, for every ε there exists a value which is an ε away from the optimal solution. Meaning that for each ε iteration the true optimal value is approached more closely, but not necessarily reached. These ε optimal strategies sometimes contain difficult discontinuities complicating calculations. For non zero-sum stochastic games the same holds, since zero-sum stochastic games are general sum stochastic games (non zero-sum games) with a positive payoff for one player and an equally high negative payoff for the other. In addition to only the ε equilibria holding for the non zero-sum games, a strategy-pair must be found for which optimal value holds for both players with regards to the other player. Meaning one-sided deviation is not equitable.

The literature from which these limited average reward criterion assessments emerged uses the work of Brenner and Witt (2003) who formally introduce, in their paper, the idea of Frequency-Dependent games (FD-games). These are games where the aggregated frequency of strategies chosen for players in past influence the stage payoffs of those strategies now. Joosten et al. (2003) elaborate on some practical examples of stage payoffs being frequency-dependent over time and elaborate on the modelling process concerning these FD-games and develop a framework for analysis inspired by Joosten et al. (1995) and Joosten (1996). Joosten et al. (2003) also introduce the concept used in this thesis called "the set of limiting average rewards (or feasible rewards)" which are the feasible average outcomes of the game, as discussed previously. The paper therefore lays a strong foundation for what is further explored in this thesis with regards to converging short-term outcomes over the long-term.

Later Joosten (2008) explores the idea of frequency-dependency further and also introduces the idea of "externalities" introduced by advertisement influencing competition, something which also prominently features in this thesis. Additionally the concept of games with jointly frequency-dependent stage payoffs (JFD-games) is introduced, which are different from regular FD-games since they concern the frequency-dependency of action pairs, meaning the "joint" frequencies matter rather than the "marginal" frequencies. In Joosten (2008) the advertisement choice is also made independently and simultaneously and two time dimensions are discussed, similarly to this thesis where the dyadic advertisement choice is made first impacting the long-term, and the continuous outcomes of the short-term competition are partially separated but are also influenced by advertisement choice. The existence of a short-term Cournot competition (equilibrium in quantity) yielding certain immediate payoffs distributed over time which in turn yield certain limiting average rewards is assumed in Joosten (2008).

Afterwards Joosten (2015) further explores the idea of time-dependent game theory with jointly-converging frequency-dependent payoffs. The short-term competition is extensively discussed for a Bertrand short-term competition model (a step forward from Joosten (2008)), meaning the assumed short-term feasibility is now explained, and the short- and long-term effects of advertisement are discussed extensively. This thesis goes one step further and tries to elaborate how the long-term decision making influences the short-term parameter (through contest theory).

Further developments in the field of stochastic game theory introduce Joosten and Meijboom (2010) to the idea of having frequency-dependent transition probabilities rather than stage payoffs, a term coined as Endogenous Transition Probabilities (ETP). ETP influences the transition probabilities between states directly, which indirectly influences the payoffs. Instead of ESP where the average payoffs are influenced directly. Joosten and Meijboom (2010) elaborate on this through fishery games. Where over-exploitation of the fish population can induce a state of the game (and in this case the lake) where fish are a scarce in the long-term. Making it very difficult to restore the fish population to a state where they are no longer scarce (only through extremely careful fishing behaviour). This means, simply put, that the certain strategies of play, for either player, have influence on the state of the entire game in the long-term.

Later work by Joosten and Samuel (2017) combined the idea of Endogenous Transition Probabilities (ETP) and Endogenous Stage Payoffs (ESP) in another paper on fishery games. This time another stochastic fishery game is introduced here overfishing has a two effects. First it damages the fish stock, inducing lower catches in states "High and Low" and second overfishing causes the system to spend more time in the "Low" state. They analyse the two effects (or double-effect) by finding the set of jointly-convergent pure-strategy rewards through equilibria and threats and they do this under the limiting average reward criterion.

The final contribution to the field, and with that also the final contribution to the section is by Joosten and Samuel (2020), who use a computation-inspired ordering among stochastic games to properly define the different classes of games we have previously introduced through the other literature. They introduce a table of the classes of games and their attributes. A framework for their earlier coined games which have frequency-dependent stage payoffs but also frequency-dependent transition probabilities is introduced, and call this the Endogenous Transition Probability/Endogenous Stage Payoffs-framework (ETP/ESP-framework). They also elaborate on what researchers should wary of while using this framework (e.g. the applicability of the algorithms to the class types). They help understand the approach of the computing of complex game theoretical problems and therefore provide good insight into the computing/modelling process for this thesis. Implying that with the introduction of increasing complexity, even though the algorithm is generally applicable, extra calculations should be performed.

2.2 Strategic dimensions of the market and collusion

Several strategic dimensions of the market are relevant. The market under consideration possesses one of such strategic dimensions, the market can for instance be monopolistic, duopolistic, oligopolistic or several other possibilities. The market form will be discussed first. The form in which competition is chosen, the theoretical competitive behaviour/model, is also a strategic dimension of the market and will be explained afterwards. The next strategic dimension discussed is the entity influenced by the long-term advertisement behaviour. Additionally we explain how collusion introduces a new strategic element to the research.

For the short-term competition the market under consideration is duopolistic. Using Cournot, Bertrand, Stackelberg-variants of the former, and collusion variants of short-term Cournot and Bertrand competition. The actual stage payoffs, i.e., payoffs on the short-term resulting from advertisement, are the market size and the market shares of the players (determined by the agents).

The first important choice made is to analyse a duopolistic market. Researchers are attracted to oligopolistic markets, since most markets have multiple players (firms) competing in the market in some direct or indirect fashion. One specific type of oligopolistic market form which lends itself to analysis well, is the duopoly, which can be used in many contexts (e.g., Singh and Vives (1984), Benoit and Krishna (1987), Van Cayseele and Furth (1996), Joosten (2015)). Generally speaking a duopoly is a type of oligopoly where two players (firms) have dominant or even exclusive control over the market and therefore the decisions regarding it. Duopolistic markets depict a scenario where there is dependence of the two players on each other, which is a necessary condition for analysis. The two principal types are Cournot duopoly and Bertrand duopoly, which are two types of short-term competition, the reason why they are used still is because they both assume simultaneous games between two firms and are therefore suitable for computation in a duopolistic scenario.

Cournot (1838) and Bertrand (1883) competition are both fundamentally important competition forms in the field of economics and specifically with regards to duopoly (e.g., Bertrand (1883), Fisher (1898), Friedman (2000)). Cournot as a duopoly model insists that duopolists independently seek to maximise profit and that this is done through competition in quantities, for homogeneous products. Bertrand, on the other hand, believed that if competition was done through prices rather than quantities, profit maximisation would occur. In the original Bertrand model products could also not be differentiated (giving rise to the Bertrand paradox, solved by a small degree of product differentiation (e.g. Edgeworth (1925)). Both theoretical models are frequently used nowadays, due to the difference in approaching the competitive variable leading to equilibrium and information differences (e.g. Vives (1984), Okuguchi (1987)).

Now in addition to the regular forms of Cournot and Bertrand competition there are the variations by Von Stackelberg (1934), introducing a leader-follower system. These variations were invented because unlike what Cournot and Bertrand assume, which is simultaneous moves of both players, reality might have one player make its strategic decision first (the leader) and the other chooses second (the follower). There are four Stackelberg variants in this thesis, two Stackelberg-Cournot and two Stackelberg-Bertrand models where the leader and follower are switched. Our theoretical approach is very similar to Joosten (2015), who analysed the long- and short run interaction with regards to Bertrand competition, additionally however like Jin and Parcero (2010), we approach multiple market structures (as short-term competition forms).

The final strategic dimension is the entity influenced by the advertisement. The influenced entity can be many things, for instance sales, market shares or sales potential (e.g., Nerlove and Arrow (1962), Fershtman (1984), Joosten (2015)). The latter likens this notion of sales potential to goodwill (e.g. Nerlove and Arrow (1962) and Friedman 1983). Vidale and Wolfe (1957) provide empirical evidence regarding current sales being influenced by the positive effects of past advertisement and are therefore credited by Joosten (2015), Nerlove and Arrow (1962) and Friedman (1983). Sales potential is the influenced entity in this thesis too, since we intend to further explore the route taken by Joosten (2015). Using the addition of other short-term competitive forms as well as the introduction of collusion and contest theory to expand on the idea.

Now with regards to collusion it is interesting to understand what happens in practice. Here players (firms) usually compete on either the long-term or the short-term, since it is expensive to compete on both and illegal to cooperate on both (cartel-like behaviour). This has potential to be more equitable for both. We therefore think it is important to also research some degree of collusion in the short-term competition as a part of the market structures, to analyse what happens if both players collude on the short-term. Whether this collusion is actually profitable, the degree of collusion, and whether it is sustainable is discussed by e.g., Bishop (1960), Brander and Harris (1983), Fershtman and Muller (1986), and Rothschild (1992). We attempt, similarly to Jackson and Wilkie (2005) who discuss the mechanism of side-payments which occur in the short-term, to apply the collusion effects to the short-term competition form. The long-term dyadic decision making behaviour influences the profitability of the short-term competition for collusion and therefore its usefulness.

2.3 Contest theory

The next section introduces some important papers on the idea of contest theory. In *Section 4*, contest theory is used to influence the theoretical model.

To introduce contest theory we start with Corchón (2007) who in his survey provides a unified framework for contest theory. His definitions and references to older works help establish the field. He refers to Nitzan (1994) and Konrad (2007) as additional sources of information and reference. Corchón (2007) states the following, which gives insight into what it is that contests actual are: "A part of economics (e.g., general equilibrium) studies situations where property rights are well defined and agents voluntarily trade rights over goods or produce rights for new goods. This approach has produced very important insights into the role of markets in resource allocation such as the existence and efficiency of competitive equilibrium, the optimal specialization under international trade, the role of prices in providing information to the agents, etc. There are other situations, though, where agents do not trade but rather fight over property rights. In these situations agents can influence the outcome of the process by means of certain actions such as investment in weapons, bribing judges/politicians, hiring lawyers, etc. These situations are called *Contests.*" Additionally Corchón (2007) explains that the field of contests (contest theory) emerged from the seminal works in rent seeking (specifically Tullock (1967, 1980) and Krueger (1974)) and lobbying (specifically Becker (1983)), two specific types of contests.

The definition of Corchón (2007) illustrates the idea that contests occur outside the regular boundaries of play, in the case of this thesis this purely means that it is an external stochastic process in which individual agent behaviour (and preference) is influenced through advertisement in a process separate from the short-term variables. Meaning individual consumers in the market prefer either player over the other in a stochastic process, and the aggregate of their preferences determines market distribution. Homogeneous products are also required for this preference to manifest since the effect of preference must be external and not through short-term competition market mechanics.

The field of contest theory has evolved over time to gain an understanding of prices and incentives, whether that would be in competition, compensation or elimination tournaments, the works by Nalebuff and Stiglitz (1983) and Rosen (1985) on this, opened up an avenue for connection with the closely related field of game theory and strategic behaviour. Dixit (1987) considered; "The effect of precommitment in contests where rivals expend effort to win a prize." and also discussed applications to sports, oligopoly and rent seeking, was one of the first combining the fields.

More recently Vojnović (2016, 2017) elaborated on mechanisms of contest theory and its additional applications. His contributions were the main inspiration for including contest theory in the theoretical model of this thesis. Vojnović (2016, 2017) also discusses individual agents, who prefer through perceived individual rewards some outcomes others. Although the process is a separate stochastic process, the outcomes can be perceived as a non-stochastic and therefore deterministic, the deterministic nature of the outcomes in combination with a long existent theorem as explained by Hsu and Robbins (1947), the law of large numbers and its convergent nature, would allow for the creation of a model which, with sufficient iterations, would yield an aggregated effect for all the individual agents. The addition of contest theory therefore functions in this thesis as explanation for market size (growth) and distribution (market shares) in the short-term competition through aggregated individual agent (consumers) preferences resulting from the long-term advertisement behaviour history of both players.

2.4 Combination of long and short-term variables

The last aspect which our research attempts to accommodate, as shortly discussed in the third strategic dimension, is the variable which is influenced by the long-term decision making (advertising), and specifically the effect of market shares (e.g., Fershtman (1984)). Two important specifications regarding market shares are firstly, whether the long-term investment increases/decreases the total market size, and secondly, how the investment changes the distribution of the market.

An important criterion in the case of advertisement, functioning as a long-term investment, as is the case in Joosten (2008, 2015), is type of effects advertisement has on the market. For instance, cooperative advertisement or competitive advertisement. The field of advertisement is vast, with some early works like Braithwaite (1928) elaborating on the economic effects of advertisement in general, and specifically stating that competitive advertisement is most probably more expensive than cooperative advertisement. Additionally whether the games itself are cooperative or not is discussed by Li et al. (2016).

Another important aspect for the long and short-term variables is the idea of decisions and their effects operating in multiple time dimensions. The idea is not novel in this thesis, we believe the idea originates from the work of Cellini et al. (2002) and Cellini et al. (2008), who introduce the idea of one advertisement decision having a positive external effect on the underlying market (increasing it in size). It also introduces the idea that both players may benefit from the market size increasing as an effect of advertisement from either player.

There is one underlying assumption though, most of the papers following this line of thinking assume differentiated products (e.g. Joosten (2015)) for their equilibria and external effects, and the ideas in this thesis and those in Joosten (2008) are explicitly based on homogeneous products (due to the fact that that is a requirement of the short-term competition models we use and the usefulness of contest theory), which creates, next to an increase in the total market size, a competition which also distributes the market in accordance with these long-term advertisement expenditures. In this case contest theory is the deciding factor in distributing the market (i.e., estimated market shares).

3 ELEMENTS AND MODEL ASSUMPTIONS

In this section we present the core ideas and assumptions underlying our model. Elaborating the thought processes which led to the model and explaining how certain aspects of reality might influence its applicability.

3.1 Core idea

From a financial perspective maximizing profit or minimizing cost is usually the objective for modelling (i.e., linear programming solution by Kantorovich (1940)), this is also the case for our model. The competitors try to maximize their profits. A long-term dyadic decision making process, with regards to advertising, can be introduced. The long-term decision influences short-term competitive behaviour. From which optimal prices, quantities and costs arise in the short-term resulting in an infinite stream of profits inducing a set of limiting average rewards.

To elaborate on the interaction between the long-term dyadic decision making and the short-term competition we introduce, together with the law of large numbers, contest theory. Which combined provide an aggregated discrete distribution of the market share between the two players. We theorise that these discrete outcomes emerge from the aggregation of a whole series of binary choices by individual agents, advertisement in the market (by either player, or both) influences these individual agents in a separate stochastic process.

The separate stochastic process is the influences of advertisement on the preference of one agent for either player. The preference of an individual agent is discrete, therefore the aggregation of all these preferences is also discrete, leading to a mechanism with which the market share distribution is determined.

Joosten (2015) refined the idea that through a series of direct- and indirect effects, advertisement expenditures can change the balance with which the two cplayers compete. We will not only analyse this competition for several other short-term competitive models (Stackelberg-Bertrand, collusion under Bertrand, Cournot, Stackelberg-Cournot and collusion under Cournot) but also introduce through contest theory an explanation for the effect of player choices resulting in a continuous set of limiting average rewards. Additional we wish to show through the model that changes in expenditure costs, direct advertisement costs, fixed market growth rate and variable market change rate result in varying sets of limiting average rewards for the players. The findings can then be used to generalise the influence of long-term dyadic decision making on continuous short-term outcomes.

To further illustrate the effects and to make the computations more expedient an algorithmic model is used. Many game iterations in all kinds of different short-term competitive forms are engineered. The engineering process is based on the thinking of Aumann (2008). Our models are not intended to represent reality perfectly, the idea is that the theoretical models are designed to emulate some effects between certain variables and to ascertain how these relationships might have implications on reality.

In the model, random (or manual) input generates many combinations of advertisement choices. The model pertains to an infinite number of games, which results in many matrices containing combinations of actions for the players at a certain point in time, these variables will be called ρ_t. Many combinations of ρ_ts will then be used as input. Using the aforementioned contest theory model with a large number of iterations (the law of large numbers), theoretical Bertrand and Cournot models can be used to compute a set of limiting average rewards for all these ρs. The theoretical models itself will be shown in *Section 4*. These computations will result in outcomes for all the player actions and result in a set of limiting average rewards. Some of these limiting average rewards are Nash (1950) equilibria. The outcomes will then be plotted in order to discern patterns and rankings between the model structures, the investment costs, the demand changes, the degree of symmetry and other parameters.

The algorithmic approach is inspired by Samuel (2017) and Harmelink (2019). Samuel (2017) takes an algorithmic approach to compute dynamic 2-state, 2-player, 2-action stochastic competitive games and Harmelink (2019) expands on this with N-actions and allowances for frequency-dependent transition probabilities and frequency-dependent stage payoffs. Since we believe that the data-creation, -compiling and -visualization is made significantly more accurate and better using a similar Python-based approach, Harmelink modelling work in particular will be used as the foundation for the model in this thesis.

3.2 Core model assumptions

The model operates under a number of theoretical and modelling assumptions. We elaborate on why they are introduced and how they might influence the outcomes.

The first assumption was symmetry. This was introduced since theoretically the duopolistic market structures mechanisms have symmetrical costs and demand. By model definition the outcomes of two players, playing symmetrical strategies should be equal. More importantly it made the computations and outcomes significantly easier to do and interpret. Later on certain scenarios could then be approached without this symmetry to increase complexity.

The second assumption pertains to the ρ matrix. An array of random numbers, in the uniform distribution between 0 and 1, represents the advertisement behaviour of both players. Resulting in a series of joint relative frequencies. The ρ matrix and underlying distribution are as follows, if we take:

$$\hat{\rho}_{ij} \sim U(0,1)$$

We then say,

$$\rho = \frac{1}{\hat{\rho}_{11}+\hat{\rho}_{12}+\hat{\rho}_{21}+\hat{\rho}_{22}} \begin{bmatrix} \hat{\rho}_{11} & \hat{\rho}_{12} \\ \hat{\rho}_{21} & \hat{\rho}_{22} \end{bmatrix}$$

Resulting in,

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

The matrix can not be chosen randomly. The four cells it contains however, if chosen from a uniform distribution can take two values (zero or one) this can cause, when the distribution is normalised to be equal to one, a randomness of player action accumulated over the infinite stage games.

The uniform distribution introduces one problem however, this is the lack of action pairs constituting the higher end of limiting average rewards. This is counteracted by introducing a secondary drawing for joint relative frequencies, $\dot{\rho}_{ij}$, namely:

$$\dot{\rho}_{ij} \sim Beta(A,B)$$

where different values for A and B are used to create a distribution of points which is weighted towards the following intervals for the ρ matrix entries:

$$\begin{aligned} \rho_{11} =& [0.75-1] \\ \rho_{12} =& [0.01-0.10] \\ \rho_{21} =& [0.01-0.10] \\ \rho_{22} =& 0 \end{aligned}$$

These intervals are achieved by using the following distributions for $Beta$:

$$\begin{aligned} \dot{\rho}_{11} &\sim Beta(20,1) \\ \dot{\rho}_{12} &\sim Beta(2,8) \\ \dot{\rho}_{21} &\sim Beta(2,8) \\ \dot{\rho}_{22} &= 0 \end{aligned}$$

We then again normalise the four entries of $\dot{\rho}_{ij}$:

$$\rho = \frac{1}{\dot{\rho}_{11}+\dot{\rho}_{12}+\dot{\rho}_{21}+\dot{\rho}_{22}} \begin{bmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{bmatrix}$$

Resulting in an additional drawing of rho which is weighted on limiting average rewards in the higher end.

The assumption is introduced to increase ease of computation. This distribution of ρ is however never actually fully representative for player behaviour since it relies on one distribution making it a core assumption.

As discussed previously the third assumption is important since it pertains to the the convergence of outcomes through large numbers. Using the law of large numbers in combination with contest theory we assume that in a repeated games scenario, like the computations in this thesis, individual agent behaviour converges to an aggregated distribution of the market. The individual agents in this case are the consumers in the market who prefer either player through their long-term dyadic advertisement behaviour. This means that the aggregated preferences of the individual agents (with a law of large numbers) allows for the short-term competition forms to have a parameter which is influenced by previous advertisement behaviour of the players.

The reason that this is an assumption is because this might in reality not hold. Since the stochastic nature of reality might cause small outcome discrepancies, which causes the agents to not settle on one player. In turn making the market distribution invalid, yielding no results. For the sake of wanting results though, the theoretical space holds, and the aggregated individual agent preferences yield a market distribution which will be taken as valid, resulting in feasible outcomes. It is important to understand that the process of individual agents having an aggregated preference through the advertisement history of the players is a stochastic process separate from the long- and short-term parameters such as price and quantity.

Fourth is the duopolistic nature of the model. As explained in *Section 2* most markets actually hold a much more complex oligopolistic nature involving more than two players. For the sake of isolating findings however we use a duopolistic market from which the effects of the change of certain variables can be determined much easier. Further research could perhaps include additional players for which new order relationships could appear.

The combination of these assumptions might not make the outcomes very realistic, it is however along the lines of Aumann (2008) more important to establish theoretically valid models which can then later be made more complicated after establishing theoretical validity.

4 MODEL

In this section we discuss the mathematical ground work of the theoretical model and then introduce eight theoretical short-term competition variants. These variants, as partially discussed in the second section, are forms of competition (market structures) between two players. We show the theoretical equations, and operations performed to determine them, and elaborate shortly on the reasoning.

We first introduce the theoretical models of Bertrand and then those of Cournot, this is because the calculations that Joosten (2015) did were on a Bertrand market for differentiated goods (or incomplete substitutes) and that the Cournot model we intend to use, is a transformation of the Bertrand model with regards to the demand functions. Additionally sales potential, which is a type of advertisement effect, which influences quantity (in Bertrand) and price (in Cournot) through advertisement decisions per player is discussed at the end and included in the Python model.

With regards to the implications of the theoretical models is it important to understand that both players engage in one of the eight possibilities of competitive interaction in the short-term, meaning one possible competition is repeated over and over again at each stage of play to get results. For the algorithmic model results are accumulated for these eight different environments separately to try and ascertain specifications of the model.

The model operates on some mathematical principles, these principles and their derivations are taken from Joosten (2015) and will be elaborated. Joosten (2015) introduces the idea of immediate- and long-term effects of advertisement. The immediate effects of advertisement directly influence short-term competition by means of demand, this effects occurs through the decisions of both players to advertise or to not advertise, this leads to the following notation for demand of the players:

$$x_k^{i^A,i^B} = D_k^{i^A,i^B} - z_{1k}^{i^A,i^B} p_k + z_{2k}^{i^A,i^B} p_{\neg k}$$

Here $i^k \in 0, 1$ denotes whether player $k \in A, B$ advertises (1) or not (2), here $\neg k$ denotes "not" k, in one stage of the games. $D_k^{i^A,i^B}$ denotes the total demand for player k given the advertisement decisions of both players. $z_{1k}^{i^A,i^B}$ and $z_{2k}^{i^A,i^B}$ denote the direct advertisement price effects and the cross advertisement price effect. This means there are four possible combinations for advertisement of both players at each stage of the sequence of games. We do not assume advertisement costs to be zero, the following can be induced regarding advertisement decisions (i^A, i^B):

$$(D, z)^{i^A,i^B} = (D_A^{i^A,i^B}, D_B^{i^A,i^B}, z_1^{i^A,i^B}, z_2^{i^A,i^B}, z_3^{i^A,i^B}, z_3^{i^A,i^B}).$$

Where z_1 is the direct price effects for player A, z_2 the cross price effects for player A, z_3 the direct price effects for player B and z_4 the cross price effects for player B. These functions will also be used to define the short-term demand functions. For additional specifications regarding simplifying the terms and the proof of the above functions we refer to Joosten (2015).

What we can now do is establish a series of matrices denoting the long-term effects of advertising, additionally we also need the principle of jointly convergent pure strategies to explain how the effects converge over time. Important now for the long-term effects is to understand that advertisement both has a cumulative effect on the total market size as well the distribution of market share for both players. The following notations are needed for introducing these long-term effects: We choose $y^k_{t'} = (i^k_1, ..., i^k_{t'-1})$ to be the sequence of actions taken by player $k \in A, B$ until stage $t' \geq 2$. Additionally we let $\lambda^{m \times n}$ denote a set of real-numbered non-negative $m \times n$-matrices such that all components add up to unity. To illustrate:

$$\lambda^{m \times n} = \{o \in \mathbb{R}^{m \times n} \mid o_{ij} \geq 0 \text{ for all } i, j, \text{ and } \sum_{ij} o_{ij} = 1\}$$

We now let matrix $U(i', j') \in \lambda^{2 \times 2}$ be defined by:

$$U_{ij}\left(i', j'\right) = \begin{cases} 1 & \text{if } (i,j) = (i',j') \\ 0 & \text{otherwise} \end{cases}$$

Now we take $s \geq 0$, and define matrix $\rho_t \in \lambda^{2 \times 2}$ recursively for $t \leq t'$ by;

$$\begin{aligned} &\rho_1 = \tilde{\rho} \in \lambda^{2 \times 2}, \quad \text{and} \\ &\rho_t = \frac{s+t-1}{s+t}\rho_{t-1} + \frac{1}{s+t}U\left(i^A_{t-1}, i^B_{t-1}\right) \end{aligned}$$

The interpretation of this matrix is that entry ij of ρ_t "approximates" the relative frequency with which the action pair ij was used before stage $t \geq 2$.

We now additionally introduce the idea of jointly-convergent pure-strategies from Joosten (2015). Jointly-convergent pure-strategies yield a set of jointly-convergent pure-strategy rewards which is the principle behind converging joint frequency pairs for the players. We follow the same approach as Joosten (2015) in explaining the jointly-convergent pure-strategy rewards and refer to it for the proof of the theorem.

