BSc Thesis Applied Mathematics

Infinitely rolling polyhedra extending a model for rolling polyhedra

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February, 2022

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Preface

For decades the properties of rolling polyhedra have been studied by mathematicians, giving beautiful insights into geometry. It started with requirements for finding when polyhedra could rest on certain faces, trying to find convex polyhedra which were either stable on only one face, or maybe only two. Unistability and bistability. This lead to some nice results using computer search to try and optimize when unistability could be achieved. The model then got extended to actively rolling with concepts such as Hamiltonian stability, meaning the polyhedron will roll through every face, given the correct starting face. This also led to research of how far the polyhedron could roll. Giving rise to the question what would happen if we were to tilt this plane on which the polyhedron is rolling.

I would like to thank my supervisor Antonios Antoniadis for his help with formulating a fitting research question for my bachelor thesis and his advice on how to attack the proofs of the theorems found in this paper.

Infinitely rolling polyhedra
extending a model for rolling polyhedra

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Abstract

When a convex polyhedron is constructed and placed upon a slanted plane it can roll from face to face. It does this obeying the laws of physics, continuously lowering its centre of gravity. In this process it is assumed the polyhedron loses all its momentum when landing on its new resting face. Under the right conditions the convex polyhedron can roll forever. I show what the minimum number of required faces is to achieve this and I show an upper bound for which it is definitely possible to have a convex polyhedron that rolls indefinitely.

Keywords: rolling, convex, polyhedron, polyhedra

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Contents

1 Introduction

In this paper I extend a model for rolling polyhedra from the horizontal plane to slanted planes. I attempt to give a lower and upper bound on the required amount of faces of polyhedron for it to roll on forever. The model has been used before to show requirements for certain properties of convex polyhedra in the horizontal plane. I proof a lower bound and show an upper bound for the required number of faces for the convex polyhedron to roll forever. It is possible to improve these boundaries, but the problem is rather complicated with the added variable of what direction the polyhedron is rolling at every step, making it hard to generalise the shape of the polyhedron and using the nice symmetries given by the problem when applied to the horizontal plane.

2 The model

The origin of the model on which the research question is based on comes from the paper "On the cover of the rolling stone" [1]. In this model the main idea is to emulate rolling convex polyhedra by ignoring friction and inertia and only focusing on the fact that for the polyhedron to roll from the face it is standing on to the next face, it needs to roll over an edge and it can only do this when it is lowering its centre of gravity. On the horizontal plane several results have already been shown:

- For a convex polyhedron of unit diameter to roll a distance L; the amount of faces required is of order $\Theta(L^6)$ [1].
- It is possible to create a unistable polyhedron with at most 14 faces [2].

Unistable polyhedra are defined such that the polyhedra are only stable on one face, from any other face it will roll from face to face until it reaches its only stable face.
I extend this model by looking into what can be said about rolling convex polyhedra when the plane, the convex polyhedron is placed upon, is slanted.

I will define the planes according to their location in the Cartesian coordinate system.

Definition 2.1 (horizontal plane)**.** The horizontal plane H is defined as the following $H = \{(x, y, 0) \mid \forall x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$.

Because we want the horizontal plane to be perpendicular to the direction of gravity, it also follows from this that the direction of gravity is the vector $-\hat{z}$, but to be rigorous and unambiguous I will also define the direction of gravity.

Definition 2.2 (direction of gravity)**.** The direction of gravity is $-\hat{z}$

Definition 2.3 (slanted plane)**.** The slanted plane is a copy of the horizontal plane H rotated clockwise around the y-axis with angle α

The slanted plane is defined like this so the direction "down the slope" is in the positive x direction.

Definition 2.4 (projection plane)**.** The projection plane P is defined as $P = \{(x, 0, z) | \forall x, z \in \mathbb{R}$ in $\mathbb{R}^3$

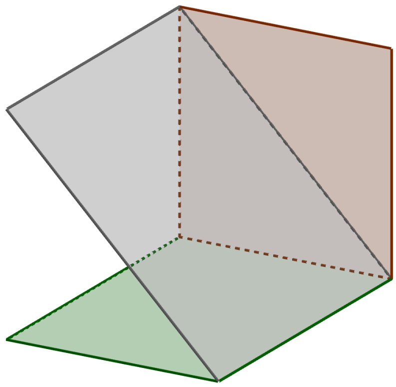
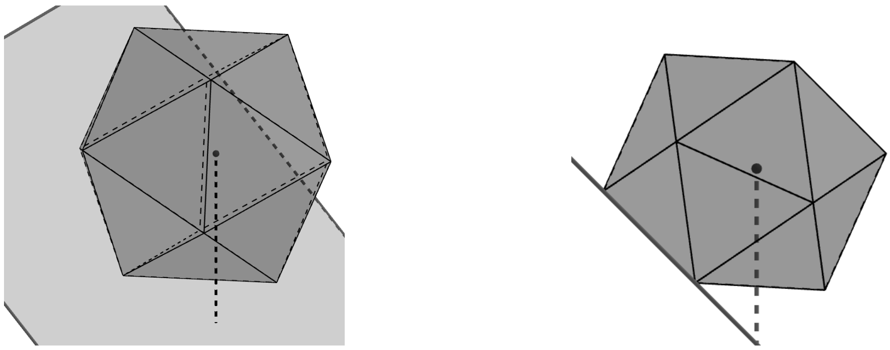