We first introduce the sets of all strategies for A, respectively B, which we will denote by Δ^A respectively Δ^B, with $\Delta \equiv \Delta^A \times \Delta^B$. Here a *strategy* prescribes for any state and history of play. and at all stages t, a mixed action to be used by a player. For this to hold both players know, at any stage t, the state visited and actions chosen at stage $u < t$ denoted by (ρ_u, i^A_u, i^B_u).

We also state that the stage payoff for player k ($k \in A, B$), at stage t, is stochastic and depends on the strategy-pair $(\mu, \sigma) \in \Delta$; and the *expected stage payoff* is denoted by $E_t^k(\mu, \sigma)$.

Since we have an infinite number of stage games, the players also receive an infinite stream of stage payoffs during the play. We also assume players wish to maximise their average rewards (maximising profit). We therefore state that for a given strategy-pair (μ, σ), the *average reward* of player k is given by $\theta^k(\mu, \sigma) = \liminf\limits_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E_t^k(\mu, \sigma)$; $\theta(\mu, \sigma) \equiv (\theta^A(\mu, \sigma), \theta^B(\mu, \sigma))$.

Joosten (2015) mentions that it difficult to determine the set of feasible (average) rewards, F, directly. He mentions that it is not uncommon in analysis of repeated or stochastic games to limit the scope of strategies and focus on rewards. He did both and so will we. The focus being on rewards from pure strategies that are pure and jointly-convergent, we can then extend the analysis to obtain more feasible rewards. This leads to the following definition by Joosten (2015: First the definition of the set of pure strategies for player k, which is Ω^k, where $\Omega \equiv \Omega^A \times \Omega^B$. Here a strategy is *pure* if an action is chosen with probability 1 at *each* stage t. Now the following can be said regarding jointly-convergent strategies; a strategy-pair $(\mu, \sigma) \in \Delta$ is jointly-convergent if, and only if, $o^{\mu,\sigma} \in \lambda^{m \times n}$ exists such that for all $\varepsilon > 0$:

$$\limsup_{t \to \infty} Pr_{\mu,\sigma} \left(\left| \frac{\#\{i_u^A = i \text{ and } i_u^B = j | 1 \leq u \leq t\}}{t} - o_{i,j}^{\mu,\sigma} \right| \geq \varepsilon \right) = 0$$

for all $(i, j) \in I$,

where $Pr_{\mu,\sigma}$ denotes the probability under strategy-pair (μ, σ). We let χ denote the set of jointly-convergent strategy-pairs. Under a pair of jointly-convergent strategies, the relative frequency of each action pair $(i, j) \in I$ converges with probability 1 to $o_{i,j}^{\mu,\sigma}$ (i.e. $\lim\limits_{t \to \infty} Q_{\mu,\sigma}\{U\left(y_t^A, y_t^B\right)\} = o^{\mu,\sigma}$, which in turn implies $\lim\limits_{t \to \infty} Q_{\mu,\sigma}\{\rho_t\} = o^{\mu,\sigma}$). From this follows the *set of jointly-convergent pure-strategy rewards*, which is given by;

$$P^{\chi} \equiv cl\{(x^1, x^2) \in \mathbb{R}^2 \exists_{(\mu,\sigma) \in \Omega \cap \chi} : (\theta^k(\mu, \sigma), \theta^k(\mu, \sigma)) = (x^1, x^2)\},$$

where *cl* is the closure of the set. For any pair of rewards in this set, we can find a pair of jointly-convergent pure strategies that yield rewards arbitrarily close to original pair of rewards.

Due to the convergent nature of the joint action pairs and the joint relative frequencies can now induce the following general matrix for ρ, which is as follows:

$$\begin{bmatrix} \rho_{11}^t & \rho_{12}^t \\ \rho_{21}^t & \rho_{22}^t \end{bmatrix} \lim_{t\to\infty} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

In the theoretical model the values of the ρ matrix are used as input in several other variables, which are then used in each short-term competitive model. In addition to ρ there are several other variables used in all eight variants, for Cournot there are some transformations to them which will be explained later. The choice for some of the values in the additional variables is trivial. These variables can be adjusted as an input variable in the algorithmic model, since the effects of changing these numbers has distinct effects on the limiting average rewards of the models. The variables are as follows:

$$\begin{aligned} &q(\rho) \\ &D(\rho) \\ D_{A,0} = D_{B,0} &= 100 \\ c_A = c_B &= 3 \\ &z_1(\rho) \\ &z_2(\rho) \\ &z_3(\rho) \\ &z_4(\rho) \end{aligned}$$

$q(\rho)$ is the aggregated market share distribution depending on the advertisement action by player k. It is used to divide the demand, meaning that for a certain combination of joint relative frequencies (ρ) the market is distributed more in favor of one player. Here $D_{A,0}$ and $D_{B,0}$ are the fixed demands of both players, they are trivially selected values and quantify how much demand there is regardless of investment. $D(\rho)$ is the variable demand which is dependent on advertisement actions, since those grow the market, and another parameter u. u is the positive demand intercept and is a value which can be changed in the algorithm to reflect how advertisement behaviour influences the flexibility of demand. c_A and c_B are the variable cost parameters, they are slightly influenced by the investment choices of the players and chosen at a fixed number. The last four variables are the factors which influence demand depending on advertisement behaviour, these are the variables $z_1(\rho)$, $z_2(\rho)$, $z_3(\rho)$ and $z_4(\rho)$. They indicate the rate with which certain joint relative frequencies influence demand, cost and market distribution. They are therefore a degree of sensitivity to certain advertisement behaviour by the players.

In the following sections the above expressions will dispense with ρ, since it highly conveniences notational and visual clarity to do so. It is important to understand however that all expressions using q, D and z_i are using functions dependent on ρ. A short disclaimer will be placed at the start of each section for every short-term competition form to remind the reader. To illustrate the visual difference we will take a demand function from the next section (short-form) and show how the variables are actually functions of ρ, showing the notational difference. The demand is expressed in x_A for Bertrand competition. The short-form notation (omitting ρ) represents the exact same as the long-form (with ρ) and the functions look as follows:

$$
\begin{aligned}
x_A =& D_{A,0} + q(\rho)D(\rho) - z_1(\rho)p_A + z_2(\rho)p_B \\
x_A =& D_{A,0} + qD - z_1 p_A + z_2 p_B
\end{aligned}
$$

4.1 Bertrand competition

The first four forms are types of Bertrand competition. Bertrand is a very well known competition type, which comes down to a description of the interaction among competitors that set prices from which competitive quantities arise under the assumption of homogeneous products. In the following sections we introduce the principles of the standard Bertrand model, which is our first model. We then introduce two different Stackelberg-Bertrand models, our second and third model. And then introduce collusion under Bertrand, which is our fourth and last Bertrand model. All models use the core principles of the regular Bertrand model as the order of operation and way through which optimal profits are determined, this will become evident throughout the sections.

4.1.1 Regular Bertrand short-term competition

In the first short-term competition variation of Bertrand we try and create a theoretical model which determines the optimal profit per player. The framework of the model is heavily inspired on the model used in Joosten (2015). The starting point of the Bertrand model is therefore an expression used in his paper.

As previously mentioned the following equations using q, D or z_i are all functions of ρ. We dispense with ρ for notational convenience.

First we define two expressions for demand with which Bertrand starts and define the profit maximisation problems with which we wish to determine optimal price, optimal quantity and optimal profit respectively.

First the demand expressions for Bertrand (x is quantity and p is price):

$$x_A = D_{A,0} + qD - z_1 p_A + z_2 p_B \tag{1}$$

$$x_B = D_{B,0} + (1-q)D - z_3 p_B + z_4 p_A \tag{2}$$

Where the variables used are as explained before. The maximisation problems for regular Bertrand are (where $c_{A,0}$ and $c_{B,0}$ are the fixed costs):

$$\max_{p_A} x_A p_A - c_A x_A - c_{A,0} \tag{3}$$

$$\max_{p_B} x_B p_B - c_B x_B - c_{B,0} \tag{4}$$

Now using Equations (1) and (2) and entering them into Equations (3) and (4), we get the following two equations for the maximisation problems with demand:

$$\begin{aligned} &\max_{p_A} x_A(p_A - c_A) - c_{A,0} \\ &= \max_{p_A} (D_{A,0} + qD - z_1 p_A + z_2 p_B)(p_A - c_A) - c_{A,0} \end{aligned} \tag{5}$$

$$\max_{p_B} x_B(p_B - c_B) - c_{B,0}$$

$$= \max_{p_B} (D_{B,0} + (1-q)D - z_3 p_B + z_4 p_A)(p_B - c_B) - c_{B,0} \tag{6}$$

Now we determine the following first order conditions (FOC):

$$\begin{aligned} D_{A,0} + qD - z_1 p_A + z_2 p_B - z_1(p_A - c_A) = 0 \\ D_{B,0} + (1-q)D - z_3 p_B + z_4 p_A - z_3(p_B - c_B) = 0 \end{aligned}$$

Which can then be simplified to:

$$D_{A,0} + qD - 2z_1 p_A + z_2 p_B + z_1 c_A = 0 \tag{7}$$

$$D_{B,0} + (1-q)D - 2z_3 p_B + z_4 p_A + z_3 c_B = 0 \tag{8}$$

Now the optimal price function for player B is first determined. In order to determine the optimal price function for B, we aim to eliminate p_A from the expression. We first multiply the FOC for A with z_4 and then multiply the FOC for B with $2z_1$, which leads to the following:

$$\begin{aligned} z_4 D_{A,0} + z_4 qD - 2z_1 z_4 p_A + z_2 z_4 p_B + z_1 z_4 c_A) &= 0 \\ 2z_1 D_{B,0} + (1-q)2z_1 D - 4z_1 z_3 p_B + 2z_1 z_4 p_A + \\ 2z_1 z_3 c_B &= 0 \end{aligned}$$

We sum both expressions and simplify to get the expression for p_B^*, the optimal price for player B:

$$\begin{aligned} 0 = z_4 D_{A,0} + 2z_1 D_{B,0} + ((1-q)2z_1 + z_4 q)D - \\ (4z_1 z_3 - z_2 z_4)p_B + z_1 z_4 c_A + 2z_1 z_3 c_B \end{aligned}$$

$$p_B^* = \frac{z_4(D_{A,0} + qD) + 2z_1(D_{B,0} + (1-q)D) + z_1 z_4 c_A + 2z_1 z_3 c_B}{4z_1 z_3 - z_2 z_4} \tag{9}$$

Now we similarly approach the optimal price function for A. Now we aim eliminate p_B from the expression. First we multiply the FOC of A with $2z_3(\rho)$ and then multiply the FOC of B with $z_2(\rho)$, which leads to the following:

$$2z_3 D_{A,0} + 2z_3 qD - 4z_1 z_3 p_A + 2z_2 z_3 p_B + 2z_1 z_3 c_A) = 0$$

$$z_2 D_{B,0} + (1-q)z_2 D - 2z_2 z_3 p_B + z_2 z_4 p_A + z_2 z_3 c_B = 0$$

Again both expressions are summed and simplified to get the expression of p_A^*, the optimal price for player A:

$$\begin{aligned} 0 = 2z_3 D_{A,0} + z_2 D_{B,0} + ((1-q)z_2 + 2z_3 q)D - (4z_1 z_3 - z_2 z_4)p_A \\ + 2z_1 z_3 c_A + z_2 z_3 c_B \end{aligned}$$

$$p_A^* = \frac{z_2(D_{B,0} + (1-q)D) + 2z_3(D_{A,0} + qD) + 2z_1 z_3 c_A + z_2 z_3 c_B}{4z_1 z_3 - z_2 z_4} \tag{10}$$

Now we compare Equations (9) and (10) to check for similarity and symmetry, which if present implies, given our core assumption, that the prices are theoretically correct:

$$p_A^* = \frac{z_2(D_{B,0} + (1-q)D) + 2z_3(D_{A,0} + qD) + 2z_1 z_3 c_A + z_2 z_3 c_B}{4z_1 z_3 - z_2 z_4}$$

$$p_B^* = \frac{z_4(D_{A,0} + qD) + 2z_1(D_{B,0} + (1-q)D) + z_1 z_4 c_A + 2z_1 z_3 c_B}{4z_1 z_3 - z_2 z_4}$$

From which, using Equations (1) and (2), we can determine the equations for the optimal quantity per player. Namely x_A^* and x_B^*. Which are as follows:

$$x_A^* = D_{A,0} + qD - z_1 p_A^* + z_2 p_B^* \tag{11}$$

$$x_B^* = D_{B,0} + (1-q)D - z_3 p_B^* + z_4 p_A^* \tag{12}$$

Then lastly for our regular Bertrand model we provide the equations for the optimal profit per player

$$\pi_A^* = x_A^* \left(p_A^* - c_A\right) - c_{A,0} = \left(D_{A,0} + qD - z_1 p_A^* + z_2 p_B^*\right)\left(p_A^* - c_A\right) - c_{A,0} \tag{13}$$

$$\pi_B^* = x_B^* \left(p_B - c_B\right) - c_{B,0} = \left(D_{B,0} + (1-q)D - z_3 p_B^* + z_4 p_A^*\right)\left(p_B^* - c_B\right) - c_{B,0} \tag{14}$$

For all the functions of Bertrand, we state the following with regards to ρ, as an addition to the disclaimer regarding functions of ρ as input:

$$\begin{aligned} q &= \frac{\rho_{11} + \rho_{12}}{2\rho_{11} + \rho_{12} + \rho_{21}} \\ D &= 100\left(3\rho_{11} + \rho_{12} + \rho_{21}\right) \\ z_1 &= 24 - 6\left(\rho_{11} + \rho_{12}\right) \\ z_2 &= 8 - 4\left(\rho_{11} + \rho_{21}\right) \\ z_3 &= 24 - 6\left(\rho_{11} + \rho_{21}\right) \\ z_4 &= 8 - 4\left(\rho_{11} + \rho_{12}\right) \end{aligned}$$

An important side note for q is that with a certain pure strategy in which both players do not advertise, namely ρ_{22}, q has a division by zero, which is not possible. To counteract this it is taken as 0 in this scenario.

4.1.2 Stackelberg-Bertrand short-term competition

The next short-term competition form is a variation on Bertrand which results in two similar models, the Stackelberg variation. Stackelberg uses a leader-follower mechanic to make the price of one player dependant on the price of the other. This is used to represent the practical case in which, in contrast to regular Bertrand, one player decides on their optimal price before the other rather than simultaneously. The model set up is similar since the core model is still Bertrand.

As previously mentioned the following equations using q, D or z_i are all functions of ρ. We dispense with ρ for notational convenience.

The demand, maximisation problem and resulting two formulas are therefore exactly the same. We end up again with Equations (1), (2), (3) and (4). Which again leads to Equations (7) and (8). The calculations only apply for the cases where player A and B are *followers*. From this we continue by adding the responses and maximisation problems in case of *leadership*. To again determine the optimal profit for player A and B, both as leader and as follower.

Since we assume symmetry in *Section 3* we will only regard the scenario in which player A is the leader and player B is the follower.

If A is the leader the best response for B is determined by taking the maximisation with demand of B, and rewriting it for an expression in p_B:

$$\begin{aligned} D_{B,0}+(1-q)D-2z_3p_B+z_4p_A+z_3c_B &= 0 \to \\ D_{B,0}+(1-q)D+z_4p_A+z_3c_B &= 2z_3p_B \to \\ \frac{D_{B,0}+(1-q)D+z_4p_A+z_3c_B}{2z_3} &= p_B \end{aligned} \tag{15}$$

Now we take take Equation (5), the previous maximisation problem with demand of regular Bertrand and substitute the new expression Equation (15), with the response of **follower B** into it. Which will lead to a new maximisation problem for **leader A**, the computation is as follows:

$$\max_{p_A}\left(D_{A,0}+qD-z_1p_A+z_2p_B\right)\left(p_A-c_A\right)-c_{0A}$$

and,

$$p_B=\frac{D_{B,0}+(1-q)D+z_4p_A+z_3c_B}{2z_3}$$

Substitution then leads to the following expression:

$$\max_{p_A}\left(D_{A,0}+qD-z_1p_A+z_2\frac{D_{B,0}+(1-q)D+z_4p_A+z_3c_B}{2z_3}\right)\left(p_A-c_A\right)-c_{A,0} \tag{16}$$

Now to determine the optimal price for **leader A**, we take the First Order Conditions again:

$$
\begin{aligned}
D_{A,0}+qD-z_1p_A+z_2\frac{D_{B,0}+(1-q)D+z_4p_A+z_3c_B}{2z_3}+\frac{-2z_1z_3+z_2z_4}{2z_3}(p_A-c_A)=0\rightarrow\\
D_{A,0}+qD-2\left(\frac{2z_1z_3-z_2z_4}{2z_3}\right)p_A+z_2\frac{D_{B,0}+(1-q)D+z_3c_B}{2z_3}+\frac{2z_1z_3-z_2z_4}{2z_3}c_A=0\rightarrow\\
D_{A,0}+qD+z_2\frac{D_{B,0}+(1-q)D+z_3c_B}{2z_3}+\frac{2z_1z_3-z_2z_4}{2z_3}c_A=\frac{4z_1z_3-2z_2z_4}{2z_3}p_A\\
\frac{2z_3}{4z_1z_3-2z_2z_4}\frac{2z_3(D_{A,0}+qD)+z_2D_{B,0}+(1-q)z_2D+z_2z_3c_B}{2z_3}+\frac{1}{2}c_A=p_A\rightarrow\\
\frac{2z_3(D_{A,0}+qD)+z_2(D_{B,0}+(1-q)D)+z_2z_3c_B}{4z_1z_3-2z_2z_4}+\frac{1}{2}c_A=p_A^{LB}
\end{aligned}
\tag{17}
$$

So now we have an equation for the optimal price of **leader A** in Bertrand, meaning we can take Equation (15) and plug in p_A^{LB}, to get the equation for the optimal price of **follower B**, this results in:

$$\frac{D_{B,0}+(1-q)D+z_4p_A^{LB}+z_3c_B}{2z_3}=p_B^{FB} \tag{18}$$

Then if we plug in Equations (17) and (18) into Equations (1) and (2), we get, similarly to regular Bertrand, the expressions for the optimal quantity with **leader A** and **follower B**:

$$x_A^{LB}=D_{A,0}+qD-z_1p_A^{LB}+z_2p_B^{FB} \tag{19}$$

$$x_B^{FB}=D_{B,0}+(1-q)D-z_3p_B^{FB}+z_4p_A^{LB} \tag{20}$$

Which then finally leads to the profit equations for Stackelberg-Bertrand with **leader A** and **follower B**:

$$\pi_A^{LB}=x_A^{LB}\left(p_A^{LB}-c_A\right)-c_{A,0} \tag{21}$$

$$\pi_B^{FB}=x_B^{FB}\left(p_B^{FB}-c_B\right)-c_{B,0} \tag{22}$$

To which the same logic is applied as Equations (13) and (14) with regards to input and ρs.

In the exact same order and line of logic the scenario for **follower A** and **leader B** is approached, it is however mirrored with regards to response and dictation of price. The demand and maximisation problem with and without demand is approached from a followers perspective again, which is entirely similar to Equations (1), (2), (3), (4), (7) and (8) again.

Finally we can compare Equations (17), (18) with the mirrored equations of **leader B and follower A** to see the equations are similar and symmetrical, so we compare:

$$p_A^{LB} = \frac{2z_3\left(D_{A,0}+qD\right)+z_2\left(D_{B,0}+(1-q)D\right)+z_2z_3c_B}{4z_1z_3-2z_2z_4}+\frac{1}{2}c_A$$
$$p_B^{FB} = \frac{D_{B,0}+(1-q)D+z_4p_A^{LB}+z_3c_B}{2z_3}$$

With;

$$p_B^{LB} = \frac{2z_1\left(D_{B,0}+(1-q)D\right)+z_4\left(D_{A,0}+qD\right)+z_1z_4c_A}{4z_1z_3-2z_2z_4}+\frac{1}{2}c_B$$
$$p_A^{FB} = \frac{D_{A,0}+qD+z_2p_B^{LB}+z_1c_A}{2z_1}$$

Both equations check out with regards to the prior discussed symmetry of advertisement behaviour and the regular Bertrand model.

4.1.3 Short-term collusion under Bertrand

The final variation on Bertrand short-term competition is collusion. In this variation players collude in one dimension. Here we approach the model from a combined maximisation problem. In the previous Bertrand model both players tried to separately maximise their profit, in this model both players try to maximise their combined profit.

As previously mentioned the following equations using q, D or z_i are all functions of ρ. We dispense with ρ for notational convenience.

The profit maximisation problem for collusion is different since it is a combination of the player A and player B problem for regular Bertrand.

$$\max_{p_A,p_B} x_A\left(p_A-c_A\right)-c_{A,0}+x_B\left(p_B-c_B\right)-c_{B,0} \tag{23}$$

Now similarly to the previous model we plug Equations (1) and (2) into Equation (23) to acquire the maximisation problem with demand which is as follows:

$$\begin{aligned}= \max_{p_A,p_B} &\left(D_{A,0}+qD-z_1p_A+z_2p_B\right)\left(p_A-c_A\right)-c_{A,0}+\\ &\left(D_{B,0}+(1-q)D-z_3p_B+z_4p_A\right)\left(p_B-c_B\right)-c_{B,0}\end{aligned} \tag{24}$$

To determine the optimal profit for player A and B, we take the FOC again resulting in two equations:

$$\begin{gathered}D_{A,0}+qD-z_1p_A+z_2p_B-z_1\left(p_A-c_A\right)+z_4\left(p_B-c_B\right)=0\rightarrow\\ D_{A,0}+qD-2z_1p_A+\left(z_2+z_4\right)p_B+z_1c_A-z_4c_B=0\end{gathered} \tag{25}$$

$$
\begin{gathered}
D_{B,0}+(1-q)D-z_3p_B+z_4p_A-z_3\left(p_B-c_B\right)+z_2\left(p_A-c_A\right)=0\rightarrow \\
D_{B,0}+(1-q)D-2z_3p_B+\left(z_2+z_4\right)p_A+z_3c_B-z_2c_A=0
\end{gathered} \tag{26}
$$

To find expressions for p_A^{CB}, the optimal price of collusion under Bertrand for player A, and for p_B^{CB}, the optimal price for collusion under Bertrand for player B, we perform several mathematical operations on Equations (25) and (26) respectively.
For determining p_A^{CB}, here the superscript CB stands for *Collusion Bertrand*, we must eliminate p_B from Equations (25) and (26). To do this we multiply Equation (25) with $2z_3$, and multiply Equation (26) with (z_2+z_4) resulting in the following computations:

$$
2z_3\left(D_{A,0}+qD\right)-4z_1z_3p_A+2z_3\left(z_2+z_4\right)p_B+2z_1z_3c_A-2z_3z_4c_B=0 \tag{27}
$$

$$
\begin{aligned}
&\left(z_2+z_4\right)\left(D_{B,0}+(1-q)D\right)-2z_3\left(z_2+z_4\right)p_B+\left(z_2+z_4\right)^2p_A \\
&\qquad +z_3\left(z_2+z_4\right)c_B-z_2\left(z_2+z_4\right)c_A=0
\end{aligned} \tag{28}
$$

Then both Equations (27) and (28) are summed leading to the following expression for p_A^{CB}:

$$
\begin{aligned}
&-\left(4z_1z_3-\left(z_2+z_4\right)^2\right)p_A+2z_3\left(D_{A,0}+qD+z_1c_A-z_4c_B\right) \\
&\qquad +\left(z_2+z_4\right)\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)=0\rightarrow
\end{aligned}
$$

$$
\frac{2z_3\left(D_{A,0}+qD+z_1c_A-z_4c_B\right)+\left(z_2+z_4\right)\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)}{4z_1z_3-\left(z_2+z_4\right)^2}=p_A^{CB} \tag{29}
$$

For determining p_B^{CB}, we must eliminate p_A from Equation (25) and (26). To do this we multiply Equation (25) with (z_2+z_4), and multiply Equation (26) with $2z_3$, which is the mirrored operation for p_A^{CB}, this results in the following computations:

$$
\begin{aligned}
&-2z_1\left(z_2+z_4\right)p_A+\left(z_2+z_4\right)^2p_B \\
&\qquad +\left(z_2+z_4\right)\left(D_{A,0}+qD+z_1c_A-z_4c_B\right)=0
\end{aligned} \tag{30}
$$

$$
-4z_1z_3p_B+2z_1\left(z_2+z_4\right)p_A+2z_1\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)=0 \tag{31}
$$

Then both Equations (30) and (31) are summed leading to the following expression for p_B^{CB}:

$$
\begin{aligned}
&-\left(4z_1z_3-\left(z_2+z_4\right)^2\right)p_B+\left(z_2+z_4\right)\left(D_{A,0}+qD+z_1c_A-z_4c_B\right) \\
&\qquad +2z_1\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)=0\rightarrow
\end{aligned}
$$

$$\frac{2z_1\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)+\left(z_2+z_4\right)\left(D_{A,0}+qD+z_1c_A-z_4c_B\right)}{4z_1z_3-\left(z_2+z_4\right)^2}=p_B^{CB} \tag{32}$$

Then we separate the two optimal prices again and plug Equations (29) and (32), into Equations (1) and (2), and get the optimal quantities x_A^{CB} and x_B^{CB} in the following equations:

$$x_A^{CB}=D_{A,0}+qD-z_1p_A^{CB}+z_2p_B^{CB} \tag{33}$$

$$x_B^{CB}=D_{B,0}+(1-q)D-z_3p_B^{CB}+z_4p_A^{CB} \tag{34}$$

Then the profit equations for collusion under Bertrand with can be solved:

$$\pi_A^{CB}=x_A^{CB}\left(p_A^{CB}-c_A\right)-c_{A,0} \tag{35}$$

$$\pi_B^{CB}=x_B^{CB}\left(p_B^{CB}-c_B\right)-c_{B,0} \tag{36}$$

Finally we again compare the optimal price Equations (29) and (32), to see how the equations are similar and symmetrical, the comparison is as follows:

$$p_A^{CB}=\frac{2z_3\left(D_{A,0}+qD+z_1c_A-z_4c_B\right)+\left(z_2+z_4\right)\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)}{4z_1z_3-\left(z_2+z_4\right)^2}$$
$$p_B^{CB}=\frac{2z_1\left(D_{B,0}+(1-q)D+z_3c_B-z_2c_A\right)+\left(z_2+z_4\right)\left(D_{A,0}+qD+z_1c_A-z_4c_B\right)}{4z_1z_3-\left(z_2+z_4\right)^2}$$

Again the equations are similar and symmetrical.

4.2 Cournot competition

As mentioned in the Bertrand introduction it is the start for all our other models . The Cournot competition we use is a transformation from the Bertrand approach. A transformation takes place in the matrix for advertisement flexibility since these variables are inverse when demand is switched from quantity to price.

As previously mentioned the following equations using q, D or z_i are all functions of ρ, additionally the expressions of g_i are also functions of ρ, since they are transformations of the z_i functions. So ρ is again dispensed with for notational convenience.

To start, we break down the Bertrand model, first the demand functions, (1) and (2), which are as follows:

$$\begin{aligned} x_A &= D_{A,0} + qD - z_1 p_A + z_2 p_B \\ x_B &= D_{B,0} + (1-q)D - z_3 p_B + z_4 p_A \end{aligned}$$

From this we try to transform the z vector to work for Cournot, hence the Bertrand demand functions lead to the following:

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} D_{A,0} + qD \\ D_{B,0} + (1-q)D \end{bmatrix} - \begin{bmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{bmatrix} \begin{bmatrix} p_A \\ p_B \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} D_{A,0} + qD \\ D_{B,0} + (1-q)D \end{bmatrix} - \begin{bmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

From this it is interesting to try and understand the $\begin{pmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{pmatrix}^{-1}$ matrix. Some further transformation yields the following:

$$\begin{bmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{z_3}{z_1 z_3 - z_2 z_4} & \frac{z_2}{z_1 z_3 - z_2 z_4} \\ \frac{z_4}{z_1 z_3 - z_2 z_4} & \frac{z_1}{z_1 z_3 - z_2 z_4} \end{bmatrix} = \frac{1}{z_1 z_3 - z_2 z_4} \begin{bmatrix} z_3 & z_2 \\ z_4 & z_1 \end{bmatrix}$$

Now this equation leads to another possible formulation which results in some new variables, similarly to those introduced in the Bertrand section, the z_s. These new g variables serve as the factors influencing demand for Cournot, which rather than price functions on quantities, which in turn lead to a competitive price. The g_s are as follows:

$$\frac{1}{z_1 z_3 - z_2 z_4} \begin{bmatrix} z_3 & z_2 \\ z_4 & z_1 \end{bmatrix} = \begin{bmatrix} g_1 & -g_2 \\ -g_4 & g_3 \end{bmatrix} \rightarrow$$

$$\begin{aligned} g_1 &= \frac{z_3}{z_1 z_3 - z_2 z_4} \\ g_2 &= -\frac{z_2}{z_1 z_3 - z_2 z_4} \\ g_3 &= \frac{z_1}{z_1 z_3 - z_2 z_4} \\ g_4 &= -\frac{z_4}{z_1 z_3 - z_2 z_4} \end{aligned}$$

An additional exception for the Cournot model is that the variables $D_{A,0}$, q, and D are combined into a the variable Y_A and $D_{B,0}$, $(1-q)$, and D are combined into the variable Y_B. The is done since Y_A and Y_B are shorter-form notations. The transformation is based on the above and the inverse demand functions in price which is how Cournot differs from Bertrand. The inverse demand functions are as follows:

$$p_A = Y_A - g_1 x_A + g_2 x_B$$
$$p_B = Y_B - g_3 x_B + g_4 x_A$$

We can now break-down the demand functions as follows:

$$\begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} Y_A \\ Y_B \end{bmatrix} - \begin{bmatrix} g_1 & -g_2 \\ -g_4 & g_3 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

Since as we mentioned before Y_A and Y_B must be equal to the inverse of the z matrix multiplied by the demand vector the following expressions for Y_A and Y_B emerge, which hold for all Cournot models in the following sections, they are as follows:

$$\begin{aligned} \begin{bmatrix} Y_A \\ Y_B \end{bmatrix} &= \begin{bmatrix} z_1 & -z_2 \\ -z_4 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} D_{A,0} + qD \\ D_{B,0} + (1-q)D \end{bmatrix} \\ &= \frac{1}{z_1 z_3 - z_2 z_4} \begin{bmatrix} z_3 & z_2 \\ z_4 & z_1 \end{bmatrix} \begin{bmatrix} D_{A,0} + qD \\ D_{B,0} + (1-q)D \end{bmatrix} \end{aligned}$$

Which if computed and simplified leads to the follows two functions:

$$Y_A = \frac{z_3 (D_{A,0} + qD) + z_2 (D_{B,0} + (1-q)D)}{z_1 z_3 - z_2 z_4}$$
$$Y_B = \frac{z_4 (D_{A,0} + qD) + z_1 (D_{B,0} + (1-q)D)}{z_1 z_3 - z_2 z_4}$$

An important assumption is made with regards to the inverse of the z matrix. This is the fact that it even exists at all. Since if it does not exist the functions become infeasible, we therefore assume that the inverse always exists. This also eliminates a problem with the numerators of the Y_A and Y_B equations since a feasible inverse of z ensures the values of the numerators do not equate to 0.

4.2.1 Regular Cournot short-term competition

The fifth model is the first major alteration since it is the fundamental Cournot short-term competitive model. It, similarly to Bertrand, runs on a demand function, the difference however is that demand for Cournot is in price rather than in quantity.

As previously mentioned the following equations using q, D and g_i are all functions of ρ. We dispense with ρ for notational convenience.

The system for Cournot is, as mentioned above in the computation for Y_A and Y_B, as follows:

$$p_A = Y_A - g_1 x_A + g_2 x_B \tag{37}$$

$$p_B = Y_B - g_3 x_B + g_4 x_A \tag{38}$$

The regular Cournot model also has maximisation problems similar to Bertrand, but this time maximising on x_A and x_B rather than on p_A and p_B, which are as follows:

$$\max_{x_A} p_A x_A - c_A x_A - c_{A,0} \tag{39}$$

$$\max_{x_B} p_B x_B - c_B x_B - c_{B,0} \tag{40}$$

Again, plugging in the demand into the maximisation problems we get the following expressions:

$$\max_{x_A} (Y_A - g_1 x_A + g_2 x_B - c_A) x_A - c_{A,0} \tag{41}$$

$$\max_{x_B} (Y_B - g_3 x_B + g_4 x_A - c_B) x_B - c_{B,0} \tag{42}$$

The FOC are as follows:

$$Y_A - 2g_1 x_A + g_2 x_B - c_A = 0 \tag{43}$$

$$Y_B - 2g_3 x_B + g_4 x_A - c_B = 0 \tag{44}$$

Now similarly to Bertrand we determine the first optimal variable, which is x_B^*, the optimal quantity for player B in regular Cournot, it requires some mathematical operations to eliminate the x_A variable from the FOC, the first Equation (43) must be multiplied with g_4 and the second Equation (44) must be multiplied with $2g_1$, and then both must be summed. This looks as follows:

$$g_4 Y_A - 2g_1 g_4 x_A + g_2 g_4 x_B - g_4 c_A = 0$$
$$2g_1 Y_B - 4g_1 g_3 x_B + 2g_1 g_4 x_A - 2g_1 c_B = 0$$

Which if added lead to x_B^*;

$$\begin{aligned} g_4 Y_A + 2g_1 Y_B - (4g_1 g_3 - g_2 g_4)\, x_B - g_4 c_A - 2g_1 c_B = 0 \rightarrow \\ \frac{g_4 Y_A + 2g_1 Y_B - g_4 c_A - 2g_1 c_B}{4g_1 g_3 - g_2 g_4} = x_B^* \end{aligned} \tag{45}$$

The optimal quantity for player A, x_A^*, is determined similarly. Equation (43) is multiplied by $2g_3$ and Equation (44) is then multiplied by g_2, they are added up to eliminate x_B^*. The calculation is as follows:

$$\begin{aligned} 2g_3 Y_A - 4g_1 g_3 x_A + 2g_2 g_3 x_B - 2g_3 c_A = 0 \\ g_2 Y_B - 2g_2 g_3 x_B + g_2 g_4 x_A - g_2 c_B = 0 \end{aligned}$$

Which if added lead to x_A^*;

$$\begin{aligned} g_2 Y_B + 2g_3 Y_A - (4g_1 g_3 - g_2 g_4)\, x_A - 2g_3 c_A - g_2 c_B = 0 \rightarrow \\ \frac{g_2 Y_B + 2g_3 Y_A - 2g_3 c_A - g_2 c_B}{4g_1 g_3 - g_2 g_4} = x_A^* \end{aligned} \tag{46}$$

From which, using Equations (37) and (38), we can determine the equations for the optimal price per player. Namely p_A^* and p_B^*. Which are as follows:

$$p_A^* = Y_A - g_1 x_A^* + g_2 x_B^* \tag{47}$$

$$p_B^* = Y_B - g_3 x_B^* + g_4 x_A^* \tag{48}$$

Then lastly for our regular Cournot model, we provide the equations for the optimal profit per player:

$$\pi_A^* = (p_A^* - c_A)\, x_A^* - c_{A,0} = (Y_A - g_1 x_A^* + g_2 x_A^* - c_A)\, x_A^* - c_{A,0} \tag{49}$$

$$\pi_B^* = (p_B^* - c_B)\, x_A^* - c_{B,0} = (Y_B - g_3 x_A^* + g_4 x_A^* - c_B)\, x_A^* - c_{B,0} \tag{50}$$

This final profit equations have fixed input with regards to the variables mentioned in the start of this section. The values that enter the top of the equation however are fully dependant of the ρs.

4.2.2 Stackelberg-Cournot short-term competition

For the Cournot markets structure the Stackelberg variation also exist. A similar set up to Stackelberg-Bertrand is used, this means the demand, maximisation problem and resulting two formula's are the same for regular Cournot as they are for Stackelberg-Cournot.

As previously mentioned the following equations using q, D and g_i are all functions of ρ. We dispense with ρ for notational convenience.

The Equations (37), (38), (39) and (40) are used again. For regular Cournot this leads to Equations (43) and (44). It is very important however to know that this only leads to the follower responses for A and B and that the equations for leadership come later. The same calculation steps as used in determining Equation (15). The difference being again that the demand functions are on quantity rather than price.

The response for A and B are as follows:

$$\frac{Y_A + g_2 x_B - c_A}{2g_1} = x_A \tag{51}$$

$$\frac{Y_B + g_4 x_A - c_B}{2g_3} = x_B \tag{52}$$

In this case the FC stands for "Follower Cournot", similarly so FB for "Follower Bertrand".

Similarly to Stackelberg-Bertrand, the assumption of symmetry completely mirrors both Stackelberg-Variants, we will therefore only depict the calculations for *follower A and leader B*

If **A is the follower** and **B is the leader**, we determine the maximisation problem for the **leader B**, we take Equation (42), which is the maximisation problem with demand for B for regular Cournot and Equation (51) which is the new best response for **follower A**, substituting it in leads to the optimal quantity for player B in Stackelberg-Cournot for **leader A** and **follower B**, which is x_B^{FC}, the computations are as follows:

$$\max_{x_B} (Y_B - g_3 x_B + g_4 x_A - c_B) x_B - c_{B,0}$$

And,

$$\frac{Y_A + g_2 x_B - c_A}{2g_1} = x_A$$

Substitution then leads to:

$$\max_{x_B} \left(Y_B - g_3 x_B + g_4 \frac{Y_A + g_2 x_B - c_A}{2g_1} - c_B \right) x_B - c_{B,0} \tag{53}$$

Now taking FOC we get the expression for x_B^{LC}:

$$\begin{aligned}
Y_B - 2g_3x_B + g_4\frac{Y_A + 2g_2x_B - c_A}{2g_1} - c_B = 0 \rightarrow \\
Y_B + g_4\frac{Y_A - c_A}{2g_1} - c_B &= \left(2g_3 - \frac{g_2g_4}{g_1}\right)x_B \rightarrow \\
\frac{2g_1Y_B + g_4\left(Y_A - c_A\right)}{2g_1} - c_B &= \left(\frac{2g_1g_3 - g_2g_4}{g_1}\right)x_B \rightarrow \\
\frac{2g_1Y_B + g_4\left(Y_A - c_A\right)}{2g_1} - c_B &= \left(\frac{2g_1g_3 - g_2g_4}{g_1}\right)x_B \rightarrow \\
\frac{2g_1\left(Y_B - c_B\right) + g_4\left(Y_A - c_A\right)}{2g_1} &= x_B \rightarrow \\
\frac{2g_1\left(Y_B - c_B\right) + g_4\left(Y_A - c_A\right)}{4g_1g_3 - 2g_2g_4} &= x_B^{LC}
\end{aligned} \tag{54}$$

Now after having determined an equation for the optimal quantity of **leader B** in Cournot, we can take Equation (52) and plug into it Equation (54), x_B^{LC}, to get the equation for the optimal quantity of **follower A**, resulting in:

$$\frac{Y_A + g_2x_B^{LC} - c_A}{2g_1} = x_A^{FC} \tag{55}$$

Plugging Equations (54) and (55) back into the demand functions for the Cournot system, which are Equations (37) and (38), we can now determine the optimal prices for Stackelberg-Cournot for **follower A** and **leader B**, which are the following expression:

$$p_A^{FC} = Y_A - g_1x_A^{FC} + g_2x_B^{LC} \tag{56}$$

$$p_B^{LC} = Y_B - g_3x_B^{LC} + g_4x_A^{FC} \tag{57}$$

Again leading me to the final variable, the optimal profit equations for Stackelberg-Cournot with **follower A** and **leader B**:

$$\pi_A^{FC} = \left(p_A^{FC} - c_A\right)x_A^{FC} - c_{A,0} \tag{58}$$

$$\pi_B^{LC} = \left(p_B^{LC} - c_B\right)x_B^{LC} - c_{B,0} \tag{59}$$

As we have done with all the models we will now compare the prices of both leadership structures as a double check, we compare:

$$\begin{aligned}
\frac{2g_1\left(Y_B - c_B\right) + g_4\left(Y_A - c_A\right)}{4g_1g_3 - 2g_2g_4} &= x_B^{LC} \rightarrow \\
\frac{Y_A + g_2x_B^{LC} - c_A}{2g_1} &= x_A^{FC}
\end{aligned}$$

With:

$$\frac{2g_3(Y_A - c_A) + g_2(Y_B - c_B)}{4g_1g_3 - 2g_2g_4} = x_A^{LC} \rightarrow$$
$$\frac{Y_B + g_4x_A^{LC} - c_B}{2g_3} = x_B^{FC}$$

4.2.3 Short-term collusion under Cournot

The last theoretical model we will talk about is the scenario in which both players collude under Cournot. In this scenario the players again collude similarly to short-term collusion under Bertrand but they collude on price in order to maximise their profits.

As previously mentioned the following equations using q, D and g_i are all functions of ρ. We dispense with ρ for notational convenience.

The same system of Cournot is used here therefore we reintroduce Equations (37) and (38), the demand functions:

$$p_A = Y_A - g_1x_A + g_2x_B$$
$$p_B = Y_B - g_3x_B + g_4x_A$$

Now since there is collusion a new maximisation problem can be written, similarly to collusion under Bertrand, where the maximisation occurs for both players together rather than individually, the maximisation problem is similar to Equation (32), we additionally incorporate the demand Equations (37) and (38) to get the maximisation problem with demand for collusion under Cournot:

$$\max_{x_A, x_B} (Y_A - g_1x_A + g_2x_B - c_A)x_A + (Y_B - g_3x_B + g_4x_A - c_B)x_B - c_{A,0} - c_{B,0} \tag{60}$$

Now we can use FOC on the new maximisation problem to try and determine the optimal quantity functions for player A and B:

$$\begin{aligned} Y_A - 2g_1x_A + (g_2 + g_4)x_B - c_A = 0 \\ Y_B - 2g_3x_B + (g_2 + g_4)x_A - c_B = 0 \end{aligned} \tag{61}$$

We will first solve to determine x_A^{CC}, where the CC stands for *Collusion Cournot*, in order to do that we multiply the first line of Equation (61) with $2g_3$ and the second line of Equation (61) with $(g_2 + g_4)$ and continue by adding both outcomes, this eliminates x_B from the equation and leaves x_A, which is the optimal quantity for player A in collusion under Cournot, x_A^{CC}, the computation is as follows:

$$2g_3Y_A - 4g_1g_3x_A + 2g_3(g_2 + g_4)x_B - 2g_3c_A = 0$$
$$(g_2 + g_4)Y_B - 2g_3(g_2 + g_4)x_B + (g_2 + g_4)^2x_A - (g_2 + g_4)c_B = 0$$

We then add the equations to get the optimal quantity:

$$
\begin{aligned}
2g_3\left(Y_A - c_A\right) + \left(g_2 + g_4\right)\left(Y_B - c_B\right) - \left(4g_1g_3 - \left(g_2 + g_4\right)^2\right)x_A = 0 \rightarrow \\
\frac{2g_3\left(Y_A - c_A\right) + \left(g_2 + g_4\right)\left(Y_B - c_B\right)}{4g_1g_3 - \left(g_2 + g_4\right)^2} = x_A^{CC}
\end{aligned}
\tag{62}
$$

Similarly, we solve to determine x_B^{CC}, which requires similar operations but swapped, we multiply the first line of Equation (61) with $(g_2 + g_4)$ and the second line of Equation (61) with $2g_3$ and continue by adding both outcomes again to eliminate x_A, leaving x_B, which is the optimal quantity for player B in collusion under Cournot, x_B^{CC}, the computation is as follows:

$$
\begin{aligned}
\left(g_2 + g_4\right)Y_A - 2g_1\left(g_2 + g_4\right)x_A + \left(g_2 + g_4\right)^2 x_B - \left(g_2 + g_4\right)c_A = 0 \\
2g_1Y_B - 4g_1g_3x_B + 2g_1\left(g_2 + g_4\right)x_A - 2g_1c_B = 0
\end{aligned}
$$

We then add the equations to get the optimal quantity:

$$
\begin{aligned}
\left(g_2 + g_4\right)\left(Y_A - c_A\right) + 2g_1\left(Y_B - c_B\right) - \left(4g_1g_3 - \left(g_2 + g_4\right)^2\right)x_B = 0 \\
\frac{2g_1\left(Y_B - c_B\right) + \left(g_2 + g_4\right)\left(Y_A - c_A\right)}{4g_1g_3 - \left(g_2 + g_4\right)^2} = x_B^{CC}
\end{aligned}
\tag{63}
$$

Finally we introduce Equations (37) and (38) again as they are the demand function for Cournot and plug in Equations (62) and (63) to get the optimal price for players A and B in collusion under Cournot, which results in the following two functions:

$$p_A^{CC} = Y_A - g_1x_A^{CC} + g_2x_B^{CC} \tag{64}$$

$$p_B^{CC} = Y_B - g_3x_B^{CC} + g_4x_A^{CC} \tag{65}$$

In turn both Equations (64) and (65) can be used together with Equations (62) and (63) to get the final optimal profit functions for collusion under Cournot:

$$\pi_A^{CC} = x_A^{CC}\left(p_A^{CC} - c_A\right) - c_{A,0} \tag{66}$$

$$\pi_B^{CC} = x_B^{CC}\left(p_B^{CC} - c_B\right) - c_{B,A} \tag{67}$$

4.3 Sales potential

The idea of sales potential is an idea introduced by Joosten (2015), where the long-term effects of advertising on the actual sales in the stage game are incorporated. The function on which sales potential is dependent is the demand functions as introduced in the previous sections. It incorporates some trivially chosen parameters into demand for which advertisement decision the players have made, important however is that even though the numbers are trivially chosen the ordinal difference is important, since there needs to be some ranking among the scores the reflect advertisement benefits. Directly influencing the optimal quantity for the Bertrand competition and the optimal price for the Cournot competition.

By incorporating a degree of influence of the advertisement decisions the games become more dynamic. The theoretical system is therefore altered, instead of optimal price for Bertrand and optimal quantity for Cournot immediately being input into the demand functions of the models they are now input into the sales potential functions, which are then altered by the advertisement choices. Meaning ρ determines the sales potential directly. For Bertrand the sales potential functions which influences the actual games therefore look as follows:

$$\begin{aligned}\mathrm{SP}_t^A &= D_{A,0} + qD - z_1(\rho_t)p_A^* + z_2(\rho_t)p_B^* \\ \mathrm{SP}_t^B &= D_{B,0} + (1-q)D - z_3(\rho_t)p_B^* + z_4(\rho_t)p_A^*\end{aligned} \tag{68}$$

Where ρ_t is the advertising decision as a factor in the function, and the t subscript implies it is during the game, as is with the matrix for ρ_t. Further explanation of ρ was given in *Section 4*. This then leads to the following adjustment for the optimal **quantity** adjustments:

$$\begin{aligned}
x_{A,t}^{*(1,1)} &= 1 \cdot \mathrm{SP}_t^A \quad \text{and} \quad & x_{B,t}^{*(1,1)} &= 1 \cdot \mathrm{SP}_t^B \\
x_{A,t}^{*(1,2)} &= \frac{7}{8} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & x_{B,t}^{*(1,2)} &= \frac{5}{8} \cdot \mathrm{SP}_t^B \\
x_{A,t}^{*(2,1)} &= \frac{5}{8} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & x_{B,t}^{*(2,1)} &= \frac{7}{8} \cdot \mathrm{SP}_t^B \\
x_{A,t}^{*(2,2)} &= \frac{1}{2} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & x_{B,t}^{*(2,2)} &= \frac{1}{2} \cdot \mathrm{SP}_t^B
\end{aligned}$$

Analysing the functions we see that using fractions a weight is assigned to certain advertisement behaviour and that behaviour in which both players advertise favors the sales potential. This is to incorporate some momentary effects of advertisement. If both players advertise the market is fully utilised, whilst if both players do not advertise demand (optimal quantity for Bertrand) is lost. The fact that it is halved specifically is trivial but the fact that with less advertisement behaviour the demand decreases should always occur.

The following line is a line of code in the algorithm in which the way sales potential is used is illustrated in the algorithm:

```
xa = SPxa * (self.rho11 * 1 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
xb = SPxb * (self.rho11 * 1 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))
```

Figure 1. Example of sales potential in optimal quantity for Bertrand

The sales potential functions for Cournot are similar to Bertrand but have slight differences. The optimal price is influenced in the long-term by advertisement choices of the players, the market will change in accordance with how both players advertise as well. The market size will grow, shrink and/or be redistributed between the players. The functions for Cournot look as follows:

$$\mathrm{SP}_t^A = D_{A,0} + qD - z_1(\rho_t)\, x_A^* + z_2(\rho_t)\, x_B^* \tag{69}$$

$$\mathrm{SP}_t^B = D_{B,0} + (1-q)D - z_3(\rho_t)\, x_B^* + z_4(\rho_t)\, x_A^* \tag{70}$$

Where ρ_t is the advertising decision as a factor in the function, and the t subscript implies it is during the game, this then leads to the following adjustment for the optimal **price** adjustments:

$$\begin{aligned}
p_{A,t}^{*(1,1)} &= 1 \cdot \mathrm{SP}_t^A \quad \text{and} \quad & p_{B,t}^{*(1,1)} &= 1 \cdot \mathrm{SP}_t^B \\
p_{A,t}^{*(1,2)} &= \frac{7}{8} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & p_{B,t}^{*(1,2)} &= \frac{5}{8} \cdot \mathrm{SP}_t^B \\
p_{A,t}^{*(2,1)} &= \frac{5}{8} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & p_{B,t}^{*(2,1)} &= \frac{7}{8} \cdot \mathrm{SP}_t^B \\
p_{A,t}^{*(2,2)} &= \frac{1}{2} \cdot \mathrm{SP}_t^A \quad \text{and} \quad & p_{B,t}^{*(2,2)} &= \frac{1}{2} \cdot \mathrm{SP}_t^B
\end{aligned}$$

The equations are inserted into the model for Cournot the exact same way as *Figure 1* but for the price.

As previously mentioned the equations using q, D and g_i are all functions of ρ, so also the sales potential equations. We dispense with ρ for notational convenience again.

5 RESULTS

In this section we present and discuss the results of the algorithm. As discussed in *Section 3* the theoretical model was inserted into Excel (*Appendix A*) and then a computation was done in Python (*Appendix B*). The Python code yields certain figures as results, with which we want to compare the different types of short-term competitions, the different model inputs, changes in symmetry and so on. The algorithm will be included in the GTToolbox of Rogier Harmelink [1].

5.1 Baseline results

In order to compare the models adequately we first establish a baseline for all the models. These baseline models are also called the main models. This baseline is established by taking a random sample of a million ρs for the player input, which are uniformly distributed between 0 and 1, for joint relative frequencies. We additionally draw another random sample of a million ρs for the player input, on a beta distribution in order to increase density of limiting average rewards in sensitive areas.

We then take, for all short-term competition, the exact same input for advertisement costs (both variable and fixed) and positive intercept for demand. The results are shown below in six different visualisations, all representing a different type of short-term competition. The first being the regular Bertrand model (displayed in cyan), the second (displayed in aqua) is the Stackelberg-Bertrand (Leader A - Follower B model), the third is the visualisation of the results for collusion under Bertrand (displayed in teal) and the fourth, fifth and sixth (displayed in pink, orchid and violet respectively) are the same competitions as the Bertrand models but for Cournot (with the exception that Stackelberg-Cournot has a Leader B - Follower A structure).