FIGURE 1: Red plane: Projection plane

Definition 2.5 (step)**.** A step is the polyhedron rolling from its resting face, over an edge, to the next resting face. The next resting face is adjacent to the current resting face, separated by the edge the polyhedron rolls over. To complete a step, the vertical line through centre of gravity needs to not pass through the current resting face.



(A) 3d view

(B) Side profile

FIGURE 2: An icosahedron taking a step

Later on in the proof it turns out that defining a concept like "a full rotation" is very useful so I will do that here.

Definition 2.6 (step size)**.** The step size $size(s)$ is defined based on the direction of the step in the projection plane P. Up-hill, in the negative x direction, it is defined as: $size(s) = -(\pi - \beta_s)$. Down-hill, in the positive x direction, it is defined as: $size(s) = \pi - \beta_s$. Where β_s is the internal angle in the projection plane of step s.

Definition 2.7 (full rotation)**.** A full rotation is a set of steps S of which the sum of the sizes of $s \in S$ is greater or equal to 2π.

$$\sum_{s \in S} size(\beta_s) \geq 2\pi$$

3 Theorems

I would like to look at the slanted plane and rolling polyhedra, this of course makes it possible for the polyhedra to roll on forever, but when is it possible for the polyhedron to keep on rolling forever; what are the properties the polyhedron must possess?

Theorem 3.1 (minimal interior angle)**.** *Let e be the edge the polyhedron will roll over, then the projection of the face-edge-face angle onto the projection plane P needs to have an angle* $\beta > \frac{\pi}{2} - \alpha$*, where* α *is the angle of the slanted plane to the horizontal plane.*

Proof. For the polyhedron to roll, the vertical line through the centre of gravity may not pass through the current resting face, otherwise the polyhedron will not roll. We can also find what edge the polyhedron will roll over, but we can, without loss of generality, assume we know what edge the polyhedron will roll over. Now we can draw a line segment from the middle of the edge to the centre of gravity. This line can be projected onto the projection plane, and it has angle γ to the horizontal plane. We have three cases:

- The polyhedron rolls up the hill, in this case the proof of the polyhedron rolling down hill will hold
- The polyhedron rolls in perpendicular direction of the projection plane, the angle γ will be $\frac{\pi}{2}$.
- The polyhedron rolls down hill, in that case we have the following proof:

Angle γ needs to be bigger than $\frac{\pi}{2}$, otherwise the vertical line through the centre of gravity passes through the face. Now angle $\beta + \alpha$ needs to be bigger than γ, otherwise we can find points on the line which are outside the convex polyhedron on the line from the centre of the edge to the centre of gravity, two points which are inside the polyhedron, making the polyhedron non-convex, a contradiction. So we find $\alpha + \beta > \gamma > \frac{\pi}{2}$ □

In the proof it is stated that we can have the polyhedron rolling sideways, giving us that the angle γ may equal $\frac{\pi}{2}$, we find that this does not matter too much, since we can ignore the steps where this happens, because to roll infinitely, the polyhedron needs to roll down the hill, else the polyhedron cannot roll indefinitely, see [1].

Lemma 3.2 (full rotation downhill)**.** *The polyhedron needs to complete a full rotation 2.7 downhill at some point in its rotation.*

Proof. The polyhedron cannot roll uphill forever, since the centre of gravity needs to lower at every step. So given a situation it is possible to show that there is a maximal distance the polyhedron can roll uphill. Given the height h of the centre of gravity with respect to the slanted plane, the polyhedron can roll at most:

$$\frac{h}{sin(\alpha)}$$

Most importantly, this value is finite and since by [1] the polyhedron can also not roll horizontally indefinitely the polyhedron needs to roll down, and at some point complete a full rotation downhill.