For all the baseline figures the positive intercept for demand, u, is 100, the variable advertisement investment cost for player A and B, $ac_{0,A}$ and $ac_{0,B}$, are 150, the fixed advertisement investment cost for player A and B, $c_{0,A}$ and $c_{0,B}$, is 30. The baseline visualisations are shown in the figures below:

[1] https://github.com/Rogierr/GTToolbox

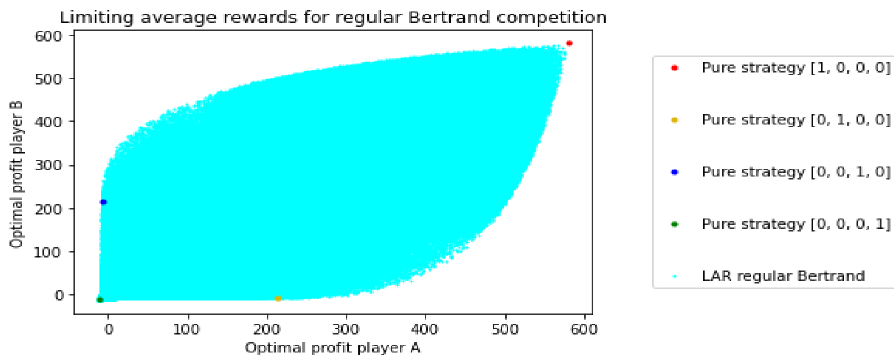


Figure 2. Baseline limiting average rewards for regular Bertrand competition.

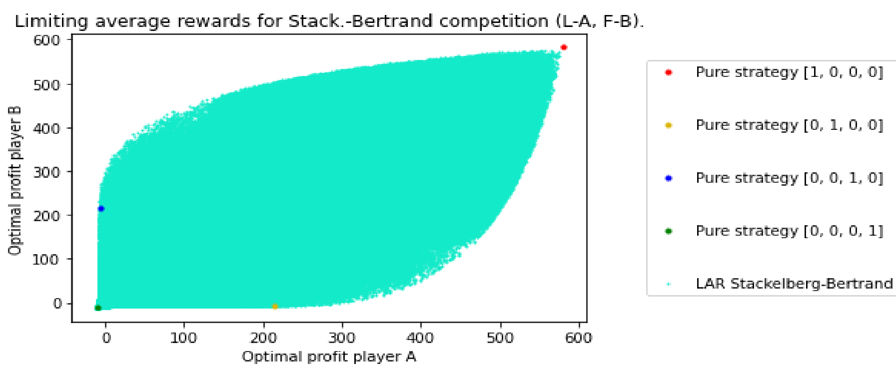


Figure 3. Baseline limiting average rewards for Stackelberg-Bertrand competition with Leader A and Follower B.

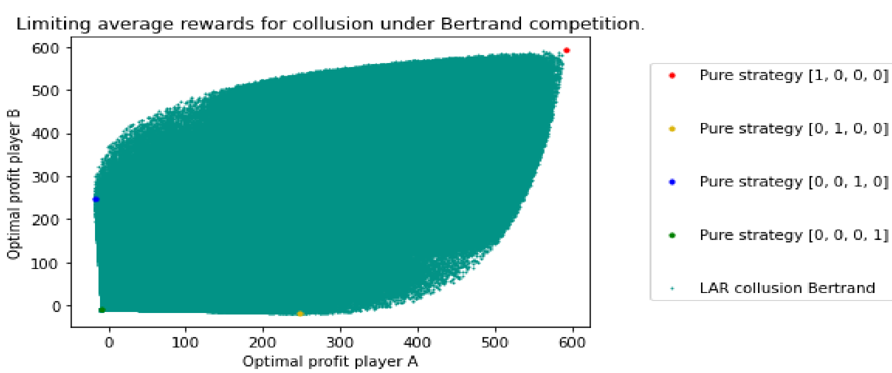


Figure 4. Baseline limiting average rewards for collusion under Bertrand competition.

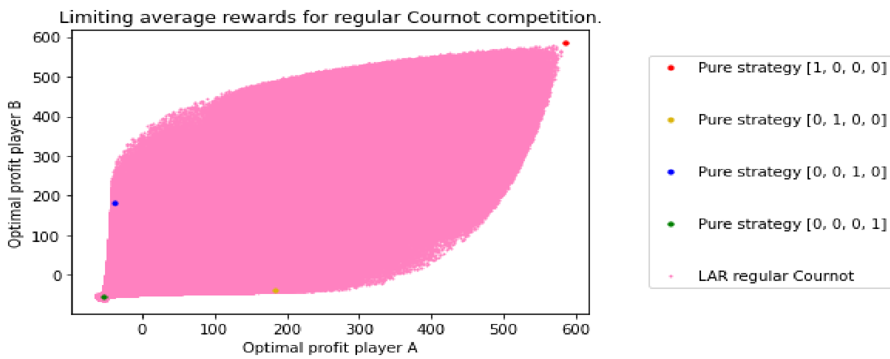


Figure 5. Baseline limiting average rewards for regular Cournot competition.

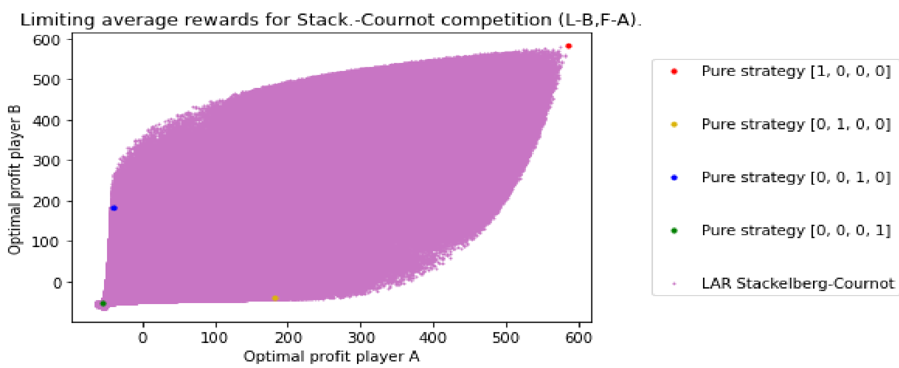


Figure 6. Baseline limiting average rewards for Stackelberg-Cournot competition with Leader B and Follower A.

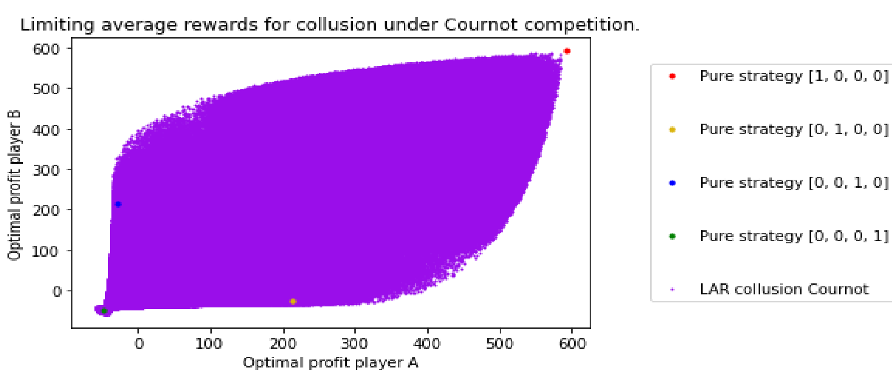


Figure 7. Baseline limiting average rewards for collusion under Cournot competition.

The figures above all contain visualisations of the limiting average rewards points for certain ρs, it is interesting to know what the minimum and maximum values are for these limiting average rewards points per short-term competition model, since it allows us to compare general outcomes. This method is also used for later result comparisons. The algorithm provides these maxima and minima per model, which regretfully are displayed quite cluttered. I therefore accumulate the outcomes in *Table 1* and *Table 2*, the figure numbers reference to the above figures.

Table 1. Minima for all short-term competition figures

Figures	**Minimum values**
2	-13.53787245, -13.24469243
3	(-13.4708611, -13.08185065), -13.1998692, -13.32994336
4	-18.22704459, -18.16854123
5	-64.48420254, -64.50905918
6	(-64.86361991, -66.03099224), -64.86361991, -66.03099224
7	-59.36278173, -59.13278624

Table 2. Maxima for all short-term competition figures

Figures	**Maximum values**
2	580.5, 580.5
3	580.61884669, 582.64074489, (582.64074489, 580.61884669)
4	592.57142857, 592.57142857
5	584.84571429, 584.84571429
6	584.9652401, 582.69576253, (582.69576253, 584.9652401)
7	592.57142857, 592.57142857

Here the bracketed values in *Table 1* and *Table 2* represent the values for Leader A - Follower B in *Figure 3* and Follower A - Leader B in *Figure 6* respectively. Comparing the tables with the six figures we can state that with the current input the short-term competitions forms are different most significantly on the minima and only slightly on the maxima.

What we can see is that all the Cournot models have lower minima and higher maxima, meaning the potential yield for Cournot is slightly more volatile than for Bertrand. The strategy outcomes sit between the extremes of the pure strategy outcomes for all Bertrand models except for collusion under Bertrand. We explain this by looking at *Figure 4* and seeing that the leaf is a little wider, meaning the minimum values in *Table 1* are close to pure strategies $[0, 1, 0, 0]$ and $[0, 0, 1, 0]$ and not at the stem of the leaf.

For the models in Cournot competition, the finding regarding maximum/minimum values in the pure strategies does not hold, since for all Cournot models there are some strategies which result in lower limiting average rewards than the pure strategies. This means that some combinations of advertisement decisions of both players result in lower optimal quantities, and therefore price and profit, than both players strictly not advertising. Another interesting finding is that for Bertrand and Cournot collusion yields the potential highest profit, but for Bertrand it also yields the lowest minimum (of the Bertrand models) whilst for Cournot it yield the highest minimum (of the Cournot models). For the Stackelberg-variants we can see that in the Bertrand-variant being the follower both yields a higher maximum as well as a lower minimum. Whilst for Cournot the exact opposite is true, here being the leader yields a higher maximum as well as a lower minimum.

With regards to the accuracy of the data we discuss the density of limiting average rewards. The density of limiting average rewards in the visualisations is correlated to the joint relative frequencies for certain distributions. Two methods were used in creating a higher density of limiting average rewards. By increasing the sample size for ρ we get a denser shape at the extremes. Additionally a secondary drawing for ρ using a Beta distribution was used to increase density in tip. The reason for using one million as the current sample size for uniform ρ is purely time-efficiency, since for instance quintupling the sample size will also quintuple the algorithm run time. To illustrate the above an example of the regular Bertrand model with five million uniform ρs is given here in *Figure 8*:

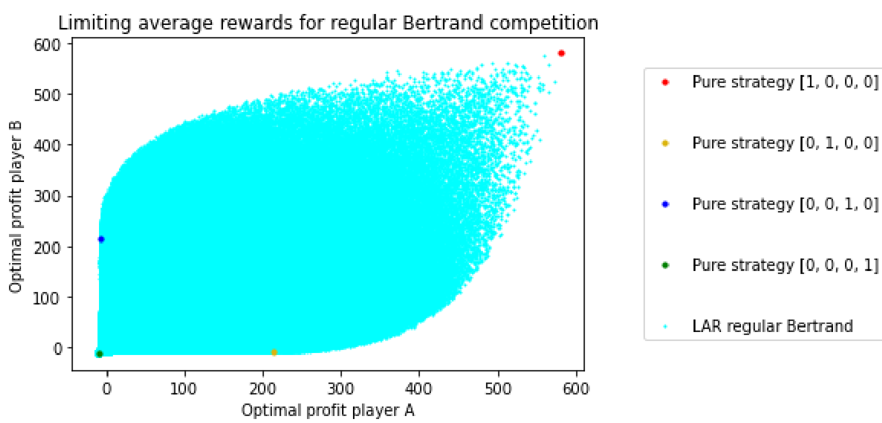


Figure 8. Limiting average rewards outcomes for regular Bertrand competition, with sample size 5 million.

The beta distribution significantly improves density at the tip, we visualise it by taking the visualisations for limiting average rewards for regular Bertrand competition and labeling the secondary drawings using a Beta distribution with the color *green*. This results in the following visualisation shown in *Figure 9*.

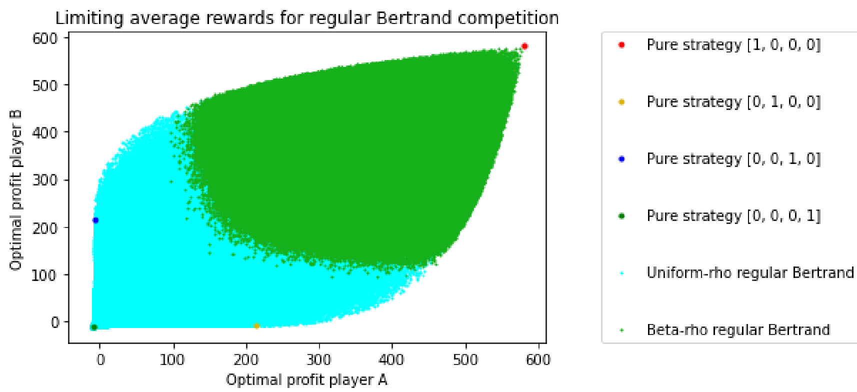


Figure 9. Limiting average rewards outcomes for regular Bertrand competition, with a secondary ρ drawing of 1 million.

Now after having ascertained some similarities and differences between all the base models we aim to isolate some variables and investigate what the effect of tweaking them has on specific, but also the different competition forms.

5.2 Symmetric variable cost changes

The first changes to the main (or baseline) models are changes to the variable costs. Now as explained in *Section 5.1*, there are several input parameters which can manually be changed in the algorithm (i.e., $c_{A,0}$, $c_{0,B}$ or u). This is done as input and changes the visualisations and outcomes of the model.

By showing a series of model outcomes we try to discern how changes in variable cost influence the model ($c_{A,0}$ and $c_{0,B}$), for the first few changes we adjust the variables symmetrically, since this provides a greater opportunity to isolate effects.

Before visualising the cost adjustment we explain which exact parameter we will change and why. Looking at the profit equations for all the short-term competition forms the equations all end in $c_{A,0}$ or $C_{A,0}$. These are the costs which occur when the stage game is played. It should however be specified now that this expression also contains some degree of variable and fixed costs. If we were to take the profit equations for regular Bertrand in the algorithm they would look as follows:

$$\pi_A^* = x_A^* (p_A^* - c_A) - (\rho_{11} + \rho_{12}) a c_{0,A} + c_{0,A}$$
$$\pi_B^* = x_B^* (p_A^* - c_B) - (\rho_{11} + \rho_{21}) a c_{0,B} + c_{0,B}$$

What can be seen is that the expressions for $c_{A,0}$ and $c_{A,0}$ are replaced by $(\rho_{11}+\rho_{12})ac_{0,A}+c_{0,A}$ and $(\rho_{11}+\rho_{21})ac_{0,B}+c_{0,B}$. These expressions contain an expression which represents the influence of variable advertisement costs ($ac_{0,A}$ and $ac_{0,B}$ on the advertisement decisions which influence the cost of either player ($\rho_{11}+\rho_{12}$ for player A and $\rho_{11}+\rho_{21}$ for player B). Where $c_{0,A}$ and $c_{0,B}$ simply represent the fixed costs.

Relevant to the changes in the structures, are the changes in variable advertisement costs, since these are directly influenced by the advertisement decisions and therefore dynamic. They can be adjusted in two ways. Either reducing the cost towards zero, where going negative is not relevant since it is not realistic nor feasible in the model (negative costs would be equivalent to subsidies, which will be omitted in this research), and increasing the cost to see how expensive advertising influences the limiting average rewards.

First we show visualisations of downward cost adjustments (decreasing costs). We only show four visualisations since the change of the limiting average rewards only occurs very insignificantly in the extremes, and is therefore not as interesting. The visualisations are as follows:

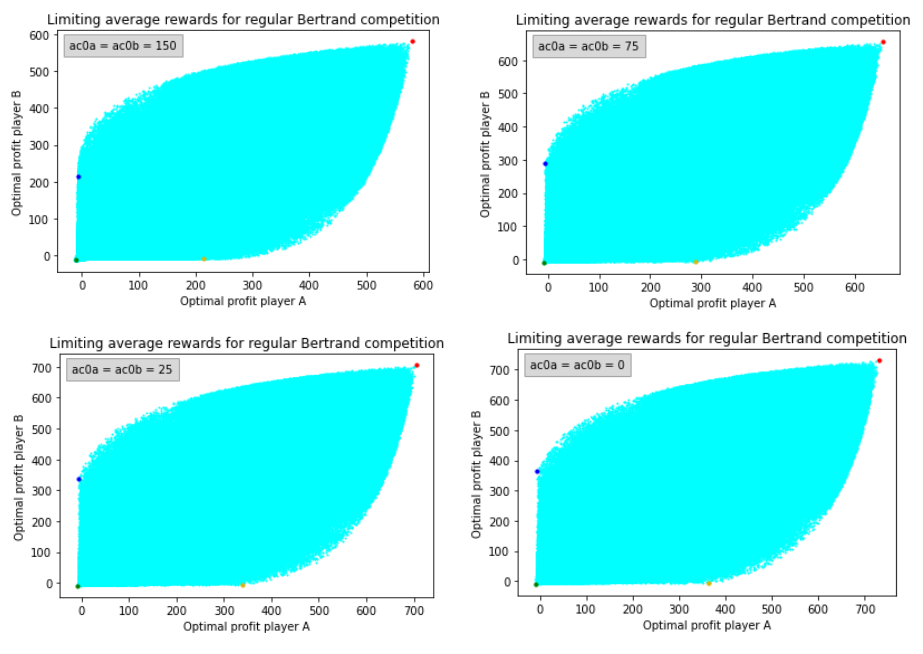


Figure 10. Regular Bertrand competition with decrease over $ac_{0,A}$ and $ac_{0,B}$.

Figure 10 shows us that even with relatively drastic decreases and $ac_{0,A}$ and $ac_{0,B}$ at 0 the only major changes occur in the steepness of the limiting average rewards around the x- and y-axis, additionally a minor increase of the maximum limiting average rewards occurs (the shape becomes pointier).

Now we show the changes which occur when the variable advertisement costs are increased, first slightly and then more drastically:

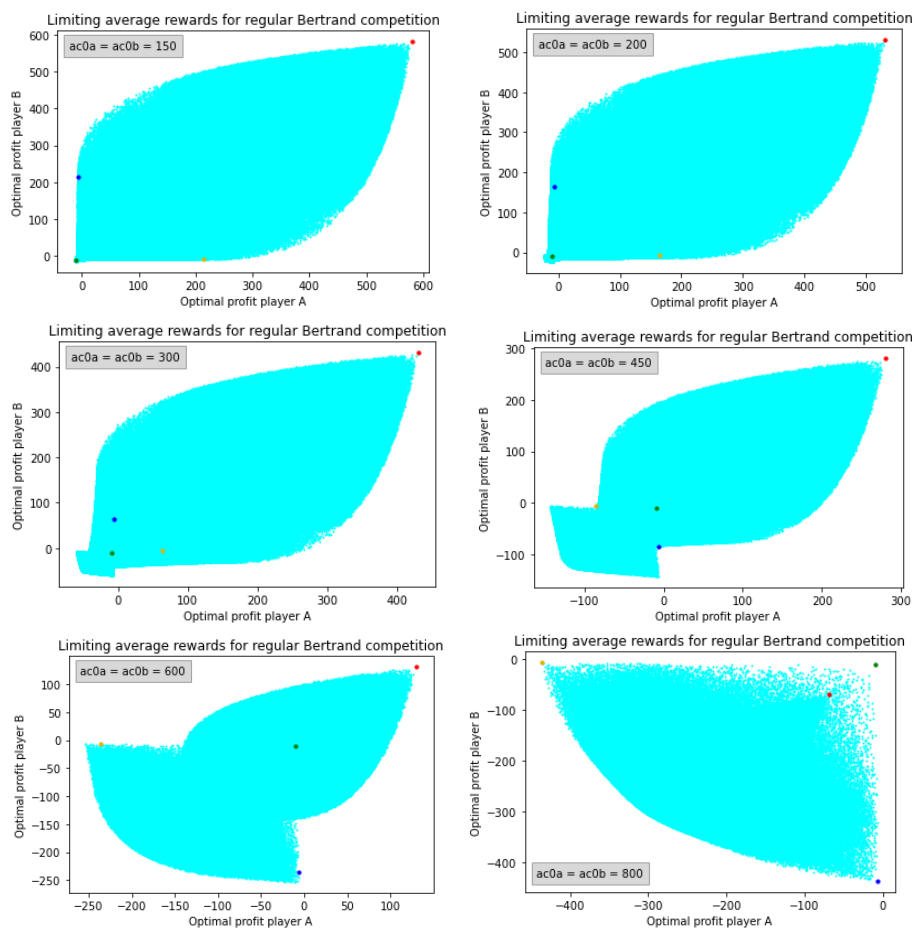


Figure 11. Regular Bertrand competition with increase over $ac_{0,A}$ and $ac_{0,B}$.

Observing the increase of $ac_{0,A}$ and $ac_{0,B}$ in *Figure 11*, we ascertain that for the players the decision to advertise becomes less and less attractive. This is visible in the pure strategies (yellow and blue, [0, 1, 0, 0] and [0, 0, 1, 0] respectively) since they at one point pass each other. This implies that for a certain threshold cost, the pure strategies in which player A only advertises become less favorable than the pure strategy in which he does not, and vise versa for player B. Eventually the cost of advertisement becomes so significant that even the previous maximum (pure strategy [1, 0, 0, 0], the red dot) where both players always advertise becomes less equitable than both players not advertising (pure strategy [0, 0, 0, 1], the green dot).

These findings currently only hold for the regular Bertrand competition, it is now interesting to see what happens when we apply the exact same variable advertisement cost changes to the Cournot competition. To do so we start off with the baseline model for regular Cournot and take $ac_{0,A} = ac_{0,B} = 150$. We then similarly approach the variable advertisement costs, by incrementally decreasing and increasing. First we visualise a decrease variable advertisement costs:

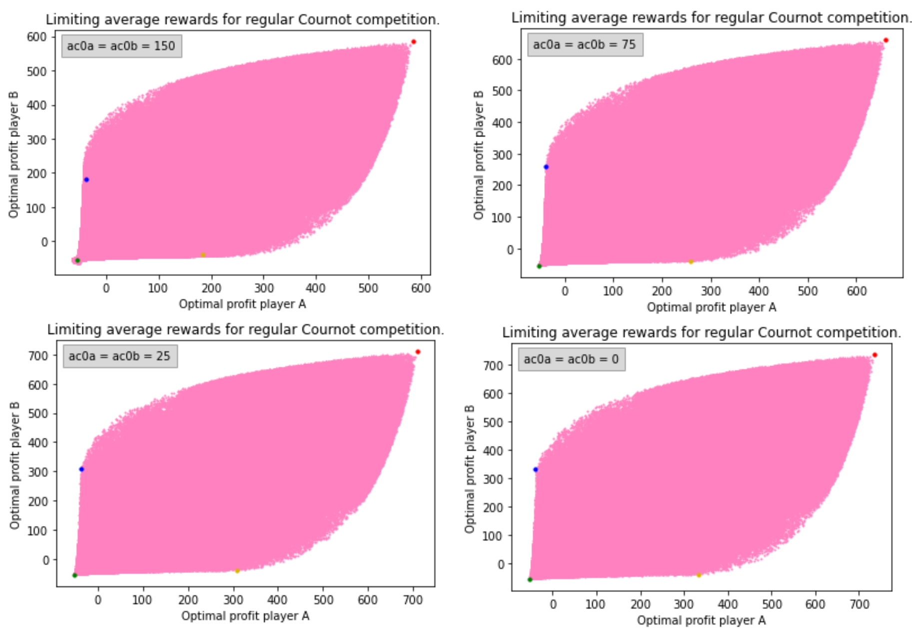


Figure 12. Regular Cournot competition with decrease over $ac_{0,A}$ and $ac_{0,B}$.

A similar effect to *Figure 10* can be observed in *Figure 12*, where the baseline model is only changed very slightly, with the pure strategies becoming steeper and more extreme, and the shape becoming more edge-like at the lower limiting average rewards, and the maximum limiting average rewards outcome increasing. Comparing the two market structures the effect of decreasing the variable advertisement cost is consistent and seems to be logically explicable for both.

Additionally we now show the changes which occur when the variable advertisement costs are increased for Cournot, in the exact same increments used for Bertrand (starting off with $ac_{0,A} = ac_{0,B} = 150$ again):

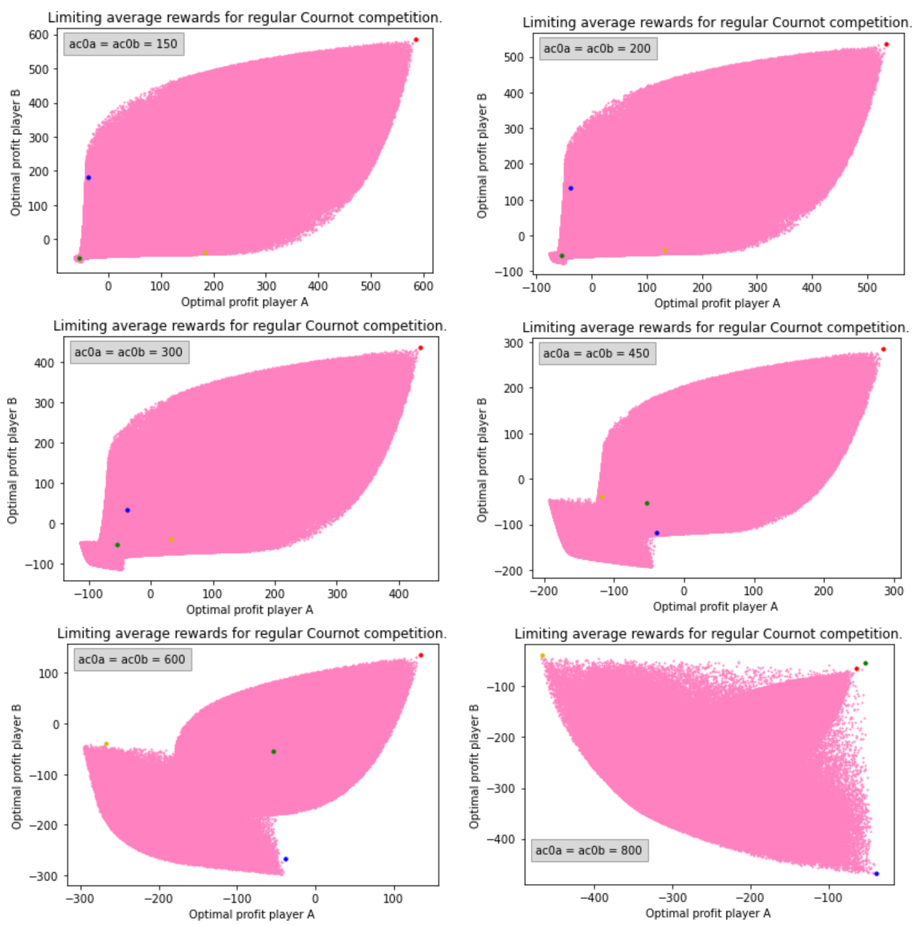


Figure 13. Regular Cournot competition with increase over $ac_{0,A}$ and $ac_{0,B}$.

We compare the movement of the changes in limiting average rewards when $ac_{0,A}$ and $ac_{0,B}$ are increased in *Figure 13*. We see the change in the shape to be very similar to those of the Bertrand model when exposed to increasing variable advertisement cost. This therefore implies that similarly to decreasing the cost, increasing the cost is equally impactful, making us able to generalise the effect for both short-term competition forms. The model structure is identical only the decision-parameter (quantity instead of price) is different so this finding seems realistic.

For the sake of keeping the research compact we will only include the visualisations of the other short-term competition variants for $ac_{0,A} = ac_{0,B} = 600$. This is mostly due to the fact that their general outlines and shapes are very similar to the movements of regular Bertrand and Cournot competition, with some slight outliers in both collusion models, which seemed to only apply to the extreme cases, illustrated in *Figure 14*.

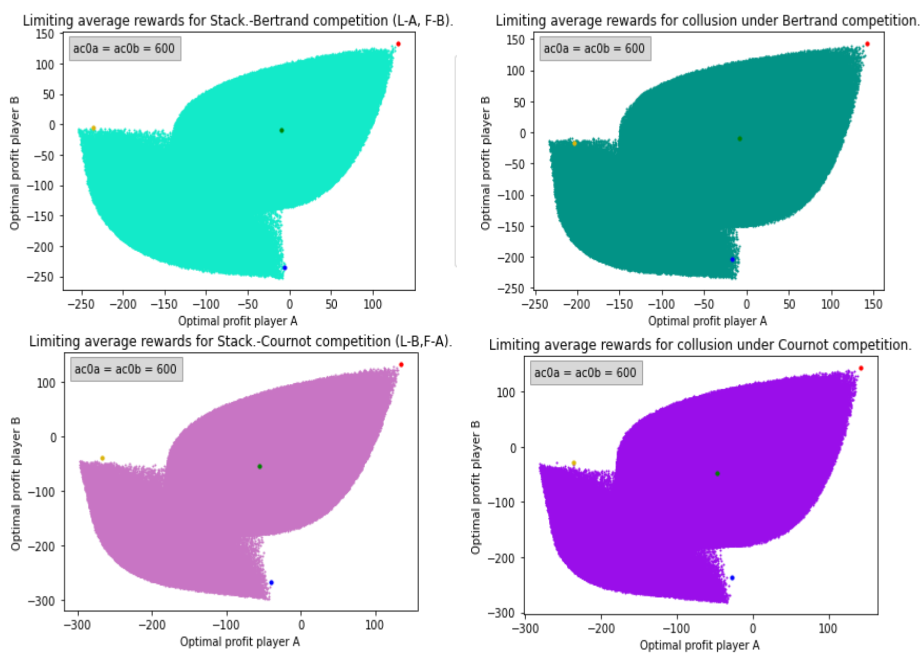


Figure 14. Variable advertisement cost at $ac_{0,A} = ac_{0,B} = 600$ for the other short-term competitions.

5.3 Symmetrical positive demand intercept changes

Similarly to the changes to the theoretical model which were implemented to adjust the variable advertisement costs, a minor addition can be made to the demand function in *Section 4.1*. Which is as follows:

$$D = 100\,(3\rho_{11} + \rho_{12} + \rho_{21}) \tag{71}$$

As shown in Equation (71 the function in the baseline model has a value of $u = 100$. This number is the positive demand intercept. In order to see the variables effects we replaced the 100 with u where it functions as an input variable as follows:

$$D = u\,(3\rho_{11} + \rho_{12} + \rho_{21}) \tag{72}$$

The changes in this new variable u (positive demand intercept influencing market growth rate) are interesting, since it represents the rate with which the market grows if players advertise and therefore represents the flexibility of a short-term competition with regards to market growth. In this section we change the positive demand intercept in a similar fashion as we previously did for variable advertisement costs. First we decrease u and visualise the effect, and then do the same for an increase in u. To do this we take the baseline model for regular Bertrand, *Figure 2*, and decrease u. The visualisations are in *Figure 15*.

We will show more visualisations for decreasing u than for increasing u since the changes in the model shape are more significant. It shows, in a similar manner to increasing variable advertisement cost, $ac_{0,A}$ and $ac_{0,B}$, that the leaf shape transforms into a squid shape. Additionally the squid shape fans out and becomes much steeper and edgier with the pure strategies completely inverted compared to the start, in a kite shape. We argue for these cases, that the cost a player makes, with regards to advertising, has so little influence on the total demand of that market (none at all for $u = 0$) that advertising is simply money down the drain for the players. It therefore also makes sense that the convex shape is so steep, since there are no strategies which are more expensive than the pure strategies. To visualise the effects of increasing u on the limiting average rewards, we theorise that the shape will not change much, since the relationship between the variables is less effected for higher numbers and that it is mostly going to be the limiting average rewards outcomes which are going to increase. The visualisation can be found in *Figure 16*.

The figure shows that the shape remains the exact same when u is increased, the only thing which significantly increases is the limiting average rewards outcomes. The sixth figure, which is an extreme value for u, illustrates this since the shape does not change. This is a logical outcome, since an exponential increase in market growth rate would also warrant an equal increase in profit for both players and therefore exponentially increase the limiting average rewards.

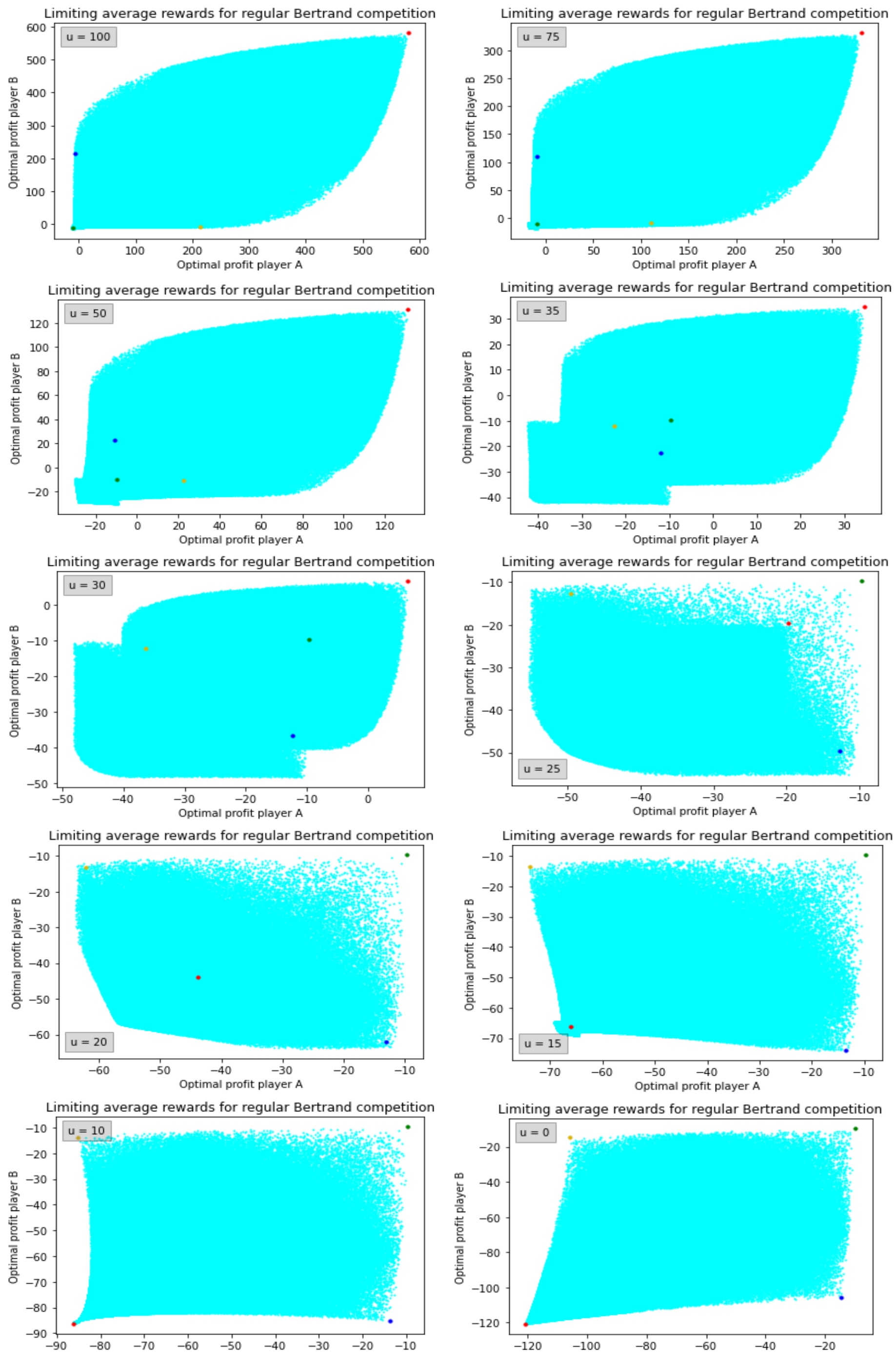


Figure 15. Regular Bertrand competition with decrease over u.

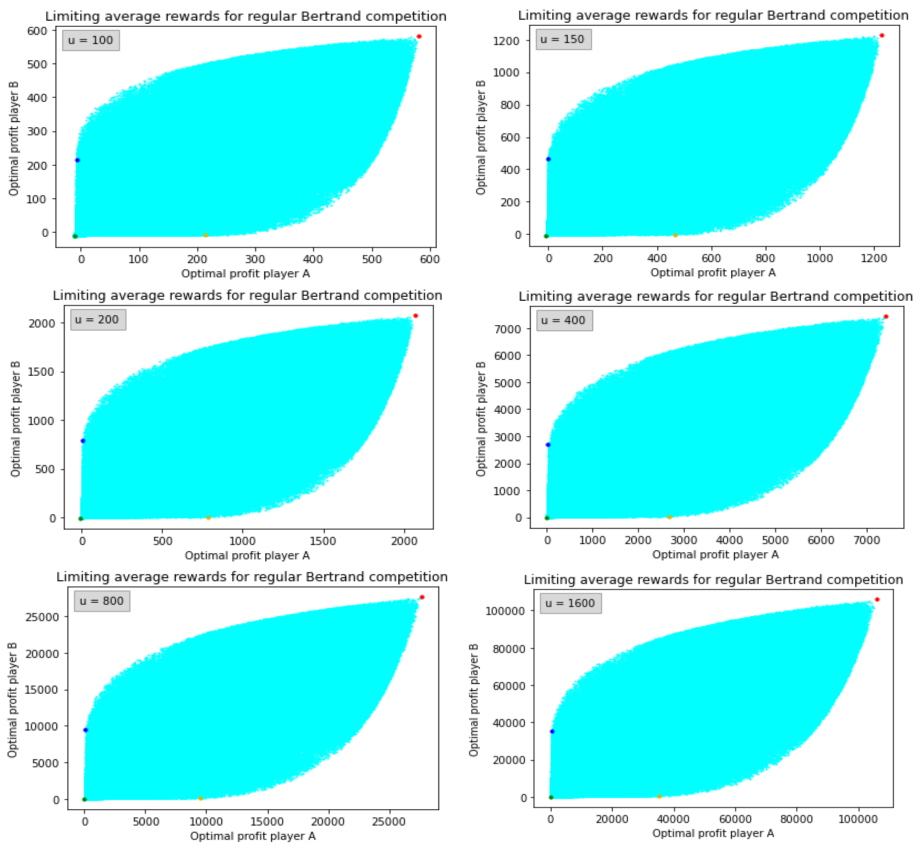


Figure 16. Regular Bertrand competition with increase over u.

Now similarly to *Section 5.2* we wish to compare the changes of the positive demand intercept, u, for the Cournot competition. For this we again take the baseline model for regular Cournot, and decrease the u parameter in a similar fashion. Multiple visualisations will again be shown in the figure since we expect a similar shape-change for Cournot as for Bertrand. The visualisations are shown in *Figure 17*.

A shape-change similar to that of Bertrand in *Figure 15* occurs for Cournot in *Figure 17*. In contrast to our speculation however there are slight differences when the shape transitions from the leaf, it looks more like a jellyfish than a squid. It becomes similarly sharp to the shape for Bertrand competition but is significantly wider. We speculate that this is due to the fact that Cournot is slightly more prone to strategies with lower limiting average rewards, there are therefore more limiting average rewards located near the extremes of the pure strategies.

Additionally we see that by decreasing u, Cournot similarly turns into a rigid model in which advertising becomes directly related to a loss of profit. Where the pure strategies in which both players advertise, both simultaneously as well as individually, create profit minima for that player. This is due to the fact that we again observe the pure strategies for either player (yellow and blue, [0, 1, 0, 0] and [0, 0, 1, 0] respectively) passing each other around the (0,0) mark.

Now additionally an increase in u can also be visualised for Cournot. The visualisation however are exactly the same as those for Bertrand with an increase in u as can be seen in *Figure 16*, we therefore exclude them from the report since they do not contribute new findings, they simply confirm the relationship between an increase in u, an increase in limiting average rewards maxima, and the maintaining of the leaf shape for both short-term competition forms.

Finally this section can be concluded by drawing a comparison between the increase in variable advertisement costs ($ac_{0,A}$ and $ac_{0,B}$) and a decrease in the positive demand intercept (u), as can be seen in *Figure 13* and *Figure 17*. Both figures show a similar shape and limiting average rewards change, however it looks like the shape is a little less sensitive to $ac_{0,A}$ and $ac_{0,B}$ than it is to u. This means the shape-change by decreasing u is more extreme, we speculate that if $ac_{0,A}$ and $ac_{0,B}$ could be changed further (below 0, which is not realistic) it might look exactly like u.

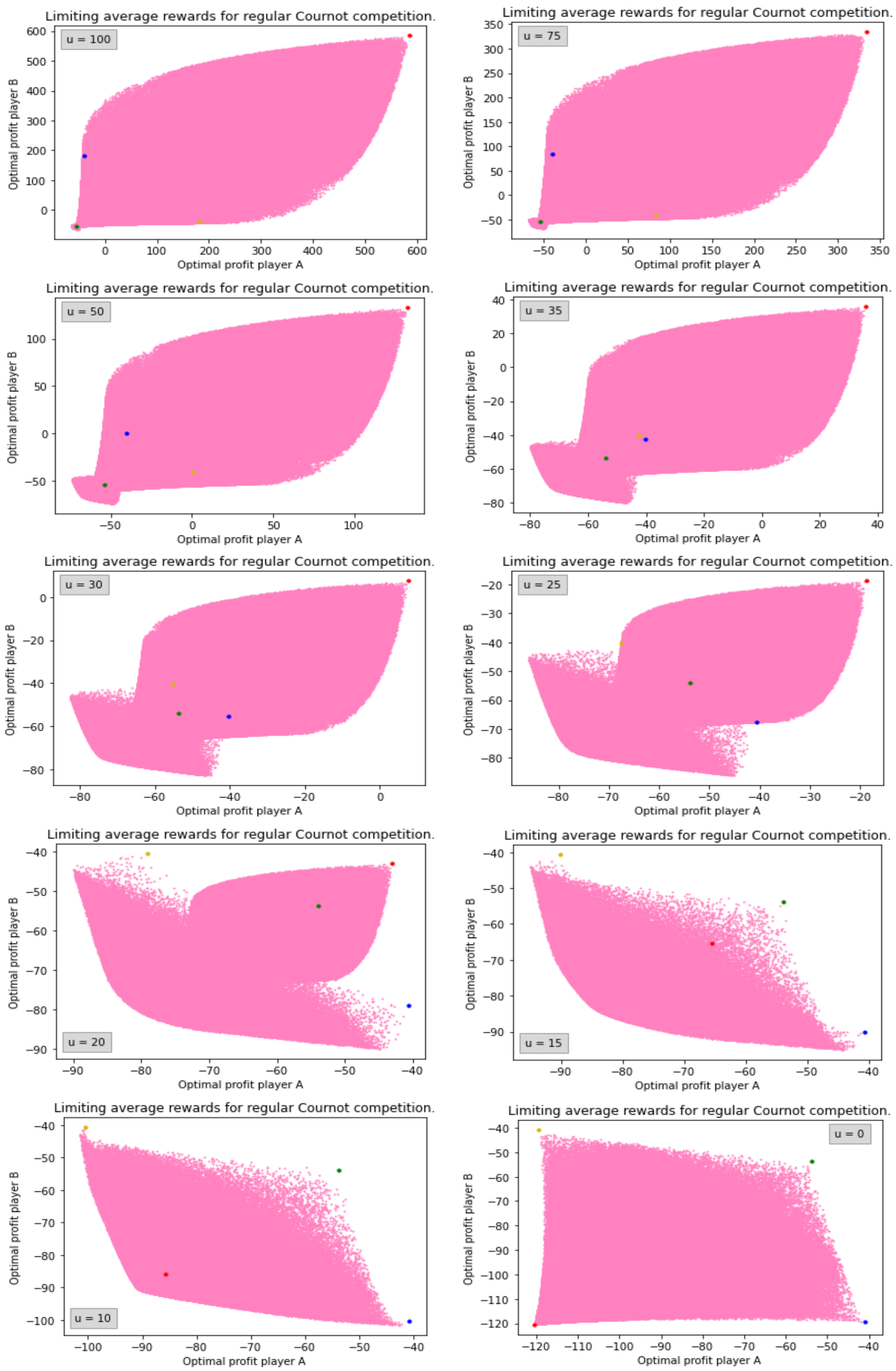


Figure 17. Regular Cournot competition with decrease over u.

5.4 Asymmetric variable cost changes

As previously discussed it is also interesting to observe what happens when we let go of one of our core assumptions and that is the symmetrical nature of both players. In this section we analyse what happens when the cost for one player is significantly different than that for the other player. Due to the symmetrical nature of the model we will put the visualisation where we keep the variable advertisement cost constant for player A and increase the variable advertisement cost for player B on the left and the mirrored results (player B constant, player A increase) on the right.

Additionally we know, through the previous sections, that the structures are not as sensitive to decrease in variable cost as they are to increases. This means that in the visualisation for isolating the effect of asymmetry in variable costs we will only be using an increase in the variable advertisement cost. An effect like this might be exactly how realistic long-term competition in advertisement occurs since one party in the competition might have significantly different cost structures for funding advertisement than their competition, making it more/less expensive for them.

The visualisation will first be done on the baseline model for regular Bertrand, after which the variable cost for advertisement for player B (left) and player A (right) is steadily increased. The results are the outcomes shown in *Figure 18*

Figure 18 shows us that the assymetrical change to variable advertisement costs achieves the same effect for both players. By increasing the cost for one player its limiting average rewards steadily decrease whilst the rewards for the other player (with constant cost) keeps their optimal value. Advertisement profit, for the player who has their cost increased, still exists in the scenarios where they completely abstain from advertisement since their counter party increases the total market size.

We observed that with the previous changes to variable advertisement cost the models for Cournot had an identical shape change, this occurs again for asymmetrical changes. To illustrate the similarities *Figure 19* shows the visualisations of a cost increase for both players, asymmetrically, for the extreme values of $ac_{0,A} = 750$ and $ac_{0,B} = 750$ for Stackelberg-Bertrand (only leader A and follower B), collusion under Bertrand, regular Cournot, Stackelberg-Cournot (only leader B and follower A) and collusion under Cournot. This is purely to illustrate the similarities for both players and for all the short-term competition forms. We again show the increase in $ac_{0,A}$ on the right and $ac_{0,B}$ on the left. The slight difference which can be seen is a minor inward tilt for the extreme minima for all Cournot competitive forms.

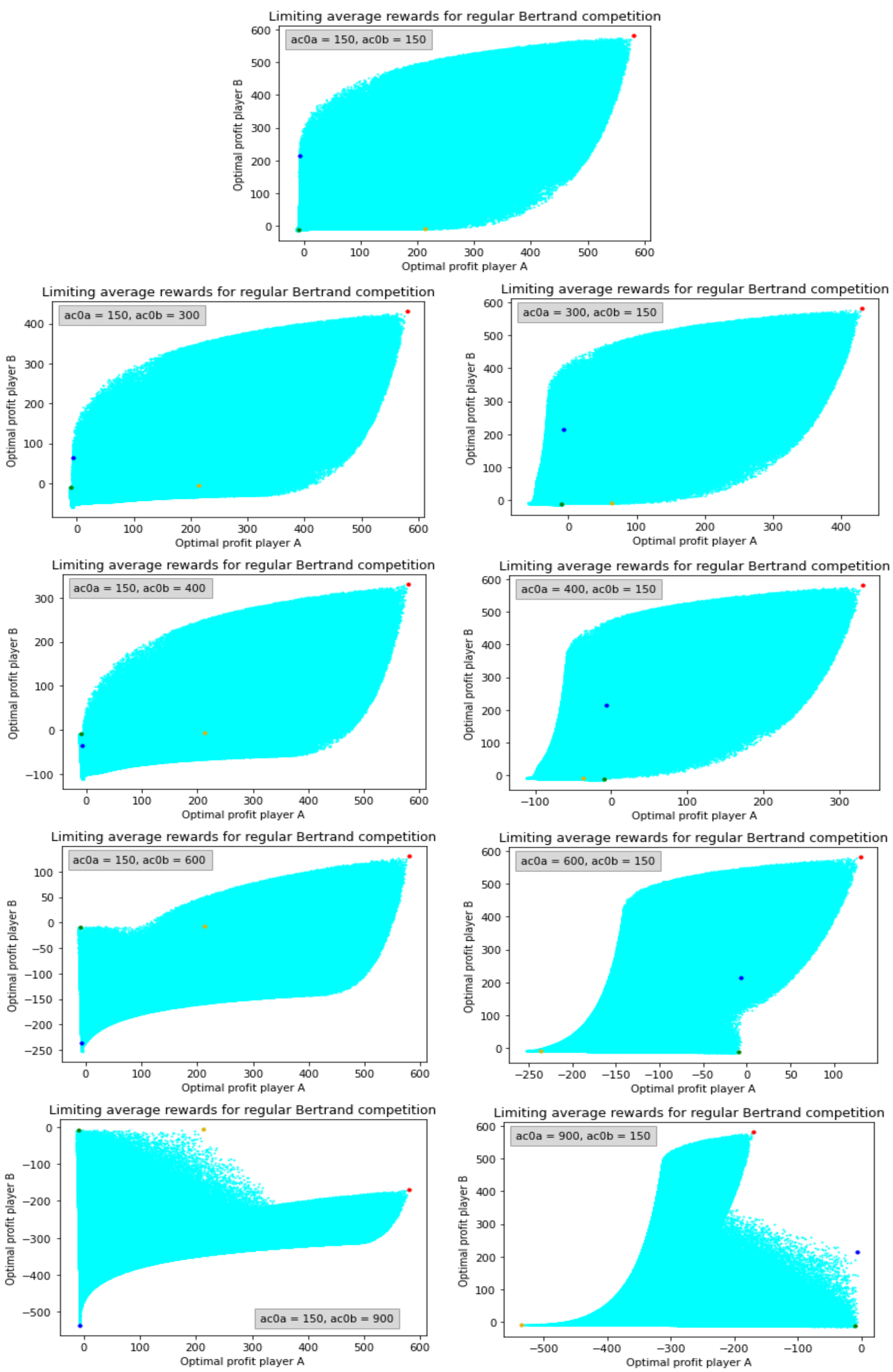


Figure 18. Regular Bertrand competition with asymmetrical increase over $ac_{0,A}$ and $ac_{0,B}$.