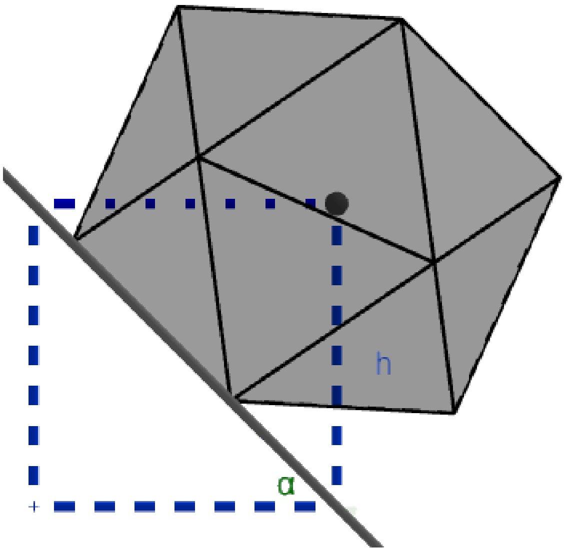


FIGURE 3: The polyhedron cannot roll higher than the point at equal height on the slope

□

Theorem 3.3 (minimum number of required faces)**.** *The minimum number of required faces for a slanted plane with angle α to the horizontal plane is given by the formula:*

$$n \geq \lceil \frac{2\pi}{\alpha + \frac{\pi}{2}} \rceil \tag{1}$$

Proof. We use 3.1. From 3.2 we have that at some point the polyhedron needs to have completed an entire rotation down the hill. We know from 3.1 that in the projection plane P every angle $\beta > \frac{\pi}{2} - \alpha$.

$$\begin{aligned} 2\pi &\leq n \cdot (\pi - \beta) \\ 2\pi - n\pi &\leq -n\beta \\ \pi - \frac{2\pi}{n} &\geq \beta \\ \pi - \frac{2\pi}{n} &> \frac{\pi}{2} - \alpha \\ -\frac{2\pi}{n} &> -\alpha - \frac{\pi}{2} \\ \frac{1}{n} &< \frac{\alpha + \frac{\pi}{2}}{2\pi} \\ n &> \frac{2\pi}{\alpha + \frac{\pi}{2}} \end{aligned} \tag{2}$$

□

It should be noted that the lower bound given by this theorem is quite weak, the theorem in effect shows that for all angles $0 < \alpha \leq \frac{\pi}{2}$ the minimum required number of faces ranges from two to four, even though we already know that to have a convex polyhedron we already need at least four faces. This theorem gives however a good basis for a stronger lower bound proof.

Theorem 3.4 (Upper bound of required number of faces)**.** *For any given angle α, the minimal required number of faces is at most N, given by:*

$$N = \left\lceil \frac{\pi}{\alpha} \right\rceil + 2$$

Proof. For any angle α of the slanted plane, we can always construct a prism with as base a regular polygon as our convex polyhedron. We then angle the polyhedron such that the base of the prism is parallel with the projection plane. This makes it so we can solve for the required number of faces only in the projection plane, making this a 2 dimensional problem. We know for a regular polygon that the centre of gravity will be the point where all angle bisectors cross. Just like previously, we still need the centre of gravity on the other side of the edge, not above the current face. We can now require that the angle between current face and bisector is bigger than $\frac{\pi}{2}$ this angle is exactly $\frac{beta}{2}$, since the bisector divides the internal angle in two.

$$\begin{aligned} \frac{\beta}{2} + \alpha &> \frac{\pi}{2} \\ \frac{\beta}{2} &> \frac{\pi}{2} - \alpha \\ \beta &> \pi - 2\alpha \end{aligned} \tag{3}$$

The internal angle of a regular polygon is given by:

$$\begin{aligned} \beta &= \frac{n\pi - 2\pi}{n} \\ \frac{n\pi - 2\pi}{n} &> \pi - 2\alpha \\ \pi - \frac{2\pi}{n} &> \pi - 2\alpha \\ -\frac{2\pi}{n} &> -2\alpha \\ \frac{2\pi}{n} &< 2\alpha \\ \frac{n}{2\pi} &> \frac{1}{2\alpha} \\ n &> \frac{\pi}{\alpha} \end{aligned} \tag{4}$$

We round up to get to the nearest integer and we add two, since in the two-dimensional case we have not accounted for the two base faces of the prism, giving us the formula as given in 3.4. □

4 Summary

4.1 Discussion

The lower bound can be improved upon quite a bit, it does not account for the location of the centre of gravity, since for every step the proof assumes that the centre of gravity can be placed exactly so that the polyhedron rolls, and the next step the centre of gravity moves again. So to improve the lower bound, the centre of gravity needs to be included.
If it is possible to show that in the 2 dimensional case the optimal solution possible is a regular polygon than the lower bound can be improved to $n \geq \left\lceil \frac{\pi}{\alpha} \right\rceil$ since the projection of our three dimensional polyhedron has the same requirements as a polygon in two dimensional space.

4.2 Conclusion

It is also possible to show that an infinitely rolling polyhedron of four faces can be constructed for a slanted plane with angle $\alpha > \frac{\pi}{4}$, after that it becomes unclear and the upper bound gives 7 faces for $\alpha < \frac{\pi}{4}$, so the question becomes if polyhedra with 5 or 6 faces are optimal in any case.
In this paper I tried to expand a known model for rolling polyhedra from the horizontal plane to the slanted plane. I found mainly that giving a strong relation between the angle of the slanted plane to the horizontal plane and the amount of faces is rather complicated, mainly because the slanted plane adds the additional difficulty that the direction of rolling also matters.

References

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