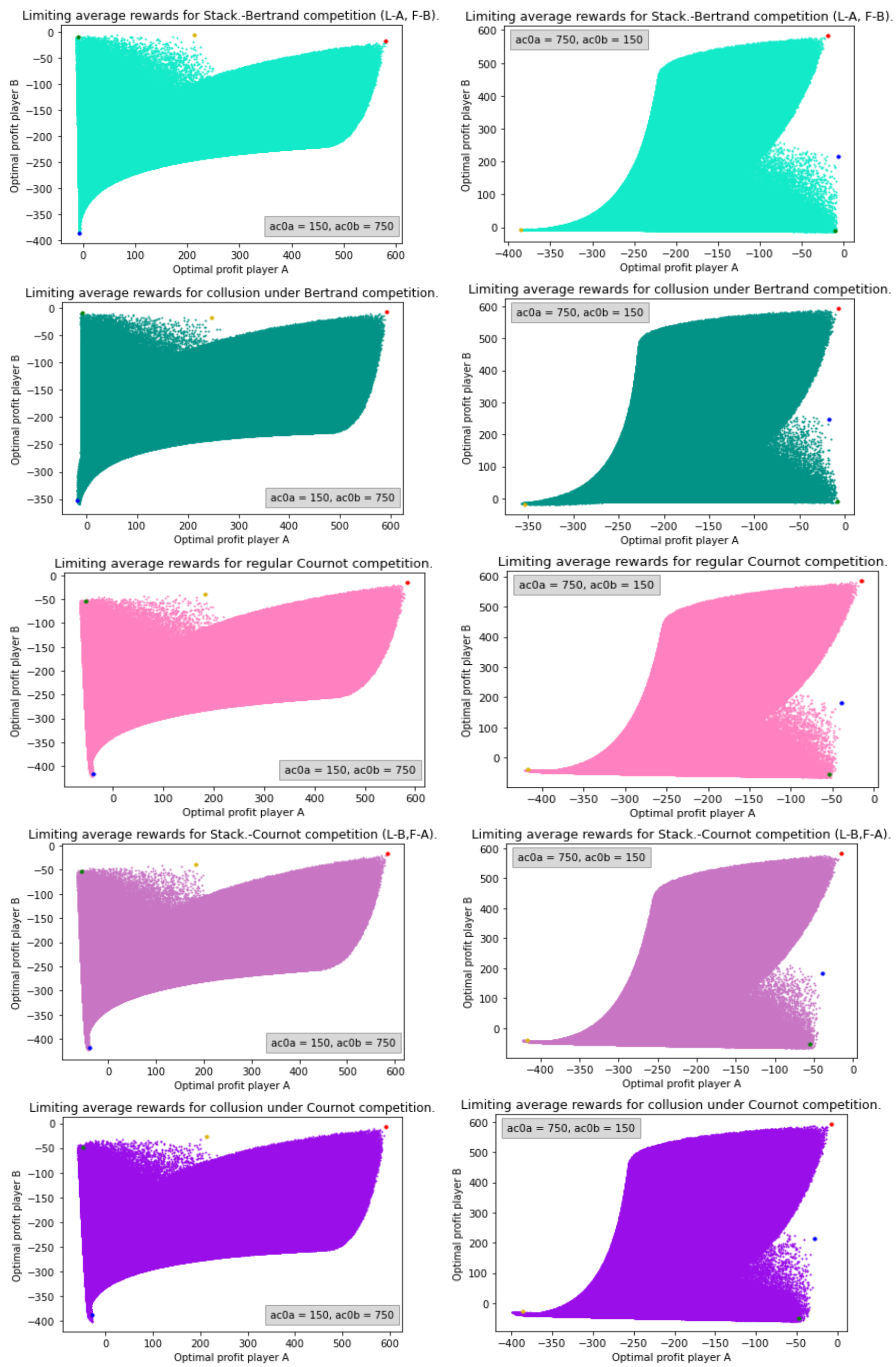


Figure 19. Extreme cases of asymmetrical $ac_{0,A}$ and $ac_{0,A}$ for other models of competition.

We have now tried to isolate a lot of different relationships by adjusting some core variables of the model. The effects can be summarised to some significant shape changes. We believe that the short-term competition forms have similar attributes for the baseline models as well as their reaction to some variable changes. Generally speaking we see that the Cournot models have lower lows and higher highs than the Bertrand models. We also see that for the Stackelberg-Bertrand competition the outcomes are higher for the follower, whilst for Stackelberg-Cournot the outcomes are higher for the leader (see *Table 1* and *Table 2*). There is also a distinct effect where the collusion models for both short-term competition forms have the highest maxima and lowest minima.

Additionally both symmetrical and assymetrical changes in variable advertisement cost have similar effects on most short-term competitions forms, where an increase greatly changes the shape in all cases. As previously mentioned there are parallels to the effect of increasing the positive demand intercept and decreasing the variable advertisement cost.

6 CONCLUSIONS

This thesis models the connection between the long-term decision making and the short-term competitive behaviour through the introduction of contest theory in combination with the law of large numbers. It formalises, through contest theory an explanation for the effect of advertisement choices on short-term competition, since the aggregated effect of a large number of randomised individual decisions of the agents would, for an infinitely iterated repeated game (law of large numbers), lead to a market division based on the advertisement expenditures of both players and also explain how the market grows when advertisement expenditure is made.

Additionally I conclude that the approach of the model in Python, although potentially less efficient than other coding-languages, allows for better generalisation. The Python model and the results it yielded, provide insights into the general effects of long-term dyadic advertisement behaviour on a series of short-term competition models.

It shows, through the visualisations, that by decreasing advertisement cost symmetrically, the profits grow significantly for both players and that the distribution of market shares remains the same. It also shows in the visualisations, that when advertisement cost is increased symmetrically the market situation for all short-term competition models changes significantly. They change into a scenario in which both players are deterred from spending money on advertisement. Asymmetric decrease to the variable advertisement cost favor both players, since the market size is increased for both players. The distribution of market shares does not change much in this case either, a slight favoring of the player who has less cost occurs. It is different however in the case of assymetrical increase of variable advertisement costs. Here the player which has cheaper advertisement cost dominates the market and the player who has their advertisement cost increased is slowly deterred from advertising.

Increase in the positive demand intercept increases the market size significantly and retains the market share distribution in a similar fashion to decreasing the variable advertisement costs. Decreasing the positive demand intercept has similar effects to market size and market share distribution as increasing the variable advertisement costs, it however seems to effect the market a little less significantly than variable advertisement cost increases. We believe this to be the case since the factor of decrease is limited to zero, which it can not exceed and factor of increase can be sustained infinitely (theoretically).

7 DISCUSSION AND RECOMMENDATIONS

First we will discuss the difficulties which were encountered when writing this thesis. Even though computation and calculating average rewards could potentially have been more efficient and faster in VBA, we decided to approach the algorithm in Python, this was a hurdle since the programming aspect of modelling was foreign. This introduces several complications in the thesis, since we believe that the approach to the algorithm is crude. In the draft version of the paper the limiting average reward visualisations had a data point distribution which was not distributed as well, leading me to revise the underlying distribution of the ρ matrix.

Additionally some of the distribution problems could have been due to the low number of ρ generations. We used one million points for ρ for the sake of time efficiency, this however also meant that some of the data was distributed unevenly. We partially solved this issue by drawing another million ρs in a beta distribution. Some density issues now ocurred at the extremes where only one player advertised whilst the other did not. We believe this can be solved by drawing even more ρs in beta distribution, this time weighted towards the pure strategies $[0, 1, 0, 0]$ and $[0, 0, 1, 0]$. We did not do this due to time constraints

Additionally density issues can be solved by scaling the generation of points up to for instance ten millions points, which would not be too difficult (it would be time restricting). Scaling up the point generation to one billion however would mean new means of more efficient vector calculation and perhaps drawing points not simultaneously but in turns would be required. Due to time constraints we therefore limited the sample size.

With regards to recommendations for future research we believe the next step for research in this field and area, which is stochastic game theory (and FD-games), is the inclusion of solving equilibria not only for the short-term (limiting average rewards), as we did in this thesis, but also in the long-term. Now the long-term is a dyadic decision taken on a separate time-horizon than the short-term, whilst in reality the advertisement choices and the strategies employed by the players can also lead to equilibria.

In the paper which might follow this thesis, we wish to include the calculation of threat points for the long-term advertisement behaviour. We believe that by finding the first value from a zero-sum game, through a mini-max strategy, we determine the minimum Nash-equilibrium. After determining this minimum Nash-equilibrium the values can converge using a maxi-min strategy. These insights add another layer of complexity to this thesis and approaches fields like ETP/ESP.

These threat point calculations can be done for what is known as "grim trigger" threats, which are the situations in which one players forces the other to the absolute worst possible limiting average rewards when they transgress even once, most literature is on these threats. But future research could also explore "valid" threats, which are situation where players do not know the worst outcome with which they can threaten their opponent, but know a strategy which at least punishes the opponent which makes the threat valid and not weak. There are also what is known as "forgiving" threat strategies, where one player does not immediately punish their opponent for breaking an agreement as long as the aggregated average of outcomes is sufficient. The above are all possible avenues to explore with contest theory and the multiple short-term competitions as sources of information.

Another possibility for future research is the inclusion of a section where the avenue of combinations of short-term competitions was explored. We contemplated the possibility of introducing a mechanism where in the short-term, instead of what we do in the current models, where in the infinitely repeated stage games the same short-term competitive model is used, we introduce the idea that for certain relative frequencies of player choice different short-term competitive forms are used.

For instance, if at a certain time of play both players chose to advertise (long-term dyadic advertisement choice), leading to the joint relative frequency $\rho = [1, 0, 0, 0]$, the short-term model under which this decision would be assessed would be collusion under Cournot. This is due to the fact that collusion under Cournot yields the highest limiting average rewards for this advertisement decision. And that if players decided to only partially advertise ($\rho = [0, 1/2, 1/2, 0]$), the short-term competition form under which the advertisement decision would be assessed would be regular Bertrand, since it is again provides the highest limiting average rewards.

REFERENCES

Aumann, R. J. (2008). Game engineering. *Mathematical programming and game theory for decision making* (pp. 279–285). World Scientific.

Becker, G. S. (1983). A Theory of Competition Among Pressure Groups for Political Influence. *The Quarterly Journal of Economics*, *98*(3), 371–400. https://doi.org/10.2307/1886017

Benoit, J.-P., & Krishna, V. (1987). Dynamic Duopoly: Prices and Quantities. *The Review of Economic Studies*, *54*(1), 23. https://doi.org/10.2307/2297443

Bertrand, J. L. F. (1883). Book review of theorie mathematique de la richesse sociale and of recherches sur les principles mathematiques de la theorie des richesses. *Journal de Savants*, *67*, 499–508.

Bishop, R. L. (1960). Duopoly: Collusion or Warfare? *The American Economic Review*, *50*(5), 933–961. http://www.jstor.org/stable/1810954

Braithwaite, D. (1928). The economic effects of advertisement. *The Economic Journal*, *38*(149), 16–37. http://www.jstor.org/stable/2224394

Brander, J. A., & Harris, R. (1983). *Anticipated Collusion and Excess Capacity* (Working Paper No. 530). Economics Department, Queen's University. https://ideas.repec.org/p/qed/wpaper/530.html

Brenner, T., & Witt, U. (2003). Melioration learning in games with constant and frequency-dependent pay-offs. *Journal of Economic Behavior and Organization*, *50*(4), 429–448. https://doi.org/https://doi.org/10.1016/S0167-2681(02)00034-3

Cellini, R., Lambertini, L. et al. (2002). *Advertising with spillover effects in a differential oligopoly game with differentiated goods* (Vol. 11). Università degli Studie, Dipartimento di Scienze Economiche.

Cellini, R., Lambertini, L., & Mantovani, A. (2008). Persuasive advertising under Bertrand competition: A differential game. *Operations Research Letters*, *36*(3), 381–384. https://doi.org/10.1016/j.orl.2007.08.003

Corchón, L. C. (2007). The theory of contests: A survey. *Review of Economic Design*, *11*(2), 69–100. https://doi.org/10.1007/s10058-007-0032-5

Cournot, A. A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses*. L. Hachette.

Dixit, A. (1987). Strategic behavior in contests. *The American Economic Review*, *77*(5), 891–898. http://www.jstor.org/stable/1810215

Edgeworth, F. Y. (1925). The pure theory of monopoly. *REPRINT IN: Collected papers relating to political economy*, *1*, 111–142.

Fershtman, C. (1984). Goodwill and Market Shares in Oligopoly. *Economica*, *51*(203), 271. https://doi.org/10.2307/2554545

Fershtman, C., & Muller, E. (1986). Capital Investments and Price Agreements in Semicollusive Markets. *The RAND Journal of Economics*, *17*(2), 214. https://doi.org/10.2307/2555385

Fisher, I. (1898). Cournot and Mathematical economics. *The Quarterly Journal of Economics*, *12*(2), 119–138. https://doi.org/10.2307/1882115

Friedman, J. W. (1983). Advertising and Oligopolistic Equilibrium. *The Bell Journal of Economics*, *14*(2), 464. https://doi.org/10.2307/3003647

Friedman, J. W. (2000). The legacy of Augustin Cournot. *Cahiers d'économie politique*, *37*(1), 31–46. https://doi.org/10.3406/cep.2000.1287

Harmelink, R. (2019). Computing threat points in two-player ETP-ESP games (master's thesis). *University of Twente, essay code = 77278.*

Hsu, P. L., & Robbins, H. (1947). Complete convergence and the law of large numbers. *Proceedings of the National Academy of Sciences of the United States of America*, *33*(2), 25–31. http://www.jstor.org/stable/87477

Jackson, M. O., & Wilkie, S. (2005). Endogenous Games and Mechanisms: Side Payments Among Players. *The Review of Economic Studies*, *72*(2), 543–566. https://doi.org/10.1111/j.1467-937x.2005.00342.x

Jin, J. Y., & Parcero, O. J. (2010). Asymmetric Duopoly under Different Market Structures. *International Journal of Business and Economics*, *9*(1), 45–56.

Joosten, R. (2008). Strategic Advertisement with Externalities: A New Dynamic Approach. *Modeling, Computation and Optimization*, 21–43. https://doi.org/10.1142/9789814273510_0003

Joosten, R. (2015). Long-run Strategic Advertising and Short-run Bertrand Competition. *International Game Theory Review*, *17*(02), 1540014. https://doi.org/10.1142/s0219198915400149

Joosten, R., Brenner, T., & Witt, U. (2003). Games with frequency-dependent stage payoffs. *International Journal of Game Theory*, *31*(4), 609–620. https://doi.org/10.1007/s001820300143

Joosten, R., & Meijboom, R. (2010). Stochastic Games with Endogenous Transitions. *Mathematical programming and game theory* (pp. 205–226). Springer Singapore. https://doi.org/10.1007/978-981-13-3059-9_12

Joosten, R., Peters, H., & Thuijsman, F. (1995). Unlearning by not doing: Repeated games with vanishing actions. *Games and economic behavior*, *9*(1), 1–7.

Joosten, R., & Samuel, L. (2017). On stochastic fishery games with endogenous stage-payoffs and transition probabilities. In D.-F. Li, X.-G. Yang, M. Uetz, & G.-J. Xu (Eds.), *Game theory and applications* (pp. 115–133). Springer Singapore.

Joosten, R., & Samuel, L. (2020). On Finding Large Sets of Rewards in Two-Player ETP–ESP Games. *International Game Theory Review*, *22*(02), 2040002. https://doi.org/10.1142/s0219198920400022

Joosten, R. (1996). Deterministic evolutionary dynamics: A unifying approach. *Journal of evolutionary economics*, *6*(3), 313–324. https://doi.org/10.1007/BF01193636

Kantorovich, L. V. (1940). A new method of solving of some classes of extremal problems. *Dokl. Akad. Nauk SSSR*, *28*, 211–214.

Konrad, K. A. (2007). Strategy in contests. An introduction. *SSRN Electronic Journal.* https://doi.org/10.2139/ssrn.960458

Krueger, A. O. (1974). The political economy of the rent-seeking society. *The American Economic Review*, *64*(3), 291–303. http://www.jstor.org/stable/1808883

Li, D.-F., Yang, X.-G., Uetz, M., & Xu, G.-J. (2016). Game theory and applications, 115–133.
Nalebuff, B. J., & Stiglitz, J. E. (1983). Prizes and incentives: Towards a general theory of compensation and competition. *The Bell Journal of Economics*, *14*(1), 21–43. http://www.jstor.org/stable/3003535
Nash, J. F. (1950). Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences of the United States of America*, *36*(1), 48–49. http://www.jstor.org/stable/88031
Nerlove, M., & Arrow, K. J. (1962). Optimal Advertising Policy under Dynamic Conditions. *Economica*, *29*(114), 129. https://doi.org/10.2307/2551549
Nitzan, S. (1994). Modelling rent-seeking contests. *European Journal of Political Economy*, *10*(1), 41–60. https://doi.org/10.1016/0176-2680(94)90061-2
Okuguchi, K. (1987). Equilibrium prices in the Bertrand and Cournot Oligopolies. *Journal of Economic Theory*, *42*(1), 128–139. https://doi.org/10.1016/0022-0531(87)90106-2
Rosen, S. (1985). *Prizes and incentives in elimination tournaments* (Working Paper No. 1668). National Bureau of Economic Research. https://doi.org/10.3386/w1668
Rothschild, R. (1992). On the sustainability of collusion in differentiated duopolies. *Economics Letters*, *40*(1), 33–37. https://doi.org/https://doi.org/10.1016/0165-1765(92)90241-P
Samuel, L. N. (2017). Computations in Stochastic Game Theory: Large sets of rewards in games with communicating states and frequency-dependent transition probabilities and stage payoffs (master's thesis). *University of Twente, essay code = 72328*.
Shapley, L. S. (1953). Stochastic games. *Proceedings of the national academy of sciences*, *39*(10), 1095–1100.
Singh, N., & Vives, X. (1984). Price and quantity competition in a differentiated duopoly. *The RAND Journal of Economics*, *15*(4), 546–554. https://doi.org/10.2307/2555525
Tullock, G. (1967). The welfare costs of tariffs, monopolies, and theft. *Economic Inquiry*, *5*(3), 224–232. https://doi.org/10.1111/j.1465-7295.1967.tb01923.x
Tullock, G. (1980). Efficient rent-seeking. In: Buchanan JM, Tollison RD, Tullock G (eds) Towards a theory of a rent-seeking society. *Texas AM University Press College Station*, 97–112.
Van Cayseele, P., & Furth, D. (1996). Bertrand–Edgeworth duopoly with buyouts or first refusal contracts. *Games and Economic Behavior*, *16*(2), 153–180. https://doi.org/10.1006/game.1996.0080
Vidale, M. L., & Wolfe, H. B. (1957). An operations-research study of sales response to advertising. *Operations Research*, *5*(3), 370–381. https://doi.org/10.1287/opre.5.3.370
Vives, X. (1984). Duopoly information equilibrium: Cournot and Bertrand. *Journal of Economic Theory*, *34*(1), 71–94. https://doi.org/10.1016/0022-0531(84)90162-5

Vojnović, M. (2016). *Contest theory incentive mechanisms and ranking methods*. Cambridge University Press.

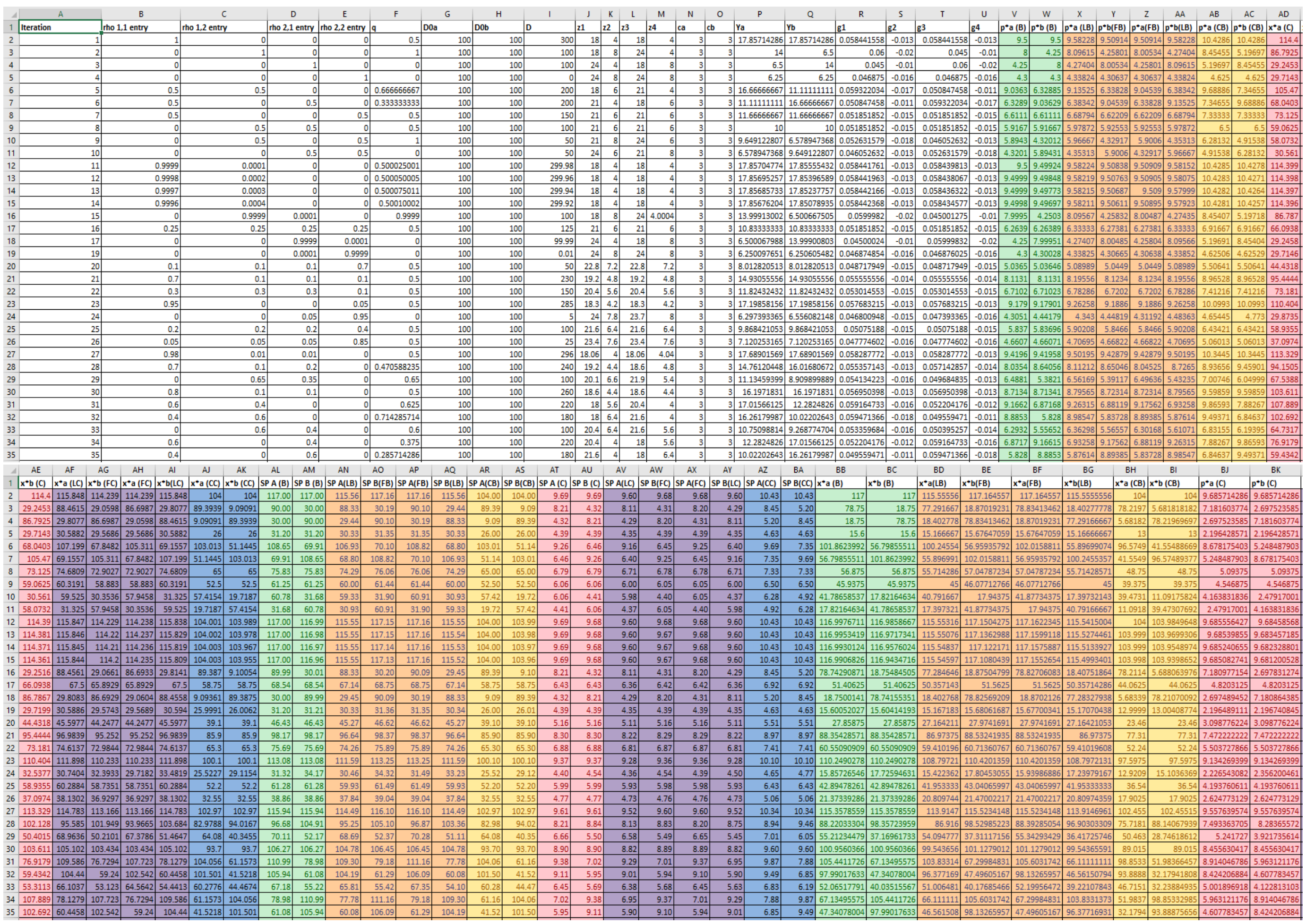
Vojnović, M. (2017). Contest Theory. *Commun. ACM*, *60*(5), 70–80. https://doi.org/10.1145/3012008

Von Stackelberg, H. (1934). Marktform und gleichgewicht (Market Structure and Equilibrium). *J. Springer*.

Appendices

A PRELIMINARY EXCEL MODEL

As a preliminary process I made eight versions of Excel models where I manually imputed the ρs to visualise some effects for the short-term competition. The Excel model will be shown in the following visualisations, the first few screenshots being the variables and their values and the others being the preliminary visualisations. The model is displayed only for 35 iterations since those conveniently fit the space, in reality there are 50 manual iterations. The variables look as follows, the visualisations are consecutive, meaning all four figures are horizontally next to each-other in the model.



	BL	BM	BN	BO	BP	BQ	BR	BS
1	p*a (LC)	p*b (FC)	p*a (FC)	p*b (LC)	p*a (CC)	p*b (CC)	c*a (B)	c*b (B)
2	9.603174603	9.676311031	9.676311031	9.603174603	10.42857143	10.42857143	351	351
3	7.097222222	2.692307692	7.176682692	2.682291667	7.397727273	3.248106061	236.25	56.25
4	2.682291667	7.176682692	2.692307692	7.097222222	3.248106061	7.397727273	56.25	236.25
5	2.177083333	2.193014706	2.193014706	2.177083333	2.3125	2.3125	46.8	46.8
6	8.585069444	5.240551002	8.669313525	5.203869048	9.083303824	5.969074599	305.5871976	170.396
7	5.203869048	8.669313525	5.240551002	8.585069444	5.969074599	9.083303824	170.3956653	305.587
8	5.035714286	5.085106383	5.085106383	5.035714286	5.5	5.5	170.625	170.625
9	4.5	4.539893617	4.539893617	4.5	4.875	4.875	137.8125	137.813
10	4.113800125	2.473797286	4.159224232	2.460320724	4.318406593	2.764903846	125.3597561	53.4649
11	2.460320724	4.159224232	2.473797286	4.113800125	2.764903846	4.318406593	53.46493902	125.36
12	9.603012728	9.6751825	9.676152783	9.60205304	10.42832599	10.42745365	350.9930133	350.958
13	9.602850836	9.674054081	9.675994518	9.600931589	10.42808055	10.42633597	350.9860257	350.915
14	9.602688925	9.672925772	9.675836234	9.599810249	10.42783509	10.4252184	350.9790373	350.873
15	9.602526998	9.671797574	9.675677932	9.598689021	10.42758963	10.42410093	350.9720479	350.83
16	7.096602303	2.692614778	7.176055216	2.682594412	7.397101337	3.248365559	236.2287261	56.2645
17	4.767857143	4.8125	4.8125	4.767857143	5.1875	5.1875	154.21875	154.219
18	2.682256456	7.17594322	2.692273275	7.096488819	3.248006434	7.396969546	56.25004231	236.225
19	2.177143717	2.193326701	2.193074698	2.177390541	2.312587305	2.312819741	46.80156082	46.8124
20	3.066396761	3.09339538	3.09339538	3.066396761	3.303846154	3.303846154	83.57625	83.5763
21	7.39765625	7.462600806	7.462600806	7.39765625	8.06875	8.06875	265.0628571	265.063
22	5.445257022	5.495388237	5.495388237	5.445257022	5.92972973	5.92972973	181.6527273	181.653
23	9.052502616	9.124632519	9.124632519	9.052502616	9.846808511	9.846808511	330.7470833	330.747
24	2.207131035	2.352648433	2.222861888	2.334260891	2.356313378	2.475994293	47.57179637	53.1778
25	4.147807018	4.186640112	4.186640112	4.147807018	4.503947368	4.503947368	128.6843478	128.684
26	2.599066861	2.620366171	2.620366171	2.599066861	2.78306962	2.78306962	64.12017857	64.1202
27	9.475413971	9.548206674	9.548206674	9.475413971	10.29278531	10.29278531	346.0735678	346.074
28	7.423048516	8.274050184	7.484067245	8.202764977	8.154615177	8.867823025	264.6609991	295.072
29	5.183447819	3.914960533	5.234897004	3.885620438	5.518374248	4.310617067	165.6370344	111.509
30	8.37720733	8.446087558	8.446087558	8.37720733	9.118661972	9.118661972	302.8681098	302.868
31	8.82156677	5.954753676	8.904712525	5.911069219	9.372637727	6.700270223	316.3235177	201.405
32	8.331215824	4.600323852	8.415962194	4.570206775	8.781684779	5.305936848	293.970529	142.022
33	4.947464737	4.115932485	4.994974051	4.083206156	5.29445433	4.490613879	156.1955337	120.105
34	5.911069219	8.904712525	5.954753676	8.82156677	6.700270223	9.372637727	201.4048673	316.324
35	4.570206775	8.415962194	4.600323852	8.331215824	5.305936848	8.781684779	142.0223401	293.971

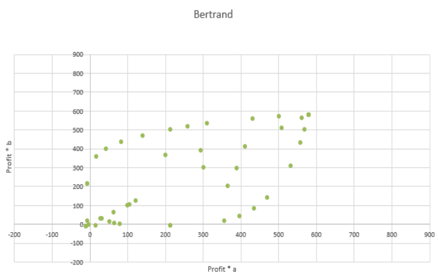
	BT	BU	BV	BW	BX	BY	BZ	CA	CB	CC	CD	CE	CF	CG
1	c*a(LB)	c*b(FB)	c*a(FB)	c*b(LB)	c*a (CB)	c*b (CB)	c*a (C)	c*b (C)	c*a (LC)	c*b (FC)	c*a (FC)	c*b (LC)	c*a (CC)	c*b (CC)
2	346.667	351.494	351.494	346.667	312	312	343.2	343.2	347.544	342.717	342.717	347.544	312	312
3	231.875	56.6106	236.502	55.2083	234.659	17.0455	260.377	87.7358	265.385	87.1795	260.096	89.4231	268.182	27.2727
4	55.2083	236.502	56.6106	231.875	17.0455	234.659	87.7358	260.377	89.4231	260.096	87.1795	265.385	27.2727	268.182
5	45.5	47.0294	47.0294	45.5	39	39	89.1429	89.1429	91.7647	88.7059	88.7059	91.7647	78	78
6	300.737	170.878	306.048	167.691	289.725	124.665	316.411	204.121	321.598	203.545	315.933	207.467	309.04	153.433
7	167.691	306.048	170.878	300.737	124.665	289.725	204.121	316.411	207.467	315.933	203.545	321.598	153.433	309.04
8	167.143	171.144	171.144	167.143	146.25	146.25	219.375	219.375	224.043	218.708	218.708	224.043	195	195
9	135	138.231	138.231	135	118.125	118.125	177.188	177.188	180.957	176.649	176.649	180.957	157.5	157.5
10	122.375	53.8313	125.632	52.192	118.419	33.2753	174.22	91.6829	178.575	91.0607	173.838	93.975	172.246	59.156
11	52.192	125.632	53.8313	122.375	33.2753	118.419	91.6829	174.22	93.975	173.838	91.0607	178.575	59.156	172.246
12	346.659	351.451	351.487	346.625	311.999	311.955	343.197	343.171	347.541	342.688	342.714	347.515	312.003	311.967
13	346.652	351.409	351.48	346.582	311.997	311.91	343.194	343.142	347.539	342.659	342.711	347.486	312.005	311.933
14	346.645	351.367	351.473	346.54	311.996	311.865	343.191	343.113	347.536	342.63	342.708	347.457	312.008	311.9
15	346.638	351.324	351.466	346.498	311.994	311.82	343.188	343.084	347.533	342.601	342.705	347.428	312.01	311.866
16	231.854	56.6251	236.481	55.2226	234.634	17.0642	260.361	87.7547	265.368	87.1983	260.08	89.4423	268.161	27.3016
17	151.071	154.688	154.688	151.071	132.188	132.188	198.281	198.281	202.5	197.679	197.679	202.5	176.25	176.25
18	55.2083	236.477	56.6106	231.85	17.0502	234.632	87.7375	260.36	89.4249	260.079	87.1811	265.367	27.2808	268.162
19	45.5015	47.0419	47.031	45.5121	38.9996	39.0123	89.1439	89.1598	91.7657	88.7228	88.7068	91.7821	77.9972	78.0187
20	81.4926	83.9225	83.9225	81.4926	70.38	70.38	133.295	133.295	136.793	132.743	132.743	136.793	117.3	117.3
21	260.921	265.597	265.597	260.921	231.93	231.93	286.333	286.333	290.952	285.756	285.756	290.952	257.7	257.7
22	178.231	182.141	182.141	178.231	156.72	156.72	219.543	219.543	223.841	218.953	218.953	223.841	195.9	195.9
23	326.392	331.26	331.26	326.392	292.793	292.793	331.213	331.213	335.693	330.699	330.699	335.693	300.3	300.3
24	46.2671	53.4136	47.8196	51.7194	38.7626	45.3109	89.6206	97.6132	92.2212	97.1798	89.1545	100.446	76.5681	87.3463
25	125.86	129.122	129.122	125.86	109.62	109.62	176.806	176.806	180.865	176.205	176.205	180.865	156.6	156.6
26	62.4292	64.4101	64.4101	62.4292	53.7075	53.7075	111.292	111.292	114.391	110.789	110.789	114.391	97.65	97.65
27	341.744	346.57	346.57	341.744	307.365	307.365	339.986	339.986	344.348	339.498	339.498	344.348	308.91	308.91
28	260.748	295.59	265.179	290.709	227.154	264.422	282.452	306.385	286.755	305.847	281.9	311.053	248.936	282.05
29	162.284	111.934	166.029	109.252	151.389	86.2386	202.616	151.205	206.891	150.63	202.136	154.394	192.24	121.037
30	298.631	303.384	303.384	298.631	267.045	267.045	310.832	310.832	315.305	310.303	310.303	315.305	281.1	281.1
31	311.499	201.9	316.81	198.333	296.56	155.951	323.666	230.754	328.758	230.188	323.168	234.384	312.168	183.472
32	289.132	142.488	294.398	139.685	281.666	96.5383	308.077	178.303	313.321	177.72	307.627	181.338	304.504	124.565
33	153.019	120.531	156.599	117.663	140.145	96.7165	194.195	159.934	198.311	159.369	193.693	163.324	180.833	133.402
34	198.333	316.81	201.9	311.499	155.951	296.56	230.754	323.666	234.384	323.168	230.188	328.758	183.472	312.168
35	139.685	294.398	142.488	289.132	96.5383	281.666	178.303	308.077	181.338	307.627	177.72	313.321	124.565	304.504

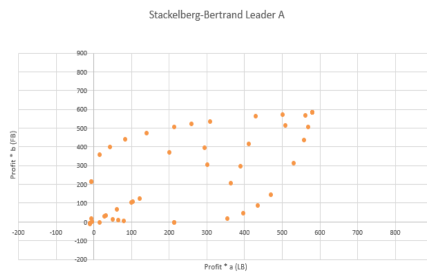
	CH	CI	CJ	CK	CL	CM	CN	CO	CP
1	ac0a	ac0b	c0a	c0b	profit*a (B)	profit*b (B)	profit*a (LB)	profit*b (FB)	profit*a (FB)
2	150	150	30	30	580.5	580.5	580.6188467	582.6407449	582.6407449
3	150	150	30	30	213.75	-6.5625	213.8902244	-6.26105615	214.5917956
4	150	150	30	30	-6.5625	213.75	-6.554153312	214.5917956	-6.26105615
5	150	150	30	30	-9.72	-9.72	-9.703431373	-9.52068916	-9.52068916
6	150	150	30	30	434.8710145	84.07404321	435.0310121	85.14613656	436.7253337
7	150	150	30	30	84.07404321	434.8710145	84.12324281	436.7253337	85.14613656
8	150	150	30	30	100.3819444	100.3819444	100.4711246	101.6323644	101.6323644
9	150	150	30	30	28.984375	28.984375	29.04255319	29.80010751	29.80010751
10	150	150	30	30	15.94328773	-6.473253458	16.01527778	-6.14976563	16.46922387
11	150	150	30	30	-6.473253458	15.94328773	-6.459249442	16.46922387	-6.14976563
12	150	150	30	30	580.4792306	580.3344466	580.5980901	582.4749922	582.6200437
13	150	150	30	30	580.4584575	580.1689265	580.5773298	582.3092728	582.5993387
14	150	150	30	30	580.4376806	580.0034397	580.5565657	582.1435867	582.5786298
15	150	150	30	30	580.4168999	579.837986	580.5357978	581.9779337	582.5579172
16	150	150	30	30	213.692216	-6.565737524	213.8324264	-6.2641938	214.5341071
17	150	150	30	30	62.78428819	62.78428819	62.85714286	63.80580357	63.80580357
18	150	150	30	30	-6.561995986	213.6842894	-6.553647984	214.5260425	-6.26053031
19	150	150	30	30	-9.719154285	-9.72524399	-9.702585819	-9.52587896	-9.51980902
20	150	150	30	30	-3.266816406	-3.266816406	-3.229703851	-2.79575025	-2.79575025
21	150	150	30	30	301.7638776	301.7638776	301.8777293	303.5873424	303.5873424
22	150	150	30	30	104.6576343	104.6576343	104.740488	105.8665537	105.8665537
23	150	150	30	30	508.7301037	508.7301037	508.8514309	510.8463035	510.8463035
24	150	150	30	30	-9.304290722	-11.94285109	-9.287820075	-11.7157443	-9.08811364
25	150	150	30	30	31.69063327	31.69063327	31.75194405	32.51973616	32.51973616
26	150	150	30	30	-9.504901148	-9.504901148	-9.478841319	-9.18322828	-9.18322828
27	150	150	30	30	562.0485302	562.0485302	562.1674058	564.1756917	564.1756917
28	150	150	30	30	294.2252946	389.789768	294.3253588	391.7387401	295.9643851
29	150	150	30	30	65.08646561	6.041565491	65.16869895	6.717240925	65.9985628
30	150	150	30	30	411.8036968	411.8036968	411.9198395	413.7692355	413.7692355
31	150	150	30	30	470.1661329	139.9251605	470.3221053	141.2035515	472.16552
32	150	150	30	30	396.7011806	43.87989572	396.862764	44.75955343	398.379512
33	150	150	30	30	51.46000954	12.35081032	51.53390915	13.076606	52.34627181
34	150	150	30	30	139.9251605	470.1661329	139.9875156	472.16552	141.2035515
35	150	150	30	30	43.87989572	396.7011806	43.91740695	398.379512	44.75955343

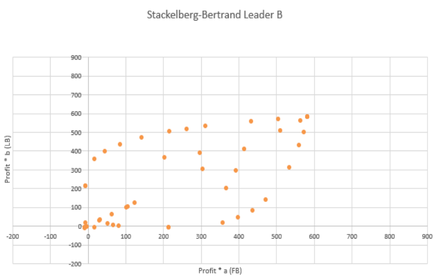
CQ	CR	CS	CT	CU	CV	CW	CX	CY	CZ	DA
profit*b (LB)	profit*a (CB)	profit*b (CB)	profit*a (C)	profit*b (C)	profit*a (LC)	profit*b (FC)	profit*a (FC)	profit*b (LC)	profit*a (CC)	profit*b(CC)
580.6188467	592.5714286	592.5714286	584.845714	584.845714	584.9652401	582.6957625	582.6957625	584.9652401	592.5714286	592.5714286
-6.55415331	246.6528926	-17.5172176	182.931648	-38.846008	182.4465812	-38.9414859	182.1130347	-39.4701522	213.1301653	-27.7444904
213.8902244	-17.5172176	246.6528926	-38.846008	182.931648	-39.4701522	182.1130347	-38.9414859	182.4465812	-27.7444904	213.1301653
-9.70343137	-8.875	-8.875	-53.877551	-53.877551	-55.1715686	-53.8614475	-53.8614475	-55.1715686	-47.875	-47.875
84.12324281	465.9756941	75.62053202	418.879602	47.9878423	418.7163925	47.01737234	417.041559	47.41018979	446.6607153	46.85176432
435.0310121	75.62053202	465.9756941	47.9878423	418.879602	47.41018979	417.041559	47.01737234	418.7163925	46.85176432	446.6607153
100.4711246	106.25	106.25	48.1054688	48.1054688	47.02887538	47.00995926	47.00995926	47.02887538	57.5	57.5
29.04255319	32.8125	32.8125	-13.637695	-13.637695	-14.5212766	-14.3264769	-14.3264769	-14.5212766	-6.5625	-6.5625
-6.45924944	24.52374472	-8.75501691	-37.412595	-45.917073	-38.7010475	-45.9721317	-37.8277858	-46.9054533	-29.3031784	-34.6357861
16.01527778	-8.75501691	24.52374472	-45.917073	-37.412595	-46.9054533	-37.8277858	-45.9721317	-38.7010475	-34.6357861	-29.3031784
580.4532812	592.5560097	592.3991687	584.820798	584.666892	584.9402827	582.5171578	582.670783	584.7862429	592.5521097	592.3874699
580.2877489	592.5405864	592.2269433	584.795878	584.488105	584.9153214	582.3385884	582.6457997	584.6072811	592.5327863	592.2035483
580.1222498	592.5251588	592.0547525	584.770955	584.309354	584.8903564	582.1600545	582.6208126	584.4283546	592.5134585	592.0196638
579.956784	592.5097268	591.8825962	584.746027	584.130638	584.8653875	581.981556	582.5958219	584.2494636	592.4941263	591.8358162
-6.55738594	246.5854076	-17.5173166	182.869585	-38.853906	182.3843911	-38.949488	182.0508916	-39.4781657	213.0585879	-27.7547397
62.85714286	67.578125	67.578125	15.3112793	15.3112793	14.33035714	14.43080357	14.43080357	14.33035714	23.515625	23.515625
213.8245041	-17.5140778	246.5794318	-38.847173	182.858262	-39.4713918	182.0397042	-38.9426509	182.3729957	-27.7447269	213.0490673
-9.70866824	-8.87445672	-8.87955262	-53.87602	-53.887806	-55.1699847	-53.8717773	-53.8599261	-55.1819324	-47.8720942	-47.8859644
-3.22970385	-1.19961538	-1.19961538	-55.611193	-55.611193	-56.9724626	-55.8674658	-55.8674658	-56.9724626	-48.1196154	-48.1196154
301.8777293	311.175625	311.175625	276.848765	276.848765	276.5017263	275.071724	275.071724	276.5017263	285.405625	285.405625
104.740488	110.4913514	110.4913514	63.2253953	63.2253953	62.44958484	62.12441102	62.12441102	62.44958484	71.31135135	71.31135135
508.8514309	520.3730319	520.3730319	504.750405	504.750405	504.7605087	502.636999	502.636999	504.7605087	512.8655319	512.8655319
-11.9225115	-8.62312127	-10.721236	-53.105893	-58.447785	-54.3731105	-58.4698318	-53.0951187	-59.7902234	-46.4286205	-52.7566591
31.75194405	35.48605263	35.48605263	-19.645141	-19.645141	-20.8005758	-20.3026278	-20.3026278	-20.8005758	-11.4939474	-11.4939474
-9.47884132	-8.11858386	-8.11858386	-58.919951	-58.919951	-60.2876668	-59.0197667	-59.0197667	-60.2876668	-52.0610839	-52.0610839
562.1674058	573.982653	573.982653	564.667692	564.667692	564.7654481	562.533632	562.533632	564.7654481	572.438103	572.438103
389.9147374	299.5056409	404.3016377	273.052663	374.61046	272.7769879	372.6838088	271.3521506	374.4445792	277.7237096	386.6735018
6.079372243	74.72840076	5.175547631	23.9035844	-36.04314	23.0783407	-36.5597233	23.08423493	-36.9217791	33.87740502	-29.6224671
411.9198395	422.3736268	422.3736268	400.261015	400.261015	400.1542764	398.3116734	398.3116734	400.1542764	408.3186268	408.3186268
139.9875156	498.7204828	133.8191237	458.058919	107.917119	457.9613003	106.7163314	456.0712568	107.4356234	483.1120623	106.2983601
43.91740695	429.6866648	33.77395242	377.024223	5.55730759	376.793477	4.803141764	375.3658475	4.912460973	406.8488591	5.746717314
12.39492372	58.99152113	12.96927861	9.58609681	-30.141327	8.734642861	-30.7182959	8.803997512	-31.0288137	18.30415458	-23.7163086
470.3221053	133.8191237	498.7204828	107.917119	458.058919	107.4356234	456.0712568	106.7163314	457.9613003	106.2983601	483.1120623
396.862764	33.77395242	429.6866648	5.55730759	377.024223	4.912460973	375.3658475	4.803141764	376.793477	5.746717314	406.8488591

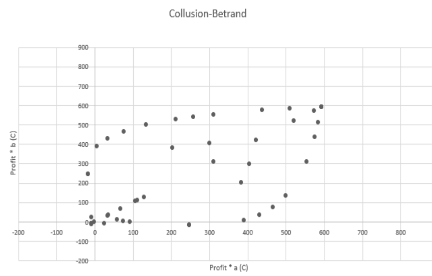
DC	DD	DE
Bertrand		
Stackelberg-Betrand	LB = Leader Betrand	FB = Follower Betrand
Collusion-Betrand		
Cournot		
Stackelberg-Cournot	LC = Leader Cournot	FC = Follower Cournot
Collusion-Cournot		

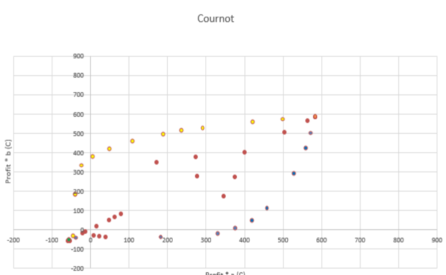
Now the eight preliminary visualisations, the parameters are the exact same as the baseline models.

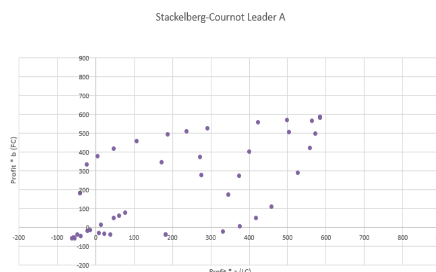


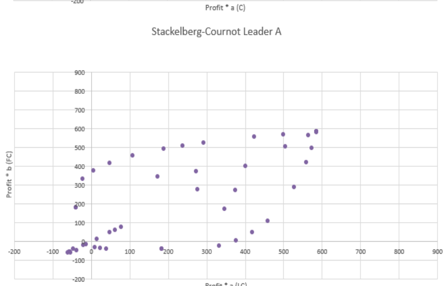


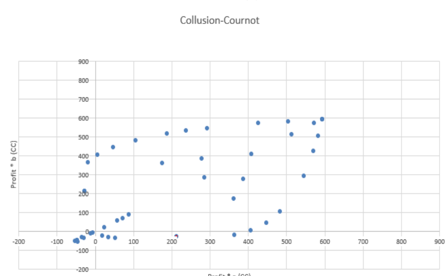












B PYTHON ALGORITHM

The main creation and backbone of this thesis is the algorithmic model in Python. The model contains several components and requires some manual input.The model is shown first and the input field is shown next. For the use of Python one should always first download the packages required to run the code. Additionally the code comes with some remarks and side-notes, some pertaining to the use and input. The functions used in the Python code are cryptic descriptions of their use, the importance is mostly in the results which they yield.

B.1 Import packages

First I import the required packages to run the model.

```
class ETPGame:
    """In this code all the game types are combined, we have 6 types.
    Consisting of: 1. Regular Bertrand, 2. Stackelberg-Bertrand, 3. Collusion under Bertrand, 4. Regular Cournot, 5. Stackelberg-Cournot,
    6. Collusion under Cournot."""

    def __init__(self, game_type, x = 100, ad_varA = 150, ad_varB = 150, ad_fixA = 30, ad_fixB = 30, rho11 = 1, rho12 = 0, rho21 = 0, rho22 = 0):
        """Here we initlialize the game by storing the game type and the matrix player entries,
        the assigned values are default values in case of non-assignment by user."""
        self.game_type = game_type             # type of game
        self.x = x                             # positive intercept for demand
        self.ad_varA = ad_varA                 # the variable advertisement investment cost for player A
        self.ad_varB = ad_varB                 # the variable advertisement investment cost for player B
        self.ad_fixA = ad_fixA                 # the fixed advertisement investment cost for player A
        self.ad_fixB = ad_fixB                 # the fixed advertisement investment cost for player B
        self.rho11 = rho11                     # top-left entry of matrix
        self.rho12 = rho12                     # bottom-left entry of matrix
        self.rho21 = rho21                     # top-right entry of matrix
        self.rho22 = rho22                     # bottom-right entry of matrix

        # Here we add a check to ensure the user inputs a valid game type.
        if self.game_type != 0 and self.game_type != 1 and self.game_type != 2 and self.game_type != 3 and self.game_type != 4 and self.game_type != 5:
            print("Please enter a valid game type. The valid game types are 0, 1, 2, 3, 4 and 5.")

```

```
        # We must add a check for non-positive intercepts for demand
        if self.x < 0:
            self.x = 0
            print("The positive intercept for demand can not be negative, the value has therefore been changed to 0.")

        # Additionally we must add a check to ensure the values input for all advertisement investment costs (both var and fix)
        # does not take values below 0 since they are realistically unfeasible, and the game changes them to 0, which is the
        # feasible minimum. Below this is done for all 4 variables.
        if self.ad_varA < 0:
            self.ad_varA = 0
            print("The variable investment costs for advertisement of player A can not be negative, the value has therefore been changed to 0.")

        if self.ad_varB < 0:
            self.ad_varB = 0
            print("The variable investment costs for advertisement of player B can not be negative, the value has therefore been changed to 0.")

        if self.ad_fixA < 0:
            self.ad_fixA = 0
            print("The fixed investment costs for advertisement of player A can not be negative, the value has therefore been changed to 0.")

        if self.ad_fixB < 0:
            self.ad_fixB = 0
            print("The fixed investment costs for advertisement of player B can not be negative, the value has therefore been changed to 0.")

        # Lastly Rho's needs to sum to 1.
        if (self.rho11 + self.rho12 + self.rho21 + self.rho22) != 1:
            print("The player rho inputs must sum to 1.")

    def rho_generator(self):
        "Generate rho vectors which can be used as mixed strategy input for the players."

        # Define the way the random rho's for the mixed strategies are generated.
        draw_rho11 = np.random.uniform() # draw a number from the uniform distribution
        draw_rho12 = np.random.uniform()
        draw_rho21 = np.random.uniform()
        draw_rho22 = np.random.uniform()
        #print(draw_rho11, draw_rho12, draw_rho21, draw_rho22)
        sum_draw = draw_rho11 + draw_rho12 + draw_rho21 + draw_rho22 # Sum the rho's
        gen_rho11 = draw_rho11 / sum_draw # divide single entries by sum to get values between 0 and 1
```

```
        gen_rho12 = draw_rho12 / sum_draw
        gen_rho21 = draw_rho21 / sum_draw
        gen_rho22 = draw_rho22 / sum_draw
        sum_check = gen_rho11 + gen_rho12 + gen_rho21 + gen_rho22 # check to see if sum does not exceed 1
        self.rho11 = gen_rho11 # assign generated rho's to corresponding variable
        self.rho12 = gen_rho12
        self.rho21 = gen_rho21
        self.rho22 = gen_rho22
        #print(gen_rho11, gen_rho12, gen_rho21, gen_rho22)

    def extra_rho_generator(self):
        "Generate rho vectors which can be used as mixed strategy input for the players."

        # Define the way the random rho's for the mixed strategies are generated.
        draw_rho11 = np.random.beta(20,1) # draw a number from the binomial distribution
        draw_rho12 = np.random.beta(2,8)
        draw_rho21 = np.random.beta(2,8)
        draw_rho22 = 0
        #print(draw_rho11, draw_rho12, draw_rho21, draw_rho22)
        sum_draw = draw_rho11 + draw_rho12 + draw_rho21 + draw_rho22 # Sum the rho's
        gen_rho11 = draw_rho11 / sum_draw # divide single entries by sum to get values between 0 and 1
        gen_rho12 = draw_rho12 / sum_draw
        gen_rho21 = draw_rho21 / sum_draw
        gen_rho22 = draw_rho22 / sum_draw
        sum_check = gen_rho11 + gen_rho12 + gen_rho21 + gen_rho22 # check to see if sum does not exceed 1
        self.rho11 = gen_rho11 # assign generated rho's to corresponding variable
        self.rho12 = gen_rho12
        self.rho21 = gen_rho21
        self.rho22 = gen_rho22
        #print(gen_rho11, gen_rho12, gen_rho21, gen_rho22)

    def pure_strategies(self, counter_ps):
        "Function which includes the four pure strategies in which the players only stick to one option."

        # Assign the rho's to the variables and count how many pure strategies there are, so that only 4 options are used.
        if counter_ps == 0: # if there are 0 pure strategies
            self.rho11 = 1
            self.rho12 = 0
            self.rho21 = 0
            self.rho22 = 0

        if counter_ps == 1: # if there is 1 pure strategies
            self.rho11 = 0
            self.rho12 = 1

        if counter_ps == 2: # if there are 2 pure strategies
```

```
            self.rho12 = 0
            self.rho21 = 1

        if counter_ps == 3: # if there are 3 pure strategies
            self.rho21 = 0
            self.rho22 = 1

    def equilibrium_computation(self, print_text = False):
        "This function contains the calculations of all variables to determine the equilibrium points."

        # Here we calculate the first attributes of used in both the Bertrand & Cournot models which we can later use to
        # calculate prices.

        # In this case we use fixed formula's (possibly add them to input)
        z1 = 24 - 6 * (self.rho11 + self.rho12)  # demand modifiers dependent on advertisement behavior
        z2 = 8 - 4 * (self.rho11 + self.rho21)
        z3 = 24 - 6 * (self.rho11 + self.rho21)
        z4 = 8 - 4 * (self.rho11 + self.rho12)
        D = self.x * (3 * self.rho11 + self.rho12 + self.rho21) # the variable demand function containing the positive intercept
        DA0 = 100 # fixed demand
        DB0 = 100
        ca = 3  # cost modifiers
        cb = 3

        # For q we add a special statement to ensure no devision by 0 occurs.
        if self.rho11 == 0 and self.rho12 == 0 and self.rho21 == 0:
            q = 0
        else:
            q = (self.rho11 + self.rho12)/(2 * self.rho11 + self.rho12 + self.rho21)

        # The following four variables are fixed at 150, 150, 30 and 30 by default and can be changed as user input.
        ac0a = self.ad_varA # advertisement variable investment cost
        ac0b = self.ad_varB
        c0a = self.ad_fixA # advertisement fixed investment cost
        c0b = self.ad_fixB

        # Now additionally for Cournot we need a transformation on D, the D0's and the z's which fits the Cournot model.

        # The calculations are as follows.
        g1 = z3/(z1 * z3 - z2 * z4)
        g2 = -z2/(z1 * z3 - z2 * z4)
        g3 = z1/(z1 * z3 - z2 * z4)
        g4 = -z4/(z1 * z3 - z2 * z4)
        Ya = (z3 * (DA0 + q * D) + z2 * (DB0 + (1-q) * D))/(z1 * z3 - z2 * z4)
        Yb = (z4 * (DA0 + q * D) + z1 * (DB0 + (1-q) * D))/(z1 * z3 - z2 * z4)
```

```python

        # Now using the type of ETP game and the above parameters generated we can calculate the optimal price for Bertrand type games
        # and quantities for Cournot type games

        # For regular Bertrand (game type 0)
        if self.game_type == 0:
            pa = (2 * z3 * (DA0 + q * D) + z2 * (DB0 + (1-q) * D) + 2 * z1 * z3 * ca + z2 * z3 * cb)/(4 * z1 * z3 - z2 * z4)
            pb = (2 * z1 * (DB0 + (1-q) * D) + z4 * (DA0 + q * D) + 2 * z1 * z3 * cb + z1 * z4 * ca)/(4 * z1 * z3 - z2 * z4)

        # For Stackelberg-Bertrand (game type 1)
        if self.game_type == 1:
            paL = (2 * z3 * (DA0 + q * D) + z2 * (DB0 + (1-q) * D) + z2 * z3 * cb)/(4 * z1 * z3 - 2 * z2 * z4) + ca/2
            pbF = (DB0 + (1-q) * D + z4 * paL + z3 * cb)/(2 * z3)
            pbL = (2 * z1 * (DB0 + (1-q) * D) + z4 * (DA0 + q * D) + z1 * z4 * ca)/(4 * z1 * z3 - 2 * z2 * z4) + cb/2
            paF = (DA0 + q * D + z2 * pbL + z1 * ca)/(2 * z1)

        # For collusion under Bertrand (game type 2)
        if self.game_type == 2:
            pa = (2 * z3 * (DA0 + q * D + z1 * ca - z4 * cb) + (z2 + z4) * (DB0 + (1-q) * D + z3 * cb - z2 * ca))/(4 * z1 * z3 - (z2 + z4)**2)
            pb = (2 * z1 * (DB0 + (1-q) * D + z3 * cb - z2 * ca) + (z2 + z4) * (DA0 + q * D + z1 * ca - z4 * cb))/(4 * z1 * z3 - (z2 + z4)**2)

        # For regular Cournot (game type 3)
        if self.game_type == 3:
            xa = (g2 * Yb + 2 * g3 * Ya - 2 * g3 * ca - g2 * cb)/(4 * g1 * g3 - g2 * g4)
            xb = (g4 * Ya + 2 * g1 * Yb - g4 * ca - 2 * g1 * cb)/(4 * g1 * g3 - g2 * g4)

        # For Stackelberg-Cournot (game type 4)
        if self.game_type == 4:
            xaL = (2 * g3 * (Ya - ca) + g2 * (Yb - cb))/(4 * g1 * g3 - 2 * g2 * g4)
            xbF = (Yb + g4 * xaL - cb)/(2 * g3)
            xbL = (2 * g1 * (Yb - cb) + g4 * (Ya - ca))/(4 * g1 * g3 - 2 * g2 * g4)
            xaF = (Ya + g2 * xbL - ca)/(2 * g1)

        # For collusion under Cournot (game type 5)
        if self.game_type == 5:
            xa = (2 * g3 * (Ya - ca) + (g2 + g4) * (Yb - cb))/(4 * g1 * g3 - (g2 + g4)**2)
            xb = (2 * g1 * (Yb - cb) + (g2 + g4) * (Ya - ca))/(4 * g1 * g3 - (g2 + g4)**2)

        # Now we calculate the sales potentials for quantity and price here in order to determine the optimal quantity and price.
```

```python
        # For Bertrand and Cournot respectively.

        # For regular Bertrand and for collusion under Bertrand
        if self.game_type == 0 or self.game_type == 2:
            SPxa = DA0 + q * D - z1 * pa + z2 * pb
            SPxb = DB0 + (1-q) * D - z3 * pb + z4 * pa

        # For Stackelberg-Bertrand (since we have leader and follower here)
        if self.game_type == 1:
            SPxaL = DA0 + q * D - z1 * paL + z2 * pbF
            SPxbF = DB0 + (1-q) * D - z3 * pbF + z4 * paL
            SPxaF = DA0 + q * D - z1 * paF + z2 * pbL
            SPxbL = DB0 + (1-q) * D - z3 * pbL + z4 * paF

        # For regular Cournot and for collusion under Cournot
        if self.game_type == 3 or self.game_type == 5:
            SPpa = Ya - g1 * xa + g2 * xb
            SPpb = Yb - g3 * xb + g4 * xa

        # For Stackelberg-Cournot (since we have leader and follower here)
        if self.game_type == 4:
            SPpaL = Ya - g1 * xaL + g2 * xbF
            SPpbF = Yb - g3 * xbF + g4 * xaL
            SPpaF = Ya - g1 * xaF + g2 * xbL
            SPpbL = Yb - g3 * xbL + g4 * xaF

        # Calculate the optimal quantities and prices through the sales potential, which we can use to find the profit.
        # Where the margins imposed by advertisement behavior,
        # by which the actual prices and quanitities are adjusted due to the rho's, are FIXED

        # For regular Bertrand and collusion under Bertrand
        if self.game_type == 0 or self.game_type == 2:
            xa = SPxa * (self.rho11 * 1 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
            xb = SPxb * (self.rho11 * 1 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))

        # For Stackelberg-Bertrand
        if self.game_type == 1:
            xaL = SPxaL * (self.rho11 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
            xbF = SPxbF * (self.rho11 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))
            xaF = SPxaF * (self.rho11 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
            xbL = SPxbL * (self.rho11 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))

        # For regular Cournot and collusion under Cournot
        if self.game_type == 3 or self.game_type == 5:
            pa = SPpa * (self.rho11 * 1 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
```

```python
        pb = SPpb * (self.rho11 * 1 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))

    # For Stackelberg-Cournot
    if self.game_type == 4:
        paL = SPpaL * (self.rho11 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
        pbF = SPpbF * (self.rho11 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))
        paF = SPpaF * (self.rho11 + self.rho12 * (7/8) + self.rho21 * (5/8) + self.rho22 * (1/2))
        pbL = SPpbL * (self.rho11 + self.rho12 * (5/8) + self.rho21 * (7/8) + self.rho22 * (1/2))

    # Finally we find the optimal profit for the player entries given the above parameters and a cost-parameter
    # We again seperate regular and collusion Bertrand and Cournot from Stackelberg-Bertrand and Stackelberg-Bertrand,
    # due to parameter differences.

    # Now we can use the quanitity, price and cost plus the variable and fixed costs of advertisement to calculate the profit
    if self.game_type == 0 or self.game_type == 2:
        PIa = xa * (pa - ca) - (self.rho11 + self.rho12) * ac0a - c0a
        PIb = xb * (pb - cb) - (self.rho11 + self.rho21) * ac0b - c0b

    # Include statement that if I want printing the seperate equilibrium points are printed for all game-types.
        if print_text:
            if self.game_type == 0:
                print("The regular Bertrand profit equilibrium is at:", PIa, PIb)
            else:
                print("The collusion under Bertrand profit equilibrium is at:", PIa, PIb)

    if self.game_type == 1:
        PIaL = xaL * (paL - ca) - (self.rho11 + self.rho12) * ac0a - c0a
        PIbF = xbF * (pbF - cb) - (self.rho11 + self.rho21) * ac0b - c0b
        PIaF = xaF * (paF - ca) - (self.rho11 + self.rho12) * ac0a - c0a
        PIbL = xbL * (pbL - cb) - (self.rho11 + self.rho21) * ac0b - c0b

        if print_text:
            print("The Stackelberg-Bertrand profit equilibrium, with Leader A is at:", PIaL, PIbF)
            print("The Stackelberg-Bertrand profit equilibrium, with Leader B is at:", PIaF, PIbL)


    if self.game_type == 3 or self.game_type == 5:
```

```
            PIa = xa * (pa - ca) - (self.rho11 + self.rho12) * ac0a - c0a
            PIb = xb * (pb - cb) - (self.rho11 + self.rho21) * ac0b - c0b

            if print_text is True:
                if self.game_type == 3:
                    print("The regular Cournot profit equilibrium is at:", PIa, PIb)
                else:
                    print("The collusion under Cournot profit equilibrium is at:", PIa, PIb)

        if self.game_type == 4:
            PIaL = xaL * (paL - ca) - (self.rho11 + self.rho12) * ac0a - c0a
            PIbF = xbF * (pbF - cb) - (self.rho11 + self.rho21) * ac0b - c0b
            PIaF = xaF * (paF - ca) - (self.rho11 + self.rho12) * ac0a - c0a
            PIbL = xbL * (pbL - cb) - (self.rho11 + self.rho21) * ac0b - c0b

            if print_text is True:
                print("The Stackelberg-Cournot profit equilibrium, with Leader A is at:", PIaL, PIbF)
                print("The Stackelberg-Cournot profit equilibrium, with Leader B is at:", PIaF, PIbL)

        if self.game_type == 0 or self.game_type == 2 or self.game_type == 3 or self.game_type == 5:
            return[PIa, PIb] # return equilibrium profit for regular- and collusion Bertrand and Cournot
        if self.game_type == 1 or self.game_type == 4:
            return[PIaL, PIbF, PIaF, PIbL] # return equilibrium profit for Stackelberg-Bertrand and Stackelberg-Cournot

    def optimal_profit(self, rho_generated, extra_rho_generated, total_points = 4, extra_points = 0):
        "We calculate the optimum profit for all game types using a lot of options."

        #We start off by timing the optimal profit iteration process
        start_time = time.time() #uses the time package to start timer

        # First define the boundries of the equilibrium matrices for both pure strategies and mixed strategies.
        if self.game_type == 0 or self.game_type == 2 or self.game_type == 3 or self.game_type == 5:
            equilibrium_matrix_total = np.zeros((total_points, 2))
            pure_equilibrium_total = np.zeros((4, 2))
            extra_equilibrium_total = np.zeros((extra_points, 2))

        if self.game_type == 1 or self.game_type == 4:
            equilibrium_matrix_total = np.zeros((total_points, 4))
```

```
            pure_equilibrium_total = np.zeros((4, 4))
            extra_equilibrium_total = np.zeros((extra_points, 4))

        # Now iterate first over the four pure strategies and assign them to the previously bounded matrix.
        for i in range(0, 4):
            self.pure_strategies(i)

            pure_equilibrium = self.equilibrium_computation()
            pure_equilibrium_total[i, :] = pure_equilibrium

        self.pure_equilibrium_total = pure_equilibrium_total

        # For a total_points amount of iterations loop through the calculations (by default 1)
        for i in range(0, total_points):
            if rho_generated:
                self.rho_generator()

            random_equilibrium = self.equilibrium_computation()
            equilibrium_matrix_total[i, :] = random_equilibrium

        self.equilibrium_matrix_total = equilibrium_matrix_total

        # For a extra_points amount of iterations loop through the rho's which increase tail-weight.
        for i in range(0, extra_points):
            if extra_rho_generated:
                self.extra_rho_generator()

            extra_equilibrium = self.equilibrium_computation()
            extra_equilibrium_total[i, :] = extra_equilibrium

        self.extra_equilibrium_total = extra_equilibrium_total

        # Lastly we end the timer and print how long the process took
        end_time = time.time() #stops the timer
        print("Running time for equilibrium points generation:", end_time-start_time)

    def plot_equilibrium_outcomes(self, leader = 0):
        "Plot the equilibrium outcomes of the different type games, and additionally print the investment costs for advertisements used."

        if self.game_type == 0:
            # Plots first two entries of equilibrium point, does not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="
```

```python
Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,0], self.equilibrium_matrix_total[:,1], color='xkcd:cyan', zorder=1, s=1, label="LAR regular Bertrand")
            plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:cyan', zorder=1, s=1)
            plt.title("Limiting average rewards for regular Bertrand competition")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
            plt.figtext(0.15, 0.85, "ac0a = 900, ac0b = 150", horizontalalignment ="left", verticalalignment ="top", wrap = True, fontsize = 10, bbox ={'facecolor':'grey', 'alpha':0.3, 'pad':5})
            plt.show()
            print("With the positive intercept for demand at:", self.x)
            print("With variable advertisement investment cost for player A at:", self.ad_varA)
            print("With variable advertisement investment cost for player B at:", self.ad_varB)
            print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
            print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
            print("")

        if self.game_type == 1 and leader == 0:
            # Plots first two entries of equilibrium point, does not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,0], self.equilibrium_matrix_total[:,1], color='xkcd:aqua', zorder=1, s=1, label="LAR Stackelberg-Bertrand")
            plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:aqua', zorder=1, s=1)
            plt.title("Limiting average rewards for Stack.-Bertrand competition (L-A, F-B).")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
```

```
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
            plt.show()
            print("With the positive intercept for demand at:", self.x)
            print("With variable advertisement investment cost for player A at:", self.ad_varA)
            print("With variable advertisement investment cost for player B at:", self.ad_varB)
            print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
            print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
            print("")

        if self.game_type == 1 and leader == 1:
            # Plots first two entries of equilibrium point, does not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,2], self.equilibrium_matrix_total[:,3], color='xkcd:aqua', zorder=1, s=1, label="LAR Stackelberg-Bertrand")
            plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:aqua', zorder=1, s=1)
            plt.title("Limiting average rewards for Stackelberg-Bertrand competition with Leader B and Follower A.")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
            plt.figtext(0.15, 0.85, "ac0a = ac0b = 150", horizontalalignment ="left", verticalalignment ="top", wrap = True, fontsize = 10, bbox ={'facecolor':'grey', 'alpha':0.3, 'pad':5})
            plt.show()
            print("With the positive intercept for demand at:", self.x)
            print("With variable advertisement investment cost for player A at:", self.ad_varA)
            print("With variable advertisement investment cost for player B at:", self.ad_varB)
            print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
            print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
            print("")
```

```
        if self.game_type == 2:
            # Plots first two entries of equilibrium point, does not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,0], self.equilibrium_matrix_total[:,1], color='xkcd:teal', zorder=1, s=1, label="LAR collusion Bertrand")
            plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:teal', zorder=1, s=1)
            plt.title("Limiting average rewards for collusion under Bertrand competition.")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
            plt.show()
            print("With the positive intercept for demand at:", self.x)
            print("With variable advertisement investment cost for player A at:", self.ad_varA)
            print("With variable advertisement investment cost for player B at:", self.ad_varB)
            print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
            print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
            print("")

        if self.game_type == 3:
            # Plots first two entries of equilibrium point, does not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,0], self.equilibrium_matrix_total[:,1], color='xkcd:pink', zorder=1, s
```

```
    =1, label="LAR regular Cournot")
            plt.scatter(self.extra_equilibrium_total[:,0], self.
    extra_equilibrium_total[:,1], color='xkcd:pink', zorder=1, s=1)
            plt.title("Limiting average rewards for regular Cournot
     competition.")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5)
    , labelspacing=3)
            plt.figtext(0.80, 0.85, "u = 0", horizontalalignment ="
    left", verticalalignment ="top", wrap = True, fontsize = 10,
    bbox ={'facecolor':'grey', 'alpha':0.3, 'pad':5})
            #plt.figtext(0.80, 0.85, "u = 0", horizontalalignment
    ="left", verticalalignment ="top", wrap = True, fontsize = 10,
    box ={'facecolor':'grey', 'alpha':0.3, 'pad':5})
            plt.show()
            print("With the positive intercept for demand at:",
    self.x)
            print("With variable advertisement investment cost for
    player A at:", self.ad_varA)
            print("With variable advertisement investment cost for
    player B at:", self.ad_varB)
            print("With fixed advertisement investment cost for
    player A at:", self.ad_fixA)
            print("With fixed advertisement investment cost for
    player B at:", self.ad_fixB)
            print("")

        if self.game_type == 4 and leader == 0:
            # Plots first two entries of equilibrium point, does
    not require axis assigning.
            plt.scatter(self.pure_equilibrium_total[0,0], self.
    pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="
    Pure strategy [1, 0, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[1,0], self.
    pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10,
     label="Pure strategy [0, 1, 0, 0]")
            plt.scatter(self.pure_equilibrium_total[2,0], self.
    pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="
    Pure strategy [0, 0, 1, 0]")
            plt.scatter(self.pure_equilibrium_total[3,0], self.
    pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="
    Pure strategy [0, 0, 0, 1]")
            plt.scatter(self.equilibrium_matrix_total[:,0], self.
    equilibrium_matrix_total[:,1], color='xkcd:orchid', zorder=1, s
    =1, label="LAR Stackelberg-Cournot")
            plt.scatter(self.extra_equilibrium_total[:,0], self.
    extra_equilibrium_total[:,1], color='xkcd:orchid', zorder=1, s
    =1)
            plt.title("Limiting average rewards for Stackelberg-
    Cournot competition with Leader A and Follower B.")
            plt.xlabel("Optimal profit player A")
            plt.ylabel("Optimal profit player B")
            plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5)
    , labelspacing=3)
            plt.show()
```

```
        print("With the positive intercept for demand at:", self.x)
        print("With variable advertisement investment cost for player A at:", self.ad_varA)
        print("With variable advertisement investment cost for player B at:", self.ad_varB)
        print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
        print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
        print("")

    if self.game_type == 4 and leader == 1:
        # Plots first two entries of equilibrium point, does not require axis assigning.
        plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="Pure strategy [1, 0, 0, 0]")
        plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
        plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
        plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
        plt.scatter(self.equilibrium_matrix_total[:,2], self.equilibrium_matrix_total[:,3], color='xkcd:orchid', zorder=1, s=1, label="LAR Stackelberg-Cournot")
        plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:orchid', zorder=1, s=1)
        plt.title("Limiting average rewards for Stack.-Cournot competition (L-B,F-A).")
        plt.xlabel("Optimal profit player A")
        plt.ylabel("Optimal profit player B")
        plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
        plt.show()
        print("With the positive intercept for demand at:", self.x)
        print("With variable advertisement investment cost for player A at:", self.ad_varA)
        print("With variable advertisement investment cost for player B at:", self.ad_varB)
        print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
        print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
        print("")

    if self.game_type == 5:
        # Plots first two entries of equilibrium point, does not require axis assigning.
        plt.scatter(self.pure_equilibrium_total[0,0], self.pure_equilibrium_total[0,1], color='r', zorder=2, s=10, label="
```

```
Pure strategy [1, 0, 0, 0]")
        plt.scatter(self.pure_equilibrium_total[1,0], self.pure_equilibrium_total[1,1], color='xkcd:gold', zorder=2, s=10, label="Pure strategy [0, 1, 0, 0]")
        plt.scatter(self.pure_equilibrium_total[2,0], self.pure_equilibrium_total[2,1], color='b', zorder=2, s=10, label="Pure strategy [0, 0, 1, 0]")
        plt.scatter(self.pure_equilibrium_total[3,0], self.pure_equilibrium_total[3,1], color='g', zorder=2, s=10, label="Pure strategy [0, 0, 0, 1]")
        plt.scatter(self.equilibrium_matrix_total[:,0], self.equilibrium_matrix_total[:,1], color='xkcd:violet', zorder=1, s=1, label="LAR collusion Cournot")
        plt.scatter(self.extra_equilibrium_total[:,0], self.extra_equilibrium_total[:,1], color='xkcd:violet', zorder=1, s=1)
        plt.title("Limiting average rewards for collusion under Cournot competition.")
        plt.xlabel("Optimal profit player A")
        plt.ylabel("Optimal profit player B")
        plt.legend(loc='center left', bbox_to_anchor=(1.1, 0.5), labelspacing=3)
        plt.show()
        print("With the positive intercept for demand at:", self.x)
        print("With variable advertisement investment cost for player A at:", self.ad_varA)
        print("With variable advertisement investment cost for player B at:", self.ad_varB)
        print("With fixed advertisement investment cost for player A at:", self.ad_fixA)
        print("With fixed advertisement investment cost for player B at:", self.ad_fixB)
        print("")

    def check_extremes(self):
        "Check extreme values in Pure and mixed strategies."

        # Check minimum values for pure- and mixed strategies
        print("The minimum value for the pure strategy equilibrium values:")
        print(np.min(self.pure_equilibrium_total, axis=0))
        print("The minimum value for the mixed strategy equilibrium values:")
        print(np.min(self.equilibrium_matrix_total, axis=0))
        print("")

        # Check maximum values for pure- and mixed strategies
        print("The maximum value for the pure strategy equilibrium values:")
        print(np.max(self.pure_equilibrium_total, axis=0))
        print("The maximum value for the mixed strategy equilibrium values:")
        print(np.max(self.equilibrium_matrix_total, axis=0))
```

B.2 Getting output

The last part of the Python code is the input needed from the user of the code to get visualisations. I will first explain what the fields mean. The first entry is game type choosing the integers 0, ..., 5 for this yields the eight different models.Where 0 is regular Bertrand, 1 is both Stackelberg-Bertrand variations, 2 is collusion under Bertrand, 3 is regular Cournot, 4 is both Stackelberg-Cournot variations and 5 is collusion under Cournot.

Then the user defines a variable which is this run of games. It equals to the class name and is provided with some input variables, as defined in the main body of code the ETP class must always have game type as input, additionally the user can add in the next spaces manual input for u, $ac_{0,A}$, $ac_{0,B}$, $c_{0,A}$ and $ac_{0,A}$, they will have default values if left empty. The next line runs the optimal profit calculations and therefore the actual computations.

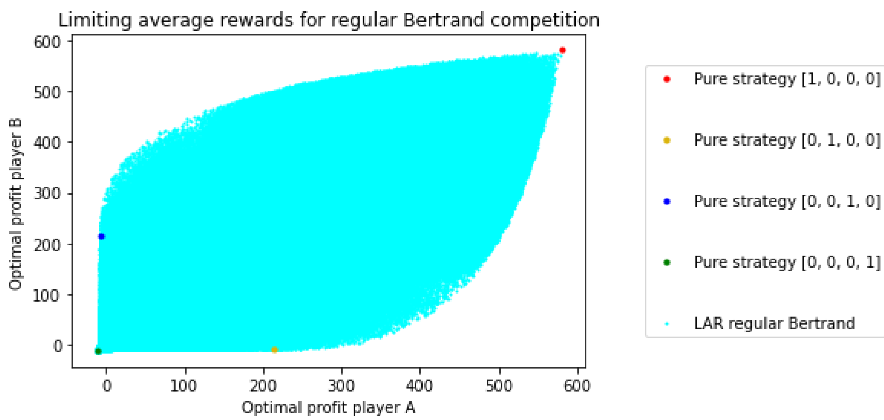
The input variables are whether all limited average rewards are printed (TRUE for not printing, FALSE for printing) and the number of iterations for which the calculations are done (1 million in this case). The next line is creating the visualisations, where the input variable can be 0 or 1, this simply determines for the four Stackelberg models if the structure is Leader A - Follower B (input is 0) or Follower A - Leader B (input is 1). Lastly the code checks for extremes, which does not require any input.

```
game_type = 0

FirstTryETP = ETPGame(game_type, 100, 150, 150, 30, 30)
FirstTryETP.optimal_profit(True, True, 1000000, 1000000)
FirstTryETP.plot_equilibrium_outcomes(0)
FirstTryETP.check_extremes()
```

This then yields the following visualisation as a result (due to input this is the baseline model for regular Bertrand), results are shown in *Figure 20*:

```
Running time for equilibrium points generation: 32.090252161026
```




```
With the positive intercept for demand at: 100
With variable advertisement investment cost for player A at: 150
With variable advertisement investment cost for player B at: 150
With fixed advertisement investment cost for player A at: 30
With fixed advertisement investment cost for player B at: 30

The minimum value for the pure strategy equilibrium values:
[-9.72 -9.72]
The minimum value for the mixed strategy equilibrium values:
[-13.40633673 -13.47127745]

The maximum value for the pure strategy equilibrium values:
[580.5 580.5]
The maximum value for the mixed strategy equilibrium values:
[553.76552644 561.67775973]
```

Figure 20. Baseline equilibrium outcomes for regular Bertrand, with additional code